Chapter 20
Centralizers in PBW Extensions

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Abstract In this article we give a description for the centralizer of the coefficient ring R in the skew PBW extension $\sigma(R) < x_1, x_2, \ldots, x_n >$. We give an explicit description in the quasi-commutative case and state a necessary condition in the general case. We also consider the PBW extension $\sigma(\mathcal{A}) < x_1, x_2, \ldots, x_n >$ of the algebra of functions with finite support on a countable set, describing the centralizer of $\mathcal{A}$ and the center of the skew PBW extension.

Keywords Ring extensions · Skew PBW extensions · Contralizer · Center

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20.1 Introduction

Skew PBW (Poincare-Birkoff-Witt) extensions also known as $\sigma-$PBW extensions are a wide class of non commutative rings which were introduced in [4]. Skew PBW extensions include many rings and algebras arising in quantum mechanics such as the classical PBW extensions, Weyl algebras, enveloping algebras of finite dimensional Lie algebras, iterated Ore extensions of injective type and many others. See for

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example [1–5, 7] for examples of rings and algebras which are skew PBW and some ring theory properties that have been investigated.

In this article we describe the centralizer of the coefficient ring R in the skew PBW extension $\sigma(R) < x_1, x_2, \ldots, x_n >$. Specifically, we extend some of the results in [8, 9] in the setting of Ore extensions, to the more general setting of skew PBW extensions. We also describe the center of the skew PBW extension $\tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, x_2, \ldots, x_n >$ where $\mathbb{R}^{\Omega}$ is the algebra of real valued functions on a finite set Ω. Centers of many algebras that can be interpreted as skew PBW extensions have been described in [6], but that is in a different setting to the one here. The paper is arranged as follows.

In Sect. 20.2 we state definitions and preliminaries of skew PBW extensions. Most of the work in this section is based on [4]. In Sect. 20.3, we give a description of the centralizer of the coefficient ring R in the skew PBW extension for an integral domain R. We give a full description of the centralizer in the quasi-commutative case and state a necessary condition in the general case. In Sect. 20.4, we turn attention to the skew PBW extension for the algebra of real-valued functions $\mathbb{R}^{\Omega}$ on a finite set Ω. We prove that this algebra is isomorphic to the algebra $\mathcal{A}$ of piecewise constant functions on the real line with a finite number of jumps. We then give a full description of the centralizer of the coefficient algebra $\mathbb{R}^{\Omega}$ and the center of the PBW extension in the quasi-commutative case, and state a necessary condition in the general case. We finish the section by describing the centralizer of $\mathcal{A}$ in the skew PBW extension $\tilde{\sigma}(\mathcal{A}) < x_1, x_2, \ldots, x_n >$ in terms of $Sep^{\alpha}(\Omega)$ via the isomorphism between $\mathcal{A}$ and $\mathbb{R}^{\Omega}$.

20.2 Definitions and Preliminary Notions

In this section we define skew PBW extensions and state some preliminary results concerning skew PBW extensions.

Definition 20.1 Let R and A be rings. We say that A is a $\sigma-$PBW extension of R (or skew PBW extension), if the following conditions hold:

(a) $R \subseteq A$.
(b) There exist finite elements $x_1, \ldots, x_n$ such that A is a left $R-$free module with basis
$$Mon(A) := \{x^{\alpha} = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \ : \ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}.$$
(c) For every $1 \leqslant i \leqslant n$ and $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that
$$x_i r - c_{i,r} x_i \in R.$$
(d) For every $1 \leqslant i, j \leqslant n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that
$$x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n.$$

Under these conditions we write $A = \sigma(R)\langle x_1, \dots, x_n\rangle$.

The following result [4, Proposition 3] is crucial in establishing the link between skew PBW extensions and many well known algebras.

Proposition 20.1 *Let A be a $\sigma-$PBW extension of R. Then for every $1 \leqslant i \leqslant n$, there exists an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that*

$$x_i r = \sigma_i(r)x + \delta_i(r)$$

for each $r \in R$.

A particular case of $\sigma-$PBW extension is when all derivations δ_i are zero. Another interesting case is when all σ_i are bijective. This gives motivation to the next definition.

Definition 20.2 Let A be a $\sigma-$PBW extension.

1. A is quasi-commutative if the conditions (c) and (d) in Definition 20.1 are replaced by:

(c') For every $1 \leqslant i \leqslant n$ and $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r = c_{i,r} x_i.$$

(d') For every $1 \leqslant i, j \leqslant n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j.$$

2. A is bijective if σ_i is bijective for every $1 \leqslant i \leqslant n$ and $c_{i,j}$ is invertible for any $1 \leqslant i < j \leqslant n$.

In the next definition, we state some useful notation.

Definition 20.3 Let A be a $\sigma-$PBW extension of R with endomorphisms σ_i, $1 \leqslant i \leqslant n$, as in Proposition 20.1.

(a) For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.
(b) For $X = x^\alpha \in Mon(A)$, $exp(X) := \alpha$ and $deg(X) = |\alpha|$.
(c) Let $0 \neq f \in A$ such that $f = c_1 X_1 + \cdots + c_t X_t$ with $X_i \in Mon(A)$ and $c_i \in R \setminus \{0\}$ then $deg(f) = \max\{deg(X_i)\}_{i=1}^t$.

20.3 Centralizers in Skew PBW Extensions

In this section we give a description of the centralizer $C(R)$ of the (commutative) coefficient ring R in the skew PBW extension $\sigma(R) < x_1, x_2, \ldots, x_n >$. We start by giving a full description of the centralizer in the quasi commutative case and then give a necessary condition in the general case.

Theorem 20.1 *Let R be a commutative ring and suppose that for all $1 \leqslant i \leqslant n,\ \delta_i = 0$. Then the centralizer $C(R)$ of R in the skew PBW extension $\sigma(R) < x_1, \ldots, x_n >$ is given by*

$$C(R) = \left\{ \sum_{\alpha} f_\alpha x^\alpha \ : \ (\forall\, r \in R),\ (\sigma^\alpha(r) - r) f_\alpha = 0 \right\}.$$

Proof An element $f = \sum_{\alpha} f_\alpha x^\alpha \in \sigma(R) < x_1, \ldots, x_n >$ belongs to $C(R)$ if and only if for every $r \in R,\ rf = fr.$

$$rf = r \sum_{\alpha} f_\alpha x^\alpha = \sum_{\alpha} r f_\alpha x^\alpha.$$

On the other hand, if $\delta_i = 0$ for $1 \leqslant i \leqslant n$, then for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and every $r \in R$ we have;

$$x^\alpha r = \sigma^\alpha(r) x^\alpha.$$

Therefore

$$\begin{aligned} fr &= \left(\sum_{\alpha} f_\alpha x^\alpha \right) r \\ &= \sum_{\alpha} f_\alpha x^\alpha r \\ &= \sum_{\alpha} f_\alpha \sigma^\alpha(r) x^\alpha. \end{aligned}$$

Since R is commutative, it follows that $rf = fr$ if and only if

$$(\sigma^\alpha(r) - r) f_\alpha = 0.$$

Therefore

$$C(R) = \left\{ \sum_{\alpha} f_\alpha x^\alpha \ : \ (\forall\, r \in R),\ (\sigma^\alpha(r) - r) f_\alpha = 0 \right\}.$$

In the general case, we have the following necessary condition.

Theorem 20.2 *Let R be a commutative ring. If an element $\sum_\alpha f_\alpha x^\alpha \in \sigma(R) < x_1, \ldots, x_n >$ belongs to the centralizer $C(R)$, then $(\sigma^\alpha(r) - r)f_\alpha = 0$ for all $\alpha \in \mathbb{N}^n$.*

Proof Suppose an element $f = \sum_\alpha f_\alpha x^\alpha \in \sigma(R) < x_1, \ldots, x_n >$ belongs to the centralizer of R. Then $fr = rf$ for every $r \in R$. Now,

$$rf = r\sum_\alpha f_\alpha x^\alpha = \sum_\alpha rf_\alpha x^\alpha.$$

On the other hand, by [4, Theorem 7], for every $x^\alpha \in Mon(\sigma(R) < x_1, \ldots, x_n >)$ and every $r \in R$ we have

$$x^\alpha r = \sigma^\alpha(r)x^\alpha + p_{\alpha,r}$$

where $p_{\alpha,r} \in R[x_1, \ldots, x_n]$ such that $p_{\alpha,r} = 0$ or $deg(p_{\alpha,r}) < |\alpha|$. Therefore;

$$\begin{aligned} fr &= \sum_\alpha \Big(f_\alpha x^\alpha\Big) r \\ &= \sum_\alpha f_\alpha \Big(x^\alpha r\Big) \\ &= \sum_\alpha f_\alpha \Big(\sigma^\alpha(r)x^\alpha + p_{\alpha,r}\Big). \end{aligned}$$

Comparing the leading coefficients and using the fact that R is commutative, we see that if $fr = rf$, then

$$(\sigma^\alpha(r) - r)f_\alpha = 0 \text{ for all } \alpha.$$

As a result, we have the following Corollary which is the extension of [8, Proposition 3.3] to the skew PBW extension case.

Corollary 20.1 *Let R be a commutative ring. If for every $\alpha \in \mathbb{N}^n$ there exists $r \in R$ such that $(\sigma^\alpha(r) - r)$ is a regular element, then $C(R) = R$.*

Proof Suppose $f = \sum_\alpha f_\alpha x^\alpha \in \sigma(\mathcal{A}) < x_1, \ldots, x_n >$ is a non-constant element of degree α which belongs to the centralizer of R. Then $fr = rf$ for every $r \in R$. Now,

$$rf = r\sum_\alpha f_\alpha x^\alpha = \sum_\alpha rf_\alpha x^\alpha.$$

On the other hand, by [4, Theorem 7], for every $x^\alpha \in Mon(A)$ and every $r \in \mathcal{A}$ we have

$$x^\alpha r = \sigma^\alpha(r) + p_{\alpha,r}$$

where $p_{\alpha,r} = 0$ or $deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. Therefore;

$$\begin{aligned} fr &= \sum_{\alpha} \left(f_\alpha x^\alpha \right) r \\ &= \sum_{\alpha} f_\alpha \left(x^\alpha r \right) \\ &= \sum_{\alpha} f_\alpha \left(\sigma^\alpha(r) x^\alpha + p_{\alpha,r} \right) \end{aligned}$$

Equating coefficients and using commutativity of R, we get

$$r f_\alpha = \sigma^\alpha(r) f_\alpha, \quad \text{or equivalently } (\sigma^\alpha(r) - r) f_\alpha = 0.$$

Since $\sigma^\alpha(r) - r$ is a regular element, then we $f_\alpha = 0$ for all α, which is a contradiction.

20.4 Skew PBW Extensions of Function Algebras

In this section we treat skew PBW extensions for the algebra of functions on a finite set. In [10], the commutant of the coefficient algebra in the crossed product algebra for the algebra of piecewise constant functions on the real line was described. However, as we show in Proposition 20.2 below, the algebra of piecewise constant functions on the real line is isomorphic to the algebra of real-valued functions on some finite set.

Let $\mathbb{P} = \bigcup_{k=0}^{2N} I_k$ be a partition of $\mathbb{R}$, where $I_k = (t_k, t_{k+1})$, for $k = 0, 1, \ldots, N$ with $t_0 = -\infty$ and $t_{N+1} = \infty$ and $I_{N+k} = \{t_k\}$, $k = 1, \ldots, N$ and let $\mathcal{A}$ be the algebra of functions which are constant on the intervals I_k, $k = 0, 1, \ldots, 2N$. Then $\mathcal{A}$ is the algebra of piecewise constant functions $h : \mathbb{R} \to \mathbb{R}$ with N fixed jumps at points $t_1, \ldots, t_N$.

Let $\Omega = \{0, 1, \ldots, 2N\}$ be a finite set and let $\mathbb{R}^\Omega$ denote the algebra of all functions $f : \Omega \to \mathbb{R}$.

Proposition 20.2 *The algebra $\mathcal{A}$ is isomorphic to the algebra $\mathbb{R}^\Omega$.*

Proof Define a function $\mu : \mathbb{R}^\Omega \to \mathcal{A}$ as follows: For every $f \in \mathbb{R}^\Omega$,

$$\mu(f)(x) = f(\omega) \text{ if } x \in I_\omega, \quad \omega = 0, 1, \ldots, 2N. \tag{20.1}$$

We need to prove that μ is an algebra isomorphism.

Let $f, g \in \mathbb{R}^\Omega$ and let $a, b \in \mathbb{R}$. Then we have the following.

- If $x \in \mathbb{R}$, then $x \in I_\omega$ for some $\omega \in \{0, 1, \ldots, 2N\}$. Therefore

$$\begin{aligned}\mu(af+bg)(x) &= (af+bg)(\omega)\\ &= af(\omega)+bg(\omega)\\ &= a\mu(f)(x)+b\mu(g)(x)\\ &= [a\mu(f)+b\mu(g)](x)\end{aligned}$$

That is, μ is $\mathbb{R}-$linear.

•

$$\begin{aligned}\mu(fg)(x) &= (fg)(\omega)\\ &= f(\omega)g(\omega)\\ &= \mu(f)(x)\mu(g)(x)\\ &= [\mu(f)\mu(g)](x)\end{aligned}$$

Therefore, μ is multiplicative and hence an algebra homomorphism.

- If for all $x \in \mathbb{R}$, $\mu(f)(x) = \mu(g)(x)$, then $f(\omega) = g(\omega)$ for all $\omega \in \{0, 1, \ldots, 2N\}$. That is, $f = g$ and hence μ is injective.
- Finally, let $h \in \mathcal{A}$. Then for every $x \in \mathbb{R}$ such that $x \in I_\omega$, $h(x) = c_\omega$ for some $c_\omega \in \mathbb{R}$. Define $f \in \mathbb{R}^\Omega$ by $f(\omega) = c_\omega$, $\omega = 0, 1, \ldots, 2N$. If $y \in \mathbb{R}$ such that $y \in I_\theta$ for some $\theta \in \{0, 1, \ldots, 2N\}$, then

$$\mu(f)(y) = f(\theta) = c_\theta = h(y).$$

Since y is arbitrary, we conclude that μ is onto.

Therefore μ is an isomorphism.

Now, let $\sigma : \mathbb{R} \to \mathbb{R}$ be a bijection such that $\mathcal{A}$ is invariant under σ (and σ^{-1}). In [10, Lemma 1], it was proved that such a σ is a permutation of the partition intervals I_ω, $\omega = 0, 1, \ldots, 2N$. Let $\tau : \Omega \to \Omega$ be a bijection (permutation) such that $\tau(\omega) = \theta$ if and only if $\sigma(I_\omega) = I_\theta$. Suppose $\tilde{\sigma} : \mathcal{A} \to \mathcal{A}$ is the automorphism induced by σ and $\tilde{\tau} : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ is the automorphism induced by τ, that is, for every $h \in \mathcal{A}$ and every $f \in \mathbb{R}^\Omega$,

$$\tilde{\sigma}(h) = h \circ \sigma^{-1} \text{ and } \tilde{\tau}(f) = f \circ \tau^{-1}. \tag{20.2}$$

The automorphisms $\tilde{\sigma}$ and $\tilde{\tau}$ satisfy the following intertwining relation.

Proposition 20.3 *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a bijection such that $\mathcal{A}$ is invariant under σ (and σ^{-1}) and let $\tau : \Omega \to \Omega$ be a bijection (permutation) such that $\tau(\omega) = \theta$ if and only if $\sigma(I_\omega) = I_\theta$. Suppose $\tilde{\sigma} : \mathcal{A} \to \mathcal{A}$ is the automorphism induced by σ and $\tilde{\tau} : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ is the automorphism induced by τ. Then*

$$\tilde{\sigma} \circ \mu = \mu \circ \tilde{\tau}, \tag{20.3}$$

where μ is given by (20.1). Moreover, for every $n \in \mathbb{Z}$,

$$\tilde{\sigma}^n \circ \mu = \mu \circ \tilde{\tau}^n. \tag{20.4}$$

Proof Let $f \in \mathbb{R}^{\Omega}$ and $x \in \mathbb{R}$. Suppose $x \in I_\omega$ and that $\sigma^{-1}(I_\theta) = I_\omega$ for some $\theta \in \{0, 1, \ldots, 2N.\}$ Then,

$$\begin{aligned}\tilde{\sigma} \circ \mu(f)(x) &= \tilde{\sigma}(\mu(f))(x) \\ &= \mu(f)(\sigma^{-1}(x)) \\ &= f(\tau^{-1}(\omega)) \\ &= \tilde{\tau}(f)(\omega) \\ &= \mu \circ \tilde{\tau}(f)(x),\end{aligned}$$

which proves (20.3). The relation (20.4) follows by induction.

In the next Lemma we prove an equivalence between $Sep^n_{\mathcal{A}}(\mathbb{R})$ and $Sep^n(\Omega)$, two sets which will be important in the description of the centralizer of the coefficient algebra in the skew PBW extension. First we give the definitions.

Definition 20.4 For every $n \in \mathbb{Z}$ set,

$$Sep^n_{\mathcal{A}}(\mathbb{R}) := \{x \in \mathbb{R} \ : \ (\exists\, h \in \mathcal{A}),\ h(x) \neq \tilde{\sigma}^n(h)(x)\}, \tag{20.5}$$

$$Sep^n_{\mathbb{R}^\Omega}(\Omega) := \{\omega \in \Omega \ : \ (\exists\, f \in \mathbb{R}^\Omega),\ f(\omega) \neq \tilde{\tau}^n(f)(\omega)\}, \tag{20.6}$$

and

$$Sep^n(\Omega) := \{\omega \in \Omega \ : \ \tau^n(\omega) \neq \omega\}. \tag{20.7}$$

We have the following.

Lemma 20.1 *Let $x \in I_\omega \subset \mathbb{R}$. Then $x \in Sep^n_{\mathcal{A}}(\mathbb{R})$ if and only if $\omega \in Sep^n(\Omega)$.*

Proof Since the algebra $\mathbb{R}^\Omega$ separates points, then $Sep^n_{\mathbb{R}^\Omega}(\Omega) = Sep^n(\Omega)$ for every $n \in \mathbb{Z}$. Therefore, it suffices to prove that $x \in I_\omega \subset \mathbb{R}$ belongs to $Sep^n_{\mathcal{A}}(\mathbb{R})$ if and only if $\omega \in Sep^n_{\mathbb{R}^\Omega}(\Omega)$. To this end, we have the following.

Suppose $\omega \in Sep^n_{\mathbb{R}^\Omega}(\Omega)$. Then there exists $f \in \mathbb{R}^\Omega$ such that $\tilde{\tau}^n(f)(\omega) \neq f(\omega)$. Since μ is injective, then $\tilde{\tau}^n(f)(\omega) \neq f(\omega)$ implies that

$$\mu \circ \tilde{\tau}^n(f)(x) \neq \mu(f)(x) \ \ \forall\, x \in I_\omega.$$

But from (20.4), $\mu \circ \tilde{\tau}^n = \tilde{\sigma}^n \circ \mu$. Therefore,

$$\tilde{\sigma}^n(\mu(f))(x) \neq \mu(f)(x).$$

That is, $x \in Sep^n_{\mathcal{A}}(\mathbb{R})$.

Conversely, suppose $x \in Sep^n_{\mathcal{A}}(\mathbb{R})$. Then there exists $h \in \mathcal{A}$ such that $\tilde{\sigma}^n(h)(x) \neq h(x)$. Using injectivity of μ^{-1}, we get

$$(\mu^{-1} \circ \tilde{\sigma}^n)(h)(\omega) \neq \mu^{-1}(h)(\omega).$$

Again, using (20.4), we get that $\mu^{-1} \circ \tilde{\sigma}^n = \tilde{\tau}^n \circ \mu^{-1}$. Therefore

$$\tilde{\tau}^n(\mu^{-1}(h))(\omega) \neq (\mu^{-1}(h))(\omega),$$

and hence $\omega \in Sep^n_{\mathbb{R}^\Omega}(\Omega)$.

From Proposition 20.2 and Lemma 20.1 above, it follows that we can consider the algebra of functions on a finite set. Indeed in the following section we consider the skew PBW extension of the algebra $\mathbb{R}^\Omega$ of functions on a finite set Ω and then deduce the corresponding results in the case of the skew PBW extension of the algebra of piecewise constant functions on $\mathbb{R}$ via the isomorphism μ.

20.4.1 Algebra of Functions on a Finite Set

Let $\Omega = \{0, 1, \ldots, 2N\}$ be a finite set and let $\mathbb{R}^\Omega = \{f : \Omega \to \mathbb{R}\}$ denote the algebra of real-valued functions on Ω with respect to the usual pointwise operations. By writing $f_k := f(k)$, $\mathbb{R}^\Omega$ can be identified with $\mathbb{R}^{2N+1}$ where $\mathbb{R}^{2N+1}$ is equipped with the usual operations of pointwise addition, scalar multiplication and multiplication defined by

$$xy = (x_1y_1, x_2y_2, \ldots, x_ny_n)$$

for every $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

Now, for $1 \leqslant i \leqslant n$, let $\tau_i : \Omega \to \Omega$ be a bijection such that $\mathbb{R}^\Omega$ is invariant under τ_i and τ_i^{-1}, (that is both τ_i and τ_i^{-1} are permutations on Ω). For $1 \leqslant i \leqslant n$ let $\tilde{\tau}_i : \mathcal{A} \to \mathcal{A}$ be the automorphism induced by τ_i, that is

$$\tilde{\tau}_i(f) = f \circ \tau_i^{-1} \tag{20.8}$$

for every $f \in \mathbb{R}^\Omega$ and let δ_i, $1 \leqslant i \leqslant n$ be a $\tilde{\tau}_i-$derivation. Consider the skew-PBW extension $\tilde{\tau}(\mathbb{R}^\Omega) < x_1, \ldots, x_n >$.

The following definition is important in the description of the centralizer of $\mathbb{R}^\Omega$ in the skew PBW extension $\tilde{\tau}(\mathbb{R}^\Omega) < x_1, \ldots, x_n >$.

Definition 20.5 For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, define

(a) $Sep^\alpha(\Omega) := \{\omega \in \Omega \ : \ \tau^\alpha(\omega) \neq \omega\}$;
(b) $Per^\alpha(\Omega) := \{\omega \in \Omega \ : \ \tau^\alpha(\omega) = \omega\}$.

20.4.2 Centralizers in Skew PBW Extensions for Function Algebras

In this section the centralizer of $\mathbb{R}^{\Omega}$ in the skew PBW extension $\tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, \ldots, x_n >$ is described. We start by describing the centralizer in the quasi-commutative case and then state a necessary condition for an element to belong to the centralizer of $\mathbb{R}^{\Omega}$ in the general case. We finish by giving the description of the center of the skew PBW extension in the quasi-commutative case.

20.4.2.1 The Centralizer of $\mathbb{R}^{\Omega}$

Theorem 20.3 *Suppose that for* $1 \leqslant i \leqslant n,\ \delta_i = 0$. *Then the centralizer* $C(\mathbb{R}^{\Omega})$, *of* $\mathbb{R}^{\Omega}$ *in the skew* PBW *extension* $\tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, \ldots, x_n >$ *is given by*

$$C\left(\mathbb{R}^{\Omega}\right) = \left\{ \sum_{\alpha} f_{\alpha} x^{\alpha} \ : \ f_{\alpha} = 0 \ on \ Sep^{\alpha}(\Omega) \right\}.$$

Proof Using the results of Theorem 20.1, an element $f = \sum_{\alpha} f_{\alpha} x^{\alpha} \in \tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, \ldots, x_n >$ belongs to the centralizer of $\mathbb{R}^{\Omega}$ if and only if for every $r \in \mathbb{R}^{\Omega}$,

$$f_{\alpha}(\tilde{\tau}^{\alpha}(r) - r) = 0.$$

Since $(\tilde{\tau}^{\alpha}(r) - r)(y) = 0$ for every $y \in Per^{\alpha}_{\mathbb{R}^{\Omega}}(\Omega)$, then $fr = rf$ for every $r \in \mathbb{R}^{\Omega}$ if and only if $f_{\alpha} = 0$ on $Sep^{\alpha}_{\mathbb{R}^{\Omega}}(\Omega)$. Since $Sep^{\alpha}_{\mathbb{R}^{\Omega}}(\Omega) = Sep^{\alpha}(\Omega)$, we have,

$$C\left(\mathbb{R}^{\Omega}\right) = \left\{ \sum_{\alpha} f_{\alpha} x^{\alpha} \ : \ f_{\alpha} = 0 \text{ on } Sep^{\alpha}(\Omega) \right\}.$$

Now consider the case when $\delta_i \neq 0$. In [9] a necessary condition for an element $\sum_{k=0}^{m} f_k x^k$ in the Ore extension $\mathbb{R}^{\Omega}[x, \tilde{\tau}, \delta]$ to belong to the centralizer of $\mathbb{R}^{\Omega}$ was stated and the following Theorem was proved.

Theorem 20.4 *If an element of degree* m, $\sum_{k=0}^{m} f_k x^k \in \mathbb{R}^{\Omega}[x, \tilde{\tau}, \delta]$ *belongs to the centralizer of* $\mathbb{R}^{\Omega}$*, then* $f_m = 0$ *on* $Sep^m(\Omega)$.

We aim to extend this theorem to the skew PBW extension $\tilde{\tau}(\mathbb{R}) < x_1, \ldots, x_n >$ of which the Ore extension $\mathbb{R}^{\Omega}[x, \tilde{\tau}, \delta]$ is a special case. This extension is given in the following Theorem.

Theorem 20.5 *If an element* $\sum_{\alpha} f_{\alpha} x^{\alpha} \in \tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, \ldots, x_n >$ *belongs to the centralizer of* $\mathbb{R}^{\Omega}$, *then* $f_{\alpha} = 0$ *on* $Sep^{\alpha}(\Omega)$.

Proof Again, using Theorem 20.2, we see that if an element $f = \sum_{\alpha} f_{\alpha} x^{\alpha} \in \tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, \ldots, x_n >$ belongs to the centralizer of $\mathbb{R}^{\Omega}$, then

$$(r - \tilde{\tau}^r(r)) f_{\alpha} = 0 \;\; \forall \, \alpha \in \mathbb{N}^n. \tag{20.9}$$

Equation (20.9) holds on $Per^{\alpha}_{\mathbb{R}^{\Omega}}(\Omega)$ and holds on $Sep^{\alpha}_{\mathbb{R}^{\Omega}}(\Omega)$ if $f_{\alpha} = 0$. The conclusion follows from the fact that $Sep^{\alpha}_{\mathbb{R}^{\Omega}}(\Omega) = Sep^{\alpha}(\Omega)$ for all $\alpha \in \mathbb{N}^n$.

20.4.2.2 Center in the Quasi-commutative Case

In this section we give the description of the center of the skew PBW extension $\tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, \ldots, x_n >$ in the quasi-commutative case, Definition 20.2. We start with a result which will be important in the description of the center.

Lemma 20.4.1 *Let* $\tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, \ldots, x_n >$ *be a quasi-commutative PBW extension. Then for every* $1 \leqslant i, j \leqslant n$ *and every* $m \in \mathbb{N}$,

(a) $x_j x_i^m = \left(\prod_{k=0}^{m-1} \tilde{\tau}_i^k(c_{ij}) \right) x_i^m x_j.$

(b) $x_j^m x_i = \left(\prod_{k=0}^{m-1} \tilde{\tau}_j^k(c_{ij}) \right) x_i x_j^m.$

Proof (a) The case $m = 1$ corresponds to condition (c') in Definition 20.2 and for $m = 2$ we have

$$\begin{aligned} x_j x_i^2 &= (x_j x_i) x_i \\ &= (c_{ij} x_i x_j) x_i \\ &= c_{ij} x_i (x_j x_i) \\ &= c_{ij} x_i (c_{ij}) x_i x_j \\ &= c_{ij} \tilde{\tau}_i (c_{ij}) x_i^2 x_j . \end{aligned}$$

Suppose the formula holds for all positive integers up to and including m. Then

$$\begin{aligned} x_j x_i^{m+1} &= (x_j x_i^m) x_i \\ &= \left(\left(\prod_{k=0}^{m-1} \tilde{\tau}_i^k (c_{ij}) \right) x_i^m x_j \right) x_i \\ &= \left(\prod_{k=0}^{m-1} \tilde{\tau}_i^k (c_{ij}) \right) \left(x_i^m c_{ij} \right) x_i x_j \\ &= \left(\prod_{k=0}^{m-1} \tilde{\tau}_i^k (c_{ij}) \right) \tilde{\tau}_i^m x_i^{m+1} x_j \\ &= \left(\prod_{k=0}^{m} \tilde{\tau}_i^k (c_{ij}) \right) x_i^{m+1} x_j . \end{aligned}$$

A similar proof can be done for part (b).

Using these formulas we can derive necessary and sufficient conditions for an element to belong to the center. We state these conditions in the following Theorem.

Theorem 20.6 *An element* $f = \sum_{\alpha} f_\alpha x^\alpha$ *belongs to the center of the quasi-commutative skew PBW extension* $\tilde{\tau}(\mathbb{R}^\Omega) < x_1, \ldots, x_n >$ *if and only if* $f_\alpha = 0$ *on* $Sep^\alpha(\Omega)$ *and for every* $1 \leqslant i \leqslant n$

$$\tilde{\tau}_i (f_\alpha) \prod_{j=1}^{i-1} \left(\prod_{k_j=0}^{\alpha_j - 1} \tilde{\tau}_1^{\alpha_1} \tilde{\tau}_2^{\alpha_2} \cdots \tilde{\tau}_{j-2}^{\alpha_{j-2}} \tilde{\tau}_j^{k_j} (c_{j,i}) \right) =$$

$$f_\alpha \prod_{j=1}^{i+2} \left(\prod_{k_{n-j+1}=0}^{\alpha_{n-j+1}-1} \tilde{\tau}_1^{\alpha_1} \cdots \tilde{\tau}_i^{\alpha_1} \tilde{\tau}_{n-i+1}^{\alpha_{n-i+1}} \cdots \tilde{\tau}_{n-j}^{\alpha_{n-j}} \tilde{\tau}_{n-j+1}^{k_{n-j+1}} (c_{i,n-j+1}) \right)$$

Proof An element $f = \sum_{\alpha} f_\alpha x^\alpha$ belongs to the center of the quasi-commutative skew PBW extension $\tilde{\tau}(\mathbb{R}^\Omega) < x_1, \ldots, x_n >$ if and only if $f \in C(\mathbb{R}^\Omega)$ and, for every $1 \leqslant i \leqslant n,\ x_i f = f x_i$. So we compute.

$$\begin{aligned}
x_i f &= \sum_\alpha x_i f_\alpha x_1^{\alpha_1} \cdots x_i^{\alpha_i} \cdots x_n^{\alpha_n} \\
&= \sum_\alpha \tau_i(f_\alpha) x_i x_1^{\alpha_1} \cdots x_i^{\alpha_i} \cdots x_n^{\alpha_n} \\
&= \sum_\alpha \tau_i(f_\alpha) \left(\prod_{k_1=0}^{\alpha_1-1} \tilde{\tau}_1^{k_1}(c_{1,i}) \right) x_1^{\alpha_1} x_i x_2^{\alpha_2} \cdots x_i^{\alpha_i} \cdots x_n^{\alpha_n} \\
&= \sum_\alpha \tilde{\tau}_i(f_\alpha) \left(\prod_{k_1=0}^{\alpha_1-1} \tilde{\tau}_1^{k_1}(c_{1,i}) \right) \left(\prod_{k_2=0}^{\alpha_2-1} \tilde{\tau}_1^{\alpha_1} \tilde{\tau}_2^{k_2}(c_{2,i}) \right) x_1^{\alpha_1} x_2^{\alpha_2} x_i \cdots x_i^{\alpha_i} \cdots x_n^{\alpha_n} \\
&\ \vdots \\
&= \sum_\alpha \tilde{\tau}_i(f_\alpha) \left(\prod_{k_1=0}^{\alpha_1-1} \tilde{\tau}_1^{k_1}(c_{1,i}) \right) \left(\prod_{k_2=0}^{\alpha_2-1} \tilde{\tau}_1^{\alpha_1} \tilde{\tau}_2^{k_2}(c_{2,i}) \right) \cdots \left(\prod_{k_{i-1}}^{\alpha_{i-1}-1} \tilde{\tau}_1^{\alpha_1} \cdots \tilde{\tau}_{i-2}^{\alpha_{i-2}} \tilde{\tau}_{i-1}^{k_{i-1}}(c_{i-1,i}) \right) \\
&\quad x_1^{\alpha_1} \cdots x_i^{\alpha_i+1} \cdots x_n^{\alpha_n} \\
&= \sum_\alpha \tilde{\tau}_i(f_\alpha) \prod_{j=1}^{i-1} \left(\prod_{k_j=0}^{\alpha_j-1} \tilde{\tau}_1^{\alpha_1} \tilde{\tau}_2^{\alpha_2} \cdots \tilde{\tau}_{j-2}^{\alpha_{j-2}} \tilde{\tau}_j^{k_j}(c_{j,i}) \right) x_1^{\alpha_1} \cdots x_i^{\alpha_i+1} \cdots x_n^{\alpha_n}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
f x_i &= \left(\sum_\alpha f_\alpha x_1^{\alpha_1} \cdots x_i^{\alpha^{\alpha_i}} \cdots x_n^{\alpha_n} \right) \\
&= \sum_\alpha f_\alpha x_1^{\alpha_1} \cdots x_i^{\alpha^{\alpha_i}} \cdots x_{n-1}^{\alpha_{n-1}} \left(\prod_{k_n=0}^{\alpha_n-1} \tilde{\tau}_n^{k_n}(c_{i,n}) \right) x_i x_n^{\alpha_n} \\
&= \sum_\alpha f_\alpha x_1^{\alpha_1} \cdots x_i^{\alpha^{\alpha_i}} \cdots x_{n-2}^{\alpha_{n-2}} \left(\prod_{k_n=0}^{\alpha_n-1} \tilde{\tau}_{n-1}^{\alpha_{n-1}} \tilde{\tau}_n^{k_n}(c_{i,n}) \right) \left(\prod_{k_{n-1}=0}^{\alpha_{n-1}-1} \tilde{\tau}_{n-1}^{k_{n-1}}(c_{i,n-1}) \right) x_i x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\
&\ \vdots \\
&= \sum_\alpha f_\alpha x_1^{\alpha_1} \cdots x_i^{\alpha_i} \left(\prod_{k_n=0}^{\alpha_n-1} \tilde{\tau}_{n-i+1}^{\alpha_{n-i+1}} \tilde{\tau}_{n-i+2}^{\alpha_{n-i+2}} \cdots \tilde{\tau}_{n-1}^{\alpha_{n-1}} \tilde{\tau}_n^{k_n}(c_{i,n}) \right) \\
&\left(\prod_{k_{n-1}=0}^{\alpha_{n-1}-1} \tilde{\tau}_{n-i+1}^{\alpha_{n-i+1}} \cdots \tilde{\tau}_{n-2}^{\alpha_{n-2}} \tilde{\tau}_{n-1}^{\alpha_{n-1}}(c_{i,n-1}) \right) \left(\prod_{k_{n-i-1}=0}^{\alpha_{n-i-1}-1} \tilde{\tau}_{n-i-1}^{k_{n-i-1}}(c_{n-i-1,i}) \right) x_i x_{n-i-1}^{\alpha_{n-i-1}} \cdots x_n^{\alpha_n} \\
&= \sum_\alpha f_\alpha \prod_{j=1}^{i+2} \left(\prod_{k_{n-j+1}=0}^{\alpha_{n-j+1}-1} \tilde{\tau}_1^{\alpha_1} \cdots \tilde{\tau}_i^{\alpha_1} \tilde{\tau}_{n-i+1}^{\alpha_{n-i+1}} \cdots \tilde{\tau}_{n-j}^{\alpha_{n-j}} \tilde{\tau}_{n-j+1}^{k_{n-j+1}}(c_{i,n-j+1}) \right) x_1^{\alpha_1} \cdots x_i^{\alpha_i+1} \cdots x_n^{\alpha_n}
\end{aligned}$$

Comparing coefficients of $x_1^{\alpha_1} \cdots x_i^{\alpha_{i+1}} \cdots x_n^{\alpha_n}$ completes the proof of the theorem.

20.4.2.3 Some Examples

In the special case when when $n = 2$ we have the following.

Let $A = \tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, x_2 >$ and suppose an element $f = \sum_{\alpha_1,\alpha_2} f_{\alpha_1,\alpha_2} x_1^{\alpha_1} x_2^{\alpha_2} \in Z(A)$. Then $x_1 f = f x_1$ and $x_2 f = f x_2$. Now

$$\begin{aligned} x_1 f &= x_1 \Bigg(\sum_{\alpha_1,\alpha_2} f_{\alpha_1,\alpha_2} x_1^{\alpha_1} x_2^{\alpha_2} \Bigg) \\ &= \sum_{\alpha_1,\alpha_2} \tilde{\tau}_1(f_{\alpha_1,\alpha_2}) x_1^{\alpha_1+1} x_2^{\alpha_2}, \end{aligned}$$

and

$$\begin{aligned} f x_1 &= \Bigg(\sum_{\alpha_1,\alpha_2} f_{\alpha_1,\alpha_2} x_1^{\alpha_1} x_2^{\alpha_2} \Bigg) x_1 \\ &= \sum_{\alpha_1,\alpha_2} f_{\alpha_1,\alpha_2} x_1^{\alpha_1} \Big(x_2^{\alpha_2} x_1 \Big) \\ &= \sum_{\alpha_1,\alpha_2} f_{\alpha_1,\alpha_2} x_1^{\alpha_1} \Bigg(\prod_{k=0}^{\alpha_2-1} \tilde{\tau}_2^k(c_{12}) \Bigg) x_1 x_2^{\alpha_2} \\ &= \sum_{\alpha_1,\alpha_2} f_{\alpha_1,\alpha_2} \Bigg(\prod_{k=0}^{\alpha_2-1} \tilde{\tau}_1{}^{\alpha_1} \tilde{\tau}_2{}^{k}(c_{12}) \Bigg) x_1^{\alpha_1+1} x_2^{\alpha_2}. \end{aligned}$$

Therefore $x_1 f = f x_1$ if and only if

$$\tilde{\tau}_1(f_{\alpha_1,\alpha_2}) = f_{\alpha_1,\alpha_2} \Bigg(\prod_{k=0}^{\alpha_2-1} \tilde{\tau}_1{}^{\alpha_1} \tilde{\tau}_2{}^{k}(c_{12}) \Bigg).$$

On the other hand,

$$\begin{aligned} f x_2 &= \Bigg(\sum_{\alpha_1,\alpha_2} f_{\alpha_1,\alpha_2} x_1^{\alpha_1} x_2^{\alpha_2} \Bigg) x_2 \\ &= \sum_{\alpha_1,\alpha_2} f_{\alpha_1,\alpha_2} x_1^{\alpha_1} x_2^{\alpha_2+1}, \end{aligned}$$

and

$$\begin{aligned}x_2 f &= x_2\Bigg(\sum_{\alpha_1,\alpha_2} f_{\alpha_1,\alpha_2} x_1^{\alpha_1} x_2^{\alpha_2}\Bigg)\\ &= \sum_{\alpha_1,\alpha_2} \tilde{\tau}_2\Big(f_{\alpha_1,\alpha_2}\Big) x_2 x_1^{\alpha_1} x_2^{\alpha_2}\\ &= \sum_{\alpha_1,\alpha_2} \tilde{\tau}_2\Big(f_{\alpha_1,\alpha_2}\Big)\Bigg(\prod_{k=1}^{\alpha_1-1} \tilde{\tau}_1^k(c_{12})\Bigg) x_1^{\alpha_1} x_2^{\alpha_2+1}.\end{aligned}$$

Therefore $x_2 f = f x_2$ if and only if

$$f_{\alpha_1,\alpha_2} = \tilde{\tau}_2\Big(f_{\alpha_1,\alpha_2}\Big)\Bigg(\prod_{k=1}^{\alpha_1-1} \tilde{\tau}_1^k(c_{12})\Bigg).$$

We conclude that $f \in Z(A)$ if and only if

$$\tilde{\tau}_1(f_{\alpha_1,\alpha_2}) = \tilde{\tau}_2\Big(f_{\alpha_1,\alpha_2}\Big)\Bigg(\prod_{k=1}^{\alpha_1-1} \tilde{\tau}_1^k(c_{12})\Bigg)\Bigg(\prod_{k=0}^{\alpha_2-1} \tilde{\tau}_1{}^{\alpha_1}\tilde{\tau}_2{}^k(c_{12})\Bigg).$$

In the next example we give an explicit description of the centralizer of $\mathbb{R}^{\Omega}$ and the center for a particular quasi-commutative skew PBW-extension $\tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, x_2 >$. Recall that for $m = 2$ the algebra $\mathbb{R}^{\Omega}$ is isomorphic to $\mathbb{R}^2$.

Example 20.4.1 Consider the quasi-commutative skew PBW extension $A = \tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, x_2 >$ with the following conditions.

- The automorphisms $\tilde{\tau}_1, \tilde{\tau}_2 : \mathbb{R}^2 \to \mathbb{R}^2$ are defined as follows: $\tilde{\tau}_1 = id,\ \tilde{\tau}_2(e_1) = e_2$ and $\tilde{\tau}_2(e_2) = e_1$, where e_1, e_2 are the standard basis vectors in $\mathbb{R}^2$.
- $x_2x_1 = (1,2)x_1x_2 \ \Big(\Leftrightarrow\ x_1x_2 = \Big(1, \frac{1}{2}\Big)x_2x_1\Big)$

From Theorem 20.3, the centralizer of $\mathbb{R}^{\Omega}$ in the skew PBW extension $\tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, x_2 >$ is given by

$$C\Big(\mathbb{R}^{\Omega}\Big) = \Bigg\{\sum_{\alpha} f_\alpha x^\alpha \ :\ f_\alpha = 0 \text{ on } Sep^\alpha(\Omega)\Bigg\}.$$

In this case

$$Sep^\alpha(\Omega) = Sep^{\alpha_1,\alpha_2}(\Omega) = Sep^{\alpha_2} = \begin{cases}\Omega & \text{if } \alpha_2 \text{ is odd}\\ \emptyset & \text{if } \alpha_2 \text{ is even.}\end{cases}$$

Therefore

$$C\left(\mathbb{R}^{\Omega}\right) = \left\{ \sum_{j,k} f_{j,2k} x_1^j x_2^{2k} \right\}.$$

Now let us consider the center.

Suppose an element $f = \sum_{\alpha} f_\alpha x^\alpha \in Z(A)$. Then $f \in C(\mathbb{R}^{\Omega})$, and $x_i f = f x_i$ for $i = 1, 2$. Since $f \in C(\mathbb{R}^{\Omega})$, then $f = \sum_{j,k} f_{j,2k} x_1^j x_2^{2k}$.

Now

$$\begin{aligned} x_1 f &= x_1 \left(\sum_{j,k} f_{j,2k} x_1^j x_2^{2k} \right) \\ &= \sum_{j,k} \tilde{\tau}_1(f_{j,2k}) x_1^{j+1} x_2^{2k} \\ &= \sum_{j,k} f_{j,2k} x_1^{j+1} x_2^{2k} \end{aligned}$$

and

$$\begin{aligned} f x_1 &= \left(\sum_{j,k} f_{j,2k} x_1^j x_2^{2k} \right) x_1 \\ &= \sum_{j,k} f_{j,2k} x_1^j \left(x_2^{2k} x_1 \right) \\ &= \sum_{j,k} f_{j,2k} x_1^j \left(\prod_{l=0}^{2k-1} \tilde{\tau}_2^l(1, 2) \right) x_1 x_2^{2k} \\ &= \sum_{j,k} f_{j,2k} \tilde{\tau}_1^j \left(2^k, 2^k \right) x_1^{j+1} x_2^{2k} \\ &= \sum_{j,k} f_{j,2k} \left(2^k, 2^k \right) x_1^{j+1} x_2^{2k}. \end{aligned}$$

Therefore $x_1 f = f x_1$ if and only if

$$f_{j,2k} = f_{j,2k}(2^k, 2^k)$$

from which we obtain that either $f_{j,2k} = 0$ for all j, k or $k = 0$.

Also $f x_2 = x_2 f$ and since $k = 0$, we get $f = \sum_{j} f_j x_1^j$. Therefore

$$f x_2 = \Big(\sum_j f_j x_1^j\Big) x_2 = \sum_j f_j x_1^j x_2,$$

on the other hand

$$\begin{aligned} x_2 f &= x_2\Big(\sum_j f_j x_1^j\Big) \\ &= \sum_j \tilde{\tau}_2(f_j) x_2 x_1^j \\ &= \sum_j \tilde{\tau}_2(f_j)\Big(\prod_{l=0}^{j-1} \tilde{\tau}_1^l(1,2)\Big) x_1^j x_2 \\ &= \sum_j \tilde{\tau}_2(f_j)(1, 2^j) x_1^j x_2 \end{aligned}$$

from which we obtain that $x_2 f = f x_2$ if and only if

$$\tilde{\tau}_2(f)(1, 2^j) = f_j.$$

If we suppose $f_j = (a, b)$ then we obtain that $x_2 f = f x_2$ if and only if

$$(b, a)(1, 2^i) = (a, b)$$

that is, $a = b$ and $2^j = 1$ ($\Rightarrow j = 0$ for all j). Therefore

$$Z(A) = \Big\{ \sum_j f_j x_1^j \ : \ f_j = k(1,1) \text{ for some } k \in \mathbb{R} \Big\}.$$

In following example, we investigate what happens to the center $Z(\tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, x_2 \geqslant)$ if we make a choice of constants $c_{12} = (c_1, c_2)$ for arbitrary $c_1, c_2 \in \mathbb{R}$.

Example 20.4.2 Consider the quasi-commutative skew PBW extension $A = \tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, x_2 >$ with the following conditions.

- The automorphisms $\tilde{\tau}_1, \tilde{\tau}_2 : \mathcal{A} \to \mathcal{A}$ are defined as follows: $\tilde{\tau}_1 = id$, $\tilde{\tau}_2(e_1) = e_2$ and $\tilde{\tau}_2(e_2) = e_1$, where e_1, e_2 are the standard basis vectors in $\mathbb{R}^2$.
- $x_2 x_1 = (c_1, c_2) x_1 x_2 \left(\Leftrightarrow \ x_1 x_2 = \left(\frac{1}{c_1}, \frac{1}{c_2}\right) x_2 x_1 \right)$ where $c_1, c_2 \in \mathbb{R}$ with $c_1 \neq 0 \neq c_2$.

From Theorem 20.3, the centralizer of $\mathbb{R}^{\Omega}$ in the skew PBW extension $\tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, x_2 >$ is given by

$$C\left(\mathbb{R}^{\Omega}\right) = \left\{ \sum_{\alpha} f_\alpha x^\alpha \ : \ f_\alpha = 0 \text{ on } Sep^{\alpha}(\Omega) \right\}.$$

In this case

$$Sep^{\alpha}(\Omega) = Sep^{\alpha_1,\alpha_2}(\Omega) = Sep^{\alpha_2}(\Omega) = \begin{cases} \Omega & \text{if } \alpha_2 \text{ is odd} \\ \emptyset & \text{if } \alpha_2 \text{ is even.} \end{cases}$$

Therefore

$$C\left(\mathbb{R}^{\Omega}\right) = \left\{ \sum_{j,k} f_{j,2k} x_1^j x_2^{2k} \right\}.$$

Now suppose an element $f = \sum\limits_{\alpha} f_\alpha x^\alpha \in Z(A)$. Then $f \in C(\mathbb{R}^{\Omega})$, and $x_i f = f x_i$ for $i = 1, 2$. Since $f \in C(\mathbb{R}^{\Omega})$, then $f = \sum\limits_{j,k} f_{j,2k} x_1^j x_2^{2k}$.

Now

$$\begin{aligned} x_1 f &= x_1 \left(\sum_{j,k} f_{j,2k} x_1^j x_2^{2k} \right) \\ &= \sum_{j,k} \tilde{\tau}_1(f_{j,2k}) x_1^{j+1} x_2^{2k} \\ &= \sum_{j,k} f_{j,2k} x_1^{j+1} x_2^{2k} \end{aligned}$$

and

$$\begin{aligned} f x_1 &= \left(\sum_{j,k} f_{j,2k} x_1^j x_2^{2k} \right) x_1 \\ &= \sum_{j,k} f_{j,2k} x_1^j \left(x_2^{2k} x_1 \right) \\ &= \sum_{j,k} f_{j,2k} x_1^j \left(\prod_{l=0}^{2k-1} \tilde{\tau}_2^l(c_1, c_2) \right) x_1 x_2^{2k} \\ &= \sum_{j,k} f_{j,2k} \tilde{\tau}_1^j \left((c_1 c_2)^k, (c_1 c_2)^k \right) x_1^{j+1} x_2^{2k} \\ &= \sum_{j,k} f_{j,2k} \left((c_1 c_2)^k, (c_1 c_2)^k \right) x_1^{j+1} x_2^{2k}. \end{aligned}$$

Therefore $x_1 f = f x_1$ if and only if

$$f_{j,2k} = f_{j,2k}\Big((c_1c_2)^k, (c_1c_2)^k\Big)$$

from which we obtain that either $f_{j,2k} = 0$ for all j, k or $(c_1c_2)^k = 1$ for all k. That is $c_2 = \frac{1}{c_1}$. Therefore we have the following;

If $c_2 = c_1^{-1}$, then from $f = \sum_{j,k} f_{j,2k}x_1^jx_2^{2k}$, we have,

$$fx_2 = \Big(\sum_{j,k} f_{j,2k}x_1^jx_2^{2k}\Big)x_2 = \sum_{j,k} f_{j,2k}x_1^jx_2^{2k+1}.$$

On the other hand

$$\begin{aligned} x_2f &= x_2\Big(\sum_{j,k} f_{j,2k}x_1^jx_2^{2k}\Big) \\ &= \sum_{j,k} \tilde{\tau}_2(f_{j,2k})x_2x_1^jx_2^{2k} \\ &= \sum_{j,k} \tilde{\tau}_2(f_{j,2k})\Big(\prod_{l=0}^{j-1} \tilde{\tau}_1^l(c_1, c_2)\Big)x_1^jx_2^{2k+1} \\ &= \sum_{j,k} \tilde{\tau}_2(f_{j,2k})(c_1^j, c_1^{-j})x_1^jx_2^{2k+1}, \end{aligned}$$

from which we obtain that $x_2f = fx_2$ if and only if $\tilde{\tau}_2(f_{j,2k})(c_1^j, c_1^{-j}) = f_{j,2k}$. If we suppose $f_{j,2k} = (a_{j,2k}, b_{j,2k})$ then we obtain that $x_2f = fx_2$ if and only if

$$(b_{j,2k}, a_{j,2k})(c_1^j, c_1^{-j}) = (a_{j,2k}, b_{j,2k}).$$

That is, $b_{j,2k} = c_1^{-j}a_{j,2k}$, and hence ,

$$Z\Big(\tilde{\tau}\Big(\mathbb{R}^{\Omega}\Big) < x_1, x_2 > \Big) = \Big\{\sum_{j,k} f_{j,2k}x_1^jx_2^{2k} \;:\; f_{j,2k} = a_{j,2k}(1, c_1^{-j}) \text{ for some } a_{j,2k} \in \mathbb{R}\Big\}.$$

20.4.3 PBW Extensions for the Algebra of Piecewise Constant Functions

In Sect. 20.4, the algebra $\mathcal{A}$ of piecewise constant functions $h : \mathbb{R} \to \mathbb{R}$ with N fixed jumps at points $t_1, t_2, \ldots, t_N$ was introduced and we proved that this algebra is isomorphic to $\mathbb{R}^{\Omega}$, the algebra of all functions $f : \Omega \to \mathbb{R}$ indexed by $\Omega = \{0, 1, \ldots, 2N\}$. In Sect. 20.4.2, we gave a description of the centralizer of $\mathbb{R}^{\Omega}$

in the skew PBW extension $\tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, x_2, \ldots, x_n >$. Therefore in this section, we give the centralizer of the coefficient algebra $\mathcal{A}$ in the skew PBW extension $\tilde{\sigma}(\mathcal{A}) < x_1, x_2, \ldots, x_n >$ in terms of the isomorphism $\mu : \mathbb{R}^{\Omega} \to \mathcal{A}$ as given in Eq. (20.1) and $Sep^{\alpha}(\Omega)$, as given in Definition 20.5. We start with the following definition.

Definition 20.6 For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, define

(a) $Sep^{\alpha}_{\mathcal{A}}(\mathbb{R}) := \{x \in \mathbb{R} \ : \ (\exists\, h \in \mathcal{A}) \ : \ \tilde{\sigma}^{\alpha}(h)(x) \neq h(x)\}$;
(b) $Per^{\alpha}_{\mathcal{A}}(\mathbb{R}) := \{x \in \mathbb{R} \ : \ \tilde{\sigma}^{\alpha}(h)(x) = h(x)\}$.

Using methods similar to the proof of Theorem 20.3, it can be shown that the centralizer of $\mathcal{A}$ in the quasi-commutative skew PBW extension $\tilde{\sigma}(\mathcal{A}) < x_1, x_2, \ldots, x_n >$ is given by the following.

Proposition 20.4 *Suppose that for* $1 \leqslant i \leqslant n, \ \delta_i = 0$. *Then the centralizer* $C(\mathcal{A})$, *of* $\mathcal{A}$ *in the skew* PBW *extension* $\tilde{\sigma}(\mathcal{A}) < x_1, \ldots, x_n >$ *is given by*

$$C(\mathcal{A}) = \left\{ \sum_{\alpha} h_{\alpha} x^{\alpha} \ : \ h_{\alpha} = 0 \text{ on } Sep^{\alpha}_{\mathcal{A}}(\mathbb{R}) \right\}.$$

In the next theorem, we give the description of the centralizer of the $\mathcal{A}$ in terms of the isomorphism μ and $Sep^{\alpha}(X)$, in the quasi-commutative case.

Theorem 20.7 *The centralizer* $C(\mathcal{A})$ *of* $\mathcal{A}$ *in the quasi-commutative* $\sigma-PBW$ *extension* $\tilde{\sigma}(\mathcal{A}) < x_1, x_2, \ldots, x_n >$ *is given by;*

$$C(\mathcal{A}) = \left\{ \sum_{\alpha} h_{\alpha} x^{\alpha} \ : \ \mu^{-1}(h_{\alpha}) = 0 \text{ on } Sep^{\alpha}(\Omega) \right\}.$$

where μ *is given by* (20.1).

Proof Define a map $\gamma : \mathbb{R} \to \Omega$ such that

$$\gamma(x) = \omega \ \text{ if } x \in I_{\omega}, \ \omega \in \{0, 1, \ldots, 2N\}.$$

Then for every $f \in \mathbb{R}^{\Omega}$ and every $x \in I_{\omega}$,

$$\mu(f)(x) = f(\omega) = (f \circ \gamma)(x),$$

and for every $h \in \mathcal{A}$,

$$\mu^{-1}(h)(\omega) = h(x) \ \text{ for all } x \in \gamma^{-1}(\omega),$$

where $\gamma^{-1}(\omega)$ denotes the pre-image of ω. Observe that

$$\left\{ \sum_{\alpha} h_{\alpha} x^{\alpha} \ : \ \mu^{-1}(h_{\alpha}) = 0 \text{ on } Sep^{\alpha}(\Omega) \right\} = \left\{ \sum_{\alpha} h_{\alpha} x^{\alpha} \ : \ h_{\alpha}(x) = 0 \ \forall\, x \in \gamma^{-1}\Big(Sep^{\alpha}(\Omega) \Big) \right\}.$$

Therefore it remains to prove that $\gamma^{-1}\Big(Sep^{\alpha}(\Omega)\Big) = Sep^{\alpha}_{\mathcal{A}}(\mathbb{R})$. To this end we have the following;

$$\begin{aligned}\gamma^{-1}\Big(Sep^{\alpha}(\Omega)\Big) &= \gamma^{-1}\Big(\{\omega \in \Omega \ : \ \tau^{\alpha}(\omega) \neq \omega\}\Big)\\ &= \{\gamma^{-1}(\omega) \ : \ \tau^{\alpha}(\omega) \neq \omega\}\\ &= \{x \in \mathbb{R} \ : \ (x \in I_{\omega}) \ : \ \tau^{\alpha}(\omega) \neq \omega\}\\ &= \{x \in \mathbb{R} \ : \ x \in I_{\omega} \ \text{and}\ \sigma^{\alpha}(I_{\omega}) \cap I_{\omega} = \emptyset\}\\ &= \{x \in \mathbb{R} \ : \ (\exists\, h \in \mathcal{A}) \ : \ \tilde{\sigma}^{\alpha}(h)(x) \neq h(x)\}\\ &= Sep^{\alpha}_{\mathcal{A}}(\mathbb{R}).\end{aligned}$$

This completes the proof.

Using the same methods in the proof of Theorem 20.5 and from Theorem 20.7 above, we have the following necessary condition in the general case.

Theorem 20.8 *If an element* $\sum_{\alpha} f_{\alpha}x^{\alpha} \in \tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, \ldots, x_n >$ *belongs to the centralizer of* $\mathbb{R}^{\Omega}$*, then* $\mu^{-1}(h_{\alpha}) = 0$ *on* $Sep^{\alpha}(\Omega)$.

Example 20.4.3 Consider the quasi-commutative skew PBW extension $A = \tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, x_2 >$ with the following conditions.

- The automorphisms $\tilde{\tau}_1, \tilde{\tau}_2 : \mathcal{A} \to \mathcal{A}$ are defined as follows: $\tilde{\tau}_1 = id,\ \tilde{\tau}_2(e_1) = e_2$ and $\tilde{\tau}_2(e_2) = e_1$, where e_1, e_2 are the standard basis vectors in $\mathbb{R}^2$.
- $x_2x_1 = (c_1, c_2)x_1x_2 \left(\Leftrightarrow\ x_1x_2 = \left(\frac{1}{c_1}, \frac{1}{c_2}\right)x_2x_1\right)$ where $c_1, c_2 \in \mathbb{R}$ with $c_1 \neq 0 \neq c_2$.

This corresponds to the algebra $\mathcal{A}$ of piecewise constant functions with one fixed jump point t_1 with $\mathbb{R}$ partitioned into intervals $I_0 = (-\infty, t_1),\ \ I_1 = (t_1, \infty)$ and $I_3 = t_1$. Invariance of $\mathcal{A}$ under any bijection $\sigma : \mathbb{R} \to \mathbb{R}$ implies that $\sigma(t_1) = t_1$.

From the definition of the automorphisms $\tilde{\tau}_1, \tilde{\tau}_2$ we see that the corresponding bijections $\sigma_1, \sigma_2 : \mathbb{R} \to \mathbb{R}$ be have as follows

- $\sigma_1(t_1) = \sigma_2(t_1) = t_1$.
- $\sigma_1(I_0) = I_0$ and hence $\sigma_1(I_1) = I_1$.
- $\sigma_2(I_0) = I_1$ which implies $\sigma_2(I_1) = I_0$.

From Theorem 20.7, it follows that for every $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$,

$$\gamma^{-1}\Big(Sep^{\alpha}(\Omega)\Big) = Sep^{\alpha}_{\mathcal{A}}(\mathbb{R}) = \begin{cases} I_0 \cup I_1 & \text{if } \alpha_2 \text{ is odd}\\ \emptyset, & \text{if } \alpha_2 \text{ is even}\end{cases}.$$

Therefore the centralizer of $\mathcal{A}$ in the skew PBW extension $\tilde{\sigma}(\mathcal{A}) < x_1, x_2 >$ is given by

$$C(\mathcal{A}) = \left\{ \sum_{j,k} h_{j,k} x_1^j x_2^k \ : \ h_{j,2k+1} = 0 \text{ on } I_0 \cup I_1 \right\}.$$

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