Chapter 13
On Hom-Yetter-Drinfeld Category

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Abstract Let (H, β) be a Hom-Hopf algebra. Recently we introduced the Hom-Yetter-Drinfeld category ${}_H^H\mathcal{YD}$ via Radford biproduct Hom-Hopf algebra, and proved that ${}_H^H\mathcal{YD}$ is a braided tensor category. Let $(H, \beta, \mathrm{R}(\text{or } \sigma))$ be a quasitriangular (or cobraided) Hom-Hopf algebra. In this paper, we prove that the category ${}_H\mathcal{M}$ (or ${}^H\mathcal{M}$) of left (H, β)-Hom-modules comodules) is a braided tensor subcategory of ${}_H^H\mathcal{YD}$. As a generalization of Radford biproduct Hom-Hopf algebra, we derive necessary and sufficient conditions for R-smash product Hom-algebra $(A\natural_R H, \alpha \otimes \beta)$ and T-smash coproduct Hom-coalgebra $(A \diamond_T H, \alpha \otimes \beta)$ to be a Hom-Hopf algebra. At last, two nontrivial examples are given.

Keywords Hom-Hopf algebra · Hom-coalgebra · Hom-modules comodule · Smash product · Braided tensor category

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13.1 Introduction

Hom-structures (Lie algebras, algebras, coalgebras, Hopf algebras) have been intensively investigated in the literature recently. Hom-algebras are generalizations of algebras obtained by a twisting map, which have been introduced for the first time in

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[6] by Makhlouf and Silvestrov. The associativity is replaced by Hom-associativity, Hom-coassociativity for a Hom-coalgebra can be considered in a similar way.

In [9, 12], Yau introduced and characterized the concept of module Hom-algebras as a twisted version of usual module algebras and the dual version (i.e. comodule Hom-coalgebras) was studied by Zhang in [13]. Based on Yau's definition of module Hom-algebras, Ma, Li and Yang in [3] constructed smash product Hom-Hopf algebra $(A\natural H, \alpha \otimes \beta)$ generalizing the Molnar's smash product (see [4]), and gave the cobraided structure (in the sense of Yau's definition in [11]) on $(A\natural H, \alpha \otimes \beta)$, and also considered the case of twist tensor product Hom-Hopf algebra. Makhlouf and Panaite defined and studied a class of Yetter-Drinfeld modules over Hom-bialgebras in [5] via the "twisting principle" introduced by Yau for Hom-algebras and since then extended to various Hom-type algebras. In [2], the authors introduced the notion of Hom-Yetter-Drinfeld category ${}^H_H\mathcal{YD}$ via Radford biproduct Hom-Hopf algebra, and proved that the Hom-Yetter-Drinfeld modules can provide solutions of the Hom-Yang-Baxter equation (in the sense of Yau's definition in [10–12]) and ${}^H_H\mathcal{YD}$ is a braided tensor category.

It is well-known that the category ${}_H\mathcal{M}$ (or ${}^H\mathcal{M}$) of left H-modules (or comodules) is a braided tensor subcategory of ${}^H_H\mathcal{YD}$, where $(H, \mathrm{R}$ (or σ)) is a quasitriangular (or cobraided) Hopf algebra. Radford biproduct plays an important role in the lifting method for the classification of finite dimensional pointed Hopf algebras. In [4], Ma and Wang generalized the Radford biproduct to the twist tensor biproduct.

The main purpose of this article is to consider the above results in the Hom-setting.

This article is organized as follows. In Sect. 13.2, we recall some definitions and results which will be used later. In Sect. 13.3, we construct a subcategory of the Hom-Yetter-Drinfeld category via quasitriangular Hom-Hopf algebra (see Theorem 13.1). On the other hand, we give a second braided tensor structure on ${}^H_H\mathcal{YD}$ (see Theorem 13.3). Yau's results in [10, 11] can be obtained as a corollary (see Corollaries 13.1 and 13.2). In [2], we defined the Hom-Yetter-Drinfeld module by Radford biproduct Hom-Hopf algebra. In Sect. 13.4, we consider a generalized version of Radford's biproduct Hom-Hopf algebra, named twisted tensor biproduct Hom-Hopf algebra (see Theorems 13.6 and 13.7). And two nontrivial examples are given (see Examples 13.1 and 13.2).

13.2 Preliminaries

Throughout this paper, we follow the definitions and terminologies in [1–3, 10, 11, 13], with all algebraic systems supposed to be over the field K. Given a K-space M, we write id_M for the identity map on M.

We now recall some useful definitions throughout this paper. We point out that we will be using in the special contexts we consider a simplified terminology for involved Hom-algebra structures just for convenience of exposition in this article.

Definition 13.1 A **Hom-algebra** Hom-algebra, or more exactly, a unital multiplicative Hom-associative algebra, is a quadruple $(A, \mu, 1_A, \alpha)$ (abbr. (A, α)), where A is a K-linear space, $\mu : A \otimes A \longrightarrow A$ is a K-linear map, $1_A \in A$ and α is an automorphism of A, such that

$$(HA1)\ \alpha(aa') = \alpha(a)\alpha(a');\quad \alpha(1_A) = 1_A,$$
$$(HA2)\ \alpha(a)(a'a'') = (aa')\alpha(a'');\quad a1_A = 1_Aa = \alpha(a)$$

are satisfied for $a, a', a'' \in A$. Here we use the notation $\mu(a \otimes a') = aa'$.

Let (A, α) and (B, β) be two Hom-algebras. Then $(A \otimes B, \alpha \otimes \beta)$ is a Hom-algebra (called **tensor product Hom-algebra**) with the multiplication $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ and unit $1_A \otimes 1_B$.

Definition 13.2 A **Hom-coalgebra** is a quadruple $(C, \Delta, \varepsilon_C, \beta)$ (abbr. (C, β)), where C is a K-linear space, $\Delta : C \longrightarrow C \otimes C$, $\varepsilon_C : C \longrightarrow K$ are K-linear maps, and β is an automorphism of C, such that

$$(HC1)\ \beta(c)_1 \otimes \beta(c)_2 = \beta(c_1) \otimes \beta(c_2);\quad \varepsilon_C \circ \beta = \varepsilon_C$$
$$(HC2)\ \beta(c_1) \otimes c_{21} \otimes c_{22} = c_{11} \otimes c_{12} \otimes \beta(c_2);\quad \varepsilon_C(c_1)c_2 = c_1\varepsilon_C(c_2) = \beta(c)$$

are satisfied for $c \in A$. Here we use the notation $\Delta(c) = c_1 \otimes c_2$ (summation implicitly understood).

Let (C, α) and (D, β) be two Hom-coalgebras. Then $(C \otimes D, \alpha \otimes \beta)$ is a Hom-coalgebra (called **tensor product Hom-coalgebra**) with the comultiplication $\Delta(c \otimes d) = c_1 \otimes d_1 \otimes c_2 \otimes d_2$ and counit $\varepsilon_C \otimes \varepsilon_D$.

Definition 13.3 A **Hom-bialgebra** is a sextuple $(H, \mu, 1_H, \Delta, \varepsilon, \gamma)$ (abbr. (H, γ)), where $(H, \mu, 1_H, \gamma)$ is a Hom-algebra and $(H, \Delta, \varepsilon, \gamma)$ is a Hom-coalgebra, such that Δ and ε are morphisms of Hom-algebras, i.e.

$$\Delta(hh') = \Delta(h)\Delta(h');\quad \Delta(1_H) = 1_H \otimes 1_H,$$

$$\varepsilon(hh') = \varepsilon(h)\varepsilon(h');\quad \varepsilon(1_H) = 1.$$

Furthermore, if there exists a linear map $S : H \longrightarrow H$ such that

$$S(h_1)h_2 = h_1S(h_2) = \varepsilon(h)1_H \text{ and } S(\gamma(h)) = \gamma(S(h)),$$

then we call $(H, \mu, 1_H, \Delta, \varepsilon, \gamma, S)$(abbr. (H, γ, S)) a **Hom-Hopf algebra**.

Let (H, γ) and (H', γ') be two Hom-bialgebras. The linear map $f : H \longrightarrow H'$ is called a **Hom-bialgebra map** if $f \circ \gamma = \gamma' \circ f$ and at the same time f is a bialgebra map in the usual sense.

Definition 13.4 Let (A, β) be a Hom-algebra. A **left** (A, β)**-Hom-module** is a triple $(M, \triangleright, \alpha)$, where M is a linear space, $\triangleright : A \otimes M \longrightarrow M$ is a linear map, and α is an automorphism of M, such that

$$(HM1)\ \alpha(a \triangleright m) = \beta(a) \triangleright \alpha(m),$$
$$(HM2)\ \beta(a) \triangleright (a' \triangleright m) = (aa') \triangleright \alpha(m);\quad 1_A \triangleright m = \alpha(m)$$

are satisfied for $a, a' \in A$ and $m \in M$.

Let $(M, \triangleright_M, \alpha_M)$ and $(N, \triangleright_N, \alpha_N)$ be two left (A, β)-Hom-modules. Then a linear morphism $f : M \longrightarrow N$ is called a **morphism of left** (A, β)**-Hom-modules** if $f(h \triangleright_M m) = h \triangleright_N f(m)$ and $\alpha_M \circ f = f \circ \alpha_N$.

Definition 13.5 Let (H, β) be a Hom-bialgebra and (A, α) a Hom-algebra. If $(A, \triangleright, \alpha)$ is a left (H, β)-Hom-module and for all $h \in H$ and $a, a' \in A$,

$$(HMA1)\ \beta^2(h) \triangleright (aa') = (h_1 \triangleright a)(h_2 \triangleright a'),$$
$$(HMA2)\ h \triangleright 1_A = \varepsilon_H(h) 1_A,$$

then $(A, \triangleright, \alpha)$ is called an (H, β) **-module Hom-algebra**.

Definition 13.6 Let (C, β) be a Hom-coalgebra. A left (C, β)**-Hom-comodule** is a triple (M, ρ, α), where M is a linear space, $\rho : M \longrightarrow C \otimes M$ (write $\rho(m) = m_{-1} \otimes m_0,\ \forall m \in M$) is a linear map, and α is an automorphism of M, such that

$$(HCM1)\ \alpha(m)_{-1} \otimes \alpha(m)_0 = \beta(m_{-1}) \otimes \alpha(m_0),$$
$$(HCM2)\ \beta(m_{-1}) \otimes m_{0-1} \otimes m_{00} = m_{-11} \otimes m_{-12} \otimes \alpha(m_0);\quad \varepsilon_C(m_{-1}) m_0 = \alpha(m)$$

are satisfied for all $m \in M$.

Let (M, ρ^M, α_M) and (N, ρ^N, α_N) be two left (C, β)-Hom-comodules. Then a linear map $f : M \longrightarrow N$ is called a **morphism of left** (C, β)**-Hom-comodules** if $f(m)_{-1} \otimes f(m)_0 = m_{-1} \otimes f(m_0)$ and $\alpha_M \circ f = f \circ \alpha_N$.

Definition 13.7 Let (H, β) be a Hom-bialgebra and (C, α) a Hom-coalgebra. If (C, ρ, α) is a left (H, β)-Hom-comodule and for all $c \in C$,

$$(HCMC1)\ \beta^2(c_{-1}) \otimes c_{01} \otimes c_{02} = c_{1-1} c_{2-1} \otimes c_{10} \otimes c_{20},$$
$$(HCMC2)\ c_{-1} \varepsilon_C(c_0) = 1_H \varepsilon_C(c),$$

then (C, ρ, α) is called an (H, β)**-comodule Hom-coalgebra**.

Definition 13.8 Let (H, β) be a Hom-bialgebra and (C, α) a Hom-coalgebra. If $(C, \triangleright, \alpha)$ is a left (H, β)-Hom-module and for all $h \in H$ and $c \in A$,

$$(HMC1)\ m(h \triangleright c)_1 \otimes (h \triangleright c)_2 = (h_1 \triangleright c_1) \otimes (h_2 \triangleright c_2),$$
$$(HMC2)\ \varepsilon_C(h \triangleright c) = \varepsilon_H(h) \varepsilon_C(c),$$

then $(C, \triangleright, \alpha)$ is called an (H, β)**-module Hom-coalgebra**.

Definition 13.9 Let (H, β) be a Hom-bialgebra and (A, α) a Hom-algebra. If (A, ρ, α) is a left (H, β)-Hom-comodule and for all $a, a' \in A$,

$$(HCMA1)\ \rho(aa') = a_{-1}a'_{-1} \otimes a_0 a'_0,$$
$$(HCMA2)\ \rho(1_A) = 1_H \otimes 1_A,$$

then (A, ρ, α) is called an (H, β)**-comodule Hom-algebra**.

Definition 13.10 Let (H, β) be a Hom-bialgebra and $(A, \triangleright, \alpha)$ an (H, β)-module Hom-algebra. Then $(A\natural H, \alpha \otimes \beta)$ ($A\natural H = A \otimes H$ as a linear space) with the multiplication

$$(a \otimes h)(a' \otimes h') = a(h_1 \triangleright \alpha^{-1}(a')) \otimes \beta^{-1}(h_2)h',$$

where $a, a' \in A$, $h, h' \in H$, and unit $1_A \otimes 1_H$ is a Hom-algebra, we call it **smash product Hom-algebra** denoted by $(A\natural H, \alpha \otimes \beta)$.

Definition 13.11 Let (H, β) be a Hom-bialgebra, $(M, \triangleright_M, \alpha_M)$ a left (H, β)-module with action $\triangleright_M : H \otimes M \longrightarrow M, h \otimes m \mapsto h \triangleright_M m$ and (M, ρ^M, α_M) a left (H, β)-comodule with coaction $\rho^M : M \longrightarrow H \otimes M, m \mapsto m_{-1} \otimes m_0$. Then we call $(M, \triangleright_M, \rho^M, \alpha_M)$ a **(left-left) Hom-Yetter-Drinfeld module over** (H, β) if the following condition holds:

$$(HYD)\quad h_1\beta(m_{-1}) \otimes (\beta^3(h_2) \triangleright_M m_0) = (\beta^2(h_1) \triangleright_M m)_{-1}h_2 \otimes (\beta^2(h_1) \triangleright_M m)_0,$$

where $h \in H$ and $m \in M$.

Definition 13.12 Let $(A, \mu_A, 1_A, \alpha)$ and $(H, \mu_H, 1_H, \beta)$ be two Hom-algebras, $R : H \otimes A \longrightarrow A \otimes H$ a linear map such that for all $a \in A, h \in H$,

$$(R)\qquad \alpha(a)_R \otimes \beta(h)_R = \alpha(a_R) \otimes \beta(h_R).$$

Then $(A\natural_R H, \alpha \otimes \beta)$ ($A\natural_R H = A \otimes H$ as a linear space) with the multiplication

$$(a \otimes h)(b \otimes g) = a\alpha^{-1}(b)_R \otimes \beta^{-1}(h_R)g,$$

where $a, b \in A, h, g \in H$, and unit $1_A \otimes 1_H$ becomes a Hom-algebra if and only if the following conditions hold:

$$(RS1)\ a_R \otimes 1_{BR} = \alpha(a) \otimes 1_H;\ \ 1_{AR} \otimes h_R = 1_A \otimes \beta(h),$$
$$(RS2)\ \alpha(a)_R \otimes (hg)_R = a_{Rr} \otimes \beta^{-1}(\beta(h)_r)g_R,$$
$$(RS3)\ \alpha((ab)_R) \otimes \beta(h)_R = \alpha(a_R)\alpha(b)_r \otimes h_{Rr},$$

where $a, b \in A, h, g \in H$. We call this Hom-algebra R**-smash product Hom-algebra** and denote it by $(A\natural_R H, \alpha \otimes \beta)$.

Definition 13.13 Let $(C, \Delta_C, \varepsilon_C, \alpha)$ and $(H, \Delta_H, \varepsilon_H, \beta)$ be two Hom-coalgebras, $T: C \otimes H \longrightarrow H \otimes C$ (write $T(c \otimes h) = h_T \otimes c_T, \forall c \in C, h \in H$) a linear map such that for all $c \in C, h \in H$,

$$(T) \qquad \alpha(c)_T \otimes \beta(h)_T = \alpha(c_T) \otimes \beta(h_T).$$

Then $(C \diamond_T H, \alpha \otimes \beta)$ ($C \diamond_T H = C \otimes H$ as a linear space) with the comultiplication

$$\Delta_{C \diamond_T H}(c \otimes h) = c_1 \otimes \beta^{-1}(h_1)_T \otimes \alpha^{-1}(c_{2T}) \otimes h_2,$$

and counit $\varepsilon_C \otimes \varepsilon_H$ becomes a Hom-coalgebra if and only if the following conditions hold:

$$(TS1)\ \varepsilon_H(h_T)c_T = \varepsilon_H(h)\alpha(c);\quad h_T\varepsilon_C(c_T) = \beta(h)\varepsilon_C(c),$$
$$(TS2)\ h_{T1} \otimes h_{T2} \otimes \alpha(c_T) = \beta(\beta^{-1}(h_1)_T) \otimes h_{2t} \otimes c_{Tt},$$
$$(TS3)\ \beta(h_T) \otimes \alpha(c)_{T1} \otimes \alpha(c)_{T2} = h_{Tt} \otimes \alpha(c_1)_t \otimes \alpha(c_{2T}),$$

where $c \in C, h \in H$ and t is a copy of T. We call this Hom-coalgebra **T-smash coproduct Hom-coalgebra** and denote it by $(C \diamond_T H, \alpha \otimes \beta)$.

Definition 13.14 A **quasitriangular Hom-Hopf algebra** is a octuple $(H, \mu, 1_H, \Delta, \varepsilon, S, \beta, \mathrm{R})$ (abbr.(H, β, R)) in which $(H, \mu, 1_H, \Delta, \varepsilon, S, \beta)$ is a Hom-Hopf algebra and $\mathrm{R} = \mathrm{R}^1 \otimes \mathrm{R}^2 \in H \otimes H$, satisfying the following axioms (for all $h \in H$ and $\mathrm{R} = \mathrm{r}$):

$$(QHA1)\ \varepsilon(\mathrm{R}^1)\mathrm{R}^2 = \mathrm{R}^1\varepsilon(\mathrm{R}^2) = 1,$$
$$(QHA2)\ \mathrm{R}^1{}_1 \otimes \mathrm{R}^1{}_2 \otimes \beta(\mathrm{R}^2) = \beta(\mathrm{R}^1) \otimes \beta(\mathrm{r}^1) \otimes \mathrm{R}^2\mathrm{r}^2,$$
$$(QHA3)\ \beta(\mathrm{R}^1) \otimes \mathrm{R}^2{}_1 \otimes \mathrm{R}^2{}_2 = \mathrm{R}^1\mathrm{r}^1 \otimes \beta(\mathrm{r}^2) \otimes \beta(\mathrm{R}^2),$$
$$(QHA4)\ h_2\mathrm{R}^1 \otimes h_1\mathrm{R}^2 = \mathrm{R}^1 h_1 \otimes \mathrm{R}^2 h_2,$$
$$(QHA5)\ \beta(\mathrm{R}^1) \otimes \beta(\mathrm{R}^2) = \mathrm{R}^1 \otimes \mathrm{R}^2.$$

Definition 13.15 A **cobraided Hom-Hopf algebra** is a octuple $(H, \mu, 1_H, \Delta, \varepsilon, S, \beta, \sigma)$ (abbr.(H, β, σ)) in which $(H, \mu, 1_H, \Delta, \varepsilon, S, \beta)$ is a Hom-Hopf algebra and σ is a bilinear form on H (i.e., $\sigma \in Hom(H \otimes H, K)$), satisfying the following axioms (for all $h, g, l \in H$):

$$(CHA1)\ \sigma(h, 1_H) = \sigma(1_H, h) = \varepsilon(h),$$
$$(CHA2)\ \sigma(hg, \beta(l)) = \sigma(\beta(h), l_1)\sigma(\beta(g), l_2),$$
$$(CHA3)\ \sigma(\beta(h), gl) = \sigma(h_1, \beta(l))\sigma(h_2, \beta(g)),$$
$$(CHA4)\ \sigma(h_1, g_1)h_2g_2 = g_1h_1\sigma(h_2, g_2),$$
$$(CHA5)\ \sigma(\beta(h), \beta(g)) = \sigma(h, g).$$

13.3 A Class of Braided Tensor Category

In this section, we construct a subcategory of the Hom-Yetter-Drinfeld category. On the other hand, we give a second braided tensor structure on ${}^H_H\mathcal{YD}$. Yau's results in [10, 11] can be obtained as a corollary.

First we recall the structure of Hom-Yetter-Drinfeld category in [2].

Proposition 13.1 ([2]) *Let (H, β) be a Hom-bialgebra. Then the Hom-Yetter-Drinfeld category ${}^H_H\mathcal{YD}$ is a braided tensor category, with tensor product defined by*

$$\triangleright_{M\otimes N} : H \otimes M \otimes N \longrightarrow M \otimes N, h \otimes m \otimes n \mapsto (h_1 \triangleright_M m) \otimes (h_2 \triangleright_N n),$$

and

$$\rho^{M\otimes N} : M \otimes N \longrightarrow H \otimes M \otimes N, m \otimes n \mapsto \beta^{-2}(m_{-1}n_{-1}) \otimes m_0 \otimes n_0,$$

where $h \in H$, $m \in M$ and $n \in N$, associativity constraints defined by

$$a_{M,N,P} : (M \otimes N) \otimes P \longrightarrow M \otimes (N \otimes P), \ \ (m \otimes n) \otimes p \mapsto \alpha_M^{-1}(m) \otimes (n \otimes \alpha_P(p)),$$

where $m \in M$, $n \in N$ and $p \in P$, the braiding defined by

$$c_{M,N} : M \otimes N \longrightarrow N \otimes M, \ \ m \otimes n \mapsto (\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n)) \otimes \alpha_M^{-1}(m_0),$$

where $m \in M$ and $n \in N$ and the unit (K, id_K).

Proposition 13.2 *Let (H, β, R) be a quasitriangular Hom-Hopf algebra and (M, α_M) a left (H, β)-Hom-module with action $\bar{\triangleright}_M : H \otimes M \longrightarrow M, h \otimes m \mapsto h\bar{\triangleright}_M m$. Define the linear map*

$$\bar{\rho}^M : M \longrightarrow H \otimes M, m \mapsto \beta^{-3}(\mathrm{R}^2) \otimes (\mathrm{R}^1\bar{\triangleright}_M m),$$

Then $(M, \bar{\triangleright}_M, \bar{\rho}^M, \alpha_M)$ is a Hom-Yetter-Drinfeld module over (H, β).

Proof The condition $(HCM1)$ is easy to be proved by $(QHA5)$ and $(HAM1)$. We check $(HCM2)$ as follows.

$$\begin{aligned}
\text{LHS} &= \beta^{-2}(\mathrm{R}^2) \otimes \beta^{-3}(\mathrm{r}^2) \otimes \mathrm{r}^1\bar{\triangleright}_M(\mathrm{R}^1\bar{\triangleright}_M m) \\
&\overset{(QHA5)}{=} \beta^{-2}(\mathrm{R}^2) \otimes \beta^{-2}(\mathrm{r}^2) \otimes \beta(\mathrm{r}^1)\bar{\triangleright}_M(\mathrm{R}^1\bar{\triangleright}_M m) \\
&\overset{(HM2)}{=} \beta^{-2}(\mathrm{R}^2) \otimes \beta^{-2}(\mathrm{r}^2) \otimes (\mathrm{r}^1\mathrm{R}^1)\bar{\triangleright}_M m) \\
&\overset{(QHA3)}{=} \beta^{-3}(\mathrm{R}^2{}_1) \otimes \beta^{-3}(\mathrm{R}^2{}_2) \otimes (\beta(\mathrm{R}^1)\bar{\triangleright}_M\alpha_M(m)) \\
&\overset{(HC1)(HM1)}{=} \beta^{-3}(\mathrm{R}^2)_1 \otimes \beta^{-3}(\mathrm{R}^2)_2 \otimes \alpha_M(\mathrm{R}^1\bar{\triangleright}_M m) = \text{RHS},
\end{aligned}$$

and it is obvious that $\varepsilon_H(m_{-1})m_0 = \alpha_M(m)$ by $(QHA1)$, $(HC1)$ and $(HM1)$. Thus $(M, \bar{\rho}^M, \alpha_M)$ is a (H, β)-Hom-comodule.

Next we check that the condition (HYD) holds.

$$\begin{aligned}
\text{LHS} &= h_1\beta^{-2}(\mathrm{R}^2) \otimes (\beta^3(h_2)\bar{\triangleright}_M(\mathrm{R}^1\bar{\triangleright}_M m))\\
&\overset{(HM2)}{=} h_1\beta^{-2}(\mathrm{R}^2) \otimes ((\beta^2(h_2)\mathrm{R}^1)\bar{\triangleright}_M\alpha_M(m))\\
&\overset{(HA1)(HC1)}{=} \beta^{-2}(\beta^2(h)_1\mathrm{R}^2) \otimes ((\beta^2(h)_2\mathrm{R}^1)\bar{\triangleright}_M\alpha_M(m))\\
&\overset{(QHA4)}{=} \beta^{-2}(\mathrm{R}^2\beta^2(h)_2) \otimes ((\mathrm{R}^1\beta^2(h)_1)\bar{\triangleright}_M\alpha_M(m))\\
&\overset{(HA1)(HC1)}{=} \beta^{-2}(\mathrm{R}^2)h_2 \otimes ((\mathrm{R}^1\beta^2(h_1))\bar{\triangleright}_M\alpha_M(m))\\
&\overset{(QHA5)}{=} \beta^{-3}(\mathrm{R}^2)h_2 \otimes ((\beta(\mathrm{R}^1)\beta^2(h_1))\bar{\triangleright}_M\alpha_M(m))\\
&\overset{(HM2)}{=} \beta^{-3}(\mathrm{R}^2)h_2 \otimes (\mathrm{R}^1\bar{\triangleright}_M(\beta^2(h_1)\bar{\triangleright}_M m)) = \text{RHS},
\end{aligned}$$

finishing the proof. □

Proposition 13.3 *Let (H, β, R) be a quasitriangular Hom-Hopf algebra, $(M, \bar{\triangleright}_M, \bar{\rho}^M, \alpha_M)$ and $(N, \bar{\triangleright}_N, \bar{\rho}^N, \alpha_N)$ two Hom-Yetter-Drinfeld module over (H, β) with the structure defined in Proposition 13.2. We regard $(M \otimes N, \bar{\triangleright}_{M\otimes N}, \alpha_M \otimes \alpha_N)$ as a left (H, β)-Hom-module via the standard action*

$$h\bar{\triangleright}_{M\otimes N}(m \otimes n) = (h_1\bar{\triangleright}_M m) \otimes (h_2\bar{\triangleright}_N n)$$

and we regard $(M \otimes N, \bar{\triangleright}_{M\otimes N}, \bar{\rho}^{M\otimes N}, \alpha_M \otimes \alpha_N)$ as a Hom-Yetter-Drinfeld module over (H, β) with the structure defined in Proposition 13.2. Then this Hom-Yetter-Drinfeld $(M \otimes N, \bar{\triangleright}_{M\otimes N}, \bar{\rho}^{M\otimes N}, \alpha_M \otimes \alpha_N)$ coincides with the Hom-Yetter-Drinfeld module defined in Proposition 13.1.

Proof We only need to check that the two comodule structures on $M \otimes N$ coincide, i.e., for all $m \in M$ and $n \in N$,

$$\beta^{-2}(m_{-1}n_{-1}) \otimes (m_0 \otimes n_0) = \beta^{-3}(\mathrm{R}^2) \otimes (\mathrm{R}^1\bar{\triangleright}_{M\otimes N}(m \otimes n)).$$

While

$$\begin{aligned}
\text{LHS} &= \beta^{-2}(\beta^{-3}(\mathrm{R}^2)\beta^{-3}(\mathrm{r}^2)) \otimes ((\mathrm{R}^1\bar{\triangleright}_M m) \otimes (\mathrm{r}^1\bar{\triangleright}_N n))\\
&\overset{(HA1)(QHA5)}{=} \beta^{-4}(\mathrm{R}^2\mathrm{r}^2) \otimes ((\beta(\mathrm{R}^1)\bar{\triangleright}_M m) \otimes (\beta(\mathrm{r}^1)\bar{\triangleright}_N n))\\
&\overset{(QHA2)}{=} \beta^{-3}(\mathrm{R}^2) \otimes ((\mathrm{R}^1{}_1\bar{\triangleright}_M m) \otimes (\mathrm{R}^1{}_2\bar{\triangleright}_N n)) = \text{RHS},
\end{aligned}$$

finishing the proof. □

Proposition 13.4 ([12]) *Let (H, β) be a Hom-bialgebra. If $(M, \bar{\triangleright}_M, \alpha_M)$ and $(N, \bar{\triangleright}_N, \alpha_N)$ are two (H, β)-Hom-modules, then $(M \otimes N, \bar{\triangleright}_{M\otimes N}, \alpha_M \otimes \alpha_N)$ is a (H, β)-Hom-module with the action defined by*

$$\bar{\triangleright}_{M\otimes N} : H \otimes (M \otimes N) \longrightarrow M \otimes N, \quad h\bar{\triangleright}_{M\otimes N}(m \otimes n) = (h_1\bar{\triangleright}_M m) \otimes (h_2\bar{\triangleright}_N n).$$

By Propositions 13.1–13.4, we have

Theorem 13.1 *Let (H, β, R) be a quasitriangular Hom-Hopf algebra. Denote by ${}_H\mathcal{M}$ the category whose objects are left (H, β)-Hom-modules $(M, \bar{\triangleright}_M, \alpha_M)$ and morphisms are morphisms of left-(H, β)-Hom-modules. Then ${}_H\mathcal{M}$ is a braided tensor subcategory of ${}_H^H\mathcal{YD}$, with tensor product defined as in Proposition 13.4, associativity constraints defined by the formula $a_{M,N,P} : (M \otimes N) \otimes P \longrightarrow M \otimes (N \otimes P)$, $(m \otimes n) \otimes p \mapsto \alpha_M^{-1}(m) \otimes (n \otimes \alpha_P(p))$, where $m \in M$, $n \in N$ and $p \in P$, the braiding defined by $c_{M,N} : M \otimes N \longrightarrow N \otimes M$, $m \otimes n \mapsto \alpha_N^{-1}(\mathrm{R}^2 \bar{\triangleright}_N n) \otimes \alpha_M^{-1}(\mathrm{R}^1 \bar{\triangleright}_M n)$, where $m \in M$ and $n \in N$ and the unit (K, id_K).*

Remark 13.1 Let $m_{-1} \otimes m_0 = \beta^{-3}(\mathrm{R}^2) \otimes (\mathrm{R}^1 \bar{\triangleright}_M m)$ in Proposition 13.1, we can get the the braiding in Theorem 13.1.

Proposition 13.5 ([2]) *Let (H, β) be a Hom-bialgebra and $(M, \triangleright_M, \rho^M, \alpha_M)$, $(N, \triangleright_N, \rho^N, \alpha_N) \in_H^H \mathcal{YD}$. Define the linear map*

$$\tau_{M,N} : M \otimes N \longrightarrow N \otimes M, \quad m \otimes n \mapsto \beta^3(m_{-1}) \triangleright_N n \otimes m_0,$$

where $m \in M$ and $n \in N$. Then, we have $\tau_{M,N} \circ (\alpha_M \otimes \alpha_N) = (\alpha_N \otimes \alpha_M) \circ \tau_{M,N}$ and, if $(P, \triangleright_P, \rho^P, \alpha_P) \in_H^H \mathcal{YD}$, the maps $\tau_{\_,\,\_}$ satisfy the Hom-Yang-Baxter equation:

$$(\alpha_P \otimes \tau_{M,N}) \circ (\tau_{M,P} \otimes \alpha_N) \circ (\alpha_M \otimes \tau_{N,P}) = (\tau_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes \tau_{M,P}) \circ (\tau_{M,N} \otimes \alpha_P).$$

Corollary 13.1 ([10]) *Let (H, β, R) be a quasitriangular Hom-Hopf algebra and $(M, \bar{\triangleright}_M, \alpha_M)$ a left (H, β)-Hom-module. Then the linear map*

$$B : M \otimes M \longrightarrow M \otimes M, \quad B(m \otimes m') = (\mathrm{R}^2 \bar{\triangleright}_M m') \otimes (\mathrm{R}^1 \bar{\triangleright}_M m)$$

is a solution of the Hom-Yang-Baxter equation for $(M, \bar{\triangleright}_M, \alpha_M)$.

Proof By Theorem 13.1, and let $m_{-1} \otimes m_0 = \beta^{-3}(\mathrm{R}^2) \otimes (\mathrm{R}^1 \bar{\triangleright}_M m)$ in Proposition 13.5, we can obtain the result. □

We have seen that Hom-modules over quasitriangular Hom-Hopf algebras become Hom-Yetter-Drinfeld modules. Similarly, Hom-comodules over cobraided Hom-Hopf algebras become Hom-Yetter-Drinfeld modules. In the following, we introduce another braided tensor category structure on Hom-Yetter-Drinfeld category.

Similar to [2, Lemma 4.4], we have

Proposition 13.6 *With notations as above. Let (H, β) be a Hom-bialgebra and $(M, \bullet_M, \psi^M, \alpha_M)$, $(N, \bullet_N, \psi^N, \alpha_N) \in_H^H \mathcal{YD}$. Define the linear maps*

$$\bullet_{M \otimes N} : H \otimes M \otimes N \longrightarrow M \otimes N, h \otimes (m \otimes n) \mapsto (\beta^{-2}(h_1) \bullet_M m) \otimes (\beta^{-2}(h_2) \bullet_N n),$$

and

$$\psi^{M\otimes N}: M\otimes N \longrightarrow H\otimes M\otimes N, m\otimes n\mapsto m_{-1}n_{-1}\otimes m_0\otimes n_0,$$

where $h\in H$, $m\in M$ and $n\in N$. Then $(M\otimes N, \bullet_{M\otimes N}, \psi^{M\otimes N}, \alpha_M\otimes\alpha_N)$ is a Hom-Yetter-Drinfeld module.

Theorem 13.2 *Let (H,β) be a Hom-bialgebra. Then the Hom-Yetter-Drinfeld category ${}^H_H\mathcal{YD}$ is a braided tensor category, with tensor product defined as in Proposition 13.6, associativity constraints defined by*

$$\mathbf{a}_{M,N,P}: (M\otimes N)\otimes P\longrightarrow M\otimes(N\otimes P),\ \ (m\otimes n)\otimes p\mapsto \alpha_M(m)\otimes(n\otimes\alpha_P^{-1}(p)),$$

where $m\in M$, $n\in N$ and $p\in P$, the braiding defined by

$$\mathbf{c}_{M,N}: M\otimes N\longrightarrow N\otimes M,\ \ m\otimes n\mapsto(\beta^2(m_{-1})\bullet_N\alpha_N^{-1}(n))\otimes\alpha_M^{-1}(m_0),$$

where $m\in M$ and $n\in N$ and the unit (K, id_K).

Proof Same to [2, Theorem 4.7]. □

Similar to Propositions 13.2, 13.3, we have

Proposition 13.7 *Let (H,β,σ) be a cobraided Hom-Hopf algebra.*

(1) Let (M,α_M) a left (H,β)-Hom-comodule with coaction $\bar\psi^M: M\longrightarrow H\otimes M, m\mapsto m_{-1}\otimes m_0$. Define the linear map

$$\bar\bullet_M: H\otimes M\longrightarrow M, h\bar\bullet_M m=\sigma(m_{-1},\beta^{-3}(h))m_0$$

Then $(M,\bar\bullet_M,\bar\psi^M,\alpha_M)$ is a Hom-Yetter-Drinfeld module over (H,β).

(2) Let $(N,\bar\psi^N,\alpha_N)$ be another left (H,β)-Hom-comodule with coaction $\bar\psi^N: M\longrightarrow H\otimes N, n\mapsto n_{-1}\otimes n_0$, regarded as a Hom-Yetter-Drinfeld module over (H,β) with the structure defined as above, via the map $\bar\bullet_N: H\otimes N\longrightarrow N, h\otimes n\mapsto h\bar\bullet_N n=\sigma(n_{-1},\beta^{-3}(h))n_0$. We regard $(M\otimes N,\bar\psi^{M\otimes N},\alpha_M\otimes\alpha_N)$ as a left (H,β)-Hom-comodule via the standard coaction $M\otimes N\longrightarrow H\otimes(M\otimes N)$, $m\otimes n\mapsto m_{-1}n_{-1}\otimes(m_0\otimes n_0)$ and then we get $(M\otimes N,\bar\bullet_{M\otimes N},\bar\psi^{M\otimes N},\alpha_M\otimes\alpha_N)$ as a Hom-Yetter-Drinfeld module defined as above, then this Yetter-Drinfeld module coincides with the Hom-Yetter-Drinfeld module defined in Theorem 13.2.

Proposition 13.8 ([12]) *Let (H,β) be a Hom-bialgebra. If $(M,\bar\psi^M,\alpha_M)$ and $(N,\bar\psi^N,\alpha_N)$ are two (H,β)-Hom-comodules, then $(M\otimes N,\bar\psi^{M\otimes N},\alpha_M\otimes\alpha_N)$ is a (H,β)-Hom-comodule with the coaction defined by*

$$\bar\psi^{M\otimes N}: M\otimes N\longrightarrow H\otimes(M\otimes N),\quad m\otimes n\mapsto m_{-1}n_{-1}\otimes(m_0\otimes n_0).$$

By Propositions 13.6, 13.7, 13.8 and Theorem 13.2, we have

Theorem 13.3 *Let* (H, β, σ) *be a cobraided Hom-Hopf algebra. Denote by* ${}^{H}\mathcal{M}$ *the category whose objects are left* (H, β)*-Hom-comodules* $(M, \bar{\psi}^{M}, \alpha_M)$ *and morphisms are morphisms of left-*(H, β)*-Hom-comodules. Then* ${}^{H}\mathcal{M}$ *is a braided tensor subcategory of* ${}^{H}_{H}\mathcal{YD}$*, with tensor product defined as in Proposition 13.8, associativity constraints defined by the formula* $\bar{\mathbf{a}}_{M,N,P} : (M \otimes N) \otimes P \longrightarrow M \otimes (N \otimes P)$*,* $(m \otimes n) \otimes p \mapsto \alpha_M(m) \otimes (n \otimes \alpha_P^{-1}(p))$*, where* $m \in M$*,* $n \in N$ *and* $p \in P$*, the braiding defined by* $\bar{\mathbf{c}}_{M,N} : M \otimes N \longrightarrow N \otimes M$*,* $m \otimes n \mapsto \sigma(n_{-1}, m_{-1})\alpha_N^{-1}(n_0) \otimes \alpha_M^{-1}(m_0)$*, where* $m \in M$ *and* $n \in N$ *and the unit* (K, id_K)*.*

Corollary 13.2 ([11]) *Let* (H, β, σ) *be a cobraided Hom-Hopf algebra. If* $(M, \bar{\psi}^{M}, \alpha_M)$ *and* $(N, \bar{\psi}^{N}, \alpha_N)$ *are two* (H, β)*-Hom-comodules, we define the linear map*

$$B_{M,N} : M \otimes N \longrightarrow N \otimes M, \quad m \otimes n \mapsto \sigma(n_{-1}, m_{-1})(n_0 \otimes m_0).$$

Then, we have $B_{M,N} \circ (\alpha_M \otimes \alpha_N) = (\alpha_N \otimes \alpha_M) \circ B_{M,N}$ *and, if* $(P, \bar{\psi}^{P}, \alpha_P)$ *is another* (H, β)*-Hom-comodule, the maps* $B_{\_,\_}$ *satisfy the Hom-Yang-Baxter equation:*

$$(\alpha_P \otimes B_{M,N}) \circ (B_{M,P} \otimes \alpha_N) \circ (\alpha_M \otimes B_{N,P}) = (B_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes B_{M,P}) \circ (B_{M,N} \otimes \alpha_P).$$

Proof By Theorem 13.3 and let $h \bullet_N n = \sigma(n_{-1}, \beta^{-3}(h))n_0$ in Proposition 13.5, we can obtain the result. □

Theorem 13.4 *Let* (H, β, σ) *be a cobraided Hom-Hopf algebra. Assume that* (A, ρ^A, α) *is a Hom-Hopf algebra in the category* ${}^{H}\mathcal{M}$*. Define* $\triangleright_A : H \otimes A \longrightarrow A$ *by*

$$h \triangleright_A a = \sigma(a_{-1}, \beta^{-3}(h))a_0,$$

where $h \in H$*,* $a \in A$ *and* $\rho^A(a) = a_{-1} \otimes a_0$*. Then* $(A \natural_\diamond H, \alpha \otimes \beta)$ *is a Radford biproduct Hom-Hopf algebra.*

Proof By Theorem 13.3, we only need to prove that the conditions $(HM1)$, $(HM2)$, $(HMA1)$, $(HMA2)$ and (HYD) hold. And $(HM1)$ and $(HMA2)$ are easy. While

$$\begin{aligned}
\beta(h) \triangleright_A (g \triangleright_A a) &= \sigma(a_{-1}, \beta^{-3}(g))\sigma(a_{0-1}, \beta^{-2}(h))a_{00} \\
&\overset{(HCM2)}{=} \sigma(\beta^{-1}(a_{-11}), \beta^{-3}(g))\sigma(a_{-12}, \beta^{-2}(h))\alpha(a_0) \\
&\overset{(CHA5)}{=} \sigma(a_{-11}, \beta^{-2}(g))\sigma(a_{-12}, \beta^{-2}(h))\alpha(a_0) \\
&\overset{(CHA3)}{=} \sigma(\beta(a_{-1}), \beta^{-3}(h)\beta^{-3}(g))\alpha(a_0) \\
&\overset{(HCM1)(HA1)}{=} \sigma(\alpha(a)_{-1}), \beta^{-3}(hg))\alpha(a)_0 \\
&= hg \triangleright_A \alpha(a),
\end{aligned}$$

$$\begin{array}{rcl}\beta^2(h)\rhd_A(ab) & = & \sigma((ab)_{-1},\beta^{-1}(h))(ab)_0\\ & \overset{(HCMA1)}{=} & \sigma(a_{-1}b_{-1},\beta^{-1}(h))a_0b_0\\ & \overset{(CHA2)(HC1)}{=} & \sigma(\beta(a_{-1}),\beta^{-2}(h_1))\sigma(\beta(b_{-1}),\beta^{-2}(h_2))a_0b_0\\ & \overset{(CHA5)}{=} & \sigma(a_{-1},\beta^{-3}(h_1))\sigma(b_{-1},\beta^{-3}(h_2))a_0b_0\\ & = & (h_1\rhd_A a)(h_2\rhd_A b),\end{array}$$

and

$$\begin{array}{rcl}(\beta^2(h_1)\rhd_A a)_{-1}h_2\otimes(\beta^2(h_1)\rhd_A a)_0 & = & \sigma(a_{-1},\beta^{-1}(h_1))a_{0-1}h_2\otimes a_{00}\\ & \overset{(HCM2)}{=} & \sigma(\beta^{-1}(a_{-11}),\beta^{-1}(h_1))a_{-12}h_2\otimes\alpha(a_0)\\ & \overset{(CHA5)}{=} & \sigma(a_{-11},h_1)a_{-12}h_2\otimes\alpha(a_0)\\ & \overset{(CHA4)}{=} & h_1a_{-11}\sigma(a_{-12},h_2)\otimes\alpha(a_0)\\ & \overset{(HCM2)}{=} & h_1\beta(a_{-1})\otimes\sigma(a_{0-1},h_2)a_{00}\\ & = & h_1\beta(a_{-1})\otimes(\beta^3(h_2)\rhd_A a_0),\end{array}$$

finishing the proof. □

Dually, we have

Theorem 13.5 *Let (H,β,R) be a quasitriangular Hom-Hopf algebra. Assume that $(A,\rhd_A,\alpha)$ is a Hom-Hopf algebra in the category ${}_H\mathcal{M}$. Define $\rho^A: A\longrightarrow H\otimes A$ by*

$$\rho^A(a)=\beta^{-3}(\mathrm{R}^2)\otimes(\mathrm{R}^1\rhd_A a),$$

where $a\in A$. Then $(A^{\natural}_{\diamond}H,\alpha\otimes\beta)$ is a Radford biproduct Hom-Hopf algebra.

13.4 Twisted Tensor Biproduct Hom-Hopf Algebra

In this section, we consider the twisted tensor biproduct Hom-Hopf algebra generalizing the Radford's biproduct Hom-Hopf algebra. And two nontrivial examples are given.

Theorem 13.6 *Let (H,β) be a Hom-bialgebra, (A,α) a Hom-algebra and a Hom-coalgebra. Let $R: H\otimes A\longrightarrow A\otimes H$ and $T: A\otimes H\longrightarrow H\otimes A$ be two linear maps such that the conditions (R) and (T) hold. Assume that $(A\natural_R H,\alpha\otimes\beta)$ is a R-smash product Hom-algebra and $(A\diamond_T H,\alpha\otimes\beta)$ is a T-smash coproduct Hom-coalgebra. Then the following are equivalent:*

- *$(A^{\natural_R}_{\diamond_T}H,\mu_{\natural_R H},1_A\otimes 1_H,\Delta_{A\diamond_T H},\varepsilon_A\otimes\varepsilon_H,\alpha\otimes\beta)$ is a Hom-bialgebra.*
- *The following conditions hold ($\forall\, a,b\in A$ and $h,g\in H$):*
  (B1) $1_{AT}\otimes 1_{HT}=1_A\otimes 1_H$ and $\Delta_A(1_A)=1_A\otimes 1_A$,
  (B2) $(ab)_1\otimes 1_{HT}\otimes(ab)_{2T}=a_1\alpha^{-1}(b_1)_R\otimes\beta^{-1}(1_{HTR})1_{Ht}\otimes a_{2T}b_{2t}$,
  (B3) $h_T\otimes a_T=1_{HT}\beta^{-1}(h)_t\otimes\alpha^{-1}(a)_T1_{At}$,

(B4) $(h_1g_1)_T \otimes 1_{AT} \otimes \beta(h_2)\beta(g_2)$
$= h_{1T}g_{1t} \otimes 1_{AT}\alpha(\alpha^{-2}(1_{At})_R) \otimes \beta^{-1}(\beta(h_2)_R)\beta(g_2)$,
(B5) $\alpha^{-1}(a)_{R1} \otimes \beta^{-1}(h_{R1})_T \otimes \alpha(\alpha^{-1}(a)_{R2})_T \otimes h_{R2}$
$= \alpha^{-1}(a_1)_R \otimes \beta^{-1}(\beta^{-1}(h_1)_{TR})1_{Ht} \otimes 1_{AT}\alpha(\alpha^{-2}(a_{2t})_r) \otimes h_{2r}$. *(B6)* $\varepsilon_A(a_R)\varepsilon_H(h_R) = \varepsilon_A(a)\varepsilon_H(h)$ *and* ε_A *is a Hom-algebra map.*

In this case, we call this Hom-bialgebra twisted tensor biproduct Hom-bialgebra and denote it by $(A^{\natural_R}_{\diamond_T} H, \alpha \otimes \beta)$.

Proof ($\Longleftarrow$) It is easy to prove that $\varepsilon_{A^{\natural_R}_{\diamond_T} H} = \varepsilon_A \otimes \varepsilon_H$ is a morphism of Hom-algebras. Next we check $\Delta_{A^{\natural_R}_{\diamond_T} H} = \Delta_{A\diamond_T H}$ is a morphism of Hom-algebras as follows. For all $a, b \in A$ and $h, g \in H$, we have

$$
\begin{aligned}
&\Delta_{A^{\natural_R}_{\diamond_T} H}((a \otimes h)(b \otimes g))\\
&= (a\alpha^{-1}(b)_R)_1 \otimes \beta^{-1}((\beta^{-1}(h_R)g)_1)_T \otimes \alpha^{-1}((a\alpha^{-1}(b)_R)_{2T}) \otimes (\beta^{-1}(h_R)g)_2\\
&\overset{(B3)}{=} (a\alpha^{-1}(b)_R)_1 \otimes 1_{HT}\beta^{-2}((\beta^{-1}(h_R)g)_1)_t \otimes \alpha^{-1}(\alpha^{-1}((a\alpha^{-1}(b)_R)_2)_T 1_{At})\\
&\quad \otimes (\beta^{-1}(h_R)g)_2\\
&\overset{(T)}{=} (a\alpha^{-1}(b)_R)_1 \otimes \beta^{-1}(1_{HT})\beta^{-2}((\beta^{-1}(h_R)g)_1)_t \otimes \alpha^{-1}(\alpha^{-1}((a\alpha^{-1}(b)_R)_{2T})1_{At})\\
&\quad \otimes (\beta^{-1}(h_R)g)_2\\
&\overset{(B2)}{=} a_1\alpha^{-1}(\alpha^{-1}(b)_{R1})_r \otimes \beta^{-1}(\beta^{-1}(1_{H\bar{T}r})1_{H\bar{t}})\beta^{-2}((\beta^{-1}(h_R)g)_1)_t\\
&\quad \otimes \alpha^{-1}((\alpha^{-1}(a_{2\bar{T}})\alpha^{-1}(\alpha^{-1}(b)_{R2\bar{t}}))1_{At}) \otimes (\beta^{-1}(h_R)g)_2\\
&\overset{(HA1)}{=} a_1\alpha^{-1}(\alpha^{-1}(b)_{R1})_r \otimes \beta^{-1}(\beta^{-1}(1_{H\bar{T}r})1_{H\bar{t}})(\beta^{-3}(h_R)_1\beta^{-2}(g)_1)_t\\
&\quad \otimes \alpha^{-1}((\alpha^{-1}(a_{2\bar{T}})\alpha^{-1}(\alpha^{-1}(b)_{R2\bar{t}}))1_{At}) \otimes \beta(\beta(\beta^{-3}(h_R)_2)\beta(\beta^{-2}(g)_2))\\
&\overset{(B4)}{=} a_1\alpha^{-1}(\alpha^{-1}(b)_{R1})_r \otimes \beta^{-1}(\beta^{-1}(1_{H\bar{T}r})1_{H\bar{t}})(\beta^{-3}(h_R)_{1T}\beta^{-2}(g)_{1t})\\
&\quad \otimes \alpha^{-1}((\alpha^{-1}(a_{2\bar{T}})\alpha^{-1}(\alpha^{-1}(b)_{R2\bar{t}}))(1_{AT}\alpha(\alpha^{-2}(1_{At})_{\bar{R}})))\\
&\quad \otimes \beta(\beta^{-1}(\beta(\beta^{-3}(h_R)_2)_{\bar{R}})\beta(\beta^{-2}(g)_2))\\
&\overset{(HA1)}{=} a_1\alpha^{-1}(\alpha^{-1}(b)_{R1})_r \otimes (\beta^{-2}(1_{H\bar{T}r})\beta^{-1}(1_{H\bar{t}}))(\beta^{-3}(h_R)_{1T}\beta^{-2}(g)_{1t})\\
&\quad \otimes \alpha^{-1}((\alpha^{-1}(a_{2\bar{T}})\alpha^{-1}(\alpha^{-1}(b)_{R2\bar{t}}))(1_{AT}\alpha(\alpha^{-2}(1_{At})_{\bar{R}})))\\
&\quad \otimes \beta(\beta^{-1}(\beta(\beta^{-3}(h_R)_2)_{\bar{R}})\beta(\beta^{-2}(g)_2))\\
&\overset{(HA2)}{=} a_1\alpha^{-1}(\alpha^{-1}(b)_{R1})_r \otimes (\beta^{-2}(1_{H\bar{T}r})\beta^{-1}(\beta^{-1}(1_{H\bar{t}})\beta^{-3}(h_R)_{1T}))\beta(\beta^{-2}(g)_{1t})\\
&\quad \otimes \alpha^{-1}((\alpha^{-1}(a_{2\bar{T}})\alpha^{-1}(\alpha^{-1}(\alpha^{-1}(b)_{R2\bar{t}})1_{AT}))\alpha^2(\alpha^{-2}(1_{At})_{\bar{R}})))\\
&\quad \otimes \beta(\beta^{-1}(\beta(\beta^{-3}(h_R)_2)_{\bar{R}})\beta(\beta^{-2}(g)_2))\\
&\overset{(HA1)}{=} a_1\alpha^{-1}(\alpha^{-1}(b)_{R1})_r \otimes (\beta^{-2}(1_{H\bar{T}r})(\beta^{-2}(1_{H\bar{t}})\beta^{-1}(\beta^{-3}(h_R)_{1T})))\beta(\beta^{-2}(g)_{1t})\\
&\quad \otimes (\alpha^{-2}(a_{2\bar{T}})\alpha^{-1}(\alpha^{-2}(\alpha^{-1}(b)_{R2\bar{t}})\alpha^{-1}(1_{AT})))\alpha(\alpha^{-2}(1_{At})_{\bar{R}})\\
&\quad \otimes \beta(\beta^{-1}(\beta(\beta^{-3}(h_R)_2)_{\bar{R}})\beta(\beta^{-2}(g)_2))\\
&\overset{(T)}{=} a_1\alpha^{-1}(\alpha^{-1}(b)_{R1})_r \otimes (\beta^{-2}(1_{H\bar{T}r})(1_{H\bar{t}}\beta^{-1}(\beta^{-3}(h_R)_1)_T))\beta(\beta^{-2}(g)_{1t})\\
&\quad \otimes (\alpha^{-2}(a_{2\bar{T}})\alpha^{-1}(\alpha^{-2}(\alpha^{-1}(b)_{R2})_{\bar{t}}1_{AT}))\alpha(\alpha^{-2}(1_{At})_{\bar{R}})\\
&\quad \otimes \beta(\beta^{-1}(\beta(\beta^{-3}(h_R)_2)_{\bar{R}})\beta(\beta^{-2}(g)_2))\\
&\overset{(B3)}{=} a_1\alpha^{-1}(\alpha^{-1}(b)_{R1})_r \otimes (\beta^{-2}(1_{H\bar{T}r})\beta^{-3}(h_R)_{1T})\beta(\beta^{-2}(g)_{1t})
\end{aligned}
$$

$$
\begin{aligned}
&\otimes(\alpha^{-2}(a_{2\bar{T}})\alpha^{-1}(\alpha^{-1}(\alpha^{-1}(b)_{R2})_T))\alpha(\alpha^{-2}(1_{At})_{\bar{R}})\\
&\otimes\beta(\beta^{-1}(\beta(\beta^{-3}(h_R)_2)_{\bar{R}})\beta(\beta^{-2}(g)_2))\\
&\overset{(HC1)}{=} a_1\alpha^{-1}(\alpha^{-1}(b)_{R1})_r\otimes(\beta^{-2}(1_{H\bar{T}r})\beta^{-3}(h_{R1})_T)\beta(\beta^{-2}(g_1)_t)\\
&\otimes(\alpha^{-2}(a_{2\bar{T}})\alpha^{-1}(\alpha^{-1}(\alpha^{-1}(b)_{R2})_T))\alpha(\alpha^{-2}(1_{At})_{\bar{R}})\otimes\beta(\beta^{-1}(\beta^{-2}(h_{R2})_{\bar{R}})\beta^{-1}(g_2))\\
&\overset{(T)}{=} a_1\alpha^{-1}(\alpha^{-1}(b)_{R1})_r\otimes(\beta^{-2}(1_{H\bar{T}r})\beta^{-2}(\beta^{-1}(h_{R1})_T))\beta(\beta^{-2}(g_1)_t)\\
&\otimes(\alpha^{-2}(a_{2\bar{T}})\alpha^{-3}(\alpha(\alpha^{-1}(b)_{R2})_T))\alpha(\alpha^{-2}(1_{At})_{\bar{R}})\otimes\beta(\beta^{-1}(\beta^{-2}(h_{R2})_{\bar{R}})\beta^{-1}(g_2))\\
&\\
&\overset{(B5)(HA1)}{=} a_1\alpha^{-1}(\alpha^{-1}(b_1)_R)_r\otimes(\beta^{-2}(1_{H\bar{T}r})\beta^{-2}(\beta^{-1}(\beta^{-1}(h_1)_{TR})1_{H\bar{t}}))\beta(\beta^{-2}(g_1)_t)\\
&\otimes(\alpha^{-2}(a_{2\bar{T}})\alpha^{-3}(1_{AT}\alpha(\alpha^{-2}(b_{2\bar{t}})_{\bar{r}})))\alpha(\alpha^{-2}(1_{At})_{\bar{R}})\otimes\beta^{-2}(h_{2\bar{r}})_{\bar{R}}g_2\\
&\overset{(HA2)(HA1)}{=} a_1\alpha^{-1}(\alpha^{-1}(b_1)_R)_r\otimes\beta^{-2}(1_{H\bar{T}r}\beta^{-1}(h_1)_{TR})(\beta^{-1}(1_{H\bar{t}})\beta^{-2}(g_1)_t)\\
&\otimes\alpha^{-2}(a_{2\bar{T}}1_{AT})(\alpha^{-1}(\alpha^{-2}(b_{2\bar{t}})_{\bar{r}})\alpha^{-2}(1_{At})_{\bar{R}})\otimes\beta^{-2}(h_{2\bar{r}})_{\bar{R}}g_2\\
&\overset{(R)}{=} a_1\alpha^{-1}(\alpha^{-1}(b_1)_{Rr})\otimes\beta^{-2}(\beta^{-1}(\beta(1_{H\bar{T}})_r)\beta^{-1}(h_1)_{TR})(\beta^{-1}(1_{H\bar{t}})\beta^{-2}(g_1)_t)\\
&\otimes\alpha^{-2}(a_{2\bar{T}}1_{AT})(\alpha^{-1}(\alpha^{-2}(b_{2\bar{t}})_{\bar{r}})\alpha^{-2}(1_{At})_{\bar{R}})\otimes\beta^{-2}(h_{2\bar{r}})_{\bar{R}}g_2\\
&\overset{(RS2)}{=} a_1\alpha^{-1}(b_{1R})\otimes\beta^{-2}((1_{H\bar{T}}\beta^{-1}(h_1)_T)_R)(\beta^{-1}(1_{H\bar{t}})\beta^{-2}(g_1)_t)\\
&\otimes\alpha^{-2}(\alpha^{-1}(\alpha(a_2))_{\bar{T}}1_{AT})(\alpha^{-1}(\alpha^{-2}(b_{2\bar{t}})_{\bar{r}})\alpha^{-2}(1_{At})_{\bar{R}})\otimes\beta^{-2}(h_{2\bar{r}})_{\bar{R}}g_2\\
&\overset{(B3)}{=} a_1\alpha^{-1}(b_{1R})\otimes\beta^{-2}(h_{1TR})\beta^{-1}(1_{H\bar{t}}\beta(\beta^{-2}(g_1)_t))\\
&\otimes\alpha^{-2}(\alpha(a_2)_T)(\alpha^{-1}(\alpha^{-2}(b_{2\bar{t}})_{\bar{r}})\alpha^{-2}(1_{At})_{\bar{R}})\otimes\beta^{-2}(h_{2\bar{r}})_{\bar{R}}g_2\\
&\overset{(HA1)}{=} a_1\alpha^{-1}(b_{1R})\otimes\beta^{-2}(h_{1TR})(\beta^{-1}(1_{H\bar{t}})\beta^{-2}(g_1)_t)\\
&\otimes\alpha^{-2}(\alpha(a_2)_T)(\alpha^{-1}(\alpha^{-2}(b_{2\bar{t}})_{\bar{r}})\alpha^{-2}(1_{At})_{\bar{R}})\otimes\beta^{-2}(h_{2\bar{r}})_{\bar{R}}g_2\\
&\overset{(T)}{=} a_1\alpha^{-1}(b_{1R})\otimes\beta^{-2}(h_{1TR})(1_{H\bar{t}}\beta^{-2}(g_1)_t)\\
&\otimes\alpha^{-2}(\alpha(a_2)_T)(\alpha^{-1}(\alpha^{-1}(\alpha^{-1}(b_2)_{\bar{t}})_{\bar{r}})\alpha^{-2}(1_{At})_{\bar{R}})\otimes\beta^{-2}(h_{2\bar{r}})_{\bar{R}}g_2\\
&\overset{(R)}{=} a_1\alpha^{-1}(b_{1R})\otimes\beta^{-2}(h_{1TR})(1_{H\bar{t}}\beta^{-2}(g_1)_t)\\
&\otimes\alpha^{-2}(\alpha(a_2)_T)(\alpha^{-2}(\alpha^{-1}(b_2)_{\bar{t}})_{\bar{r}}\alpha^{-1}(\alpha^{-1}(1_{At})_{\bar{R}}))\otimes\beta^{-1}(\beta^{-1}(h_2)_{\bar{r}\bar{R}})g_2\\
&\overset{(R)}{=} a_1\alpha^{-1}(b_1)_R\otimes\beta^{-1}(\beta^{-1}(h_1)_{TR})(1_{H\bar{t}}\beta^{-2}(g_1)_t)\\
&\otimes\alpha^{-1}(a_{2T})(\alpha^{-2}(\alpha^{-1}(b_2)_{\bar{t}})_{\bar{r}}\alpha^{-1}(\alpha^{-1}(1_{At})_{\bar{R}}))\otimes\beta^{-1}(\beta^{-1}(h_2)_{\bar{r}\bar{R}})g_2\\
&\overset{(HA1)}{=} a_1\alpha^{-1}(b_1)_R\otimes\beta^{-1}(\beta^{-1}(h_1)_{TR})(1_{H\bar{t}}\beta^{-2}(g_1)_t)\\
&\otimes\alpha^{-1}(a_{2T})\alpha^{-1}(\alpha(\alpha^{-2}(\alpha^{-1}(b_2)_{\bar{t}})_{\bar{r}})\alpha(\alpha^{-2}(1_{At}))_{\bar{R}})\otimes\beta^{-1}(\beta^{-1}(h_2)_{\bar{r}\bar{R}})g_2\\
&\overset{(RS3)}{=} a_1\alpha^{-1}(b_1)_R\otimes\beta^{-1}(\beta^{-1}(h_1)_{TR})(1_{H\bar{t}}\beta^{-2}(g_1)_t)\\
&\otimes\alpha^{-1}(a_{2T})(\alpha^{-2}(\alpha^{-1}(b_2)_{\bar{t}})\alpha^{-2}(1_{At}))_r)\otimes\beta^{-1}(h_{2r})g_2\\
&\overset{(HA1)}{=} a_1\alpha^{-1}(b_1)_R\otimes\beta^{-1}(\beta^{-1}(h_1)_{TR})(1_{H\bar{t}}\beta^{-2}(g_1)_t)\\
&\otimes\alpha^{-1}(a_{2T})\alpha^{-2}(\alpha^{-1}(b_2)_{\bar{t}}1_{At})_r\otimes\beta^{-1}(h_{2r})g_2\\
&\overset{(B3)}{=} a_1\alpha^{-1}(b_1)_R\otimes\beta^{-1}(\beta^{-1}(h_1)_{TR})\beta^{-1}(g_1)_t\otimes\alpha^{-1}(a_{2T})\alpha^{-2}(b_{2t})_r\otimes\beta^{-1}(h_{2r})g_2\\
&=\Delta_{A\diamond_T^{\natural_R}H}(a\otimes h)\Delta_{A\diamond_T^{\natural_R}H}(b\otimes g),
\end{aligned}
$$

and $\Delta_{A\diamond_T^{\natural_R}H}(1_A\otimes 1_H)=1_A\otimes 1_H\otimes 1_A\otimes 1_H$ can be proved directly.

($\Longrightarrow$) It is easy to prove that the conditions $(B1)$ and $(B6)$ hold. Next we check the conditions $(B2)$–$(B5)$ are satisfied as follows.

For all $a, b \in A$ and $h, g \in H$, since $\Delta_{A_{\diamond T}^{\natural R} H}((a \otimes h)(b \otimes g)) = \Delta_{A_{\diamond T}^{\natural R} H}(a \otimes h)\Delta_{A_{\diamond T}^{\natural R} H}(b \otimes g)$, we have

$$
\begin{aligned}
(*)\quad & (a\alpha^{-1}(b)_R)_1 \otimes \beta^{-1}((\beta^{-1}(h_R)g)_1)_T \otimes \alpha^{-1}((a\alpha^{-1}(b)_R)_{2T}) \otimes (\beta^{-1}(h_R)g)_2 \\
& = a_1\alpha^{-1}(b_1)_R \otimes \beta^{-1}(\beta^{-1}(h_1)_{TR})\beta^{-1}(g_1)_t \otimes \alpha^{-1}(a_{2T})\alpha^{-2}(b_{2t})_r \otimes \beta^{-1}(h_{2r})g_2
\end{aligned}
$$

Apply $id_A \otimes id_H \otimes id_A \otimes \varepsilon_H$ to Eq.$(*)$ and then set $h = g = 1_H$, we get $(B2)$.

$(B3)$ can be obtained by using $\varepsilon_A \otimes id_H \otimes id_A \otimes \varepsilon_H$ to Eq.$(*)$ and setting $b = 1_A, h = 1_H$.

Similarly, we apply $\varepsilon_A \otimes id_H \otimes id_A \otimes id_H$ to Eq.$(*)$ and set $a = b = 1_A$, then $(B4)$ holds.

$(B5)$ can be derived by letting $a = 1_A$ and $g = 1_H$ in Eq.$(*)$. □

Remark 13.2 If $\alpha = id_A$ and $\beta = id_H$, then we can get the twisted tensor biproduct bialgebra in [4].

Corollary 13.3 ([2]) *Let $(C, \alpha), (H, \beta)$ be two Hom-bialgebras, and $T : C \otimes H \longrightarrow H \otimes C$ a linear map such that the condition (T) holds. Then the T-smash coproduct Hom-coalgebra $(C \diamond_T H, \alpha \otimes \beta)$ endowed with the tensor product Hom-algebra structure becomes a Hom-bialgebra if and only if T is a map of Hom-algebras.*

Proof Let $R(h \otimes c) = \alpha(c) \otimes \beta(h)$ in Theorem 13.6. Then, by $(B2)$–$(B5)$, we have

$$
\begin{aligned}
&(C1)\ 1_{HT} \otimes (ab)_T = 1_{HT}1_{Ht} \otimes a_T b_t, \\
&(C2)\ h_T \otimes a_T = 1_{hT}\beta^{-1}(h)_t \otimes \alpha^{-1}(a)_T 1_{At} = \beta^{-1}(h)_t 1_{hT} \otimes 1_{At}\alpha^{-1}(a)_T, \\
&(C3)\ (hg)_T \otimes 1_{AT} = h_T g_t \otimes 1_{AT}1_{At}.
\end{aligned}
$$

Next we only prove that $(hg)_T \otimes (ab)_T = h_T g_t \otimes a_T b_t$ as follows. And the rest are easy.

$$
\begin{aligned}
&(hg)_T \otimes (ab)_T \\
&\overset{(C2)}{=} 1_{HT}\beta^{-1}(hg)_t \otimes \alpha^{-1}(ab)_T 1_{At} \\
&\overset{(HA1)}{=} 1_{HT}(\beta^{-1}(h)\beta^{-1}(g))_t \otimes (\alpha^{-1}(a)\alpha^{-1}(b))_T 1_{At} \\
&\overset{(C1)(C3)}{=} (1_{H\bar{\bar{T}}}1_{HT}(\beta^{-1}(h)_{\bar{T}}\beta^{-1}(g)_t) \otimes (\alpha^{-1}(a)_{\bar{\bar{T}}}\alpha^{-1}(b)_T)(1_{A\bar{T}}1_{At}) \\
&\overset{(HA2)}{=} (1_{H\bar{\bar{T}}}\beta^{-1}(1_{HT}\beta^{-1}(h)_{\bar{T}}))\beta(\beta^{-1}(g)_t) \otimes (\alpha^{-1}(a)_{\bar{\bar{T}}}\alpha^{-1}(\alpha^{-1}(b)_T 1_{A\bar{T}}))\alpha(1_{At}) \\
&\overset{(C2)}{=} (1_{H\bar{\bar{T}}}\beta^{-1}(\beta^{-1}(h)_{\bar{T}}1_{HT}))\beta(\beta^{-1}(g)_t) \otimes (\alpha^{-1}(a)_{\bar{\bar{T}}}\alpha^{-1}(1_{A\bar{T}}\alpha^{-1}(b)_T))\alpha(1_{At}) \\
&\overset{(HA2)}{=} (1_{H\bar{\bar{T}}}\beta^{-1}(h)_{\bar{T}})(1_{HT}\beta^{-1}(g)_t) \otimes (\alpha^{-1}(a)_{\bar{\bar{T}}}1_{A\bar{T}})(\alpha^{-1}(b)_T 1_{At}) \\
&\overset{(C2)}{=} h_T g_t \otimes a_T b_t,
\end{aligned}
$$

finishing the proof. □

Corollary 13.4 ([3]) *Let $(A, \alpha), (H, \beta)$ be two Hom-bialgebras, and $R : H \otimes A \longrightarrow A \otimes H$ a linear map such that the condition (R) holds. Then the R-smash product Hom-algebra $(A\natural_R H, \alpha \otimes \beta)$ endowed with the tensor product Hom-coalgebra structure becomes a Hom-bialgebra if and only if R is a map of Hom-coalgebras.*

Proof Let $T(a \otimes h) = \beta(h) \otimes \alpha(a)$ in Theorem 13.6. □

Corollary 13.5 ([2]) *Let (H, β) be a Hom-bialgebra, (A, α) a left (H, β)-module Hom-algebra with module structure $\triangleright : H \otimes A \longrightarrow A$ and a left (H, β)-comodule Hom-coalgebra with comodule structure $\rho : A \longrightarrow H \otimes A$. Then the following are equivalent:*

- $(A^{\natural}_{\diamond} H, \mu_{A\natural H}, 1_A \otimes 1_H, \Delta_{A\diamond H}, \varepsilon_A \otimes \varepsilon_H, \alpha \otimes \beta)$ *is a Hom-bialgebra, where $(A\natural H, \alpha \otimes \beta)$ is a smash product Hom-algebra and $(A \diamond H, \alpha \otimes \beta)$ is a smash coproduct Hom-coalgebra.*
- *The following conditions hold ($\forall\, a, b \in A$ and $h \in H$):*
  (R1) (A, ρ, α) is an (H, β)-comodule Hom-algebra,
  (R2) $(A, \triangleright, \alpha)$ is an (H, β)-module Hom-coalgebra,
  (R3) ε_A is a Hom-algebra map and $\Delta_A(1_A) = 1_A \otimes 1_A$,
  (R4) $\Delta_A(ab) = a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}(b_1)) \otimes \alpha^{-1}(a_{20})b_2$,
  (R5) $h_1\beta(a_{-1}) \otimes (\beta^3(h_2) \triangleright a_0) = (\beta^2(h_1) \triangleright a)_{-1}h_2 \otimes (\beta^2(h_1) \triangleright a)_0$.

In this case, we call $(A^{\natural}_{\diamond} H, \alpha \otimes \beta)$ **Radford biproduct bialgebra**.

Proof Let $R(h \otimes a) = (h_1 \triangleright a) \otimes h_2$ and $T(a \otimes h) = a_{-1}h \otimes a_0$ in Theorem 13.6. □

Theorem 13.7 *Let (H, β, S_H) be a Hom-Hopf algebra, and (A, α) be a Hom-algebra and a Hom-coalgebra. Let $R : H \otimes A \longrightarrow A \otimes H$ and $T : A \otimes H \longrightarrow H \otimes A$ be two linear maps such that the conditions (R) and (T) hold. Assume that $(A^{\natural_R}_{\diamond_T} H, \alpha \otimes \beta)$ is a twisted tensor biproduct Hom-bialgebra defined as above, and $S_A : A \longrightarrow A$ is a linear map such that $S_A(a_1)a_2 = a_1 S_A(a_2) = \varepsilon_A(a)1_A$ and $\alpha \circ S_A = S_A \circ \alpha$ hold. Then $(A^{\natural_R}_{\diamond_T} H, \alpha \otimes \beta, S_{A^{\natural_R}_{\diamond_T} H})$ is a Hom-Hopf algebra, where*

$$S_{A^{\natural_R}_{\diamond_T} H}(a \otimes h) = S_A(\alpha^{-2}(a_T))_R \otimes \beta^{-1}(S_H(\beta^{-1}(h)_T)_R).$$

Proof We can compute that $(A^{\natural}_{\diamond} H, \alpha \otimes \beta, S_{A^{\natural}_{\diamond} H})$ is a Hom-Hopf algebra as follows. For all $a \in A$ and $h \in H$, we have

$$
\begin{aligned}
&(S_{A^{\natural R}_{\diamond T}H} * id_{A^{\natural R}_{\diamond T}H})(a\otimes h)\\
&= S_A(\alpha^{-2}(a_{1t}))_R\alpha^{-2}(a_{2T})_r \otimes \beta^{-1}(\beta^{-1}(S_H(\beta^{-1}(\beta^{-1}(h_1)_T)_t)_R)_r)h_2\\
&\overset{(T)}{=} S_A(\alpha^{-2}(a_{1t}))_R\alpha^{-3}(\alpha(a_2)_T)_r \otimes \beta^{-1}(\beta^{-1}(S_H(\beta^{-2}(h_{1T})_t)_R)_r)h_2\\
&\overset{(T)}{=} S_A(\alpha^{-4}(\alpha^2(a_1)_t))_R\alpha^{-3}(\alpha(a_2)_T)_r \otimes \beta^{-1}(\beta^{-1}(S_H(\beta^{-2}(h_{1Tt}))_R)_r)h_2\\
&\overset{(T)}{=} S_A(\alpha^{-4}(\alpha(\alpha(a)_1)_t))_R\alpha^{-4}(\alpha(\alpha(a)_{2T}))_r \otimes \beta^{-1}(\beta^{-1}(S_H(\beta^{-2}(h_{1Tt}))_R)_r)h_2\\
&\overset{(TS3)}{=} S_A(\alpha^{-4}(\alpha^2(a)_{T1}))_R\alpha^{-4}(\alpha^2(a)_{T2})_r \otimes \beta^{-1}(\beta^{-1}(S_H(\beta^{-1}(h_{1T}))_R)_r)h_2\\
&= \alpha(S_A(\alpha^{-5}(\alpha^2(a)_{T1}))_R)\alpha(\alpha^{-5}(\alpha^2(a)_{T2}))_r \otimes \beta^{-1}(S_H(\beta^{-2}(h_{1T}))_{Rr})h_2\\
&\overset{(RS3)}{=} \alpha((S_A(\alpha^{-5}(\alpha^2(a)_{T1}))\alpha^{-5}(\alpha^2(a)_{T2}))_R) \otimes \beta^{-1}(S_H(\beta^{-1}(h_{1T}))_R)h_2\\
&\overset{(HA1)}{=} \alpha(\alpha^{-5}(S_A(\alpha^2(a)_{T1})\alpha^2(a)_{T2})_R) \otimes \beta^{-1}(S_H(\beta^{-1}(h_{1T}))_R)h_2\\
&\overset{(HA1)}{=} (\alpha(1_{AR}) \otimes \beta^{-1}(S_H(\beta^{-1}(h_{1T}))_R)h_2)\varepsilon_A(\alpha^2(a)_T)\\
&\overset{(TS1)}{=} (\alpha(1_{AR}) \otimes \beta^{-1}(S_H(h_1)_R)h_2)\varepsilon_A(a)\\
&\overset{(RS1)(HA1)}{=} (1_A \otimes S_H(h_1)h_2)\varepsilon_A(a)\\
&= (1_A\otimes 1_H)\varepsilon_A(a)\varepsilon_H(h).
\end{aligned}
$$

Similarly, we have $(id_{A^{\natural R}_{\diamond T}H} * S_{A^{\natural R}_{\diamond T}H})(a\otimes h) = (1_A\otimes 1_H)\varepsilon_A(a)\varepsilon_H(h)$.

Finally,

$$
\begin{aligned}
(\alpha\otimes\beta)\circ S_{A^{\natural R}_{\diamond T}H}(a\otimes h) &= \alpha(S_A(\alpha^{-2}(a_T))_R) \otimes S_H(\beta^{-1}(h)_T)_R\\
&\overset{(R)}{=} \alpha(S_A(\alpha^{-2}(a_T)))_R \otimes \beta^{-1}(\beta(S_H(\beta^{-1}(h)_T))_R)\\
&= S_A(\alpha^{-1}(a_T))_R \otimes \beta^{-1}(S_H(\beta(\beta^{-1}(h)_T))_R)\\
&\overset{(T)}{=} S_A(\alpha^{-2}(\alpha(a)_T))_R \otimes \beta^{-1}(S_H(h_T)_R)\\
&\overset{(T)}{=} S_{A^{\natural R}_{\diamond T}H} \circ (\alpha(a)\otimes\beta(h)),
\end{aligned}
$$

finishing the proof. □

Corollary 13.6 ([2]) *Let (H, β, S_H) be a Hom-Hopf algebra, and (A, α) be a Hom-algebra and a Hom-coalgebra. Assume that $(A^{\natural}_{\diamond}H, \alpha\otimes\beta)$ is a Radford biproduct Hom-bialgebra defined in Corollary 13.5, and $S_A : A \longrightarrow A$ is a linear map such that $S_A(a_1)a_2 = a_1S_A(a_2) = \varepsilon_A(a)1_A$ and $\alpha\circ S_A = S_A\circ\alpha$ hold. Then $(A^{\natural}_{\diamond}H, \alpha\otimes\beta, S_{A^{\natural}_{\diamond}H})$ is a Hom-Hopf algebra, where*

$$
S_{A^{\natural}_{\diamond}H}(a\otimes h) = (S_H(a_{-1}\beta^{-1}(h))_1 \triangleright S_A(\alpha^{-2}(a_0))) \otimes \beta^{-1}(S_H(a_{-1}\beta^{-1}(h))_2).
$$

Proof Let $R(h\otimes a)=(h_1\triangleright a)\otimes h_2$ and $T(a\otimes h)=a_{-1}h\otimes a_0$ in Theorem 13.7. □

Example 13.1 Let $K\mathbf{Z}_2 = K\{1, a\}$ be Hopf group algebra (see [8]). Then $(K\mathbf{Z}_2, id_{K\mathbf{Z}_2})$ is a Hom-Hopf algebra.

Let $T_{2,-1} = K\{1, g, x, gx | g^2 = 1, x^2 = 0\}$ be Taft's Hopf algebra (see [4]), its coalgebra structure and antipode are given by

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes g + 1 \otimes x,\ \Delta(gx) = gx \otimes 1 + g \otimes gx;$$

$$\varepsilon(g) = 1, \varepsilon(x) = 0, \varepsilon(gx) = 0;$$

and

$$S(g) = g,\ S(x) = gx,\ S(gx) = -x.$$

Define a linear map α: $T_{2,-1} \longrightarrow T_{2,-1}$ by

$$\alpha(1) = 1,\ \alpha(g) = g,\ \alpha(x) = kx,\ \alpha(gx) = kgx$$

where $0 \neq k \in K$. Then α is an automorphism of Hopf algebras.

So we can get a Hom-Hopf algebra $H_\alpha = (T_{2,-1}, \alpha \circ \mu_{T_{2,-1}}, 1_{T_{2,-1}}, \Delta_{T_{2,-1}} \circ \alpha, \varepsilon_{T_{2,-1}}, \alpha)$ (see [7]).

With notations above. Define two linear maps as follows:

$$\begin{aligned} R : K\mathbf{Z}_2 \otimes H_\alpha &\longrightarrow H_\alpha \otimes K\mathbf{Z}_2 \\ 1_{K\mathbf{Z}_2} \otimes 1_{H_\alpha} &\mapsto 1_{H_\alpha} \otimes 1_{K\mathbf{Z}_2} \\ 1_{K\mathbf{Z}_2} \otimes g &\mapsto g \otimes 1_{K\mathbf{Z}_2} \\ 1_{K\mathbf{Z}_2} \otimes x &\mapsto kx \otimes 1_{K\mathbf{Z}_2} \\ 1_{K\mathbf{Z}_2} \otimes gx &\mapsto kgx \otimes 1_{K\mathbf{Z}_2} \\ a \otimes 1_{H_\alpha} &\mapsto 1_{H_\alpha} \otimes a \\ a \otimes g &\mapsto g \otimes a \\ a \otimes x &\mapsto kx \otimes a \\ a \otimes gx &\mapsto kgx \otimes a \end{aligned}$$

and

$$\begin{aligned} T : H_\alpha \otimes K\mathbf{Z}_2 &\longrightarrow K\mathbf{Z}_2 \otimes H_\alpha \\ 1_{H_\alpha} \otimes 1_{K\mathbf{Z}_2} &\mapsto 1_{K\mathbf{Z}_2} \otimes 1_{H_\alpha} \\ g \otimes 1_{K\mathbf{Z}_2} &\mapsto 1_{K\mathbf{Z}_2} \otimes g \\ x \otimes 1_{K\mathbf{Z}_2} &\mapsto ka \otimes x \\ gx \otimes 1_{K\mathbf{Z}_2} &\mapsto ka \otimes gx \\ 1_{H_\alpha} \otimes a &\mapsto a \otimes 1_{H_\alpha} \\ g \otimes a &\mapsto a \otimes g \\ x \otimes a &\mapsto k1_{K\mathbf{Z}_2} \otimes x \\ gx \otimes a &\mapsto k1_{K\mathbf{Z}_2} \otimes gx. \end{aligned}$$

By a direct computation, we have $(H_\alpha {}^{\natural_R}_{\diamond_T} K\mathbf{Z}_2, \mu_{H_\alpha \natural_R K\mathbf{Z}_2}, 1_{H_\alpha} \otimes 1_{K\mathbf{Z}_2}, \Delta_{H_\alpha \diamond_T K\mathbf{Z}_2}, \varepsilon_{H_\alpha} \otimes \varepsilon_{K\mathbf{Z}_2}, \alpha \otimes id_{K\mathbf{Z}_2})$ is a twisted tensor biproduct Hom-bialgebra. Furthermore,

$(H_\alpha {}^{\natural_R}_{\diamond_T} K\mathbf{Z}_2, \alpha \otimes id_{K\mathbf{Z}_2}, S_{H_\alpha {}^{\natural_R}_{\diamond_T} K\mathbf{Z}_2})$ is a Hom-Hopf algebra, where $S_{H_\alpha {}^{\natural}_{\diamond} K\mathbf{Z}_2}$ is defined by

$$S_{H_\alpha {}^{\natural_R}_{\diamond_T} K\mathbf{Z}_2}(1_{H_\alpha} \otimes 1_{K\mathbf{Z}_2}) = 1_{H_\alpha} \otimes 1_{K\mathbf{Z}_2}; \quad S_{H_\alpha {}^{\natural_R}_{\diamond_T} K\mathbf{Z}_2}(1_{H_\alpha} \otimes a) = 1_{H_\alpha} \otimes a$$
$$S_{H_\alpha {}^{\natural_R}_{\diamond_T} K\mathbf{Z}_2}(g \otimes 1_{K\mathbf{Z}_2}) = g \otimes 1_{K\mathbf{Z}_2}; \quad S_{H_\alpha {}^{\natural_R}_{\diamond_T} K\mathbf{Z}_2}(g \otimes a) = g \otimes a$$
$$S_{H_\alpha {}^{\natural_R}_{\diamond_T} K\mathbf{Z}_2}(x \otimes 1_{K\mathbf{Z}_2}) = y \otimes a; \quad S_{H_\alpha {}^{\natural_R}_{\diamond_T} K\mathbf{Z}_2}(x \otimes a) = y \otimes 1_{K\mathbf{Z}_2}$$
$$S_{H_\alpha {}^{\natural_R}_{\diamond_T} K\mathbf{Z}_2}(y \otimes 1_{K\mathbf{Z}_2}) = -x \otimes a; \quad S_{H_\alpha {}^{\natural_R}_{\diamond_T} K\mathbf{Z}_2}(y \otimes a) = -x \otimes 1_{K\mathbf{Z}_2}.$$

Example 13.2 Let $K\mathbf{Z}_2 = K\{1, a\}$ be Hopf group algebra (see [8]). Then $(K\mathbf{Z}_2, id_{K\mathbf{Z}_2})$ is a Hom-Hopf algebra. Let $A = K\{1, x\}$ be a vector space. Define the multiplication μ_A by

$$1x = x1 = lx, \quad x^2 = 0$$

and the automorphism $\beta : A \longrightarrow A$ by

$$\beta(1) = 1, \quad \beta(x) = lx$$

where $0 \neq l \in K$. Then (A, β) is a Hom-algebra.

Define the comultiplication Δ_A by

$$\Delta_A(1) = 1 \otimes 1, \quad \Delta_A(x) = lx \otimes 1 + l1 \otimes x, \quad \text{and} \quad \varepsilon_A(1) = 1, \quad \varepsilon_A(x) = 0.$$

Then (A, β) is a Hom-coalgebra.

With notations above. Define two linear maps as follows:

$$\begin{aligned} R : K\mathbf{Z}_2 \otimes A &\longrightarrow A \otimes K\mathbf{Z}_2 \\ 1_{K\mathbf{Z}_2} \otimes 1_A &\mapsto 1_A \otimes 1_{K\mathbf{Z}_2} \\ 1_{K\mathbf{Z}_2} \otimes x &\mapsto lx \otimes 1_{K\mathbf{Z}_2} \\ a \otimes 1_A &\mapsto 1_A \otimes a \\ a \otimes x &\mapsto -lx \otimes a \end{aligned}$$

and

$$\begin{aligned} T : A \otimes K\mathbf{Z}_2 &\longrightarrow K\mathbf{Z}_2 \otimes A \\ 1_A \otimes 1_{K\mathbf{Z}_2} &\mapsto 1_{K\mathbf{Z}_2} \otimes 1_A \\ x \otimes 1_{K\mathbf{Z}_2} &\mapsto la \otimes x \\ 1_A \otimes a &\mapsto a \otimes 1_A \\ x \otimes a &\mapsto l1_{K\mathbf{Z}_2} \otimes x. \end{aligned}$$

By a direct computation, we have $(A {}^{\natural_R}_{\diamond_T} K\mathbf{Z}_2, \mu_{A\natural_R K\mathbf{Z}_2}, 1_A \otimes 1_{K\mathbf{Z}_2}, \Delta_{A\diamond_T K\mathbf{Z}_2}, \varepsilon_A \otimes \varepsilon_{K\mathbf{Z}_2}, \alpha \otimes id_{K\mathbf{Z}_2})$ is a twisted tensor biproduct Hom-bialgebra. Furthermore,

$(A^{\natural_R}_{\diamond_T} K\mathbf{Z}_2, \alpha \otimes id_{K\mathbf{Z}_2}, S_{A^{\natural_R}_{\diamond_T} K\mathbf{Z}_2})$ is a Hom-Hopf algebra, where $S_{A^{\natural}_{\diamond} K\mathbf{Z}_2}$ is defined by

$$S_{A^{\natural}_{\diamond} K\mathbf{Z}_2}(1_A \otimes 1_{K\mathbf{Z}_2}) = 1_A \otimes 1_{K\mathbf{Z}_2}; \quad S_{A^{\natural}_{\diamond} K\mathbf{Z}_2}(1_A \otimes a) = 1_A \otimes a$$
$$S_{A^{\natural}_{\diamond} K\mathbf{Z}_2}(x \otimes 1_{K\mathbf{Z}_2}) = x \otimes a; \quad S_{A^{\natural}_{\diamond} K\mathbf{Z}_2}(x \otimes a) = -x \otimes 1_{K\mathbf{Z}_2}.$$

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