Chapter 2
Algebras with Ternary Composition Law Combining Z_2 and Z_3 Gradings

Viktor Abramov, Richard Kerner and Olga Liivapuu

Abstract We investigate the possibility of combining the usual Grassmann algebras with their ternary $\mathbb{Z}_3$-graded counterparts, thus creating a more general algebra with quadratic and cubic constitutive relations coexisting together. We recall the classification of ternary and cubic algebras according to the symmetry properties of ternary products under the action of the S_3 permutation group. Instead of only two kinds of binary algebras, symmetric or antisymmetric, here we get *four* different generalizations of each of those cases. Then we study a particular case of algebras generated by two types of variables, ξ^α and θ^A, satisfying quadratic and cubic relations respectively, $\xi^\alpha \xi^\beta = -\xi^\beta \xi^\alpha$ and $\theta^A \theta^B \theta^C = j \theta^B \theta^C \theta^A$, $j = e^{\frac{2\pi i}{3}}$. Differential calculus of the first order is defined on these algebras, and its fundamental properties investigated. The invariance properties of the generalized algebras are also considered.

Keywords Grassmann algebra · Clifford algebra · Ternary algebra · Cubic algebra · Minkowskian space-time metric · Lorentz group · Cubic matrices

2.1 Classification of Ternary and Cubic Algebras

In [1, 9, 10, 13] certain types of $\mathbb{Z}_3$-graded ternary and cubic algebras have been introduced and investigated. In [19] a general classification of n-ary algebras was

V. Abramov (✉)
Institute of Mathematics and Statistics, University of Tartu, Liivi 2, Tartu, 50409, Estonia
e-mail: viktor.abramov@ut.ee

R. Kerner
LPTMC, Tour 23-13, 5-e étage, Boîte 121, Université P. et M. Curie, 4, Place Jussieu, 75252 Paris Cedex 05, France
e-mail: richard.kerner@upmc.fr

O. Liivapuu
Institute of Technology, Estonian University of Life Sciences, Fr. R. Kreutzwaldi 56, Tartu 51006, Estonia
e-mail: olga.liivapuu@emu.ee

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given. Ternary generalizations of Banach algebras and their derivations have been studied in [2, 7, 14, 15, 17]. Our aim in this section is to find relations between *triple* products of generators of an associative unital algebra, which can be considered as analogs of (binary) commutativity or anti-commutativity. Since we consider triple products of generators these analogs of commutativity and anti-commutativity will be referred to as ternary analogs.

2.1.1 Algebras and Superalgebras with Binary Law of Composition

The usual definition of an algebra involves a linear space $\mathcal{A}$ (over real or complex numbers) endowed with a *binary* constitutive relations

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}. \tag{2.1}$$

In a finite-dimensional case, dim $\mathcal{A}$ =N, in a chosen basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$, the constitutive relations (2.1) can be encoded in *structure constants* f^k_{ij} as follows:

$$\mathbf{e}_i \mathbf{e}_j = f^k_{ij}\, \mathbf{e}_k. \tag{2.2}$$

With the help of these structure constants all essential properties of a given algebra can be expressed, e.g. they will define a *Lie algebra* if they are antisymmetric and satisfy the Jacobi identity:

$$f^k_{ij} = -f^k_{ji}, \quad f^k_{im} f^m_{jl} + f^k_{jm} f^m_{li} + f^k_{lm} f^m_{ij} = 0, \tag{2.3}$$

whereas an abelian algebra will have its structure constants symmetric, $f^k_{ij} = f^k_{ji}$.

Usually, when we speak of algebras, we mean *binary algebras*, understanding that they are defined via *quadratic* constitutive relations (2.2). On such algebras the notion of $\mathbb{Z}_2$-grading can be naturally introduced. The $\mathbb{Z}_2$-graded algebraic structures were abundantly exploited in physics, both in the supersymmetric field theories as well as in models based on the non-commutative geometry [5, 20]. We recall that an algebra $\mathcal{A}$ is called a $\mathbb{Z}_2$-graded algebra if it is a direct sum of two parts, with symmetric (abelian) and anti-symmetric product respectively,

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1, \tag{2.4}$$

with *grade* of an element being 0 if it belongs to $\mathcal{A}_0$, and 1 if it belongs to $\mathcal{A}_1$. Under the multiplication in a $\mathbb{Z}_2$-graded algebra the grades add up reproducing the composition law of the $\mathbb{Z}_2$ permutation group: if the grade of an element A is a, and that of the element B is b, then the grade of their product will be $a + b$ *modulo 2*:

$$\text{grade}(AB) = \text{grade}(A) + \text{grade}(B). \tag{2.5}$$

A $\mathbb{Z}_2$-graded algebra is called a $\mathbb{Z}_2$-graded commutative if for any two homogeneous elements A, B we have

$$AB = (-1)^{ab} BA. \tag{2.6}$$

It is worthwhile to notice at this point that the above relationship can be written in an alternative form, with all the expressions on the left side as follows:

$$AB - (-1)^{a\,b} BA = 0, \quad \text{or} \quad AB + (-1)^{(a\,b+1)} BA = 0 \tag{2.7}$$

The equivalence between these two alternative definitions of commutation (anticommutation) relations inside a $\mathbb{Z}_2$-graded algebra is no more possible if by analogy we want to impose *cubic* relations on algebras with $\mathbb{Z}_3$-symmetry properties, in which the non-trivial cubic root of unity, $j = e^{\frac{2\pi i}{3}}$ plays the role similar to that of -1 in the binary relations displaying a $\mathbb{Z}_2$-symmetry.

2.1.2 Ternary Analog of Commutativity

The $\mathbb{Z}_3$ cyclic group is an abelian subgroup of the S_3 symmetry group of permutations of three objects. The S_3 group contains *six* elements, including the group unit e (the identity permutation, leaving all objects in place: $(abc) \to (abc)$), the two cyclic permutations

$$(abc) \to (bca) \quad \text{and} \quad (abc) \to (cab),$$

and three odd permutations,

$$(abc) \to (cba), \quad (abc) \to (bac) \quad \text{and} \quad (abc) \to (acb).$$

The group $\mathbb{Z}_2$ has the representation $\mathbb{Z}_2 \to \{1, -1\}$. Just like the group $\mathbb{Z}_2$, the group of cyclic permutations $\mathbb{Z}_3$ also has the faithful representation $\mathbb{Z}_3 \to \{1, j, j^2\}$ by cubic roots of unity and this representation is an important tool that we use to find ternary analogs of commutativity and anti-commutativity.

Let an algebra be generated by the finite set of *commutative* variables x^μ, $\mu = 1, 2, \ldots, n$. The *quadratic* relations of *commutativity* of generators given in two equivalent forms,

$$x^\mu y^\nu + (-1) y^\nu x^\mu = 0 \quad \text{or} \quad x^\mu y^\nu = y^\nu x^\mu. \tag{2.8}$$

lead in the case of algebras with *cubic* relations [19] to the following *four* generalizations of the notion of *commutativity*:

(a) Generalizing the first form of the commutativity relation (2.8), which amounts to replacing the -1 generator of $\mathbb{Z}_2$ by j-generator of $\mathbb{Z}_3$ and binary products by products of three elements, we get

$$S: \quad x^\mu x^\nu x^\lambda + j\, x^\nu x^\lambda x^\mu + j^2\, x^\lambda x^\mu x^\nu = 0, \tag{2.9}$$

where $j = e^{\frac{2\pi i}{3}}$ is a primitive third root of unity.

(b) Another primitive third root, $j^2 = e^{\frac{4\pi i}{3}}$ can be used in place of the former one; this will define the conjugate algebra $\bar{S}$, satisfying the following cubic constitutive relations:

$$\bar{S}: \quad x^\mu x^\nu x^\lambda + j^2\, x^\nu x^\lambda x^\mu + j\, x^\lambda x^\mu x^\nu = 0. \tag{2.10}$$

Clearly enough, both algebras are infinitely-dimensional and have the same structure. Each of them is a possible generalization of infinitely-dimensional algebra of usual commuting variables with a finite number of generators. In the usual $\mathbb{Z}_2$-graded case such algebras are just polynomials in variables $x^1,\ x^2, \ldots x^N$; the algebras S and $\bar{S}$ defined above are also spanned by polynomials, but with different symmetry properties, and as a consequence, with different dimensions corresponding to a given power.

(c) Then we can impose the following "weak" commutation, valid only for cyclic permutations of factors:

$$S_1: \quad x^\mu x^\nu x^\lambda = x^\nu x^\lambda x^\mu \neq x^\nu x^\mu x^\lambda, \tag{2.11}$$

(d) Finally, we can impose the following "strong" commutation, valid for arbitrary (even or odd) permutations of three factors:

$$S_0: \quad x^\mu x^\nu x^\lambda = x^\nu x^\lambda x^\mu = x^\nu x^\mu x^\lambda \tag{2.12}$$

The four different associative algebras with cubic commutation relations can be represented in the following diagram, in which all arrows correspond to *surjective homomorphisms*. The commuting generators can be given the common grade 0.

$$\begin{array}{ccc} S & & \bar{S} \\ & \searrow \quad \swarrow & \\ & S_1 & \\ & \downarrow & \\ & S_0 & \end{array}$$

2.1.3 Ternary Analogs of Anti-commutativity

Let us now turn to a cubic generalization of the notion of anti-commutativity. We recall that an algebra generated by anti-commuting variables ξ^α, $\alpha = 1, 2, \ldots, n$, which obey the following quadratic relations

$$\xi^\alpha \xi^\beta + \xi^\beta \xi^\alpha = 0 \text{ or } \xi^\alpha \xi^\beta = -\xi^\beta \xi^\alpha, \tag{2.13}$$

is referred to as a *Grassmann algebra*. Here, too, we have four different choices. Indeed let θ^A, $A = 1, 2, \ldots, N$ be generators of an associative unital algebra over $\mathbb{C}$. We get *four* different cubic analogs of the relations (2.13) as follows:

(a) Generalizing the first relation in (2.13), we get the following "strong" cubic analog of anti-commutativity relation

$$\Lambda_0 : \quad \Sigma_{\pi \in S_3} \, \theta^{\pi(A)} \theta^{\pi(B)} \theta^{\pi(C)} = 0, \tag{2.14}$$

i.e. the sum of *all* permutations of three factors, even and odd ones, must vanish.

(b) We can get another cubic analog of the first relation in (2.13) if we take the subgroup of cyclic permutations $\mathbb{Z}_3$. In this case we get somewhat weaker "cyclic" cubic analog of anti-commutation relation

$$\Lambda_1 : \quad \theta^A \theta^B \theta^C + \theta^B \theta^C \theta^A + \theta^C \theta^A \theta^B = 0, \tag{2.15}$$

i.e. the sum of *cyclic* permutations of three elements must vanish. The same independent relation for the odd combination $\theta^C \theta^B \theta^A$ holds separately.

(c) Generalizing the second relation in (2.13) by means of the representation $\mathbb{Z}_3 \rightarrow \{1, j, j^2\}$, we get the following cubic analog of anti-commutativity relation:

$$\Lambda : \theta^A \theta^B \theta^C = j \; \theta^B \theta^C \theta^A. \tag{2.16}$$

and its conjugate algebra $\bar{\Lambda}$, isomorphic with Λ, which we distinguish by putting a bar on the generators and using dotted indices:

(d) The conjugate relations $\bar{\Lambda}$, which we distinguish by putting a bar on the generators and using dotted indices, are the following ones:

$$\bar{\Lambda} : \quad \bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} = j^2 \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} \tag{2.17}$$

Both these algebras are finite-dimensional. For j or j^2-skew-symmetric algebras with N generators the dimensions of their subspaces of given polynomial order are given by the following generating function:

$$H(t) = 1 + Nt + N^2 t^2 + \frac{N(N-1)(N+1)}{3} \, t^3, \tag{2.18}$$

where we include pure numbers (dimension 1), the N generators θ^A (or $\bar{\theta}^{\dot{B}}$), the N^2 independent quadratic combinations $\theta^A \theta^B$ and $N(N-1)(N+1)/3$ products of three generators $\theta^A \theta^B \theta^C$.

The above four cubic generalizations of Grassmann algebra are represented in the following diagram, in which all the arrows are surjective homomorphisms.

$$\begin{array}{ccc} & \Lambda_0 & \\ & \downarrow & \\ & \Lambda_1 & \\ \swarrow & & \searrow \\ \Lambda & & \bar{\Lambda} \end{array}$$

2.2 Examples of $\mathbb{Z}_3$-Graded Ternary Algebras

2.2.1 The Z_3-Graded Analogue of Grassman Algebra

Let us introduce N generators spanning a linear space over complex numbers, satisfying the following cubic relations [9, 10]:

$$\theta^A \theta^B \theta^C = j\, \theta^B \theta^C \theta^A = j^2\, \theta^C \theta^A \theta^B, \tag{2.19}$$

with $j = e^{2i\pi/3}$, the primitive root of 1. We have $1 + j + j^2 = 0$ and $\bar{j} = j^2$.

Let us denote the algebra spanned by the θ^A generators by $\mathcal{A}$ [9, 10].

We shall also introduce a similar set of *conjugate* generators, $\bar{\theta}^{\dot{A}}$, $\dot{A}, \dot{B}, \ldots = 1, 2, \ldots, N$, satisfying similar condition with j^2 replacing j:

$$\bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} = j^2\, \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} = j\, \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}}, \tag{2.20}$$

Let us denote this algebra by $\bar{\mathcal{A}}$.

We shall endow the algebra $\mathcal{A} \oplus \bar{\mathcal{A}}$ with a natural $\mathbb{Z}_3$ grading, considering the generators θ^A as grade 1 elements, their conjugates $\bar{\theta}^{\dot{A}}$ being of grade 2.

The grades add up modulo 3, so that the products $\theta^A \theta^B$ span a linear subspace of grade 2, and the cubic products $\theta^A \theta^B \theta^C$ being of grade 0. Similarly, all quadratic expressions in conjugate generators, $\bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}}$ are of grade $2 + 2 = 4_{mod\,3} = 1$, whereas their cubic products are again of grade 0, like the cubic products of θ^A's [12].

Combined with the associativity, these cubic relations impose finite dimension on the algebra generated by the $\mathbb{Z}_3$ graded generators. As a matter of fact, cubic expressions are the highest order that does not vanish identically. The proof is immediate:

$$\theta^A \theta^B \theta^C \theta^D = j\, \theta^B \theta^C \theta^A \theta^D = j^2\, \theta^B \theta^A \theta^D \theta^C = j^3\, \theta^A \theta^D \theta^B \theta^C = j^4\, \theta^A \theta^B \theta^C \theta^D, \tag{2.21}$$

and because $j^4 = j \neq 1$, the only solution is $\theta^A \theta^B \theta^C \theta^D = 0$.

2.2.2 Ternary Clifford Algebra

Let us consider the algebra $M_3(\mathbb{C})$ of complex 3rd order square matrices. In accordance with our approach to ternary analog of anti-commutativity we define the ternary j- and j^2-commutator of three matrices as follows

$$[A,B,C]_j = ABC + j\ BCA + j^2\ CAB,\ \ [A,B,C]_{j^2} = ABC + j^2\ BCA + j\ CAB,\quad A,B,C \in M_3(\mathbb{C}). \tag{2.22}$$

It is easy to see that the ternary j-commutator is j-skew symmetric and the ternary j^2-commutator is j^2-skew-symmetric, i.e.

$$[A,B,C]_j = j\ [B,C,A]_j,\ \ [A,B,C]_{j^2} = j^2\ [B,C,A]_{j^2},$$

and from these symmetries it follows that both ternary commutators satisfy the relation Λ_1 (sum of cyclic permutations is zero) as well as more general relation Λ_0 (sum of all permutations is zero).

It is useful to stress a difference between the commutators (2.22) and a ternary Lie bracket used in a theory of 3-Lie algebras initiated by Filippov [6] and Nambu [16]. A ternary Lie bracket of a theory of 3-Lie algebras is totally skew-symmetric, i.e. the permutation of any two elements in a ternary Lie bracket implies the change of sign from plus to minus. Obviously a ternary Lie bracket satisfies the relation Λ_0, but it does not satisfy the relation Λ_1. Hence if there are at least two equal elements in a ternary Lie bracket of a theory of 3-Lie algebras then ternary Lie bracket identically vanishes while the ternary j-commutator (or j^2-commutator) may not be zero. The ternary j-commutator (or j^2-commutator) (2.22) vanishes identically only when all *three* elements are equal.

The matrix algebra $M_3(\mathbb{C})$ has the natural structure of $\mathbb{Z}_3$-graded algebra which can be defined by attaching the degrees 0, 1, 2 to the following matrices respectively

$$\begin{pmatrix} x & 0 & 0\\ 0 & y & 0\\ 0 & 0 & z \end{pmatrix},\quad \begin{pmatrix} 0 & x & 0\\ 0 & 0 & y\\ z & 0 & 0 \end{pmatrix},\quad \begin{pmatrix} 0 & 0 & x\\ y & 0 & 0\\ 0 & z & 0 \end{pmatrix}.$$

Let us introduce the following three 3×3 matrices:

$$Q_1 = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & j\\ j^2 & 0 & 0 \end{pmatrix},\quad Q_2 = \begin{pmatrix} 0 & j & 0\\ 0 & 0 & 1\\ j^2 & 0 & 0 \end{pmatrix},\quad Q_3 = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0 \end{pmatrix}, \tag{2.23}$$

and their hermitian conjugates

$$Q_1^\dagger = \begin{pmatrix} 0 & 0 & j\\ 1 & 0 & 0\\ 0 & j^2 & 0 \end{pmatrix},\quad Q_2^\dagger = \begin{pmatrix} 0 & 0 & j\\ j^2 & 0 & 0\\ 0 & 1 & 0 \end{pmatrix},\quad Q_3^\dagger = \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0 \end{pmatrix}. \tag{2.24}$$

We have

$$\text{degree}(Q_k) = 1, \quad \text{degree}(Q_k^\dagger) = 2, \tag{2.25}$$

The above matrices span a very interesting ternary algebra. Out of three independent $\mathbb{Z}_3$-graded ternary combinations, only one leads to a non-vanishing result. One can check without much effort that both ternary j- and j^2-commutators do vanish:

$$[Q_1, Q_2, Q_3]_j = Q_1 Q_2 Q_3 + j\, Q_2 Q_3 Q_1 + j^2 Q_3 Q_1 Q_2 = 0,$$

$$[Q_1, Q_2, Q_3]_{j^2} = Q_1 Q_2 Q_3 + j^2 Q_2 Q_3 Q_1 + j\, Q_3 Q_1 Q_2 = 0,$$

and similarly for the odd permutation, $Q_2 Q_1 Q_3$. On the contrary, the totally symmetric combination does not vanish; it is proportional to the 3×3 identity matrix $\mathbf{1}$:

$$Q_a Q_b Q_c + Q_b Q_c Q_a + Q_c Q_a Q_b = 3\, \eta_{abc}\, \mathbf{1}, \quad a, b, \ldots = 1, 2, 3. \tag{2.26}$$

with η_{abc} given by the following non-zero components:

$$\begin{aligned} \eta_{111} &= \eta_{222} = \eta_{333} = 1, \\ \eta_{123} &= \eta_{231} = \eta_{312} = 1, \\ \eta_{213} &= \eta_{321} = \eta_{132} = j^2. \end{aligned} \tag{2.27}$$

all other components vanishing. The relation (2.26) may serve as the definition of *ternary Clifford algebra*.

Let us denote $Q_{\dot{a}} = Q_a^\dagger$. Then analogously to (2.26) the matrices $Q_{\dot{a}}$ satisfy the conjugate identities

$$Q_{\dot{a}} Q_{\dot{b}} Q_{\dot{c}} + Q_{\dot{b}} Q_{\dot{c}} Q_{\dot{a}} + Q_{\dot{c}} Q_{\dot{a}} Q_{\dot{b}} = 3\, \eta_{\dot{a}\dot{b}\dot{c}}\, \mathbf{1}, \quad \dot{a}, \dot{b}, \ldots = 1, 2, 3. \tag{2.28}$$

with $\eta_{\dot{a}\dot{b}\dot{c}} = \bar{\eta}_{abc}$.

It is obvious that any similarity transformation of the generators Q_a will keep the ternary anti-commutator (2.26) invariant. As a matter of fact, if we define $\tilde{Q}_b = P^{-1} Q_b P$, with P a non-singular 3×3 matrix, the new set of generators will satisfy the same ternary relations, because

$$\tilde{Q}_a \tilde{Q}_b \tilde{Q}_c = P^{-1} Q_a P P^{-1} Q_b P P^{-1} Q_c P = P^{-1} (Q_a Q_b Q_c) P,$$

and on the right-hand side we have the unit matrix which commutes with all other matrices, so that $P^{-1}\, \mathbf{1}\, P = \mathbf{1}$.

It is also worthwhile to note that the six matrices displayed in (2.23), (2.24) together with two traceless diagonal matrices

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \quad B^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}$$

form the basis for certain representation of the SU(3), which was shown in the nineties by Kac [8].

2.3 Generalized $\mathbb{Z}_2 \times \mathbb{Z}_3$-Graded Ternary Algebra

Let us suppose that we have binary and ternary skew-symmetric products defined by corresponding structure constants:

$$\xi^{\alpha}\xi^{\beta} = -\xi^{\beta}\xi^{\alpha} \tag{2.29}$$

$$\theta^A\theta^B\theta^C = j\ \theta^B\theta^C\theta^A \tag{2.30}$$

The unifying ternary relation is of the type Λ_0, i.e.

$$X^iX^jX^k + X^jX^kX^i + X^kX^iX^j + X^kX^jX^i + X^jX^iX^k + X^iX^kX^j = 0. \tag{2.31}$$

It is obviously satisfied by both types of variables; the θ^A's by definition of the product, for which at this stage the associativity property can be not decided yet; the product of grassmannian ξ^{α} variables (2.29) on the contrary, should be associative in order to make the formula (2.31) applicable.

It can be added that the cubic constitutive relation (2.30) satisfies a simpler condition with cyclic permutations only,

$$\theta^A\theta^B\theta^C + \theta^B\theta^C\theta^A + \theta^C\theta^A\theta^B = 0,$$

but the cubic products of grassmannian variables are invariant under even (cyclic) permutations, so that only the combination of all six permutations of $\xi^{\alpha}\xi^{\beta}\xi^{\gamma}$, like in (2.31) will vanish.

Now, if we want to merge the two algebras into a common one, we must impose the general condition (2.31) to the mixed cubic products. These are of two types: $\theta^A\xi^{\alpha}\theta^B$ and $\xi^{\alpha}\theta^B\xi^{\beta}$, with two θ's and one ξ, or with two ξ's and one θ. These identities, all like (2.31) should follow from *binary* constitutive relations imposed on the *associative* products between one θ and one ξ variable.

Let us suppose that one has

$$\xi^{\alpha}\theta^B = \omega\ \theta^B\xi^{\alpha} \quad \text{and consequently} \quad \theta^A\xi^{\beta} = \omega^{-1}\ \xi^{\beta}\theta^A. \tag{2.32}$$

A simple exercise leads to the conclusion that in order to satisfy the general condition (2.31), the unknown factor ω must verify the equation $\omega + \omega^{-1} + 1 = 0$, or equivalently, $\omega + \omega^2 + \omega^3 = 0$. Indeed, we have, assuming the associativity:

$$\theta^A \xi^\alpha \theta^B = \omega^{-1}\, \xi^\alpha \theta^A \theta^B = \omega\, \theta^A \theta^B \xi^\alpha,$$

$$\theta^B \xi^\alpha \theta^A = \omega^{-1}\, \xi^\alpha \theta^B \theta^A = \omega\, \theta^B \theta^A \xi^\alpha.$$

From this we get, by transforming all the six products so that ξ^α should appear always in front of the monomials:

$$\theta^A \xi^\alpha \theta^B = \omega^{-1}\, \xi^\alpha \theta^A \theta^B, \quad \theta^A \theta^B \xi^\alpha = \omega^{-2}\, \xi^\alpha \theta^A \theta^B,$$

$$\theta^B \xi^\alpha \theta^A = \omega^{-1}\, \xi^\alpha \theta^B \theta^A, \quad \theta^B \theta^A \xi^\alpha = \omega^{-2}\, \xi^\alpha \theta^B \theta^A.$$

Adding up all permutations, even (cyclic) and odd alike, we get the following result:

$$\theta^A \xi^\alpha \theta^B + \xi^\alpha \theta^B \theta^A + \theta^B \theta^A \xi^\alpha + \theta^B \xi^\alpha \theta^A + \xi^\alpha \theta^A \theta^B + \theta^A \theta^B \xi^\alpha =$$

$$(1 + \omega + \omega^{-1})\, \xi^\alpha \theta^A \theta^B + (1 + \omega + \omega^{-1})\, \xi^\alpha \theta^B \theta^A. \tag{2.33}$$

The expression in (2.33) will identically vanish if $\omega = j = e^{\frac{2\pi i}{3}}$ (or j^2, which satisfies the same relation $j + j^2 + 1 = 0$).

The second type of cubic monomials, $\xi^\alpha \theta^B \xi^\beta$, satisfies the identity

$$\xi^\alpha \theta^B \xi^\delta + \theta^B \xi^\delta \xi^\alpha + \xi^\delta \xi^\alpha \theta^B + \xi^\delta \theta^B \xi^\alpha + \theta^B \xi^\alpha \xi^\delta + \xi^\alpha \xi^\delta \theta^B = 0 \tag{2.34}$$

no matter what the value of ω is chosen in the constitutive relation (2.32), the anti-symmetry of the product of two ξ's suffices. As a matter of fact, because we have $\xi^\alpha \xi^\delta = -\xi^\delta \xi^\alpha$, in the formula (2.34) the second term cancels the fifth term, and the third term is canceled by the sixth one. What remains is the sum of the first and the fourth terms:

$$\xi^\alpha \theta^B \xi^\delta + \xi^\delta \theta^B \xi^\alpha.$$

Now we can transform both terms so as to put the factor θ in front; this will give

$$\xi^\alpha \theta^B \xi^\delta + \xi^\delta \theta^B \xi^\alpha = \omega \theta^B \xi^\alpha \xi^\delta + \omega \theta^B \xi^\delta \xi^\alpha = 0 \tag{2.35}$$

because of the anti-symmetry of the product between the two ξ's.

This completes the construction of the $\mathbb{Z}_3 \times \mathbb{Z}_2$-graded extension of Grassman algebra.

The existence of *two* cubic roots of unity, j and j^2, suggests that one can extend the above algebraic construction by introducing a set of *conjugate* generators, denoted for convenience with a bar and with dotted indices, satisfying conjugate ternary constitutive relation (2.20). The unifying condition of vanishing of the sum of all permutations (algebra of Λ_0-type) will be automatically satisfied.

But now we have to extend this condition to the triple products of the type $\theta^A \bar{\theta}^{\dot{B}} \theta^C$ and $\bar{\theta}^{\dot{A}} \theta^B \bar{\theta}^{\dot{C}}$. This will be achieved if we impose the obvious condition, similar to the one proposed already for binary combinations $\xi\theta$:

$$\theta^A \bar{\theta}^{\dot{B}} = j \bar{\theta}^{\dot{B}} \theta^A, \quad \bar{\theta}^{\dot{B}} \theta^A = j^2 \, \theta^A \bar{\theta}^{\dot{B}} \tag{2.36}$$

The proof of the validity of the condition (2.31) for the above combinations is exactly the same as for the triple products $\xi^\alpha \theta^B \xi^\gamma$ and $\theta^A \xi^\delta \theta^B$.

We have also to impose commutation relations on the mixed products of the type

$$\xi^\alpha \bar{\theta}^{\dot{B}} \xi^\beta \quad \text{and} \quad \bar{\theta}^{\dot{B}} \xi^\beta \bar{\theta}^{\dot{C}}.$$

It is easy to see that like in the former case, it is enough to impose the commutation rule similar to the former one with θ's, namely

$$\xi^\alpha \bar{\theta}^{\dot{B}} = j^2 \, \bar{\theta}^{\dot{B}} \xi^\alpha \tag{2.37}$$

Although we could stop at this point the extension of our algebra, for the sake of symmetry it seems useful to introduce the new set of conjugate variables $\bar{\xi}^{\dot{\alpha}}$ of the $\mathbb{Z}_2$-graded type. We shall suppose that they anti-commute, like the ξ^β's, and not only between themselves, but also with their conjugates, which means that we assume

$$\bar{\xi}^{\dot{\alpha}} \bar{\xi}^{\dot{\beta}} = -\bar{\xi}^{\dot{\beta}} \bar{\xi}^{\dot{\alpha}}, \quad \xi^\alpha \bar{\xi}^{\dot{\beta}} = -\bar{\xi}^{\dot{\beta}} \xi^\alpha. \tag{2.38}$$

This ensures that the condition (2.31) will be satisfied by any ternary combination of the $\mathbb{Z}_2$-graded generators, including the mixed ones like

$$\bar{\xi}^{\dot{\alpha}} \xi^\beta \bar{\xi}^{\dot{\delta}} \quad \text{or} \quad \xi^\beta \bar{\xi}^{\dot{\alpha}} \xi^\gamma.$$

The dimensions of classical Grassmann algebras with n generators are well known: they are equal to 2^n, with subspaces spanned by the products of k generators having the dimension $C^n_k = n!/(n-k)!k!$. With $2n$ anticommuting generators, ξ^α and $\bar{\xi}^{\dot{\beta}}$ we shall have the dimension of the corresponding Grassmann algebra equal to 2^{2n}.

It is also quite easy to determine the dimension of the $\mathbb{Z}_3$-graded generalizations of Grassmann algebras constructed above (see, e.g. in [1, 10, 11]). The $\mathbb{Z}_3$-graded algebra with N generators θ^A has the total dimension $N + N^2 + (N^3 - N)/3 = (N^3 + N^2 + 2N)/3$. The conjugate algebra, with the same number of generators, has the identical dimension. However, the dimension of the extended algebra unifying both these algebras is not equal to the square of the dimension of one of them because of the extra conditions on the mixed products between the generators and their conjugates, $\theta^A \bar{\theta}^{\dot{B}} = \bar{\theta}^{\dot{B}} \theta^A$.

2.4 Two Distinct Gradings: $\mathbb{Z}_3 \times \mathbb{Z}_2$ Versus $\mathbb{Z}_6$

In the previous section we combined a Grassmann algebra and a ternary Grassmann algebra into a single algebra called the ternary extension of Grassmann algebra. But a ternary Grassmann algebra can be endowed with the $\mathbb{Z}_3$ grading and a Grassmann algebra with the $\mathbb{Z}_2$ grading. In this section we show that these gradings can be combined into a single $\mathbb{Z}_6$ grading of ternary extension of Grassmann algebra.

The $\mathbb{Z}_2$-grading of ordinary (binary) algebras is well known and widely studied and applied (e.g. in the super-symmetric field theories in Physics). The Grassmann algebra is perhaps the oldest and the best known example of a $\mathbb{Z}_2$-graded structure. Other gradings are much less popular. The $\mathbb{Z}_3$-grading was introduced and studied in the paper [9]; the $\mathbb{Z}_N$ grading was discussed in [4]. An approach to ternary Clifford algebra based on ternary triples and a successive process of ternary Galois extensions is proposed in [18].

In the case of ternary algebras of type Λ_1 or Λ_2, the grade 1 is attributed to the generators θ^A and the grade 2 to the conjugate generators $\bar{\theta}^{\dot{B}}$. Consequently, their products acquire the grade which is the sum of grades of the factors modulo 3. When we consider an algebra including a ternary $\mathbb{Z}_3$-graded subalgebra and a binary $\mathbb{Z}_2$-graded one, we can quite naturally introduce a combination of the two gradings considered as a pair of two numbers, say (a, λ), with $a = 0, 1, 2$ representing the $\mathbb{Z}_3$-grade, and $\lambda = 0, 1$ representing the $\mathbb{Z}_2$ grade, $\lambda = 0, 1$. The first grades add up modulo 3, the second grades add up modulo 2. The six possible combined grades are then

$$(0,0), \quad (1,0), \quad (2,0), \quad (0,1), \quad (1,1) \quad \text{and} \quad (2,1). \tag{2.39}$$

To add up two of the combined grades amounts to adding up their first entries modulo 3, and their second entries modulo 2. Thus, we have

$$(2,1) + (1,1) = (3,2) \simeq (0,0), \quad \text{or} \quad ((2,1) + (1,0) = (3,1) \simeq (0,1), \quad \text{and so forth.}$$

It is well known that the cartesian product of two cyclic groups $\mathbb{Z}_N \times \mathbb{Z}_n$, N and n being two prime numbers, is the cyclic group $\mathbb{Z}_{Nn}$ corresponding to the product of those prime numbers. This means that there is an isomorphism between the cyclic group $\mathbb{Z}_6$, generated by the *sixth* primitive root of unity, $q^6 = 1$, satisfying the equation

$$q + q^2 + q^3 + q^4 + q^5 + q^6 = 0.$$

This group can be represented on the complex plane, with $q = e^{\frac{2\pi i}{6}}$, as shown on the diagram (Fig. 2.1):

The elements of the group $\mathbb{Z}_6$ represented by complex numbers multiply modulo 6, e. g. $q^4 \cdot q^5 = q^9 \simeq q^3$, etc. The six elements of $\mathbb{Z}_6$ can be put in the one-to-one correspondence with the pairs defining six elements of $\mathbb{Z}_3 \times \mathbb{Z}_2$ according to the following scheme:

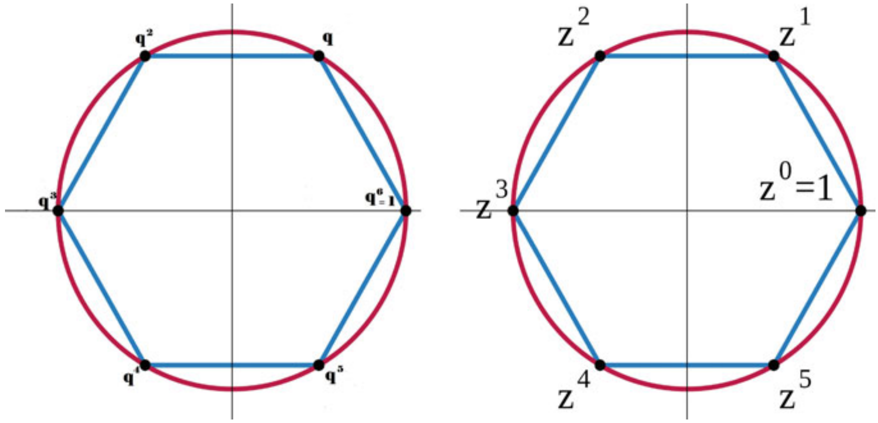

Fig. 2.1 Representation of the cyclic group $\mathbf{Z_6}$ in the complex plane with three colors and three "anti-colors" attributed to even and odd powers of q, accordingly with colors attributed in quantum chromodynamics to quarks and to anti-quarks

$$(0,0) \simeq q^0 = 1, \quad (2,1) \simeq q, \quad (1,0) \simeq q^2, \quad (0,1) \simeq q^3, \quad (2,0) \simeq q^4, \quad (1,1) \simeq q^5. \tag{2.40}$$

The same result can be obtained directly using the representations of $\mathbb{Z}_3$ and $\mathbb{Z}_2$ in the complex plane. Taken separately, each of these cyclic groups is generated by one non-trivial element, the third root of unity $j = e^{\frac{2\pi i}{3}}$ for $\mathbb{Z}_3$ and $-1 = e^{\pi i}$ for $\mathbb{Z}_2$. It is enough to multiply these complex numbers and take their different powers in order to get all the six elements of the cyclic group $\mathbb{Z}_6$. One easily identifies then

$$-j^2 = q, \quad j = q^2, \quad -1 = q^3, \quad j^2 = q^4, \quad -j = q^5, \quad 1 = q^6.$$

The colors attributed to the powers of the complex generator q can be used to modelize the exclusion principle used in Quantum Chromodynamics, where exclusively the "white" combinations of three quarks and three anti-quarks, as well as the "white" quark-anti-quark pairs are declared observable. Replacing the word "white" by 0, we see that there are *two* vanishing linear combinations of *three* powers of q, and *three* pairs of powers of q that are also equal to zero. Indeed, we have:

$$q^2 + q^4 + q^6 = j + j^2 + 1 = 0, \quad \text{and} \quad q + q^3 + q^5 = -j^2 - 1 - j = 0, \tag{2.41}$$

as well as

$$q + q^4 = 0, \quad q^2 + q^5 = 0, \quad q^3 + q^6 = 0. \tag{2.42}$$

The $\mathbb{Z}_6$-grading should unite both $\mathbb{Z}_2$ and $\mathbb{Z}_3$ gradings, reproducing their essential properties. Obviously, the $\mathbb{Z}_3$ subgroup is formed by the elements 1, q^2 and q^4, while the $\mathbb{Z}_2$ subgroup is formed by the elements 1 and $q^3 = -1$. In what follows, we shall see that the associativity imposes many restrictions which can be postponed in the case of non-associative ternary structures.

The natural choice for the $\mathbb{Z}_3$-graded algebra with cubic relations was to attribute the grade 1 to the generators θ^A, and grade 2 to their conjugates $\bar{\theta}^{\dot{B}}$. All other expressions formed by products and powers of those got the well defined grade, the sum of the grades of factors modulo 3. In a simple Cartesian product of two algebras, a $\mathbb{Z}_3$-graded with a $\mathbb{Z}_2$-graded one, the generators of the latter will be given grade 1, and their products will get automatically the grade which is the sum of the grades of factors modulo 2, which means that the all products and powers of generators ξ^α will acquire grade 1 or 0 according to the number and character of factors involved. The mixed products of the type $\theta^A\xi^\beta$, $\xi^\beta\theta^B\theta^C$, etc. can be given the double $\mathbb{Z}_3 \times \mathbb{Z}_2$ grade according to (2.39). According to the isomorphism defined by (2.40), this is equivalent to a $\mathbb{Z}_6$-grading of the product algebra.

As long as the algebra is supposed to be *homogeneous* in the sense that all the constitutive relations contain exclusively terms of *one and the same type*, like in the extension of Grassmann algebra discussed above, the supposed associativity does not impose any particular restrictions. However, this is not the case if we consider the possibility of *non-homogeneous* constitutive equations, including terms of different nature, but with the same $\mathbb{Z}_6$-grade. The grading defined by (2.40) suggests a possibility of extending the constitutive relations by comparing terms of the type $\theta^A\theta^B\theta^C$, whose $\mathbb{Z}_6$-grade is 3, to the generators ξ^α having the same $\mathbb{Z}_6$-grade. This will lead to the following constitutive relations:

$$\theta^A\theta^B\theta^C = \rho^{ABC}{}_\alpha\, \xi^\alpha \quad \text{and} \quad \bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}\bar{\theta}^{\dot{C}} = \bar{\rho}^{\dot{A}\dot{B}\dot{C}}{}_{\dot{\alpha}}\, \bar{\xi}^{\dot{\alpha}} \tag{2.43}$$

with the coefficients (structure constants) $\rho^{ABC}{}_\alpha$ and $\bar{\rho}^{\dot{A}\dot{B}\dot{C}}{}_{\dot{\alpha}}$ displaying obvious symmetry properties mimicking the properties of ternary products of θ-generators with respect to cyclic permutations:

$$\rho^{ABC}{}_\alpha = j\ \rho^{BCA}{}_\alpha = j^2\ \rho^{CAB}{}_\alpha \quad \text{and} \quad \bar{\rho}^{\dot{A}\dot{B}\dot{C}}{}_{\dot{\alpha}} = j^2\ \bar{\rho}^{\dot{B}\dot{C}\dot{A}}{}_{\dot{\alpha}} = j\ \bar{\rho}^{\dot{C}\dot{A}\dot{B}}{}_{\dot{\alpha}}. \tag{2.44}$$

If all products are supposed to be associative, then we see immediately that the products between θ and ξ generators, as well as those between $\bar{\theta}$ and $\bar{\xi}$ generators must vanish identically, because of the vanishing of quadric products $\theta\theta\theta\theta = 0$ and $\bar{\theta}\bar{\theta}\bar{\theta}\bar{\theta} = 0$. This means that we must set

$$\theta^A\ \xi^\beta = 0, \quad \xi^\beta\theta^A = 0, \quad \text{as well as} \quad \bar{\theta}^{\dot{B}}\bar{\xi}^{\dot{\alpha}} = 0, \quad \bar{\xi}^{\dot{\alpha}}\bar{\theta}^{\dot{B}} = 0. \tag{2.45}$$

But now we want to unite the two gradings into a unique common one. Let us start by defining a ternary product of generators, not necessarily derived from an ordinary associative algebra. We shall just suppose the existence of ternary product of generators, displaying the j-skew symmetry property:

$$\{\theta^A, \theta^B, \theta^C\} = j\{\theta^B, \theta^C, \theta^A\} = j^2\{\theta^C, \theta^A, \theta^B\}. \tag{2.46}$$

and similarly, for the conjugate generators,

$$\{\bar{\theta}^{\dot{A}}, \bar{\theta}^{\dot{B}}, \bar{\theta}^{\dot{C}}\} = j^2 \{\bar{\theta}^{\dot{B}}, \bar{\theta}^{\dot{C}}, \bar{\theta}^{\dot{A}}\} = j \{\bar{\theta}^{\dot{C}}, \bar{\theta}^{\dot{A}}, \bar{\theta}^{\dot{B}}\}. \tag{2.47}$$

Let us attribute the $\mathbb{Z}_6$-grade 1 to the generators θ^A. Then it is logical to attribute the $\mathbb{Z}_6$ grade 5 to the conjugate generators $\bar{\theta}^{\dot{B}}$, so that mixed products $\theta^A\bar{\theta}^{\dot{B}}$ would be of $\mathbb{Z}_6$ grade 0. Ternary products (2.46) are of grade 3, and ternary products of conjugate generators (2.47) are also of grade 3, because $5 + 5 + 5 = 15$, and 15 $modulo$ $6 = 3$. But we have also $q^3 = -1$, which is the generator of the $\mathbb{Z}_2$-subalgebra of $\mathbb{Z}_6$. Therefore we should attribute the $\mathbb{Z}_6$-grade 3 to both kinds of the anti-commuting variables, ξ^α and $\bar{\xi}^{\dot{\beta}}$, because we can write their constitutive relations using the root q as follows:

$$\xi^\alpha\xi^\beta = -\xi^\beta\xi^\alpha = q^3\ \xi^\beta\xi^\alpha, \quad \bar{\xi}^{\dot{\alpha}}\bar{\xi}^{\dot{\beta}} = -\bar{\xi}^{\dot{\beta}}\bar{\xi}^{\dot{\alpha}} = q^3\ \bar{\xi}^{\dot{\beta}}\bar{\xi}^{\dot{\alpha}}, \quad \xi^\alpha\bar{\xi}^{\dot{\beta}} = -\bar{\xi}^{\dot{\beta}}\xi^\alpha = q^3\ \bar{\xi}^{\dot{\beta}}\xi^\alpha, \tag{2.48}$$

On the other hand, the expressions containing products of θ with $\bar{\xi}$ and $\bar{\theta}$ with ξ:

$$\theta^A\bar{\xi}^{\dot{\alpha}} \quad \text{and} \quad \bar{\theta}^{\dot{B}}\xi^\beta.$$

The first expression has the $\mathbb{Z}_6$-grade $1 + 3 = 4$, and the second product has the $\mathbb{Z}_6$-grade $5 + 3 = 8$ *modulo* $6 = 2$. Other products endowed with the same grade in our associative $\mathbb{Z}_6$-grade algebra are $\bar{\theta}^{\dot{A}}\,\bar{\theta}^{\dot{B}}$ (grade 4, because $5 + 5 = 10$ *modulo*, 6 $= 4$, and $\theta^A\theta^B$ of grade 2, because $1 + 1 = 2$).

This suggests that the following non-homogeneous constitutive relations can be proposed:

$$\theta^A\bar{\xi}^{\dot{\alpha}} = f^{A\dot{\alpha}}{}_{\dot{C}\dot{D}}\ \bar{\theta}^{\dot{C}}\bar{\theta}^{\dot{D}}, \quad \text{and} \quad \bar{\theta}^{\dot{A}}\xi^\alpha = \bar{f}^{\dot{A}\alpha}{}_{CD}\ \theta^C\theta^D, \tag{2.49}$$

where the coefficients should display the symmetry properties contravariant to those of the generators themselves, which means that we should have

$$f^{A\dot{\alpha}}{}_{\dot{C}\dot{D}} = j^2\ f^{\dot{\alpha}A}{}_{\dot{C}\dot{D}} \quad \text{and} \quad \bar{f}^{\dot{A}\alpha}{}_{CD} = j\ \bar{f}^{\alpha\dot{A}}{}_{CD} \tag{2.50}$$

2.5 First Order Differential Calculus Over $\mathbb{Z}_2$ and $\mathbb{Z}_3$ Skew-Symmetric Algebras

Given an algebra generated by a finite number of generators, which are subjected to relations, one can develop a first order differential calculus over this algebra. Our aim in this section is to develop a first order differential calculus over $\mathbb{Z}_2$ and $\mathbb{Z}_3$ skew-symmetric algebras. We will use a coordinate first order differential calculus proposed in [3].

Let us briefly remind a notion of a coordinate first order differential calculus over an algebra. A first order differential calculus over an algebra $\mathscr{A}$ is a triple $(\mathscr{A}, \mathscr{M}, d)$, where $\mathscr{A}$ is a unital associative algebra, $\mathscr{M}$ is $\mathscr{A}$-bimodule and $d : \mathscr{A} \to \mathscr{M}$ is a differential, i.e. linear mapping satisfying Leibniz rule. Let us denote by $\mathscr{M}_R$

the right $\mathscr{A}$-module of $\mathscr{M}$. If algebra $\mathscr{A}$ is generated by a finite set of variables $x^1, x^2, \ldots, x^n$ which obey relations $f^\alpha(x^1, x^2, \ldots, x^n) = 0,\ \alpha = 1, 2, \ldots m$ and $\mathscr{M}_R$ is freely generated by $dx^1, dx^2, \ldots, dx^n$, then a first order differential calculus $(\mathscr{A}, \mathscr{M}, d)$ is called a coordinate first order differential calculus over $\mathscr{A}$. In this case one can introduce partial derivatives with respect to generators $x^1, x^2, \ldots, x^n$ by means of $df = dx^k\ \frac{\partial f}{\partial x^k}$ and prove that partial derivatives satisfy the twisted Leibniz rule

$$\frac{\partial}{\partial x^k}(f\,g) = \frac{\partial f}{\partial x^k}\ g + A^m_k(f)\frac{\partial g}{\partial x^k}, \quad f, g \in \mathscr{A} \tag{2.51}$$

where a homomorphism $A : \mathscr{A} \to \mathrm{Mat}_n(\mathscr{A})$ from an algebra $\mathscr{A}$ to the algebra of square matrices of order n over $\mathscr{A}$ is defined by $f\ dx^k = dx^m\ A^k_m(f)$.

Assume that we have a first order differential calculus over an algebra generated by variables $x^1, x^2, \ldots, x^n$ which are subjected to relations $f^\alpha(x^1, x^2, \ldots, x^n) = 0$, where $\alpha = 1, 2, \ldots m$ and $f^\alpha(x^1, x^2, \ldots, x^n)$ are homogeneous polynomials of variables $x^1, x^2, \ldots, x^n$. Then differentiating the both sides of relations with the help of a differential d, we get a consistency conditions of algebra relations with a first order differential calculus. Hence $df^\alpha = 0$, and applying the definition of partial derivatives we can write the consistency condition as follows

$$\frac{\partial f^\alpha}{\partial x^k} = 0,\ k = 1, 2, \ldots, n,\ \alpha = 1, 2, \ldots, m. \tag{2.52}$$

Let $\mathscr{A}$ be a unital associative algebra over $\mathbb{C}$ generated by $x^1, x^2, \ldots, x^n$. For any triple of integers i, j, k, where $1 \le i \le j \le k \le n$, we denote the symmetric polynomial of the third degree by

$$f^{(ijk)}(x) = \sum_{S_3} x^i x^j x^k, \tag{2.53}$$

where at the right-hand side of the above formula we mean the sum of products generated by all permutations of i, j, k in $x^i x^j x^k$, i.e

$$\sum_{S_3} x^i x^j x^k = x^i x^j x^k + x^j x^k x^i + x^k x^i x^j + x^k x^j x^i + x^j x^i x^k + x^i x^k x^j.$$

By analogy with the binary anti-commutator $\{a, b\} = a \cdot b + b \cdot a$ it is useful in ternary case to introduce a notion of ternary anti-commutator, but in this case for a structure of ternary anti-commutator we have a choice between the whole group S_3 of all permutations and its subgroup $\mathbb{Z}_3$ of cyclic permutations. Hence it is useful to introduce two ternary anti-commutators

$$\{x^i, x^j, x^k\}_{S_3} = \sum_{S_3} x^i x^j x^k, \tag{2.54}$$

$$\{x^i, x^j, x^k\}_{\mathbb{Z}_3} = \sum_{\mathbb{Z}_3} x^i x^j x^k = x^i x^j x^k + x^j x^k x^i + x^k x^i x^j. \tag{2.55}$$

Making use of these notations we can write

$$f^{(ijk)}(x) = \{x^i, x^j, x^k\}_{S_3}.$$

Obviously the polynomial $f^{(ijk)}(x)$ is invariant under any permutation of integers i, j, k. An algebra $\mathscr{A}$, whose generators $x^1, x^2, \ldots, x^n$ obey the relations

$$f^{(ijk)}(x) = \sum_{S_3} x^i x^j x^k = x^i x^j x^k + x^j x^k x^i + x^k x^i x^j + x^k x^j x^i + x^j x^i x^k + x^i x^k x^j = 0, \tag{2.56}$$

will be referred to as a *3-Grassmann algebra*. Now our aim is to construct a coordinate first order differential calculus $(\mathscr{A}, \mathscr{M}, d)$ over 3-Grassmann algebra $\mathscr{A}$. To this end, we assume that a differential d maps each generator x^i to its differential dx^i, the right $\mathscr{A}$-module $\mathscr{M}_R$ is freely generated by the differentials $dx^1, dx^2, \ldots, dx^n$ and for any element $f \in \mathscr{A}$ we have $f\,dx^i = dx^j\, A^i_j(f)$, where A^i_j are the entries of a matrix of homomorphism of algebras $A : \mathscr{A} \to \mathrm{Mat}_n(\mathscr{A})$. Making use of twisted Leibniz rule for partial derivatives we compute

$$\frac{\partial f^{(ijk)}}{\partial x^l} = \sum_{S_3} \delta^i_l\, x^j\, x^k + \sum_{S_3} A^i_l(x^j)\, x^k + \sum_{S_3} A^i_l(x^j\, x^k). \tag{2.57}$$

It is worth to mention that $A : \mathscr{A} \to \mathrm{Mat}_n(\mathscr{A})$ is a homomorphism of algebras and the last term of the right-hand side of the above formula can be written as

$$A^i_l(x^j\, x^k) = A^m_l(x^j)\, A^i_m(x^k),$$

where at the right-hand side we take a sum over m. Hence the entries of matrix of a homomorphism A of first order differential calculus over 3-Grassmann algebra must satisfy the condition

$$\sum_{S_3} \delta^i_l\, x^j\, x^k + \sum_{S_3} A^i_l(x^j)\, x^k + \sum_{S_3} A^i_l(x^j\, x^k) = 0. \tag{2.58}$$

In this section we will consider first order noncommutative differential calculuses over algebras, which are particular cases of 3-Grassmann algebra. Firstly we will assume that for any pair of integers i, j there is a homogeneous polynomial $g^{(ij)}(x)$ of second degree of generators $x^1, x^2, \ldots, x^n$ such that

$$f^{(ijk)}(x) = \lambda^{ijk}_{lrs}\, g^{(lr)}(x)\, a^{(s)}(x) + \mu^{ijk}_{lrs}\, b^{(l)}(x)\, g^{(rs)}(x), \tag{2.59}$$

where $a^{(s)}(x), b^{(l)}(x)$ are linear polynomials of generators and $\lambda^{ijk}_{lrs}, \mu^{ijk}_{lrs}$ are complex numbers. If for any pair of integers i, j we impose the condition $g^{(ij)}(x) = 0$ on generators of algebra, we get a particular case of 3-Grassmann algebra because from (2.59) it follows directly that $f^{(ijk)}(x) = 0$. Obviously in this case we get an algebra with quadratic relations because polynomials $g^{(ij)}(x)$ are of second degree. A first order differential calculus over an algebra with quadratic relations $g^{(ij)}(x) = 0$ must satisfy the conditions

$$\frac{\partial g^{(ij)}(x)}{\partial x^k} = 0, \tag{2.60}$$

which give the equations for the entries A^i_j of a matrix of homomorphism A, when we know an exact form of polynomials $g^{(ij)}(x)$. Differentiating (2.59) we get

$$\begin{aligned}\frac{\partial f^{(ijk)}(x)}{\partial x^m} = \lambda^{ijk}_{lrs}\Big(&\frac{\partial g^{(lr)}(x)}{\partial x^m}\, a^{(s)}(x) + A^p_m(g^{(lr)}(x))\, \frac{\partial a^{(s)}(x)}{\partial x^p}\Big)\\ +\mu^{ijk}_{lrs}\Big(&\frac{\partial b^{(l)}(x)}{\partial x^m}\, g^{(ij)}(x) + A^p_m(b^{(l)}(x))\, \frac{\partial g^{(rs)}(x)}{\partial x^p}\Big).\end{aligned}$$

The quadratic relations of algebra $g^{(ij)}(x) = 0$ implies $A(g^{(ij)}(x)) = 0$, and together with (2.60) it shows that the right-hand side of the above formula vanishes, which means that calculus of partial derivatives over an algebra with quadratic relations $g^{(ij)}(x) = 0$ is consistent with a calculus of partial derivatives over a 3-Grassmann algebra.

Secondly we will consider a particular case of 3-Grassmann algebra, where for any triple of integers i, j, k there is a homogeneous polynomial $g^{(ijk)}(x)$ of third degree of generators $x^1, x^2, \ldots, x^n$ such that the basic polynomial of 3-Grassmann algebra $f^{(ijk)}(x)$ can be represented as

$$f^{(ijk)}(x) = \sum_{S_3} g^{(ijk)}(x). \tag{2.61}$$

Then requiring $g^{(ijk)}(x) = 0$ (and assuming no quadratic relations between generators) we get a particular case of 3-Grassmann algebra whose generators are subjected to cubic relations. Then

$$\frac{\partial g^{(ijk)}(x)}{\partial x^l} = 0, \tag{2.62}$$

will yield equations for the entries A^i_j of a homomorphism A, and, similarly to the case of quadratic relations, guarantee a consistency of calculus of partial derivatives over an algebra with cubic relations with a calculus of partial derivatives over a 3-Grassmann algebra.

Now we consider three particular cases of 3-Grassmann algebra and develop a first order differential calculus over these algebras. We begin with a case of quadratic relations. Let us denote the generators of algebra in the first case by $\xi^1, \xi^2, \ldots, \xi^n$.

For any pair of integers i, j let

$$g^{(ij)}(\xi) = \xi^i \xi^j + \xi^j \xi^i.$$

Obviously each polynomial is homogeneous second degree polynomial symmetric with respect to permutations of superscripts $g^{(ij)}(\xi) = g^{(ji)}(\xi)$. Next it is easy to see that this is particular case of (2.59) because

$$f^{(ijk)}(\xi) = \sum_{\mathbb{Z}_3} \xi^i \, g^{(jk)}(\xi).$$

Consequently if we require $g^{(ij)}(\xi) = \xi^i \xi^j + \xi^j \xi^i = 0$ then we get a particular case of 3-Grassmann algebra, which is well known under the name of Grassmann algebra. The Eq. (2.60), where unknowns are the entries A^i_j of matrix of a homomorphism A, takes on the form

$$A^j_k(\xi^i) + A^i_k(\xi^j) = -\delta^i_k \, \xi^j - \delta^j_k \, \xi^i. \tag{2.63}$$

It should be mentioned that (2.63) together with $g^{(ij)}(\xi) = 0$ implies (2.58). Indeed the first sum can be written as

$$\sum_{S_3} \delta^i_l \, \xi^j \, \xi^k = \sum_{\mathbb{Z}_3} \delta^i_l \, g^{(jk)}(\xi) = 0,$$

the second by means of (2.63) can be put into the form

$$\sum_{S_3} A^i_l(\xi^j) \, \xi^k = -\sum_{\mathbb{Z}_3} \delta^i_l \; g^{(jk)}(\xi) = 0,$$

and finally the third sum is

$$\sum_{S_3} A^i_l(\xi^j \, \xi^k) = \sum_{\mathbb{Z}_3} A^i_l(g^{(jk)}(\xi)) = 0.$$

It is natural for first order noncommutative differential calculus to seek a solution of this equation in the form $A^i_k(\xi^j) = p \, \delta^i_k \, \xi^j$, where p is a complex number. Substituting this into the above equation we get immediately $p = -1$. This solution induces a well known calculus of partial derivatives over a Grassmann algebra, where a derivative of a monomial $\xi^{i_1} \xi^{i_2} \ldots \xi^{i_k}$ $(1 \leq i_1 < i_2 < \cdots < i_k \leq n)$ can be calculated by means of the formula

$$\frac{\partial}{\partial \xi^j}(\xi^{i_1} \xi^{i_2} \ldots \xi^{i_k}) = \sum_m (-1)^{m-1} \delta^{i_m}_j \; \xi^{i_1} \xi^{i_2} \ldots \hat{\xi}^{i_m} \ldots \xi^{i_k},$$

and hat over a generator ξ^{i_m} means that this generator is omitted.

In order to describe a next particular case of 3-Grassmann algebra with quadratic relations we denote generators of algebra by $\eta^1, \eta^2, \ldots, \eta^n$. Let j be a primitive 3rd order root of unity. We define homogeneous second degree polynomials by

$$g^{(ij)}(\eta) = \eta^i \eta^j - j\ \eta^j \eta^i, \qquad i < j, \tag{2.64}$$
$$g^{(ij)}(\eta) = \eta^i \eta^j - j^{-1}\ \eta^j \eta^i, \quad i > j. \tag{2.65}$$

Then for any triple of integers $1 \le i < j < k \le n$ we have

$$f^{(ijk)}(\eta) = \{\eta^i, \eta^j, \eta^k\}_{S_3} = g^{(ij)}(\eta)\, \eta^k + \eta^k\, g^{(ji)}(\eta) + g^{(ik)}(\eta)\, \eta^j + \eta^j\, g^{(ki)}(\eta), \tag{2.66}$$

i.e. this is a particular case of (2.59), where

$$\lambda^{ijk}_{lrs} = \delta^i_l\, \delta^j_r\, \delta^k_s + \delta^i_l\, \delta^k_r\, \delta^j_s, \quad \mu^{ijk}_{lrs} = \delta^k_l\, \delta^j_r\, \delta^i_s + \delta^j_l\, \delta^k_r\, \delta^i_s.$$

Let us mention that the polynomials (2.64), (2.65) do not solve the basic relation of 3-Grassmann algebra (2.56) in the particular case of $i = j = k$ (if two of them are equal, for instant $i = j < k$, then the group of all permutations S_3 reduces to the group of cyclic permutations $\mathbb{Z}_3$ and the basic relation (2.56) is satisfied). Hence for every generator η^i we have to add an additional relation $(\eta^i)^3 = 0$ which guarantees that (2.56) holds for any combination of generators. An algebra generated by $\eta^1, \eta^2, \ldots, \eta^n$ with the relations

$$g^{(ij)}(\eta) = 0, \ \Rightarrow\ \eta^i \eta^j = j\ \eta^j \eta^i,\ (i < j), \quad (\eta^i)^3 = 0, \tag{2.67}$$

is a generalized Grassmann algebra at cubic root of unity. Our aim is to show that applying previously developed method of differential calculus over an algebra with relations we can construct a differential calculus over a generalized Grassmann algebra. We will do this in a most general form considering a generalized Grassmann algebra generated by $\eta^1, \eta^2, \ldots, \eta^n$, which obey the relations

$$\eta^i \eta^j = q\ \eta^j \eta^i,\ (i < j), \quad (\eta^i)^N = 0, \tag{2.68}$$

where q is a primitive Nth root of unity. First the entries of the matrix of a homomorphism of first order differential calculus over a generalized Grassmann algebra are given by

$$A^j_k(x^i) = 0\ (j \ne k),\ A^j_j(x^i) = q\ x^i\ (i < j),\ A^j_j(x^i) = q^{-1}\ x^i\ (i > j). \tag{2.69}$$

Differentiating the left-hand side of the relation $(\eta^i)^N = 0$, applying the Leibniz rule and the definition of a homomorphism A we get

$$d\ (\eta^i)^N = \sum_{k=0}^{N-1} (\eta^i)^k\, d\eta^i\ (\eta^i)^{N-k-1} = \sum_{k=0}^{N-1} d\eta^m\ A^i_m((\eta^i)^k)\ (\eta^i)^{N-k-1}.$$

Substituting this into $d(\eta^i)^N = 0$ we get for any integer m the equation

$$\sum_{k=0}^{N-1} A^i_m((\eta^i)^k)\,(\eta^i)^{N-k-1} = 0, \tag{2.70}$$

which becomes an identity

$$(1 + q + q^2 + \cdots + q^{n-1})\,\delta^i_k(\eta^i)^{n-1} \equiv 0,$$

if we extend the solution for first order differential calculus (2.69) to equal values of superscripts by

$$A^i_k(\eta^i) = q\,\delta^i_k\eta^i. \tag{2.71}$$

2.6 Graded Partial Derivatives

Analogously to the case of Grassmann algebra the solution (2.69) yields the calculus of partial derivatives over a generalized Grassmann algebra, where a partial derivative of any monomial $(\eta^{i_1})^{k_1}(\eta^{i_2})^{k_2}\ldots(\eta^{i_m})^{k_m},\ 1 \le i_1 < i_2 < \cdots < i_m \le n$ can by calculated with the help of the formula

$$\frac{\partial}{\partial\eta^j}\big((\eta^{i_1})^{k_1}(\eta^{i_2})^{k_2}\ldots(\eta^{i_m})^{k_m}\big) = \sum_{r=1}^{m} q^{|k_r|}[k_r]_q\,\delta^{i_r}_j\ (\eta^{i_1})^{k_1}(\eta^{i_2})^{k_2}\ldots(\eta^{i_r})^{k_r-1}\ldots(\eta^{i_m})^{k_m},$$

where $[k_r]_q = (1 + q + \cdots + q^{k_r-1})$, $|k_r| = k_1 + k_2 + \cdots + k_{r-1}$.

Next we consider an algebra with cubic commutation relations. The generators of this algebra we will denote by $\theta^1, \theta^2, \ldots, \theta^N$ and for a superscript of a generator we will use capital letters $A, B, C, \ldots$. For any triple of integers A, B, C let us introduce the polynomials

$$g^{(ABC)}(\theta) = \theta^A\theta^B\theta^C - j\ \theta^B\theta^C\theta^A, \tag{2.72}$$

where j is a primitive cubic root of unity. These polynomials have the property

$$\sum_{\mathbb{Z}_3} g^{(ABC)}(\theta) = (1 - j)\ \sum_{\mathbb{Z}_3}\theta^A\theta^B\theta^C. \tag{2.73}$$

From this we directly get

$$f^{(ABC)}(\theta) = \sum_{S_3}\theta^A\theta^B\theta^C = \frac{1}{1-j}\big(\sum_{\mathbb{Z}_3} g^{(ABC)}(\theta) + \sum_{\mathbb{Z}_3} g^{(CBA)}(\theta)\big).$$

If for any triple of integers A, B, C we impose the cubic relations $g^{(ABC)}(\theta) = 0$ then the previous relation implies $f^{(ABC)}(\theta) = 0$, and we conclude that our algebra is a particular case of 3-Grassmann algebra. This algebra will be denoted by $\mathscr{G}$, and it will be referred to as a 3-Grassmann algebra with cyclic commutation relations. The explicit form of commutation relations of this algebra is

$$\theta^A\theta^B\theta^C = j\ \theta^B\theta^C\theta^A. \tag{2.74}$$

We remind that it can be proved by means of cyclic commutation relations of algebra that any product of four (or more than four) generators vanishes. Hence a 3-Grassmann algebra with cyclic commutation relations is spanned by the monomials

$$\mathbb{1},\ \theta^A,\ \theta^A\theta^B, \tag{2.75}$$
$$\theta^A(\theta^B)^2,\ (\theta^A)^2\theta^B\ \ (A < B), \tag{2.76}$$
$$\theta^A\theta^B\theta^C,\ \theta^C\theta^B\theta^A\ \ (A < B < C). \tag{2.77}$$

Now our aim is to construct a first order differential calculus over this algebra. To this end, we introduce the differentials of generators $d\theta^A$, $A = 1, 2, \ldots, N$ and a bimodule $\mathfrak{M}$ over a 3-Grassmann algebra with cyclic commutation relations. We define $\mathfrak{M}$ as a vector space spanned by all products $f(\theta)\ d\theta^C\ h(\theta)$, where $f(\theta), h(\theta)$ are two monomials from the set of linear independent monomials (2.75)–(2.77). Obviously $\mathfrak{M}$ can be considered as the bimodule over $\mathscr{G}$. Next we define a differential $d : \mathscr{G} \to \mathfrak{M}$ by $d(\mathbb{1}) = 0$, $d(\theta^A) = d\theta^A$ and extend it to any monomial with the help of Leibniz rule.

A differential calculus $(\mathscr{G}, \mathfrak{M}, d)$ must be consistent with the cyclic commutation relations (2.74). Differentiating these relations we get the consistency conditions

$$\begin{aligned} d\theta^A(\theta^B\theta^C) + \theta^A d\theta^B\theta^C + (\theta^A\theta^B)d\theta^C \\ = j\ d\theta^B(\theta^C\theta^A) + j\ \theta^B d\theta^C\theta^A + j\ (\theta^B\theta^C)\, d\theta^A. \end{aligned} \tag{2.78}$$

Assume that we wish to solve it by analogy with a differential calculus over an algebra with quadratic relations supposing relations between generators and their differentials

$$\theta^A\, d\theta^B = d\theta^C\ \phi^B_C(\theta^A),$$

where $\phi^B_C(\theta^A)$ are unknown entries of a matrix of homomorphism. Then the consistency condition (2.78) takes on the form

$$\delta^A_E\theta^B\theta^C - j\ \phi^A_E(\theta^B\theta^C) + \phi^B_E(\theta^A)\theta^C - j\ \delta^B_E\theta^C\theta^A + \phi^C_E(\theta^A\theta^B) - j\ \phi^C_E(\theta^B)\theta^A = 0.$$

A structure of differential calculus suggests that a general form for solution of this equation should be $\phi^A_E(\theta^B) = \lambda^{AB}_{CD}\theta^D$, where λ^{AB}_{CD} are complex numbers. Substituting this into consistency condition we get the equation for unknowns λ^{AB}_{CD}

$$\delta^A_E\theta^B\theta^C - j\ \lambda^{DB}_{EL}\lambda^{AC}_{DK}\theta^L\theta^K + \lambda^{BA}_{ED}\theta^D\theta^C - j\ \delta^B_E\theta^C\theta^A + \lambda^{DA}_{EL}\lambda^{CB}_{DK}\theta^L\theta^K - j\ \lambda^{CB}_{ED}\theta^D\theta^A = 0.$$

If we could solve this equation, this would mean that there are quadratic relations between generators of 3-Grassmann algebra with cyclic relations, but it is worth to remind that in order to construct a 3-Grassmann algebra with cyclic relations we proceed from the assumption that there are no quadratic relations between generators θ^A. Thus in a case of differential calculus over 3-Grassmann algebra with cyclic relations we have to use an approach different from the one used in a case of algebra with quadratic relations.

The consistency condition (2.78) suggests that we have to assume ternary relations between generators and their differentials analogous to the cyclic relations of algebra. Indeed we can solve the consistency condition (2.78) by assuming the following relations between the generators and their differentials

$$(\theta^A\theta^B)\, d\theta^C = j^2\, d\theta^C\, (\theta^A\theta^B), \tag{2.79}$$

$$\theta^A\, d\theta^B\, \theta^C = j\, d\theta^B\, (\theta^C\theta^A). \tag{2.80}$$

We see that these relations are very similar to the cyclic relations of generators. Shortly for any ternary product of two generators and one differential of generator any cyclic permutation of the factors in this product is accompanied by appearance of the coefficient j. From this it immediately follows that any product of three generators and one differential of generator vanishes. This can be proved in the same way as in the case of generators. This is consistent with the structure of 3-Grassmann algebra with cyclic relations because the only way to get a product of three generators and one differential of generator by means of d is to apply a differential d to a product of four generators, but any such product is zero.

Taking all this into account we conclude that the vector space of bimodule $\mathfrak{M}$ is spanned by the products

$$\begin{aligned}&d\theta^A,\\ &d\theta^A\,\theta^B,\ \ \theta^A\, d\theta^B,\\ &d\theta^A\,\theta^B\,\theta^C,\, \theta^A\, d\theta^B\,\theta^C,\, \theta^A\theta^B\, d\theta^C,\end{aligned}$$

where there are no relations between binary products $d\theta^A\,\theta^B$, $\theta^A\, d\theta^B$, and ternary products obey cyclic type relations (2.79), (2.80), which allow to choose a basis for the vector space spanned by ternary products. It is important here that unlike a coordinate differential calculus over an algebra with quadratic relations, in the case of 3-Grassmann algebra with cyclic relations the right $\mathscr{G}$-module of bimodule $\mathfrak{M}$ is not freely generated by differentials of generators. Hence the approach, which is based on the assumption of freely generated right $\mathscr{G}$-module, is not applicable in the case of a 3-Grassmann algebra with cyclic commutation relations. But we can slightly modify it in order to have partial derivatives over a 3-Grassmann algebra with cyclic relations. Let f be an element of an algebra $\mathscr{G}$. We say that the right (left) partial derivatives of this element are defined if there exists an element $g \in \mathscr{G}$ such that

$$df \cdot g = d\theta^K \cdot R_K(f, g)\ \ \big(g \cdot df = R_K(f, g) \cdot d\theta^K\big), \tag{2.81}$$

where $R_K(f, g) \in \mathcal{G}$. Then a right (left) partial derivative of an element f is defined by

$$df = d\theta^K \, \frac{\overrightarrow{\partial}}{\partial\theta^K}(f) \quad \left(df = \frac{\overleftarrow{\partial}}{\partial\theta^K}(f)\, d\theta^K\right)$$

and it is given by the implicit equation

$$\frac{\overrightarrow{\partial}}{\partial\theta^K}(f) \cdot g = R_K(f, g) \quad \left(g \cdot \frac{\overleftarrow{\partial}}{\partial\theta^K}(f) = R_K(f, g)\right). \tag{2.82}$$

Particularly if g is invertible then one can solve the above equation by multiplying both sides by $g^{(-1)}$ and get a derivative in an explicit form. Now we apply this definition to the monomials (2.75)–(2.77) which form the basis for the vector space of $\mathcal{G}$. Obviously

$$\frac{\overrightarrow{\partial}}{\partial\theta^K}(\mathbb{1}) = 0, \quad \frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^A) = \delta^A_K,$$

and the same for left partial derivatives. Next we consider a binary product $\theta^A\theta^B$. According to a peculiar property of 3-Grassmann algebra with cyclic relations, there are no relations between generators and their differentials, thus one can not write the differential of a binary product of two generators $d(\theta^A\theta^B) = d\theta^A \cdot \theta^B + \theta^A \cdot d\theta^B$ in the form $d\theta^K \cdot g_K,\; g_K \in \mathcal{G}$. But we can apply the above definition for partial derivatives taking $g = \theta^C$, and, making use of cyclic relations (2.79), (2.80), we obtain

$$\begin{aligned} d(\theta^A\theta^B) \cdot \theta^C &= d\theta^A \cdot \theta^B\theta^C + \theta^A \cdot d\theta^B \cdot \theta^C = d\theta^A \cdot \theta^B\theta^C + j\; d\theta^B \cdot \theta^C\theta^A \\ &= d\theta^K \cdot (\delta^A_K\; \theta^B\theta^C + j\; \delta^B_K\; \theta^C\theta^A). \end{aligned}$$

From (2.81) it follows that $R_K(f, g) = \delta^A_K\; \theta^B\theta^C + j\; \delta^B_K\; \theta^C\theta^A$. Consequently the right partial derivatives of any binary product of generators $\theta^A\theta^B$ are defined, and they are given by the implicit equation

$$\frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^A\theta^B)\; \theta^C = \delta^A_K\; \theta^B\theta^C + j\; \delta^B_K\; \theta^C\theta^A. \tag{2.83}$$

Analogously we find the left partial derivatives for binary products of generators

$$\theta^C \, \frac{\overleftarrow{\partial}}{\partial\theta^K}(\theta^A\theta^B) = \delta^B_K\; \theta^C\theta^A + j^2\; \delta^A_K\; \theta^B\theta^C. \tag{2.84}$$

Thus

$$\theta^C \, \frac{\overleftarrow{\partial}}{\partial\theta^K}(\theta^A\theta^B) = j^2\; \frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^A\theta^B)\; \theta^C. \tag{2.85}$$

Next we consider ternary products of generators. In this case we can take $g \equiv \mathbb{1}$, and this means that we will obtain the partial derivatives (right and left) in an explicit form. Indeed

$$\begin{aligned} d(\theta^A\theta^B\theta^C) &= d\theta^A \cdot \theta^B\theta^C + \theta^A \cdot d\theta^B \cdot \theta^C + \theta^A\theta^B \cdot d\theta^C \\ &= d\theta^A \cdot \theta^B\theta^C + j\; d\theta^B \cdot \theta^C\theta^A + j^2\; d\theta^C \cdot \theta^A\theta^B \\ &= d\theta^K \cdot \big(\delta^A_K\theta^B\theta^C + j\;\delta^B_K\theta^C\theta^A + j^2\;\delta^C_K\theta^A\theta^B\big). \end{aligned} \tag{2.86}$$

Hence

$$\frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^A\theta^B\theta^C) = \delta^A_K\theta^B\theta^C + j\;\delta^B_K\theta^C\theta^A + j^2\;\delta^C_K\theta^A\theta^B, \tag{2.87}$$

and analogously

$$\frac{\overleftarrow{\partial}}{\partial\theta^K}(\theta^A\theta^B\theta^C) = \delta^C_K\theta^A\theta^B + j^2\;\delta^B_K\theta^C\theta^A + j\;\delta^A_K\theta^B\theta^C. \tag{2.88}$$

Similarly to (2.85) in this case we have

$$\frac{\overleftarrow{\partial}}{\partial\theta^K}(\theta^A\theta^B\theta^C) = j\;\frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^A\theta^B\theta^C). \tag{2.89}$$

As in the case of Grassmann algebra and generalized Grassmann algebra, we see that our calculus of partial derivatives is consistent with cyclic commutation relations of a 3-Grassmann algebra. Indeed (2.87) and (2.88) show that in order to compute a partial derivative of a triple product we put each generator of this product by means of cyclic relations to the first position and then replace it with the corresponding Kronecker symbol.

The relation (2.83) can be interpreted as an analog of j-twisted Leibniz rule for right partial derivatives of binary product of generators. Indeed we can write it as follows

$$\frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^A\theta^B)\,\theta^C = \frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^A)\theta^B\;\theta^C + j\;\theta^C\;\theta^A\frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^B), \tag{2.90}$$

We would like to point out an essential and active role of a generator θ^C in differentiation with respect to θ^K. Clearly θ^C can not be removed from this analog of j-twisted Leibniz rule because the second term in (2.90) is obtained not only by carrying a partial derivative (with respect to θ^K) through a generator θ^A (accompanied by appearance of j) but also by moving a generator θ^C from the last position to the first. In order to stress this peculiar property of j-twisted Leibniz rule (2.90) we write it in the form

$$\frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^A\theta^B)\,\theta^C = \frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^A)\theta^B\theta^C + j\;\theta^A\frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^B)\,\theta^C + j\;\delta^B_K\,[\theta^C,\theta^A], \tag{2.91}$$

where $[\theta^C, \theta^A] = \theta^C\,\theta^A - \theta^A\,\theta^C$. The formula (2.91) clearly shows that an origin of the peculiar form of Leibniz rule (which is not only j-twisted, but also involves in non-trivial way in the orbit of its action a generator θ^C) lies in our assumption about absence of relations between binary products of generators.

Now making use of (2.83) we can write the formula for right partial derivative of triple product (2.87) in a form of j^2-twisted Leibniz rule

$$\frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^A\theta^B\theta^C) = \frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^A\theta^B)\;\theta^C + j^2\;\theta^A\theta^B\frac{\overrightarrow{\partial}}{\partial\theta^K}(\theta^C).$$

It is worth to mention that there is no analog of Leibniz rule for right partial derivative of a triple product $\theta^A\theta^B\theta^C$ if we split it into two parts as θ^A and $\theta^B\theta^C$. In this case we have to use left partial derivatives. Indeed (2.84) and (2.88) give

$$\frac{\overleftarrow{\partial}}{\partial\theta^K}(\theta^A\theta^B\theta^C) = \theta^A\;\frac{\overleftarrow{\partial}}{\partial\theta^K}(\theta^B\theta^C) + j\;\frac{\overleftarrow{\partial}}{\partial\theta^K}(\theta^A)\theta^B\theta^C.$$

2.7 Invariance Groups of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ Skew-Symmetric Algebras

Before discussing the merger of a $\mathbb{Z}_2$-graded algebra with a $\mathbb{Z}_3$-graded algebra, let us explore the invariance properties of each type separately. First, let us ask what kind of linear transformations preserves the ordinary Grassmann algebra spanned by anti-commuting generators ξ^α $(\alpha = 1, 2, \ldots n)$. Any linearly independent n combinations of anti-commuting variables ξ^α will span another anti-commuting basis: indeed, if $\eta^{\alpha'} = S^{\alpha'}_\beta\,\xi^\beta$, and take on purely numerical values, i.e. do commute with all other generators, then we can write

$$\eta^{\alpha'}\eta^{\delta'} = S^{\alpha'}_\alpha\,\xi^\alpha\;S^{\delta'}_\delta\,\xi^\delta = S^{\alpha'}_\alpha\;S^{\delta'}_\delta\;\xi^\alpha\xi^\delta = -S^{\alpha'}_\alpha\;S^{\delta'}_\delta\;\xi^\delta\xi^\alpha = -S^{\delta'}_\delta\;\xi^\delta S^{\alpha'}_\alpha\;\xi^\alpha = -\eta^{\delta'}\eta^{\alpha'} \tag{2.92}$$

Let us consider the simplest case of a $\mathbb{Z}_2$-graded algebra spanned by two generators ξ^α, $\alpha, \beta = 1, 2$. The anti-commutation property can be encoded in the invariant 2-form $\varepsilon_{\alpha\beta}$. We can obviously write

$$\varepsilon_{\alpha\beta}\xi^\alpha\xi^\beta = \varepsilon_{\beta\alpha}\xi^\beta\xi^\alpha = -\varepsilon_{\beta\alpha}\xi^\alpha\xi^\beta,$$

from which we conclude that $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$. We can choose the basis in which

$$\varepsilon_{11} = 0, \quad \varepsilon_{22} = 0, \quad \varepsilon_{12} = -\varepsilon_{21} = 1.$$

After a change of basis, $\xi^\beta \to S^{\alpha'}_\beta\,\xi^\beta = \eta^{\alpha'}$ the 2-form $\varepsilon_{\alpha\beta}$, as any tensor, also undergoes the inverse transformation:

$$S^\alpha_{\alpha'}S^\beta_{\beta'}\,\varepsilon_{\alpha\beta},$$

with $S^{\alpha}_{\beta'}$ the inverse matrix of the matrix $S^{\beta'}_{\beta}$. Whatever non-singular linear transformation $S^{\beta}_{\beta'}$ is chosen, the new components $\varepsilon_{\alpha'\beta'}$ remain anti-symmetric, but they have not necessarily the same values as those of $\varepsilon_{\alpha\beta}$ However, if we require that also in new basis

$$\varepsilon_{1'1'} = 0, \quad \varepsilon_{2'2'} = 0, \quad \varepsilon_{1'2'} = -\varepsilon_{2'1'} = 1,$$

then it is easy to show that this imposes extra condition on the 2×2 matrix $S^{\alpha'}_{\beta}$, namely that $\det S = 1$. This defines the $SL(2, \mathbf{C})$ group as the group of invariance of the subalgebra spanned by two anti-commuting generators $\xi^{\alpha}, \alpha, \beta, \ldots = 1, 2$.

Now let us turn to the invariance properties of the ternary subalgebra spanned by two generators $\theta^1, \ \theta^2$, satisfying homogeneous cubic j-anticommutation relations $\theta^A\theta^B\theta^C = j \ \theta^B\theta^C\theta^A$, and their conjugate counterparts $\bar{\theta}^{\dot{1}}, \ \bar{\theta}^{\dot{2}}$ satisfying homogeneous cubic j^2 anti-commutation relations $\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}\bar{\theta}^{\dot{C}} = j^2 \ \bar{\theta}^{\dot{B}}\bar{\theta}^{\dot{C}}\bar{\theta}^{\dot{A}}$.

We shall also impose *binary* constitutive relations between the generators θ^A and their conjugate counterparts $\bar{\theta}^{\dot{B}}$, making the choice consistent with the introduced $\mathbb{Z}_6$-grading

$$\theta^A\bar{\theta}^{\dot{B}} = -j \ \bar{\theta}^{\dot{B}}\theta^A, \quad \bar{\theta}^{\dot{B}}\theta^A = -j^2 \ \theta^A\bar{\theta}^{\dot{B}}.$$

Consider a tri-linear form ρ^{α}_{ABC}. We shall call this form $\mathbb{Z}_3$-invariant if we can write:

$$\rho^{\alpha}_{ABC}\, \theta^A\theta^B\theta^C = \frac{1}{3}\Big[\rho^{\alpha}_{ABC}\, \theta^A\theta^B\theta^C + \rho^{\alpha}_{BCA}\, \theta^B\theta^C\theta^A + \rho^{\alpha}_{CAB}\, \theta^C\theta^A\theta^B\Big] =$$

$$= \frac{1}{3}\Big[\rho^{\alpha}_{ABC}\, \theta^A\theta^B\theta^C + \rho^{\alpha}_{BCA}\, (j^2\, \theta^A\theta^B\theta^C) + \rho^{\alpha}_{CAB}\, j\, (\theta^A\theta^B\theta^C)\Big], \tag{2.93}$$

by virtue of the commutation relations (2.19).

From this it follows that we should have

$$\rho^{\alpha}_{ABC}\ \theta^A\theta^B\theta^C = \frac{1}{3}\Big[\rho^{\alpha}_{ABC} + j^2\, \rho^{\alpha}_{BCA} + j\, \rho^{\alpha}_{CAB}\Big]\theta^A\theta^B\theta^C, \tag{2.94}$$

from which we get the following properties of the ρ-cubic matrices:

$$\rho^{\alpha}_{ABC} = j^2\, \rho^{\alpha}_{BCA} = j\, \rho^{\alpha}_{CAB}. \tag{2.95}$$

Even in this minimal and discrete case, there are covariant and contravariant indices: the lower and the upper indices display the inverse transformation property. If a given cyclic permutation is represented by a multiplication by j for the upper indices, the same permutation performed on the lower indices is represented by multiplication by the inverse, i.e. j^2, so that they compensate each other.

Similar reasoning leads to the definition of the conjugate forms $\bar{\rho}^{\dot{\alpha}}_{\dot{C}\dot{B}\dot{A}}$ satisfying the relations similar to (2.95) with j replaced be its conjugate, j^2:

$$\bar{\rho}^{\dot{\alpha}}_{\dot{A}\dot{B}\dot{C}} = j\,\bar{\rho}^{\dot{\alpha}}_{\dot{B}\dot{C}\dot{A}} = j^2\,\bar{\rho}^{\dot{\alpha}}_{\dot{C}\dot{A}\dot{B}} \tag{2.96}$$

In the simplest case of two generators, the j-skew-invariant forms have only two independent components:

$$\rho^1_{121} = j\,\rho^1_{211} = j^2\,\rho^1_{112},$$

$$\rho^2_{212} = j\,\rho^2_{122} = j^2\,\rho^2_{221},$$

and we can set

$$\rho^1_{121} = 1,\ \ \rho^1_{211} = j^2,\ \ \rho^1_{112} = j,$$

$$\rho^2_{212} = 1,\ \ \rho^2_{122} = j^2,\ \ \rho^2_{221} = j.$$

The constitutive cubic relations between the generators of the $\mathbb{Z}_3$ graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.

Let $U^{A'}_A$ denote a non-singular $N \times N$ matrix, transforming the generators θ^A into another set of generators, $\theta^{B'} = U^{B'}_B\,\theta^B$. In principle, the generators of the $\mathbb{Z}_2$-graded subalgebra ξ^α may or may not undergo a change of basis. Uniting the two subalgebras in one $\mathbb{Z}_6$-graded algebra suggests that a change of basis should concern all generators at once, both ξ^α and θ^A, This means that under the simultaneous change of basis,

$$\xi^\alpha \to \tilde{\xi}^{\beta'} = S^{\beta'}_\alpha\,\xi^\alpha, \quad\ \theta^A \to \tilde{\theta}^{B'} = U^{B'}_A\,\theta^A, \tag{2.97}$$

It seems natural to identify the upper indices α, β appearing in the ρ-tensors with the indices appearing in the generators ξ^α of the $\mathbb{Z}_2$-graded subalgebra. Therefore, we are looking for the solution of the simultaneous invariance condition for the $\varepsilon_{\alpha\beta}$ and ρ^α_{ABC} tensors:

$$\varepsilon_{\alpha'\beta'} = S^\alpha_{\alpha'} S^\beta_{\beta'}\,\varepsilon_{\alpha\beta}, \qquad S^{\alpha'}_\beta\,\rho^\beta_{ABC} = U^{A'}_A\,U^{B'}_B\,U^{C'}_C\,\rho^{\alpha'}_{A'B'C'}, \tag{2.98}$$

so that in new basis the numerical values of both tensors remain the same as before, just like the components of the Minkowskian space-time metric tensor $g_{\mu\nu}$ remain unchanged under the Lorentz transformations. Notice that in the last formula above, (2.98), the matrix $S^{\alpha'}_\alpha$ is the inverse matrix for $S^\alpha_{\alpha'}$ appearing in the transformation of the basis ξ^β.

Now, $\rho^1_{121} = 1$, and we have two equations corresponding to the choice of values of the index α' equal to 1 or 2. For $\alpha' = 1'$ the ρ-matrix on the right-hand side is $\rho^{1'}_{A'B'C'}$, which has only three components,

$$\rho^{1'}_{1'2'1'} = 1, \quad \rho^{1'}_{2'1'1'} = j^2, \quad \rho^{1'}_{1'1'2'} = j,$$

which leads to the following equation:

$$S_1^{1'} = U_1^{1'} U_2^{2'} U_1^{1'} + j^2 U_1^{2'} U_2^{1'} U_1^{1'} + j U_1^{1'} U_2^{1'} U_1^{2'} = U_1^{1'} (U_2^{2'} U_1^{1'} - U_1^{2'} U_2^{1'}), \qquad (2.99)$$

because $j^2 + j = -1$. For the alternative choice $\alpha' = 2'$ the ρ-matrix on the right-hand side is $\rho^{2'}_{A'B'C'}$, whose three non-vanishing components are

$$\rho^{2'}_{2'1'2'} = 1, \quad \rho^{2'}_{1'2'2'} = j^2, \quad \rho^{2'}_{2'2'1'} = j.$$

The corresponding equation becomes now:

$$S_1^{2'} = U_1^{2'} U_2^{1'} U_1^{2'} + j^2 U_1^{1'} U_2^{2'} U_1^{2'} + j U_1^{2'} U_2^{2'} U_1^{1'} = U_1^{2'} (U_2^{1'} U_1^{2'} - U_1^{1'} U_2^{2'}), \qquad (2.100)$$

The two remaining equations are obtained in a similar manner. We choose now the three lower indices on the left-hand side equal to another independent combination, (212). Then the ρ-matrix on the left hand side must be ρ^2 whose component ρ^2_{212} is equal to 1. This leads to the following equation when $\alpha' = 1'$:

$$S_2^{1'} = U_2^{1'} U_1^{2'} U_2^{1'} + j^2 U_2^{2'} U_1^{1'} U_2^{1'} + j U_2^{1'} U_1^{1'} U_2^{2'} = U_2^{1'} (U_2^{1'} U_1^{2'} - U_1^{1'} U_2^{2'}), \qquad (2.101)$$

and the fourth equation corresponding to $\alpha' = 2'$ is:

$$S_2^{2'} = U_2^{2'} U_1^{1'} U_2^{2'} + j^2 U_2^{1'} U_1^{2'} U_2^{2'} + j U_2^{2'} U_1^{2'} U_2^{1'} = U_2^{2'} (U_1^{1'} U_2^{2'} - U_1^{2'} U_2^{1'}). \qquad (2.102)$$

The determinant of the 2×2 complex matrix $U_B^{A'}$ appears everywhere on the right-hand side

$$S_1^{2'} = -U_1^{2'} [\det(U)]. \qquad (2.103)$$

The remaining two equations are obtained in a similar manner, resulting in the following:

$$S_2^{1'} = -U_2^{1'} [\det(U)], \qquad S_2^{2'} = U_2^{2'} [\det(U)]. \qquad (2.104)$$

The determinant of the 2×2 complex matrix $U_B^{A'}$ appears everywhere on the right-hand side. Taking the determinant of the matrix $S_\beta^{\alpha'}$ one gets immediately

$$\det(S) = [\det(U)]^3. \qquad (2.105)$$

However, the U-matrices on the right-hand side are defined only up to the phase, which due to the cubic character of the covariance relations (2.99)–(2.104), and they can take on three different values: 1, j or j^2, i.e. the matrices $j\, U_B^{A'}$ or $j^2\, U_B^{A'}$ satisfy the same relations as the matrices $U_B^{A'}$ defined above. The determinant of U can take on the values 1, j or j^2 if $det(S) = 1$.

Another reason to impose the unitarity condition is as follows. It can be derived if we require the same behavior for the duals, ρ_β^{DEF}. This extra condition amounts to the invariance of the anti-symmetric tensor ε^{AB}, and this is possible only if the determinant of U-matrices is 1 (or j or j^2), because only cubic combinations of these matrices appear in the transformation law for ρ-forms.

We have determined the invariance group for the simultaneous change of the basis in our $\mathbb{Z}_6$-graded algebra. However, these transformations based on the $SL(2, \mathbf{C})$ groups combined with complex representation of the $\mathbb{Z}_3$ cyclic group keep invariant the binary constitutive relations between the $\mathbb{Z}_2$-graded generators ξ^α and the ternary constitutive relations between the $\mathbb{Z}_3$-graded generators alone, without mentioning their conjugates $\bar{\xi}^{\dot{\alpha}}$ and $\bar{\theta}^{\dot{B}}$.

Let us put aside for the moment the conjugate $\mathbb{Z}_2$-graded variables, and concentrate our attention on the conjugate $\mathbb{Z}_3$-graded generators $\bar{\theta}^{\dot{A}}$ and their commutation relations with θ^B generators, which we shall modify as:

$$\theta^A \, \bar{\theta}^{\dot{B}} = -j \, \bar{\theta}^{\dot{B}} \, \theta^A, \quad \bar{\theta}^{\dot{B}} \, \theta^A = -j^2 \, \theta^A \, \bar{\theta}^{\dot{B}}. \tag{2.106}$$

A similar covariance requirement can be formulated with respect to the set of 2-forms mapping the quadratic $\theta^A \, \bar{\theta}^{\dot{B}}$ combinations into a four-dimensional linear real space.

It is easy to see, by counting the independent combinations of dotted and undotted indices, that the symmetry (2.106) imposed on these expressions reduces their number to four: $(1\dot{1}),\ (1, \dot{2}),\ (2\ \dot{1}),\ (2, \dot{2})$, the conjugate combinations of the type $(\dot{A}\ B)$ being dependent on the first four because of the imposed symmetry properties.

Let us define two quadratic forms, $\pi^\mu_{A\dot{B}}$ and its conjugate $\bar{\pi}^\mu_{\dot{B}A}$

$$\pi^\mu_{A\dot{B}} \, \theta^A \bar{\theta}^{\dot{B}} \quad \text{and} \quad \bar{\pi}^\mu_{\dot{B}A} \, \bar{\theta}^{\dot{B}} \theta^A. \tag{2.107}$$

The Greek indices $\mu, \nu \ldots$ take on four values, and we shall label them 0, 1, 2, 3.

The four tensors $\pi^\mu_{A\dot{B}}$ and their hermitian conjugates $\bar{\pi}^\mu_{\dot{B}A}$ define a bi-linear mapping from the product of quark and anti-quark cubic algebras into a linear four-dimensional vector space, whose structure is not yet defined.

Let us impose the following invariance condition:

$$\pi^\mu_{A\dot{B}} \, \theta^A \bar{\theta}^{\dot{B}} = \bar{\pi}^\mu_{\dot{B}A} \bar{\theta}^{\dot{B}} \theta^A. \tag{2.108}$$

It follows immediately from (2.106) that

$$\pi^\mu_{A\dot{B}} = -j^2 \, \bar{\pi}^\mu_{\dot{B}A}. \tag{2.109}$$

Such matrices are non-hermitian, and they can be realized by the following substitution:

$$\pi^\mu_{A\dot{B}} = j^2 \, i \, \sigma^\mu_{A\dot{B}}, \quad \bar{\pi}^\mu_{\dot{B}A} = -j \, i \, \sigma^\mu_{\dot{B}A} \tag{2.110}$$

where $\sigma^\mu_{A\dot{B}}$ are the unit 2 matrix for $\mu = 0$, and the three hermitian Pauli matrices for $\mu = 1, 2, 3$.

Again, we want to get the same form of these four matrices in another basis. Knowing that the lower indices A and $\dot{B}$ undergo the transformation with matrices $U^{A'}_B$ and $\bar{U}^{\dot{A}'}_{\dot{B}}$, we demand that there exist some 4×4 matrices $\Lambda^{\mu'}_\nu$ representing the transformation of lower indices by the matrices U and $\bar{U}$:

$$\Lambda^{\mu'}_{\nu}\,\pi^{\nu}_{\ A\dot{B}} = U^{A'}_{A}\,\bar{U}^{\dot{B}'}_{\dot{B}}\,\pi^{\mu'}_{\ A'\dot{B}'}, \tag{2.111}$$

It is clear that we can replace the matrices $\pi^{\nu}_{\ A\dot{B}}$ by the corresponding matrices $\sigma^{\nu}_{\ A\dot{B}}$, and this defines the vector (4×4) representation of the Lorentz group.

The first four equations relating the 4×4 real matrices $\Lambda^{\mu'}_{\nu}$ with the 2×2 complex matrices $U^{A'}_{B}$ and $\bar{U}^{\dot{A}'}_{\dot{B}}$ are as follows:

$$\Lambda^{0'}_{0} - \Lambda^{0'}_{3} = U^{1'}_{2}\,\bar{U}^{\dot{1}'}_{\dot{2}} + U^{2'}_{2}\,\bar{U}^{\dot{2}'}_{\dot{2}}$$

$$\Lambda^{0'}_{0} + \Lambda^{0'}_{3} = U^{1'}_{1}\,\bar{U}^{\dot{1}'}_{\dot{1}} + U^{2'}_{1}\,\bar{U}^{\dot{2}'}_{\dot{1}}$$

$$\Lambda^{0'}_{0} - i\Lambda^{0'}_{2} = U^{1'}_{1}\,\bar{U}^{\dot{1}'}_{\dot{2}} + U^{2'}_{1}\,\bar{U}^{\dot{2}'}_{\dot{2}}$$

$$\Lambda^{0'}_{0} + i\Lambda^{0'}_{2} = U^{1'}_{2}\,\bar{U}^{\dot{1}'}_{\dot{1}} + U^{2'}_{2}\,\bar{U}^{\dot{2}'}_{\dot{1}}$$

The next four equations relating the 4×4 real matrices $\Lambda^{\mu'}_{\nu}$ with the 2×2 complex matrices $U^{A'}_{B}$ and $\bar{U}^{\dot{A}'}_{\dot{B}}$ are as follows:

$$\Lambda^{1'}_{0} - \Lambda^{1'}_{3} = U^{1'}_{2}\,\bar{U}^{\dot{2}'}_{\dot{2}} + U^{2'}_{2}\,\bar{U}^{\dot{1}'}_{\dot{2}}$$

$$\Lambda^{1'}_{0} + \Lambda^{1'}_{3} = U^{1'}_{1}\,\bar{U}^{\dot{2}'}_{\dot{1}} + U^{2'}_{1}\,\bar{U}^{\dot{1}'}_{\dot{1}}$$

$$\Lambda^{1'}_{1} - i\Lambda^{1'}_{2} = U^{1'}_{1}\,\bar{U}^{\dot{2}'}_{\dot{2}} + U^{2'}_{1}\,\bar{U}^{\dot{1}'}_{\dot{2}}$$

$$\Lambda^{1'}_{1} + i\Lambda^{1'}_{2} = U^{1'}_{2}\,\bar{U}^{\dot{2}'}_{\dot{1}} + U^{2'}_{2}\,\bar{U}^{\dot{1}'}_{\dot{1}}$$

We skip the next two groups of four equations corresponding to the "spatial" indices 2 and 3, reproducing the same scheme as the last four equations with the space index equal to 1.

It can be checked that now $\det\,(\Lambda) = [\det U]^2\,\left[\det \bar{U}\right]^2$.

The group of transformations thus defined is $SL(2, \mathbf{C})$, which is the covering group of the Lorentz group.

With the invariant "spinorial metric" in two complex dimensions, ε^{AB} and $\varepsilon^{\dot{A}\dot{B}}$ such that $\varepsilon^{12} = -\varepsilon^{21} = 1$ and $\varepsilon^{\dot{1}\dot{2}} = -\varepsilon^{\dot{2}\dot{1}}$, we can define the contravariant components $\pi^{\nu\,A\dot{B}}$. It is easy to show that the Minkowskian space-time metric, invariant under the Lorentz transformations, can be defined as

$$g^{\mu\nu} = \frac{1}{2}\left[\pi^{\mu}_{\ A\dot{B}}\,\pi^{\nu\,A\dot{B}}\right] = diag(+, -, -, -) \tag{2.112}$$

Together with the anti-commuting spinors ψ^α the four real coefficients defining a Lorentz vector, $x_\mu\, \pi^\mu_{A\dot{B}}$, can generate now the supersymmetry via standard definitions of super-derivations. Let us then choose the matrices $\Lambda^{\alpha'}_\beta$ to be the usual spinor representation of the $SL(2, \mathbf{C})$ group, while the matrices $U^{A'}_B$ will be defined as follows:

$$U^{1'}_1 = j\Lambda^{1'}_1, \quad U^{1'}_2 = -j\Lambda^{1'}_2, \quad U^{2'}_1 = -j\Lambda^{2'}_1, \quad U^{2'}_2 = j\Lambda^{2'}_2, \tag{2.113}$$

the determinant of U being equal to j^2. Obviously, the same reasoning leads to the conjugate cubic representation of $SL(2, \mathbf{C})$ if we require the covariance of the conjugate tensor

$$\bar{\rho}^{\dot{\beta}}_{\dot{D}\dot{E}\dot{F}} = j\,\bar{\rho}^{\dot{\beta}}_{\dot{E}\dot{F}\dot{D}} = j^2\,\bar{\rho}^{\dot{\beta}}_{\dot{F}\dot{D}\dot{E}},$$

by imposing the equation similar to (2.98)

$$\Lambda^{\dot{\alpha}'}_{\dot{\beta}}\,\bar{\rho}^{\dot{\beta}}_{\dot{A}\dot{B}\dot{C}} = \bar{\rho}^{\dot{\alpha}'}_{\dot{A}'\dot{B}'\dot{C}'}\bar{U}^{\dot{A}'}_{\dot{A}}\,\bar{U}^{\dot{B}'}_{\dot{B}}\,\bar{U}^{\dot{C}'}_{\dot{C}}. \tag{2.114}$$

The matrix $\bar{U}$ is the complex conjugate of the matrix U, with determinant equal to j.

In conclusion, we have found the way to derive the covering group of the Lorentz group acting on spinors via the usual spinorial representation, and on vectors via the 4×4 real matrices. Spinors are obtained as a homomorphic image of tri-linear combinations of three $\mathbb{Z}_3$-graded generators θ^A (or their conjugates $\bar{\theta}^{\dot{B}}$). The $\mathbb{Z}_3$-graded generators transform with matrices U (or $\bar{U}$ for the conjugates), but these matrices are not unitary: their determinants are equal to j^2 or j, respectively.

In our forthcoming paper we shall investigate similar $\mathbb{Z}_6$-graded generalization extended to the differential forms $d\xi^\alpha$ and $d\theta^B$.

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