Chapter 27
On the Exponential and Trigonometric q, ω-Special Functions

Thomas Ernst

Abstract The purpose of this article is to continue the study of q, ω-special functions in the spirit of Wolfgang Hahn from the previous papers by Annaby et al. and Varma et al. By introducing the new variable ω, we develop a quite similar calculus consisting of two dual exponential, hyperbolic and trigonometric functions. The concept even and odd functions is replaced by x, ω-even and odd, since a change of sign in x is always accompanied by a change of sign in ω. In the same way, formulas for chain rule, Leibniz theorem, q, ω-additions for the three above functions are introduced. Graphs for these functions are shown, which closely resemble the original ones. To enable trigonometric formulas with half argument and de Moivre theorem, Ward numbers and q, ω-rational numbers are introduced.

Keywords q, ω-special functions · q, ω-difference operator · q, ω-rational number · Similar graphs · Rules for zeros

Mathematics Subject Classification (2010) 05A30

27.1 Introduction

Our first aim is to generalize the q-difference operator by introducing the variable ω. Then the ε operator with the same name as before is used to compute $\mathrm{D}_{q,\omega}$ of the second exponential and for the Leibniz theorem. Let $\omega \in \mathbb{R}, \ 0 < \omega < 1$. Put $\omega_0 \equiv \frac{\omega}{1-q}, \ 0 < q < 1$. The x, ω even and odd functions are used to simplify q, ω trigonometric formulas, which were proved by generalizations of Euler's formula. The two basic sequences are x, ω-even for even exponents and x, ω-odd for odd exponents, which motivates the definitions of two new exponential functions. The first exponential function can be given as power series in the first basic sequence with domain a subset of three variables x, q, ω in $\mathbb{R}^3$, bounded in x by two hyperspheres,

T. Ernst (✉)
Department of Mathematics, Uppsala University, P.O. Box 480, 751 06 Uppsala, Sweden
e-mail: thomas@math.uu.se; thomas.ernst948@gmail.com

S. Silvestrov et al. (eds.), *Algebraic Structures and Applications*,
Springer Proceedings in Mathematics & Statistics 317,
https://doi.org/10.1007/978-3-030-41850-2_27

the first one centered at ω_0, with radius $\frac{1}{1-q}$, the second one with radius 1. The second exponential function can be given as power series in the second basic sequence with domain $|\omega| < 1$. The corresponding meromorphic continuations are equal to quotients of infinite q-shifted factorials, similar to q-calculus. Limit formulas for $x \to \infty$ and inequalities valid for $x > \omega_0$ can then be proved in a straightforward way. In the next section we define the q,ω-addition followed by the q,ω-real numbers, the exposition closely resembles the related q-addition. The two q,ω-additions have absolute maximum only for ω in a small interval starting at 0. This interval should be dependent of x and q. Then the q,ω trigonometric and hyperbolic functions are defined and some of their graphs are shown. In a remarkable way, Euler's formulas appear again, and the differentiation formulas are quite similar. By introducing q,ω-rational numbers, we are able to prove trigonometric and hyperbolic formulas corresponding to half arguments.

27.2 Preliminary Definitions and Theorems

We first generalize some definitions and theorems from q-calculus by simply adding an index ω.

Definition 27.1 The automorphism ε on the vector space of polynomials is defined by

$$\varepsilon f(x) \equiv f(qx+\omega). \tag{27.1}$$

This automorphism is a generalization of the operator with the same name in q-calculus [2]. In [1, p. 136] it is proved that

$$\varepsilon^k f(x) = f(q^k x + \omega\{k\}_q). \tag{27.2}$$

Definition 27.2 Let φ be a continuous real function of x. Then we define the q,ω-difference operator $\mathrm{D}_{q,\omega}$ as follows:

$$\mathrm{D}_{q,\omega}(\varphi)(x) \equiv \begin{cases} \frac{\varphi(qx+\omega)-\varphi(x)}{(q-1)x+\omega}, & \text{if } x \neq \omega_0; \\ \frac{d\varphi}{dx}(x) & \text{if } x = \omega_0. \end{cases} \tag{27.3}$$

We say that a function $f(x)$ is n times q,ω-differentiable if $\mathrm{D}^n_{q,\omega} f(x)$ exists. If we want to point out that this operator operates on the variable x, we write $\mathrm{D}_{q,\omega,x}$ for the operator. Furthermore, $\mathrm{D}_{q,\omega}(K) = 0$, like for the derivative.

This operator interpolates between two well-known operators, the Nørlund difference operator

$$\triangle_\omega[f(x)] \equiv \frac{f(x+\omega)-f(x)}{\omega}, \tag{27.4}$$

and the Jackson q-derivative

$$\left(\mathrm{D}_q\varphi\right)(x) \equiv \begin{cases} \frac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{if } q \in \mathbb{C}\backslash\{1\},\ x \neq 0; \\ \frac{d\varphi}{dx}(x) & \text{if } q = 1; \end{cases} \tag{27.5}$$

Furthermore, we need the following chain rule:

Definition 27.3

$$\mathrm{D}_{q,\omega}\left(\varepsilon^k\varphi\right)(x) \equiv q^k \frac{\varepsilon^{k+1}\varphi(x) - \varepsilon^k\varphi(x)}{(q-1)x+\omega}. \tag{27.6}$$

The motivation for formula (27.6) is that it is identical with the q-calculus case and enables smooth proofs of the following formulas, like the Leibniz formula. It also follows from the chain rule (27.16).

Definition 27.4 A q, ω-analogue of the mathematical object G is a mathematical function $F(q, \omega)$, with the property $\lim_{\omega\to 0} F(q, \omega) = G_q$, the q-analogue of G. Both F and G can depend on more, common variables. They can also be operators.

Theorem 27.1 *The q, ω-difference operator is linear*

$$D_{q,\omega} \sum_{k=0}^{\infty} a_k f_k(x) = \sum_{k=0}^{\infty} a_k D_{q,\omega} f_k(x). \tag{27.7}$$

Proof This is obvious, since the definition of $\mathrm{D}_{q,\omega}$ is linear in the function.

Theorem 27.2 ([1, (16), p. 137]) *The q, ω-difference operator for a product of functions.*

$$\mathrm{D}_{q,\omega}(fg)(x) = \mathrm{D}_{q,\omega}(f(x))g(x) + f(qx+\omega)\mathrm{D}_{q,\omega}(g(x)). \tag{27.8}$$

Theorem 27.3 ([1, (17), p. 137]) *The q, ω-difference operator for a quotient of functions.*

$$\mathrm{D}_{q,\omega}\left(\frac{f}{g}\right)(x) = \frac{\mathrm{D}_{q,\omega}(f)(x)g(x) - f(x)\mathrm{D}_{q,\omega}(g)(x)}{g(x)g(qx+\omega)}, \tag{27.9}$$

where $g(x)g(qx+\omega) \neq 0$.

We now introduce two basic sequences, which generalize the Ciglerian polynomials in [2, (5.5)].

Definition 27.5

$$(x)^k_{q,\omega} \equiv \prod_{m=0}^{k-1}(x - \omega\{m\}_q). \quad \text{(see [6, (16)])} \tag{27.10}$$

$$[x]^k_{q,\omega} \equiv \prod_{m=0}^{k-1}(q^m x + \omega\{m\}_q). \quad \text{(see [6, (15)])} \tag{27.11}$$

Theorem 27.4 *We have the following special cases of the basic sequences.*

$$(x)^0_{q,\omega} = 1,\ (x)^1_{q,\omega} = x,\ (x)^k_{q,x} = 0,\ k \geq 2. \tag{27.12}$$

$$[x]^0_{q,\omega} = 1,\ [x]^1_{q,\omega} = x,\ [x]^k_{q,\omega} = 0,\ k \geq 2,\ \omega = -\frac{q^m x}{\{m\}_q},\ 0 < m < k. \tag{27.13}$$

This will be used in the comments to the four q,ω-additions. The following names will be used for the ensuing q,ω-trigonometric and hyperbolic functions.

Definition 27.6 A function f of two variables x,ω is called x,ω-even if $f(-x,-\omega) = f(x,\omega)$. A function f of two variables x,ω is called x,ω-odd if $f(-x,-\omega) = -f(x,\omega)$.

Lemma 27.1 *Products and sums of any number of x,ω-even functions are x,ω-even. The product and quotient of an x,ω-even function and an x,ω-odd function are x,ω-odd.*

Lemma 27.2 *The two functions $(x)^{2k}_{q,\omega}$ and $[x]^{2k}_{q,\omega}$ are x,ω-even. The two functions $(x)^{2k+1}_{q,\omega}$ and $[x]^{2k+1}_{q,\omega}$ are x,ω-odd.*

The two following formulas correspond to the formula $\mathrm{D}x^n = nx^{n-1}$:

$$\mathrm{D}_{q,\omega}(x)^n_{q,\omega} = \{n\}_q(x)^{n-1}_{q,\omega}. \quad \text{(see [5, 2.5], [6, (17)])} \tag{27.14}$$

$$\mathrm{D}_{q,\omega}[x]^n_{q,\omega} = \{n\}_q[qx+\omega]^{n-1}_{q,\omega}. \quad \text{(see [6, (18)])} \tag{27.15}$$

Theorem 27.5 *The chain rule for the q,ω-difference operator.*

$$\mathrm{D}_{q,\omega}\left((ax)^n_{q,a\omega}\right) = a\{n\}_q(ax)^{n-1}_{q,a\omega}. \tag{27.16}$$

$$\mathrm{D}_{q,\omega}\left([ax]^n_{q,a\omega}\right) = a\{n\}_q[aqx+a\omega]^{n-1}_{q,a\omega}. \tag{27.17}$$

Proof We prove (27.16) by induction. The formula (27.16) is true for $n = 1,2$. Assume that it is true for $n-1$. Then we have

$$\begin{aligned} &\mathrm{D}_{q,\omega}\left[(ax)^{n-1}_{q,a\omega}(ax - \{n-1\}_q a\omega)\right] \\ &\overset{\text{by}(27.8)}{=} a(ax)^{n-1}_{q,a\omega} + a^2\left[qx+\omega-\{n-1\}_q\right]\{n-1\}_q(ax)^{n-2}_{q,a\omega} \\ &= a(ax)^{n-1}_{q,a\omega}\left[1+q\{n-1\}_q\right] = \text{RHS}. \end{aligned} \tag{27.18}$$

Formula (27.17) is proved in a similar style.

We now give an improved proof of the following crucial theorem.

Theorem 27.6 (The Leibniz' q, ω-theorem [1, (25), p. 138]) *Let $f(x)$ and $g(x)$ be n times q, ω-differentiable functions on I. Then the product $fg(x)$ is also n times q, ω-differentiable and*

$$\mathrm{D}_{q,\omega}^{n}(fg)(x) = \sum_{k=0}^{n} \binom{n}{k}_q \mathrm{D}_{q,\omega}^{k}(f)(\varepsilon^{n-k}x)\mathrm{D}_{q,\omega}^{n-k}(g)(x), \ x \neq \omega_0. \tag{27.19}$$

Proof For $n = 1$ the formula above becomes (27.8). Assume that the formula is proved for $n = m$. Then it is also true for $n = m + 1$, because

$$\begin{aligned}
&\mathrm{D}_{q,\omega}^{m+1}(fg)(x) = \mathrm{D}_{q,\omega}(\mathrm{D}_{q,\omega}^{m}(fg)(x)) = \\
&= \mathrm{D}_{q,\omega} \sum_{k=0}^{m} \binom{m}{k}_q \mathrm{D}_{q,\omega}^{k}(f)(\varepsilon^{m-k}x)\mathrm{D}_{q,\omega}^{m-k}(g)(x) = \\
&\overset{\text{by (27.6), (27.8)}}{=} \sum_{k=0}^{m} \binom{m}{k}_q \big(q^{m-k}\mathrm{D}_{q,\omega}^{k+1}(f)(\varepsilon^{m-k}x)\mathrm{D}_{q,\omega}^{m-k}(g)(x)+ \\
&+ \mathrm{D}_{q,\omega}^{k}(f)(\varepsilon^{m+1-k}x)\mathrm{D}_{q,\omega}^{m+1-k}(g)(x)\big) = \\
&= \sum_{k=0}^{m} \binom{m}{k}_q \mathrm{D}_{q,\omega}^{k}(f)(\varepsilon^{m+1-k}x)\mathrm{D}_{q,\omega}^{m+1-k}(g)(x)+ \\
&+ \sum_{k=1}^{m+1} \binom{m}{k-1}_q q^{m+1-k}\mathrm{D}_{q,\omega}^{k}(f)(\varepsilon^{m+1-k}x)\mathrm{D}_{q,\omega}^{m+1-k}(g)(x) = \\
&= f(\varepsilon^{m+1}x)\mathrm{D}_{q,\omega}^{m+1}(g)(x) + \sum_{k=1}^{m} \left(\binom{m}{k}_q + q^{m+1-k}\binom{m}{k-1}_q\right) \times \\
&\times \mathrm{D}_{q,\omega}{}^{k}(f)(\varepsilon^{m+1-k}x)\mathrm{D}_{q,\omega}^{m+1-k}(g)(x) + \mathrm{D}_{q,\omega}^{m+1}(f)(x)g(x) = \\
&\overset{\text{by [2, 6.90]}}{=} \sum_{k=0}^{m+1} \binom{m+1}{k}_q \mathrm{D}_{q,\omega}^{k}(f)(\varepsilon^{m+1-k}x)\mathrm{D}_{q,\omega}^{m+1-k}(g)(x).
\end{aligned} \tag{27.20}$$

We next introduce two q, ω-analogues of the exponential function:

Definition 27.7 The q, ω-exponential function $\mathrm{E}_{q,\omega}(z)$ (see [6, (21)]) is defined by

$$\mathrm{E}_{q,\omega}(z) \equiv \sum_{k=0}^{\infty} \frac{(z)_{q,\omega}^{k}}{\{k\}_q!}, \ |(1-q)z - \omega| < 1. \tag{27.21}$$

The complementary q, ω-exponential function $\mathrm{E}_{\frac{1}{q},\omega}(z)$ (see [6, (26)]) is defined by

$$\mathrm{E}_{\frac{1}{q},\omega}(z) \equiv \sum_{k=0}^{\infty} \frac{[z]_{q,\omega}^{k}}{\{k\}_q!}, \ |\omega| < 1. \tag{27.22}$$

Theorem 27.7 *The q, ω-exponential function is the unique solution of the first order initial value problem* [6, (19)]:

$$\mathrm{D}_{q,\omega} f(z) = f(z), \ f(0) = 1. \tag{27.23}$$

The complementary q, ω-exponential function is the unique solution of the first order initial value problem [6, (24)]:

$$\mathrm{D}_{q,\omega} f(z) = f(qz + \omega), \ f(0) = 1. \tag{27.24}$$

Theorem 27.8 *The meromorphic continuation of the q, ω-exponential function $\mathrm{E}_{q,\omega}(z)$ is given by* [6, (21)]*:*

$$\mathrm{E}_{q,\omega}(z) = \frac{(-\omega; q)_\infty}{((1-q)z - \omega; q)_\infty}. \tag{27.25}$$

The meromorphic continuation of the complementary q, ω-exponential function $\mathrm{E}_{\frac{1}{q},\omega}(z)$ is given by [6, (26)]*:*

$$\mathrm{E}_{\frac{1}{q},\omega}(z) = \frac{((q-1)z + \omega; q)_\infty}{(\omega; q)_\infty}. \tag{27.26}$$

Corollary 27.1 ([6])

$$\mathrm{E}_{q,\omega}(z)\mathrm{E}_{\frac{1}{q},-\omega}(-z) = 1 \tag{27.27}$$

Theorem 27.9 *We have the following limit:*

$$\lim_{x\to\infty} \mathrm{E}_{\frac{1}{q},\omega}(x) = \infty, \ 0 < q < 1. \tag{27.28}$$

Proof We put $\mathrm{E}_{n,q}(x) \equiv (-x(1-q) + \omega; q)_n$.

Then $\mathrm{E}_{\frac{1}{q},\omega}(x) = (\omega; q)_\infty^{-1} \lim_{n\to\infty} \mathrm{E}_{n,q}(x)$. For q fixed, $0 < q < 1$, choose x so big that

$$|1 + q^k[x(1-q) - \omega]| > a > 1, \ k \in \mathbb{N}. \tag{27.29}$$

This means $|\mathrm{E}_{n,q}(x)| > a^n$ and

$$\lim_{n\to\infty} |\mathrm{E}_{n,q}(x)| > \lim_{n\to\infty} a^n = \infty. \tag{27.30}$$

Theorem 27.10 *The function $\mathrm{E}_{q,\omega}(x)$ oscillates between $\pm\infty$ if*

$$\lim_{x\to\infty}, \; 0 < q < 1.$$

Proof Consider the function

$$f(x) \equiv \mathrm{E}_{\frac{1}{q},-\omega}(-x) = \frac{(x(1-q)-\omega; q)_\infty}{(-\omega; q)_\infty}. \tag{27.31}$$

This function has infinitely many zeros $x_m = (1-q)^{-1}(\omega + q^{-m})$ with accumulation point $+\infty$. Our original function $\mathrm{E}_{q,\omega}(x) = \frac{1}{f(x)}$ then goes $\to \pm\infty$.

We now prove two inequalities for the two q, ω-exponential functions.

Theorem 27.11 (A generalization of [2, (6.160)]). *An inequality for* $\mathrm{E}_{q,\omega}(-x)$ *holds:*

$$\mathrm{E}_{q,-\omega}(-x) > e^{-x+\omega_0}(\omega; q)_\infty, \; 0 < q < 1, \;\; x > \omega_0. \tag{27.32}$$

Proof Denote

$$P_N(x) \equiv \prod_{k=0}^{N} \frac{1}{1 + [x(1-q)-\omega]q^k}. \tag{27.33}$$

Then we have

$$P_N(x) > \exp\left(-\sum_{k=0}^{N}[x(1-q)-\omega]q^k\right) = \exp\left(-(x-\omega_0)(1-q^{N+1})\right) \tag{27.34}$$

which implies

$$\mathrm{E}_{q,-\omega}(-x) = (\omega; q)_\infty \lim_{N\to\infty} P_N(x) > e^{-x+\omega_0}(\omega; q)_\infty. \tag{27.35}$$

Corollary 27.2 (A generalization of [2, (6.164)]) *An inequality for* $\mathrm{E}_{\frac{1}{q},\omega}(x)$ *holds:*

$$E_{\frac{1}{q},\omega}(x) < e^{x-\omega_0}(\omega; q)_\infty^{-1}, \; x > \omega_0, \; 0 < q < 1. \tag{27.36}$$

27.3 On the q, ω-Addition with Applications to q, ω-Special Functions

In order to use these functions, we need to generalize the q-addition. The ordinary q-addition is the special case $\omega = 0$. Just like for the q-addition, we use letters in an alphabet for the q, ω-additions. Equality between letters is denoted by $\sim$. In the following, beware of the fact that whenever we multiply the function argument x in $(x)_{q,\omega}^{\nu}$ or in $[x]_{q,\omega}^{\nu}$ by the constant a, we must also multiply ω by a.

Definition 27.8 The NWA q, ω-addition is defined as follows:

$$(x \oplus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} (y)_{q,\omega}^k. \tag{27.37}$$

The NWA q, ω-subtraction is defined as follows:

$$(x \ominus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} (-y)_{q,-\omega}^k. \tag{27.38}$$

The JHC q, ω-addition is defined as follows:

$$(x \boxplus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} [y]_{q,\omega}^k. \tag{27.39}$$

The JHC q, ω-subtraction is defined as follows:

$$(x \boxminus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} [-y]_{q,-\omega}^k. \tag{27.40}$$

Theorem 27.12 *The NWA q, ω-addition is commutative and associative.*

Proof Similar to the proof for NWA q-addition.

Example 27.1 We have the following special cases for $x = y = \omega$:

$$\begin{aligned} &(x \oplus_{q,x} x)^1 = (x \boxplus_{q,x} x)^1 = 2x, \\ &(x \oplus_{q,x} x)^2 = (1+q)x^2, \ (x \boxplus_{q,x} x)^2 = 2(1+q)x^2, \\ &(x \oplus_{q,x} x)^n = 0, \ n \geq 3. \end{aligned} \tag{27.41}$$

Proof Use formulas (27.12) and (27.13).

Definition 27.9 For an arbitrary set M, let $\langle M \rangle$ denote the set generated by M together with the four operations (27.37)–(27.40). For a given letter $a_{q,\omega}$ in (27.44), assume that

1. The sums (27.37)–(27.40) do not change sign as functions of the exponent n.
2. These sums do not first decrease to a minimum and then increase.

Then the set $\mathbb{R}_{q,\omega}$ is defined as follows.

$$\mathbb{R}_{q,\omega} \equiv \langle \mathbb{R} \rangle. \tag{27.42}$$

Like for the q-addition [2, p. 25], it turns out that these two q, ω-additions, as functions of the exponent n, first increase to a maximum value and then decrease to zero.

Like for the q-real numbers, this happens only in a certain set $J_{m_0,\ldots,m_{j-1}}$, defined as follows.

Definition 27.10 Given an integer k, the formula

$$m_0 + m_1 + \cdots + m_{j-1} = k \tag{27.43}$$

determines a set $J_{m_0,\ldots,m_{j-1}} \in \mathbb{N}^j$. If $a_{q,\omega}$ is the q, ω-real number with j letters $\oplus_{q,\omega,l=0}^{j-1} a_l$, $|a_l| < 1$, $\forall l$, its k'th power is given by

$$(\oplus_{q,\omega,l=0}^{j-1} a_l)^k \equiv (a_0 \oplus_{q,\omega} a_1 \oplus_{q,\omega} \ldots)^k \equiv \sum_{|m|=k} \prod_{m_l \in J_{m_0,\ldots,m_{j-1}}} (a_l)_{q,\omega}^{m_l} \binom{k}{\mathbf{m}}_q, \tag{27.44}$$

where for each JHC-addition in a_i, we change from $(a_l)_{q,\omega}$ to $[a_l]_{q,\omega}$.

Conjecture 27.1 *For certain values $\{a_l\}_{l=0}^{j-1}$, within a convex set inside the hypercube with length 1 in $\mathbb{R}^j$, the q, ω-real number $a_{q,\omega}$, with j letters, and with k'th power given by (27.44) has an absolute maximum.*

If there is no absolute maximum, and the function in (27.44) is infinite for some exponent k, we say that there is no q, ω-real number $a_{q,\omega}$. As further confirmation of our hypothesis, we show the following tables, which display n-values for this maximum.

Example 27.2 Define the following function:

$$F(n)_{a,0.88,\omega} : n \to (a \oplus_{0.88,\omega} a)^n, 0 < a < 1, \ 0 < \omega < 1. \tag{27.45}$$

The table gives the n-values for the maximum of F

a, ω	n	a, ω	n	a, ω	n	a, ω	n
0.54, 0.05	1	0.54, 0.1	3	0.54, 0.15	1	0.54, 0.2	3
0.55, 0.05	2	0.55, 0.1	4	0.55, 0.15	1	0.55, 0.2	4
0.6, 0.05	3	0.6, 0.1	7	0.6, 0.15	2	0.6, 0.2	7
0.65, 0.05	4	0.65, 0.1	11	0.65, 0.15	2	0.65, 0.2	11
0.7, 0.05	6	0.7, 0.1	14	0.7, 0.15	3	0.7, 0.2	14
0.8, 0.05	9	0.8, 0.1	24	0.8, 0.15	4	0.8, 0.2	24

Example 27.3 Define the following function:

$$G(n)_{a,0.88,\omega} : n \to (a \boxplus_{0.88,\omega} a)^n, 0 < a < 1, \ 0 < \omega < 1. \tag{27.46}$$

The table gives the n-values for the maximum of G

a,ω	n	a,ω	n	a,ω	n	a,ω	n
0.54, 0.025	2	0.54, 0.05	2	0.54, 0.075	2	0.54, 0.1	2
0.55, 0.025	2	0.55, 0.05	2	0.55, 0.075	2	0.55, 0.1	2
0.6, 0.025	4	0.6, 0.05	4	0.6, 0.075	4	0.6, 0.1	4
0.65, 0.025	5	0.65, 0.05	5	0.65, 0.075	5	0.65, 0.1	6
0.7, 0.025	7	0.7, 0.05	6	0.7, 0.075	7	0.7, 0.1	8
0.8, 0.025	11	0.8, 0.05	9	0.8, 0.075	11	0.8, 0.1	12

Corollary 27.3 *An extension of the formula [2, (4.29)]*

$$\mathrm{D}_{q,\omega,x}(x \oplus_{q,\omega} y)^n=\{n\}_q(x \oplus_{q,\omega} y)^{n-1},\ \oplus_{q,\omega} \equiv \oplus_{q,\omega} \vee \boxplus_{q,\omega}. \tag{27.47}$$

Proof

$$\mathrm{D}_{q,\omega,x}(x \oplus_{q,\omega} y)^n \overset{\text{by}(27.14)}{=} \sum_{k=0}^{n-1} \binom{n}{k}_q \{n-k\}_q (x)_{q,\omega}^{n-k-1}(y)_{q,\omega}^k = \text{RHS}. \tag{27.48}$$

Corollary 27.4 *Four q,ω-additions for the q,ω-exponential function.*

$$\mathrm{E}_{q,\omega}(x \oplus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{q,\omega}(y). \tag{27.49}$$

$$\mathrm{E}_{q,\omega}(x \ominus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{q,-\omega}(-y). \tag{27.50}$$

$$\mathrm{E}_{q,\omega}(x \boxplus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{\frac{1}{q},\omega}(y). \tag{27.51}$$

$$\mathrm{E}_{q,\omega}(x \boxminus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{\frac{1}{q},-\omega}(-y). \tag{27.52}$$

Definition 27.11 The corresponding q,ω-trigonometric functions are:

$$\mathrm{Sin}_{q,\omega}(x) \equiv \sum_{k=0}^{\infty}(-1)^k \frac{(x)_{q,\omega}^{2k+1}}{\{2k+1\}_q!},\ |(1-q)x-\omega|<1. \tag{27.53}$$

$$\mathrm{Cos}_{q,\omega}(x) \equiv \sum_{k=0}^{\infty}(-1)^k \frac{(x)_{q,\omega}^{2k}}{\{2k\}_q!},\ |(1-q)x-\omega|<1. \tag{27.54}$$

$$\mathrm{Sin}_{\frac{1}{q},\omega}(x) \equiv \sum_{k=0}^{\infty} \frac{(-1)^k[x]_{q,\omega}^{2k+1}}{\{2k+1\}_q!},\ |\omega|<1. \tag{27.55}$$

$$\mathrm{Cos}_{\frac{1}{q},\omega}(x) \equiv \sum_{k=0}^{\infty} \frac{(-1)^k[x]_{q,\omega}^{2k}}{\{2k\}_q!},\ |\omega|<1. \tag{27.56}$$

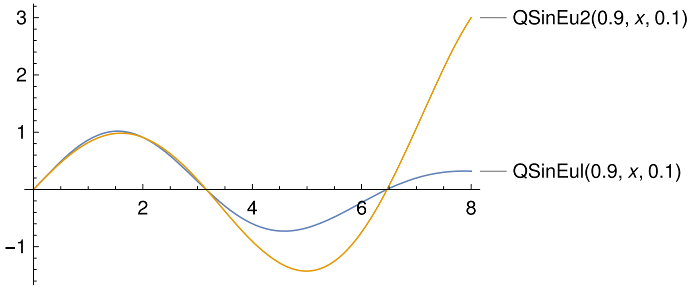

Fig. 27.1 $\mathrm{Sin}_{q,\omega}(x),\ \mathrm{Sin}_{\frac{1}{q},\omega}(x)$

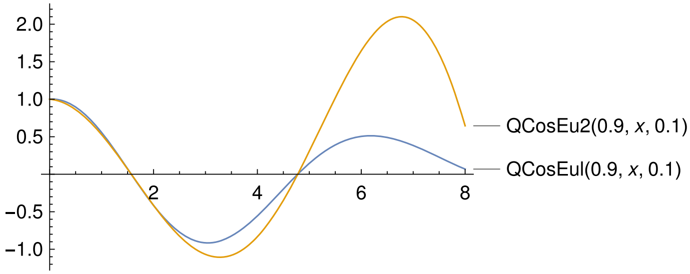

Fig. 27.2 $\mathrm{Cos}_{q,\omega}(x),\ \mathrm{Cos}_{\frac{1}{q},\omega}(x)$

Like in [2, p. 222], assume $q = 0.9$ and $0 \leq x \leq 8$. Figure 27.1 shows $\mathrm{Sin}_{q,\omega}(x)$, $\mathrm{Sin}_{\frac{1}{q},\omega}(x)$ and Fig. 27.2 shows $\mathrm{Cos}_{q,\omega}(x),\ \mathrm{Cos}_{\frac{1}{q},\omega}(x)$ for $\omega = 0.1$.

Definition 27.12 The corresponding q, ω-hyperbolic functions are:

$$\mathrm{Sinh}_{q,\omega}(x) \equiv \sum_{k=0}^{\infty} \frac{(x)_{q,\omega}^{2k+1}}{\{2k+1\}_q!},\ |(1-q)x-\omega| < 1. \tag{27.57}$$

$$\mathrm{Cosh}_{q,\omega}(x) \equiv \sum_{k=0}^{\infty} \frac{(x)_{q,\omega}^{2k}}{\{2k\}_q!},\ |(1-q)x-\omega| < 1. \tag{27.58}$$

$$\mathrm{Sinh}_{\frac{1}{q},\omega}(x) \equiv \sum_{k=0}^{\infty} \frac{[x]_{q,\omega}^{2k+1}}{\{2k+1\}_q!},\ |\omega| < 1. \tag{27.59}$$

$$\mathrm{Cosh}_{\frac{1}{q},\omega}(x) \equiv \sum_{k=0}^{\infty} \frac{[x]_{q,\omega}^{2k}}{\{2k\}_q!},\ |\omega| < 1. \tag{27.60}$$

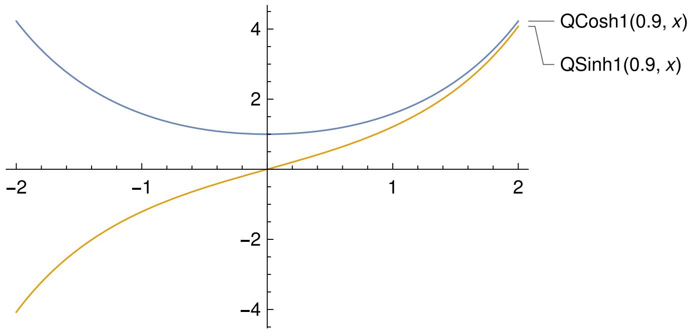

Fig. 27.3 $[\mathrm{Sinh}_q(x),\ \mathrm{Cosh}_q(x)]$

$$\mathrm{Tanh}_{q,\omega}(x) \equiv \frac{\mathrm{Sinh}_{q,\omega}(x)}{\mathrm{Cosh}_{q,\omega}(x)},\ |(1-q)x-\omega| < 1. \tag{27.61}$$

$$\mathrm{Coth}_{q,\omega}(x) \equiv \frac{\mathrm{Cosh}_{q,\omega}(x)}{\mathrm{Sinh}_{q,\omega}(x)},\ |(1-q)x-\omega| < 1. \tag{27.62}$$

$$\mathrm{Tanh}_{\frac{1}{q},\omega}(x) \equiv \frac{\mathrm{Sinh}_{\frac{1}{q},\omega}(x)}{\mathrm{Cosh}_{\frac{1}{q},\omega}(x)},\ |\omega| < 1. \tag{27.63}$$

$$\mathrm{Coth}_{\frac{1}{q},\omega}(x) \equiv \frac{\mathrm{Cosh}_{\frac{1}{q},\omega}(x)}{\mathrm{Sinh}_{\frac{1}{q},\omega}(x)},\ |\omega| < 1. \tag{27.64}$$

We have chosen not to use the names for inverse ratios of hyperbolic functions.

Our next aim is to show pictures of q,ω-hyperbolic functions which resemble the four basic graphs for hyperbolic functions and their inverse ratios. Each of these pictures contain two functions, just like in the elementary textbooks. We choose five examples; in order to show the similarity with the q-hyperbolic functions from [2, p. 229 f.], we begin with five graphs of the latter functions. Everywhere we have $q = 0.9$. Figures 27.3, 27.4, 27.5, 27.6 and 27.7 show $[\mathrm{Sinh}_q(x),\ \mathrm{Cosh}_q(x)]$, $[\mathrm{Sinh}_{\frac{1}{q}}(x),\ \mathrm{Cosh}_{\frac{1}{q}}(x)]$, $[\mathrm{Cosh}_q(x),\ (\mathrm{Cosh}_q(x))^{-1}]$, $[\mathrm{Sinh}_q(x),\ (\mathrm{Sinh}_q(x))^{-1}]$ and $[\mathrm{Tanh}_q(x),\ \mathrm{Coth}_q(x)]$, respectively.

Figures 27.8, 27.9, 27.10, 27.11 and 27.12 show $[\mathrm{Sinh}_{q,.1}(x),\ \mathrm{Cosh}_{q,.1}(x)]$, $[\mathrm{Sinh}_{\frac{1}{q},.1}(x),\ \mathrm{Cosh}_{\frac{1}{q},.1}(x)]$, $[\mathrm{Cosh}_{q,.2}(x),\ (\mathrm{Cosh}_{q,.2}(x))^{-1}]$, $[\mathrm{Sinh}_{q,.2}(x),\ (\mathrm{Sinh}_{q,.2}(x))^{-1}]$ and $[\mathrm{Tanh}_{q,.1}(x),\ \mathrm{Coth}_{q,.1}(x)]$, respectively.

The above definitions are obviously equivalent to the following formulas:

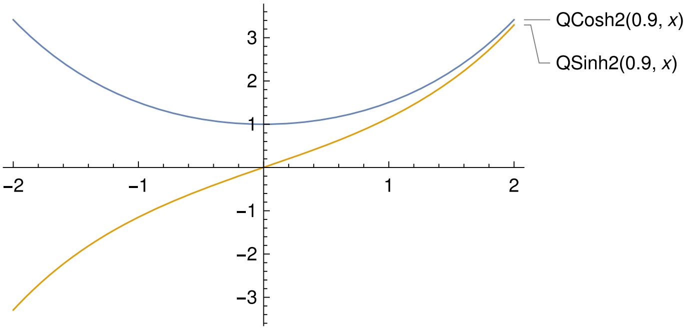

Fig. 27.4 $[\mathrm{Sinh}_{\frac{1}{q}}(x),\ \mathrm{Cosh}_{\frac{1}{q}}(x)]$

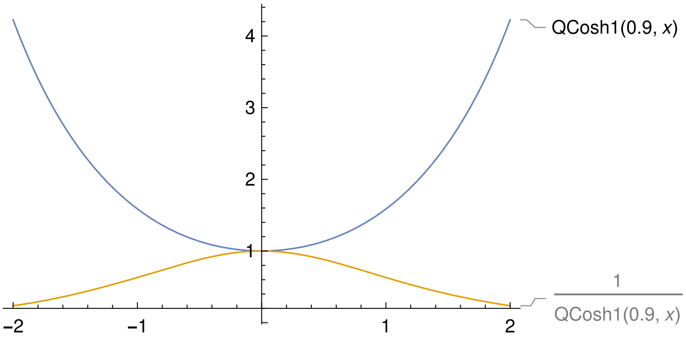

Fig. 27.5 $[\mathrm{Cosh}_q(x),\ (\mathrm{Cosh}_q(x))^{-1}]$

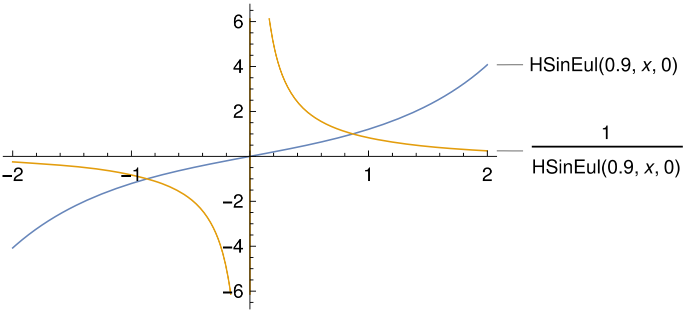

Fig. 27.6 $[\mathrm{Sinh}_q(x),\ (\mathrm{Sinh}_q(x))^{-1}]$

$$\mathrm{Sin}_{q,\omega}(x) = \frac{1}{2i}(\mathrm{E}_{q,i\omega}(ix) - \mathrm{E}_{q,-i\omega}(-ix)), \tag{27.65}$$

$$\mathrm{Cos}_{q,\omega}(x) = \frac{1}{2}(\mathrm{E}_{q,i\omega}(ix) + \mathrm{E}_{q,-i\omega}(-ix)), \tag{27.66}$$

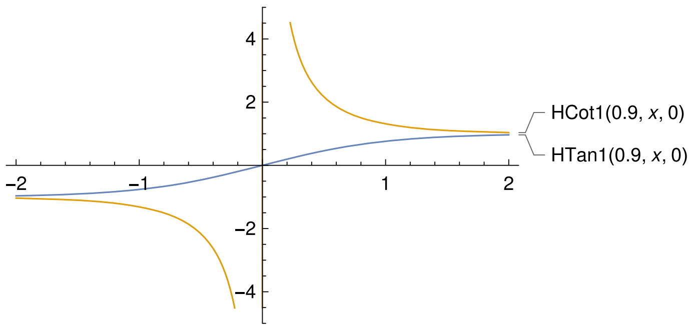

Fig. 27.7 $[\mathrm{Tanh}_q(x),\ \mathrm{Coth}_q(x)]$

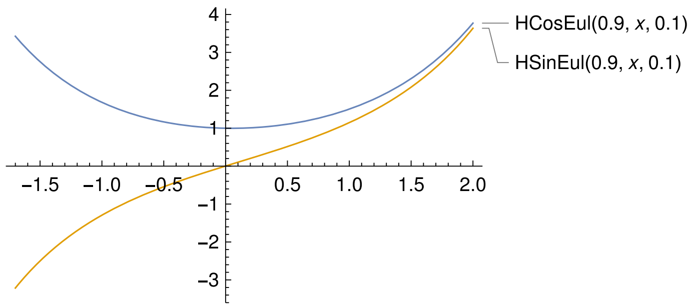

Fig. 27.8 $[\mathrm{Sinh}_{q,.1}(x),\ \mathrm{Cosh}_{q,.1}(x)]$

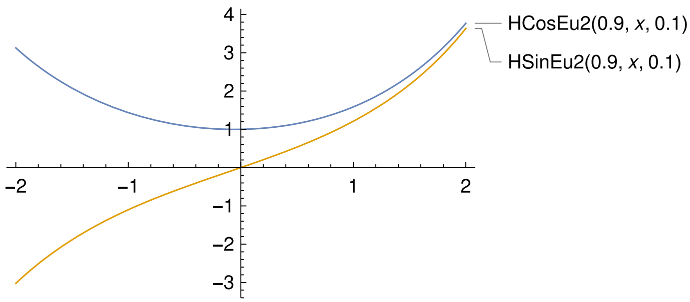

Fig. 27.9 $[\mathrm{Sinh}_{\frac{1}{q},.1}(x),\ \mathrm{Cosh}_{\frac{1}{q},.1}(x)]$

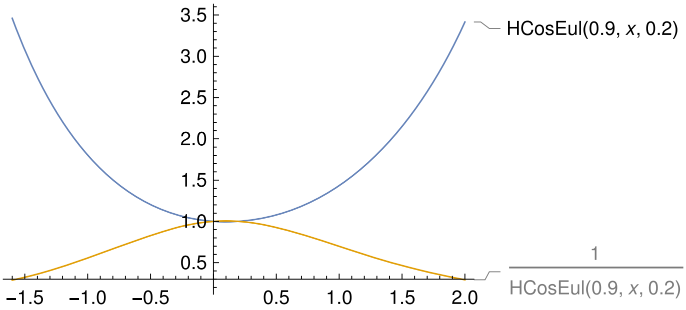

Fig. 27.10 $[\mathrm{Cosh}_{q,.2}(x),\ (\mathrm{Cosh}_{q,.2}(x))^{-1}]$

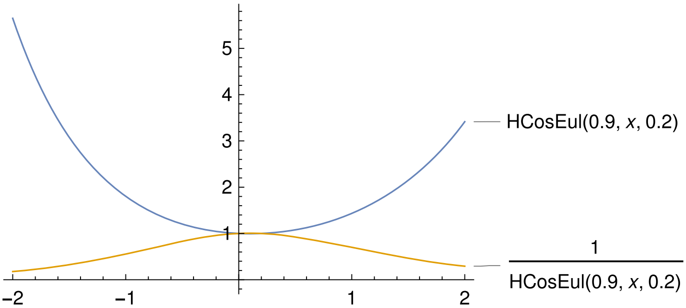

Fig. 27.11 $[\mathrm{Sinh}_{q,.2}(x),\ (\mathrm{Sinh}_{q,.2}(x))^{-1}]$

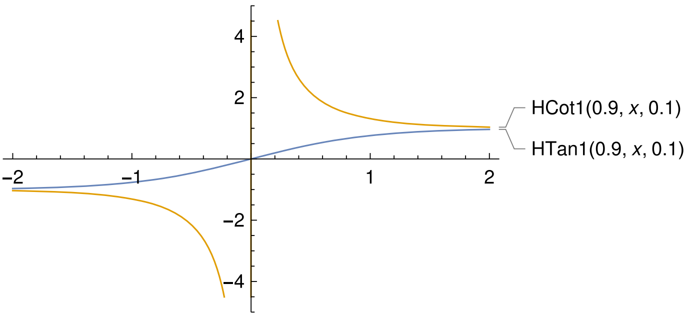

Fig. 27.12 $[\mathrm{Tanh}_{q,.1}(x),\ \mathrm{Coth}_{q,.1}(x)]$

$$\mathrm{Sinh}_{q,\omega}(x) = \frac{1}{2}(\mathrm{E}_{q,\omega}(x) - \mathrm{E}_{q,-\omega}(-x)), \tag{27.67}$$

$$\mathrm{Cosh}_{q,\omega}(x) = \frac{1}{2}(\mathrm{E}_{q,\omega}(x) + \mathrm{E}_{q,-\omega}(-x)), \tag{27.68}$$

The following theorem is needed for the correct formulation of the next q,ω-addition formulas.

Theorem 27.13 *The two functions $Cos_{q,\omega}$ and $Cos_{\frac{1}{q},\omega}$ are x,ω-even. The six functions $Sin_{q,\omega}$, $Sin_{\frac{1}{q},\omega}$, $Tan_{q,\omega}$, $Cot_{q,\omega}$, $Tan_{\frac{1}{q},\omega}$, $Cot_{\frac{1}{q},\omega}$ are x,ω-odd. The same applies to the corresponding q,ω-hyperbolic functions.*

Proof This follows from the corresponding properties of the two functions $(x)_{q,\omega}^k$ and $[x]_{q,\omega}^k$.

Theorem 27.14 *New q,ω Euler formulas:*

$$\mathrm{E}_{q,i\omega}(ix) = Cos_{q,\omega}(x) + iSin_{q,\omega}(x), \tag{27.69}$$

$$\mathrm{E}_{q,\omega}(x) = Cosh_{q,\omega}(x) + Sinh_{q,\omega}(x), \tag{27.70}$$

Proof Add formulas (27.65), (27.66), and (27.67), (27.68), respectively.

Theorem 27.15 *The q,ω-differences for the q,ω-exponential functions are:*

$$\mathrm{D}_{q,\omega}\,\mathrm{E}_{q,a\omega}(ax) = a\,\mathrm{E}_{q,a\omega}(ax), \tag{27.71}$$

$$\mathrm{D}_{q,\omega}\,\mathrm{E}_{\frac{1}{q},a\omega}(ax) = a\,\mathrm{E}_{\frac{1}{q},a\omega}(aqx + a\omega), \tag{27.72}$$

The q,ω-differences for the q,ω-trigonometric functions are:

$$\mathrm{D}_{q,\omega}\,Cos_{q,a\omega}(ax) = -a\,Sin_{q,a\omega}(ax), \tag{27.73}$$

$$\mathrm{D}_{q,\omega}Sin_{q,b\omega}(bx) = b\,Cos_{q,b\omega}(bx). \tag{27.74}$$

The q,ω-differences for the q,ω-hyperbolic functions are:

$$\mathrm{D}_{q,\omega}\,Cosh_{q,a\omega}(ax) = a\,Sinh_{q,a\omega}(ax), \tag{27.75}$$

$$\mathrm{D}_{q,\omega}\,Sinh_{q,b\omega}(bx) = b\,Cosh_{q,b\omega}(bx). \tag{27.76}$$

The functions $Cos_{q,a\omega}(ax)$ and $Sin_{q,a\omega}(ax)$ are solutions to the q,ω-difference equation

$$\mathrm{D}_{q,\omega}^2 f(x) = -a^2 f(x), \tag{27.77}$$

with initial values $f(0) = 1$ and $f(0) = 0$ respectively.

The functions $Cosh_{q,a\omega}(ax)$ and $Sinh_{q,a\omega}(ax)$ are solutions to the q,ω-difference equation

$$\mathrm{D}_{q,\omega}^2 f(x) = a^2 f(x), \tag{27.78}$$

with initial values $f(0) = 1$ *and* $f(0) = 0$ *respectively.*

Again, the following formulas closely resemble the ordinary ones. The following umbral numbers can only be function arguments in formal power series.

In the book [4] we introduced several new q-deformed number systems, semiring, biring etc., each with an extra index q. By a miracle, we can extend these number systems by adding another index ω. The proofs will be very similar, and we just state the definitions and corresponding theorems.

Definition 27.13 The Ward-ω number $\overline{n}_{q,\omega}$ is defined by

$$\overline{n}_{q,\omega} \sim 1 \oplus_{q,\omega} 1 \oplus_{q,\omega} \ldots \oplus_{q,\omega} 1, \tag{27.79}$$

where the number of 1 on the RHS is n.

Definition 27.14 Let $(\mathbb{N}_{\oplus_{q,\omega}}, \oplus_{q,\omega}, \odot_{q,\omega})$ denote the semiring of Ward-ω numbers $\overline{k}_{q,\omega},\ k \geq 0$ together with two binary operations: $\oplus_{q,\omega}$ is the Ward q, ω-addition. The multiplication $\odot_{q,\omega}$ is defined as follows:

$$\overline{n}_{q,\omega} \odot_{q,\omega} \overline{m}_{q,\omega} \sim \overline{nm}_{q,\omega}, \tag{27.80}$$

where $\sim$ denotes the equivalence in the alphabet.

Theorem 27.16 *Functional equations for Ward-ω numbers operating on the q, omega-exponential function. First assume that the letters $\overline{m}_{q,\omega}$ and $\overline{n}_{q,\omega}$ are independent, i.e. come from two different functions, when operating with the functional. Furthermore, $mnt < \frac{1+\omega}{1-q}$. Then we have*

$$E_{q,\omega}(\overline{m}_{q,\omega}\overline{n}_{q,\omega}t) = E_{q,\omega}(\overline{mn}_{q,\omega}t). \tag{27.81}$$

Furthermore,

$$E_{q,\omega}(\overline{jm}_{q,\omega}) = E_{q,\omega}(\overline{j}_{q,\omega})^m = E_{q,\omega}(\overline{m}_{q,\omega})^j = E_{q,\omega}(\overline{j}_{q,\omega} \odot_{q,\omega} \overline{m}_{q,\omega}). \tag{27.82}$$

We will now extend this semiring to a graded commutative biring.

Definition 27.15 Let the q, ω-integers $(\mathbb{Z}_{q,\omega}, \oplus_{q,\omega}, \boxplus_{q,\omega}, \odot_{q,\omega}, \overline{0}_{q,\omega})$ denote $\pm$ the Ward ω numbers, i.e. $\mathbb{Z}_{q,\omega} \equiv \mathbb{N}_{\oplus_{q,\omega}} \cup -\mathbb{N}_{\oplus_{q,\omega}}$, where there are two inverse q-additions $\oplus_{q,\omega}$ and $\boxplus_{q,\omega}$. $\overline{0}_{q,\omega}$ denotes the zero θ, and $\overline{1}_{q,\omega}$ denotes the multiplicative identity. The dual addition is defined by

$$\overline{n}_{q,\omega} \boxplus_{q,\omega} -\overline{m}_{q,\omega} \sim \overline{n-m}_{q,\omega}, n \geq m. \tag{27.83}$$

Furthermore, the multiplication $\odot_{q,\omega}$ is defined by (27.80) and

$$\overline{n}_{q,\omega} \odot_{q,\omega} -\overline{m}_{q,\omega} \sim -\overline{nm}_{q,\omega}. \tag{27.84}$$

Finally, we define

$$\overline{-m}_{q,\omega} \equiv -\overline{m}_{q,\omega}. \tag{27.85}$$

Theorem 27.17 *An extension of [2, p. 167]. Assume that $\mathbb{Z}_{q,\omega}$ is defined by the previous definition. Then $(\mathbb{Z}_{q,\omega}, \oplus_{q,\omega}, \boxplus_{q,\omega}, \odot_{q,\omega}, \overline{0}_{q,\omega})$ is a graded commutative biring.*

Definition 27.16 An extension of [3, p. 4]. Let the q,ω-rational numbers $\mathbb{Q}_{\oplus_{q,\omega}}$ be defined as follows:

$$\mathbb{Q}_{\oplus_{q,\omega}} \equiv \left\{ \frac{\overline{m}_{q,\omega}}{\overline{n}_{q,\omega}},\ m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N},\ m \neq n,\ \frac{\overline{0}_{q,\omega}}{\overline{n}_{q,\omega}} \sim \theta,\ \frac{\overline{n}_{q,\omega}}{\overline{n}_{q,\omega}} \sim 1 \right\}, \tag{27.86}$$

together with a linear functional

$$v,\ \mathbb{R}[x] \times \mathbb{Q}_{\oplus_{q,\omega}} \to \mathbb{R}, \tag{27.87}$$

called the evaluation. If $v(x) = \sum_{k=0}^{n} a_k x^k$, then

$$v\left(\frac{\overline{m}_{q,\omega}}{\overline{n}_{q,\omega}}\right) \equiv \sum_{k=0}^{n} a_k \frac{(\overline{m}_{q,\omega})^k}{(\overline{n}_{q,\omega})^k}. \tag{27.88}$$

Definition 27.17 An extension of [2, (4.70)]:

$$(\overline{n}_{q,\omega})^k \equiv \sum_{m_1+\cdots+m_n=k} \binom{k}{m_1, \ldots, m_n}_q \prod_{i=1}^{n} (1)_{q,\omega}^{m_i}, \tag{27.89}$$

where each partition of k is multiplied with its number of permutations. We have the following special cases:

$$(\overline{0}_{q,\omega})^k = \delta_{k,0};\ (\overline{n}_{q,\omega})^0 = 1;\ (\overline{n}_{q,\omega})^1 = n. \tag{27.90}$$

All of the following exponential, trigonometric and hyperbolic formulas are only valid for ω far away from 1.

Theorem 27.18

$$\mathrm{E}_{q,\omega}(\overline{n}_{q,\omega}) = (\mathrm{E}_{q,\omega}(1))^n. \tag{27.91}$$

Lemma 27.3 *Eight q,ω addition theorems for q,ω trigonometric functions.*

$$Cos_{q,\omega}(x)Cos_{q,-\omega}(-y) - Sin_{q,\omega}(x)Sin_{q,-\omega}(-y) = Cos_{q,\omega}(x \ominus_{q,\omega} y). \tag{27.92}$$

$$Cos_{q,\omega}(x)Cos_{q,\omega}(y) - Sin_{q,\omega}(x)Sin_{q,\omega}(y) = Cos_{q,\omega}(x \oplus_{q,\omega} y). \tag{27.93}$$

$$Sin_{q,\omega}(x)Cos_{q,\omega}(y) + Sin_{q,\omega}(y)Cos_{q,\omega}(x) = Sin_{q,\omega}(x \oplus_{q,\omega} y). \tag{27.94}$$

$$Sin_{q,\omega}(x)Cos_{q,-\omega}(-y) + Sin_{q,-\omega}(-y)Cos_{q,\omega}(x) = Sin_{q,\omega}(x \ominus_{q,\omega} y). \tag{27.95}$$

$$Cos_{q,\omega}(x)Cos_{\frac{1}{q},-\omega}(-y) - Sin_{q,\omega}(x)Sin_{\frac{1}{q},-\omega}(-y) = Cos_{q,\omega}(x \boxminus_{q,\omega} y). \tag{27.96}$$

$$Cos_{q,\omega}(x)Cos_{\frac{1}{q},\omega}(y) - Sin_{q,\omega}(x)Sin_{\frac{1}{q},\omega}(y) = Cos_{q,\omega}(x \boxplus_{q,\omega} y). \tag{27.97}$$

$$Sin_{q,\omega}(x)Cos_{\frac{1}{q},\omega}(y) + Sin_{\frac{1}{q},\omega}(y)Cos_{q,\omega}(x) = Sin_{q,\omega}(x \boxplus_{q,\omega} y). \tag{27.98}$$

$$Sin_{q,\omega}(x)Cos_{\frac{1}{q},-\omega}(-y) + Sin_{\frac{1}{q},-\omega}(-y)Cos_{q,\omega}(x) = Sin_{q,\omega}(x \boxminus_{q,\omega} y). \tag{27.99}$$

Proof Use formulas (27.49)–(27.52). We prove (27.96):

$$\begin{aligned}
&\mathrm{Cos}_{q,\omega}(x \boxminus_{q,\omega} y) \overset{\text{by}(27.66)}{=} \frac{1}{2}(\mathrm{E}_{q,i\omega}(ix \boxminus_{q,i\omega} iy) + \mathrm{E}_{q,-i\omega}(-ix \boxplus_{q,i\omega} iy)) \\
&\overset{\text{by}(27.52)}{=} \frac{1}{2}\left[\mathrm{E}_{q,i\omega}(ix)\mathrm{E}_{\frac{1}{q},-i\omega}(-iy) + \mathrm{E}_{q,-i\omega}(-ix)\mathrm{E}_{\frac{1}{q},i\omega}(iy)\right] \overset{\text{by}(27.66)}{=} \text{LHS}.
\end{aligned} \tag{27.100}$$

These formulas can be simplified by using Theorem 27.13.

Theorem 27.19 *Eight q,ω addition theorems for q,ω trigonometric functions with only positive function arguments on the LHS.*

$$Cos_{q,\omega}(x)Cos_{q,\omega}(y) + Sin_{q,\omega}(x)Sin_{q,\omega}(y) = Cos_{q,\omega}(x \ominus_{q,\omega} y). \tag{27.101}$$

$$Cos_{q,\omega}(x)Cos_{q,\omega}(y) - Sin_{q,\omega}(x)Sin_{q,\omega}(y) = Cos_{q,\omega}(x \oplus_{q,\omega} y). \tag{27.102}$$

$$Sin_{q,\omega}(x)Cos_{q,\omega}(y) + Sin_{q,\omega}(y)Cos_{q,\omega}(x) = Sin_{q,\omega}(x \oplus_{q,\omega} y). \tag{27.103}$$

$$Sin_{q,\omega}(x)Cos_{q,\omega}(y) - Sin_{q,\omega}(y)Cos_{q,\omega}(x) = Sin_{q,\omega}(x \ominus_{q,\omega} y). \tag{27.104}$$

$$Cos_{q,\omega}(x)Cos_{\frac{1}{q},\omega}(y) + Sin_{q,\omega}(x)Sin_{\frac{1}{q},\omega}(y) = Cos_{q,\omega}(x \boxminus_{q,\omega} y). \tag{27.105}$$

$$Cos_{q,\omega}(x)Cos_{\frac{1}{q},\omega}(y) - Sin_{q,\omega}(x)Sin_{\frac{1}{q},\omega}(y) = Cos_{q,\omega}(x \boxplus_{q,\omega} y). \tag{27.106}$$

$$Sin_{q,\omega}(x)Cos_{\frac{1}{q},\omega}(y) + Sin_{\frac{1}{q},\omega}(y)Cos_{q,\omega}(x) = Sin_{q,\omega}(x \boxplus_{q,\omega} y). \tag{27.107}$$

$$Sin_{q,\omega}(x)Cos_{\frac{1}{q},\omega}(y) - Sin_{\frac{1}{q},\omega}(y)Cos_{q,\omega}(x) = Sin_{q,\omega}(x \boxminus_{q,\omega} y). \tag{27.108}$$

Corollary 27.5

$$Cos^2_{q,\omega}(x) - Sin^2_{q,\omega}(x) = Cos_{q,\omega}(\overline{2}_{q,\omega}x). \tag{27.109}$$

$$2Cos_{q,\omega}(x)Sin_{q,\omega}(x) = Sin_{q,\omega}(\overline{2}_{q,\omega}x). \tag{27.110}$$

$$Cos_{q,\omega}(x)Cos_{\frac{1}{q},\omega}(x) + Sin_{q,\omega}(x)Sin_{\frac{1}{q},\omega}(x) = 1. \tag{27.111}$$

Remark 27.1 If the series expansion is not defined, we can use the corresponding q, ω-addition formula to define the q, ω-addition for the product expansion.

Definition 27.18 Denote the k:th zero of $\mathrm{Sin}_{q,\omega}(x),\ x > 0$ by $\xi(q, k, \omega)$. Denote the k:th zero of $\mathrm{Cos}_{q,\omega}(x),\ x > 0$ by $\tau(q, k, \omega)$.

Theorem 27.20 *First equality between q, ω-trigonometric zeros.*

$$\xi(q, k, \omega) = \xi\left(\frac{1}{q}, k, \omega\right),\ k > 0. \tag{27.112}$$

Theorem 27.21 *Second equality between q, ω-trigonometric zeros.*

$$\tau(q, k, \omega) = \tau\left(\frac{1}{q}, k, \omega\right),\ k > 0. \tag{27.113}$$

Proof We prove (27.112). Use

$$\mathrm{E}_{q,i\omega}(ix)\mathrm{E}_{\frac{1}{q},-i\omega}(-ix) = 1, \tag{27.114}$$

and put $x = \xi(q, k, \omega)$. This implies $\mathrm{Sin}_{\frac{1}{q},\omega}(\xi(q, k, \omega)) = 0$, since the right hand side is real. The second equation is proved in a similar way.

Theorem 27.22

$$Sin_{q,\omega}(\tau(q, k, \omega))Sin_{\frac{1}{q},\omega}(\tau(q, k, \omega)) = 1,\ k > 0. \tag{27.115}$$

$$Cos_{q,\omega}(\xi(q, k, \omega))Cos_{\frac{1}{q},\omega}(\xi(q, k, \omega)) = 1,\ k > 0. \tag{27.116}$$

Proof Put $x = \xi(q, k, \omega)$ in (27.114).

Theorem 27.23 *The function $f(x) : x \mapsto Sin_{q,\omega}(x)Sin_{\frac{1}{q},\omega}(x)$ has extreme values for $x = \tau(q, k, \omega),\ k > 0$. The function $g(x) : x \mapsto Cos_{q,\omega}(x)Cos_{\frac{1}{q},\omega}(x)$ has extreme values for $x = \xi(q, k, \omega),\ k > 0$. These extreme values are both* 1.

Proof We prove the first statement. Differentiate formula (27.111) with respect to x and put $x = \tau(q, k, \omega)$. The second term on the left is zero, as well as the right hand side. This proves the first part. The second part follows from Theorem 27.22.

Remark 27.2 Numerical computations show that both functions $f(x)$ and $g(x)$ are positive, which means that

$$\mathrm{Sin}_{q,\omega}(x),\ \mathrm{Sin}_{\frac{1}{q},\omega}(x) \text{ and } \mathrm{Cos}_{q,\omega}(x),\ \mathrm{Cos}_{\frac{1}{q},\omega}(x)$$

have the same signs for a fixed value of x, respectively. They have the same zeros by Theorems 27.20 and 27.21. This means that the extreme values in Theorem 27.23 are maxima.

Remark 27.3 Theorem 27.23 does not mean that the maxima and minima of

$$\mathrm{Sin}_{q,\omega}(x),\ \mathrm{Sin}_{\frac{1}{q},\omega}(x) \text{ and } \mathrm{Cos}_{q,\omega}(x),\ \mathrm{Cos}_{\frac{1}{q},\omega}(x)$$

occur for $x = \tau(q, k, \omega)$ and $x = \xi(q, k, \omega)$, respectively.

Compare with the pictures in [2, p. 222]. On the basis of Theorem 27.22 and numerical computations we make a guess:

Conjecture 27.2 *The function* $f(x) = Sin_{q,\omega}(x) \vee Cos_{q,\omega}(x)$ *for fixed* $q < 1$ *and fixed* $0 < \omega < 1$*, far away from 1, oscillates between decreasing positive maximum values and increasing negative minimum values as function of* $x > 0$.

Conjecture 27.3 *The function* $f(x) = Sin_{\frac{1}{q},\omega}(x) \vee Cos_{\frac{1}{q},\omega}(x)$ *for fixed* $q < 1$ *and fixed* $0 < \omega < 1$*, far away from 1, oscillates between increasing positive maximum values and decreasing negative minimum values as function of* $x > 0$.

Definition 27.19 The q, ω-tangens and cotangens are defined by

$$\mathrm{Tan}_{q,\omega}(x) \equiv \frac{\mathrm{Sin}_{q,\omega}(x)}{\mathrm{Cos}_{q,\omega}(x)},\ x \neq \tau(q, k, \omega). \tag{27.117}$$

$$\mathrm{Cot}_{q,\omega}(x) \equiv \frac{\mathrm{Cos}_{q,\omega}(x)}{\mathrm{Sin}_{q,\omega}(x)},\ x \neq \xi(q, k, \omega). \tag{27.118}$$

Theorem 27.24 *Formulas for* q, ω*-Tangens and Cotangens.*

$$Tan_{q,\omega}(x \oplus_{q,\omega} y) = \frac{Tan_{q,\omega}(x) + Tan_{q,\omega}(y)}{1 - Tan_{q,\omega}(x)Tan_{q,\omega}(y)}. \tag{27.119}$$

$$Tan_{q,\omega}(x \ominus_{q,\omega} y) = \frac{Tan_{q,\omega}(x) - Tan_{q,\omega}(y)}{1 + Tan_{q,\omega}(x)Tan_{q,\omega}(y)}. \tag{27.120}$$

$$Cot_{q,\omega}(x \oplus_{q,\omega} y) = \frac{Cot_{q,\omega}(x)Cot_{q,\omega}(y) - 1}{Cot_{q,\omega}(x) + Cot_{q,\omega}(y)}. \tag{27.121}$$

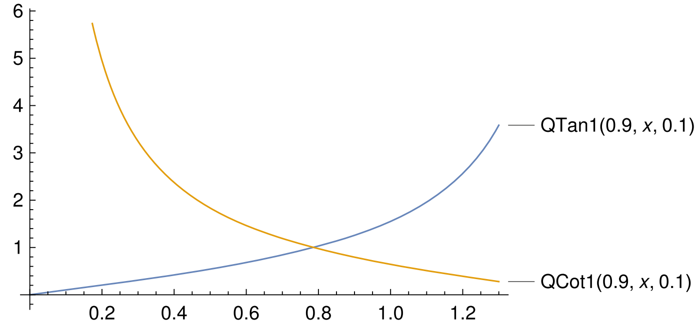

Fig. 27.13 $\mathrm{Tan}_{q,\omega}(x),\ \mathrm{Cot}_{q,\omega}(x)$

$$Cot_{q,\omega}(x \ominus_{q,\omega} y) = \frac{Cot_{q,\omega}(x)Cot_{q,\omega}(y) + 1}{Cot_{q,\omega}(y) - Cot_{q,\omega}(x)}. \tag{27.122}$$

$$Tan_{q,\omega}(\overline{2}_{q,\omega}x) = \frac{2Tan_{q,\omega}(x)}{1 - Tan^2_{q,\omega}(x)}. \tag{27.123}$$

$$Cot_{q,\omega}(\overline{2}_{q,\omega}x) = \frac{Cot^2_{q,\omega}(x) - 1}{2Cot_{q,\omega}(x)}. \tag{27.124}$$

Figure 27.13 shows $\mathrm{Tan}_{q,\omega}(x),\ \mathrm{Cot}_{q,\omega}(x)$ for $q = 0.9$ and $\omega = 0.1$.

The graphs for $\mathrm{Tan}_{\frac{1}{q},\omega}(x),\ \mathrm{Cot}_{\frac{1}{q},\omega}(x)$ closely resemble the previous figure. Again, the following formulas closely resemble the ordinary ones. They are also valid for q-analogues.

Theorem 27.25 *More q, ω-differences for quotient q, ω-trigonometric functions.*

$$\mathrm{D}_{q,\omega}Tan_{q,\omega}(x) = \frac{Cos_{q,\omega}(x \ominus_{q,\omega} x)}{Cos_{q,\omega}(x)\varepsilon Cos_{q,\omega}(x)}, \tag{27.125}$$

$$\mathrm{D}_{q,\omega}Cot_{q,\omega}(x) = -\frac{Cos_{q,\omega}(x \ominus_{q,\omega} x)}{Sin_{q,\omega}(x)\varepsilon Sin_{q,\omega}(x)}, \tag{27.126}$$

$$\mathrm{D}_{q,\omega}\left(\frac{1}{Sin_{q,\omega}(x)}\right) = -\frac{Cos_{q,\omega}(x)}{Sin_{q,\omega}(x)\varepsilon Sin_{q,\omega}(x)}, \tag{27.127}$$

$$\mathrm{D}_{q,\omega}\left(\frac{1}{Cos_{q,\omega}(x)}\right) = \frac{Sin_{q,\omega}(x)}{Cos_{q,\omega}(x)\varepsilon Cos_{q,\omega}(x)}. \tag{27.128}$$

Proof Use formulas (27.9), (27.73), and (27.74).

Remark 27.4 The numerators in (27.125) and (27.126) are q, ω-analogues of 1.

We now turn to q, ω-hyperbolic functions. All of the following hyperbolic formulas are only valid for ω far away from 1.

Lemma 27.4 *Eight q, ω addition theorems for q, ω hyperbolic functions.*

$$Cosh_{q,\omega}(x)Cosh_{q,-\omega}(-y) + Sinh_{q,\omega}(x)Sinh_{q,-\omega}(-y) = Cosh_{q,\omega}(x \ominus_{q,\omega} y). \tag{27.129}$$

$$Cosh_{q,\omega}(x)Cosh_{q,\omega}(y) + Sinh_{q,\omega}(x)Sinh_{q,\omega}(y) = Cosh_{q,\omega}(x \oplus_{q,\omega} y). \tag{27.130}$$

$$Sinh_{q,\omega}(x)Cosh_{q,\omega}(y) + Sinh_{q,\omega}(y)Cosh_{q,\omega}(x) = Sinh_{q,\omega}(x \oplus_{q,\omega} y). \tag{27.131}$$

$$Sinh_{q,\omega}(x)Cosh_{q,-\omega}(-y) + Sinh_{q,-\omega}(-y)Cosh_{q,\omega}(x) = Sinh_{q,\omega}(x \ominus_{q,\omega} y). \tag{27.132}$$

$$Cosh_{q,\omega}(x)Cosh_{\frac{1}{q},-\omega}(-y) + Sinh_{q,\omega}(x)Sinh_{\frac{1}{q},-\omega}(-y) = Cosh_{q,\omega}(x \boxminus_{q,\omega} y). \tag{27.133}$$

$$Cosh_{q,\omega}(x)Cosh_{\frac{1}{q},\omega}(y) + Sinh_{q,\omega}(x)Sinh_{\frac{1}{q},\omega}(y) = Cosh_{q,\omega}(x \boxplus_{q,\omega} y). \tag{27.134}$$

$$Sinh_{q,\omega}(x)Cosh_{\frac{1}{q},\omega}(y) + Sinh_{\frac{1}{q},\omega}(y)Cosh_{q,\omega}(x) = Sinh_{q,\omega}(x \boxplus_{q,\omega} y). \tag{27.135}$$

$$Sinh_{q,\omega}(x)Cosh_{\frac{1}{q},-\omega}(-y) + Sinh_{\frac{1}{q},-\omega}(-y)Cosh_{q,\omega}(x) = Sinh_{q,\omega}(x \boxminus_{q,\omega} y). \tag{27.136}$$

These formulas can be simplified by using Theorem 27.13.

Theorem 27.26 *Eight q, ω-addition theorems for q, ω hyperbolic functions with only positive function arguments on the LHS.*

$$Cosh_{q,\omega}(x)Cosh_{q,\omega}(y) - Sinh_{q,\omega}(x)Sinh_{q,\omega}(y) = Cosh_{q,\omega}(x \ominus_{q,\omega} y). \tag{27.137}$$

$$Cosh_{q,\omega}(x)Cosh_{q,\omega}(y) + Sinh_{q,\omega}(x)Sinh_{q,\omega}(y) = Cosh_{q,\omega}(x \oplus_{q,\omega} y). \tag{27.138}$$

$$Sinh_{q,\omega}(x)Cosh_{q,\omega}(y) + Sinh_{q,\omega}(y)Cosh_{q,\omega}(x) = Sinh_{q,\omega}(x \oplus_{q,\omega} y). \tag{27.139}$$

$$Sinh_{q,\omega}(x)Cosh_{q,\omega}(y) - Sinh_{q,\omega}(y)Cosh_{q,\omega}(x) = Sinh_{q,\omega}(x \ominus_{q,\omega} y). \tag{27.140}$$

$$Cosh_{q,\omega}(x)Cosh_{\frac{1}{q},\omega}(y) - Sinh_{q,\omega}(x)Sinh_{\frac{1}{q},\omega}(y) = Cosh_{q,\omega}(x \boxminus_{q,\omega} y). \tag{27.141}$$

$$Cosh_{q,\omega}(x)Cosh_{\frac{1}{q},\omega}(y) + Sinh_{q,\omega}(x)Sinh_{\frac{1}{q},\omega}(y) = Cosh_{q,\omega}(x \boxplus_{q,\omega} y). \tag{27.142}$$

$$Sinh_{q,\omega}(x)Cosh_{\frac{1}{q},\omega}(y) + Sinh_{\frac{1}{q},\omega}(y)Cosh_{q,\omega}(x) = Sinh_{q,\omega}(x \boxplus_{q,\omega} y). \tag{27.143}$$

$$Sinh_{q,\omega}(x)Cosh_{\frac{1}{q},\omega}(y) - Sinh_{\frac{1}{q},\omega}(y)Cosh_{q,\omega}(x) = Sinh_{q,\omega}(x \boxminus_{q,\omega} y). \tag{27.144}$$

Theorem 27.27 *The following formulas for q, ω-Tanghyp and Cothyp hold:*

$$Tanh_{q,\omega}(x \oplus_{q,\omega} y) = \frac{Tanh_{q,\omega}(x) + Tanh_{q,\omega}(y)}{1 + Tanh_{q,\omega}(x)Tanh_{q,\omega}(y)}. \tag{27.145}$$

$$Tanh_{q,\omega}(x \ominus_{q,\omega} y) = \frac{Tanh_{q,\omega}(x) - Tanh_{q,\omega}(y)}{1 - Tanh_{q,\omega}(x)Tanh_{q,\omega}(y)}. \tag{27.146}$$

$$Coth_{q,\omega}(x \oplus_{q,\omega} y) = \frac{Coth_{q,\omega}(x)Coth_{q,\omega}(y) + 1}{Coth_{q,\omega}(x) + Coth_{q,\omega}(y)}. \tag{27.147}$$

$$Coth_{q,\omega}(x \ominus_{q,\omega} y) = \frac{1 - Coth_{q,\omega}(x)Coth_{q,\omega}(y)}{Coth_{q,\omega}(x) - Coth_{q,\omega}(y)}. \tag{27.148}$$

$$Tanh_{q,\omega}(\overline{2}_{q,\omega}x) = \frac{2Tanh_{q,\omega}(x)}{1 + Tanh^2_{q,\omega}(x)}. \tag{27.149}$$

$$Coth_{q,\omega}(\overline{2}_{q,\omega}x) = \frac{Coth^2_{q,\omega}(x) + 1}{2Coth_{q,\omega}(x)}. \tag{27.150}$$

Corollary 27.6 *The following formulas hold:*

$$Cosh^2_{q,\omega}(x) + Sinh^2_{q,\omega}(x) = Cosh_{q,\omega}(\overline{2}_{q,\omega}x). \tag{27.151}$$

$$2Cosh_{q,\omega}(x)Sinh_{q,\omega}(x) = Sinh_{q,\omega}(\overline{2}_{q,\omega}x). \tag{27.152}$$

$$Cosh_{q,\omega}(x)Cosh_{\frac{1}{q},\omega}(x) - Sinh_{q,\omega}(x)Sinh_{\frac{1}{q},\omega}(x) = 1. \tag{27.153}$$

Theorem 27.28 *For q, ω-differences for quotient q, ω-hyperbolic functions*

$$\mathrm{D}_{q,\omega}Tanh_{q,\omega}(x) = \frac{Cosh_{q,\omega}(x \ominus_{q,\omega} x)}{Cosh_{q,\omega}(x)\ \varepsilon\ Cos_{q,\omega}(x)}, \tag{27.154}$$

$$\mathrm{D}_{q,\omega}Coth_{q,\omega}(x) = -\frac{Cosh_{q,\omega}(x \ominus_{q,\omega} x)}{Sinh_{q,\omega}(x)\ \varepsilon\ Sinh_{q,\omega}(x)}, \tag{27.155}$$

Remark 27.5 The numerators in (27.154) and (27.155) are q, ω-analogues of 1.

Theorem 27.29 (q, ω-de Moivre's theorem)

$$Cos_{q,\omega}(\overline{n}_{q,\omega}x) + iSin_{q,\omega}(\overline{n}_{q,\omega}x) = (Cos_{q,\omega}(x) + iSin_{q,\omega}(x))^n, \tag{27.156}$$

Proof Use formula (27.91).

We now prove some more q,ω-analogues of trigonometric formulas. These formulas are, of course, also valid for q-analogues, i.e. for $\omega = 0$. It seems that these formulas have not been published before. The q,ω-rational numbers will be used when we divide the function argument by '2'.

Theorem 27.30 *The following formulas hold:*

$$\mathrm{Cos}^2_{q,\omega}(x) = \frac{1}{2}\left(\mathrm{Cos}_{q,\omega}(x \ominus_{q,\omega} x) + \mathrm{Cos}_{q,\omega}(\overline{2}_{q,\omega}x)\right), \tag{27.157}$$

$$\mathrm{Sin}^2_{q,\omega}(x) = \frac{1}{2}\left(\mathrm{Cos}_{q,\omega}\left(x \ominus_{q,\omega} x\right) - \mathrm{Cos}_{q,\omega}\left(\overline{2}_{q,\omega}x\right)\right), \tag{27.158}$$

$$\mathrm{Sin}^2_{q,\omega}(x) + \mathrm{Cos}^2_{q,\omega}(x) = \mathrm{Cos}_{q,\omega}(x \ominus_{q,\omega} x). \tag{27.159}$$

Proof To prove (27.157), use (27.101) twice with $y = -x$ and $y = x$ and add the results.

Corollary 27.7 *The following formulas hold:*

$$\mathrm{Cos}_{q,\omega}\left(\frac{x}{\overline{2}_{q,\omega}}\right) = \pm\sqrt{\frac{\mathrm{Cos}_{q,\omega}(\frac{x\ominus_{q,\omega}x}{\overline{2}_{q,\omega}}) + \mathrm{Cos}_{q,\omega}(x)}{2}}, \tag{27.160}$$

$$\mathrm{Sin}_{q,\omega}\left(\frac{x}{\overline{2}_{q,\omega}}\right) = \pm\sqrt{\frac{\mathrm{Cos}_{q,\omega}(\frac{x\ominus_{q,\omega}x}{\overline{2}_{q,\omega}}) - \mathrm{Cos}_{q,\omega}(x)}{2}}, \tag{27.161}$$

$$\mathrm{Tan}_{q,\omega}\left(\frac{x}{\overline{2}_{q,\omega}}\right) = \pm\sqrt{\frac{\mathrm{Cos}_{q,\omega}(\frac{x\ominus_{q,\omega}x}{\overline{2}_{q,\omega}}) - \mathrm{Cos}_{q,\omega}(x)}{\mathrm{Cos}_{q,\omega}(\frac{x\ominus_{q,\omega}x}{\overline{2}_{q,\omega}}) + \mathrm{Cos}_{q,\omega}(x)}}, \tag{27.162}$$

$$\mathrm{Cot}_{q,\omega}\left(\frac{x}{\overline{2}_{q,\omega}}\right) = \pm\sqrt{\frac{\mathrm{Cos}_{q,\omega}(\frac{x\ominus_{q,\omega}x}{\overline{2}_{q,\omega}}) + \mathrm{Cos}_{q,\omega}(x)}{\mathrm{Cos}_{q,\omega}(\frac{x\ominus_{q,\omega}x}{\overline{2}_{q,\omega}}) - \mathrm{Cos}_{q,\omega}(x)}}. \tag{27.163}$$

Proof To prove (27.160), take square root of formula (27.157) and replace x by $\frac{x}{\overline{2}_{q,\omega}}$.

27.3.1 New q,ω-Hyperbolic Formulas

Analogues of three well-known formulas.

Theorem 27.31 *The following formulas hold:*

$$Cosh^2_{q,\omega}(x) = \frac{1}{2}\left(Cosh_{q,\omega}(x \ominus_{q,\omega} x) + Cosh_{q,\omega}(\overline{2}_{q,\omega}x)\right), \tag{27.164}$$

$$Sinh^2_{q,\omega}(x) = \frac{1}{2}\left(Cosh_{q,\omega}\left(\overline{2}_{q,\omega}x\right) - Cosh_{q,\omega}\left(x \ominus_{q,\omega} x\right)\right), \tag{27.165}$$

$$Cosh^2_{q,\omega}(x) - Sinh^2_{q,\omega}(x) = Cosh_{q,\omega}(x \ominus_{q,\omega} x). \tag{27.166}$$

Corollary 27.8 *The following formulas hold:*

$$Cosh_{q,\omega}\left(\frac{x}{\overline{2}_{q,\omega}}\right) = \pm\sqrt{\frac{Cosh_{q,\omega}(\frac{x\ominus_{q,\omega}x}{\overline{2}_{q,\omega}}) + Cosh_{q,\omega}(x)}{2}}, \tag{27.167}$$

$$Sinh_{q,\omega}\left(\frac{x}{\overline{2}_{q,\omega}}\right) = \sqrt{\frac{Cosh_{q,\omega}(x) - Cosh_{q,\omega}(\frac{x\ominus_{q,\omega}x}{\overline{2}_{q,\omega}})}{2}}, \tag{27.168}$$

$$Tanh_{q,\omega}\left(\frac{x}{\overline{2}_{q,\omega}}\right) = \pm\sqrt{\frac{Cosh_{q,\omega}(x) - Cosh_{q,\omega}(\frac{x\ominus_{q,\omega}x}{\overline{2}_{q,\omega}})}{Cosh_{q,\omega}(\frac{x\ominus_{q,\omega}x}{\overline{2}_{q,\omega}}) + Cosh_{q,\omega}(x)}}, \tag{27.169}$$

$$Coth_{q,\omega}\left(\frac{x}{\overline{2}_{q,\omega}}\right) = \pm\sqrt{\frac{Cosh_{q,\omega}(\frac{x\ominus_{q,\omega}x}{\overline{2}_{q,\omega}}) + Cosh_{q,\omega}(x)}{Cosh_{q,\omega}(x) - Cosh_{q,\omega}(\frac{x\ominus_{q,\omega}x}{\overline{2}_{q,\omega}})}}. \tag{27.170}$$

27.4 Conclusion

We have constructed a solid basis for the further development of finite differences and the corresponding q,ω-Appell polynomials in the same vein. At the same time this subject is also very popular among other authors and it will hopefully reach new heights in the near future. Other attempts have been made to produce new trigonometric formulas, but our approach clearly shows close resemblance to the original. This umbral approach in the spirit of Rota assumes a knowledge of the corresponding alphabets in q-calculus.

References

1. Annaby, M.H., Hamza, A.E., Aldwoah, K.A.: Hahn difference operator and associated Jackson-Nørlund integrals. J. Optim. Theory Appl. **154**(1), 133–153 (2012)
2. Ernst, T.: A Comprehensive Treatment of q-calculus. Birkhäuser, Basel (2012)
3. Ernst, T.: Multiplication formulas for q-Appell polynomials and the multiple q-power sums. Ann. Univ. Marie Curie **70**(1), 1–18 (2016)
4. Ernst, T.: A new semantics for special functions. Submitted
5. Johnson, W.P.: q-extension of identities of Abel-Rothe type. Discret. Math. **159**(1–3), 161–177 (1996)
6. Varma., S., Yasar, B., Özarslan, M.: Hahn-Appell polynomials and their d-orthogonality. Revista de la Real Academia de Ciencias Ex. (2018)