Chapter 21
Reordering, Centralizers and Centers in an Algebra with Three Generators and Lie Type Relations

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Abstract Simple and explicit formulas for reordering elements in an algebra with three generators and Lie type relations are derived. Centralizers and centers are computed as an example of an application of the formulas.

Keywords Centralizers · Center · Commutation relations · Reordering formulas

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21.1 Introduction

This paper is about reordering of elements in noncommutative algebras defined by commutation relations. We follow the nice exposition by Mansour and Schork [8]. A commutation relation is a relation that describes the discrepancy between different orders of operation of two operations, say A and B . To describe it, we use the commutator $[A, B] \equiv AB - BA$. If A and B commute, then the commutator vanishes. How far a given structure deviates from the commutative case is described by the right-hand side of the commutation relation. For example, in a complex Lie algebra $\mathfrak{g}$ one has a set of generators $\{A_w\}_{w \in W}$ with the Lie bracket $\left[A_j A_k\right] = \sum_{l \in W} c^l_{jk} A_l$, where the coefficients $c^l_{jk} \in \mathbb{C}$ are called the structure constants of the Lie algebra $\mathfrak{g}$.

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The associated universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is an associative algebra generated by $\{A_w\}_{w\in W}$, and the above bracket becomes

$$\left[A_j, A_k\right] = \sum_{l\in W} c^l_{jk} A_l.$$

One of the earliest instances of a noncommutative structure was recognized in the context of operational calculus. If $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the ordinary derivative, then the Leibniz rule (the product rule) states that

$$D(xf(x)) = xD(f(x)) + D(x)f(x).$$

Interpreting the multiplication with the independent variable x as an application of the multiplication operator M_x, and suppressing the operand f, this equation can be written as the commutation relation

$$DM_x - M_x D = \mathbb{1},$$

where $\mathbb{1}$ is the identity operator: $\mathbb{1}f(x) = f(x)$.

The main object studied in this paper is the associative $\mathbb{C}$-algebra $\mathcal{A}_{R,J,Q}$ generated by three elements R, J and Q satisfying the commutation relations

$$QR - RQ = J, \quad QJ - JQ = -R, \quad RJ - JR = 0. \tag{21.1}$$

In the sequel, we consider the effect of adding the constraint $J^2 + R^2 = \mathbb{1}$, where $\mathbb{1}$ is the identity element. A concrete representation is given by the operators

$$\begin{aligned} R_i(f)(x) &= \frac{f(x+i) + f(x-i)}{2}, \\ J_i(f)(x) &= \frac{f(x+i) - f(x-i)}{2i}, \\ Q_x(f)(x) &= xf(x) \end{aligned}$$

acting on complex functions (see [11, p. 61], [10, p. 14], or [14, pp. 468–469]). The main goal of this paper is to compute the centralizers of elements and thus the center in these algebras using some simple and explicit reordering formulas.

In this paper, reordering an element in R, J and Q means to bring it, using the commutation relations, into a form where all elements Q stand to the right. For example,

$$Q^2R = RQ^2 + 2JQ - R.$$

Similarly, one can use commutation relations (21.1) successively and transform for any positive integer n the element Q^nR into a form where all elements Q stand to the right. The coefficients which appear upon reordering in this case are the binomial coefficients. In general, as demonstrated in this example, one can use commutation relations (21.1) successively and transform each element ω in R, J and Q into its normal ordered form

$$\omega = \sum_{j,k,l\in\mathbb{N}_0} A_{jkl}(\omega) R^j J^k Q^l,$$

where the coefficients $A_{jkl}(\omega)$, depending on the exponents j, k and l, are called the normal ordering coefficients of ω. Since R and J commute, this can be written as

$$\omega = \sum_{k=0}^{n} p_k(R, J) Q^k, \tag{21.2}$$

where the coefficients $p_k(R, J)$ are polynomials in R and J. Writing $S = R + iJ$ and $T = R - iJ$, we have $R = (S + T)/2$, $J = (S - T)/2i$ and $ST = R^2 + J^2$. Therefore, denoting $f(S, T) = p((S+T)/2, (S-T)/2i)$, we see that polynomials in R and J can also be written as polynomials in S and T, and in this case an arbitrary element α in S, T and Q can be transformed into its normal ordered form

$$\alpha = \sum_{k=0}^{n} p_k(S, T) Q^k. \tag{21.3}$$

Reordering of arbitrary elements in noncommutative algebras defined by commutation relations is important in many research directions, open problems and applications of the algebras and their operator representations. For a broader view of this active area of research, see, for example, [1–9, 13, 15–18, 22, 24–27, 29] and the references therein.

In investigation of the structure, representations and applications of noncommutative algebras, an important role is played by the explicit description of suitable normal forms for noncommutative expressions or functions of generators. These normal forms are particularly important for computing commutative subalgebras or commuting families of operators which are a key ingredient in representation theory of many important algebras [12, 19–21, 23, 28]. In this paper the norm forms (21.2) and (21.3) are used to compute the centralizers of elements and thus the center.

Definition 21.1 Let $\mathcal{A}$ be any algebra. The centralizer of $g \in \mathcal{A}$, denoted by $\mathrm{Cen}(g)$, is the set of all elements of $\mathcal{A}$ that commute with g. That is,

$$\mathrm{Cen}(g) = \big\{h \in \mathcal{A} : gh - hg = 0\big\}.$$

The center of $\mathcal{A}$, denoted by $\mathrm{Z}(\mathcal{A})$, is the set of all elements of $\mathcal{A}$ that commute with every element of $\mathcal{A}$. That is,

$$\mathrm{Z}(\mathcal{A}) = \big\{g \in \mathcal{A} : gh - hg = 0 \ \text{ for all } h \in \mathcal{A}\}.$$

It follows that the center of an algebra is the intersection of the centralizers of every element in the algebra. Note that it suffices to find the centralizers for a set of generators. In the case of the algebra $\mathcal{A}_{R,J,Q}$, we first compute the centralizers of R, J, and Q, and then obtain the center of the algebra $\mathcal{A}_{R,J,Q}$ as the intersection of these centralizers. Since every element in the algebra $\mathcal{A}_{R,J,Q}$ can be transformed into its normal ordered form, it suffices to find the elements $\omega = \sum_{k=0}^{n} p_k(R, J)Q^k$ for which the commutators $[Q, \omega]$, $[\omega, R]$, and $[\omega, J]$ vanish. And in order to do this, we need to compute reordered expressions for the commutators $[Q, p(R, J)]$, $[Q^n, R]$ and $[Q^n, J]$. This is done in Sect. 21.2, and it turns out that the commutator of the element Q and the polynomials $p(R, J)$ in R and J is the partial differential operator

$$[Q, p(R, J)] = J\frac{\partial p(R, J)}{\partial R} - R\frac{\partial p(R, J)}{\partial J}.$$

For example, for any nonnegative integer n, one obtains

$$[Q, R^n] = nR^{n-1}J \quad \text{and} \quad [Q, J^n] = -nRJ^{n-1}.$$

For the monomials in Q, we have that for any nonnegative integer k,

$$[Q^k, R] = \sum_{j=1}^{k}(-1)^{j(j-1)/2}\binom{k}{j} I_j Q^{k-j},$$
$$[Q^k, J] = \sum_{j=1}^{k}(-1)^{j(j+1)/2}\binom{k}{j} I_{j+1} Q^{k-j},$$

where $I_m = R$ for m even, and $I_m = J$ for m odd. These formulas are used in Sect. 21.3 to compute the centralizers of R, J and Q, and thus the center of $\mathcal{A}_{R,J,Q}$. One can also compute the center of the associative algebra $\mathcal{A}_{R,J,Q}$ by first making the transformations $S = R + iJ$ and $T = R - iJ$. This is done is Sect. 21.4. The centralizers of Q, S, and T are then computed as the elements $\alpha = \sum_{k=0}^{n} p_k(S, T)Q^k$ for which the commutators $[Q, \alpha]$, $[\alpha, S]$, and $[\alpha, T]$ vanish. The reordered expressions for the commutators $[Q, p(S, T)]$, $[Q^n, S]$ and $[Q^n, T]$ are derived and proved in Sect. 21.5. It turns out that for all nonnegative integers k, m and n,

$$Q^k S^m T^n = S^m T^n \left(Q + (n-m)i\mathbb{1}\right)^k.$$

For $k = 1$, this reduces to the commutator formula

$$[Q, S^m T^n] = (n-m)iS^m T^n,$$

which shows that $S^m T^n$ is an eigenvector of $[Q, \cdot]$ with eigenvalue $(n-m)i$, something which, perhaps, is interesting in its own right. The centralizers of Q, S, and T, and thus the center, are then computed in Sect. 21.6.

21.2 Some Commutator Formulas in $\mathcal{A}_{R,J,Q}$

The commutator of two elements A and B of an algebra $\mathcal{A}$ is defined by

$$[A, B] = AB - BA.$$

When $\mathcal{A}$ is unital, $\mathbb{1}$ denotes the unit element. It is easy to see that for all $A, B, C \in \mathcal{A}$, $p, q \in \mathbb{C}$.

(a) $[A, q\mathbb{1}] = 0$,
(b) $[A, A] = 0$,
(c) $[A, B] = -[B, A]$,
(d) $[A, pB + qC] = p[A, B] + q[A, C]$,
(e) $[A, BC] = [A, B]C + B[A, C]$,
(f) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$.

Commutation relations (21.1) can also be written in terms of the commutator:

$$[Q, R] = J, \quad [Q, J] = -R, \quad [R, J] = 0.$$

We now derive and prove reordered expressions for $[Q, p(R, J)]$, $[Q^n, R]$ and $[Q^n, J]$.

21.2.1 An Expression for $[Q, p(R, J)]$

We first derive expressions for the commutators $[Q, R^m]$ and $[Q, J^n]$ for all positive integers m and n. For $[Q, R^m]$, we start by looking at the effect of increasing the power of R. For $m = 1, 2, 3$, we have

$$[Q, R^1] = J = 1JR^0, \tag{21.4}$$
$$[Q, R^2] = [Q, RR] = [Q, R]R + R[Q, R] = JR + RJ = 2RJ,$$
$$[Q, R^3] = [Q, R^2R] = [Q, R^2]R + R^2[Q, R] = (2RJ)R + R^2J = 3R^2J.$$

Observing the pattern, we deduce the following result:

Lemma 21.1 *For all positive integers n,*

$$[Q, R^m] = mR^{m-1}J.$$

Proof For $m = 1$, the formula follows from (21.4). Now suppose that the formula holds for some integer $m \geq 1$, then

$$[Q, R^{m+1}] = [Q, R^m R] = [Q, R^m]R + R^m[Q, R]$$
$$= (mR^{m-1}J)R + R^m J = (m+1)R^m J,$$

and this proves the assertion. □

For $[Q, J^n]$ and $n = 1, 2, 3$, we have

$$[Q, J^1] = -R = -1RJ^0, \tag{21.5}$$
$$[Q, J^2] = [Q, J]J + J[Q, J] = (-R)J + J(-R) = -2RJ,$$
$$[Q, J^3] = [Q, JJ^2] = [Q, J]J^2 + J[Q, J^2] = (-R)J^2 + J(-2RJ) = -3RJ^2.$$

Observing the pattern, we deduce the following result:

Lemma 21.2 *For all positive integers n,*

$$[Q, J^n] = -nRJ^{n-1}.$$

Proof For $n = 1$, the formula follows from (21.5). Now suppose that the formula holds for some integer $n \geq 1$, then

$$\begin{aligned}[Q, J^{n+1}] &= [Q, JJ^n] \\ &= [Q, J]J^n + J[Q, J^n] \\ &= (-R)J^n + J(-nJ^{n-1}R) \\ &= -(n+1)RJ^n,\end{aligned}$$

and this proves the assertion. □

Proposition 21.1 *For all polynomials $p(R, J)$ in R and J,*

$$[Q, p(R, J)] = J\frac{\partial p(R, J)}{\partial R} - R\frac{\partial p(R, J)}{\partial J}. \tag{21.6}$$

Proof Combining Lemmas 21.1 and 21.2, we have

$$\begin{aligned}[Q, R^m J^n] &= [Q, R^m]J^n + R^m[Q, J^n] \\ &= (mR^{m-1}J)J^n + R^m(-nRJ^{n-1}) \\ &= J(mR^{m-1}J^n) - R(nR^m J^{n-1}) \\ &= J\frac{\partial(R^m J^n)}{\partial R} - R\frac{\partial(R^m J^n)}{\partial J}\end{aligned}$$

for all nonnegative integers m and n. By linearity of commutator and linearity of partial derivative operator, this implies (21.6). The general solution to the partial differential equation

$$J\frac{\partial p(R,J)}{\partial R} - R\frac{\partial p(R,J)}{\partial J} = 0$$

is given by $p(R, J) = f(R^2 + J^2)$ for some polynomial f. □

21.2.2 An Expression for $[Q^n, J]$

We start by looking at the effect of increasing the power of Q . For $n = 1, 2, 3, 4, 5$,

$$QJ = JQ - R,$$

$$\begin{aligned} Q^2J &= Q(QJ) = Q(JQ - R) \\ &= (JQ - R)Q - (RQ + J) \\ &= JQ^2 - 2RQ - J, \end{aligned}$$

$$\begin{aligned} Q^3J &= Q(Q^2J) = Q(JQ^2 - 2RQ - J) \\ &= (JQ - R)Q^2 - 2(RQ + J)Q - (JQ - R) \\ &= JQ^3 - 3RQ^2 - 3JQ + R \end{aligned}$$

$$\begin{aligned} Q^4J &= Q(Q^3J) = Q(JQ^3 - 3RQ^2 - 3JQ + R) \\ &= (JQ - R)Q^3 - 3(RQ + J)Q^2 - 3(JQ - R)Q + (RQ + J) \\ &= JQ^4 - 4RQ^3 - 6JQ^2 + 4RQ + J \end{aligned}$$

$$\begin{aligned} Q^5J &= Q(Q^4J) = Q(JQ^4 - 4RQ^3 - 6JQ^2 + 4RQ + J) \\ &= (JQ - R)Q^4 - 4(RQ + J)Q^3 - 6(JQ - R)Q^2 + 4(RQ + J)Q + (JQ - R) \\ &= JQ^5 - 5RQ^4 - 10JQ^3 + 10RQ^2 + 5JQ - R \end{aligned}$$

Following the pattern, we deduce that for all positive integers n,

$$\begin{aligned} Q^nJ - JQ^n = &-\binom{n}{1}RQ^{n-1} - \binom{n}{2}JQ^{n-2} + \binom{n}{3}RQ^{n-3} \\ &+ \binom{n}{4}JQ^{n-4} - \binom{n}{5}RQ^{n-5} - \binom{n}{6}JQ^{n-6} + \cdots . \end{aligned}$$

Observe that the signs of the terms alternate between negative and positive every two terms. Now the triangular numbers $1, 3, 6, 10, 15, \ldots$, given by the formula $T_n = n(n+1)/2$, have the property that T_{4k+1} and T_{4k+2} are odd and that T_{4k+3} and T_{4k+4} are even. So the expression $(-1)^{T_n}$ alternates in sign in the same way. Therefore, observing also that the terms' leading elements alternate between R and J, we have

$$Q^n J - J Q^n = \sum_{k=1}^{n} (-1)^{k(k+1)/2} \binom{n}{k} I_{k+1} Q^{n-k},$$

where $I_m = R$ for m even, and $I_m = J$ for m odd.

Proposition 21.2 *For all positive integers n,*

$$[Q^n, J] = \sum_{k=1}^{n} (-1)^{k(k+1)/2} \binom{n}{k} I_{k+1} Q^{n-k} \tag{21.7}$$

$$= \sum_{k=0}^{n-1} (-1)^{(n-k)(n-k+1)/2} \binom{n}{k} I_{n-k+1} Q^{k}, \tag{21.8}$$

where $I_m = R$ for m even, and $I_m = J$ for m odd.

Proof For formula (21.7), we prove the formula

$$Q^n J = \sum_{k=0}^{n} (-1)^{k(k+1)/2} \binom{n}{k} I_{k+1} Q^{n-k} \tag{21.9}$$

by induction on n; formula (21.8) follows by reindexing the sum. For $n = 1$, formula (21.9) becomes $QJ = JQ - R$, which is one of the defining relations of $\mathcal{A}_{R,J,Q}$ in (21.1). Now assume that (21.9) holds for some integer $n \geq 1$, then

$$\begin{aligned}
Q^{n+1} J &= Q(Q^n J) = Q \left(\sum_{k=0}^{n} (-1)^{k(k+1)/2} \binom{n}{k} I_{k+1} Q^{n-k} \right) \\
&= \left(\sum_{k=0}^{n} (-1)^{k(k+1)/2} \binom{n}{k} I_{k+1} Q^{n-k} \right) Q \\
&\quad + \sum_{k=0}^{n} (-1)^{k(k+1)/2} \binom{n}{k} [Q, I_{k+1}] Q^{n-k} \\
&= \sum_{k=0}^{n} (-1)^{k(k+1)/2} \binom{n}{k} I_{k+1} Q^{n+1-k} \\
&\quad + \sum_{k=1}^{n+1} (-1)^{k(k+1)/2} \binom{n}{k-1} I_{k+1} Q^{n+1-k} \\
&= \sum_{k=0}^{n+1} (-1)^{k(k+1)/2} \left(\binom{n}{k} + \binom{n}{k-1} \right) I_{k+1} Q^{n+1-k} \\
&= \sum_{k=0}^{n+1} (-1)^{k(k+1)/2} \binom{n+1}{k} I_{k+1} Q^{n+1-k},
\end{aligned}$$

where the fourth equality follows by reindexing the second sum and observing that

$$(-1)^{k(k-1)/2}[Q, I_k] = (-1)^{k(k+1)/2} I_{k+1},$$

and the last equality follows from Pascal's identity. □

21.2.3 An Expression for $[Q^n, R]$

We start by looking at the effect of increasing the power of Q . For $n = 1, 2, 3, 4, 5$,

$$QR = RQ + J,$$

$$\begin{aligned} Q^2R &= Q(QR) = Q(RQ + J) \\ &= (RQ + J)Q + (JQ - R) \\ &= RQ^2 + 2JQ - R, \end{aligned}$$

$$\begin{aligned} Q^3R &= Q(Q^2R) = Q(RQ^2 + 2JQ - R) \\ &= (RQ + J)Q^2 + 2(JQ - R)Q - (RQ + J) \\ &= RQ^3 + 3JQ^2 - 3RQ - J, \end{aligned}$$

$$\begin{aligned} Q^4R &= Q(Q^3R) = Q(RQ^3 + 3JQ^2 - 3RQ - J) \\ &= (RQ + J)Q^3 + 3(JQ - R)Q^2 - 3(RQ + J)Q - (JQ - R) \\ &= RQ^4 + 4JQ^3 - 6RQ^2 - 4JQ + R, \end{aligned}$$

$$\begin{aligned} Q^5R &= Q(Q^4R) = Q(RQ^4 + 4JQ^3 - 6RQ^2 - 4JQ + R) \\ &= (RQ + J)Q^4 + 4(JQ - R)Q^3 - 6(RQ + J)Q^2 - 4(JQ - R)Q + (RQ + J) \\ &= RQ^5 + 5JQ^4 - 10RQ^3 - 10JQ^2 + 5RQ + J, \end{aligned}$$

Following the pattern, we deduce that for all positive integers n,

$$\begin{aligned} Q^nR - RQ^n =& \binom{n}{1} JQ^{n-1} - \binom{n}{2} RQ^{n-2} - \binom{n}{3} JQ^{n-3} + \binom{n}{4} RQ^{n-4} \\ &+ \binom{n}{5} JQ^{n-5} - \binom{n}{6} RQ^{n-6} - \binom{n}{7} JQ^{n-7} + \cdots . \end{aligned}$$

Observe that the sign is positive on the first term and then alternates between negative and positive every two terms. Using the property of the triangular numbers $T_n = n(n+1)/2$, we see that the expression $(-1)^n(-1)^{T_n} = (-1)^{n(n+3)/2}$ alternates in sign in the same way. Therefore, observing also that the leading elements of the terms alternate between R and J, we can write

$$[Q^n, R] = \sum_{k=1}^{n}(-1)^{k(k+3)/2}\binom{n}{k} I_k Q^{n-k} = \sum_{k=0}^{n-1}(-1)^{(n-k)(n-k+3)/2}\binom{n}{k} I_{n-k} Q^k,$$

where $I_m = R$ for m even, and $I_m = J$ for m odd.

Proposition 21.3 *For all positive integers n,*

$$Q^n R - R Q^n = \sum_{k=1}^{n}(-1)^{k(k+3)/2}\binom{n}{k} I_k Q^{n-k} \tag{21.10}$$

$$= \sum_{k=0}^{n-1}(-1)^{(n-k)(n-k+3)/2}\binom{n}{k} I_{n-k} Q^k, \tag{21.11}$$

where $I_m = R$ for m even, and $I_m = J$ for m odd.

Proof For formula (21.10), we prove the formula

$$Q^n R = \sum_{k=0}^{n}(-1)^{k(k+3)/2}\binom{n}{k} I_k Q^{n-k} \tag{21.12}$$

by induction on n; formula (21.11) follows by reindexing the sum. For $n = 1$, (21.12) becomes $QR = RQ + J$, which is one of the defining relations of $\mathcal{A}_{R,J,Q}$ in (21.1). Now assume that (21.12) holds for some integer $n \geq 1$, then

$$\begin{aligned}
Q^{n+1} R &= Q(Q^n R)) = Q\left(\sum_{k=0}^{n}(-1)^{k(k+3)/2}\binom{n}{k} I_k Q^{n-k}\right) \\
&= \left(\sum_{k=0}^{n}(-1)^{k(k+3)/2}\binom{n}{k} I_k Q^{n-k}\right) Q \\
&\quad + \sum_{k=0}^{n}(-1)^{k(k+3)/2}\binom{n}{k} [Q, I_k] Q^{n-k} \\
&= \sum_{k=0}^{n}(-1)^{k(k+3)/2}\binom{n}{k} I_k Q^{n+1-k} \\
&\quad + \sum_{k=1}^{n+1}(-1)^{k(k+3)/2}\binom{n}{k-1} I_k Q^{n+1-k} \\
&= \sum_{k=0}^{n+1}(-1)^{k(k+3)/2}\left(\binom{n}{k} + \binom{n}{k-1}\right) I_k Q^{n+1-k} \\
&= \sum_{k=0}^{n+1}(-1)^{k(k+3)/2}\binom{n+1}{k} I_k Q^{n+1-k},
\end{aligned}$$

where the fourth equality follows by reindexing the second sum and observing that

$$(-1)^{(k-1)(k+2)/2}[Q, I_{k-1}] = (-1)^{k(k+3)/2} I_k,$$

and the last equality follows from Pascal's identity. □

21.3 Centralizers and the Center in $\mathcal{A}_{R,J,Q}$

Proposition 21.4 *For all nonnegative integers n,*

$$Q\left(\sum_{k=0}^{n} p_k(R, J) Q^k\right) - \left(\sum_{k=0}^{n} p_k(R, J) Q^k\right) Q = 0$$

if and only if $p_k(R, J) = p_k(R^2 + J^2)$ for all k.

Proof Writing $a = \sum p_k(R, J) Q^k$, we have

$$[Q, a] = \sum \left([Q, p_k(R, J)]Q^k + p_k(R, J)[Q, Q^k]\right).$$

Since $[Q, Q^k] = 0$, this implies that $[Q, a] = 0$ if and only if

$$0 = [Q, p_k(R, J)] = -R\frac{\partial p_k(R, J)}{\partial J} + J\frac{\partial p_k(R, J)}{\partial R},$$

which holds if and only if $p_k(R, J) = p_k(R^2 + J^2)$ for all k. □

It follows from this result that the centralizer of Q is given by

$$\text{Cen}(Q) = \left\{ a \in \mathcal{A}_{R,J,Q} : a = \sum_{k=0}^{n} p_k(R^2 + J^2) Q^k \right\}. \tag{21.13}$$

Proposition 21.5 *For all positive integers n,*

$$\left(\sum_{k=0}^{n} p_k(R, J) Q^k\right) R - R\left(\sum_{k=0}^{n} p_k(R, J) Q^k\right) \neq 0.$$

Proof Since R and J commute,

$$\left(\sum_{k=0}^{n} p_k(R, J) Q^k\right) R - R\left(\sum_{k=0}^{n} p_k(R, J) Q^k\right) = \sum_{k=1}^{n} p_k(R, J) \left(Q^k R - R Q^k\right)$$

$$= \sum_{k=1}^{n} p_k(R, J) \sum_{j=0}^{k-1} (-1)^{(k-j)(k-j+3)/2} \binom{k}{j} I_{n-j} Q^j$$

$$= \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} (-1)^{(k-j)(k-j+3)/2} \binom{k}{j} p_k(R, J) I_{k-j} Q^j$$
$$= \sum_{j=0}^{n-1} f_j(R, J) Q^j,$$

where the second equality follows from Proposition 21.3, the third equality follows after interchanging the order of summation, and the last equality after writing

$$f_j(R, J) = \sum_{k=j+1}^{n} (-1)^{(k-j)(k-j+3)/2} \binom{k}{j} p_k(R, J) I_{k-j},$$

and where as in the preceding section $I_m = R$ for m even, and $I_m = J$ for m odd. It follows that

$$f_{n-1}(R, J) = n p_n(R, J) J,$$

which is nonzero by assumption. □

It follows that no normal ordered element containing Q belongs to the centralizer of R. And since R and J commute,

$$\operatorname{Cen}(R) = \{a \in \mathcal{A}_{R,J,Q} : a = p(R, J)\}. \tag{21.14}$$

Proposition 21.6 *For all positive integers n,*

$$\left(\sum_{k=0}^{n} p_k(R, J) Q^k\right) J - J \left(\sum_{k=0}^{n} p_k(R, J) Q^k\right) \neq 0.$$

Proof Since R and J commute,

$$\left(\sum_{k=0}^{n} p_k(R, J) Q^k\right) J - J \left(\sum_{k=0}^{n} p_k(R, J) Q^k\right)$$
$$= \sum_{k=1}^{n} p_k(R, J) \left(Q^k J - J Q^k\right)$$
$$= \sum_{k=1}^{n} p_k(R, J) \sum_{j=0}^{k-1} (-1)^{(k-j)(k-j+1)/2} \binom{k}{j} I_{k-j+1} Q^j$$
$$= \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} (-1)^{(k-j)(k-j+1)/2} \binom{k}{j} p_k(R, J) I_{k-j+1} Q^j = \sum_{j=0}^{n-1} f_j(R, J) Q^j,$$

where the second equality follows from Proposition 21.2, the third equality follows after interchanging the order of summation, and the last equality after writing

$$f_j(R, J) = \sum_{k=j+1}^{n} (-1)^{(k-j)(k-j+1)/2} \binom{k}{j} p_k(R, J) I_{k-j+1},$$

and as before, $I_m = R$ for m even, and $I_m = J$ for m odd. It follows that

$$f_{n-1}(R, J) = -np_n(R, J)R,$$

which is nonzero by assumption. □

It follows that no normal ordered element containing Q belongs to the centralizer of J. And since R and J commute,

$$\mathrm{Cen}(J) = \left\{a \in \mathcal{A}_{R,J,Q} : a = p(R, J)\right\}. \tag{21.15}$$

Therefore, following (21.13), (21.14) and (21.15), we conclude that the center of the algebra generated by the elements R, J and Q satisfying (21.1) is given by

$$\mathrm{Z}(\mathcal{A}_{R,J,Q}) = \left\{a \in \mathcal{A}_{R,J,Q} : a = p(R^2 + J^2)\right\}.$$

We summarize these results in the following theorem.

Theorem 21.1 *The following hold in the unital associative $\mathbb{C}$-algebra $\mathcal{A}_{R,J,Q}$ generated by three elements R, J and Q satisfying the commutation relations*

$$\begin{aligned} QR - RQ &= J, \\ QJ - JQ &= -R, \\ RJ - JR &= 0. \end{aligned}$$

(a) The centralizer of Q is given by $\mathrm{Cen}(Q) = \left\{ \sum_{k=0}^{n} p_k(R^2 + J^2)Q^k \right\}$.
(b) The centralizer of R is given by $\mathrm{Cen}(R) = \left\{p(R, J)\right\}$.
(c) The centralizer of J is given by $\mathrm{Cen}(J) = \left\{p(R, J)\right\}$
(d) The center of the algebra $\mathcal{A}_{R,J,Q}$ is given by $\mathrm{Z}\left(\mathcal{A}_{R,J,Q}\right) = \left\{p(R^2 + J^2)\right\}$.

Corollary 21.1 *The following hold in the unital associative $\mathbb{C}$-algebra $\mathcal{A}'_{R,J,Q}$ generated by three elements R, J and Q satisfying the commutation relations*

$$\begin{aligned} QR - RQ &= J, \\ QJ - JQ &= -R, \\ JR - RJ &= 0, \\ R^2 + J^2 &= \mathbb{1}, \end{aligned}$$

where $\mathbb{1}$ *is the identity element.*

(a) The centralizer of Q *is given by* $\mathrm{Cen}(Q) = \left\{ \sum_{k=0}^{n} c_k Q^k \right\}$.

(b) The centralizer of R *is given by* $\mathrm{Cen}(R) = \left\{ p(R, J) \right\}$.

(c) The centralizer of J *is given by* $\mathrm{Cen}(J) = \left\{ p(R, J) \right\}$.

(d) The center of $\mathcal{A}'_{R,J,Q}$ *is given by* $Z(\mathcal{A}'_{R,J,Q}) = \{c\mathbb{1} : c \in \mathbb{C}\}$.

Proof This follows directly from Theorem 21.1 by subjecting the centralizers and the center to the constraint $R^2 + J^2 = \mathbb{1}$. □

21.4 Linear Transformation of Generators

Writing $S = R + iJ$ and $T = R - iJ$, we have $R = (S + T)/2$, $J = (S - T)/2i$ and $ST = R^2 + J^2$. Therefore, denoting $f(S, T) = p((S + T)/2, (S - T)/2i)$, we see that polynomials in R and J can also be written as polynomials in S and T.

Proposition 21.7 *Let* $a, b, c, d \in \mathbb{C}$.

(i) In any algebra, if $S = aR + bJ$ *and* $T = cR + dJ$*, then the commutator of* S *and* T *is given by*

$$[S, T] = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} [R, J].$$

(ii) If $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$*, that is,* $ad \neq bc$*, then* $ST = TS$ *if and only if* $RJ = JR$.

Example 21.1 In any algebra, if $S = R + iJ$ and $T = R - iJ$, then the commutator of S and T is given by

$$[S, T] = -2i[R, J],$$

and so, $ST = TS$ if and only if $RJ = JR$.

Proposition 21.8 *In any associative algebra, if* $S = R + iJ$ *and* $T = R - iJ$*, then the elements* S*,* T *and* Q *satisfy the commutation relations*

$$[Q, S] = -iS,\ [Q, T] = iT,\ [S, T] = 0, \tag{21.16}$$

if and only if the elements R*,* J *and* Q *satisfy the commutation relations*

$$[Q, R] = J,\ [Q, J] = -R,\ [R, J] = 0. \tag{21.17}$$

Proof Writing $S = R+iJ$ and $T = R-iJ$, we have $R = (S+T)/2$, $J = (S-T)/2i$ and $ST = R^2 + J^2$. Now if commutation relations (21.17) hold, then

$$[Q, S] = [Q, R + iJ] = [Q, R] + i[Q, J] = J - iR = -i(R + iJ) = -iS,$$

$$[Q, T] = [Q, R - iJ] = [Q, R] - i[Q, J] = J + iR = i(R - iJ) = iT,$$

and combining with Proposition 21.7 proves the first assertion. Conversely, if commutation relations (21.16) hold, then

$$[Q, 2R] = [Q, S + T] = -iS + iT = -i(R + iJ) + i(R - iJ) = 2J,$$

$$[Q, 2iJ] = [Q, S - T] = -iS - iT = -i(R + iJ) - i(R - iJ) = -2iR,$$

and combining with Proposition 21.7 proves the second assertion. □

21.5 An Expression for $[Q^k, S^m T^n]$

Proposition 21.9 *If S, T and Q are elements of an algebra satisfying commutation relations* (21.16)*, then for all nonnegative integers m and n.*

$$[Q, S^m] = -miS^m \tag{21.18}$$
$$[Q, T^n] = niT^n \tag{21.19}$$
$$[Q, S^m T^n] = (n - m)iS^m T^n. \tag{21.20}$$

Remark 21.1 Formula (21.20) implies that $S^m T^n$ is an eigenvector of $[Q, \cdot]$ with eigenvalue $(n - m)i$.

Proof We prove formula (21.18) by induction on m. For $m = 1$, formula (21.18) becomes $[Q, S] = -iS$, which is in the commutation relations (21.16). Now suppose that the formula holds for some integer $m \geq 1$, then

$$[Q, S^{m+1}] = [Q, S]S^m + S[Q, S^m] = (-iS)S^m + S(-miS^m) = -(m + 1)iS^{m+1},$$

and this proves the assertion. Next we prove (21.19) by induction on n. For $n = 1$, formula (21.19) becomes $[Q, T] = iT$, which is in the commutation relations (21.16). Now suppose that the formula holds for some integer $n \geq 1$, then

$$[Q, T^{n+1}] = [Q, T]T^n + T[Q, T^n] = (iT)T^n + T(niT^n) = (n + 1)iT^{n+1},$$

and this proves the assertion. Formulas (21.18) and (21.19) can now be combined:

$$[Q, S^m T^n] = [Q, S^m]T^n + S^m[Q, T^n] = (-miS^m)T^n + S^m(niT^n) = (n - m)iS^m T^n,$$

and this proves formula (21.20). □

Proposition 21.10 *If S, T and Q are elements of an algebra satisfying commutation relations (21.16), then for all nonnegative integers k, m and n,*

$$Q^k S^m T^n = S^m T^n \,(Q + (n-m)i\,\mathbb{1})^k\,.$$

Proof We proceed by induction on k. For $k = 1$, the formula follows from (21.20). Now suppose that the formula holds for some integer $k \geq 1$, then

$$\begin{aligned} Q^{k+1} S^m T^n &= Q Q^k S^m T^n = Q S^m T^n \,(Q + (n-m)i\,\mathbb{1})^k \\ &= S^m T^n \,(Q + (n-m)i\,\mathbb{1})\,(Q + (n-m)i\,\mathbb{1})^k = S^m T^n \,(Q + (n-m)i\,\mathbb{1})^{k+1}, \end{aligned}$$

and this proves the assertion. □

Corollary 21.2 *If S, T and Q are elements of an algebra satisfying commutation relations (21.16), then for all nonnegative integers k,*

$$Q^k S = S(Q - i\,\mathbb{1})^k \quad \textit{and} \quad Q^k T = T(Q + i\,\mathbb{1})^k.$$

21.6 Centralizers of S, T and Q, and the Center

Proposition 21.11 *For all positive integers n,*

$$\Big(\sum_{k=0}^{n} p_k(S,T)Q^k\Big)S - S\Big(\sum_{k=0}^{n} p_k(S,T)Q^k\Big) \neq 0.$$

Proof For all positive integers n,

$$\begin{aligned} &\Big(\sum_{k=0}^{n} p_k(S,T)Q^k\Big)S - S\Big(\sum_{k=0}^{n} p_k(S,T)Q^k\Big) = \sum_{k=1}^{n} p_k(S,T)\Big(Q^k S - S Q^k\Big) \\ &= \sum_{k=1}^{n} p_k(S,T)S\Big((Q - i\,\mathbb{1})^k - Q^k\Big) = \sum_{k=1}^{n} p_k(S,T)S \sum_{j=0}^{k-1} \binom{k}{j} (-i)^{k-j} Q^j \\ &= \sum_{j=0}^{n-1} \sum_{k=j+1}^{n} (-i)^{k-j} \binom{k}{j} p_k(S,T)SQ^j = \sum_{m=0}^{n-1} f_m(S,T)Q^m, \end{aligned}$$

where the second equality follows from Corollary 21.2, the fourth equality follows after interchanging the order of summation, and the last equality follows after writing $f_m(S,T) = \sum_{k=m+1}^{n} \binom{k}{m} (-i)^{k-m} p_k(S,T)S$. It follows that $f_{n-1}(S,T) = nip_n(S,T)S$, which is nonzero by assumption. □

It follows that no normal ordered element in $\mathcal{A}_{R,J,Q}$ containing Q belongs to the centralizer of S. And since S and T commute,

$$\text{Cen}(S) = \{a \in \mathcal{A}_{R,J,Q} : a = p(S, T)\}.$$

Similarly, one can show that no normal ordered element in $\mathcal{A}_{R,J,Q}$ containing Q belongs to the centralizer of T. And since S and T commute,

$$\text{Cen}(T) = \{a \in \mathcal{A}_{R,J,Q} : a = p(S, T)\}.$$

And formula (21.20) implies that given a polynomial $p(S, T) = \sum a_{mn} S^m T^n$, we have $[Q, p(S, T)] = \sum (n - m) i c_{mn} S^m T^n$. It follows that $[Q, p(S, T)] = 0$ if and only if for all m, n, either $n - m = 0$ or $c_{mn} = 0$. Therefore, if p belongs to the center of $\mathcal{A}_{R,J,Q}$ then $p(S, T) = \sum c_k (ST)^k$. It also follows that

$$\text{Cen}(Q) = \left\{a \in \mathcal{A}_{R,J,Q} : a = \sum p_k(ST) Q^k\right\}.$$

The conclusion is that

$$\begin{aligned} \mathrm{Z}(\mathcal{A}_{R,J,Q}) &= \{a \in \mathcal{A}_{R,J,Q} : a = p(ST)\} \\ &= \left\{a \in \mathcal{A}_{R,J,Q} : a = p(R^2 + J^2)\right\}. \end{aligned}$$

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