Chapter 22
Reordering in Noncommutative Algebras Associated with Iterated Function Systems

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Abstract A general class of multi-parametric families of unital associative complex algebras, defined by commutation relations associated with group or semigroup actions of dynamical systems and iterated function systems, is considered. A generalization of these commutation relations in three generators is also considered, modifying Lie algebra type commutation relations, typical for usual differential or difference operators, to relations satisfied by more general twisted difference operators associated with general twisting maps. General reordering and nested commutator formulas for arbitrary elements in these algebras are presented, and some special cases are considered, generalizing some well-known results in mathematics and physics.

Keywords Commutation relations · Nested commutator formulas · Noncommutative algebras · Reordering formulas

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22.1 Introduction

The main object considered in this paper is the multi-parametric family $\mathcal{A}_{\sigma_j}$ of unital associative complex algebras generated by the element Q and the finite or infinite set $\{S_j\}_{j\in\mathcal{J}}$ of elements satisfying the commutation relations

$$S_j Q = \sigma_j(Q) S_j, \tag{22.1}$$

where σ_j is a polynomial for all $j \in \mathcal{J}$. For $\mathcal{J} = \{1, 2\}$, with the notation that $S_1 = S$, $S_2 = T$, $\sigma_1 = \sigma$ and $\sigma_2 = \tau$, this reduces to the multi-parametric family $\mathcal{A}_{\sigma,\tau}$ of unital associative complex algebras generated by three elements S, T and Q satisfying the commutation relations

$$\begin{aligned} SQ &= \sigma(Q)S, \\ TQ &= \tau(Q)T. \end{aligned} \tag{22.2}$$

Writing $R = (dS - bT)/(ad - bc)$ and $J = (aT - cS)/(ad - bc)$, where a, b, c and d are complex numbers with $ad \neq bc$, we obtain and consider also a generalization of $\mathcal{A}_{\sigma,\tau}$, the multi-parametric family $\mathcal{B}_{\sigma,\tau}$ of unital associative complex algebras generated by three elements R, J and Q satisfying the commutation relations

$$\begin{aligned} RQ &= \frac{ad\sigma(Q) - bc\tau(Q)}{ad - bc} R + \frac{bd\sigma(Q) - bd\tau(Q)}{ad - bc} J, \\ JQ &= \frac{ad\tau(Q) - bc\sigma(Q)}{ad - bc} J + \frac{ac\tau(Q) - ac\sigma(Q)}{ad - bc} R. \end{aligned} \tag{22.3}$$

Observe that the relations of the form (22.2) are recovered for $b = c = 0$.

The importance of commutation relations (22.1) can be best seen from some well-known examples. Consider the case where $\mathcal{J} = \{1\}$, that is, the case

$$SQ = \sigma(Q)S. \tag{22.4}$$

If $\sigma(x) = x$, then S and Q commute, that is, $SQ = QS$. If $\sigma(x) = -x$, then S and Q anti-commute, that is, $SQ = -QS$. If $\sigma(x) = qx + c$ for some complex numbers q and c, then S and Q satisfy

$$SQ - qQS = cS.$$

This is a deformed Heisenberg–Lie commutation relation of quantum mechanics. The famous classical Heisenberg–Lie relation is obtained when $q = 1$ and $c = 1$. If $c = 0$, then S and Q are said to q-commute, that is, they satisfy the relation

$$SQ = qQS,$$

which is often called the quantum plane relation in the context of noncommutative geometry and quantum groups. If $\sigma(x) = qx^d$ for some positive integer d, then S and Q satisfy the commutation relation

$$SQ = qQ^d S.$$

This reduces to the quantum plane relation for $d = 1$ and to the relation

$$SQ = Q^d S$$

for $q = 1$, having important applications, for instance in wavelet analysis and in investigation of transfer operators [6, 11, 12], which are fundamental for statistical physics, dynamical systems and ergodic theory.

The commutation relations of the form (22.4) play a central role in the study of crossed products and their representations, in the theory of dynamical systems and in the investigation of covariant systems and systems of imprimitivity and thus in quantum mechanics, statistical physics and quantum field theory [6–8, 11, 12, 16–19, 30, 35, 45, 46]. The commutation relations of the form (22.4) arise in the investigations of nonlinear Poisson brackets, quantization and noncommutative analysis [13, 28]. Bounded and unbounded operators satisfying relation (22.4) have also been considered in the context of representations of $*$-algebras and spectral theory [34, 36, 37, 40, 41].

On the other hand, relations (22.3) generalizes Lie algebra type commutation relations, typical for usual differential or difference operators, to relations satisfied by more general twisted difference operators associated to general twisting maps.

This paper is devoted to the reordering of arbitrary elements in the algebras $\mathcal{A}_{\sigma_j}$, $\mathcal{A}_{\sigma,\tau}$ and $\mathcal{B}_{\sigma,\tau}$. Reordering of arbitrary elements in noncommutative algebras defined by commutation relations is important in many research directions, open problems and applications of the algebras and their operator representations. For a broader view of this active area of research, see, for example, [1–5, 9, 10, 20–22, 24, 26–29, 37, 39, 42–44, 47] and the references therein. In investigation of the structure, representations and applications of noncommutative algebras, an important role is played by the explicit description of suitable normal forms for noncommutative expressions or functions of generators. These normal forms are particularly important for computing commutative subalgebras or commuting families of operators which are a key ingredient in representation theory of many important algebras [15, 31–33, 38, 45].

In Sect. 22.2, we give an introduction to commutation relations and reordering. In Sect. 22.3 general reordering formulas for arbitrary elements in the family $\mathcal{A}_{\sigma_j}$ are presented, and in Sect. 22.4 some reordered expressions for corresponding nested commutators are described. In Sect. 22.5 special cases for different choices of σ_j are considered, putting in a new perspective and generalizing some well-known results in mathematics and physics. A generalization of the family $\mathcal{A}_{\sigma_j}$ in three generators is constructed in Sect. 22.6, with some reordering formulas presented in Sect. 22.7. We conclude by mentioning some operator representations of our algebras in Sect. 22.8.

We would also like to point out that some of the results in this paper are published without proofs in our recent article [27].

22.2 Commutation Relations and Reordering

This paper is about reordering of elements in noncommutative algebras defined by commutation relations. We follow the nice exposition by Mansour and Schork [20]. A commutation relation is a relation that describes the discrepancy between different orders of operation of two operations, say S and Q. To describe it, we use the commutator

$$[S, Q] \equiv SQ - QS.$$

If S and Q commute, then the commutator vanishes. How far a given structure deviates from the commutative case is described by the right-hand side of the commutation relation. For example, in a complex Lie algebra $\mathfrak{g}$ one has a set of generators $\{S_j\}_{j\in J}$ with the Lie bracket $\left[S_j S_k\right] = \sum_{l\in J} c^l_{jk} S_l$, where the coefficients $c^l_{jk} \in \mathbb{C}$ are called the structure constants of the Lie algebra $\mathfrak{g}$. The associated universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is an associative algebra generated by $\{S_j\}_{j\in J}$, and the above bracket becomes

$$\left[S_j, S_k\right] = \sum_{l\in J} c^l_{jk} S_l.$$

One of the earliest instances of a noncommutative structure was recognized in the context of operational calculus. If $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the ordinary derivative, then the Leibniz rule (the product rule) states that

$$D(xf(x)) = xD(f(x)) + D(x)f(x).$$

Interpreting the multiplication with the independent variable x as an application of the multiplication operator Q_x, and suppressing the operand f, this equation can be written as the commutation relation

$$DQ_x - Q_x D = \mathbb{1},$$

where $\mathbb{1}$ is the identity operator: $\mathbb{1} f(x) = f(x)$.

Let us first introduce the concept of an alphabet, words and letters, and thereafter explain what we mean by reordering of an element in a noncommutative algebra defined by commutation relations.

Definition 22.1 (*see Mansour and Schork, 2016* [20]) Let a finite or infinite set $\mathcal{A} = \left\{S_j\right\}_{j\in J}$ of objects be given. For all $j \in J$, we call each S_j a letter and $\mathcal{A}$ the alphabet. For some positive integer r, an element of $\mathcal{A}^r$ will be called a word of length r in the alphabet $\mathcal{A}$. A word $\omega = \big(S_{j_1}, S_{j_2}, \ldots, S_{j_r}\big)$ will be written in the form $\omega = S_{j_1} S_{j_2} \cdots S_{j_r}$, that is, as concatenation of its letters. For convenience, we also introduce the empty word $\varnothing \in \mathcal{A}^0$. If ω is a word, we denote the concatenation $\omega\omega \cdots \omega$ (n times) briefly by ω^n. In the case $\mathcal{A}$ consists of n elements, an element of $\mathcal{A}^r$ is called n-ary word of length r. The words with letters from the set of two elements ($n = 2$) are called binary words, and the words with letters from the set of three elements are called ternary words.

Example 22.1 If $\mathcal{A} = \big\{1, 2, 3\big\}$, then the 3-ary (ternary) words of length two are 11, 12, 13, 21, 22, 23, 31, 32 and 33. If $\mathcal{A} = \big\{0, 1\big\}$, then the binary words of length three are given by 000, 001, 010, 011, 100, 101, 110 and 111.

Example 22.2 Let $\mathcal{A} = \big\{S, T, U, V\big\}$ be an alphabet with four letters. Then $\omega_1 = SSSTT$, $\omega_2 = STUVS$, $\omega_3 = VTUST$ and $\omega_4 = UTUTU$ are words of length five which in general are not related. The words ω_1 and ω_4 can be written briefly as $\omega_1 = S^3T^2$ and $\omega_4 = (UT)^2U$.

Let us turn to the situation where the alphabet is given by the finite or infinite set $\mathcal{A} = \big\{S_j, Q\big\}_{j \in J}$ of elements in a unital associative algebra satisfying the commutation relation

$$S_j Q = \sigma_j(Q) S_j.$$

An arbitrary word ω in the alphabet $\mathcal{A} = \big\{S_j, Q\big\}_{j \in J}$ can be written as

$$\omega = S_{j_1}^{k_1} Q^{l_1} S_{j_2}^{k_2} Q^{l_2} \cdots S_{j_r}^{k_r} Q^{l_r} \equiv \prod_{t=1}^{r} S_{j_t}^{k_t} Q^{l_t}$$

for some $k_t, l_t \in \mathbb{N}_0$ ($\mathbb{N}_0$ denotes the set of nonnegative integers). If σ_j is given by the polynomial $\sigma_j(x) = x + 1$ for all $j \in J$, then the above commutation relation becomes the famous classical Heisenberg–Lie commutation relation

$$S_j Q - Q S_j = S_j, \tag{22.5}$$

and two adjacent letters S_j and Q in a word can be interchanged according to this relation. Each time one uses it in a word ω, two new words result. If we write the original word as $\omega = \omega_1 S_j Q \omega_2$ (where each ω_r can be the empty word), then applying (22.5) gives that $\omega = \omega_1(QS_j + S_j)\omega_2 = \omega_1 Q S_j \omega_2 + \omega_1 S_j \omega_2$.

Example 22.3 In the last sentence of the preceding paragraph, if $\omega_1 = \omega_2 = \varnothing$, the empty words, then $\omega = S_j Q$ can be written as $\omega = QS_j + S_j$. Using (22.5) again, the word $S_j Q^2$ can be written as

$$
\begin{aligned}
S_j Q^2 &= (S_j Q)Q \\
&= (QS_j + S_j)Q \\
&= QS_j Q + S_j Q \\
&= Q(QS_j + S_j) + (QS_j + S_j) \\
&= Q^2 S_j + QS_j + QS_j + S_j \\
&= Q^2 S_j + 2QS_j + S_j.
\end{aligned}
$$

As demonstrated in this example, one can use commutation relation (22.5) successively and transform each word in S_j and Q into a sum of words, where each of these words has all the powers of Q to the left. For our considerations throughout this paper, we have the following definition.

Definition 22.2 (*cf. Mansour and Schork, 2016* [20]) A word ω in the alphabet $\mathcal{A} = \{S_j, Q\}_{j\in J}$ is called normal ordered if $\omega = a_{kl_1\cdots l_r} Q^k S_{j_1}^{l_1} \cdots S_{j_r}^{l_r}$ for some $k, l_1, \ldots, l_r \in \mathbb{N}_0$, where $a_{kl_1\cdots l_r} \in \mathbb{C}$ are arbitrary coefficients depending on the exponents $k, l_1, \ldots, l_r$. An expression consisting of a sum of words is called normal ordered if each of the summands is normal ordered. The process of bringing a word (or a sum of words) into its normal ordered form is called normal ordering. Writing the word ω in its normal ordered form,

$$
\omega = \sum_{k,l_1,\ldots,l_r\in\mathbb{N}_0} A_{kl_1\cdots l_r}(\omega) Q^k \prod_{t=1}^{r} S_{j_t}^{l_t},
$$

the coefficients $A_{kl_1\cdots l_r}(\omega)$ are called the normal ordering coefficients of ω. In a similar fashion, the word $\omega = b_{k_1\cdots k_r,l} S_{j_1}^{k_1} \cdots S_{j_r}^{k_r} Q^l$ is called antinormal ordered. Writing the word ω in its antinormal ordered form,

$$
\omega = \sum_{k_1,\ldots,k_r,l\in\mathbb{N}_0} B_{k_1\cdots k_r l}(\omega) \left(\prod_{t=1}^{r} S_{j_t}^{k_t}\right) Q^l,
$$

the coefficients $B_{k_1\cdots k_r l}(\omega)$ are called the antinormal ordering coefficients of ω, and the process of doing this is called antinormal ordering. By reordering, we mean either normal ordering or antinormal ordering.

This paper is devoted to the normal ordering of arbitrary elements in the algebras $\mathcal{A}_{\sigma_j}$, $\mathcal{A}_{\sigma,\tau}$ and $\mathcal{B}_{\sigma,\tau}$ introduced in Sect. 22.1. The paper also derives reordered expressions for nested commutators using unimodal permutations.

Definition 22.3 Let n be a positive integer. A function $f\colon \{1, \ldots, n\} \to \mathbb{R}$ is said to be *unimodal* if there exists some ν such that

$$
f(1) \geq \cdots \geq f(\nu) \leq \cdots \leq f(n).
$$

A permutation of a set is a bijection from the set to itself.

For example, written as tuples, there are four unimodal permutations of the set $\{1, 2, 3\}$, namely: (3, 1, 2), (3, 2, 1), (2, 1, 3) and (1, 2, 3).

Definition 22.4 The commutator of two elements A and B of an algebra $\mathcal{A}$ is given by

$$[A, B] = AB - BA.$$

Using this definition, it is easy to see that for all $A, B, C \in \mathcal{A}$ and $p, q \in \mathbb{C}$,

(a) $[A, q\mathbb{1}] = 0 = [A, A]$,
(b) $[A, A] = 0$,
(c) $[A, B] = -[B, A]$,
(d) $[A, pB + qC] = p[A, B] + q[A, C]$,
(e) $[A, BC] = [A, B]C + B[A, C]$,
(f) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$.

22.3 Reordering Formulas for S_j, Q-Elements

In the following an algebra means a unital associative complex algebra, $\mathbb{N}_0$ the set of nonnegative integers, and $\mathbb{N}$ the set of positive integers. The basic result is the following theorem.

Theorem 22.1 *Let r be a positive integer. If Q and $\{S_j\}_{j\in J}$ are elements of an algebra satisfying (22.1), then for any nonnegative integer k and any polynomial F,*

$$S_j^k F(Q) = F\left(\sigma_j^{\circ k}(Q)\right) S_j^k, \tag{22.6}$$

$$\left(S_j^k F(Q)\right)^r = \left(\prod_{t=1}^{r} F\left(\sigma_j^{\circ tk}(Q)\right)\right) S_j^{kr}, \tag{22.7}$$

and for any nonnegative integers k_t and any polynomials F_t, where $t = 1, \dots, r$,

$$\prod_{t=1}^{r} S_{j_t}^{k_t} F_t(Q) = \left(\prod_{t=1}^{r} F_t\left((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q)\right)\right) \prod_{t=1}^{r} S_{j_t}^{k_t}, \tag{22.8}$$

where $\circ$ denotes composition of functions, $\sigma^{\circ k}$ the k-fold composition of a function σ with itself, and we adopt the convention that $\prod_{t=1}^{r} a_t = a_1 a_2 a_3 \dots a_r$.

Proof We first prove that for all positive integers l, the formula $S_j Q^l = \left(\sigma_j(Q)\right)^l S_j$ holds, and we proceed by induction. For $l = 1$, the formula follows from (22.1). Now suppose that the formula holds for some integer $l \geq 1$, then

$$\begin{aligned} S_j Q^{l+1} = (S_j Q^l) Q = \big(\sigma_j(Q)\big)^l S_j Q \\ = \big(\sigma_j(Q)\big)^l \sigma_j(Q) S_j \\ = \big(\sigma_j(Q)\big)^{l+1} S_j, \end{aligned}$$

proving the assertion. This implies that for a given polynomial $F(Q) = \sum f_l Q^l$,

$$S_j F(Q) = \sum f_l S_j Q^l = \sum f_l \big(\sigma_j(Q)\big)^l S_j = F(\sigma_j(Q)) S_j. \tag{22.9}$$

We can now prove formula (22.6) by induction on k. For $k = 1$, formula (22.6) follows from (22.9). Now suppose that (22.6) holds for some $k \geq 1$, then

$$\begin{aligned} S_j^{k+1} F(Q) = S_j \Big(S_j^k F(Q)\Big) = S_j F\Big(\sigma_j^{\circ k}(Q)\Big) S_j^k \\ = F\Big((\sigma_j^{\circ k} \circ \sigma_j)(Q)\Big) S_j S_j^k \\ = F\Big(\sigma_j^{\circ(k+1)}(Q)\Big) S_j^{k+1}, \end{aligned}$$

and this proves formula (22.6).

Next we prove formula (22.8) by induction on r. For $r = 1$, formula (22.8) follows from (22.6). Now suppose that formula (22.8) holds for some positive integer r, then

$$\begin{aligned} &\prod_{t=1}^{r+1} S_{j_t}^{k_t} F_t(Q) = \left(\prod_{t=1}^{r} S_{j_t}^{k_t} F_t(Q)\right) S_{j_{r+1}}^{k_{r+1}} F_{r+1}(Q) \\ &= \left(\prod_{t=1}^{r} F_t\Big((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q)\Big)\right)\left(\prod_{t=1}^{r} S_{j_t}^{k_t}\right) F_{r+1}\Big(\sigma_{j_{r+1}}^{\circ k_{r+1}}(Q)\Big) S_{j_{r+1}}^{k_{r+1}} \\ &= \left(\prod_{t=1}^{r} F_t\Big((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q)\Big)\right) F_{r+1}\Big((\sigma_{j_{r+1}}^{\circ k_{r+1}} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q)\Big)\left(\prod_{t=1}^{r} S_{j_t}^{k_t}\right) S_{j_{r+1}}^{k_{r+1}} \\ &= \left(\prod_{t=1}^{r+1} F_t\Big((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q)\Big)\right) \prod_{t=1}^{r+1} S_{j_t}^{k_t}, \end{aligned}$$

and this proves (22.8), which gives formula (22.7) for $j_1 = \cdots = j_r = j, k_1 = \cdots = k_r = k$ and $F_1 = \cdots = F_r = F$. □

As a corollary of Theorem 22.1, we obtain the following result for $F(x) = x^l$, a result which is useful for computing the central elements of our algebras.

Corollary 22.1 *Let r be a positive integer. If Q and $\{S_j\}_{j \in J}$ are elements of an algebra satisfying (22.1), then for any nonnegative integers k and l,*

$$S_j^k Q^l = \left(\sigma_j^{\circ k}(Q)\right)^l S_j^k, \tag{22.10}$$

$$\left(S_j^k Q^l\right)^r = \left(\prod_{t=1}^{r} \left(\sigma_j^{\circ tk}(Q)\right)^l\right) S_j^{kr}, \tag{22.11}$$

for any nonnegative integers k_t and l_t, where $t = 1, \ldots, r$,

$$\prod_{t=1}^{r} S_{j_t}^{k_t} Q^{l_t} = \left(\prod_{t=1}^{r} \left((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q)\right)^{l_t}\right) \prod_{t=1}^{r} S_{j_t}^{k_t}. \tag{22.12}$$

Theorem 22.1 also can be formulated in terms of monomials by observing that for all $k_t, N_t \in \mathbb{N}_0$, and any polynomials $F_t(Q) = \sum_{l_t=0}^{N_t} f_{l_t} Q^{l_t}$, where $t = 1, \ldots, r$,

$$\begin{aligned}\prod_{t=1}^{r} S_{j_t}^{k_t} F_t(Q) &= \prod_{t=1}^{r} \sum_{l_t=0}^{N_t} f_{l_t} S_{j_t}^{k_t} Q^{l_t} = \sum_{l_1=0}^{N_1} \sum_{l_2=0}^{N_2} \cdots \sum_{l_r=0}^{N_r} \prod_{t=1}^{r} f_{l_t} S_{j_t}^{k_t} Q^{l_t} \\ &= \sum_{(l_1,\ldots,l_r)\in I_1\times\ldots\times I_r} \left(\prod_{t=1}^{r} f_{l_t}\right) \prod_{t=1}^{r} S_{j_t}^{k_t} Q^{l_t},\end{aligned}$$

where $I_t = \{0, \ldots, N_t\}$.

We thus have the following result, which is useful for computing explicit formulas when specific polynomials are given.

Theorem 22.2 *Let r be a positive integer. If Q and $\{S_j\}_{j\in J}$ are elements of an algebra satisfying (22.1), then for any nonnegative integers k and N, and any polynomial $F(Q) = \sum_{l=0}^{N} f_l Q^l$,*

$$S_j^k F(Q) = \sum_{l=0}^{N} f_l \left(\sigma_j^{\circ k}(Q)\right)^l S_j^k, \tag{22.13}$$

$$\left(S_j^k F(Q)\right)^r = \sum_{(l_1,\ldots,l_r)\in\{0,\ldots,N\}^r} \left(\prod_{t=1}^{r} f_{l_t}\right)\left(\prod_{t=1}^{r} \left(\sigma_j^{\circ tk}(Q)\right)^{l_t}\right) S_j^{kr}, \tag{22.14}$$

and for any nonnegative integers k_t and N_t, and any polynomials $F_t(Q) = \sum_{l_t=0}^{N_t} f_{l_t} Q^{l_t}$, where $t = 1, \ldots, r$,

$$\begin{aligned}\prod_{t=1}^{r} S_{j_t}^{k_t} F_t(Q) = &\sum_{(l_1,\ldots,l_r)\in I_1\times\ldots\times I_r} \left(\prod_{t=1}^{r} f_{l_t}\right) \\ &\cdot \left(\prod_{t=1}^{r} \left((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q)\right)^{l_t}\right) \prod_{t=1}^{r} S_{j_t}^{k_t},\end{aligned} \tag{22.15}$$

where $I_t = \{0, \ldots, N_t\}$.

Example 22.4 Formula (22.6) in Theorem 22.1 implies that

$$\begin{aligned}\Big(S_{j_1}^{k_1} F_1(Q)\Big)\Big(S_{j_2}^{k_2} F_2(Q)\Big) \\ &= \Big((F_1 \circ \sigma_{j_1}^{\circ k_1})(Q) S_{j_1}^{k_1}\Big)\Big((F_2 \circ \sigma_{j_2}^{\circ k_2})(Q) S_{j_2}^{k_2}\Big) \\ &= \Big((F_1 \circ \sigma_{j_1}^{\circ k_1})(Q)\Big)\Big((F_2 \circ \sigma_{j_2}^{\circ k_2} \circ \sigma_{j_1}^{\circ k_1})(Q)\Big) S_{j_1}^{k_1} S_{j_2}^{k_2},\end{aligned}$$

as it should be with formula (22.8) for $r = 2$. For $F_t(x) = x^{l_t}$, this reduces to

$$\begin{aligned}S_{j_1}^{k_1} Q^{l_1} S_{j_2}^{k_2} Q^{l_2} &= \Big(\sigma_{j_1}^{\circ k_1}(Q)\Big)^{l_1} S_{j_1}^{k_1} \Big(\sigma_{j_2}^{\circ k_2}(Q)\Big)^{l_2} S_{j_2}^{k_2} \\ &= \Big(\sigma_{j_1}^{\circ k_1}(Q)\Big)^{l_1} \Big((\sigma_{j_2}^{\circ k_2} \circ \sigma_{j_1}^{\circ k_1})(Q)\Big)^{l_2} S_{j_1}^{k_1} S_{j_2}^{k_2},\end{aligned}$$

as it should be with formula (22.12) for $r = 2$. For $l_1 = 0$, this becomes

$$S_{j_1}^{k_1} S_{j_2}^{k_2} Q^{l_2} = \Big((\sigma_{j_2}^{\circ k_2} \circ \sigma_{j_1}^{\circ k_1})(Q)\Big)^{l_2} S_{j_1}^{k_1} S_{j_2}^{k_2},$$

and denoting $S_{j_1} = S$, $S_{j_2} = T$, $\sigma_{j_1} = \sigma$ and $\sigma_{j_2} = \tau$ yields the following instance of Theorem 22.1 for algebras generated by three generators.

Example 22.5 Let r be a positive integer, σ and τ be polynomials, and let S, T and Q be elements of an associative algebra satisfying the relations

$$\begin{aligned}SQ &= \sigma(Q)S, \\ TQ &= \tau(Q)T.\end{aligned} \tag{22.16}$$

Then for any nonnegative integers j, k, l, j_t, k_t and l_t, and any polynomials F and F_t, where $t = 1, \ldots, r$, we have

$$SQ^l = \big(\sigma(Q)\big)^l S, \tag{22.17}$$

$$TQ^l = \big(\tau(Q)\big)^l T, \tag{22.18}$$

$$SF(Q) = (F \circ \sigma)(Q)S, \tag{22.19}$$

$$TF(Q) = (F \circ \tau)(Q)T, \tag{22.20}$$

$$S^j F(Q) = (F \circ \sigma^{\circ j})(Q)S^j, \tag{22.21}$$

$$T^k F(Q) = (F \circ \tau^{\circ k})(Q)T^k, \tag{22.22}$$

$$S^j T^k F(Q) = (F \circ \tau^{\circ k} \circ \sigma^{\circ j})(Q)S^j T^k, \tag{22.23}$$

$$T^k S^j F(Q) = (F \circ \sigma^{\circ j} \circ \tau^{\circ k})(Q)T^k S^j, \tag{22.24}$$

$$\Big(S^{j_1}T^{k_1}F_1(Q)\Big)\Big(S^{j_2}T^{k_2}F_2(Q)\Big) = \Big((F_1 \circ \tau^{\circ k_1} \circ \sigma^{\circ j_1})(Q)\Big) \cdot \Big((F_2 \circ \tau^{\circ k_2} \circ \sigma^{\circ j_2} \circ \tau^{\circ k_1} \circ \sigma^{\circ j_1})(Q)\Big) S^{j_1}T^{k_1}S^{j_2}T^{k_2}, \tag{22.25}$$

$$\prod_{t=1}^{r} S^{j_t}T^{k_t}F_t(Q) = \Bigg(\prod_{t=1}^{r}(F_t \circ \tau^{\circ k_t} \circ \sigma^{\circ j_t} \circ \cdots \circ \tau^{\circ k_1} \circ \sigma^{\circ j_1})(Q)\Bigg)\prod_{t=1}^{r} S^{j_t}T^{k_t}, \tag{22.26}$$

$$\prod_{t=1}^{r} S^{j_t}T^{k_t}Q^{l_t} = \Bigg(\prod_{t=1}^{r}\Big((\tau^{\circ k_t} \circ \sigma^{\circ j_t} \circ \cdots \circ \tau^{\circ k_1} \circ \sigma^{\circ j_1})(Q)\Big)^{l_t}\Bigg)\prod_{t=1}^{r} S^{j_t}T^{k_t}, \tag{22.27}$$

$$\Big(S^{j}T^{k}F(Q)\Big)^{r} = \Bigg(\prod_{t=1}^{r}(F \circ (\tau^{\circ k} \circ \sigma^{\circ j})^{\circ t})(Q)\Bigg)(S^{j}T^{k})^{r}, \tag{22.28}$$

$$\Big(S^{j}T^{k}Q^{l}\Big)^{r} = \Bigg(\prod_{t=1}^{r}\Big((\tau^{\circ k} \circ \sigma^{\circ j})^{\circ t}(Q)\Big)^{l_t}\Bigg)(S^{j}T^{k})^{r}, \tag{22.29}$$

$$S\sigma(Q) = \sigma^{\circ 2}(Q)S, \tag{22.30}$$

$$T\sigma(Q) = (\sigma \circ \tau)(Q)T, \tag{22.31}$$

$$S\tau(Q) = (\tau \circ \sigma)(Q)S, \tag{22.32}$$

$$T\tau(Q) = \tau^{\circ 2}(Q)T. \tag{22.33}$$

Similar examples can be obtained for algebras generated by four generators, five generators, six generators and so on.

22.4 Commutator Formulas for S_j, Q-Elements

Let n be a positive integer. A function $f\colon \{1,\ldots,n\} \to \mathbb{R}$ is said to be *unimodal* if there exists some ν such that $f(1) \geq \cdots \geq f(\nu) \leq \cdots \leq f(n)$. A permutation of a set is a bijection from the set to itself. For example, written as tuples, there are four unimodal permutations of the set $\{1, 2, 3\}$, namely: (3, 1, 2), (3, 2, 1), (2, 1, 3) and (1, 2, 3). For the permutation $\rho = (3, 1, 2)$, we have $\rho(2) = 1$ and $\rho^{-1}(3) = 1$. Finally, the commutator of two elements x and y is defined by $[x, y] = xy - yx$. We now have the following proposition.

Proposition 22.1 *For all positive integers n, we have*

$$\Big[x_n, \Big[x_{n-1}, \ldots, \big[x_2, x_1\big]\ldots\Big]\Big] = \sum_{\rho \in U_n}(-1)^{n-\rho^{-1}(1)}\prod_{\nu=1}^{n} x_{\rho(\nu)}, \tag{22.34}$$

where U_n denotes the set of all unimodal permutations of the set $\{1,\ldots,n\}$.

Proof We proceed by induction. For $n = 1$, we have $x_1 = x_1$. For $n = 2$, we have

$$\left[x_2, x_1\right] = x_2x_1 - x_1x_2 = \sum_{\rho\in U_2}(-1)^{2-\rho^{-1}(1)}\prod_{\nu=1}^{2}x_{\rho(\nu)}.$$

Now suppose that (22.34) holds for some positive integer n, then

$$\begin{aligned}\Big[x_{n+1}, \big[x_n, \ldots, [x_2, x_1]\ldots\big]\Big] &= \left[x_{n+1}, \sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\prod_{\nu=1}^{n}x_{\rho(\nu)}\right]\\
&= x_{n+1}\sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\prod_{\nu=1}^{n}x_{\rho(\nu)} - \left(\sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\prod_{\nu=1}^{n}x_{\rho(\nu)}\right)x_{n+1}\\
&= \sum_{\rho\in V_{n+1}}(-1)^{n+1-\rho^{-1}(1)}\prod_{\nu=1}^{n+1}x_{\rho(\nu)} + \sum_{\rho\in W_{n+1}}(-1)^{n+1-\rho^{-1}(1)}\prod_{\nu=1}^{n+1}x_{\rho(\nu)}\\
&= \sum_{\rho\in U_{n+1}}(-1)^{n+1-\rho^{-1}(1)}\prod_{\nu=1}^{n+1}x_{\rho(\nu)},\end{aligned}$$

where V_n and W_n denotes the sets of all unimodal permutations of $\{1, \ldots, n\}$ with x_n on the left and on the right, respectively. □

Example 22.6 For $n = 3$, we have

$$\begin{aligned}\big[x_3, [x_2, x_1]\big] &= x_3(x_2x_1 - x_1x_2) - (x_2x_1 - x_1x_2)x_3\\
&= x_3x_2x_1 - x_3x_1x_2 - x_2x_1x_3 + x_1x_2x_3\\
&= \sum_{\rho\in U_3}(-1)^{3-\rho^{-1}(1)}\prod_{\nu=1}^{3}x_{\rho(\nu)}.\end{aligned}$$

Example 22.7 For $n = 4$, we have

$$\begin{aligned}\Big[x_4, \big[x_3, [x_2, x_1]\big]\Big] &= x_4(x_3x_2x_1 - x_3x_1x_2 - x_2x_1x_3 + x_1x_2x_3)\\
&\quad - (x_3x_2x_1 - x_3x_1x_2 - x_2x_1x_3 + x_1x_2x_3)x_4\\
&= +x_4x_3x_2x_1 - x_4x_3x_1x_2 - x_4x_2x_1x_3 + x_4x_1x_2x_3\\
&\quad - x_3x_2x_1x_4 + x_3x_1x_2x_4 + x_2x_1x_3x_4 - x_1x_2x_3x_4\\
&= \sum_{\rho\in U_4}(-1)^{4-\rho^{-1}(1)}\prod_{\nu=1}^{4}x_{\rho(\nu)}.\end{aligned}$$

In the following we use a more convenient notation for nested commutators:

$$\left[x_n, \ldots, x_1\right] = \Big[x_n, \big[x_{n-1}, \ldots, [x_2, x_1]\ldots\big]\Big].$$

Combining Proposition 22.1 with Theorem 22.1, we have the following reordering result.

Theorem 22.3 *Let $r_1, \ldots, r_n, n \in \mathbb{N}$. If Q and $\{S_j\}_{j \in J}$ are elements of an algebra satisfying relations (22.1), then for any $k_1, \ldots, k_n \in \mathbb{N}_0$ and any polynomials $F_1, \ldots, F_n$,*

$$\begin{aligned}&\Big[S_{j_n}^{k_n} F_n(Q), \ldots, S_{j_1}^{k_1} F_1(Q)\Big] \\ &\quad = \sum_{\rho \in U_n} (-1)^{n-\rho^{-1}(1)} \left(\prod_{\nu=1}^{n} F_{\rho(\nu)}\Big((\sigma_{j_{\rho(\nu)}}^{\circ k_{\rho(\nu)}} \circ \cdots \circ \sigma_{j_{\rho(1)}}^{\circ k_{\rho(1)}})(Q)\Big)\right) \prod_{\nu=1}^{n} S_{j_{\rho(\nu)}}^{k_{\rho(\nu)}}, \end{aligned} \tag{22.35}$$

$$\begin{aligned}&\left[\left(S_{j_n}^{k_n} F_n(Q)\right)^{r_n}, \ldots, \left(S_{j_1}^{k_1} F_1(Q)\right)^{r_1}\right] = \sum_{\rho \in U_n} (-1)^{n-\rho^{-1}(1)} \left(\prod_{\nu=1}^{n} \prod_{t=1}^{r_{\rho(\nu)}}\right. \\ &\quad \left. F_{\rho(\nu)}\Big((\sigma_{j_{\rho(\nu)}}^{\circ t k_{\rho(\nu)}} \circ \sigma_{j_{\rho(\nu-1)}}^{\circ k_{\rho(\nu-1)} r_{\rho(\nu-1)}} \circ \cdots \circ \sigma_{j_{\rho(1)}}^{\circ k_{\rho(1)} r_{\rho(1)}})(Q)\Big)\right) \prod_{\nu=1}^{n} S_{j_{\rho(\nu)}}^{k_{\rho(\nu)} r_{\rho(\nu)}}. \end{aligned} \tag{22.36}$$

Furthermore, if $r_n > \cdots > r_1 > 1$, then for $k_1, \ldots, k_{r_n} \in \mathbb{N}_0$ and polynomials $F_1, \ldots, F_{r_n}$,

$$\begin{aligned}&\left[\prod_{t=r_{n-1}+1}^{r_n} S_{j_t}^{k_t} F_t(Q) \ldots, \prod_{t=r_0+1}^{r_1} S_{j_t}^{k_t} F_t(Q)\right] = \sum_{\rho \in U_n} (-1)^{n-\rho^{-1}(1)} \left(\prod_{s=1}^{n}\right. \\ &\prod_{t=r_{\rho(s)-1}+1}^{r_{\rho(s)}} F_t\left(\left(\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_{r_{\rho(s)-1}+1}}^{\circ k_{r_{\rho(s)-1}+1}}\right) \circ \left(\sigma_{j_{\rho(s-1)}}^{\circ k_{\rho(s-1)}} \circ \cdots \circ \sigma_{j_{r_{\rho(s-1)-1}+1}}^{\circ k_{r_{\rho(s-1)-1}+1}}\right) \circ\right. \\ &\left.\left. \cdots \circ \left(\sigma_{j_{\rho(1)}}^{\circ k_{\rho(1)}} \circ \cdots \circ \sigma_{j_{r_{\rho(1)-1}+1}}^{\circ k_{r_{\rho(1)-1}+1}}\right)(Q)\right)\right) \prod_{s=1}^{n} \prod_{t=r_{\rho(s)-1}+1}^{r_{\rho(s)}} S_{j_t}^{k_t}, \end{aligned} \tag{22.37}$$

where $\rho(0) = 0$ and $r_0 = 0$.

Proof For formula (22.35), we have

$$\begin{aligned}\Big[S_{j_n}^{k_n} F_n(Q), \ldots, S_{j_1}^{k_1} F_1(Q)\Big] &= \sum_{\rho \in U_n} (-1)^{n-\rho^{-1}(1)} \prod_{\nu=1}^{n} S_{j_{\rho(\nu)}}^{k_{\rho(\nu)}} F_{\rho(\nu)}(Q) \\ &= \sum_{\rho \in U_n} (-1)^{n-\rho^{-1}(1)} \left(\prod_{\nu=1}^{n} F_{\rho(\nu)}\Big((\sigma_{j_{\rho(\nu)}}^{\circ k_{\rho(\nu)}} \circ \cdots \circ \sigma_{j_{\rho(1)}}^{\circ k_{\rho(1)}})(Q)\Big)\right) \prod_{\nu=1}^{n} S_{j_{\rho(\nu)}}^{k_{\rho(\nu)}}, \end{aligned}$$

where the first equality follows from Proposition 22.1, and the last equality follows from formula (22.8) in Theorem 22.1. For formula (22.36), we have

$$\left[\left(S_{j_n}^{k_n}F_n(Q)\right)^{r_n},\ldots,\left(S_{j_1}^{k_1}F_1(Q)\right)^{r_1}\right]=\sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\prod_{\nu=1}^{n}\left(S_{j_{\rho(\nu)}}^{k_{\rho(\nu)}}F_{\rho(\nu)}(Q)\right)^{r_{\rho(\nu)}}$$
$$=\sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\prod_{\nu=1}^{n}\left(\prod_{t=1}^{r_{\rho(\nu)}}F_{\rho(\nu)}\left(\sigma_{j_{\rho(\nu)}}^{\circ tk_{\rho(\nu)}}(Q)\right)\right)S_{j_{\rho(\nu)}}^{k_{\rho(\nu)}r_{\rho(\nu)}}$$
$$=\sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\left(\prod_{\nu=1}^{n}\prod_{t=1}^{r_{\rho(\nu)}}F_{\rho(\nu)}\left((\sigma_{j_{\rho(\nu)}}^{\circ tk_{\rho(\nu)}}\circ\sigma_{j_{\rho(\nu-1)}}^{\circ k_{\rho(\nu-1)}r_{\rho(\nu-1)}}\circ\cdots\right.\right.$$
$$\left.\left.\cdots\circ\sigma_{j_{\rho(1)}}^{\circ k_{\rho(1)}r_{\rho(1)}})(Q)\right)\right)\prod_{\nu=1}^{n}S_{j_{\rho(\nu)}}^{k_{\rho(\nu)}r_{\rho(\nu)}},$$

where the first equality follows from Proposition 22.1, the second equality follows from formula (22.7) in Theorem 22.1, and the last equality is a reordering of all the Qs to the left. For formula (22.37), we have

$$\left[\prod_{t=r_{n-1}+1}^{r_n}S_{j_t}^{k_t}F_t(Q),\ldots,\prod_{t=r_0+1}^{r_1}S_{j_t}^{k_t}F_t(Q)\right]=\sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\prod_{\nu=1}^{n}\prod_{t=r_{\rho(\nu)-1}+1}^{r_{\rho(\nu)}}S_{j_t}^{k_t}F_t(Q)$$
$$=\sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\prod_{\nu=1}^{n}\left(\prod_{t=r_{\rho(\nu)-1}+1}^{r_{\rho(\nu)}}F_t\left((\sigma_{j_t}^{\circ k_t}\circ\cdots\circ\sigma_{j_{r_{\rho(\nu)-1}+1}}^{\circ k_{r_{\rho(\nu)-1}+1}})(Q)\right)\right)\prod_{t=r_{\rho(\nu)-1}+1}^{r_{\rho(\nu)}}S_{j_t}^{k_t}$$
$$=\sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\left(\prod_{s=1}^{n}\prod_{t=r_{\rho(s)-1}+1}^{r_{\rho(s)}}F_t\left(\left(\sigma_{j_t}^{\circ k_t}\circ\cdots\circ\sigma_{j_{r_{\rho(s)-1}+1}}^{\circ k_{r_{\rho(s)-1}+1}}\right)\circ\left(\sigma_{j_{\rho(s-1)}}^{\circ k_{\rho(s-1)}}\circ\cdots\right.\right.\right.$$
$$\left.\left.\left.\cdots\circ\sigma_{j_{r_{\rho(s-1)-1}+1}}^{\circ k_{r_{\rho(s-1)-1}+1}}\right)\circ\cdots\circ\left(\sigma_{j_{\rho(1)}}^{\circ k_{\rho(1)}}\circ\cdots\circ\sigma_{j_{r_{\rho(1)-1}+1}}^{\circ k_{r_{\rho(1)-1}+1}}\right)(Q)\right)\right)\prod_{s=1}^{n}\prod_{t=r_{\rho(s)-1}+1}^{r_{\rho(s)}}S_{j_t}^{k_t},$$

where the first equality follows from Proposition 22.1, the second equality follows from formula (22.8) in Theorem 22.1, and the last equality is a reordering of all the Qs to the left. □

As a corollary of Theorem 22.3, we obtain the following result for $F(x)=x^l$.

Corollary 22.2 *Let $r_1,\ldots,r_n,n\in\mathbb{N}$. If Q and $\{S_j\}_{j\in J}$ are elements of an algebra satisfying relations (22.1), then for any $k_1,\ldots,k_n,l_1,\ldots,l_n\in\mathbb{N}_0$, one obtains*

$$\left[S_{j_n}^{k_n}Q^{l_n},\ldots,S_{j_1}^{k_1}Q^{l_1}\right]=\sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\left(\prod_{\nu=1}^{n}\left((\sigma_{j_{\rho(\nu)}}^{\circ k_{\rho(\nu)}}\circ\cdots\circ\sigma_{j_{\rho(1)}}^{\circ k_{\rho(1)}})(Q)\right)^{l_{\rho(\nu)}}\right)\prod_{\nu=1}^{n}S_{j_{\rho(\nu)}}^{k_{\rho(\nu)}},\quad(22.38)$$

$$\left[\left(S_{j_n}^{k_n}Q^{l_n}\right)^{r_n},\ldots,\left(S_{j_1}^{k_1}Q^{l_1}\right)^{r_1}\right]=\sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\Bigg(\prod_{\nu=1}^{n}\prod_{t=1}^{r_{\rho(\nu)}}$$
$$\left((\sigma_{j_{\rho(\nu)}}^{\circ tk_{\rho(\nu)}}\circ\sigma_{j_{\rho(\nu-1)}}^{\circ k_{\rho(\nu-1)}r_{\rho(\nu-1)}}\circ\cdots\circ\sigma_{j_{\rho(1)}}^{\circ k_{\rho(1)}r_{\rho(1)}})(Q)\right)^{l_{\rho(\nu)}}\Bigg)\prod_{\nu=1}^{n}S_{j_{\rho(\nu)}}^{k_{\rho(\nu)}r_{\rho(\nu)}}. \tag{22.39}$$

Furthermore, if $r_n>\cdots>r_1>1$, then for any $k_1,\ldots,k_{r_n},l_1,\ldots,l_{r_n}\in\mathbb{N}_0$, one obtains

$$\left[\prod_{t=r_{n-1}+1}^{r_n}S_{j_t}^{k_t}Q^{l_t},\ldots,\prod_{t=r_0+1}^{r_1}S_{j_t}^{k_t}Q^{l_t}\right]=\sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\Bigg(\prod_{s=1}^{n}\prod_{t=r_{\rho(s)-1}+1}^{r_{\rho(s)}}$$
$$\Bigg(\left(\sigma_{j_t}^{\circ k_t}\circ\cdots\circ\sigma_{j_{r_{\rho(s)-1}+1}}^{\circ k_{r_{\rho(s)-1}+1}}\right)\circ\left(\sigma_{j_{\rho(s-1)}}^{\circ k_{\rho(s-1)}}\circ\cdots\circ\sigma_{j_{r_{\rho(s-1)-1}+1}}^{\circ k_{r_{\rho(s-1)-1}+1}}\right)\circ \tag{22.40}$$
$$\cdots\circ\left(\sigma_{j_{\rho(1)}}^{\circ k_{\rho(1)}}\circ\cdots\circ\sigma_{j_{r_{\rho(1)-1}+1}}^{\circ k_{r_{\rho(1)-1}+1}}\right)(Q)\Bigg)^{l_t}\Bigg)\prod_{s=1}^{n}\prod_{t=r_{\rho(s)-1}+1}^{r_{\rho(s)}}S_{j_t}^{k_t},$$

where $\rho(0)=0$, $r_0=0$.

Example 22.8 By direct computation using the definition of the commutator, one obtains for any $r_1,r_2\in\mathbb{N}$ with $r_2>r_1>1$, and for any $k_1,\ldots,k_{r_2},l_1,\ldots,l_{r_2}\in\mathbb{N}_0$, that

$$\left[\prod_{t=r_1+1}^{r_2}S_{j_t}^{k_t}Q^{l_t},\prod_{t=1}^{r_1}S_{j_t}^{k_t}Q^{l_t}\right]=\left(\prod_{t=r_1+1}^{r_2}\left((\sigma_{j_t}^{\circ k_t}\circ\cdots\circ\sigma_{j_{r_1+1}}^{\circ k_{r_1+1}})(Q)\right)^{l_t}\right)$$
$$\cdot\left(\prod_{t=1}^{r_1}\left((\sigma_{j_t}^{\circ k_t}\circ\cdots\circ\sigma_{j_1}^{\circ k_1}\circ\sigma_{j_{r_2}}^{\circ k_{r_2}}\circ\cdots\circ\sigma_{j_{r_1+1}}^{\circ k_{r_1+1}})(Q)\right)^{l_t}\right)\left(\prod_{t=r_1+1}^{r_2}S_{j_t}^{k_t}\right)\left(\prod_{t=1}^{r_1}S_{j_t}^{k_t}\right)$$
$$-\left(\prod_{t=1}^{r_2}\left((\sigma_{j_t}^{\circ k_t}\circ\cdots\circ\sigma_{j_1}^{\circ k_1})(Q)\right)^{l_t}\right)\left(\prod_{t=1}^{r_1}S_{j_t}^{k_t}\right)\left(\prod_{t=r_1+1}^{r_2}S_{j_t}^{k_t}\right),$$

which agrees with formula (22.40) for $n=2$.

Example 22.9 Similarly, one can obtain for any $r_1,r_2,r_3\in\mathbb{N}$ with $r_3>r_2>r_1>1$, and for any $k_1,\ldots,k_{r_3},l_1,\ldots,l_{r_3}\in\mathbb{N}_0$, that

$$\left[\prod_{t=r_2+1}^{r_3} S_{j_t}^{k_t} Q^{l_t}, \left[\prod_{t=r_1+1}^{r_2} S_{j_t}^{k_t} Q^{l_t}, \prod_{t=1}^{r_1} S_{j_t}^{k_t} Q^{l_t}\right]\right]$$
$$= \left(\prod_{t=r_2+1}^{r_3} \left((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_{r_2+1}}^{\circ k_{r_2+1}})(Q)\right)^{l_t}\right)$$
$$\cdot \left(\prod_{t=r_1+1}^{r_2} \left((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_{r_1+1}}^{\circ k_{r_1+1}} \circ \sigma_{j_{r_3}}^{\circ k_{r_3}} \circ \cdots \circ \sigma_{j_{r_2+1}}^{\circ k_{r_2+1}})(Q)\right)^{l_t}\right)$$
$$\cdot \left(\prod_{t=1}^{r_1}\Big((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1} \circ \sigma_{j_{r_2}}^{\circ k_{r_2}} \circ \cdots \circ \sigma_{j_{r_1+1}}^{\circ k_{r_1+1}} \circ \sigma_{j_{r_3}}^{\circ k_{r_3}} \circ \ldots$$
$$\circ\, \sigma_{j_{r_2+1}}^{\circ k_{r_2+1}})(Q)\Big)^{l_t}\right)\left(\prod_{t=r_2+1}^{r_3} S_{j_t}^{k_t}\right)\left(\prod_{t=r_1+1}^{r_2} S_{j_t}^{k_t}\right)\left(\prod_{t=1}^{r_1} S_{j_t}^{k_t}\right)$$
$$- \left(\prod_{t=r_2+1}^{r_3} \left((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_{r_2+1}}^{\circ k_{r_2+1}})(Q)\right)^{l_t}\right)\left(\prod_{t=1}^{r_2}\Big((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1}\right.$$
$$\left.\circ\, \sigma_{j_{r_3}}^{\circ k_{r_3}} \circ \cdots \circ \sigma_{j_{r_2+1}}^{\circ k_{r_2+1}})(Q)\Big)^{l_t}\right)\left(\prod_{t=r_2+1}^{r_3} S_{j_t}^{k_t}\right)\left(\prod_{t=1}^{r_1} S_{j_t}^{k_t}\right)\left(\prod_{t=r_1+1}^{r_2} S_{j_t}^{k_t}\right)$$
$$- \left(\prod_{t=r_1+1}^{r_2} \left((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_{r_1+1}}^{\circ k_{r_1+1}})(Q)\right)^{l_t}\right)$$
$$\cdot \left(\prod_{t=1}^{r_1} \left((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1} \circ \sigma_{j_{r_2}}^{\circ k_{r_2}} \circ \cdots \circ \sigma_{j_{r_1+1}}^{\circ k_{r_1+1}})(Q)\right)^{l_t}\right)$$
$$\cdot \left(\prod_{t=r_2+1}^{r_3} \left((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_{r_2+1}}^{\circ k_{r_2+1}} \circ \sigma_{j_{r_1}}^{\circ k_{r_1}} \circ \cdots \circ \sigma_{j_1}^{\circ k_1} \circ \sigma_{j_{r_2}}^{\circ k_{r_2}} \circ \cdots \circ \sigma_{j_{r_1+1}}^{\circ k_{r_1+1}})(Q)\right)^{l_t}\right)$$
$$\cdot \left(\prod_{t=r_1+1}^{r_2} S_{j_t}^{k_t}\right)\left(\prod_{t=1}^{r_1} S_{j_t}^{k_t}\right)\left(\prod_{t=r_2+1}^{r_3} S_{j_t}^{k_t}\right)$$
$$+ \left(\prod_{t=1}^{r_3} \left((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q)\right)^{l_t}\right)\left(\prod_{t=1}^{r_1} S_{j_t}^{k_t}\right)\left(\prod_{t=r_1+1}^{r_2} S_{j_t}^{k_t}\right)\left(\prod_{t=r_2+1}^{r_3} S_{j_t}^{k_t}\right),$$

which again agrees with formula (22.40) for $n = 3$.

Theorem 22.3 can also be presented in terms of monomials, which is useful for computing explicit formulas when specific polynomials are given. For example, for formula (22.35) one gets the following reordering result for nested commutators.

Theorem 22.4 *Let $n \in \mathbb{N}$ with $n > 1$. If Q and $\{S_j\}_{j\in J}$ are elements of an algebra satisfying (22.1), then for any $k_t, N_t \in \mathbb{N}_0$, and any polynomials $F_t(Q) = \sum_{l_t=0}^{N_t} f_{l_t} Q^{l_t}$, where $t = 1, \ldots, n$,*

$$\Big[S_{j_n}^{k_n}F_n(Q),\ldots,S_{j_1}^{k_1}F_1(Q)\Big] = \sum_{(l_1,\ldots,l_n)\in I_1\times\ldots\times I_n}\left(\prod_{t=1}^{n} f_{l_t}\right) \sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\left(\prod_{\nu=1}^{n}\left((\sigma_{j_{\rho(\nu)}}^{\circ k_{\rho(\nu)}}\circ\cdots\circ\sigma_{j_{\rho(1)}}^{\circ k_{\rho(1)}})(Q)\right)^{l_{\rho(\nu)}}\right)\prod_{\nu=1}^{n}S_{j_{\rho(\nu)}}^{k_{\rho(\nu)}}, \tag{22.41}$$

where $I_t = \{0,\ldots,N_t\}$.

Proof For any nonnegative integers k_t and N_t, and any polynomials

$$F_t(Q) = \sum_{l_t=0}^{N_t} f_{l_t}Q^{l_t},$$

where $t = 1,\ldots,n$, we have

$$\begin{aligned}
\Big[S_{j_n}^{k_n}F_n(Q),\ldots,S_{j_1}^{k_1}F_1(Q)\Big] &= \left[\sum_{l_n=0}^{N_n} f_{l_n}S_{j_n}^{k_n}Q^{l_n},\ldots,\sum_{l_1=0}^{N_1} f_{l_1}S_{j_1}^{k_1}Q^{l_1}\right]\\
&= \sum_{l_n=0}^{N_n} f_{l_n}\ldots\sum_{l_1=0}^{N_1} f_{l_1}\Big[S_{j_n}^{k_n}Q^{l_n},\ldots,S_{j_1}^{k_1}Q^{l_1}\Big]\\
&= \sum_{(l_1,\ldots,l_n)\in I_1\times\ldots\times I_n}\left(\prod_{t=1}^{n} f_{l_t}\right)\Big[S_{j_n}^{k_n}Q^{l_n},\ldots,S_{j_1}^{k_1}Q^{l_1}\Big]\\
&= \sum_{(l_1,\ldots,l_n)\in I_1\times\ldots\times I_n}\left(\prod_{t=1}^{n} f_{l_t}\right)\\
&\qquad \sum_{\rho\in U_n}(-1)^{n-\rho^{-1}(1)}\left(\prod_{\nu=1}^{n}\left((\sigma_{j_{\rho(\nu)}}^{\circ k_{\rho(\nu)}}\circ\cdots\circ\sigma_{j_{\rho(1)}}^{\circ k_{\rho(1)}})(Q)\right)^{l_{\rho(\nu)}}\right)\prod_{\nu=1}^{n}S_{j_{\rho(\nu)}}^{k_{\rho(\nu)}},
\end{aligned}$$

where $I_t = \big\{0,\ldots,N_t\big\}$, and where the last equality follows from Corollary 22.2. □

22.5 Examples

22.5.1 When $\sigma_j(x) = -x$

Let σ_j be the polynomial $\sigma_j(x) = -x$. Then commutation relations (22.1) become

$$S_jQ = -QS_j. \tag{22.42}$$

The following lemma is useful for obtaining the reordering results.

Lemma 22.1 *For any positive integer t and any nonnegative integers $k, k_1, \dots, k_t$,*

$$\sigma_j^{\circ k}(Q) = (-1)^k Q, \tag{22.43}$$

$$(\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q) = (-1)^{\sum_{n=1}^{t} k_n} Q. \tag{22.44}$$

Proof We prove (22.43) by induction on k. For $k = 1$, the formula follows from the definition of σ_j. Now suppose that (22.43) holds for some integer $k \geq 1$, then

$$\sigma_j^{\circ(k+1)}(Q) = \sigma_j^{\circ k}\left(\sigma_j(Q)\right) = (-1)^k(-Q) = (-1)^{k+1} Q,$$

which proves (22.43). Next we prove (22.44) by induction on t. For $t = 1$, (22.44) follows from (22.43). Now suppose that (22.44) holds for some integer $t \geq 1$, then

$$\begin{aligned}(\sigma_{j_{t+1}}^{\circ k_{t+1}} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q) &= \sigma_{j_{t+1}}^{\circ k_{t+1}}\left((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q)\right) \\ &= (-1)^{k_{t+1}}(-1)^{\sum_{n=1}^{t} k_n} Q = (-1)^{\sum_{n=1}^{t} k_n + k_{t+1}} Q = (-1)^{\sum_{n=1}^{t+1} k_n} Q,\end{aligned}$$

and this proves the assertion. □

Theorem 22.5 *Let $r \in \mathbb{N}$. If Q and $\left\{S_j\right\}_{j\in J}$ are elements of an algebra satisfying (22.42), then for any nonnegative integers k and N, and any polynomial $F(Q) = \sum_{l=0}^{N} f_l Q^l$,*

$$S_j^k F(Q) = \sum_{l=0}^{N} (-1)^{kl} f_l Q^l S_j^k, \tag{22.45}$$

$$\left(S_j^k F(Q)\right)^r = \sum_{L=0}^{rN} \sum_{\substack{(l_1,\dots,l_r)\in\left\{0,\dots,N\right\}^r \\ l_1+\cdots+l_r=L}} \left(\prod_{t=1}^{r} f_{l_t}\right)(-1)^{k\sum_{t=1}^{r} t l_t} Q^L S_j^{kr}, \tag{22.46}$$

and for all $k_t, N_t \in \mathbb{N}_0$, and polynomials $F_t(Q) = \sum_{l_t=0}^{N_t} f_{l_t} Q^{l_t}$, where $t = 1, \dots, r$,

$$\prod_{t=1}^{r} S_{j_t}^{k_t} F_t(Q) = \sum_{L=0}^{N_1+\cdots+N_r} \sum_{\substack{(l_1,\dots,l_r)\in I_1\times\dots\times I_r \\ l_1+\cdots+l_r=L}} \left(\prod_{t=1}^{r} f_{l_t}\right)(-1)^{\sum_{t=1}^{r}\sum_{n=1}^{t} k_n l_t} Q^L \prod_{t=1}^{r} S_{j_t}^{k_t}, \tag{22.47}$$

where $I_t = \left\{0, \dots, N_t\right\}$ for some t.

Proof Substituting (22.43) into (22.13) and (22.14) gives (22.45) and (22.46), respectively, and substituting (22.44) into (22.15) gives (22.47). More precisely,

$$S_j^k F(Q) = \sum_{l=0}^{N} f_l \left((-1)^k Q\right)^l S_j^k = \sum_{l=0}^{N} (-1)^{kl} f_l Q^l S_j^k,$$

$$\left(S_j^k F(Q)\right)^r = \sum_{(l_1,\dots,l_r)\in\left\{0,\dots,N\right\}^r} \left(\prod_{t=1}^{r} f_{l_t}\right) \left(\prod_{t=1}^{r} \left((-1)^{kt} Q\right)^{l_t}\right) S_j^{kr}$$

$$= \sum_{L=0}^{rN} \sum_{\substack{(l_1,\dots,l_r)\in\left\{0,\dots,N\right\}^r \\ l_1+\cdots+l_r=L}} \left(\prod_{t=1}^{r} f_{l_t}\right) (-1)^{k\sum_{t=1}^{r} t l_t} Q^L S_j^{kr},$$

and for the more general formula,

$$\prod_{t=1}^{r} S_{j_t}^{k_t} F_t(Q) = \sum_{(l_1,\dots,l_r)\in I_1\times\dots\times I_r} \left(\prod_{t=1}^{r} f_{l_t}\right) \left(\prod_{t=1}^{r} \left((-1)^{\sum_{n=1}^{t} k_n} Q\right)^{l_t}\right) \prod_{t=1}^{r} S_{j_t}^{k_t}$$

$$= \sum_{L=0}^{N_1+\cdots+N_r} \sum_{\substack{(l_1,\dots,l_r)\in I_1\times\dots\times I_r \\ l_1+\cdots+l_r=L}} \left(\prod_{t=1}^{r} f_{l_t}\right) (-1)^{\sum_{t=1}^{r}\sum_{n=1}^{t} k_n l_t} Q^L \prod_{t=1}^{r} S_{j_t}^{k_t},$$

where $I_t = \left\{0, \dots, N_t\right\}$. Formula (22.46) can also be obtained from formula (22.47) by choosing $j_1 = \cdots = j_r = j$ and $k_1 = \cdots = k_r = k$. □

For the particular case where F is a monic monomial, that is, $F(Q) = Q^l$ for some nonnegative integer l, Theorem 22.5 yields the following result.

Corollary 22.3 *Let r be a positive integer. If Q and $\left\{S_j\right\}_{j\in J}$ are elements of an algebra satisfying* (22.42)*, then for all nonnegative integers k and l,*

$$S_j^k Q^l = (-1)^{kl} Q^l S_j^k, \tag{22.48}$$

$$(S_j^k Q^l)^r = (-1)^{klr(r+1)/2} Q^{lr} S_j^{kr}, \tag{22.49}$$

and for all nonnegative k_t and l_t, where $t = 1, \dots, r$,

$$\prod_{t=1}^{r} S_{j_t}^{k_t} Q^{l_t} = (-1)^{\sum_{t=1}^{r}\sum_{n=1}^{t} k_n l_t} Q^{\sum_{t=1}^{r} l_t} \prod_{t=1}^{r} S_{j_t}^{k_t}. \tag{22.50}$$

Example 22.10 For $r = 2$, we have

$$S_{j_1}^{k_1} Q^{l_1} S_{j_2}^{k_2} Q^{l_2} = (-1)^{k_1 l_1 + (k_1+k_2) l_2} Q^{l_1+l_2} S_{j_1}^{k_1} S_{j_2}^{k_2}, \tag{22.51}$$

which for $l_1 = 0$ becomes

$$S_{j_1}^{k_1} S_{j_2}^{k_2} Q^{l_2} = (-1)^{(k_1+k_2)l_2} Q^{l_2} S_{j_1}^{k_1} S_{j_2}^{k_2}, \tag{22.52}$$

and denoting $S_{j_1} = S$ and $S_{j_2} = T$, we have the following case of Corollary 22.3.

Corollary 22.4 *Let r be a positive integer. If S, T and Q are elements of an algebra satisfying the relations*

$$SQ = -QS \quad and \quad TQ = -QT, \tag{22.53}$$

then for all nonnegative integers j, k and l,

$$S^j T^k Q^l = (-1)^{(j+k)l} Q^l S^j T^k, \tag{22.54}$$
$$(S^j T^k Q^l)^r = (-1)^{lr(r+1)(j+k)/2} Q^{lr} (S^j T^k)^r, \tag{22.55}$$

and for all nonnegative integers j_t, k_t and l_t, where $t = 1, \ldots, r$,

$$\prod_{t=1}^{r} S^{j_t} T^{k_t} Q^{l_t} = (-1)^{\sum_{t=1}^{r} \sum_{n=1}^{t} (j_n+k_n)l_t} Q^{\sum_{t=1}^{r} l_t} \prod_{t=1}^{r} S^{j_t} T^{k_t}. \tag{22.56}$$

For the more general case, we have the following result.

Corollary 22.5 *Let r be a positive integer. If S, T and Q are elements of an algebra satisfying (22.53), then for any nonnegative integers j, k and N, and any polynomial $F(Q) = \sum_{l=0}^{N} f_l Q^l$, one obtains*

$$S^j T^k F(Q) = \sum_{l=0}^{N} f_l (-1)^{(j+k)l} Q^l S^j T^k, \tag{22.57}$$

$$(S^j T^k F(Q))^r = \sum_{L=0}^{rN} \sum_{\substack{(l_1,\ldots,l_r)\in\{0,\ldots,N\}^r \\ l_1+\cdots+l_r=L}} \left(\prod_{t=1}^{r} f_{l_t}\right) (-1)^{(j+k)\sum_{t=1}^{r} t l_t} Q^L (S^j T^k)^r, \tag{22.58}$$

and for any nonnegative integers j_t, k_t, N_t, and any polynomials

$$F_t(Q) = \sum_{l_t=0}^{N_t} f_{l_t} Q^{l_t},$$

where $t = 1, \ldots, r$, one obtains

$$\prod_{t=1}^{r} S^{j_t} T^{k_t} F_t(Q) = \sum_{L=0}^{N_1+\dots+N_r} \sum_{\substack{(l_1,\dots,l_r)\in I_1\times\dots\times I_r \\ l_1+\dots+l_r=L}} \left(\prod_{t=1}^{r} f_{l_t}\right) \cdot (-1)^{\sum_{t=1}^{r}\sum_{n=1}^{t}(j_n+k_n)l_t} Q^L \prod_{t=1}^{r} S^{j_t} T^{k_t}. \tag{22.59}$$

where $I_t = \{0, \dots, N_t\}$ *for some* t.

22.5.2 When $\sigma_j(x) = c_j x^{q_j}$

Let c_j be complex numbers, q_j be positive integers, and let σ_j be the polynomials $\sigma_j(x) = c_j x^{q_j}$. Then commutation relations (22.1) become

$$S_j Q = c_j Q^{q_j} S_j. \tag{22.60}$$

The following lemma is useful for obtaining the reordering results.

Lemma 22.2 *For any positive integer* t *and any nonnegative integers* $k, k_1, \dots, k_t$*,*

$$\sigma_j^{\circ k}(Q) = c_j^{\{k\}_{q_j}} Q^{q_j^k}, \tag{22.61}$$

$$(\sigma_{j_t}^{\circ k_t} \circ \dots \circ \sigma_{j_1}^{\circ k_1})(Q) = \left(\prod_{n=1}^{t} c_{j_n}^{\{k_n\}_{q_{j_n}} \prod_{m=n+1}^{t} q_{j_m}^{k_m}}\right) Q^{\prod_{n=1}^{t} q_{j_n}^{k_n}}, \tag{22.62}$$

where $\{k\}_q$ *for some complex number* q *denotes the* q*-number*

$$\{k\}_q = \sum_{j=0}^{k-1} q^j = \begin{cases} \frac{q^k-1}{q-1}, & q \neq 1, \\ k, & q = 1, \end{cases} \tag{22.63}$$

and we use the convention that $\prod_{m=n+1}^{t} q_{j_m}^{k_m} = 1$ *for* $t < n+1$.

Proof We prove (22.61) by induction on k. For $k = 1$, the formula follows from the definition of σ_j. Now suppose that (22.61) holds for some integer $k \geq 1$, then

$$\sigma_j^{\circ(k+1)}(Q) = \sigma_j^{\circ k}\left(\sigma_j(Q)\right) = c_j^{\{k\}_{q_j}} (c_j Q^{q_j})^{q_j^k} = c_j^{\{k\}_{q_j}+q_j^k} Q^{q_j^{k+1}} = c_j^{\{k+1\}_{q_j}} Q^{q_j^{k+1}},$$

proving (22.61). Next we prove (22.62) by induction on t. For $t = 1$, the formula follows from (22.61). Now suppose that (22.62) holds for some integer $t \geq 1$, then

$$(\sigma_{j_{t+1}}^{\circ k_{t+1}} \circ \dots \circ \sigma_{j_1}^{\circ k_1})(Q) = \sigma_{j_{t+1}}^{\circ k_{t+1}}\left((\sigma_{j_t}^{\circ k_t} \circ \dots \circ \sigma_{j_1}^{\circ k_1})(Q)\right)$$

$$= c_{j_{t+1}}^{\{k_{t+1}\}_{q_{j_{t+1}}}} \left(\left(\prod_{n=1}^{t} c_{j_n}^{\{k_n\}_{q_{j_n}} \prod_{m=n+1}^{t} q_{j_m}^{k_m}} \right) Q^{\prod_{n=1}^{t} q_{j_n}^{k_n}} \right)^{q_{j_{t+1}}^{k_{t+1}}}$$
$$= c_{j_{t+1}}^{\{k_{t+1}\}_{q_{j_{t+1}}}} \left(\prod_{n=1}^{t} c_{j_n}^{\{k_n\}_{q_{j_n}} \prod_{m=n+1}^{t+1} q_{j_m}^{k_m}} \right) Q^{\prod_{n=1}^{t+1} q_{j_n}^{k_n}}$$
$$= \left(\prod_{n=1}^{t+1} c_{j_n}^{\{k_n\}_{q_{j_n}} \prod_{m=n+1}^{t+1} q_{j_m}^{k_m}} \right) Q^{\prod_{n=1}^{t+1} q_{j_n}^{k_n}},$$

and this proves the assertion. □

Theorem 22.6 *Let $r \in \mathbb{N}$. If Q and $\{S_j\}_{j\in J}$ are elements of an algebra satisfying relations (22.60), then for any $k, N \in \mathbb{N}_0$ and any polynomial $F(Q) = \sum_{l=0}^{N} f_l Q^l$,*

$$S_j^k F(Q) = \sum_{l=0}^{N} f_l c_j^{\{k\}_{q_j} l} Q^{q_j^k l} S_j^k, \tag{22.64}$$
$$\left(S_j^k F(Q)\right)^r = \sum_{L=\min \Gamma_{k,r}}^{\max \Gamma_{k,r}} \sum_{\substack{(l_1,\dots,l_r)\in\{0,\dots,N\}^r \\ \sum_{t=1}^{r} q_j^{kt} l_t = L}} \left(\prod_{t=1}^{r} f_{l_t} \right) c_j^{\sum_{t=1}^{r} \{tk\}_{q_j} l_t} Q^L S_j^{kr}, \tag{22.65}$$

where as before $\{k\}_q$ for some $q \in \mathbb{C}$ denotes the q-number of k, and

$$\Gamma_{k,r} = \left\{ \sum_{t=1}^{r} q_j^{kt} l_t \,\middle|\, (l_1,\dots,l_r) \in \{0,\dots,N\}^r \right\}.$$

More generally, for all $k_t, N_t \in \mathbb{N}_0$ and all polynomials $F_t(Q) = \sum_{l_t=0}^{N_t} f_{l_t} Q^{l_t}$, where $t = 1, \dots, r$,

$$\prod_{t=1}^{r} S_{j_t}^{k_t} F_t(Q) = \sum_{L=\min \Delta_{\mathbf{k},r}}^{\max \Delta_{\mathbf{k},r}} \sum_{\substack{(l_1,\dots,l_r)\in I_1\times\dots\times I_r \\ \sum_{t=1}^{r}\left(\prod_{n=1}^{t} q_{j_n}^{k_n}\right) l_t = L}} \left(\prod_{t=1}^{r} f_{l_t}\right) \left(\prod_{n=1}^{r} c_{j_n}^{\{k_n\}_{q_{j_n}} \sum_{t=n}^{r}\left(\prod_{m=n+1}^{t} q_{j_m}^{k_m}\right) l_t} \right) Q^L \prod_{t=1}^{r} S_{j_t}^{k_t}, \tag{22.66}$$

where $\Delta_{\mathbf{k},r}$ for $I_t = \{0,\dots,N_t\}$ is the set given by

$$\Delta_{\mathbf{k},r} = \left\{ \sum_{t=1}^{r} \left(\prod_{n=1}^{t} q_{j_n}^{k_n} \right) l_t \,\middle|\, (l_1,\dots,l_r) \in I_1 \times \dots \times I_r \right\}.$$

Remark 22.1 Observe that for positive integers q_j, one obtains that $\min \Gamma_{k,r} = 0$, $\min \Delta_{\mathbf{k},r} = 0$, $\max \Gamma_{k,r} = \sum_{t=1}^{r} q_j^{kt} N$, and $\max \Delta_{\mathbf{k},r} = \sum_{t=1}^{r} \left(\prod_{n=1}^{t} q_{j_n}^{k_n}\right) N_t$. We strongly believe that formulas (22.64), (22.65), (22.66) are probably true also for negative integers q_j.

Remark 22.2 Formula (22.65) can be obtained from formula (22.66) by choosing $j_1 = \cdots = j_r = j$ and $k_1 = \cdots = k_r = k$, and observing that for all positive integers k and r,

$$\{k\}_{q_j} \sum_{n=1}^{r} \sum_{t=n}^{r} q_j^{(t-n)k} = \{k\}_{q_j} \sum_{t=1}^{r} \sum_{n=1}^{t} q_j^{(t-n)k} = \{k\}_{q_j} \sum_{t=1}^{r} \sum_{n=0}^{t-1} q_j^{nk}$$
$$= \sum_{t=1}^{r} \{k\}_{q_j} \{t\}_{q_j^k} = \sum_{t=1}^{r} \{tk\}_{q_j},$$

where the last equality is a well-known identity (see, for example [9, p. 187]).

Proof Substituting (22.61) into (22.13) and (22.14) gives (22.64) and (22.65), respectively, and substituting (22.62) into (22.15) gives (22.66). More precisely,

$$S_j^k F(Q) = \sum_{l=0}^{N} f_l \left(\sigma_j^{\circ k}(Q)\right)^l S_j^k = \sum_{l=0}^{N} f_l \left(c_j^{\{k\}_{q_j}} Q^{q_j^k}\right)^l S_j^k$$
$$= \sum_{l=0}^{N} f_l c_j^{\{k\}_{q_j} l} Q^{q_j^k l} S_j^k,$$

$$\left(S_j^k F(Q)\right)^r = \sum_{(l_1,\ldots,l_r)\in\{0,\ldots,N\}^r} \left(\prod_{t=1}^{r} f_{l_t}\right)\left(\prod_{t=1}^{r} \left(c_j^{\{tk\}_{q_j}} Q^{q_j^{kt}}\right)^{l_t}\right) S_j^{kr}$$
$$= \sum_{(l_1,\ldots,l_r)\in\{0,\ldots,N\}^r} \left(\prod_{t=1}^{r} f_{l_t}\right) c_j^{\sum_{t=1}^{r}\{tk\}_{q_j} l_t} Q^{\sum_{t=1}^{r} q_j^{kt} l_t} S_j^{kr},$$

and for the more general formula,

$$\prod_{t=1}^{r} S_{j_t}^{k_t} F_t(Q) = \sum_{(l_1,\ldots,l_r)\in I_1\times\ldots\times I_r} \left(\prod_{t=1}^{r} f_{l_t}\right)$$
$$\cdot \left(\prod_{t=1}^{r}\left(\left(\prod_{n=1}^{t} c_{j_n}^{\{k_n\}_{q_{j_n}} \prod_{m=n+1}^{t} q_{jm}^{k_m}}\right) Q^{\prod_{n=1}^{t} q_{jn}^{k_n}}\right)^{l_t}\right)\prod_{t=1}^{r} S_{j_t}^{k_t}$$
$$= \sum_{(l_1,\ldots,l_r)\in I_1\times\ldots\times I_r} \left(\prod_{t=1}^{r} f_{l_t}\right)$$
$$\cdot \left(\prod_{t=1}^{r}\prod_{n=1}^{t} c_{j_n}^{\{k_n\}_{q_{j_n}} \left(\prod_{m=n+1}^{t} q_{jm}^{k_m}\right) l_t}\right)\left(\prod_{t=1}^{r} Q^{\left(\prod_{n=1}^{t} q_{jn}^{k_n}\right) l_t}\right)\prod_{t=1}^{r} S_{j_t}^{k_t}$$
$$= \sum_{(l_1,\ldots,l_r)\in I_1\times\ldots\times I_r} \left(\prod_{t=1}^{r} f_{l_t}\right)$$
$$\cdot \left(\prod_{n=1}^{r}\prod_{t=n}^{r} c_{j_n}^{\{k_n\}_{q_{j_n}} \left(\prod_{m=n+1}^{t} q_{jm}^{k_m}\right) l_t}\right) Q^{\sum_{t=1}^{r}\left(\prod_{n=1}^{t} q_{jn}^{k_n}\right) l_t}\prod_{t=1}^{r} S_{j_t}^{k_t}$$
$$= \sum_{(l_1,\ldots,l_r)\in I_1\times\ldots\times I_r} \left(\prod_{t=1}^{r} f_{l_t}\right)$$
$$\cdot \left(\prod_{n=1}^{r} c_{j_n}^{\{k_n\}_{q_{j_n}} \sum_{t=n}^{r}\left(\prod_{m=n+1}^{t} q_{jm}^{k_m}\right) l_t}\right) Q^{\sum_{t=1}^{r}\left(\prod_{n=1}^{t} q_{jn}^{k_n}\right) l_t}\prod_{t=1}^{r} S_{j_t}^{k_t},$$

from which the results follow. □

For the particular case where F is a monomial, that is, $F(Q) = Q^l$ for some positive integer l, Theorem 22.6 yields the following result.

Corollary 22.6 *Let r be a positive integer. If Q and $\{S_j\}_{j\in J}$ are elements of an algebra satisfying*

$$S_j Q = c_j Q^{q_j} S_j, \tag{22.67}$$

then for any nonnegative integers k and l,

$$S_j^k Q^l = c_j^{\{k\}_{q_j} l} Q^{q_j^k l} S_j^k \tag{22.68}$$
$$(S_j^k Q^l)^r = c_j^{\sum_{t=1}^{r}\{tk\}_{q_j} l} Q^{\sum_{t=1}^{r} q_j^{kt} l} S_j^{kr}, \tag{22.69}$$

and for any nonnegative integers k_t and l_t, where $t = 1, \ldots, r$,

$$\prod_{t=1}^{r} S_{j_t}^{k_t} Q^{l_t} = \left(\prod_{n=1}^{r} c_{j_n}^{\{k_n\}_{q_{j_n}} \sum_{t=n}^{r}\left(\prod_{m=n+1}^{t} q_{jm}^{k_m}\right) l_t}\right) Q^{\sum_{t=1}^{r}\left(\prod_{n=1}^{t} q_{jn}^{k_n}\right) l_t}\prod_{t=1}^{r} S_{j_t}^{k_t}. \tag{22.70}$$

Example 22.11 For $r = 2$, we have

$$S_{j_1}^{k_1} Q^{l_1} S_{j_2}^{k_2} Q^{l_2} = \left(\prod_{n=1}^{2} c_{j_n}^{\{k_n\}_{q_{j_n}} \sum_{t=n}^{2}\left(\prod_{m=n+1}^{t} q_{j_m}^{k_m}\right) l_t}\right) Q^{\sum_{t=1}^{2}\left(\prod_{n=1}^{t} q_{j_n}^{k_n}\right) l_t} S_{j_1}^{k_1} S_{j_2}^{k_2},$$
$$= c_{j_1}^{\{k_1\}_{q_{j_1}} l_1 + \{k_1\}_{q_{j_1}} q_{j_2}^{k_2} l_2} c_{j_2}^{\{k_2\}_{q_{j_2}} l_2} Q^{q_{j_1}^{k_1} l_1 + q_{j_1}^{k_1} q_{j_2}^{k_2} l_2} S_{j_1}^{k_1} S_{j_2}^{k_2},$$

which for $l_1 = 0$ becomes

$$S_{j_1}^{k_1} S_{j_2}^{k_2} Q^{l_2} = c_{j_1}^{\{k_1\}_{q_{j_1}} q_{j_2}^{k_2} l_2} c_{j_2}^{\{k_2\}_{q_{j_2}} l_2} Q^{q_{j_1}^{k_1} q_{j_2}^{k_2} l_2} S_{j_1}^{k_1} S_{j_2}^{k_2}. \tag{22.71}$$

and denoting $S_{j_1} = S$, $S_{j_2} = T$, we have the following case of Corollary 22.6.

Corollary 22.7 *Let r be a positive integer, c_σ and c_τ be complex numbers, and let q_σ and q_τ be positive integers. If S, T and Q are elements of an algebra satisfying the relations*

$$\begin{aligned} SQ &= c_\sigma Q^{q_\sigma} S, \\ TQ &= c_\tau Q^{q_\tau} T, \end{aligned} \tag{22.72}$$

then for any nonnegative integers j, k and l,

$$S^j T^k Q^l = c_\sigma^{\{j\}_{q_\sigma} q_\tau^k l} c_\tau^{\{k\}_{q_\tau} l} Q^{q_\sigma^j q_\tau^k l} S^j T^k, \tag{22.73}$$

$$\begin{aligned} (S^j T^k Q^l)^r &= c_\sigma^{\{j\}_{q_\sigma} q_\tau^k \sum_{n=1}^{r} \sum_{t=n}^{r} (q_\sigma^j q_\tau^k)^{t-n} l} c_\tau^{\{k\}_{q_\tau} \sum_{n=1}^{r} \sum_{t=n}^{r} (q_\sigma^j q_\tau^k)^{t-n} l} \\ &\quad \cdot Q^{\sum_{t=1}^{r} \left(q_\sigma^j q_\tau^k\right)^t l} (S^j T^k)^r, \end{aligned} \tag{22.74}$$

and for any nonnegative integers j_t, k_t and l_t, where $t = 1, \ldots, r$,

$$\begin{aligned} \prod_{t=1}^{r} S^{j_t} T^{k_t} Q^{l_t} &= \left(\prod_{n=1}^{r} c_\sigma^{\{j_n\}_{q_\sigma} q_\tau^{k_n} \sum_{t=n}^{r}\left(\prod_{m=n+1}^{t} q_\sigma^{j_m} q_\tau^{k_m}\right) l_t}\right) \\ &\quad \cdot \left(\prod_{n=1}^{r} c_\tau^{\{k_n\}_{q_\tau} \sum_{t=n}^{r}\left(\prod_{m=n+1}^{t} q_\sigma^{j_m} q_\tau^{k_m}\right) l_t}\right) Q^{\sum_{t=1}^{r}\left(\prod_{n=1}^{t} q_\sigma^{j_n} q_\tau^{k_n}\right) l_t} \prod_{t=1}^{r} S^{j_t} T^{k_t}. \end{aligned} \tag{22.75}$$

For the more general case, we have the following result.

Corollary 22.8 *Let r be a positive integer. If S, T and Q are elements of an algebra satisfying (22.72), then for any nonnegative integers j, k and N, and any polynomial $F(Q) = \sum_{l=0}^{N} f_l Q^l$,*

$$S^jT^kF(Q) = \sum_{l=0}^{N} f_l c_\sigma^{\{j\}_{q_\sigma} q_\tau^k l} c_\tau^{\{k\}_{q_\tau} l} Q^{q_\sigma^j q_\tau^k l} S^j T^k, \tag{22.76}$$

$$\begin{aligned}(S^jT^kF(Q))^r = &\sum_{L=0}^{N\sum_{t=1}^{r}\left(q_\sigma^j q_\tau^k\right)^t} \sum_{\substack{(l_1,\ldots,l_r)\in\left\{0,\ldots,N\right\}^r \\ \sum_{t=1}^{r}\left(q_\sigma^j q_\tau^k\right)^t l_t = L}} \left(\prod_{t=1}^{r} f_{l_t}\right) \\ &\cdot c_\sigma^{\{j\}_{q_\sigma} q_\tau^k \sum_{n=1}^{r}\sum_{t=n}^{r}(q_\sigma^j q_\tau^k)^{t-n} l_t} c_\tau^{\{k\}_{q_\tau} \sum_{n=1}^{r}\sum_{t=n}^{r}(q_\sigma^j q_\tau^k)^{t-n} l_t} Q^L (S^jT^k)^r.\end{aligned} \tag{22.77}$$

More generally, for all $j_t, k_t, N_t \in \mathbb{N}_0$*, and any polynomials* $F_t(Q) = \sum_{l_t=0}^{N_t} f_{l_t} Q^{l_t}$*,*

$$\begin{aligned}\prod_{t=1}^{r} S^{j_t}T^{k_t}F_t(Q) = &\sum_{L=0}^{\sum_{t=1}^{r}\left(\prod_{n=1}^{t} q_\sigma^{j_n} q_\tau^{k_n}\right)N_t} \sum_{\substack{(l_1,\ldots,l_r)\in I_1\times\ldots\times I_r \\ \sum_{t=1}^{r}\left(\prod_{n=1}^{t} q_\sigma^{j_n} q_\tau^{k_n}\right)l_t}} \left(\prod_{t=1}^{r} f_{l_t}\right) \\ &\cdot \left(\prod_{n=1}^{r} c_\sigma^{\{j_n\}_{q_\sigma} q_\tau^{k_n} \sum_{t=n}^{r}\left(\prod_{m=n+1}^{t} q_\sigma^{j_m} q_\tau^{k_m}\right)l_t}\right)\left(\prod_{n=1}^{r} c_\tau^{\{k_n\}_{q_\tau} \sum_{t=n}^{r}\left(\prod_{m=n+1}^{t} q_\sigma^{j_m} q_\tau^{k_m}\right)l_t}\right) \\ &\cdot Q^L \prod_{t=1}^{r} S^{j_t}T^{k_t},\end{aligned} \tag{22.78}$$

where $t = 1, \ldots, r$ *and* $I_t = \left\{0, \ldots, N_t\right\}$.

Corollaries 22.7 and 22.8 can also be derived in the following way. Let c_σ, c_τ be complex numbers, q_σ, q_τ be positive integers, and let σ, τ be the polynomials

$$\sigma(x) = c_\sigma x^{q_\sigma},$$
$$\tau(x) = c_\tau x^{q_\tau}.$$

Then commutation relations (22.16) become

$$SQ = c_\sigma Q^{q_\sigma} S, \tag{22.79}$$
$$TQ = c_\tau Q^{q_\tau} T. \tag{22.80}$$

Let j and k be nonnegative integers. Lemma 22.2 implies the relations

$$\sigma^{\circ j}(Q) = c_\sigma^{\{j\}_{q_\sigma}} Q^{q_\sigma^j}, \tag{22.81}$$
$$\tau^{\circ k}(Q) = c_\tau^{\{k\}_{q_\tau}} Q^{q_\tau^k}, \tag{22.82}$$

and the relations

$$(\tau^{\circ k} \circ \sigma^{\circ j})(Q) = c_\tau^{\{k\}_{q_\tau}} c_\sigma^{\{j\}_{q_\sigma} q_\tau^k} Q^{q_\sigma^j q_\tau^k}, \tag{22.83}$$

$$(\sigma^{\circ j} \circ \tau^{\circ k})(Q) = c_\sigma^{\{j\}_{q_\sigma}} c_\tau^{\{k\}_{q_\tau} q_\sigma^j} Q^{q_\tau^k q_\sigma^j}, \tag{22.84}$$

and the corresponding formulas in Example 22.5 become

$$S^j T^k F(Q) = F\left(c_\tau^{\{k\}_{q_\tau}} c_\sigma^{\{j\}_{q_\sigma} q_\tau^k} Q^{q_\sigma^j q_\tau^k}\right) S^j T^k, \tag{22.85}$$

$$T^k S^j F(Q) = F\left(c_\sigma^{\{j\}_{q_\sigma}} c_\tau^{\{k\}_{q_\tau} q_\sigma^j} Q^{q_\tau^k q_\sigma^j}\right) T^k S^j. \tag{22.86}$$

Let us derive an expression for $(\tau^{\circ k_t} \circ \sigma^{\circ j_t} \circ \cdots \circ \tau^{\circ k_1} \circ \sigma^{\circ j_1})(Q)$ for any nonnegative integers $j_1, \ldots, j_t, k_1, \ldots, k_t$ by induction on t. For $t = 2$, (22.83) implies that

$$\begin{aligned}(\tau^{\circ k_2} \circ \sigma^{\circ j_2} \circ \tau^{\circ k_1} \circ \sigma^{\circ j_1})(Q) &= (\tau^{\circ k_2} \circ \sigma^{\circ j_2})\left(c_\sigma^{\{j_1\}_{q_\sigma} q_\tau^{k_1}} c_\tau^{\{k_1\}_{q_\tau}} Q^{q_\sigma^{j_1} q_\tau^{k_1}}\right)\\
&= c_\sigma^{\{j_2\}_{q_\sigma} q_\tau^{k_2}} c_\tau^{\{k_2\}_{q_\tau}} \left(c_\sigma^{\{j_1\}_{q_\sigma} q_\tau^{k_1}} c_\tau^{\{k_1\}_{q_\tau}} Q^{q_\sigma^{j_1} q_\tau^{k_1}}\right)^{q_\sigma^{j_2} q_\tau^{k_2}}\\
&= c_\sigma^{\{j_1\}_{q_\sigma} q_\tau^{k_1} q_\sigma^{j_2} q_\tau^{k_2}} c_\tau^{\{k_1\}_{q_\tau} q_\sigma^{j_2} q_\tau^{k_2}} c_\sigma^{\{j_2\}_{q_\sigma} q_\tau^{k_2}} c_\tau^{\{k_2\}_{q_\tau}} Q^{q_\sigma^{j_1} q_\tau^{k_1} q_\sigma^{j_2} q_\tau^{k_2}}\end{aligned}$$

In general, one has for all positive integers t the relation

$$\begin{aligned}&(\tau^{\circ k_t} \circ \sigma^{\circ j_t} \circ \cdots \circ \tau^{\circ k_1} \circ \sigma^{\circ j_1})(Q)\\
&= \left(\prod_{n=1}^{t} c_\sigma^{\{j_n\}_{q_\sigma} q_\tau^{k_n} \prod_{m=n+1}^t q_\sigma^{j_m} q_\tau^{k_m}} c_\tau^{\{k_n\}_{q_\tau} \prod_{m=n+1}^t q_\sigma^{j_m} q_\tau^{k_m}}\right) Q^{\prod_{n=1}^t q_\sigma^{j_n} q_\tau^{k_n}}.\end{aligned} \tag{22.87}$$

Substituting relation (22.87) into Example 22.5 yields Corollaries 22.7 and 22.8. In general, relation (22.87) is useful for directly obtaining reordering results for the algebra generated by relations (22.79) and (22.80).

22.5.3 When $\sigma_j(x) = c_j x$

Let a_j, b_j and c_j be complex numbers, and let q_j be positive integers. Section 22.5.2 considers the case $\sigma_j(x) = c_j x^{q_j}$ while Sect. 22.5.4 considers the case $\sigma_j(x) = a_j x + b_j$. The intersection of these two cases is the case $\sigma_j(x) = c_j x$, for which commutation relation (22.1) become the relation

$$S_j Q = c_j Q S_j, \tag{22.88}$$

often called the quantum plane relation, in the context of noncommutative geometry and quantum groups. The following result follows from Theorem 22.6 by choosing $q_j = 1$.

Corollary 22.9 *Let r be a positive integer. If Q and $\{S_j\}_{j\in J}$ are elements of an algebra satisfying (22.88), then for any nonnegative integers k and N, and any polynomial $F(Q) = \sum_{l=0}^{N} f_l Q^l$,*

$$S_j^k F(Q) = \sum_{l=0}^{N} f_l c_j^{kl} Q^l S_j^k,$$
$$\left(S_j^k F(Q)\right)^r = \sum_{L=0}^{rN} \sum_{\substack{(l_1,\ldots,l_r)\in\{0,\ldots,N\}^r \\ l_1+\cdots+l_r=L}} \left(\prod_{t=1}^{r} f_{l_t}\right) c_j^{k\sum_{t=1}^{r} t l_t} Q^L S_j^{kr},$$

and for k_t, $N_t \in \mathbb{N}_0$, and any polynomials $F_t(Q) = \sum_{l_t=0}^{N_t} f_{l_t} Q^{l_t}$, where $t = 1, \ldots, r$,

$$\prod_{t=1}^{r} S_{j_t}^{k_t} F_t(Q) = \sum_{L=0}^{N_1+\cdots+N_r} \sum_{\substack{(l_1,\ldots,l_r)\in I_1\times\ldots\times I_r \\ l_1+\cdots+l_r=L}} \left(\prod_{t=1}^{r} f_{l_t}\right)\left(\prod_{n=1}^{r} c_{j_n}^{k_n \sum_{t=n}^{r} l_t}\right) Q^L \prod_{t=1}^{r} S_{j_t}^{k_t},$$

where $I_t = \{0, \ldots, N_t\}$ for some t.

For the particular case where F is a monic monomial in Q, Corollary 22.9 yields the following result.

Corollary 22.10 *Let r be a positive integer. If Q and $\{S_j\}_{j\in J}$ are elements of an algebra satisfying (22.88), then for all nonnegative integers k and l,*

$$S_j^k Q^l = c_j^{kl} Q^l S_j^k, \tag{22.89}$$
$$\left(S_j^k Q^l\right)^r = c_j^{klr(r+1)/2} Q^{lr} S_j^{kr}, \tag{22.90}$$

and for all nonnegative integers k_t and l_t, where $t = 1, \ldots, r$,

$$\prod_{t=1}^{r} S_{j_t}^{k_t} Q^{l_t} = \left(\prod_{n=1}^{r} c_{j_n}^{k_n \sum_{t=n}^{r} l_t}\right) Q^{\sum_{t=1}^{r} l_t} \prod_{t=1}^{r} S_{j_t}^{k_t}. \tag{22.91}$$

Example 22.12 For $r = 2$, we have

$$S_{j_1}^{k_1} Q^{l_1} S_{j_2}^{k_2} Q^{l_2} = \left(\prod_{n=1}^{2} c_{j_n}^{k_n \sum_{t=n}^{2} l_t}\right) Q^{\sum_{t=1}^{2} l_t} S_{j_1}^{k_1} S_{j_2}^{k_2}, = c_{j_1}^{k_1(l_1+l_2)} c_{j_2}^{k_2 l_2} Q^{l_1+l_2} S_{j_1}^{k_1} S_{j_2}^{k_2},$$

which for $l_1 = 0$ becomes

$$S_{j_1}^{k_1} S_{j_2}^{k_2} Q^{l_2} = c_{j_1}^{k_1 l_2} c_{j_2}^{k_2 l_2} Q^{l_2} S_{j_1}^{k_1} S_{j_2}^{k_2}.$$

Denoting $S_{j_1} = S$ and $S_{j_2} = T$, we have the following case of Corollary 22.10.

Corollary 22.11 *Let r be a positive integer. Let c_σ and c_τ be complex numbers. If S, T and Q are elements of an algebra satisfying the relations*

$$SQ = c_\sigma QS \quad \text{and} \quad TQ = c_\tau QT, \tag{22.92}$$

then for all nonnegative integers j, k and l,

$$S^j T^k Q^l = c_\sigma^{jl} c_\tau^{kl} Q^l S^j T^k, \tag{22.93}$$

$$(S^j T^k Q^l)^r = c_\sigma^{jlr(r+1)/2} c_\tau^{klr(r+1)/2} Q^{lr} (S^j T^k)^r, \tag{22.94}$$

and for all nonnegative integers j_t, k_t and l_t, where $t = 1, \ldots, r$,

$$\prod_{t=1}^{r} S^{j_t} T^{k_t} Q^{l_t} = \left(\prod_{n=1}^{r} c_\sigma^{j_n \sum_{t=n}^{r} l_t} c_\tau^{k_n \sum_{t=n}^{r} l_t} \right) Q^{\sum_{t=1}^{r} l_t} \prod_{t=1}^{r} S^{j_t} T^{k_t}. \tag{22.95}$$

Corollary 22.12 *Let $r \in \mathbb{N}$ and $c_\sigma, c_\tau \in \mathbb{C}$. If S, T and Q are elements of an algebra satisfying (22.92), then for any $j, k, N \in \mathbb{N}_0$, and any polynomial $F(Q) = \sum_{l=0}^{N} f_l Q^l$,*

$$S^j T^k F(Q) = \sum_{l=0}^{N} f_l c_\sigma^{jl} c_\tau^{kl} Q^l S^j T^k, \tag{22.96}$$

$$(S^j T^k F(Q))^r = \sum_{L=0}^{rN} \sum_{\substack{(l_1,\ldots,l_r) \in \{0,\ldots,N\}^r \\ l_1+\cdots+l_r=L}} \left(\prod_{t=1}^{r} f_{l_t} \right) c_\sigma^{j \sum_{t=1}^{r} t l_t} c_\tau^{k \sum_{t=1}^{r} t l_t} Q^L (S^j T^k)^r, \tag{22.97}$$

and for $j_t, k_t, N_t \in \mathbb{N}_0$, and polynomials $F_t(Q) = \sum_{l_t=0}^{N_t} f_{l_t} Q^{l_t}$, where $t = 1, \ldots, r$,

$$\begin{aligned} \prod_{t=1}^{r} S^{j_t} T^{k_t} F_t(Q) = \sum_{L=0}^{N_1+\cdots+N_r} & \sum_{\substack{(l_1,\ldots,l_r) \in I_1 \times \ldots \times I_r \\ l_1+\cdots+l_r=L}} \left(\prod_{t=1}^{r} f_{l_t} \right) \\ & \cdot \left(\prod_{n=1}^{r} c_\sigma^{j_n \sum_{t=n}^{r} l_t} c_\tau^{k_n \sum_{t=n}^{r} l_t} \right) Q^L \prod_{t=1}^{r} S^{j_t} T^{k_t}, \end{aligned} \tag{22.98}$$

where $I_t = \{0, \ldots, N_t\}$.

22.5.4 When $\sigma_j(x) = a_j x + b_j$

Let a_j and b_j be complex numbers, and let σ_j be the polynomials

$$\sigma_j(x) = a_j x + b_j. \tag{22.99}$$

Then commutation relations (22.1) become

$$S_j Q = a_j Q S_j + b_j S_j. \tag{22.100}$$

These are deformed Heisenberg–Lie commutation relations of quantum mechanics. The classical Heisenberg–Lie relations $S_j Q - Q S_j = S_j$ are obtained when $a_j = 1$ and $b_j = 1$. If $c_j = 0$, then we get the quantum plane relations $S_j Q = q_j Q S_j$

The following lemma is useful for obtaining the reordering results.

Lemma 22.3 *For any positive integer t and any nonnegative integers k, $k_1, \ldots, k_t$,*

$$\sigma_j^{\circ k}(Q) = a_j^k Q + \{k\}_{a_j} b_j, \tag{22.101}$$

$$(\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q) = \left(\prod_{n=1}^{t} a_{j_n}^{k_n}\right) Q + \sum_{n=1}^{t} \left(\prod_{m=n+1}^{t} a_{j_m}^{k_m}\right) \{k_n\}_{a_{j_n}} b_{j_n}, \tag{22.102}$$

Proof We prove (22.101) by induction on k. For $k = 1$, the formula follows from (22.99). Now suppose that (22.101) holds for some integer $k \geq 1$, then

$$\begin{aligned}
\sigma_j^{\circ(k+1)}(Q) = \sigma_j^{\circ k}(\sigma_j(Q)) &= a_j^k (a_j Q + b_j) + \{k\}_{a_j} b_j \\
&= a_j^{k+1} Q + (a_j^k + \{k\}_{a_j}) b_j \\
&= a_j^{k+1} Q + \{k+1\}_{a_j} b_j,
\end{aligned}$$

which proves (22.101). Next we prove (22.102) by induction on t. For $t = 1$, it follows from (22.101). Now suppose that (22.102) holds for some integer $t \geq 1$, then

$$\begin{aligned}
&(\sigma_{j_{t+1}}^{\circ k_{t+1}} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q) = \sigma_{j_{t+1}}^{\circ k_{t+1}} \left((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q) \right) \\
&= a_{j_{t+1}}^{k_{t+1}} \left(\left(\prod_{n=1}^{t} a_{j_n}^{k_n}\right) Q + \sum_{n=1}^{t} \left(\prod_{m=n+1}^{t} a_{j_m}^{k_m}\right) \{k_n\}_{a_{j_n}} b_{j_n} \right) + \{k_{t+1}\}_{a_{j_{t+1}}} b_{j_{t+1}} \\
&= \left(\prod_{n=1}^{t+1} a_{j_n}^{k_n}\right) Q + \sum_{n=1}^{t} \left(\prod_{m=n+1}^{t+1} a_{j_m}^{k_m}\right) \{k_n\}_{a_{j_n}} b_{j_n} + \{k_{t+1}\}_{a_{j_{t+1}}} b_{j_{t+1}} \\
&= \left(\prod_{n=1}^{t+1} a_{j_n}^{k_n}\right) Q + \sum_{n=1}^{t+1} \left(\prod_{m=n+1}^{t+1} a_{j_m}^{k_m}\right) \{k_n\}_{a_{j_n}} b_{j_n},
\end{aligned}$$

and this proves the assertion. □

Theorem 22.7 *Let $r \in \mathbb{Z}_+$. If Q and $\{S_j\}_{j\in J}$ are elements of an algebra satisfying (22.100), then for all $k, l, N \in \mathbb{N}_0$, and any polynomial $F(Q) = \sum_{l=0}^{N} f_l Q^l$,*

$$S_j^k Q^l = \sum_{v=0}^{l} \binom{l}{v} a_j^{kv} \left(\{k\}_{a_j} b_j\right)^{l-v} Q^v S_j^k, \tag{22.103}$$

$$(S_j^k Q^l)^r = \sum_{V=0}^{lr} \sum_{\substack{(v_1,\dots,v_r)\in\{0,\dots,l\}^r \\ v_1+\dots+v_r=V}} \left(\prod_{t=1}^{r} \binom{l}{v_t} \left(\{kt\}_{a_j}\right)^{l-v_t}\right) \cdot a_j^{k\sum_{t=1}^{r} t v_t} b_j^{\sum_{t=1}^{r}(l-v_t)} Q^V S_j^{kr}, \tag{22.104}$$

$$S_j^k F(Q) = \sum_{v=0}^{N} \sum_{l=v}^{N} \binom{l}{v} a_j^{kv} \left(\{k\}_{a_j} b_j\right)^{l-v} f_l Q^v S_j^k, \tag{22.105}$$

$$(S_j^k F(Q))^r = \sum_{(l_1,\dots,l_r)\in I^r} \left(\prod_{t=1}^{r} f_{l_t}\right) \sum_{V=0}^{l_1+\dots+l_t} \sum_{\substack{(v_1,\dots,v_r)\in M_1\times\dots M_r \\ v_1+\dots+v_r=V}} \left(\prod_{t=1}^{r} \cdot \binom{l_t}{v_t} \left(\{kt\}_{a_j}\right)^{l_t-v_t}\right) a_j^{k\sum_{t=1}^{r} t v_t} b_j^{\sum_{t=1}^{r}(l_t-v_t)} Q^V S_j^{kr}, \tag{22.106}$$

where $I = \{0,\dots,N\}$ and $M_t = \{0,\dots,l_t\}$.

Proof Substituting (22.101) into (22.10), (22.11), (22.13) and (22.14) gives (22.103), (22.104), (22.105) and (22.106), respectively. For example for (22.104), we have

$$\begin{aligned}
(S_j^k Q^l)^r &= \left(\prod_{t=1}^{r} (\sigma_j^{\circ tk}(Q))^l\right) S_j^{kr} = \left(\prod_{t=1}^{r} \left(a_j^{kt} Q + \{kt\}_{a_j} b_j\right)^l\right) S_j^{kr} \\
&= \left(\prod_{t=1}^{r} \sum_{v_t=0}^{l} \binom{l}{v_t} a_j^{ktv_t} \left(\{kt\}_{a_j} b_j\right)^{l-v_t} Q^{v_t}\right) S_j^{kr} \\
&= \left(\sum_{v_1=0}^{l} \cdots \sum_{v_r=0}^{l} \prod_{t=1}^{r} \binom{l}{v_t} a_j^{ktv_t} \left(\{kt\}_{a_j} b_j\right)^{l-v_t} Q^{v_t}\right) S_j^{kr} \\
&= \sum_{(v_1,\dots,v_r)\in\{0,\dots,l\}^r} \left(\prod_{t=1}^{r} \binom{l}{v_t} \left(\{kt\}_{a_j}\right)^{l-v_t}\right) a_j^{k\sum_{t=1}^{r} t v_t} b_j^{\sum_{t=1}^{r}(l-v_t)} Q^{\sum_{t=1}^{r} v_t} S_j^{kr} \\
&= \sum_{V=0}^{lr} \sum_{\substack{(v_1,\dots,v_r)\in\{0,\dots,l\}^r \\ v_1+\dots+v_r=V}} \left(\prod_{t=1}^{r} \binom{l}{v_t} \left(\{kt\}_{a_j}\right)^{l-v_t}\right) a_j^{k\sum_{t=1}^{r} t v_t} b_j^{\sum_{t=1}^{r}(l-v_t)} Q^V S_j^{kr}.
\end{aligned}$$

Formula (22.106) can also be obtained directly from (22.104) using (22.14). □

Let $a_\sigma, a_\tau, b_\sigma$ and b_τ be complex numbers, and let σ and τ be the polynomials $\sigma(x) = a_\sigma x + b_\sigma$ and $\tau(x) = a_\tau x + b_\tau$. Then commutation relations (22.16) become

$$SQ = a_\sigma QS + b_\sigma S, \tag{22.107}$$

$$TQ = a_\tau QT + b_\tau T. \tag{22.108}$$

Let j and k be nonnegative integers. Lemma 22.3 implies the relations

$$\sigma^{\circ j}(Q) = a_\sigma^j Q + \{j\}_{a_\sigma} b_\sigma, \tag{22.109}$$

$$\tau^{\circ k}(Q) = a_\tau^k Q + \{k\}_{a_\tau} b_\tau, \tag{22.110}$$

and the relations

$$(\tau^{\circ k} \circ \sigma^{\circ j})(Q) = a_\sigma^j a_\tau^k Q + a_\tau^k \{j\}_{a_\sigma} b_\sigma + \{k\}_{a_\tau} b_\tau, \tag{22.111}$$

$$(\sigma^{\circ j} \circ \tau^{\circ k})(Q) = a_\sigma^j a_\tau^k Q + a_\sigma^j \{k\}_{a_\tau} b_\tau + \{j\}_{a_\sigma} b_\sigma, \tag{22.112}$$

and the corresponding formulas in Example 22.5 become

$$S^j T^k F(Q) = F\left(a_\sigma^j a_\tau^k Q + a_\tau^k \{j\}_{a_\sigma} b_\sigma + \{k\}_{a_\tau} b_\tau\right) S^j T^k, \tag{22.113}$$

$$T^k S^j F(Q) = F\left(a_\sigma^j a_\tau^k Q + a_\sigma^j \{k\}_{a_\tau} b_\tau + \{j\}_{a_\sigma} b_\sigma\right) T^k S^j. \tag{22.114}$$

Let us derive an expression for $(\tau^{\circ k_t} \circ \sigma^{\circ j_t} \circ \cdots \circ \tau^{\circ k_1} \circ \sigma^{\circ j_1})(Q)$ for any nonnegative integers $j_1, \ldots, j_t, k_1, \ldots, k_t$ by induction on t. For $t = 2$, relation (22.111) implies

$$\begin{aligned}
(\tau^{\circ k_2} \circ \sigma^{\circ j_2} \circ \tau^{\circ k_1} \circ \sigma^{\circ j_1})(Q) &= (\tau^{\circ k_2} \circ \sigma^{\circ j_2})\left(a_\sigma^{j_1} a_\tau^{k_1} Q + a_\tau^{k_1}\{j_1\}_{a_\sigma} b_\sigma + \{k_1\}_{a_\tau} b_\tau\right) \\
&= a_\sigma^{j_2} a_\tau^{k_2}\left(a_\sigma^{j_1} a_\tau^{k_1} Q + a_\tau^{k_1}\{j_1\}_{a_\sigma} b_\sigma + \{k_1\}_{a_\tau} b_\tau\right) + a_\tau^{k_2}\{j_2\}_{a_\sigma} b_\sigma + \{k_2\}_{a_\tau} b_\tau \\
&= a_\sigma^{j_1} a_\tau^{k_1} a_\sigma^{j_2} a_\tau^{k_2} Q + a_\tau^{k_1} a_\sigma^{j_2} a_\tau^{k_2}\{j_1\}_{a_\sigma} b_\sigma + a_\sigma^{j_2} a_\tau^{k_2}\{k_1\}_{a_\tau} b_\tau + a_\tau^{k_2}\{j_2\}_{a_\sigma} b_\sigma + \{k_2\}_{a_\tau} b_\tau.
\end{aligned}$$

In general, one has for all positive integers t the relation

$$\begin{aligned}
(\tau^{\circ k_t} \circ \sigma^{\circ j_t} \circ \cdots \circ \tau^{\circ k_1} \circ \sigma^{\circ j_1})(Q) &= \left(\prod_{n=1}^{t} a_\sigma^{j_n} a_\tau^{k_n}\right) Q \\
&\quad + \sum_{n=1}^{t} a_\tau^{k_n}\left(\prod_{m=n+1}^{t} a_\sigma^{j_m} a_\tau^{k_m}\right)\{j_n\}_{a_\sigma} b_\sigma \\
&\quad + \sum_{n=1}^{t}\left(\prod_{m=n+1}^{t} a_\sigma^{j_m} a_\tau^{k_m}\right)\{k_n\}_{a_\tau} b_\tau.
\end{aligned} \tag{22.115}$$

Relation (22.115) is useful for directly obtaining reordering results for the algebra generated by relations (22.107) and (22.108).

22.6 Linear Transformation of the S_j-Generators

Proposition 22.2 *Let* $\{R_k\}_{k\in K}$ *be a set of elements of an algebra, m and n positive integers, and* $a_{j_m k_n}$ *complex numbers. If*

$$S_{j_m} = \sum_{t=1}^{n} a_{j_m k_t} R_{k_t},$$

then the commutator of S_{j_1} *and* S_{j_2} *is given by*

$$\left[S_{j_1}, S_{j_2}\right] = \sum_{\substack{t,u\in\{1,\dots,n\}\\ t\le u}} \det\begin{pmatrix} a_{j_1k_t} & a_{j_1k_u} \\ a_{j_2k_t} & a_{j_2k_u}\end{pmatrix}\left[R_{k_t}, R_{k_u}\right]. \tag{22.116}$$

Proof We proceed by induction on n. For $n = 1$, we have

$$\left[S_{j_1}, S_{j_2}\right] = \left[a_{j_1k_1}R_{k_1}, a_{j_2k_1}R_{k_1}\right] = a_{j_1k_1}a_{j_2k_1}\left[R_{k_1}, R_{k_1}\right] = 0,$$

which agrees with formula (22.116). For $n = 2$, we have

$$\begin{aligned}
\left[S_{j_1}, S_{j_2}\right] &= \left[a_{j_1k_1}R_{k_1} + a_{j_1k_2}R_{k_2}, a_{j_2k_1}R_{k_1} + a_{j_2k_2}R_{k_2}\right] \\
&= \left(a_{j_1k_1}R_{k_1} + a_{j_1k_2}R_{k_2}\right)\left(a_{j_2k_1}R_{k_1} + a_{j_2k_2}R_{k_2}\right) \\
&\quad - \left(a_{j_2k_1}R_{k_1} + a_{j_2k_2}R_{k_2}\right)\left(a_{j_1k_1}R_{k_1} + a_{j_1k_2}R_{k_2}\right) \\
&= a_{j_1k_1}a_{j_2k_1}R_{k_1}R_{k_1} + a_{j_1k_1}a_{j_2k_2}R_{k_1}R_{k_2} \\
&\quad + a_{j_1k_2}a_{j_2k_1}R_{k_2}R_{k_1} + a_{j_1k_2}a_{j_2k_2}R_{k_2}R_{k_2} \\
&\quad - a_{j_1k_1}a_{j_2k_1}R_{k_1}R_{k_1} - a_{j_1k_2}a_{j_2k_1}R_{k_1}R_{k_2} \\
&\quad - a_{j_1k_1}a_{j_2k_2}R_{k_2}R_{k_1} - a_{j_1k_2}a_{j_2k_2}R_{k_2}R_{k_2} \\
&= \left(a_{j_1k_1}a_{j_2k_2} - a_{j_1k_2}a_{j_2k_1}\right)R_{k_1}R_{k_2} \\
&\quad + \left(a_{j_1k_2}a_{j_2k_1} - a_{j_1k_1}a_{j_2k_2}\right)R_{k_2}R_{k_1} \\
&= \left(a_{j_1k_1}a_{j_2k_2} - a_{j_1k_2}a_{j_2k_1}\right)\left(R_{k_1}R_{k_2} - R_{k_2}R_{k_1}\right) \\
&= \det\begin{pmatrix} a_{j_1k_1} & a_{j_1k_2} \\ a_{j_2k_1} & a_{j_2k_2}\end{pmatrix}\left[R_{k_1}, R_{k_2}\right].
\end{aligned}$$

Now suppose that (22.116) holds for some integer $n \ge 1$, then we have for $n + 1$ that

$$\left[S_{j_1}, S_{j_2}\right] = \left[\sum_{t=1}^{n+1} a_{j_1k_t} R_{k_t}, \sum_{t=1}^{n+1} a_{j_2k_t} R_{k_t}\right]$$
$$= \left[\sum_{t=1}^{n} a_{j_1k_t} R_{k_t} + a_{j_1k_{n+1}} R_{k_{n+1}}, \sum_{t=1}^{n} a_{j_2k_t} R_{k_t} + a_{j_2k_{n+1}} R_{k_{n+1}}\right]$$

$$= \left[\sum_{t=1}^{n} a_{j_1k_t} R_{k_t}, \sum_{t=1}^{n} a_{j_2k_t} R_{k_t}\right] + \left[\sum_{t=1}^{n} a_{j_1k_t} R_{k_t}, a_{j_2k_{n+1}} R_{k_{n+1}}\right]$$
$$+ \left[a_{j_1k_{n+1}} R_{k_{n+1}}, \sum_{t=1}^{n} a_{j_2k_t} R_{k_t}\right] + \left[a_{j_1k_{n+1}} R_{k_{n+1}}, a_{j_2k_{n+1}} R_{k_{n+1}}\right]$$
$$= \sum_{\substack{t,u\in\left\{1,\dots n\right\}, \\ t\le u}} \det\begin{pmatrix} a_{j_1k_t} & a_{j_1k_u} \\ a_{j_2k_t} & a_{j_2k_u} \end{pmatrix} \left[R_{k_t}, R_{k_u}\right]$$
$$+ \left(\sum_{t=1}^{n} a_{j_1k_t} R_{k_t}\right) a_{j_2k_{n+1}} R_{k_{n+1}} - a_{j_2k_{n+1}} R_{k_{n+1}} \sum_{t=1}^{n} a_{j_1k_t} R_{k_t}$$
$$+ a_{j_1k_{n+1}} R_{k_{n+1}} \sum_{t=1}^{n} a_{j_2k_t} R_{k_t} - \left(\sum_{t=1}^{n} a_{j_2k_t} R_{k_t}\right) a_{j_1k_{n+1}} R_{k_{n+1}} + 0$$
$$= \sum_{\substack{t,u\in\left\{1,\dots,n\right\} \\ t\le u}} \det\begin{pmatrix} a_{j_1k_t} & a_{j_1k_u} \\ a_{j_2k_t} & a_{j_2k_u} \end{pmatrix} \left[R_{k_t}, R_{k_u}\right]$$
$$+ \left(\sum_{t=1}^{n} a_{j_1k_t} R_{k_t}\right) a_{j_2k_{n+1}} R_{k_{n+1}} - \left(\sum_{t=1}^{n} a_{j_2k_t} R_{k_t}\right) a_{j_1k_{n+1}} R_{k_{n+1}}$$
$$- \left(a_{j_2k_{n+1}} R_{k_{n+1}} \sum_{t=1}^{n} a_{j_1k_t} R_{k_t} - a_{j_1k_{n+1}} R_{k_{n+1}} \sum_{t=1}^{n} a_{j_2k_t} R_{k_t}\right)$$
$$= \sum_{\substack{t,u\in\left\{1,\dots,n\right\} \\ t\le u}} \det\begin{pmatrix} a_{j_1k_t} & a_{j_1k_u} \\ a_{j_2k_t} & a_{j_2k_u} \end{pmatrix} \left[R_{k_t}, R_{k_u}\right]$$
$$+ \sum_{\substack{t\in\left\{1,\dots,n\right\} \\ u=n+1}} a_{j_1k_t} a_{j_2k_u} R_{k_t} R_{k_u} - \sum_{\substack{t\in\left\{1,\dots,n\right\} \\ u=n+1}} a_{j_1k_u} a_{j_2k_t} R_{k_t} R_{k_u}$$
$$- \left(\sum_{\substack{t\in\left\{1,\dots,n\right\} \\ u=n+1}} a_{j_1k_t} a_{j_2k_u} R_{k_u} R_{k_t} - \sum_{\substack{t\in\left\{1,\dots,n\right\} \\ u=n+1}} a_{j_1k_u} a_{j_2k_t} R_{k_u} R_{k_t}\right)$$

$$= \sum_{\substack{t,u\in\{1,\ldots,n\}\\ t\le u}} \det\begin{pmatrix} a_{j_1k_t} & a_{j_1k_u}\\ a_{j_2k_t} & a_{j_2k_u}\end{pmatrix}\left[R_{k_t}, R_{k_u}\right]$$
$$+ \sum_{\substack{t\in\{1,\ldots,n\}\\ u=n+1}} \left(a_{j_1k_t}a_{j_2k_u} - a_{j_1k_u}a_{j_2k_t}\right) R_{k_t}R_{k_u}$$
$$- \sum_{\substack{t\in\{1,\ldots,n\}\\ u=n+1}} \left(a_{j_1k_t}a_{j_2k_u} - a_{j_1k_u}a_{j_2k_t}\right) R_{k_u}R_{k_t}$$

$$= \sum_{\substack{t,u\in\{1,\ldots,n\}\\ t\le u}} \det\begin{pmatrix} a_{j_1k_t} & a_{j_1k_u}\\ a_{j_2k_t} & a_{j_2k_u}\end{pmatrix}\left[R_{k_t}, R_{k_u}\right]$$
$$+ \sum_{\substack{t\in\{1,\ldots,n\}\\ u=n+1}} \left(a_{j_1k_t}a_{j_2k_u} - a_{j_1k_u}a_{j_2k_t}\right)\left(R_{k_t}R_{k_u} - R_{k_u}R_{k_t}\right)$$
$$= \sum_{\substack{t,u\in\{1,\ldots,n\}\\ t\le u}} \det\begin{pmatrix} a_{j_1k_t} & a_{j_1k_u}\\ a_{j_2k_t} & a_{j_2k_u}\end{pmatrix}\left[R_{k_t}, R_{k_u}\right] + \sum_{\substack{t\in\{1,\ldots,n\}\\ u=n+1}} \det\begin{pmatrix} a_{j_1k_t} & a_{j_1k_u}\\ a_{j_2k_t} & a_{j_2k_u}\end{pmatrix}\left[R_{k_t}, R_{k_u}\right]$$
$$= \sum_{\substack{t,u\in\{1,\ldots,n,n+1\}\\ t\le u}} \det\begin{pmatrix} a_{j_1k_t} & a_{j_1k_u}\\ a_{j_2k_t} & a_{j_2k_u}\end{pmatrix}\left[R_{k_t}, R_{k_u}\right],$$

and this proves formula (22.116). □

Example 22.13 For $n = 3$, we have

$$\begin{aligned}
\left[S_{j_1}, S_{j_2}\right] &= \left[a_{j_1k_1}R_{k_1} + a_{j_1k_2}R_{k_2} + a_{j_1k_3}R_{k_3}, a_{j_2k_1}R_{k_1} + a_{j_2k_2}R_{k_2} + a_{j_2k_3}R_{k_3}\right]\\
&= \left(a_{j_1k_1}R_{k_1} + a_{j_1k_2}R_{k_2} + a_{j_1k_3}R_{k_3}\right)\left(a_{j_2k_1}R_{k_1} + a_{j_2k_2}R_{k_2} + a_{j_2k_3}R_{k_3}\right)\\
&\quad - \left(a_{j_2k_1}R_{k_1} + a_{j_2k_2}R_{k_2} + a_{j_2k_3}R_{k_3}\right)\left(a_{j_1k_1}R_{k_1} + a_{j_1k_2}R_{k_2} + a_{j_1k_3}R_{k_3}\right)\\
&= a_{j_1k_1}a_{j_2k_1}R_{k_1}R_{k_1} + a_{j_1k_1}a_{j_2k_2}R_{k_1}R_{k_2} + a_{j_1k_1}a_{j_2k_3}R_{k_1}R_{k_3}\\
&\quad + a_{j_1k_2}a_{j_2k_1}R_{k_2}R_{k_1} + a_{j_1k_2}a_{j_2k_2}R_{k_2}R_{k_2} + a_{j_1k_2}a_{j_2k_3}R_{k_2}R_{k_3}\\
&\quad + a_{j_1k_3}a_{j_2k_1}R_{k_3}R_{k_1} + a_{j_1k_3}a_{j_2k_2}R_{k_3}R_{k_2} + a_{j_1k_3}a_{j_2k_3}R_{k_3}R_{k_3}\\
&\quad - a_{j_1k_1}a_{j_2k_1}R_{k_1}R_{k_1} - a_{j_1k_2}a_{j_2k_1}R_{k_1}R_{k_2} - a_{j_1k_3}a_{j_2k_1}R_{k_1}R_{k_3}\\
&\quad - a_{j_1k_1}a_{j_2k_2}R_{k_2}R_{k_1} - a_{j_1k_2}a_{j_2k_2}R_{k_2}R_{k_2} - a_{j_1k_3}a_{j_2k_2}R_{k_2}R_{k_3}\\
&\quad - a_{j_1k_1}a_{j_2k_3}R_{k_3}R_{k_1} - a_{j_1k_2}a_{j_2k_3}R_{k_3}R_{k_2} - a_{j_1k_3}a_{j_2k_3}R_{k_3}R_{k_3}\\
&= \left(a_{j_1k_1}a_{j_2k_2} - a_{j_1k_2}a_{j_2k_1}\right)R_{k_1}R_{k_2} + \left(a_{j_1k_1}a_{j_2k_3} - a_{j_1k_3}a_{j_2k_1}\right)R_{k_1}R_{k_3}\\
&\quad + \left(a_{j_1k_2}a_{j_2k_1} - a_{j_1k_1}a_{j_2k_2}\right)R_{k_2}R_{k_1} + \left(a_{j_1k_2}a_{j_2k_3} - a_{j_1k_3}a_{j_2k_2}\right)R_{k_2}R_{k_3}
\end{aligned}$$

$$
\begin{aligned}
&+ \left(a_{j_1k_3}a_{j_2k_1} - a_{j_1k_1}a_{j_2k_3}\right) R_{k_3}R_{k_1} + \left(a_{j_1k_3}a_{j_2k_2} - a_{j_1k_2}a_{j_2k_3}\right) R_{k_3}R_{k_2} \\
&= \left(a_{j_1k_1}a_{j_2k_2} - a_{j_1k_2}a_{j_2k_1}\right) (R_{k_1}R_{k_2} - R_{k_2}R_{k_1}) \\
&+ \left(a_{j_1k_1}a_{j_2k_3} - a_{j_1k_3}a_{j_2k_1}\right) (R_{k_1}R_{k_3} - R_{k_3}R_{k_1}) \\
&+ \left(a_{j_1k_2}a_{j_2k_3} - a_{j_1k_3}a_{j_2k_2}\right) (R_{k_2}R_{k_3} - R_{k_3}R_{k_2}) \\
&= \det \begin{pmatrix} a_{j_1k_1} & a_{j_1k_2} \\ a_{j_2k_1} & a_{j_2k_2} \end{pmatrix} \left[R_{k_1}, R_{k_2}\right] \\
&+ \det \begin{pmatrix} a_{j_1k_1} & a_{j_1k_3} \\ a_{j_2k_1} & a_{j_2k_3} \end{pmatrix} \left[R_{k_1}, R_{k_3}\right] \\
&+ \det \begin{pmatrix} a_{j_1k_2} & a_{j_1k_3} \\ a_{j_2k_2} & a_{j_2k_3} \end{pmatrix} \left[R_{k_2}, R_{k_3}\right],
\end{aligned}
$$

which agrees with the formula. For $n = 4$, one can similarly obtain

$$
\begin{aligned}
\left[S_{j_1}, S_{j_2}\right] &= \left[\sum_{t=1}^{4} a_{j_1k_t}R_{k_t}, \sum_{t=1}^{4} a_{j_2k_t}R_{k_t}\right] \\
&= \det \begin{pmatrix} a_{j_1k_1} & a_{j_1k_2} \\ a_{j_2k_1} & a_{j_2k_2} \end{pmatrix} \left[R_{k_1}, R_{k_2}\right] \\
&+ \det \begin{pmatrix} a_{j_1k_1} & a_{j_1k_3} \\ a_{j_2k_1} & a_{j_2k_3} \end{pmatrix} \left[R_{k_1}, R_{k_3}\right] \\
&+ \det \begin{pmatrix} a_{j_1k_1} & a_{j_1k_4} \\ a_{j_2k_1} & a_{j_2k_4} \end{pmatrix} \left[R_{k_1}, R_{k_4}\right] \\
&+ \det \begin{pmatrix} a_{j_1k_2} & a_{j_1k_3} \\ a_{j_2k_2} & a_{j_2k_3} \end{pmatrix} \left[R_{k_2}, R_{k_3}\right] \\
&+ \det \begin{pmatrix} a_{j_1k_2} & a_{j_1k_4} \\ a_{j_2k_2} & a_{j_2k_4} \end{pmatrix} \left[R_{k_2}, R_{k_4}\right] \\
&+ \det \begin{pmatrix} a_{j_1k_3} & a_{j_1k_4} \\ a_{j_2k_3} & a_{j_2k_4} \end{pmatrix} \left[R_{k_3}, R_{k_4}\right],
\end{aligned}
$$

which also agrees with the formula.

Corollary 22.13 *Let $a, b, c, d \in \mathbb{C}$.*

1. *In any algebra, if*

$$
\begin{aligned}
S &= aR + bJ, \\
T &= cR + dJ,
\end{aligned}
$$

then the commutator of S and T is given by

$$[S, T] = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} [R, J].$$

2. *If* $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \neq 0$, *that is,* $ad \neq bc$, *then* $ST = TS$ *if and only if* $RJ = JR$.

Example 22.14 In any algebra, if $S = R + iJ$ and $T = R - iJ$, then the commutator of S and T is given by

$$[S, T] = -2i[R, J],$$

and so, $ST = TS$ if and only if $RJ = JR$.

Theorem 22.8 *Let* $a, b, c, d \in \mathbb{C}$ *with* $ad \neq bc$. *If*

$$R = \frac{dS - bT}{ad - bc}$$
$$J = \frac{aT - cS}{ad - bc}$$

then the elements R, J *and* Q *satisfy the relations*

$$RQ = \frac{ad\sigma(Q) - bc\tau(Q)}{ad - bc} R + \frac{bd\sigma(Q) - bd\tau(Q)}{ad - bc} J, \quad (22.117)$$
$$JQ = \frac{ad\tau(Q) - bc\sigma(Q)}{ad - bc} J + \frac{ac\tau(Q) - ac\sigma(Q)}{ad - bc} R, \quad (22.118)$$

if and only if the elements S, T *and* Q *satisfy relations (22.16).*

Proof Writing $S = aR + bJ$ and $T = cR + dJ$, we have $R = (dS - bT)/(ad - bc)$ and $J = (aT - cS)/(ad - bc)$. Therefore, if relations (22.16) hold, then

$$\begin{aligned} RQ &= \left(\frac{dS - bT}{ad - bc}\right) Q \\ &= \frac{dSQ - bTQ}{ad - bc} \\ &= \frac{d\sigma(Q)S - b\tau(Q)T}{ad - bc} \\ &= \frac{d\sigma(Q)(aR + bJ) - b\tau(Q)(cR + dJ)}{ad - bc} \\ &= \frac{ad\sigma(Q)R + bd\sigma(Q)J - bc\tau(Q)R - bd\tau(Q)J}{ad - bc} \\ &= \frac{ad\sigma(Q) - bc\tau(Q)}{ad - bc} R + \frac{bd\sigma(Q) - bd\tau(Q)}{ad - bc} J, \end{aligned}$$

and

$$
\begin{aligned}
JQ &= \left(\frac{aT - cS}{ad - bc}\right) Q \\
&= \frac{aTQ - cSQ}{ad - bc} \\
&= \frac{a\tau(Q)T - c\sigma(Q)S}{ad - bc} \\
&= \frac{a\tau(Q)(cR + dJ) - c\sigma(Q)(aR + bJ)}{ad - bc} \\
&= \frac{ac\tau(Q)R + ad\tau(Q)J - ac\sigma(Q)R - bc\sigma(Q)J}{ad - bc} \\
&= \frac{ad\tau(Q) - bc\sigma(Q)}{ad - bc} J + \frac{ac\tau(Q) - ac\sigma(Q)}{ad - bc} R.
\end{aligned}
$$

Conversely, if (22.117) and (22.118) hold, then

$$
\begin{aligned}
SQ &= (aR + bJ)Q = aRQ + bJQ \\
&= a\left(\frac{ad\sigma(Q) - bc\tau(Q)}{ad - bc} R + \frac{bd\sigma(Q) - bd\tau(Q)}{ad - bc} J\right) \\
&\quad + b\left(\frac{ad\tau(Q) - bc\sigma(Q)}{ad - bc} J + \frac{ac\tau(Q) - ac\sigma(Q)}{ad - bc} R\right) \\
&= \frac{aad\sigma(Q) - abc\tau(Q)}{ad - bc} R + \frac{abd\sigma(Q) - abd\tau(Q)}{ad - bc} J \\
&\quad + \frac{abd\tau(Q) - bbc\sigma(Q)}{ad - bc} J + \frac{abc\tau(Q) - abc\sigma(Q)}{ad - bc} R \\
&= \frac{aad\sigma(Q) - abc\sigma(Q)}{ad - bc} R + \frac{abd\sigma(Q) - bbc\sigma(Q)}{ad - bc} J \\
&= \frac{ad - bc}{ad - bc} a\sigma(Q)R + \frac{ad - bc}{ad - bc} b\sigma(Q)J \\
&= a\sigma(Q)R + b\sigma(Q)J \\
&= \sigma(Q)(aR + bJ) \\
&= \sigma(Q)S,
\end{aligned}
$$

and

$$
\begin{aligned}
TQ &= (cR + dJ)Q = cRQ + dJQ \\
&= c\left(\frac{ad\sigma(Q) - bc\tau(Q)}{ad - bc} R + \frac{bd\sigma(Q) - bd\tau(Q)}{ad - bc} J\right) \\
&\quad + d\left(\frac{ad\tau(Q) - bc\sigma(Q)}{ad - bc} J + \frac{ac\tau(Q) - ac\sigma(Q)}{ad - bc} R\right) \\
&= \frac{acd\sigma(Q) - bcc\tau(Q)}{ad - bc} R + \frac{bcd\sigma(Q) - bcd\tau(Q)}{ad - bc} J
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{add\tau(Q)-bcd\sigma(Q)}{ad-bc}J+\frac{acd\tau(Q)-acd\sigma(Q)}{ad-bc}R \\
&=\frac{acd\tau(Q)-bcc\tau(Q)}{ad-bc}R+\frac{add\tau(Q)-bcd\tau(Q)}{ad-bc}J \\
&=\frac{ad-bc}{ad-bc}c\tau(Q)R+\frac{ad-bc}{ad-bc}d\tau(Q)J \\
&=c\tau(Q)R+d\tau(Q)J \\
&=\tau(Q)(cR+dJ) \\
&=\tau(Q)T.
\end{aligned}
$$

□

22.7 Reordering Formulas for R_j, Q-Elements

Theorem 22.9 *Let $a, b, c, d \in \mathbb{C}$ with $ad \neq bc$. If R, J and Q are elements of an algebra satisfying relations* (22.117) *and* (22.118)*, then for any nonnegative integer k,*

$$RQ^k=\frac{ad\sigma(Q)^k-bc\tau(Q)^k}{ad-bc}R+\frac{bd\sigma(Q)^k-bd\tau(Q)^k}{ad-bc}J, \tag{22.119}$$

$$JQ^k=\frac{ad\tau(Q)^k-bc\sigma(Q)^k}{ad-bc}J+\frac{ac\tau(Q)^k-ac\sigma(Q)^k}{ad-bc}R. \tag{22.120}$$

Proof By Theorem 22.8, relations (22.117) and (22.118) hold if relations (22.16) hold with $R=(dS-bT)/(ad-bc)$ and $J=(aT-cS)/(ad-bc)$. Therefore,

$$
\begin{aligned}
RQ^k &= \left(\frac{dS-bT}{ad-bc}\right)Q^k \\
&=\frac{dSQ^k-bTQ^k}{ad-bc} \\
&=\frac{d\sigma(Q)^kS-b\tau(Q)^kT}{ad-bc} \\
&=\frac{d\sigma(Q)^k(aR+bJ)-b\tau(Q)^k(cR+dJ)}{ad-bc} \\
&=\frac{ad\sigma(Q)^kR+bd\sigma(Q)^kJ-bc\tau(Q)R-bd\tau(Q)^kJ}{ad-bc} \\
&=\frac{ad\sigma(Q)^k-bc\tau(Q)^k}{ad-bc}R+\frac{bd\sigma(Q)^k-bd\tau(Q)^k}{ad-bc}J,
\end{aligned}
$$

and

$$
\begin{aligned}
JQ^k &= \left(\frac{aT - cS}{ad - bc}\right) Q^k \\
&= \frac{aTQ^k - cSQ^k}{ad - bc} \\
&= \frac{a\tau(Q)^k T - c\sigma(Q)^k S}{ad - bc} \\
&= \frac{a\tau(Q)^k (cR + dJ) - c\sigma(Q)^k (aR + bJ)}{ad - bc} \\
&= \frac{ac\tau(Q)^k R + ad\tau(Q)^k J - ac\sigma(Q)^k R - bc\sigma(Q)^k J}{ad - bc} \\
&= \frac{ad\tau(Q)^k - bc\sigma(Q)^k}{ad - bc} J + \frac{ac\tau(Q)^k - ac\sigma(Q)^k}{ad - bc} R.
\end{aligned}
$$

□

Corollary 22.14 *If R, J and Q are elements of an algebra satisfying relations (22.117) and (22.118), then for any polynomial $F(\cdot)$ in one variable,*

$$
RF(Q) = \frac{adF(\sigma(Q)) - bcF(\tau(Q))}{ad - bc} R + \frac{bdF(\sigma(Q)) - bdF(\tau(Q))}{ad - bc} J, \tag{22.121}
$$

$$
JF(Q) = \frac{adF(\sigma(Q)) - bcF(\tau(Q))}{ad - bc} J + \frac{acF(\sigma(Q)) - acF(\tau(Q))}{ad - bc} R. \tag{22.122}
$$

Proof Theorem 22.9 implies that given a polynomial $F(Q) = \sum a_k Q^k$, we have

$$
\begin{aligned}
RF(Q) &= \sum a_k RQ^k \\
&= \sum a_k \left(\frac{ad\sigma(Q)^k - bc\tau(Q)^k}{ad - bc} R + \frac{bd\sigma(Q)^k - bd\tau(Q)^k}{ad - bc} J \right) \\
&= \frac{adF(\sigma(Q)) - bcF(\tau(Q))}{ad - bc} R + \frac{bdF(\sigma(Q)) - bdF(\tau(Q))}{ad - bc} J.
\end{aligned}
$$

Similarly for $JF(Q)$, that is,

$$
\begin{aligned}
JF(Q) &= \sum a_k JQ^k \\
&= \sum a_k \left(\frac{ad\tau(Q)^k - bc\sigma(Q)^k}{ad - bc} J + \frac{ac\tau(Q)^k - ac\sigma(Q)^k}{ad - bc} R \right) \\
&= \frac{adF(\sigma(Q)) - bcF(\tau(Q))}{ad - bc} J + \frac{acF(\sigma(Q)) - acF(\tau(Q))}{ad - bc} R.
\end{aligned}
$$

□

Corollary 22.15 *If R, J and Q are elements of an algebra satisfying relations* (22.117) *and* (22.118)*, then*

$$R\sigma(Q) = \frac{ad\sigma^2(Q) - bc(\sigma\circ\tau)(Q)}{ad-bc}R + \frac{bd\sigma^2(Q) - bd(\sigma\circ\tau)(Q)}{ad-bc}J, \quad (22.123)$$

$$J\sigma(Q) = \frac{ad\sigma^2(Q) + bc(\sigma\circ\tau)(Q)}{ad-bc}J + \frac{ac\sigma^2(Q) - ac(\sigma\circ\tau)(Q)}{ad-bc}R, \quad (22.124)$$

$$R\tau(Q) = \frac{ad(\tau\circ\sigma)(Q) - bc\tau^2(Q)}{ad-bc}R + \frac{bd(\tau\circ\sigma)(Q) - bd\tau^2(Q)}{ad-bc}J, \quad (22.125)$$

$$J\tau(Q) = \frac{ad(\tau\circ\sigma)(Q) - bc\tau^2(Q)}{ad-bc}J + \frac{ac(\tau\circ\sigma)(Q) - ac\tau^2(Q)}{ad-bc}R. \quad (22.126)$$

Proof This result follows directly from Corollary 22.14 by letting $F(x) = \sigma(x)$ for the first two formulas, and $F(x) = \tau(x)$ for the last two formulas. □

22.8 Some Operator Representations

We conclude by mentioning that a concrete representation of relations (22.1) is given by the operators $\alpha_{\sigma_j}(f)(x) = f(\sigma_j(x))$ and $Q_x(f)(x) = xf(x)$ acting on polynomials or other suitable functions. Furthermore, a concrete representation of relations (22.3) is given by the operators

$$R_{\sigma,\tau}(f)(x) = \frac{adf(\sigma(x)) - bcf(\tau(x))}{ad-bc}, \quad (22.127)$$

$$J_{\sigma,\tau}(f)(x) = \frac{acf(\tau(x)) - acf(\sigma(x))}{ad-bc}, \quad (22.128)$$

$$Q_x(f)(x) = xf(x) \quad (22.129)$$

also acting on polynomials or other suitable functions. For $\sigma(x) = x + i$, $\tau(x) = x - i$, $a = c = 1$, $b = i$ and $d = -i$, these operators reduce to the operators

$$R_i(f)(x) = \frac{f(x+i) + f(x-i)}{2}, \quad (22.130)$$

$$J_i(f)(x) = \frac{f(x+i) - f(x-i)}{2i}, \quad (22.131)$$

$$Q_x(f)(x) = xf(x) \quad (22.132)$$

acting on complex functions. Three systems of orthogonal polynomials belonging to the class of Meixner–Pollaczek polynomials that are connected by these operators were presented in [14, 23, 25]. Boundedness properties of the operators R_i^{-1} and

$J_i R_i^{-1}$ in the function spaces related to the three systems of orthogonal polynomials were investigated in [15, 25].

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References

1. Aleixo, A.N.F., Balantekin, A.B.: The ladder operator normal ordering problem for quantum confined systems and the generalization of the Stirling and Bell numbers. J. Phys. A: Math. Theor. **43**, 045302 (2010)
2. Al-Salam, W.A., Ismail, M.E.H.: Some operational formulas. J. Math. Anal. Appl. **51**, 208–218 (1975)
3. Bender, C.M., Dunne, G.V.: Polynomials and operator orderings. J. Math. Phys. **29**, 1727–1731 (1988)
4. Blasiak, P., Horzela, A., Penson, K.A., Duchamp, G.H.E., Solomon, A.I.: Boson normal ordering via substitutions and Sheffer-type polynomials. Phys. Lett. A **338**, 108–116 (2005)
5. Bourgeois, G.: How to solve the matrix equation $XA - AX = f(X)$. Linear Algebra Appl. **434**, 657–668 (2011)
6. Bratteli, O., Jorgensen, P.E.T.: Wavelets Through a Looking Glass. Birkhäuser Verlag (2002)
7. Bratteli, O., Robinson, D.: Operator Algebras and Statistical Mechanics. Springer (1981)
8. Davidson, K.R.: C*-Algebras by Example. American Mathematical Society (1996)
9. Hellström, L., Silvestrov, S.D.: Commuting Elements in q-Deformed Heiseberg Algebras. World Scientific (2000)
10. Hellström, L., Silvestrov, S.: Two-sided ideals in q-deformed Heisenberg algebras. Expo. Math. **23**, 99–125 (2005)
11. Holschneider, T.: Wavelets: An Analysis Tool. Clarendon Press, Oxford (1998)
12. Jorgensen, P.E.T.: Ruelle operators: functions which are harmonic with respect to a transfer operator. Mem. Am. Math. Soc. **152**(720) (2001)
13. Karasev, M.V., Maslov, V.P.: Nonlinear Poisson Brackets. Geometry and Quantization, Translations of Mathematical Monographs, vol. 119. American Mathematical Society, Providence (1993)
14. Kaijser, S.: Några nya ortogonala polynom. Normat **47**(4), 156–165 (1999)
15. Kaijser, S., Musonda, J.: L^p-boundedness of two singular integral operators of convolution type. In: Silvestrov, S., Rančić, M. (eds.), Engineering Mathematics II, Springer Proceedings in Mathematics & Statistics, vol. 179 (2016)
16. Li, B.R.: Introduction to Operator Algebras. World Scientific (1992)
17. Mackey, G.W.: Induced Representations of Groups and Quantum Mechanics. Editore Boringhieri (1968)
18. Mackey, G.W.: The Theory of Unitary Group Representations. The University of Chicago Press (1976)
19. Mackey, G.W.: Unitary Group Representations in Physics, Probability and Number Theory. Addison-Wesley (1989)

20. Mansour, T., Schork, M.: Commutation Relations, Normal Ordering, and Stirling Numbers. CRC Press (2016)
21. Mansour, T., Schork, M.: On a close relative of the quantum plane. Mediterr. J. Math. **15**, 124 (2018)
22. Meng, X.G., Wang, J.S., Liang, B.L.: Normal ordering and antinormal ordering of the operator $(fq + gp)^n$ and some of their applications. Chin. Phys. B. **18**, 1534–1538 (2009)
23. Musonda, J.: Three Systems of Orthogonal Polynomials and Associated Operators, U.U.D.M. Project Report (2012:8)
24. Musonda, J.: Orthogonal Polynomials, Operators and Commutation Relations, Mälardalen University Press Licentiate Theses Mälardalen University ISBN 978-91-7485-320-9 (2017)
25. Musonda, J., Kaijser, S.: Three systems of orthogonal polynomials and L^2-boundedness of two associated operators. J. Math. Anal. Appl. **459**, 464–475 (2018)
26. Musonda, J.: Reordering in Noncommutative Algebras, Orthogonal Polynomials and Operators, Mälardalen University Press Dissertations, Mälardalen University ISBN 978-91-748-411-4 (2018)
27. Musonda, J., Richter, J., Silvestrov, S.D.: Reordering in a multi-parametric family of algebras. J. Phys.: Conf. Ser. **1194**, 012078 (2019)
28. Nazaikinskii, V.E., Shatalov, V.E., Sternin, B.Yu.: Methods of Noncommutative Analysis. Theory and Applications, De Gruyter Studies in Mathematics, vol. 22. Walter De Gruyter & Co., Berlin (1996)
29. Ostrovskyĭ, V.L., Samoĭlenko, Yu.S.: Introduction to the theory of representations of finitely presented $*$-Algebras. I. Representations by bounded operators. Rev. Math. Math. Phys. **11**. Gordon and Breach (1999)
30. Pedersen, G.K.: C*-Algebras and their Automorphism Groups. Academic Press (1976)
31. Persson, T., Silvestrov, S.D.: From dynamical systems to commutativity in non-commutative operator algebras. In: A. Khrennikov (ed.), Dynamical systems from number theory to probability - 2, Växjö University Press, Series: Mathematical Modeling in Physics, Engineering and Cognitive Science, vol. 6, 109–143 (2003)
32. Persson, T., Silvestrov, S.D.: Commuting elements in non-commutative algebras associated with dynamical systems. In: A. Khrennikov (ed.), Dynamical systems from number theory to probability - 2, Växjö University Press, Series: Mathematical Modeling in Physics, Engineering and Cognitive Science, vol. 6, 145–172 (2003)
33. Persson, T., Silvestrov, S.D.: Commuting operators for representations of commutation relations defined by dynamical systems. Numer. Funct. Anal. Optim. **33**(7–9), 1126–1165 (2012)
34. Rynne, B.P., Youngson, M.A.: Linear Functional Analysis. Springer (2008)
35. Sakai, S.: Operator Algebras in Dynamical Systems. Cambridge University Press (1991)
36. Samoilenko, Y., Ostrovskyi, V.L.: Introduction to the theory of representations of finitely presented $*$-algebras. Representations by bounded operators. Rev. Math. Phys. **11**. The Gordon and Breach Publishing Group (1999)
37. Samoĭlenko, Y.S.: Spectral Theory of Families of Selfadjoint Operators, Kluwer Academic Publishers (1991)
38. Schmudgen, K.: Unbounded Operator Algebras and Representation Theory. Birkhauser Verlag (1990)
39. Shibukawa, G.: Operator orderings and Meixner-Pollaczek polynomials. J. Math. Phys. **54**, 033510 (2013)
40. Silvestrov, S.D., Tomiyama, J.: Topological dynamical systems of Type I. Expositiones Mathematicae **20**(2), 117–142 (2002)
41. Silvestrov, S.D.: Representations of commutation relations. A dynamical systems approach. Haddronic J. Suppl. **11** (1996)
42. Silvestrov, S.D.: On rings generalizing commutativity. Czech. J. Phys. **48**, 1495–1500 (1998)
43. Silvestrov, S.D., Tomiyama, Y.: Topological dynamical systems of Type I. Expos. Math. **20**, 117–142 (2002)
44. Suzuki, T., Hirshfeld, A.C., Leschke, H.: The role of operator ordering in quantum field theory. Prog. Theor. Phys. **63**, 287–302 (1980)

45. Svensson, C., Silvestrov, S., de Jeu, M.: Dynamical systems and commutants in crossed products. Int. J. Math. **18**(4), 455–471 (2007)
46. Tomiyama, J.: C*-Algebras and topological dynamical systems. Rev. Math. Phys. **8**, 741–760 (1996)
47. Varvak, A.: Rook numbers and the normal ordering problem. J. Comb. Theory, Ser. **112**, 292–307 (2005)