Chapter 15
Noncommutatively Graded Algebras

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Abstract Inspired by the commutator and anticommutator algebras derived from algebras graded by groups, we introduce noncommutatively graded algebras. We generalize various classical graded results to the noncommutatively graded situation concerning identity elements, inverses, existence of limits and colimits and adjointness of certain functors. In the particular instance of noncommutatively graded Lie algebras, we establish the existence of universal graded enveloping algebras and we show a graded version of the Poincaré-Birkhoff-Witt theorem.

Keywords Graded algebras · Lie algebras · Enveloping algebra

15.1 Introduction

Let R denote a ring which is unital, commutative and associative. Suppose that A is an R-algebra. By this we mean that A is a left R-module equipped with an R-bilinear map $A \times A \ni (a, b) \mapsto ab \in A$. Let G be a group. Recall that A is called G-graded if there is a family $\{A_g\}_{g \in G}$ of R-submodules of A such that

$$A = \oplus_{g \in G} A_g, \tag{15.1}$$

as R-modules, and for all $g, h \in G$ the inclusion

$$A_g A_h \subseteq A_{gh} \tag{15.2}$$

holds. The category of G-graded R-algebras, here denoted by G-GA, is obtained by taking G-graded R-algebras as objects and for the morphisms between such objects A and B we take the R-algebra homomorphisms $f : A \to B$ satisfying $f(A_g) \subseteq B_g$, for $g \in G$.

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Graded algebras include as special cases many other constructions such as polynomial and skew polynomial rings, Ore extensions, matrix rings, Morita contexts, group rings, twisted group rings, skew group rings and crossed products. Therefore, the theory of graded algebras not only gives new results for several constructions simultaneously, but also serves as a unification of known theorems. For more details concerning graded algebras, see e.g. [4] or [5] and the references therein.

The motivation for the present article is the observation that there are many natural examples of algebras which satisfy (15.1) but only a weaker form of (15.2). Namely, suppose that A is an associative G-graded algebra. Then the induced Lie algebra $(A, [\cdot, \cdot])$ (see [6, p. 3]) and the induced Jordan algebra $(A, \{\cdot, \cdot\})$ (see [6, p. 4], where

$$A \times A \ni (a, b) \mapsto [a, b] = ab - ba \in A \tag{15.3}$$

is the commutator and

$$A \times A \ni (a, b) \mapsto \{a, b\} = ab + ba \in A$$

is the anticommutator, satisfy

$$[A_g, A_h] \subseteq A_{gh} + A_{hg} \tag{15.4}$$

and

$$\{A_g, A_h\} \subseteq A_{gh} + A_{hg} \tag{15.5}$$

respectively, for all $g, h \in G$ (see also Example 15.1). Inspired by (15.4) and (15.5) we say that an R-algebra A is *noncommutatively G-graded* if there is a family $\{A_g\}_{g \in G}$ of R-submodules of A satisfying (15.1), as R-modules, and for all $g, h \in G$ the inclusion $A_g A_h \subseteq A_{gh} + A_{hg}$ holds. The aim of this article is to generalize various G-graded classical results to the noncommutatively G-graded situation. Here is an outline of the article.

In Sect. 15.2, we introduce the category G-NCGA of noncommutatively G-graded algebras (see Definition 15.1) and we show results concerning identity elements, inverses, existence of limits and colimits and adjointness of certain functors related to G-NCGA (see Propositions 15.1–15.11). In Sect. 15.3, we fix the notation concerning G-graded modules and we recall the construction of the graded tensor algebra (see Propositions 15.12–15.14). In Sect. 15.4, we study the particular instance of noncommutatively G-graded Lie algebras (see Definition 15.4), we establish the existence of universal graded enveloping algebras (see Proposition 15.15) and we show a graded version of the Poincaré-Birkhoff-Witt theorem (see Proposition 15.16).

15.2 Graded Algebras

In this section, we introduce the category of G-NCGA of noncommutatively G-graded algebras (see Definition 15.1) and we show results concerning identity elements, inverses, existence of limits and colimits and adjointness of certain functors related to G-NCGA (see Propositions 15.1–15.8).

For the rest of this article, R denotes a unital, commutative and associative ring, A denotes an R-algebra and G denotes a multiplicatively written group with identity element e. If A is unital, then we let 1 denote the multiplicative identity of A.

Definition 15.1 We say that A is noncommutatively G-graded if there is a family $\{A_g\}_{g\in G}$ of R-submodules of A such that $A = \oplus_{g\in G} A_g$, as R-modules, and for all $g, h \in G$ the inclusion $A_g A_h \subseteq A_{gh} + A_{hg}$ holds. The category of noncommutatively G-graded algebras, denoted by G-NCGA, is obtained by taking noncommutatively G-graded algebras as objects and for the morphisms between such objects A and B we take the R-algebra homomorphisms $f : A \to B$ satisfying $f(A_g) \subseteq B_g$, for $g \in G$.

Remark 15.1 Clearly, there is an inclusion of G-GA in G-NCGA. If G is abelian, then G-NCGA coincides with G-GA.

Example 15.1 Let A be a G-graded algebra. From A it is easy to construct many examples of noncommutatively G-graded algebras. Indeed, for all $g, h \in G$, take $\lambda_{g,h}, \mu_{h,g} \in R$. Define a new product $\bullet$ on A by the additive extension of the relations $a_g \bullet b_h = \lambda_{g,h} a_g a_h + \mu_{h,g} a_h a_g$, for $g, h \in G$, $a_g \in A_g$ and $b_h \in B_h$. We will denote this algebra by (A, λ, μ). Note that if we take $\lambda \equiv 1$ and $\mu \equiv -1$ (or $\mu \equiv 1$), then (A, λ, μ) coincides with the commutator (or anticommutator) algebra defined by A. The category G-NCGA$_{\lambda,\mu}$ is obtained by taking (A, λ, μ), for G-graded algebras A, as objects, and for morphisms $(A, \lambda, \mu) \to (B, \lambda, \mu)$ we take graded R-algebra morphisms $A \to B$. It is clear that the correspondence $A \mapsto (A, \lambda, \mu)$, on objects of G-NCGA, and by the identity, on graded R-algebra morphisms, defines a functor from G-GA to G-NCGA$_{\lambda,\mu}$. We will denote this functor by (λ, μ).

Remark 15.2 Suppose that A is a unital G-graded algebra. Then, from [5, Proposition 1.1.1.1] it follows that $1 \in A_e$. If for every $g \in G$, the conditions $\lambda_{g,e} + \mu_{e,g} = 1 = \lambda_{e,g} + \mu_{g,e}$ hold, then (A, λ, μ) is also unital with multiplicative identity 1. If 2 is invertible in R, then this holds if $\lambda \equiv \mu \equiv 2^{-1}$.

The next result is a generalization of [5, Proposition 1.1.1.1].

Proposition 15.1 *If A is unital and noncommutatively G-graded, then $1 \in A_e$.*

Proof Suppose that $1 = \sum_{g\in G} a_g$ for some $a_g \in R_g$ satisfying $a_g = 0$ for all but finitely many $g \in G$. Take $h \in G$. Then

$$a_h = 1a_h = \sum_{g\in G} a_g a_h = a_e a_h + \sum_{g\in G\setminus\{e\}} a_g a_h.$$

Thus

$$A_h \ni a_h - a_e a_h = \sum_{g \in G \setminus \{e\}} a_g a_h \in \sum_{g \in G \setminus \{e\}} (A_{gh} + A_{hg}) \subseteq \oplus_{g \in G \setminus \{h\}} A_g.$$

Hence, in particular, we get that

$$\sum_{g \in G \setminus \{e\}} a_g a_h = 0. \tag{15.6}$$

Also

$$a_h = a_h 1 = \sum_{g \in G} a_h a_g = a_h a_e + \sum_{g \in G \setminus \{e\}} a_h a_g.$$

Thus

$$A_h \ni a_h - a_h a_e = \sum_{g \in G \setminus \{e\}} a_h a_g \in \sum_{g \in G \setminus \{e\}} (A_{gh} + A_{hg}) \subseteq \oplus_{g \in G \setminus \{h\}} A_g.$$

Hence, in particular, we get that

$$a_h = a_h a_e. \tag{15.7}$$

From (15.6) and (15.7) it follows that $0 = \sum_{g \in G \setminus \{e\}} a_g a_e = \sum_{g \in G \setminus \{e\}} a_g$. Thus

$$1 = \sum_{g \in G} a_g = a_e + \sum_{g \in G \setminus \{e\}} a_g = a_e + 0 = a_e \in A_e.$$

□

Definition 15.2 If A is noncommutatively G-graded, then $\cup_{g \in G} A_g$ is called the set of homogeneous elements of A; if $g \in G$, then a nonzero element $a \in A_g$ is said to be homogeneous of degree g. In that case we write $\deg(a) = g$.

The following result is a generalization of [5, Proposition 1.1.1.2]

Proposition 15.2 *Suppose that A is unital and noncommutatively G-graded. If a is a non-zero homogeneous element of A such that* $\deg(a) = g$ *and a has a right (left) inverse, then a has a right (left) homogeneous inverse of degree g^{-1}.*

Proof First show the "right" part of the proof. Suppose that there is $b \in A$ such that $ab = 1$. Suppose that $b = \sum_{h \in G} b_h$ for some $b_h \in A_h$ such that $b_h = 0$ for all but finitely many $h \in G$. Then

$$1 = ab = \sum_{h \in G} ab_h = ab_{g^{-1}} + \sum_{h \in G \setminus \{g^{-1}\}} ab_h.$$

Thus, from Proposition 15.1, we get that

$$A_e \ni 1 - ab_{g^{-1}} = \sum_{h \in G \setminus \{g^{-1}\}} ab_h \in \sum_{h \in G \setminus \{g^{-1}\}} (A_{gh} + A_{hg}) \subseteq \oplus_{g \in G \setminus \{e\}} A_g.$$

Therefore $1 - ab_{g^{-1}} = 0$ and thus $ab_{g^{-1}} = 1$.

Now we show the "left" part of the proof. Suppose that there is $c \in A$ such that $ca = 1$. Suppose that $c = \sum_{h \in G} c_h$ for some $c_h \in A_h$ such that $c_h = 0$ for all but finitely many $h \in G$. Then

$$1 = ca = \sum_{h \in G} c_h a = c_{g^{-1}} a + \sum_{h \in G \setminus \{g^{-1}\}} c_h a.$$

Thus, from Proposition 15.1, we get that

$$A_e \ni 1 - c_{g^{-1}} a = \sum_{h \in G \setminus \{g^{-1}\}} c_h a \in \sum_{h \in G \setminus \{g^{-1}\}} (A_{hg} + A_{gh}) \subseteq \oplus_{g \in G \setminus \{e\}} A_g.$$

Therefore $1 - c_{g^{-1}} a = 0$ and thus $c_{g^{-1}} a = 1$. □

Proposition 15.3 *Inverse limits exist in* G-NCGA.

Proof Suppose that we are given a preordered set $(I, \leq)$ and a family of noncommutatively G-graded R-algebras $(A_\alpha)_{\alpha \in I}$. For all $\alpha, \beta \in I$ with $\alpha \leq \beta$ let $f_{\alpha\beta} : A_\beta \to A_\alpha$ be a morphism in G-NCGA. Suppose that the the morphisms $f_{\alpha\beta}$ form an inverse system, that is, that the following conditions hold:

- the relations $\alpha \leq \beta \leq \gamma$ imply that $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$;
- for every $\alpha \in I$, the equality $f_{\alpha\alpha} = \mathrm{id}_{A_\alpha}$ holds.

Let P denote the product of the sets A_α and let $p_\alpha : P \to A_\alpha$ denote the corresponding projection. Let Q denote the subset of all $x \in P$ which satisfy $p_\alpha(x) = f_{\alpha\beta}(p_\beta(x))$ for all $\alpha, \beta \in I$ such that $\alpha \leq \beta$. Take $r \in R$ and $x, y \in Q$. Put $rx = (rp_\alpha(x))_{\alpha \in I}$, $x + y = (p_\alpha(x) + p_\alpha(y))_{\alpha \in I}$ and $xy = (p_\alpha(x)p_\alpha(y))_{\alpha \in I}$. From general results concerning inverse limits of magmas with operations (see [1, §10]) we know that this defines a well defined R-algebra structure on Q making it an inverse limit in the category of R-algebras. Take $g, h \in G$ and let Q'_g denote all $x \in Q$ such that for each $\alpha \in I$, the relation $p_\alpha(x) \in (A_\alpha)_g$ holds. Put $Q' = \oplus_{g \in G} Q'_g$. Now we show that Q' is noncommutatively graded. Take $x \in Q'_g$ and $y \in Q'_h$. Take $\alpha \in I$. Then $p_\alpha(xy) = p_\alpha(x)p_\alpha(y) \in (A_\alpha)_g(A_\alpha)_h \subseteq (A_\alpha)_{gh} + (A_\alpha)_{hg}$. Therefore $Q'_g Q'_h \subseteq Q'_{gh} + Q'_{hg}$. Now we show that Q' is an inverse limit in the category G-NCGA. For each $\alpha \in I$, let f_α denote the map of noncommutatively graded algebras $Q' \to A_\alpha$ defined by restriction of p_α and suppose that $u_\alpha : F \to A_\alpha$ is a graded map for some G-noncommutatively graded R-algebra F into A_α such that $f_{\alpha\beta} \circ u_\beta = u_\alpha$ whenever $\alpha \leq \beta$. Then there exists a unique graded map $u : F \to Q'$ such that $u_\alpha = f_\alpha \circ u$ for all $\alpha \in I$. First we show uniqueness of u. Take $y \in F_g$. From the relations $u_\alpha = f_\alpha \circ u$, for $\alpha \in I$, it follows that $u(y) = (u_\alpha(y))_{\alpha \in I}$. Next we show that u is a well defined morphism in G-NCGA. To this end, suppose that $g \in G$,

$y \in F_g$ and $\alpha \leq \beta$. Then $(f_{\alpha\beta} \circ p_\beta)(u(y)) = (f_{\alpha\beta} \circ u_\beta)(y) = u_\alpha(y) = p_\alpha(u(y))$. Therefore $u(y) \in Q$. Since $y \in F_g$ we get that $u_\alpha(y) \in (A_\alpha)_g$ and thus $u(y) \in Q'_g$. Since each u_α is an R-algebra homomorphism, the same holds for u. □

Proposition 15.4 *Direct limits exist in G-*NCGA.

Proof Suppose that we are given a directed set $(I, \leq)$ and a family of noncommutatively G-graded R-algebras $(A_\alpha)_{\alpha\in I}$. For all $\alpha, \beta \in I$ with $\alpha \leq \beta$ let $f_{\beta\alpha} : A_\alpha \to A_\beta$ be a morphism in G-NCGA. Suppose that the the morphisms $f_{\beta\alpha}$ form a direct system, that is, that the following conditions hold:

- the relations $\alpha \leq \beta \leq \gamma$ imply that $f_{\gamma\alpha} = f_{\gamma\beta} \circ f_{\beta\alpha}$;
- for every $\alpha \in I$, the equality $f_{\alpha\alpha} = \mathrm{id}_{A_\alpha}$ holds.

Take $g \in G$. Let D_g denote the direct sum of the sets $((A_\alpha)_g)_{\alpha\in I}$. We will identify each $(A_\alpha)_g$ with its image in D_g. For each $x \in D_g$ let $\lambda_g(x)$ denote the unique element in I such that $x \in (A_{\lambda_g(x)})_g$. Define an equivalence relation $\sim_g$ on D_g by saying that if $x, y \in D_g$, then $x \sim_g y$ whenever there is $\gamma \in I$ with $\lambda_g(x) \leq \gamma$, $\lambda_g(y) \leq \gamma$ and $f_{\gamma\lambda_g(x)}(x) = f_{\gamma\lambda_g(y)}(y)$. Put $C_g = D_g/\sim_g$. Denote by $(f_\alpha)_g$ the restriction to $(A_\alpha)_g$ of the canonical mapping f_g of D_g onto C_g. Denote by $(f_{\beta\alpha})_g$ the restriction of $f_{\beta\alpha}$ to $(A_\alpha)_g$. Then it follows that $(f_\beta)_g \circ (f_{\beta\alpha})_g = (f_\alpha)_g$ for $\alpha \leq \beta$. Then C_g is the direct limit of the R-modules $(A_\alpha)_g$ with a well defined R-module structure defined as follows. Take $r \in R$ and $x, y \in C_g$. There is $\alpha \in I$ and $x_\alpha, y_\alpha \in D_g$ such that $x = (f_\alpha)_g(x_\alpha)$ and $y = (f_\alpha)_g(y_\alpha)$. Put $rx = (f_\alpha)_g(rx_\alpha)$ and $x+y = (f_\alpha)_g(x_\alpha + y_\alpha)$ (for details, see [1, Sect. 10]). Put $C = \oplus_{g\in G} C_g$ (external direct sum of R-modules). Now we will define a multiplication on C. By additivity it is enough to define this on graded components. Take $g, h \in G$, $x \in C_g$ and $y \in C_h$. There is $\alpha \in I$, $x_\alpha \in D_g$ and $y_\alpha \in D_h$ such that $x = (f_\alpha)_g(x_\alpha)$ and $y = (f_\alpha)_h(y_\alpha)$.

Case 1: $gh = hg$. Then put $xy = (f_\alpha)_{gh}(x_\alpha y_\alpha)$.

Case 2: $gh \neq hg$. Then put $xy = ((f_\alpha)_{gh} \circ p_{gh} + (f_\alpha)_{hg} \circ p_{hg})(x_\alpha y_\alpha)$. Here $p_{gh} : D_{gh} \oplus D_{hg} \to D_{gh}$ and $p_{hg} : D_{gh} \oplus D_{hg} \to D_{hg}$ denote the corresponding projections.

From the definition of this multiplication, it follows that it is R-bilinear and that C is noncommutatively G-graded. Now we show that it is well defined. To this end, suppose that we take $\beta \in I$, $x'_\beta \in D_g$ and $y'_\beta \in D_h$ such that $x = (f_\beta)_g(x'_\beta)$ and $y = (f_\beta)_h(y'_\beta)$. By the definition of the direct limit there exists $\gamma \in I$ such that $\alpha \leq \gamma$, $\beta \leq \gamma$, $x_\gamma := f_{\gamma\alpha}(x_\alpha) = x'_\gamma := f_{\gamma\beta}(x'_\beta)$ and $y_\gamma := f_{\gamma\alpha}(y_\alpha) = y'_\gamma := f_{\gamma\beta}(y'_\beta)$.

Case 1: $gh = hg$. Then

$$(f_\alpha)_{gh}(x_\alpha y_\alpha) = (f_\gamma)_{gh}(f_{\gamma\alpha}(x_\alpha y_\alpha)) = (f_\gamma)_{gh}(f_{\gamma\alpha}(x_\alpha) f_{\gamma\alpha}(y_\alpha))$$

$$= (f_\gamma)_{gh}(x_\gamma y_\gamma) = (f_\gamma)_{gh}(x'_\gamma y'_\gamma) = (f_\gamma)_{gh}(f_{\gamma\beta}(x'_\beta) f_{\gamma\beta}(y'_\beta))$$

$$= (f_\gamma)_{gh}(f_{\gamma\beta}(x'_\beta y'_\beta)) = (f_\beta)_{gh}(x'_\beta y'_\beta).$$

Case 2: $gh \neq hg$. Then

$$((f_\alpha)_{gh} \circ p_{gh} + (f_\alpha)_{hg} \circ p_{hg})(x_\alpha y_\alpha) =$$

$$= (f_\gamma)_{gh}(f_{\gamma\alpha}(p_{gh}(x_\alpha y_\alpha))) + (f_\gamma)_{gh}(f_{\gamma\alpha}(p_{hg}(x_\alpha y_\alpha)))$$

$$= [f_{\gamma\alpha} \text{ graded}] = (f_\gamma)_{gh}(p_{gh}(f_{\gamma\alpha}(x_\alpha y_\alpha))) + (f_\gamma)_{gh}(p_{hg}(f_{\gamma\alpha}(x_\alpha y_\alpha)))$$

$$= (f_\gamma)_{gh}(p_{gh}(f_{\gamma\alpha}(x_\alpha) f_{\gamma\alpha}(y_\alpha))) + (f_\gamma)_{gh}(p_{hg}(f_{\gamma\alpha}(x_\alpha) f_{\gamma\alpha} y_\alpha)))$$

$$= (f_\gamma)_{gh}(p_{gh}(x'_\gamma y'_\gamma)) + (f_\gamma)_{hg}(p_{hg}(x'_\gamma y'_\gamma))$$

$$= (f_\gamma)_{gh}(p_{hg}(f_{\gamma\beta}(x'_\beta) f_{\gamma\beta}(y'_\beta))) + (f_\gamma)_{hg}(p_{hg}(f_{\gamma\beta}(x'_\beta) f_{\gamma\beta}(y'_\beta)))$$

$$= (f_\gamma)_{gh}(p_{hg}(f_{\gamma\beta}(x'_\beta y'_\beta))) + (f_\gamma)_{hg}(p_{hg}(f_{\gamma\beta}(x'_\beta y'_\beta)))$$

$$= [f_{\gamma\beta} \text{ graded}] = (f_\gamma)_{gh}(f_{\gamma\beta}(p_{gh}(x'_\beta y'_\beta))) + (f_\gamma)_{hg}(f_{\gamma\beta}(p_{hg}(x'_\beta y'_\beta)))$$

$$= ((f_\beta)_{gh} \circ p_{gh} + (f_\beta)_{hg} \circ p_{hg})(x'_\beta y'_\beta).$$

Now we show that C is a direct limit of the A_α and the maps $f_{\beta\alpha}$. Suppose that $\alpha, \beta \in I$ satisfy $\alpha \leq \beta$. Define $f_\alpha : A_\alpha \to C$ in the following way. Take $a \in A_\alpha$. Then $a = \sum_{g \in G} a_g$ for some $a_g \in (A_\alpha)_g$ such that $a_g = 0$ for all but finitely many $g \in G$. Put $f_\alpha(a) = \sum_{g \in G} (f_\alpha)_g(a_g)$. From the fact that (see above) $(f_\beta)_g \circ (f_{\beta\alpha})_g = (f_\alpha)_g$, for $g \in G$, it follows that $f_\beta \circ f_{\beta\alpha} = f_\alpha$. Suppose that F is a noncommutatively G-graded algebra and that there are graded algebra maps $u_\alpha : A_\alpha \to F$, for $\alpha \in I$, satisfying $u_\beta \circ f_{\beta\alpha} = u_\alpha$, whenever $\alpha \leq \beta$. Then there is a unique graded algebra map $u : C \to F$ such that $u \circ f_\alpha = u_\alpha$ for $\alpha \in I$. In fact, take $g \in G$ and $c_g \in C_g$. Then there is $a_{\alpha,g} \in A_g$ such that $(f_\alpha)_g(a_{\alpha,g}) = c_g$. Put $u(c_g) = u_\alpha(a_{\alpha,g})$. Then $u \circ f_\alpha = u_\alpha$ and it is clear from these relations that u has to be defined in this way. Thus uniqueness of u follows. Now we show that u is well defined. Suppose that $\alpha \leq \beta$ and put $a_{\beta,g} = f_{\beta\alpha}(a_{\alpha,g})$. Then $u_\beta(a_{\beta,g}) = u_\beta(f_{\beta\alpha}(a_{\alpha,g})) = u(f_\beta(f_{\beta\alpha}(a_{\alpha,g}))) = u(f_\alpha(a_{\alpha,g})) = u_\alpha(a_{\alpha,g})$. □

Definition 15.3 A^{op} is defined to be the algebra A as a left R-module, but with a new product $\cdot_{\text{op}}$ defined by $a \cdot_{\text{op}} b = ba$ for $a, b \in A$.

The next result generalizes [5, Remark 1.2.4].

Proposition 15.5 *Suppose that A is noncommuatively G-graded. Then A^{op} is noncommutatively G-graded with $(A^{\text{op}})_g = A_{g^{-1}}$. Furthermore, the association $A \mapsto A^{\text{op}}$, on objects of G-NCGA, and $f^{\text{op}} = f$, on morphisms of G-NCGA, defines an automorphism of the category G-NCGA.*

Proof Take $g, h \in G$. Then $(A^{\text{op}})_g (A^{\text{op}})_h = A_{g^{-1}} A_{h^{-1}} \subseteq A_{g^{-1}h^{-1}} + A_{h^{-1}g^{-1}} = A_{(hg)^{-1}} + A_{(gh)^{-1}} = (A^{\text{op}})_{hg} + (A^{\text{op}})_{gh}$. Suppose that $f : A \to B$ is a morphism in G-NCGA. Then $f^{\text{op}}((A^{\text{op}})_g) = f(A_{g^{-1}}) \subseteq B_{g^{-1}} = (B^{\text{op}})_g$. Since $(A^{\text{op}})^{\text{op}} = A$, the last statement follows. □

Proposition 15.6 *Let A be noncommutatively G-graded and suppose that $\theta : H \to G$ is a monomorphism of groups. Define the H-graded additive group A_H by $(A_H)_h = A_{\theta(h)}$ for $h \in H$. Then $A_H = \oplus_{h\in H}(A_H)_h$ is a noncommutatively H-graded algebra. The correspondence $A \mapsto A_H$, on objects of G-NCGA, and by restriction $f|_{A_H}$, on morphisms $f : A \to B$ of G-NCGA, defines a functor $(\cdot)_H$: G-NCGA $\to$ H-NCGA.*

Proof Take $h_1, h_2 \in H$. Then

$$(A_H)_{h_1}(A_H)_{h_2} = A_{\theta(h_1)}A_{\theta(h_2)} \subseteq A_{\theta(h_1)\theta(h_2)} + A_{\theta(h_2)\theta(h_1)}$$

$$= A_{\theta(h_1h_2)} + A_{\theta(h_2h_1)} = (A_H)_{h_1h_2} + (A_H)_{h_2h_1}.$$

The last statement is immediate. □

Proposition 15.7 *Let A be noncommutatively H-graded and suppose that $\theta : H \to G$ is a monomorphism of groups. Define the G-graded additive group $\overline{A}$ by $\overline{A}_g = A_{\theta^{-1}(g)}$, if $g \in \theta(H)$, and $\overline{A}_g = \{0\}$, if $g \notin \theta(H)$. Then $\overline{A} = \oplus_{g\in G}\overline{A}_g$ is a noncommutatively G-graded algebra. Given a morphism $f : A \to B$ in H-NCGA, define the morphism $\overline{f} : \overline{A} \to \overline{B}$ in G-NCGA by the additive extension of the relations $\overline{f}(a) = f(a)$, for $a \in \overline{A}_g$ such that $g \in \theta(H)$. The correspondence $A \mapsto \overline{A}$, on objects of H-NCGA, and $f \mapsto \overline{f}$, on morphisms of H-NCGA, defines a functor $\overline{(\cdot)}$: H-NCGA $\to$ G-NCGA.*

Proof Take $g_1, g_2 \in G$. Case 1: $g_1 \notin \theta(H)$ or $g_2 \notin \theta(H)$. Then $\overline{A}_{g_1} = \{0\}$ or $\overline{A}_{g_2} = \{0\}$ so that $\overline{A}_{g_1}\overline{A}_{g_2} = \{0\} \subseteq \overline{A}_{g_1g_2} + \overline{A}_{g_2g_1}$. Case 2: There are $h_1, h_2 \in H$ such that $\theta(h_1) = g_1$ and $\theta(h_2) = g_2$. Then $\overline{A}_{g_1}\overline{A}_{g_2} = A_{\theta^{-1}(g_1)}A_{\theta^{-1}(g_2)} = A_{h_1}A_{h_2} \subseteq A_{h_1h_2} + A_{h_2h_1} = A_{\theta^{-1}(g_1g_2)} + A_{\theta^{-1}(g_2g_1)} = \overline{A}_{g_1g_2} + \overline{A}_{g_2g_1}$.

Take $g \in G$ and $h \in H$ such that $\theta(h) = g$. Suppose that $f : A \to B$ is a morphism in H-NCGA. Then $f(\overline{A}_g) = f(A_h) \subseteq B_h = \overline{B}_g$. If $f' : B \to C$ is another morphism in H-NCGA, then, clearly, $\overline{f' \circ f} = \overline{f'} \circ \overline{f}$ so that $\overline{(\cdot)}$ is a functor H-NCGA $\to$ G-NCGA. □

The next result is a generalization of [5, Proposition 1.2.1].

Proposition 15.8 *If $\theta : H \to G$ is a monomorphism of groups, then $(\overline{(\cdot)}, (\cdot)_H)$ is an adjoint pair of functors.*

Proof Suppose that A is a noncommutatively H-graded algebra and that B is a noncommutatively G-graded algebra. Define a map

$$\Phi_{A,B} : \hom_{G-\text{NCGA}}(\overline{A}, B) \to \hom_{H-\text{NCGA}}(A, B_H)$$

in the following way. Given a morphism $f : \overline{A} \to B$ in G-NCGA, put $\Phi_{A,B}(f) = f_H$. Then $\Phi_{A,B}$ is a bijection. In fact, define

$$\Phi_{A,B}^{-1} : \hom_{H-\text{NCGA}}(A, B_H) \to \hom_{G-\text{NCGA}}(\overline{A}, B)$$

in the following way. Given a morphism $f' : A \to B_H$ in H-NCGA, put $\Phi_{A,B}^{-1}(f') = \overline{f'}$. Given a morphism $p : A' \to A$ in H-NCGA and a morphism $q : B \to B'$ in G-NCGA, then the following diagram is commutative

$$\begin{array}{ccc} \hom_{G-\mathrm{NCGA}}(\overline{A}, B) & \xrightarrow{\Phi_{A,B}} & \hom_{H-\mathrm{NCGA}}(A, B_H) \\ \varepsilon \big\downarrow & & \big\downarrow \delta \\ \hom_{G-\mathrm{NCGA}}(\overline{A'}, B') & \xrightarrow{\Phi_{A',B'}} & \hom_{H-\mathrm{NCGA}}(A', B'_H) \end{array} \tag{15.8}$$

Here ε and δ are defined by $\varepsilon(f) = q \circ f \circ \overline{p}$ and $\delta(g) = \overline{q_H} \circ g \circ p$ for $f \in \hom_{G-\mathrm{NCGA}}(\overline{A}, B)$ and $g \in \hom_{H-\mathrm{NCGA}}(A, B_H)$. Thus $(\overline{(\cdot)}, (\cdot)_H))$ is an adjoint pair of functors. □

The next result generalizes the construction preceding [5, Proposition 1.2.2]

Proposition 15.9 *Let A be noncommutatively G-graded and suppose that $\pi : G \to H$ is an epimorphism of groups. Define the H-graded additive group A^H by $(A^H)_h = \oplus_{g\in\pi^{-1}(h)} A_g$ for $h \in H$. Then $A^H = \oplus_{h\in H}(A_H)_h$ is a noncommutatively H-graded algebra. The correspondence $A \mapsto A^H$, on objects of G-NCGA, and by the identity, on morphisms of G-NCGA, defines a functor $(\cdot)^H : G$-NCGA $\to H$-NCGA.*

Proof Take $h_1, h_2 \in H$. Then

$$(A^H)_{h_1}(A^H)_{h_2} = (\oplus_{g_1\in\pi^{-1}(h_1)} A_{g_1})(\oplus_{g_2\in\pi^{-1}(h_2)} A_{g_2})$$

$$= \sum_{g_1\in\pi^{-1}(h_1),\ g_2\in\pi^{-1}(h_2)} A_{g_1}A_{g_2} \subseteq \sum_{g_1\in\pi^{-1}(h_1),\ g_2\in\pi^{-1}(h_2)} A_{g_1g_2} + A_{g_2g_1}$$

$$\subseteq \sum_{g\in\pi^{-1}(h_1h_2)} A_g + \sum_{g'\in\pi^{-1}(h_2h_1)} A_{g'} = (A^H)_{h_1h_2} + (A^H)_{h_2h_1}.$$

Thus A^H is noncommutatively H-graded. The last part is clear. □

Proposition 15.10 *Suppose that N is a normal subgroup of G and let A be an object in G/N-NCGA$_{\lambda,\mu}$. For each $g \in G$, let $F(A)_g$ be the subset $(\oplus_{n\in N} A_{gn})g$ of the group ring $A[G]$. Put $F(A) = (\oplus_{g\in G} F(A)_g, \lambda, \mu)$. Then $F(A)$ is a noncommutatively G-graded algebra. The correspondence $A \mapsto F(A)$, on objects of G/N-NCGA$_{\lambda,\mu}$, and by $F(f)(a_{gn}g) = f(a_{gn})g$, for $g \in G$, $n \in N$ and $a_{gn} \in A_{gn}$, on morphisms f of G/N-NCGA$_{\lambda,\mu}$, defines a functor $F : G/N$-NCGA$_{\lambda,\mu} \to G$-NCGA$_{\lambda,\mu}$.*

Proof From the proof of [5, Proposition 1.2.2] it follows that $\oplus_{g\in G} F(A)_g$ is G-graded. Therefore, from the discussion in Example 15.1 we get that $F(A)$ is noncommutatively G-graded and F defines a functor G/N-NCGA$_{\lambda,\mu} \to G$-NCGA$_{\lambda,\mu}$. □

Now we will show a generalization of [5, Proposition 1.2.2] making use of the construction in Example 15.1.

Proposition 15.11 *Suppose that N is a normal subgroup of G and consider the canonical epimorphism $\pi : G \to G/N$. Let $(\cdot)^{G/N}$ denote the functor G-NCGA$_{\lambda,\mu}$ $\to H$-NCGA$_{\lambda,\mu}$, obtained from Proposition 15.9 by restriction. Then $((\cdot)^{G/N}, F)$ is an adjoint pair of functors.*

Proof Suppose that A is an object in G-NCGA$_{\lambda,\mu}$ and that B is an object in G/N-NCGA$_{\lambda,\mu}$. Define a map

$$\Psi_{A,B} : \hom_{G/N-\mathrm{NCGA}_{\lambda,\mu}}(A^{G/N}, B) \to \hom_{G-\mathrm{NCGA}_{\lambda,\mu}}(A, F(B))$$

in the following way. Given a morphism $f : A^{G/N} \to B$ in G-NCGA, put $\Psi_{A,B}(f)(a_g) = f(a_g)g$, for $a_g \in A_g$, and extended biadditively. Then $\Psi_{A,B}$ is a bijection (see the proof of [5, Proposition 1.2.2]). Now we show that $\Psi_{A,B}(f)$ respects the multiplication $\bullet$. Take $a_g \in A_g$ and $b_h \in A_h$. Then

$$\Psi_{A,B}(f)(a_g \bullet b_h) = \Psi_{A,B}(f)(\lambda_{g,h}a_g b_h + \mu_{h,g}b_h a_g)$$

$$= f(\lambda_{g,h}a_g b_h)gh + f(\mu_{h,g}b_h a_g)hg = \lambda_{g,h} f(a_g b_h)gh + \mu_{h,g} f(b_h a_g)hg$$

$$= \lambda_{g,h} f(a_g) f(b_h)gh + \mu_{h,g} f(b_h) f(a_g)hg = f(a_g)g \bullet f(b_h)h$$

$$= \Psi_{A,B}(f)(a_g) \bullet \Psi_{A,B}(f)(b_h).$$

It also follows from the G-graded case (see loc. cit.) that the map $\Psi_{A,B}$ is natural in A and B. Therefore, $((\cdot)^{G/N}, F)$ is an adjoint pair of functors. □

It is not clear to the author of the present article whether the following question can be answered in the affirmative.

question 15.1 Does the functor $(\cdot)^{G/N} : G$-NCGA $\to H$-NCGA, obtained from Proposition 15.9, have a right adjoint?

15.3 The Graded Tensor Algebra

In this section, we fix the notation concerning G-graded modules and we state some well known results (see Propositions 15.12–15.14) that will be used in the following section. Throughout this section, M denotes a left R-module. Recall that M is called G-graded if there is a family $\{M_g\}_{g\in G}$ of R-submodules of M such that $M = \oplus_{g\in G} M_g$ as R-modules. The next result follows from well-known properties of direct sums of modules (see e.g. [3, Chap. III]).

Proposition 15.12 *If $\{M^{(i)}\}_{i\in I}$ is a family of G-graded modules, then $\oplus_{i\in I} M^{(i)}$ is a graded module, where, for each $g \in G$, we put $(\oplus_{i\in I} M^{(i)})_g = \oplus_{i\in I} M_g^{(i)}$.*

We will refer to the above grading as the canonical direct sum grading. The next result follows immediately from well-known properties of tensor products of modules (see e.g. [3, Chap. XVI]). All tensors are taken over R. Let N denote the set of non-negative integers. If $n \in N \setminus \{0\}$ and $\{M^{(i)}\}_{i=1}^n$ is a family of graded modules, then we let $\otimes_{i=1}^n M^{(i)}$ denote $M^{(1)} \otimes \cdots \otimes M^{(n)}$. An element in $\otimes_{i=1}^n M^{(i)}$ of the form $m_1 \otimes \cdots \otimes m_n$ will be referred to as a monomial.

Proposition 15.13 *If n is a positive integer and $\{M^{(i)}\}_{i=1}^n$ is a family of graded modules, then $\otimes_{i=1}^n M^{(i)}$ is a graded module, where, for each $g \in G$, $(\otimes_{i=1}^n M^{(i)})_g$ is defined to be the submodule of $\otimes_{i=1}^n M^{(i)}$ generated by all monomials $m_1 \otimes \cdots \otimes m_n$, where, $m_i \in M_{g_i}^{(i)}$, for $i = 1, \ldots, n$, for some $g_i \in G$ with the property that $g_1 \cdots g_n = g$.*

Take $n \in N$ and suppose that M is a G-graded module. If $n = 0$, then put $M^{\otimes n} = R$, where the latter is trivially graded, that is $R_e = R$ and $R_g = \{0\}$, if $g \in G \setminus \{e\}$. If $n \geq 1$, then put $M^{\otimes n} = M \otimes \cdots \otimes M$ (n times) and let $M^{\otimes n}$ be equipped with the grading introduced in Proposition 15.13. We will refer to this as the canonical tensor grading on $M^{\otimes n}$. Recall that the tensor algebra $T(M)$ is defined to be the direct sum $\oplus_{n \in N} M^{\otimes n} = R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots$ as a module. The multiplication in $T(M)$ is indicated by $\otimes$ and is defined on monomials in the following way. Take $m, n \in N$. Take monomials $x \in M^{\otimes m}$ and $y \in M^{\otimes n}$. If $m = 0$ (or $n = 0$), then $x \in R$ (or $y \in R$) and we put $x \otimes y = xy$ (or $x \otimes y = yx$) as elements in the module $M^{\otimes n}$ (or $M^{\otimes m}$). If $m, n \geq 1$, then $x = v_1 \otimes \cdots \otimes v_m$ and $y = w_1 \otimes \cdots \otimes w_n$, for some $v_1, \ldots, v_m, w_1, \ldots, w_n \in M$, and we put $x \otimes y = v_1 \otimes \cdots v_m \otimes w_1 \otimes \cdots \otimes w_n$. Define the structure of a G-graded module on $T(M)$ in the following way. Let $T(M)$ be equipped with the canonical direct sum grading, defined, in turn, by the canonical tensor gradings on $\{M^{\otimes n}\}_{n \in N}$. In other words, for each $g \in G$, put $T(M)_g = \oplus_{n \in N} (M^{\otimes n})_g$. From Propositions 15.12 and 15.13 it follows that $T(M)$ is a graded module. We will refer to this grading as the canonical grading on $T(M)$.

Proposition 15.14 *If M is G-graded module, then $T(M)$ is G-graded as an algebra.*

Proof Take $g, h \in G$, $x \in T(M)_g$ and $y \in T(M)_h$. We wish to show that $x \otimes y \in T(M)_{gh}$. Since this is clear if $m = 0$ or $n = 0$, we only need to consider the case when $m, n \geq 1$. We may assume that x and y are monomials in, respectively, $(M^{\otimes m})_g$ and $(M^{\otimes n})_h$. Therefore there are $g_1, \ldots, g_m, h_1, \ldots, h_n \in G$, $v_i \in M_{g_i}$, for $i = 1, \ldots, m$, and $w_j \in M_{h_j}$, for $j = 1, \ldots, n$, such that $g = g_1 \cdots g_m$, $h = h_1 \cdots h_n$, $x = v_1 \otimes \cdots \otimes v_m$ and $y = w_1 \otimes \cdots \otimes w_n$. Then, since $g_1 g_2 \cdots g_m h_1 h_2 \cdots h_n = gh$, we get that $xy = v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_n \in (M^{\otimes(m+n)})_{gh}$. □

15.4 Noncommutatively Graded Lie Algebras

In this section, we study the particular instance of noncommutatively G-graded Lie algebras (see Definition 15.4), we establish the existence of universal graded enveloping algebras (see Proposition 15.15) and we show a graded version of the Poincaré-Birkhoff-Witt theorem (see Proposition 15.16). For the rest of the article, L denotes a Lie algebra. By this we mean that L is a left R-module which is equipped with an R-bilinear product $[\cdot, \cdot] : L \times L \to L$ such that for all $a, b, c \in L$ the relations $[a, a] = 0$ and $[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$ hold.

Definition 15.4 We say that L is a noncommutatively G-graded Lie algebra if there is a family $\{L_g\}_{g \in G}$ of R-submodules of L such that $L = \oplus_{g \in G} L_g$, as R-modules, and for all $g, h \in G$ the inclusion $[L_g, L_h] \subseteq L_{gh} + L_{hg}$ holds. We say that the category of noncommutatively G-graded Lie algebras, denoted by G-NCGLA, is obtained by taking noncommutatively G-graded Lie algebras as objects and and for the morphisms between such objects L and L' we take the Lie algebra homomorphisms $f : L \to L'$ satisfying $f(L_g) \subseteq L'_g$, for $g \in G$.

Remark 15.3 If we let G-GAA denote the subcategory of G-GA having associative G-graded algebras as objects, then the commutator (15.3) defines a functor

$$\text{Lie} : G\text{-GAA} \to G\text{-NCGLA}.$$

Now we extend the definition of a universal enveloping algebra (see e.g. [2, Chap. V]) to the noncommutatively graded situation.

Definition 15.5 Let L be a noncommutatively G-graded Lie algebra. A pair (U, i), where U is an associative unital G-graded algebra and $i : L \to \text{Lie}(U)$ is a morphism in G-NCGLA, is called a universal G-graded enveloping algebra if the following holds: if A is any associative unital G-graded algebra and $j : L \to \text{Lie}(A)$ is a morphism in G-NCGLA, then there exists a unique morphism $k : \text{Lie}(U) \to \text{Lie}(A)$ in G-NCGLA such that $j = k \circ i$.

Proposition 15.15 *Every noncommutatively G-graded Lie algebra which is free with a homogeneous basis has a universal G-graded enveloping algebra.*

Proof We proceed exactly as in classical ungraded case (see e.g. [2, Chap. V]). Let I be the ideal of $T(L)$ generated by all elements of the form

$$[a, b] - a \otimes b + b \otimes a \tag{15.9}$$

for $a, b \in L$ and put $U = T(L)/I$. Let X denote the set of all elements of the form (15.9) with a and b homogeneous. Then $I = T(L)XT(L)$, so I is a G-graded ideal. Therefore U is a unital associative G-graded algebra. Let $l : L \to T(L)$ denote the inclusion and let $q : T(L) \to U$ denote the quotient map. Let $i : L \to U$ be the composition of the graded maps $l : L \to T(L)$ and $q : T(L) \to U$. Let $B = \{b_v\}_{v \in V}$ be a a homogeneous basis for L. Suppose that A is a unital graded algebra and there is a graded homomorphism $j : L \to \text{Lie}(A)$. By ungraded universality of (U, i) there is a unique homomorphism $k : U \to A$ such that $j = k \circ i$. The map k is defined by $k(1) = 1$ and $k(b_{v_1} \otimes \cdots \otimes b_{v_n}) = j(b_{v_1}) \otimes \cdots \otimes j(b_{v_n})$. Since j is graded, k is also graded. □

Corollary 15.1 *Every noncommutatively G-graded Lie algebra over a field has a universal graded enveloping algebra.*

Proof Suppose that R is a field. For each $g \in G$ choose a basis B_g for L_g. It is clear that $B = \cup_{g \in G} B_g$ is a homogeneous basis for L. The claim now follows directly from Proposition 15.15. □

The following result is a graded analogue of the Poincaré-Birkhoff-Witt theorem.

Proposition 15.16 *Suppose that L is a noncommutatively G-graded Lie algebra which is free with a homogeneous basis $B = \{b_v\}_{v \in V}$ where the set V is equipped with a total order $\leq$. Then the cosets $1+I$ and $b_{v_1} \otimes \cdots \otimes b_{v_n} + I$, where $v_1 < \cdots < v_n$, form a homogeneous basis for U.*

Proof This follows from the ungraded Poincaré-Birkhoff-Witt theorem (see e.g. [2, Chap. V]). □

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