Chapter 14
On the Classification of f-Quandles

Indu Rasika Churchill, Mohamed Elhamdadi and Nicolas Van Kempen

Abstract We use the structural aspects of the f-quandle theory to classify, up to isomorphisms, all f-quandles of order n. The classification is based on an effective algorithm that generate and check all f-quandles for a given order. We also include a pseudocode of the algorithm.

Keywords Quandle · f-quandle · Isomorphism · Structural aspect · Classification

14.1 Introduction

Quandles are algebraic structures whose axioms come from Reidemeister moves in knot theory. They are used to construct representations of braid groups and thus give invariants of knots. In 1982, Joyce [11] and Matveev [13] independently introduced the notion of a quandle. They associated to each knot a quandle that determines the knot up to isotopy and mirror image. Since then quandles and racks have been investigated by topologists in order to construct knot and link invariants, see [1, 2, 6, 7] for more details. Quandles are also of interests to algebraists since these algebraic structures can be investigated on their own right as non-associative algebraic structures. Recently, the notion of f-quandle was introduced in [3, 5] where the identities defining a quandle were twisted by a map. This idea was inspired by the notion of a Hom-algebra structure [10, 12] which is a multiplication on a vector space where the structure is twisted by some map. When the twisting map is the identity, one

I. R. Churchill · N. Van Kempen
Department of Mathematics, State University of New York at Oswego,
Oswego, NY 13126, USA
e-mail: indurasika.churchill@oswego.edu

N. Van Kempen
e-mail: nvankemp@oswego.edu

M. Elhamdadi (✉)
Department of Mathematics, University of South Florida, Tampa, FL 33620, USA
e-mail: emohamed@math.usf.edu

S. Silvestrov et al. (eds.), *Algebraic Structures and Applications*,
Springer Proceedings in Mathematics & Statistics 317,
https://doi.org/10.1007/978-3-030-41850-2_14

recovers the original structure. Hom-algebra structures were introduced in [10] with the goal of studying the deformations of Witt algebra and the deformation of the Virasoro algebra. Since then there has been a growing interests in Hom-structures in different settings (algebras, coalgebras, Hopf algebras, Leibniz algebras, n-ary algebras). In this article, we use the structural aspects of the f-quandle theory to classify f-quandles of low orders up to isomorphisms.

The article is organized as follows. In Sect. 14.2, we review the basics of quandles, f-quandles, some of their properties and give some examples. Section 14.3 contains the main results of the paper that is the classification, up to isomorphism, of f-quandles with cardinality less than seven.

14.2 Basics of Quandles and f-Quandles

In this section we review the basics of racks, quandles, f-racks and f-quandles including definitions, examples and some properties. We will use $\rhd$ to denote the binary operation of a quandle and $*$ to denote the binary operation of an f-quandle throughout the paper. For more details on quandles we refer the interested reader, for example, to the book [7] and article [3] for details on f-quandles.

Let $(X, \rhd)$ be a set with a binary operation. For $x \in X$, the right multiplication by x is the map $R_x : X \to X$ given by $R_x(u) = u \rhd x, \forall u \in X$. Now we give the following definition of a *shelf, rack and quandle*.

Definition 14.1 A *shelf* is a pair $(X, \rhd)$ in which X is a set, $\rhd$ is a binary operation on X such that, for any $x, y, z \in X$, the identity

$$(x \rhd y) \rhd z = (x \rhd z) \rhd (y \rhd z) \tag{14.1}$$

holds. A *rack* is a shelf such that, for any $x, y \in X$, there exists a unique $z \in X$ such that

$$z \rhd y = x. \tag{14.2}$$

A *quandle* is a rack such that, for each $x \in X$, the identity

$$x \rhd x = x \tag{14.3}$$

holds.

The following are some examples of racks and quandles.

Example 14.1 A rack X is *trivial* if $\forall x \in X$, R_x is the identity map.

Example 14.2 For any abelian group G, the operation $x \rhd y = 2y - x$ defines a quandle structure on G called *Takasaki* quandle. In particular, if $G = \mathbb{Z}_n$ (integers modulo n), it is called *dihedral quandle* and denoted by R_n.

Example 14.3 Let X be a module over the ring $\Lambda = \mathbb{Z}_n[t^{\pm 1}, s]/(s^2 - (1-t)s)$. Then X is a rack with operation $x \triangleright y = tx + sy$. If $t + s \neq 1$, then this rack is not a quandle. But if $s = 1 - t$, then this rack becomes a quandle called *Alexander* quandle (also called affine quandle).

Example 14.4 The conjugation $x \triangleright y = yxy^{-1}$ in a group G makes it into a quandle, denoted $Conj(G)$.

Example 14.5 The operation $x \triangleright y = yx^{-1}y$ in a group G makes it into a quandle, denoted $Core(G)$.

Example 14.6 Let G be a group and let ψ be an automorphism of G, then the operation $x \triangleright y = \psi(xy^{-1})y$ defines a quandle structure on G. Furthermore, if H is a subgroup of G such that $\psi(h) = h$, for all $h \in H$, then the operation $Hx \triangleright Hy = H\phi(xy^{-1})y$ gives a quandle structure on G/H.

For each $x \in X$, the left multiplication by x is the map denoted by $L_x : X \to X$ and given by $L_x(y) := x \triangleright y$. A function $f : (X, \triangleright) \to (Y, \diamond)$ is a quandle *homomorphism* if for all $x, y \in X$, $f(x \triangleright y) = f(x) \diamond f(y)$. Given a quandle $(X, \triangleright)$, we will denote by Aut(X) the automorphism group of X. The subgroup of Aut(X), generated by the automorphisms R_x, is called the *inner* automorphism group of X and denoted by $\text{Inn}(X)$. The subgroup of Aut(X), generated by $R_x R_y^{-1}$, for all $x, y \in X$, is called the $transvection$ group of X denoted by $Transv(X)$. It is well known [11] that the $transvection$ group is a normal subgroup of the inner group and the latter group is a normal subgroup of the automorphism group of X. The quotient group $\text{Inn(X)}/\text{Transv}(X)$ is a cyclic group. To every quandle, there is a group associated to it which has a *universal property* and called *enveloping* group of the quandle [7]. The following are some properties and definitions of some quandles.

- A quandle X is *involutive*, or a *kei*, if, $\forall x \in X$, R_x is an involution.
- A quandle is *connected* if $\text{Inn}(X)$ acts transitively on X.
- A quandle is *faithful* if $x \mapsto R_x$ is an injective mapping from X to $\text{Inn}(X)$.
- A *Latin quandle* is a quandle such that for each $x \in X$, the left multiplication L_x by x is a bijection. That is, the multiplication table of the quandle is a Latin square.
- A quandle X is *medial* if $(x \triangleright y) \triangleright (z \triangleright w) = (x \triangleright z) \triangleright (y \triangleright w)$ for all $x, y, z, w \in X$. It is well known that a quandle is medial if and only if its tranvection group is abelian. For this reason, sometimes medial quandles are called *abelian*. For example, every Alexander quandle is medial.
- A quandle X is called $simple$ if the only surjective quandle homomorphisms on X have trivial image or are bijective.

Now, we give some background on f-quandles.

Definition 14.2 ([3, Definition 2.1]) An *f-shelf* is a triple $(X, *, f)$ in which X is a set, $*$ is a binary operation on X, and $f : X \to X$ is a map such that, for any $x, y, z \in X$, the identity

$$(x * y) * f(z) = (x * z) * (y * z) \tag{14.4}$$

holds. An f-*rack* is a f-shelf such that, for any $x, y \in X$, there exists a unique $z \in X$ such that

$$z * y = f(x). \tag{14.5}$$

An f-*quandle* is a f-rack such that, for each $x \in X$, the identity

$$x * x = f(x) \tag{14.6}$$

holds.

Remark 14.1 Using the right translation $R_a : X \to X$ defined as $R_a(x) = x * a$, the identity (14.4) can be written as $R_{f(z)}(x * y) = R_z(x) * R_z(y)$ or $R_{f(z)} R_y = R_{R_z(y)} R_z$ for all $x, y, z \in X$.

Now we give the definition of homomorphism between two f-quandles.

Definition 14.3 Let $(X_1, *_1, f_1)$ and $(X_2, *_2, f_2)$ be two f-racks (resp. f-quandles). A map $\phi : X_1 \to X_2$ is a f-rack (resp. f-quandle) morphism if it satisfies, $\forall a, b \in X_1$ $\phi(a *_1 b) = \phi(a) *_2 \phi(b)$ and $\phi \circ f_1 = f_2 \circ \phi$.

A quandle endomorphism that is bijective is called an isomorphism of the f-quandle.

Remark 14.2 The category of f-quandle is the category whose objects are tuples $(A, *, f)$ which are f-quandles and the morphism are f-quandle morphisms.

Examples of f-quandles include the following.

Example 14.7 Given any set X and map $f : X \to X$, then the operation $x * y = f(x)$ for any $x, y \in X$ gives an f-quandle. We call this a *trivial* f-quandle structure on X.

Example 14.8 For any group G and any group endomorphism f of G, the operation $x * y = f(y)xy^{-1}$ defines an f-quandle structure on G.

Example 14.9 Consider the dihedral quandle R_n, where $n \geq 2$, and let f be an automorphism of R_n. Then f is given by $f(x) = ax + b$, for some invertible element $a \in \mathbb{Z}_n$ and some $b \in \mathbb{Z}_n$ [8]. The binary operation $x * y = f(2y - x) = 2ay - ax + b \pmod{n}$ gives an f-quandle structure called the *f-dihedral quandle*.

Example 14.10 Any $\mathbb{Z}[T^{\pm 1}, S]$-module M is a f-quandle with $x * y = Tx + Sy$, $x, y \in M$ with $TS = ST$ and $f(x) = (S + T)x$, called an *Alexander f-quandle*.

Remark 14.3 Axioms (14.4) and (14.6) of Definition 14.2 give the following equation, $(x * y) * (z * z) = (x * z) * (y * z)$. We note that the two medial terms in this equation are swapped (resembling the mediality condition of a quandle). Note also that the mediality in the general context may not be satisfied for f-quandles. For example one can check that the f-quandle given in Example 14.8 is not medial.

Proposition 14.1 ([3]) *If $(X, *, f)$ is an f-quandle and $y \in X$, then the right multiplication $R_y : X \to X$ given by $R_y(x) = x * y$ is a bijection.*

Definition 14.4 A *f-crossed set* is a f-quandle $(X, *, f)$ such that $\forall x, y \in X$, we have $x * y = f(x)$ whenever $y * x = f(y)$.

Notice that the extra condition in this definition of f-crossed means that, $\forall x, y \in X$, $R_x(y) = f(y)$ is equivalent to $R_y(x) = f(x)$.

The following proposition from [3] gives a key construction providing a new f-quandle from an f-quandle and an f-quandle morphism.

Proposition 14.2 *Let $(X, *, f)$ be a finite f-quandle and $\phi : X \to X$ be an f-quandle morphism: ($\phi(x * y) = \phi(x) * \phi(y)$). Then $(X, *_\phi, f_\phi)$ is an f-quandle with $a *_\phi b = \phi(a * b)$ and $f_\phi(a) = \phi(f(a))$ if and only if ϕ is an automorphism. Moreover, if ϕ is an element of the centralizer of the automorphism group of X, then the associated f-quandle has the same f map.*

We will refer to $(X, *_\phi, f_\phi)$ as a *twist* of $(X, *, f)$.

Remark 14.4 Notice that a quandle (resp. rack, shelf) may be viewed as an f-quandle (resp. f-rack, f-shelf) for which the structure map f is the identity map.

Corollary 14.1 *In the particular case where $f = id_x$, the proposition shows that any usual quandle along with an appropriate morphism gives rise to an f-quandle.*

Example 14.11 Let G be a non-abelian group and f a group automorphism. Then one gets an example of a f-quandle by $x * y = f(y)^{-1} f(x) f(y)$. For example, in the case of the symmetric group on three letters $G = S_3 =< s, t : s^2 = t^3 = e, ts = st^2 >$ and f be a group automorphism that maps $s \to st, t \to t^2$. Then one gets an example of a f-quandle by $x * y = f^{-1}(y) f(x) f(y)$ given by the following table:

$*$	e	s	t	t^2	st	st^2
e	e	e	e	e	t	t^2
s	st	st	st^2	s	st	st
t	t^2	t	t^2	t^2	t^2	e
t^2	t	t^2	t	t	e	t
st	s	st^2	st	st^2	st^2	st^2
st^2	st^2	s	s	st	s	s

For example $s * t = st^2$.

Remark 14.5 If we consider in the previous example the operation defined as $x*y = y^{-1} x f(y)$, then we obtain an isomorphic f-quandle.

Example 14.12 Recall from [8] that any automorphism of the dihedral quandle $\mathbb{Z}_n$ is of the form $f_{a,b}(x) = ax + b$. Using the previous proposition we recover the f-dihedral quandle of Example 14.9.

14.2.1 Enveloping Groups of f-Racks

We define in the following the concept of enveloping groups of f-racks and discuss some functorial properties.

Definition 14.5 Let $(X, *, f)$ be a f-rack. Then there is a natural map ι mapping X to a group, called the enveloping group of f-rack of X, and defined as $G_X = F(X)/ < x * y = f(y)xy^{-1}, x, y \in X >$, where $F(X)$ denotes the free group generated by X.

In the following, we discuss a functoriality property between f-racks and groups, see [9] for the classical case.

Proposition 14.3 ([3]) *Let $(X, *, f)$ be a f-rack and G be a group. Given any f-rack homomorphism $\varphi : X \to G_{conj}$, where G_{conj} is a group together with a f-rack structure along a group homomorphism g, that is the multiplication is defined as $a *_G b = g(b)ab^{-1}$. Then, there exists a unique group homomorphism $\widetilde{\varphi} : G_X \to G$ which makes the following diagram commutative.*

Remark 14.6 The functor $(X, *, f) \to G_X$ is left adjoint to the forgetful functor $G \to G_{conj}$ from the category of groups to that of f-racks. That is,

$$Hom_{groups}(G_X, G) \simeq Hom_{f-racks}(X, G_{conj})$$

by natural isomorphism.

14.3 Classification of f-Quandles up to Isomorphism

To classify f-quandles up to isomorphism, we have created an algorithm that generates and checks all f-quandles for a given order n. Since a basic brute-force method would be too computationally expensive, we added three major optimizations that allowed us to get results for $n < 7$, where n is the cardinality of the f-quandle.

14.3.1 Description of the Algorithm

In this subsection, we describe in full details the algorithm which allows us to classify f-quandles of order less than seven. We represent an f-quandle X of order n by an $n \times n$ matrix (a_{ij}) where the entry $a_{ij} := i * j$, where $*$ is the binary operation on X. Generating all possible $n \times n$ matrices and then checking which are valid f-quandles would have a complexity of $\mathcal{O}(n^{n^2})$. This brute-force method is thus not reasonable when searching for all f-quandles of any order greater than 3.

14.3.1.1 Generating All f-Quandles for a Given Order

The algorithm developed for this first part is based on and extends to f-quandles the one presented in [8]. To generate all valid f-quandles for the given order, we keep a list of uncompleted (partially defined) f-quandles and procedurally fill them with variables until the list is empty. This guarantees that all valid f-quandles for the given order are found. An f-quandle on the uncompleted list is removed either when it is complete and valid (in which case it gets added to the completed list), or when we find that it already violates one or more of the f-quandle rules, in which case we know it will never result in a valid f-quandle.

Algorithm 1 Procedurally generating all valid f-quandles

```
base ← f-quandle with all values set to undefined
incomplete ← empty list
results ← empty list
add base to incomplete
while incomplete is not empty do
  fquandle ← pop first element from incomplete
  if fquandle is fully defined then
    if fquandle is valid then
      if fquandle is not isomorphic to any f-quandle in results then
        add fquandle to results
      end if
    end if
  else
    for all i ∈ X do
      copy ← copy of fquandle
      replace next unknown value of copy by i
      if copy is valid then
        fill values of copy using f-quandle rules
        add copy to incomplete
      end if
    end for
  end if
end while
```

To illustrate the process of filling up undefined values, consider this partially defined f-quandles, where blanks signal unknown values.

$*$	a	b	c
a	c		
b	a		b
c			

We can, using the definition of an f-quandle [3, Definition 2.1]), fill in some undefined values that necessarily have to be equal a defined variable in the given configuration. Here, we can use the property of *column bijectivity*, stating that each column must contain all elements of X. We also use the rule stating that for all $x, y, z \in X$, we have $(x * y) * f(z) = (x * z) * (y * z)$.

Since $a * a = c$ and $b * a = a$, we know that $c * a = b$. Further, we have:

$$\begin{aligned}(b * a) * (c * a) &= (b * c) * f(a)\\ a * b &= b * (a * a)\\ a * b &= b * c\\ a * b &= b\end{aligned}$$

*	a	b	c
a	c	b	
b	a		b
c	b		

Once we cannot logically determine any other unknown variables anymore, we add n new quandles to the uncompleted list, selecting an unknown value and setting it successively to each possible value it can have (i.e. all elements of X). In the situation above, we would add three new f-quandles to the uncompleted list, respectively where $a * c = a$, $a * c = b$, and $a * c = c$. We will later consider these, checking if they are valid and if so filling up values as explained.

This method of procedurally filling f-quandles also allows to quickly eliminate them if at any moment any of the three f-quandle rules does not apply. For example, consider the following partially defined f-quandle.

*	a	b	c
a	c	b	
b		a	
c	b		

We can find here at least one contradiction and thus throw away this partial f-quandle. Since this partial table is already invalid, there exists no way of filling unknown values that would result in a valid f-quandle. In this configuration, we have:

$$\begin{aligned}(a * b) * (a * b) &= (a * a) * f(b)\\ b * b &= c * a\\ b * b &= b\end{aligned}$$

Or, we already know from the table that $b * b = a$. Since we have a contradiction, we throw away this partially defined f-quandle and start working on the next one.

We use one final optimization, applied after finishing to fill up the f-quandle as much as possible. When we are looking to select an unknown value and replace it with all possible elements of X, we only need to consider variables already used in the f-quandle, and one unused variable. For example, say we want to fill the blank mark $a * b$ in the following f-quandle table:

$*$	a	b	c
a	a	–	
b			
c			

Here, we will only add two f-quandles to the uncompleted list, one where $a * b = a$, and one where $a * b = b$. We do not need to consider the one where $a * b = c$, since all f-quandles generated from this path will be isomorphically equivalent to the ones generated with $a * b = b$.

14.3.1.2 Checking for Isomorphisms

While we are generating valid f-quandles for the given order, we compare each of them to every other to filter out any isomorphically-equivalent f-quandles.

More precisely, every time we find a complete and valid f-quandle, we check if it is isomorphically equivalent to any f-quandle we have already found. If so, we discard it since we only need one representative for each isomorphic class. If not, we generate all isomorphisms for the given f-quandle for faster isomorphism check later and add the f-quandle to the completed list.
Recall Definition 3.1 in [3].

Definition 14.6 Let $(X, *, f)$ and $(X_0, *_0, f_0)$ be f-quandles. If there exists an automorphism $\phi : X \to X$ and corresponding twisting of $(X, *, f)$, denoted $(X, *_\phi, f_\phi)$, that is isomorphic to $(X_0, *_0, f_0)$, then $(X, *, f)$ and $(X_0, *_0, f_0)$ are said to be twisted-isomorphically equivalent.

$$\begin{array}{ccc} (X, *, f) & \xrightarrow{\ \phi\ } & (X, *_\phi, f_\phi) \\ \psi \downarrow & & \downarrow \psi \\ (X_0, *_1, f_1) & \xrightarrow[\psi^{-1}\phi\psi]{} & (X_0, *_0, f_0) \end{array}$$

From this it becomes clear that twisted-isomorphisms do define equivalence classes on the space of f-quandles.
This allows us to consider the classification of f-quandles in terms of twisted-isomorphically distinct classes.

We generate all f-quandles isomorphic to a given f-quandle by the Definition 3.1 in [3].

That is, we consider every possible pair of functions (ϕ, ψ), where ϕ is an automorphism and ψ an isomorphism, and save a copy of all f-quandles we obtain by applying the two functions successively on the original f-quandle.

Example 14.13 We include the following Cayley table of a five element f-quandle on the set $\{0, 1, 2, 3, 4\}$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 0 & 2 \\ 3 & 2 & 4 & 1 & 0 \\ 1 & 4 & 0 & 2 & 3 \\ 2 & 0 & 3 & 4 & 1 \end{bmatrix}$$

where the map f is a bijection acting on the set $\{0, 1, 2, 3, 4\}$. It is given by $f = (1\ 3\ 2\ 4)$ in a cycle notation.

14.3.2 Results

In this section, we describe the results obtained from the algorithm of doing computations with f-quandles. The algorithm was implemented using Java. We have successfully found results for orders $n < 7$. Finding results for bigger orders will require more optimizations or a different approach of generating the f-quandles.

The following table lists the number of f-quandles of order less than seven both before and after eliminating isomorphic f-quandles from the results. Note that Sect. 14.3 of the article [3] contains a list of few f-quandles without the isomorphism filtering.

Order	Without isomorphic filtering	With isomorphic filtering
2	4	2
3	24	4
4	288	12
5	2760	23
6	56880	79

References

1. Carter, J.S., Crans, A.S., Elhamdadi, M., Saito, M.: Cohomology of categorical self-distributivity. J. Homotopy Relat. Struct. **3**(1), 13–63 (2008)
2. Carter, J.S., Crans, A., Elhamdadi, M., Graña, M., Saito, M.: Cocycle knot invariants from quandle modules and generalized quandle homology. Osaka J. Math **42**(3), 499–541 (2005)

3. Churchill, I.R.U., Elhamdadi, M., Green, M., Makhlouf, A.: f-racks, f-quandles, their extensions and cohomolog. J. Algebra Appl. **16**(6) (2017)
4. Churchill, I.R.U., Elhamdadi, M., Green, M., Makhlouf, A.: Ternary and n-ary f-distributive structures. Open Math. **16**, 32–45 (2018)
5. Churchill, I.R.U., Elhamdadi, M., Fernando, N.: The cocycle structure of the Alexander f-quandles on finite fields. J. Algebra Appl. **17**(10) (2018)
6. Elhamdadi, M., Moutuou, E.M.: Foundations of topological racks and quandles. J. Knot Theory Ramif. **25**(3) (2016)
7. Elhamdadi, M., Nelson, S.: Quandles—an introduction to the algebra of knots. In: American Mathematical Society, Providence, RI, vol. 74 (2015)
8. Elhamdadi, M., Macquarrie, J., Restrepo, R.: Automorphism groups of quandles. J. Algebra Appl. 11(1) (2012)
9. Fenn, R.: Rourke, Colin: Racks and links in codimension two. J. Knot Theory Ramif. **1**(4), 343–406 (1992)
10. Hartwig, J.T., Larsson, D., Silvestrov, S.D.: Deformations of Lie algebras using σ-derivations. J. Algebra **295**(2), 314–361 (2006)
11. Joyce, D.: A classifying invariant of knots, the knot quandle. J. Pure Appl. Algebra **23**(1), 37–65 (1982)
12. Makhlouf, A., Silvestrov, S.D.: Hom-algebra structures. J. Gen. Lie Theory Appl. **2**(2), 51–64 (2008)
13. Matveev, S.V.: Distributive groupoids in knot theory. Mat. Sb. (N.S.) **119**(161), 78–88, 160 (1982)