# Chapter 17
# The Jacobian Conjecture$_{2n}$ Implies the Dixmier Problem$_n$

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**Abstract** The aim of the paper is to describe some ideas, approaches, comments, etc. regarding the Dixmier Conjecture, its generalizations and analogues.

**Keywords** Dixmier Conjecture · Jacobian Conjecture · Weyl algebra

**MSC 2010 Classification** 14R15 · 14H37 · 16S32

## 17.1 The Jacobian Conjecture$_{2n}$ implies the Dixmier Problem$_n$

Using the *inversion formula* for automorphisms of the Weyl algebras with polynomial coefficients and the *bound* on its degree [4] (see also Sect. 17.2) a short algebraic proof is given of the result of Tsuchimoto [28], A. Belov-Kanel and M. Kontsevich [15] that $JC_{2n}$ *implies* $DP_n$. The first part of Sect. 17.1 (i.e. before the short historical comment) is identical to [3]. The second part of Sect. 17.1 is about recent progress on the Dixmier Conjecture.

The *Weyl algebra* $A_n = A_n(\mathbb{Z})$ is a $\mathbb{Z}$-algebra generated by $2n$ generators $x_1, \ldots, x_{2n}$ subject to the defining relations:

$$[x_{n+i}, x_j] = \delta_{ij}, \quad [x_i, x_j] = [x_{n+i}, x_{n+j}] = 0 \ \text{ for all } \ i, j = 1, \ldots, n,$$

where $\delta_{ij}$ is the Kronecker delta, $[a, b] := ab - ba = (\mathrm{ad}\, a)(b)$. For a ring $R$, $A_n(R) := R \otimes_{\mathbb{Z}} A_n$ is the Weyl algebra over $R$.

- **The Jacobian Conjecture** ($JC_n$): Given $\sigma \in \mathrm{End}_{\mathbb{C}-alg}(\mathbb{C}[x_1, \ldots, x_n])$ such that $\det(\frac{\partial \sigma(x_i)}{\partial x_j}) \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$ then $\sigma \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, x_n])$.

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- **The Dixmier Problem** $(DP_n)$, [18]: Is a $\mathbb{C}$-algebra endomorphism of the Weyl algebra $A_n(\mathbb{C})$ an algebra automorphism?

**Theorem 17.1.1** ([4] (The Inversion Formula)) *Let $K$ be a field of characteristic zero. Then for each $\sigma \in \mathrm{Aut}_K(A_n(K))$ and $a \in A_n(K)$,*

$$\sigma^{-1}(a) = \sum_{\alpha \in \mathbb{N}^{2n}} \phi_\sigma\left(\frac{(\partial')^\alpha}{\alpha!}a\right)x^\alpha,$$

*where $x^\alpha := (x_1')^{\alpha_1}\cdots(x_{2n}')^{\alpha_{2n}}$, $(\partial')^\alpha := (\partial_1')^{\alpha_1}\cdots(\partial_{2n}')^{\alpha_{2n}}$, $\partial_i' := \mathrm{ad}(\sigma(x_{n+i}))$ and $\partial_{n+i}' := -\mathrm{ad}(\sigma(x_i))$ for $i = 1, \ldots, n$, $\phi_\sigma := \phi_{2n}\phi_{2n-1}\cdots\phi_1$ where*

$$\phi_i := \sum_{k \geq 0}(-1)^i \frac{(\sigma(x_i))^k}{k!}(\partial_i')^k.$$

**Remark 17.1.2** This result was proved when $K$ is a field of characteristic zero, but by the *Lefschetz principle* it also holds for any commutative reduced $\mathbb{Q}$-algebra.

**Theorem 17.1.3** ([4]) *Given $\sigma \in \mathrm{Aut}_K(A_n(K[x_{2n+1}, \ldots, x_{2n+m}]))$ where $K$ is a commutative reduced $\mathbb{Q}$-algebra. Then the degree*

$$\deg\, \sigma^{-1} \leq (\deg\, \sigma)^{2n+m-1}$$

*where $\deg(\sigma) := \max\{\deg(\sigma(x_i)) \mid i = 1, \ldots, 2n+m\}$ and $\deg$ means the total degree with respect to the canonical generators $x_i$.*

**Theorem 17.1.4** ([15, 28]) $JC_{2n} \Rightarrow DP_n$.

***Proof*** Let $\sigma \in \mathrm{End}_{\mathbb{C}-alg}(A_n(\mathbb{C}))$.

*Step 1.* Let $R$ be a finitely generated (over $\mathbb{Z}$) $\mathbb{Z}$-subalgebra of $\mathbb{C}$ generated by the coefficients of the elements $x_i' := \sigma(x_i)$, $i = 1, \ldots, 2n$. Localizing at finitely many primes $q \in \mathbb{Z}$ one can assume that the ring $R_p := R/(p)$ is a domain for all primes $p \gg 0$. Then $\sigma \in \mathrm{End}_{R-alg}(A_n(R))$, $x_i' \in A_n(R) = R \otimes_{\mathbb{Z}} A_n$, and the centre $Z(A_n(R)) = R$.

*Step 2.* From this moment on $p \in \mathbb{Z}$ is any (all) sufficiently large prime number and $\mathbb{Z}_p := \mathbb{Z}/(p)$.

$$\begin{aligned} A(p) &:= A_n(R)/(p) \simeq R_p \otimes_{\mathbb{Z}_p} A_n(\mathbb{Z}_p) \simeq R_p \otimes_{\mathbb{Z}_p} M_{p^n}(\mathbb{Z}_p[x_1^p, \ldots, x_{2n}^p]) \\ &\simeq M_{p^n}(R_p[x_1^p, \ldots, x_{2n}^p]) = M_{p^n}(C_p) \end{aligned}$$

where $x_i^p$ stands for $x_i^p + (p)$, and $M_{p^n}(C_p)$ is a matrix algebra (of size $p^n$) with coefficients from a polynomial algebra $C_p := R_p[x_1^p, \ldots, x_{2n}^p]$ over $R_p$. The $\sigma$ induces an $R_p$-algebra endomorphism $\sigma_p : A(p) \to A(p)$, $a + (p) \mapsto \sigma(a) + (p)$.

*Step 3.* It follows from the inversion formula (Theorem 17.1.1) and Theorem 17.1.3 that

$$\sigma \in \mathrm{Aut}_R(A_n(R)) \quad \Leftrightarrow \quad \sigma_p \in \mathrm{Aut}_{R_p}(A(p)) \;\text{ for all }\; p \gg 0.$$

*Step 4.* $\sigma_p(C_p) \subseteq C_p$ (see [27]).

*Step 5.* Since $A(p) \simeq M_{p^n}(C_p)$, $Z(A(p)) = C_p$, and $\sigma_p(C_p) \subseteq C_p$, it is obvious that

$$\sigma_p \in \mathrm{Aut}_{R_p}(A(p)) \quad \Leftrightarrow \quad \sigma_p|_{C_p} \in \mathrm{Aut}_{R_p}(C_p).$$

*Step 6. Claim*: $\sigma_p(C_p) \subseteq C_p$ and $JC_{2n}$ imply $\sigma_p|_{C_p} \in \mathrm{Aut}_{R_p}(C_p)$.

*Proof of the Claim.* $(i)$ $(C_p, \{\cdot,\cdot\})$ is a *Poisson algebra* where

$$\{a + (p), b + (p)\} := \frac{[a, b]}{p} \pmod{p}$$

is the *canonical Poisson bracket* on a polynomial algebra in $2n$ variables (a direct computation, see [15, Lemma 4]) which is obviously $\sigma_p$-*invariant*.

$(ii)$

$$\begin{aligned}\{\pm 1\} \ni \sigma_p(\det(\{x_i^p, x_j^p\})_{1\le i,j\le n}) &= \det(\sigma_p(\{x_i^p, x_j^p\}))\\ &= \det(\{\sigma_p(x_i^p), \sigma_p(x_j^p)\}) = \det(J^T(\{x_i^p, x_j^p\})J)\\ &= \det(J)^2\det(\{x_i^p, x_j^p\})) = \det(J)^2 \cdot (\pm 1).\end{aligned}$$

where $J := (\frac{\partial\sigma(x_i^p)}{\partial(x_j^p)})_{1\le i,j\le n}$. Hence, $\det(J) \in \{\pm 1\}$. Only *now* we use the assumption that $JC_{2n}$ holds: which implies $\sigma_p|_{C_p} \in \mathrm{Aut}_{R_p}(C_p)$. □

**Historical comment.** In Spring 2000, I mentioned the conjecture $JC_{2n} \Rightarrow DP_n$ in my talk 'A question of Rentschler and the Dixmier problem' at the Jussieu Mathematics Institute (Paris) based on [2]. We also discussed it during the lunch (K. Adjamagbo and R. Rentschler). At the time (before 2000), I had an incomplete proof of the conjecture $JC_{2n} \Rightarrow DP_n$, [13], based on completely different ideas (a gap was found in the proof).

**The Problem-Conjecture of Dixmier: recent progress.** In 1982, it was proved that a positive answer to the Problem-Conjecture of Dixmier for the Weyl algebra $A_n$ implies the Jacobian Conjecture for the polynomial algebra $P_n$ in $n$ variables, see Bass, Connel and Wright [1]. In 2005, it was proved independently by Tsuchimoto [28] and Belov-Kanel and Kontsevich [15], see also [3] for a short proof, that these two problems are equivalent. The Problem-Conjecture of Dixmier can be formulated as a question of whether certain modules $\mathcal{M}$ over the Weyl algebras are *simple* [2] (recall that due to *Inequality of Bernstein* [16] each simple module over the Weyl algebra $A_n$ has the Gelfand–Kirillov dimension which is one of the natural numbers $n, n+1, \dots, 2n-1$; Bernstein and Lunts [17, 22] showed that 'generically' a simple $A_n$-module has the Gelfand–Kirillov dimension $2n-1$). It is not obvious from the outset that the modules $\mathcal{M}$ are even finitely generated. In 2001, giving a positive answer to the Question of Rentschler about the Weyl algebra it was proved that the modules $\mathcal{M}$ are finitely generated and have the *smallest possible* Gelfand–Kirillov dimension, i.e. $n$ (i.e. they are *holonomic*) and as the result they have *finite length*, [2]. This means that the next step, as far as the Jacobian Conjecture and the Dixmier Conjecture are concerned, is either to prove the conjectures or to give a counter-example.

**The set $\mathcal{M}$ and holonomic modules**, [2]. A finitely generated $A_n$-module $M$ is called *holonomic* if $\mathrm{GK}\,(M) = n$ where GK is the *Gelfand–Kirillov dimension*. Holonomic modules have many nice properties, one of them is that every holonomic module has finite length.

**Theorem 17.1.5** ([2, Theorem 1.3]) *Let $\varphi : A_n \to A_n$ be an endomorphism of the n'th Weyl algebra $A_n$ and let $M$ be a holonomic $A_n$-module. Then the $A_n$-module ${}^{\varphi}M$ is also holonomic, hence it has finite length.*

By the algebra isomorphism

$$A_n \to A_n^{\mathrm{op}}, \quad x_i \to x_i, \quad \partial_i \to -\partial_i, \quad i = 1, \dots, n,$$

we can identify the Weyl algebra $A_n$ with its *opposite algebra* $A_n^{\mathrm{op}}$. Recall that $A_n^{\mathrm{op}} = A_n$, as vector spaces, but the multiplication in $A_n^{\mathrm{op}}$ is given by the rule $a \cdot b = ba$ for all elements $a, b \in A_n$. An $A_n$-bimodule is a left module over the *enveloping algebra* $A_n^{\mathrm{e}} := A_n \otimes A_n^{\mathrm{op}}$ and, in fact, is a left $A_{2n}$-module since $A_n^{\mathrm{e}} \simeq A_{2n}$ (by the isomorphism above). The Weyl algebra $A_n$ is a *simple holonomic $A_n$-bimodule* since

$$\mathrm{GK}_{A_n^{\mathrm{e}}}(A_n) = \mathrm{GK}\,(A_n) = 2n = \frac{4n}{2} = \frac{\mathrm{GK}\,(A_{2n})}{2} = \frac{\mathrm{GK}\,(A_n^{\mathrm{e}})}{2}.$$

Let $\varphi : A_n \to A_n$ be an endomorphism of the $n$'th Weyl algebra $A_n$. By restriction of scalars, we have the twisted $A_n$-bimodule ${}^{\varphi}A_n{}^{\varphi}$. As a vector space, ${}^{\varphi}A_n{}^{\varphi}$ coincides with $A_n$ but the bimodule action is defined as follows:

$$a \cdot x \cdot b := \varphi(a)x\varphi(b) \quad \text{for all} \quad a, x, b \in A_n.$$

The set $\mathcal{M}$ above is equal to $\{{}^{\varphi}A_n{}^{\varphi} \mid \varphi \text{ is an endomorphism of the Weyl algebra } A_n\}$. The $A_n$-bimodule ${}^{\varphi}A_n{}^{\varphi}$ contains a simple $A_n$-bimodule $\varphi(A_n)$, the image of the endomorphism $\varphi$. So,

- *The Dixmier Problem has a Positive Answer if and only if the $A_n$-Bimodule ${}^{\varphi}A_n{}^{\varphi}$ is Simple for Each $\varphi$.*

The next corollary shows that it has *finite length* (a good argument in favour of the positive answer to the Dixmier Problem and the Jacobian Conjecture).

**Corollary 17.1.6** ([2, Corollary 1.4]) *Let $\varphi : A_n \to A_n$ be an endomorphism of the n'th Weyl algebra $A_n$. Then the $A_n$-bimodule ${}^{\varphi}A_n{}^{\varphi}$ is holonomic, that is*

$$\mathrm{GK}_{A_n \otimes A_n^{\mathrm{op}}}({}^{\varphi}A_n{}^{\varphi}) = 2n,$$

*and it has finite length as $A_n$-bimodule.*

***Proof*** This follows immediately from Theorem 17.1.5 applied to the Weyl algebra $A_{2n}$ since ${}^{\varphi}A_n{}^{\varphi}$ is a holonomic $A_n$-bimodule which is obtained from the holonomic $A_n$-bimodule $A_n$ by restriction of scalars. □

**Somewhat commutative algebras, the holonomicity is preserved under restriction of scalars.**

**Definition 17.1.7** A $K$-algebra $R$ is called a *somewhat commutative algebra* if it has a finite-dimensional filtration $R = \cup_{i \geq 0} R_i$ such that the associated graded algebra

$$\mathrm{gr}(R) := \bigoplus_{i \geq 0} R_i / R_{i-1}$$

is a commutative affine $K$-algebra where $R_{-1} = 0$ and $R_0 = K$.

The somewhat commutative algebra $R$ is a Noetherian affine algebra since $\mathrm{gr}(R)$ is so. A finitely generated module over a somewhat commutative algebra has the Gelfand–Kirillov dimension which is a natural number. The interested reader is referred to the books [21, 24] for the properties of somewhat commutative algebras.

**Definition 17.1.8** ([2]) For a somewhat commutative algebra $R$ we define the *holonomic number*,

$$h(R) := \min\{\mathrm{GK}\,(M) \mid M \neq 0 \text{ is a finitely generated } R-\text{module}\}.$$

**Definition 17.1.9** ([2]) A finitely generated $R$-module $M$ is called a *holonomic* module if

$$\mathrm{GK}\,(M) = h(R).$$

In other words, a nonzero finitely generated $R$-module is holonomic iff it has minimal possible Gelfand–Kirillov dimension. If $h(R) = 0$ then every holonomic $R$-module is finite-dimensional and vice versa.

**Example 17.1.10** The holonomic number of the Weyl algebra $A_n$ is $n$. The polynomial algebra $P_n := K[x_1, \dots, x_n]$ is equipped with the natural action of the ring of differential operators $A_n = K[x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]$. The polynomial algebra $P_n$ is a simple holonomic $A_n$-module and $P_n \simeq A_n / \sum_{i=1}^{n} A_n \partial_i$ where $\partial_i = \frac{\partial}{\partial x_i}$.

**Example 17.1.11** Let $X$ be a smooth irreducible algebraic affine variety of dimension $n$. The ring of differential operators $\mathcal{D}(X)$ is a simple somewhat commutative algebra of Gelfand–Kirillov dimension $2n$ with holonomic number $h(\mathcal{D}(X)) = n$. The algebra $\mathcal{O}(X)$ of regular functions of the variety $X$ is a simple holonomic $\mathcal{D}(X)$-module with respect to the natural action of the algebra $\mathcal{D}(X)$. In more detail,

$$\mathcal{O}(X) \simeq \mathcal{D}(X)/\mathcal{D}(X)\mathrm{Der}_K(\mathcal{O}(X))$$

where $\mathrm{Der}_K(\mathcal{O}(X))$ is the $\mathcal{O}(X)$-module of derivations of the algebra $\mathcal{O}(X)$.

The following theorem is one the main results of the paper [2]. The results above can be easily obtained from this one.

**Theorem 17.1.12** ((Holonomicity is Preserved under Restriction of Scalars) [2, Theorem 1.5])
*Let $R$ and $T$ be somewhat commutative algebras such that $h(R) = h(T)$ and let $\varphi : R \to T$ be an algebra homomorphism. Then every holonomic $T$-module $M$, by restriction of scalars, is a holonomic $R$-module and has finite length as $R$-module.*

**A counterexample to the Dixmier Conjecture for a localization of the Weyl Algebra.** Let $k$ be the first Weyl skew field, that is the full quotient ring of the first Weyl algebra $A_1 = \langle x, \partial, \mid \partial x - x\partial = 1\rangle$. The $K$-subalgebra $\mathcal{A}_1$ of $k$ generated by $x$, $\partial$ and $h^{-1}$, where $h := \partial x$, is isomorphic to the skew Laurent polynomial ring,

$$\mathcal{A}_1 = K[h, h^{-1}, (h \pm 1)^{-1}, (h \pm 2)^{-1}, \dots][x, x^{-1}; \sigma], \quad \sigma(h) = h - 1,$$

with coefficients from $K[h, h^{-1}, (h \pm 1)^{-1}, (h \pm 2)^{-1}, \dots]$, the localization of the polynomial algebra $K[h]$ at the multiplicatively closed subset $S$ generated by $\{h + i, i \in \mathbb{Z}\}$. The algebra $\mathcal{A}_1$ is a simple affine Noetherian domain of Gelfand–Kirillov dimension 3 (not $2 = \mathrm{GK}\,(A_1)$). The algebra $\mathcal{A}_1$ is isomorphic to the localization $S^{-1}A_1$ of the first Weyl algebra at $S$, and contains the algebra $A_1$.

**Theorem 17.1.13** ([2, Theorem 1.6]) $\mathrm{End}_K(\mathcal{A}_1) \neq \mathrm{Aut}_K(\mathcal{A}_1)$*, since, for every natural number $n \geq 2$, the endomorphism*

$$\tau_n : \mathcal{A}_1 \to \mathcal{A}_1, \quad x^{\pm 1} \to x^{\pm n}, \quad h \to \frac{h}{n},$$

*is not an automorphism, since*

$$\mathrm{im}(\tau_n) = K[h, h^{-1}, (h \pm n)^{-1}, (h \pm 2n)^{-1}, \dots][x^n, x^{-n}; \sigma^n] \neq \mathcal{A}_1.$$

**The Dixmier Conjecture holds for elements that are sums of no more than two homogeneous elements of** $A_1$. Recall that the Dixmier Conjecture says that every endomorphism of the (first) Weyl algebra $A_1$ (over a field of characteristic zero) is an automorphism, i.e. if

$$PQ - QP = 1$$

for some $P, Q \in A_1$ then $A_1 = K\langle P, Q\rangle$. The Weyl algebra

$$A_1 = \bigoplus_{i\in\mathbb{Z}} A_{1,i}$$

is a $\mathbb{Z}$-graded algebra ($A_{1,i}A_{1,j} \subseteq A_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where $A_{1,0} = K[H]$, $H = YX$ and, for $i \geq 1$, $A_{1,i} = K[H]X^i$ and $A_{1,-i} = K[H]Y^i$. For a nonzero element $a$ of $A_1$, the number of *nonzero homogeneous* components is called the *mass* of $a$, denoted by $m(a)$. For example, $m(\alpha X^i) = 1$ for all $\alpha \in K[H] \setminus \{0\}$ and $i \geq 1$. Theorem 17.1.14 shows that the Dixmier Conjecture holds if the elements $P$ and $Q$ are sums of no more than two homogeneous elements of $A_1$ and there is no restriction on the total degrees of $P$ and $Q$ with respect to the canonical generators $x$ and $\partial$ of the Weyl algebra $A_1$.

**Theorem 17.1.14** ([14, Theorem 1.1]) *Let $P, Q$ be elements of the first Weyl algebra $A_1$ with $m(P) \leq 2$ and $m(Q) \leq 2$. If $[P, Q] = 1$ then $P = \tau(Y)$ and $Q = \tau(X)$ for some automorphism $\tau \in \mathrm{Aut}_K(A_1)$.*

**Meaning of the Problem-Conjecture of Dixmier and the Jacobian Conjecture, the groups of automorphisms**. The groups of automorphisms of the polynomial algebra $P_n = P_1^{\otimes n}$, the Weyl algebra $A_n = A_1^{\otimes n}$ and the algebra

$$\mathbb{I}_n := \mathbb{I}_1^{\otimes n} = K\langle x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \int_1, \dots, \int_n\rangle$$

of polynomial integro-differential operators (see [9] for details) are *huge* infinite-dimensional algebraic groups. The groups of automorphisms are known only for the polynomial algebras when $n = 1$ (trivial) and $n = 2$ (Jung (1942) [20] and Van der Kulk (1953) [29]); and for the Weyl algebra $A_1$ (Dixmier (1968) [18]) (in characteristic $p > 0$, the group $\mathrm{Aut}_{K-\mathrm{alg}}(A_1)$ was found by Makar-Limanov (1984) [23], see also [8] for further developments and another proof). In 2009, the group $\mathrm{G}_n := \mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{I}_n)$ of automorphisms of the algebra $\mathbb{I}_n$ was found for all $n \geq 1$, [10, Theorem 5.5(1)]:

$$\mathrm{G}_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n) \supseteq S_n \ltimes \mathbb{T}^n \ltimes \underbrace{\mathrm{GL}_\infty(K) \ltimes \cdots \ltimes \mathrm{GL}_\infty(K)}_{2^n-1 \text{ times}},$$

$$\mathrm{G}_1 \simeq \mathbb{T}^1 \ltimes \mathrm{GL}_\infty(K),$$

where $S_n$ is the symmetric group, $\mathbb{T}^n$ is the $n$-dimensional algebraic torus, $\mathrm{Inn}(\mathbb{I}_n)$ is the group of inner automorphisms of $\mathbb{I}_n$ (which is huge). The ideas and approach in finding the groups $\mathrm{G}_n$ are completely different from that of Jung, Van der Kulk and Dixmier: the Fredholm operators, $\mathrm{K}_1$-theory, indices. On the other hand, when we look at the groups of automorphisms of the algebras $P_2$, $A_1$ and $\mathbb{I}_1$ (the only cases where we know explicit generators) we see that they have the 'same nature': they are generated by affine automorphisms and 'transvections.'

The Jacobian Conjecture and the Problem-Conjecture of Dixmier (if true) would give the *'defining relations'* for the infinite-dimensional algebraic groups of automorphisms as infinite-dimensional varieties in the same way as the condition $\det = 1$ defines the special linear (finite-dimensional) algebraic group $\mathrm{SL}_n(K)$. If true the conjectures would tell us nothing about generators of the groups of automorphisms (i.e. about the solutions of the defining relations, in the same way and the defining relation $\det = 1$ tells nothing about generators for the group $\mathrm{SL}_n(K)$).

More obvious meaning of the Problem-Conjecture of Dixmier is that the Weyl algebras $A_n$, which are simple *infinite-dimensional* algebras, behave like simple *finite-dimensional* algebras (each algebra endomorphism of a simple finite-dimensional algebra is, by a trivial reason, an automorphism). For a polynomial algebra $P_n$ there are plenty algebra endomorphisms that are not automorphisms. Recall that the *Jacobian Conjecture* states *that each algebra endomorphism $\sigma$ of the polynomial algebra $P_n$ with the Jacobian* $\mathrm{Jac}(\sigma) := \det(\frac{\partial\sigma(x_i)}{\partial x_j}) \in K^* := K\backslash\{0\}$ *is necessarily an automorphism*. The Jacobian condition is obviously holds for all automorphisms of $P_n$ and the Jacobian condition implies that $\sigma$ is a *monomorphism* but *not all* monomorphisms satisfy the Jacobian condition. So, the Jacobian Conjecture (if true) means that each algebra monomorphism of $P_n$ which is as *close as possible* to be an automorphism *is*, in fact, an automorphism.

**The (JD) Conjecture.** One can amalgamate **the Jacobian Conjecture** and **the Dixmier Conjecture** into a single question, [4],

**(JD)**: *Is a $K$-algebra endomorphism $\sigma : A_n \otimes P_m \to A_n \otimes P_m$ an algebra automorphism provided $\sigma(P_m) \subseteq P_m$ and* $\det\left(\frac{\partial\sigma(x_i)}{\partial x_j}\right) \in K^* := K\backslash\{0\}$? $(P_m = K[x_1, \dots, x_m])$.

The Theorem 17.1.15 follows from the inversion formula.

**Theorem 17.1.15** ([4, Corollary 2.5]) *The (JD) Conjecture, the Jacobian Conjecture and the Dixmier Conjecture are equivalent.*

Note that an algebra endomorphism $\sigma$ of the algebra $A_n \otimes P_m$ satisfying $\sigma(P_m) \subseteq P_m$ and $\det\left(\frac{\partial\sigma(x_i)}{\partial x_j}\right) \in K^*$ is automatically an algebra *monomorphism*: $\sigma|_{P_m}$ is an algebra monomorphism, it induces an algebra monomorphism, say $\sigma$, on the field of fractions $Q_m$ of $P_m$, hence $\sigma$ can be extended to an algebra endomorphism of the *simple* algebra $A_n \otimes Q_m$, hence $\sigma$ is an algebra *monomorphism*.

## 17.2 The Inversion Formula for the Weyl Algebras, the Polynomial Algebras and Their Tensor Products

In this section the following notation will remain fixed (if it is not stated otherwise): $K$ is a field of characteristic zero (not necessarily algebraically closed), module means a *left* module, $A_n = \bigoplus_{\alpha \in \mathbb{N}^{2n}} Kx^\alpha$ is the $n$'th *Weyl algebra* over $K$ (the commutator $[x_{n+i}, x_j] = \delta_{ij}$, $1 \le i, j \le n$, where $\delta_{ij}$ is the *Kronecker* delta), $P_m = \bigoplus_{\alpha \in \mathbb{N}^m} Kx^\alpha$ is a *polynomial* algebra over $K$ (in $m$ variables $x_{2n+1}, \ldots, x_{2n+m}$),

$$A := A_n \otimes P_m = \bigoplus_{\alpha \in \mathbb{N}^s} Kx^\alpha, \quad x^\alpha := x_1^{\alpha_1} \cdots x_s^{\alpha_s}, \quad s := 2n + m,$$

is the Weyl algebra with polynomial coefficients where $x_1, \ldots, x_s$ are the canonical generators of the algebra $A$.

Any $K$-algebra automorphism $\sigma \in \mathrm{Aut}_K(A)$ is uniquely determined by the elements

$$x_i' := \sigma(x_i) = \sum_{\alpha \in \mathbb{N}^s} \lambda_{i,\alpha} x^\alpha, \quad \lambda_\alpha, \in K, \quad i = 1, \ldots, s,$$

and so does its inverse,

$$\sigma^{-1}(x_i) = \sum_{\alpha \in \mathbb{N}^s} \lambda'_{i,\alpha} x^\alpha, \quad i = 1, \ldots, s.$$

Let $A$ be an algebra over a field $K$ and let $\delta$ be a $K$-derivation of the algebra $A$. For any elements $a, b \in A$ and a natural number $n$, an easy induction argument gives

$$\delta^n(ab) = \sum_{i=0}^{n} \binom{n}{i} \delta^i(a) \delta^{n-i}(b).$$

It follows that the kernel $A^\delta := \ker \delta$ of $\delta$ is a subalgebra (of *constants* for $\delta$) of $A$ and the union of the vector spaces

$$N := N(\delta, A) = \bigcup_{i \ge 0} N_i, \quad N_i := \ker(\delta^{i+1}),$$

is a positively *filtered* algebra ($N_i N_j \subseteq N_{i+j}$ for all $i, j \ge 0$). Clearly, $N_0 = A^\delta$ and $N = \{a \in A \mid \delta^n(a) = 0 \text{ for some natural } n\}$.

A $K$-derivation $\delta$ of the algebra $A$ is a *locally nilpotent* derivation if for each element $a \in A$ there exists a natural number $n$ such that $\delta^n(a) = 0$. A $K$-derivation $\delta$ is locally nilpotent iff $A = N(\delta, A)$.

Given a ring $R$ and its derivation $d$. The *Ore extension* $R[x; d]$ of $R$ is a ring freely generated over $R$ by $x$ subject to the defining relations: $xr = rx + d(r)$ for all $r \in R$.

$$R[x;d] = \bigoplus_{i\geq 0} Rx^i = \bigoplus_{i\geq 0} x^i R$$

is a left and right free $R$-module. Given $r \in R$, a derivation $(\mathrm{ad}\, r)(s) := [r, s] = rs - sr$ of $R$ is called an *inner* derivation of $R$.

**Theorem 17.2.1** ([4, Theorem 2.2]) *Let $A$ be an algebra over a field $K$ of characteristic zero, $\delta$ be a locally nilpotent $K$-derivation of the algebra $A$ such that $\delta(x) = 1$ for some $x \in A$. Then the $K$-linear map $\phi := \sum_{i\geq 0}(-1)^i \frac{x^i}{i!}\delta^i : A \to A$ satisfies the following properties:*

1. *$\phi$ is a homomorphism of right $A^\delta$-modules.*
2. *$\phi$ is a projection onto the algebra $A^\delta$:*

$$\phi : A = A^\delta \oplus xA \to A^\delta \oplus xA, \quad a + xb \mapsto a, \quad \text{where} \quad a \in A^\delta, \; b \in A.$$

   *In particular,* $\mathrm{im}(\phi) = A^\delta$ *and $\phi(y) = y$ for all $y \in A^\delta$.*
3. *$\phi(x^i) = 0$, $i \geq 1$.*
4. *$\phi$ is an algebra homomorphism provided $x \in Z(A)$, the centre of the algebra $A$.*

Recall that $A_n \otimes P_m = \bigoplus_{\alpha\in\mathbb{N}^s} Kx^\alpha$ where $s = 2n + m$, $x^\alpha := x_1^{\alpha_1}\cdots x_s^{\alpha_s}$, the *order* of the $x$'s is *fixed*. The centre of the algebra $A_n \otimes P_m$ is $P_m$. The algebra $A_n \otimes P_m$ admits the finite set of *commuting locally nilpotent* derivations, namely, the 'partial derivatives':

$$\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_s := \frac{\partial}{\partial x_s}.$$

Clearly,

$$\partial_i = \mathrm{ad}(x_{n+i}) \;\; \text{and} \;\; \partial_{n+i} = -\mathrm{ad}(x_i) \;\; \text{for} \;\; i = 1, \dots, n$$

where $\mathrm{ad}(a) : A \to A, b \mapsto [a, b]$, is the *inner* derivation of the algebra $A$ associated with $a \in A$.

For each $i = 1, \dots, s$, consider the maps from Theorem 17.2.1,

$$\phi_i := \sum_{k\geq 0}(-1)^k \frac{x_i^k}{k!}\partial_i^k : A_n \otimes P_m \to A_n \otimes P_m.$$

For each $i = 2n + 1, \dots, s$, the map $\phi_i$ *commutes* with *all* the maps $\phi_j$. For each $i = 1, \dots, n$, the map $\phi_i$ commutes with all the maps $\phi_j$ but $\phi_{n+i}$, and the map $\phi_{n+i}$ commutes with all the maps $\phi_j$ but $\phi_i$. Note that $A_n \otimes P_m = K \oplus V$ where $V := \bigoplus_{0\neq\alpha\in\mathbb{N}^s} Kx^\alpha$. Using Theorem 17.2.1, we see that the map (the order is important)

$$\phi := \phi_s\phi_{s-1}\cdots\phi_1 : A_n \otimes P_m \to A_n \otimes P_m, \quad a = \sum_{\alpha\in\mathbb{N}^s} \lambda_\alpha x^\alpha \mapsto \phi(a) = \lambda_0, \quad (17.1)$$

is a *projection* onto $K$.

The next result is a kind of a *Taylor* formula (note though that even for polynomials this is *not* the Taylor formula. The both formulae are essentially a formula for the *identity map* and they give a presentation of an element as a series but the formula below has one obvious advantage—it is 'more economical', i.e. there is no evaluation at $x = 0$ as in the Taylor formula).

**Theorem 17.2.2** ([4, Theorem 2.3]) *For any* $a \in A_n \otimes P_m$,

$$a = \sum_{\alpha \in \mathbb{N}^s} \phi\left(\frac{\partial^\alpha}{\alpha!} a\right) x^\alpha$$

*where* $s = 2n + m$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_s^{\alpha_s}$ *and* $\alpha! := \alpha_1! \cdots \alpha_s!$.

***Proof*** If $a = \sum \lambda_\alpha x^\alpha$, $\lambda_\alpha \in K$, then, by (17.1), $\phi(\frac{\partial^\alpha}{\alpha!} a) = \lambda_\alpha$. □

By Theorem 17.2.2, the identity map id : $A_n \otimes P_m \to A_n \otimes P_m$ can be written as follows:

$$\mathrm{id}(\cdot) = \sum_{\alpha \in \mathbb{N}^s} \phi\left(\frac{\partial^\alpha}{\alpha!}(\cdot)\right) x^\alpha. \tag{17.2}$$

Let $\mathrm{Aut}_K(A_n \otimes P_m)$ be the group of $K$-algebra automorphisms of the algebra $A_n \otimes P_m$. Given an automorphism $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. It is uniquely determined by the elements

$$x_1' := \sigma(x_1), \ldots, x_s' := \sigma(x_s) \tag{17.3}$$

of the algebra $A_n \otimes P_m$. The centre $Z := Z(A_n \otimes P_m)$ of the algebra $A_n \otimes P_m$ is equal to $P_m$. Therefore, the restriction of the automorphism $\sigma$ to the centre, $\sigma|_{P_m} \in \mathrm{Aut}_K(P_m)$, is an automorphism of the polynomial algebra $P_m$. Hence,

$$\mathrm{d} := \det\left(\frac{\partial x_{2n+i}'}{\partial x_{2n+j}}\right) \in K^*$$

where $i, j = 1, \ldots, n$. The corresponding (to the elements $x_1', \ldots, x_s'$) 'partial derivatives' (the set of commuting locally nilpotent derivations of the algebra $A_n \otimes P_m$)

$$\partial_1' := \frac{\partial}{\partial x_1'}, \ldots, \partial_s' := \frac{\partial}{\partial x_s'} \tag{17.4}$$

are equal to

$$\partial_i' := \mathrm{ad}(\sigma(x_{n+i})), \quad \partial_{n+i}' := -\mathrm{ad}(\sigma(x_i)), \quad i = 1, \ldots, n, \tag{17.5}$$

$$\partial'_{2n+j} := \mathrm{d}^{-1} \det \begin{pmatrix} \frac{\partial \sigma(x_{2n+1})}{\partial x_{2n+1}} & \cdots & \frac{\partial \sigma(x_{2n+1})}{\partial x_{2n+m}} \\ \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_{2n+1}} & \cdots & \frac{\partial}{\partial x_{2n+m}} \\ \vdots & \vdots & \vdots \\ \frac{\partial \sigma(x_{2n+m})}{\partial x_{2n+1}} & \cdots & \frac{\partial \sigma(x_{2n+m})}{\partial x_{2n+m}} \end{pmatrix}, \quad j = 1, \dots, m, \tag{17.6}$$

where we 'drop' $\sigma(x_{2n+j})$ in the determinant $\det\left(\frac{\partial \sigma(x_{2n+k})}{\partial x_{2n+l}}\right)$.

For each $i = 1, \dots, s$, let

$$\phi'_i := \sum_{k \geq 0} (-1)^k \frac{(x'_i)^k}{k!} (\partial'_i)^k : A_n \otimes P_m \to A_n \otimes P_m \tag{17.7}$$

and (the order is important)

$$\phi_\sigma := \phi'_s \phi'_{s-1} \cdots \phi'_1. \tag{17.8}$$

**Theorem 17.2.3** ((The Inversion Formula) [4, Theorem 2.4]) *For each* $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$ *and* $a \in A_n \otimes P_m$,

$$\sigma^{-1}(a) = \sum_{\alpha \in \mathbb{N}^s} \phi_\sigma\left(\frac{(\partial')^\alpha}{\alpha!} a\right) x^\alpha$$

*where* $(\partial')^\alpha := (\partial'_1)^{\alpha_1} \cdots (\partial'_s)^{\alpha_s}$ *and* $s = 2n + m$.

***Proof*** By Theorem 17.2.2, $a = \sum_{\alpha \in \mathbb{N}^s} \phi_\sigma\left(\frac{(\partial')^\alpha}{\alpha!} a\right) (x')^\alpha$. Applying $\sigma^{-1}$ we have the result

$$\sigma^{-1}(a) = \sum_{\alpha \in \mathbb{N}^s} \phi_\sigma\left(\frac{(\partial')^\alpha}{\alpha!} a\right) \sigma^{-1}((x')^\alpha) = \sum_{\alpha \in \mathbb{N}^s} \phi_\sigma\left(\frac{(\partial')^\alpha}{\alpha!} a\right) x^\alpha. \quad \square$$

**Gurjar's Inversion Formula and Abhyankar's Inversion Formula for polynomial automorphisms.** Let $\sigma : P_n \to P_n$, $x_i \mapsto y_i := \sigma(x_i)$ be a polynomial automorphism. Let us cite Moh, [26, page 109]: "There are two inversion formulae due to Gurjar and Abhyankar based on a formula of Goursat":

*Gurjar's formula*

$$x_i = \sum \frac{1}{\prod r_j! \prod s_j!} \frac{\partial^{r+s}}{(\partial_{x_1})^{r_1} \cdots (\partial_{x_n})^{r_n} (\partial_{y_1})^{s_1} \cdots (\partial_{y_n})^{y_n}} \left( x_i J \prod (x_i - y_i)^{r_i} \prod y_j^{s_j} \right).$$

*Abhyankar's formula*

$$x_i = \sum \frac{1}{\prod r_j!} \frac{\partial^r}{(\partial_{x_1})^{r_1} \cdots (\partial_{x_n})^{r_n}} \left( x_i J \prod (x_i - y_i)^{r_i} \right)$$

where $J$ is the Jacobian of $\sigma$.

## 17.3 Every Endomorphism of the Algebra $\mathbb{I}_1$ Is an Automorphism

In this section, main steps of the statement in the title of this section are explained (Theorem 17.3.1).

Recall that $A_1 := K\langle x, \partial\rangle$ is the Weyl algebra and $\mathbb{I}_1 := K\langle x, \partial, \int\rangle$ is the algebra of polynomial integro-differential operators over a field $K$ of characteristic zero where $\partial = \frac{d}{dx}$ and

$$\int : \ K[x] \to K[x], \quad x^n \mapsto (n+1)^{-1} x^{n+1} \quad \text{for all} \quad n \geq 0.$$

**Six Problems of Dixmier,** [18]**, for the Weyl algebra** $A_1$: In 1968, Dixmier [18] posed six problems for the Weyl algebra $A_1$.

**The First Problem-Conjecture of Dixmier,** [18]: *is an algebra endomorphism of the Weyl algebra* $A_1$ *an automorphism?*

Dixmier writes in his paper [18], p. 242: "*A. A. Kirillov informed me that the Moscow school also considered this problem*".

In 1975, the Third Problem of Dixmier was solved by Joseph [19] (using results of Stein [19], and McConnel and Robson [25]); and using his (difficult) polarization theorem for the Weyl algebra $A_1$ Joseph [19] solved the Sixth Problem of Dixmier (a short proof to this problem is given in [6]). An analogue of the Sixth Problem of Dixmier is true for the ring of differential operators on an arbitrary smooth irreducible algebraic curve [7]. In 2005, the Fifth Problem of Dixmier was solved in [5]. Problems 1, 2, and 4 are still open. The Fourth Problem of Dixmier has positive solution for *all homogeneous* elements of the Weyl algebra $A_1$, [5, Theorem 2.3].

**Conjecture** ([11]) *Each algebra endomorphism of* $\mathbb{I}_n$ *is an automorphism.*

For $n = 1$, the Conjecture above is Theorem 17.3.1.

Below, we explain the main steps and ideas of the proof of Theorem 17.3.1. The proof consists of nine steps. The proof is not straightforward and several key results of the papers [9, 10, 12] are used.

**Theorem 17.3.1** ([11, Theorem 1.1]) *Each algebra endomorphism of* $\mathbb{I}_1$ *is an automorphism.*

**Structure of the proof**, [11]. Let $\sigma$ be an algebra endomorphism of $\mathbb{I}_1$. Since $\mathbb{I}_1 = K\langle H, \int, \partial\rangle$ where $H := \partial x$ (notice that $x = \int H$), the endomorphism $\sigma$ is uniquely determined by the elements

$$H' := \sigma(H), \quad \int' := \sigma(\int), \quad \partial' := \sigma(\partial).$$

*Step 1*. $\sigma$ is a monomorphism.

*Step 2*. $\sigma(F) \subseteq F$, where $F$ is the only proper ideal of the algebra $\mathbb{I}_1$. Therefore, there is a commutative diagram of algebra homomorphisms:

$$\begin{array}{ccc} \mathbb{I}_1 & \xrightarrow{\sigma} & \mathbb{I}_1 \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ B_1 & \xrightarrow{\overline{\sigma}} & B_1 \end{array}$$

where $B_1 := \mathbb{I}_1/F \simeq K[H][\partial, \partial^{-1}; \tau]$, $\tau(H) = H + 1$, is a simple algebra, and so $\overline{\sigma}$ is an algebra monomorphism.

*Step 3*. $H' = \lambda H + \mu + h$ for some elements $\lambda \in K^* := K\backslash\{0\}$, $\mu \in K$ and $h \in F$.

*Step 4*. $H' = \frac{1}{n}H + \mu + h$, $\int' = \nu \int^n + f$ and $\partial' = \nu^{-1}\partial^n + g$ for some elements $\nu \in K^*$, $n \geq 1$ and $h, f, g \in F$.

*Step 5*. ${}^{\sigma}K[x] \simeq K[x]^n$, an isomorphism of $\mathbb{I}_1$-modules where $n$ is as in Step 4 and ${}_{\mathbb{I}_1}K[x] := \mathbb{I}_1/\mathbb{I}_1\partial$, ${}^{\sigma}K[x]$ is the twisted $\mathbb{I}_1$-module $K[x]$ by the algebra endomorphism $\sigma$.

*Step 6*. $n = 1$, i.e. ${}^{\sigma}K[x] \simeq K[x]$.

*Step 7*. Up to the algebraic torus action $\mathbb{T}^1$ ($\subseteq \text{Aut}_{K-\text{alg}}(\mathbb{I}_1)$), $\nu = 1$, i.e.

$$H' = H + \mu + h, \quad \int' = \int + f, \quad \partial' = \partial + g.$$

*Step 8*. $\mu = 0$.

*Step 9*. $\sigma$ is an inner automorphism $\omega_u$ of the algebra $\mathbb{I}_1$ for some unit $u \in (1 + F)^*$ of the algebra $\mathbb{I}_1$. □

**Ideas behind the proof of Theorem 17.3.1.** This is a combination of old ideas of approach due to Dixmier [18] of using the eigenvalues of certain inner derivations (this was a key moment in finding the group $\text{Aut}_{K-\text{alg}}(A_1)$ in [18] modulo many technicalities) and new ideas/approach of using (i) the Fredholm operators and their indices based on the fact that for the algebra $\mathbb{I}_1$ the (Strong) Compact-Fredholm Alternative holds [12] (which says that the action of each polynomial integro-differential operator of $\mathbb{I}_1$ on each simple $\mathbb{I}_1$-module is either compact or Fredholm) and (ii) the structure of the centralizers of elements of $\mathbb{I}_1$ [12].

**Remark 17.3.2** The algebra $B_1$ (see Step 2) is the left and right localization of the Weyl algebra $A_1$ at the powers of the element $\partial$, i.e. the algebra $B_1$ is obtained from $A_1$ by adding the *two-sided* inverse $\partial^{-1}$ of the element $\partial$ (the algebra $B_1$ is also a *left* (but not right) localization of the algebra $\mathbb{I}_1$ at the powers of the element $\partial$, [9], but in contrast to the Weyl algebra $A_1$ the element $\partial$ is not regular in $\mathbb{I}_1$). The algebra $B_1 = K[H][\partial, \partial^{-1}; \tau]$ is a skew Laurent polynomial algebra where $\tau(H) = H + 1$.

An analogue of the Dixmier Conjecture fails for the algebra $B_1$: for each natural number $n \geq 2$, the algebra monomorphism

$$\sigma_n : B_1 \to B_1, \quad H \mapsto \frac{1}{n}H, \quad \partial \mapsto \partial^n,$$

is obviously not an automorphism (use the $\mathbb{Z}$-grading of the algebra

$$B_1 = \bigoplus_{i\in\mathbb{Z}} K[H]\partial^i, \ \partial^i\alpha = \tau^i(\alpha)\partial^i$$

for all $\alpha \in K[H]$ and $i \in \mathbb{Z}$). In view of existence of this counterexample for the algebra $B_1$ it looks surprising that Theorem 17.3.1 is true as the algebra $\mathbb{I}_1$ is obtained from the Weyl algebra $A_1$ by adding a *right*, but not two-sided, inverse of the element $\partial$: $\partial \int = 1$ but $\int \partial \neq 1$. Theorem 17.3.1 can be seen as a sign that the of Dixmier Conjecture is true.

## References

1. Bass, H., Connel, E.H., Wright, D.: The Jacobian conjecture: reduction of degree and formal expansion of the inverse. Bull. Am. Math. Soc. (New Ser.) **7**, 287–330 (1982)
2. Bavula, V.V.: A question of Rentschler and the Dixmier problem. Ann. Math. (2) **154**(3), 683–702 (2001)
3. Bavula, V.V.: The Jacobian conjecture$_{2n}$ implies the Dixmier problem$_n$. arXiv:math/0512250
4. Bavula, V.V.: The inversion formula for automorphisms of the Weyl algebras and polynomial algebras. J. Pure Appl. Algebra **210**, 147–159 (2007). arXiv:math.RA/0512215
5. Bavula, V.V.: Dixmier's problem 5 for the Weyl algebra. J. Algebra **283**(2), 604–621 (2005)
6. Bavula, V.V.: Dixmier's problem 6 for the Weyl algebra (the generic type problem). Commun. Algebra **34**(4), 1381–1406 (2006)
7. Bavula, V.V.: Dixmier's problem 6 for somewhat commutative algebras and Dixmier's problem 3 for the ring of differential operators on a smooth irreducible affine curve. J. Algebra Its Appl. **4**(5), 577–586 (2005)
8. Bavula, V.V.: The group of automorphisms of the first Weyl algebra in prime characteristic and the restriction map. Glasg. Math. J. **51**, 263–274 (2009). arXiv:0708.1620

9. Bavula, V.V.: The algebra of integro-differential operators on a polynomial algebra. J. Lond. Math. Soc. **83**(2), 517–543 (2011). arXiv:0912.0723
10. Bavula, V.V.: The group of automorphisms of the algebra of polynomial integro-differential operators. J. Algebra **348**, 233–263 (2011). arXiv:0912.2537
11. Bavula, V.V.: An analogue of the conjecture of Dixmier is true for the algebra of polynomial integro-differential operators. J. Algebra **372**, 237–250 (2012). arXiv:1011.3009
12. Bavula, V.V.: The algebra of integro-differential operators on an affine line and its modules. J. Pure Appl. Algebra **217**, 495–529 (2013). arXiv:1011.2997
13. Bavula, V.V.: The Jacobian conjecture and the problem of Dixmier are equivalent
14. Bavula, V.V., Levandovskyy, V.: A remark on the Dixmier conjecture. Canad. Math. Bull. *63*(1), 6–12 (2020). arXiv:1812.00042
15. Belov-Kanel, A., Kontsevich, M.: The Jacobian conjecture is stably equivalent to the Dixmier conjecture. Mosc. Math. J. **7**(2), 209–218 (2007). arXiv:math.RA/0512171
16. Bernstein, I.N.: The analitic continuation of generalized functions with respect to a parameter. Funct. Anal. Its Appl. **6**(4), 26–40 (1972)
17. Bernstein, J., Lunts, V.: On nonholonomic irreducible $D$-modules. Invent. Math. **94**(2), 223–243 (1988)
18. Dixmier, J.: Sur les algèbres de Weyl. Bull. Soc. Math. Fr. **96**, 209–242 (1968)
19. Joseph, A.: The Weyl algebra–semisimple and nilpotent elements. Am. J. Math. **97**(3), 597–615 (1975)
20. Jung, H.W.E.: Uber ganze birationale Transformationen der Ebene. J. Reine Angew. Math. **184**, 161–174 (1942)
21. Krause, G.R., Lenagan, T.H.: Growth of Algebras and Gelfand-Kirillov Dimension, Graduate Studies in Mathematics, vol. 22. American Mathematical Society, Providence, RI (2000)
22. Lunts, V.: Algebraic varieties preserved by generic flows. Duke Math. J. **58**(3), 531–554 (1989)
23. Makar-Limanov, L.: On automorphisms of Weyl algebra. Bull. Soc. Math. Fr. **112**, 359–363 (1984)
24. McConnell, J.C., Robson, J.C.: Noncommutative Noetherian Rings. Wiley, Chichester (1987)
25. McConnell, J.C., Robson, J.C.: Homomorphisms and extensions of modules over certain differential polynomial rings. J. Algebra **26**, 319–342 (1973)
26. Moh, T.T.: Jacobian Conjecture. In: Proceedings of the International Conference of Algebra and Geometry, 103–116 (1995)
27. Tsuchimoto, Y.: Preliminaries on Dixmier Conjecture. Mem. Fac. Sci. Kochi Univ. Ser. A Math. **24**, 43–59 (2003)
28. Tsuchimoto, Y.: Endomorphisms of Weyl algebra and p-curvatures. Osaka J. Math. **42**(2), 435–452 (2005)
29. Van der Kulk, W.: On polynomial rings in two variables. Nieuw Arch. Wiskd. (3) **1**, 33–41 (1953)