Chapter 8
Solutions to rough differential equations

We show how to solve differential equations driven by rough paths by a simple Picard iteration argument. This yields a pathwise solution theory mimicking the standard solution theory for ordinary differential equations. We start with the simple case of differential equations driven by a signal that is sufficiently regular for Young's theory of integration to apply and then proceed to the case of more general rough signals.

8.1 Introduction

We now turn our attention to (rough) differential equations of the form

$$dY_t = f(Y_t)\,dX_t\;, \qquad Y_0 = \xi \in W\;. \tag{8.1}$$

Here, $X : [0,T] \to V$ is the driving or input signal, while $Y : [0,T] \to W$ is the output signal. As usual V and W are Banach spaces, and $f : W \to \mathcal{L}(V,W)$. When $\dim V = d < \infty$, one may think of f as a collection of vector fields $(f_1,\ldots,f_d)$ on W. As usual, the reader is welcome to think $V = \mathbf{R}^d$ and $W = \mathbf{R}^n$ but there is really no difference in the argument. Such equations are familiar from the theory of ODEs, and more specifically, control theory, where X is typically assumed to be absolutely continuous so that $dX_t = \dot{X}_t\,dt$. The case of SDEs, *stochastic differential equations*, with dX interpreted as Itô or Stratonovich differential of Brownian motion, is also well known. Both cases will be seen as special examples of RDEs, *rough differential equations*.

We may consider (8.1) on the unit time interval. Indeed, equation (8.1) is invariant under time-reparametrisation so that any (finite) time horizon may be rescaled to $[0,1]$. Alternatively, global solutions on a larger time horizon are constructed successively, i.e. by concatenating $Y|_{[0,1]}$ (started at Y_0) with $Y|_{[1,2]}$ (started at Y_1) and so on. As a matter of fact, we shall construct solutions by a variation of the classical *Picard iteration* on intervals $[0,T]$, where $T \in (0,1]$ will be chosen sufficiently small to guarantee invariance of suitable balls and the contraction property. Our key

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ingredients are estimates for rough integrals (cf. Theorem 4.10) and the composition of controlled paths with smooth maps (Lemma 7.3). Recall that, for rather trivial reasons (of the sort $|t-s|^{2\alpha} \le |t-s|$, when $0 \le s \le t \le T \le 1$), all constants in these estimates were seen to be uniform in $T \in (0,1]$.

8.2 Review of the Young case: a priori estimates

Let us postulate that there exists a solution to a differential equation in Young's sense and let us derive an a-priori estimate. (In finite dimension, this can actually be used to prove the existence of solutions. Note that the regularity requirement here is "one degree less" than what is needed for the corresponding uniqueness result.)

Proposition 8.1. *Assume $X, Y \in \mathcal{C}^\beta([0,1], V)$ for some $\beta \in (1/2, 1]$ such that, given $\xi \in W$, $f \in \mathcal{C}^1_b(W, \mathcal{L}(V, W))$, we have*

$$dY_t = f(Y_t) dX_t \,, \qquad Y_0 = \xi \,,$$

in the sense of a Young integral equation. Then

$$\|Y\|_\beta \le C\Big[\Big(\|f\|_{\mathcal{C}^1_b}\|X\|_\beta\Big) \vee \Big(\|f\|_{\mathcal{C}^1_b}\|X\|_\beta\Big)^{1/\beta}\Big].$$

Proof. By assumption, for $0 \le s < t \le 1$, $Y_{s,t} = \int_s^t f(Y_r) dX_r$. Using Young's inequality (4.3), with $C = C(\beta)$,

$$\begin{aligned} |Y_{s,t} - f(Y_s)X_{s,t}| &= \left| \int_s^t (f(Y_r) - f(Y_s)) dX_r \right| \\ &\le C\|Df\|_\infty \|Y\|_{\beta;[s,t]} \|X\|_{\beta;[s,t]} |t-s|^{2\beta} \end{aligned}$$

so that

$$|Y_{s,t}|/|t-s|^\beta \le \|f\|_\infty \|X\|_\beta + C\|Df\|_\infty \|Y\|_{\beta;[s,t]} \|X\|_{\beta;[s,t]} |t-s|^\beta.$$

Write $\|Y\|_{\beta;h} \equiv \sup |Y_{s,t}|/|t-s|^\beta$ where the sup is restricted to times $s, t \in [0,1]$ for which $|t-s| \le h$. Clearly then,

$$\|Y\|_{\beta;h} \le \|f\|_\infty \|X\|_\beta + C\|Df\|_\infty \|Y\|_{\beta;h} \|X\|_\beta h^\beta$$

and upon taking h small enough, s.t. $\delta h^\beta \asymp 1$, with $\delta = \|X\|_\beta$, more precisely s.t.

$$C\|Df\|_\infty \|X\|_\beta h^\beta \le C\Big(1 + \|f\|_{\mathcal{C}^1_b}\Big) \|X\|_\beta h^\beta \le 1/2$$

(we will take h such that the second $\le$ becomes an equality; adding 1 avoids trouble when $f \equiv 0$)

$$\frac{1}{2}\|Y\|_{\beta;h} \le \|f\|_\infty \|X\|_\beta .$$

It then follows from Exercise 4.5 that, with $h \propto \|X\|_\beta^{-1/\beta}$,

$$\begin{aligned}\|Y\|_\beta &\le \|Y\|_{\beta;h}\left(1 \vee h^{-(1-\beta)}\right) \le C\|X\|_\beta\left(1 \vee h^{-(1-\beta)}\right)\\ &= C\left(\|X\|_\beta \vee \|X\|_\beta^{1/\beta}\right) .\end{aligned}$$

Here, we have absorbed the dependence on $f \in \mathcal{C}_b^1$ into the constants. By scaling (any non-zero f may be normalised to $\|f\|_{\mathcal{C}_b^1} = 1$ at the price of replacing X by $\|f\|_{\mathcal{C}_b^1} \times X$) we then get immediately the claimed estimate. □

8.3 Review of the Young case: Picard iteration

The reader may be helped by first reviewing the classical Picard argument in a Young setting, i.e. when $\beta \in (1/2, 1]$. Given $\xi \in W$, $f \in \mathcal{C}_b^2(W, \mathcal{L}(V, W))$, $X \in \mathcal{C}^\beta([0,1], V)$ and $Y : [0,T] \to W$ of suitable Hölder regularity, $T \in (0,1]$, one defines the map $\mathcal{M}_T$ by

$$\mathcal{M}_T(Y) := \left(\xi + \int_0^t f(Y_s)dX_s : t \in [0,T]\right) .$$

Following a classical pattern of proof, we shall establish invariance of suitable balls, and then a contraction property upon taking $T = T_0$ small enough. The resulting unique fixed point is then obviously the unique solution to (8.1) on $[0, T_0]$. The unique solution on $[0,1]$ is then constructed successively, i.e. by concatenating the solution Y on $[0, T_0]$, started at $Y_0 = \xi$, with the solution Y on $[T_0, 2T_0]$ started at Y_{T_0} and so on. Care is necessary to ensure that T_0 can be chosen uniformly; for instance, if f were only $\mathcal{C}^2$ (without the boundedness assumption) one can still obtain local existence on $[0, T_1]$, and then $[T_1, T_2]$, etc, but the resulting maximal solution (with respect to extension of solutions) may only be exist on $[0, \tau)$, for some $\lim_n T_n = \tau \le T = 1$. In finite dimension, τ can be identified as explosion time, see also Exercise 8.4. (The situation here is completely analogous to the theory of Banach valued ODEs.)

We will need the Hölder norm of X over $[0,T]$ to tend to zero as $T \downarrow 0$. Now, as the example of the map $t \mapsto t$ and $\beta = 1$ shows, this may not be true relative to the β-Hölder norm; the (cheap) trick is to take $\alpha \in (1/2, \beta)$ and to view $\mathcal{M}_T$ as map from the Banach space $\mathcal{C}^\alpha([0,T], W)$, rather than $\mathcal{C}^\beta([0,T], W)$, into itself. Young's inequality is still applicable since all paths involved will be (at least) α-Hölder continuous with $\alpha > 1/2$. On the other hand,

$$\|X\|_{\alpha;[0,T]} \le T^{\beta-\alpha}\|X\|_{\beta;[0,T]} ,$$

and so the α-Hölder norm of X has the desired behaviour. As previously, when no confusion is possible, we write $\|\cdot\|_\alpha \equiv \|\cdot\|_{\alpha;[0,T]}$.

To avoid norm versus seminorm considerations, it is convenient to work on the space of paths started at ξ, namely $\{Y \in \mathcal{C}^\alpha([0,T],W) : Y_0 = \xi\}$. This affine subspace is a complete metric space under $(Y,\tilde{Y}) \mapsto \|Y-\tilde{Y}\|_\alpha$ and so is the closed unit ball

$$\mathcal{B}_T = \{Y \in \mathcal{C}^\alpha([0,T],W) : Y_0 = \xi, \|Y\|_\alpha \le 1\} .$$

Young's inequality (4.41) shows that there is a constant C which only depends on α (thanks to $T \le 1$) such that for every $Y \in \mathcal{B}_T$,

$$\begin{aligned}\|\mathcal{M}_T(Y)\|_\alpha &\le C(|f(Y_0)| + \|f(Y)\|_\alpha)\|X\|_\alpha \\ &\le C(|f(\xi)| + \|Df\|_\infty \|Y\|_\alpha)\|X\|_\alpha \\ &\le C(|f|_\infty + \|Df\|_\infty)\|X\|_\alpha \le C|f|_{\mathcal{C}_b^1}\|X\|_\beta T^{\beta-\alpha} .\end{aligned}$$

Similarly, for $Y,\tilde{Y} \in \mathcal{B}_T$, using Young, $f(Y_0) = f(\tilde{Y}_0)$ and Lemma 7.5 (with $K=1$)

$$\begin{aligned}\left\|\mathcal{M}_T(Y) - \mathcal{M}_T(\tilde{Y})\right\|_\alpha &= \left\|\int_0^\cdot f(Y_s)dX_s - \int_0^\cdot f(\tilde{Y}_s)dX_s\right\|_\alpha \\ &\le C\Big(\big|f(Y_0) - f(\tilde{Y}_0)\big| + \big\|f(Y) - f(\tilde{Y})\big\|_\alpha\Big)\|X\|_\alpha \\ &\le C\|f\|_{\mathcal{C}_b^2}\|X\|_\beta T^{\beta-\alpha}\big\|Y-\tilde{Y}\big\|_\alpha .\end{aligned}$$

It is clear from the previous estimates that a small enough $T_0 = T_0(f,\alpha,\beta,X) \le 1$ can be found such that $\mathcal{M}_{T_0}(\mathcal{B}_{T_0}) \subset \mathcal{B}_{T_0}$ and, for all $Y,\tilde{Y} \in \mathcal{B}_{T_0}$,

$$\big\|\mathcal{M}_{T_0}(Y) - \mathcal{M}_{T_0}(\tilde{Y})\big\|_{\alpha;[0,T_0]} \le \frac{1}{2}\big\|Y-\tilde{Y}\big\|_{\alpha;[0,T_0]} .$$

Therefore, $\mathcal{M}_{T_0}(\cdot)$ admits a unique fixed point $Y \in \mathcal{B}_{T_0}$ which is the unique solution Y to (8.1) on the (small) interval $[0,T_0]$. Noting that the choice $T_0 = T_0(f,\alpha,\beta,X)$ can indeed be done uniformly (in particular it does not change when the starting point ξ is replaced by Y_{T_0}), the unique solution on $[0,1]$ is then constructed iteratively, as explained in the beginning.

8.4 Rough differential equations: a priori estimates

We now consider a priori estimates for rough differential equations, similar to Section 8.2. Recall that the homogeneous rough path norm $|\!|\!|\mathbf{X}|\!|\!|_\alpha$ was introduced in (2.4).

Proposition 8.2. *Let* $\xi \in W,\ f \in \mathcal{C}_b^2(W,\mathcal{L}(V,W))$ *and a rough path* $\mathbf{X} = (X,\mathbb{X}) \in \mathscr{C}^\alpha$ *with* $\alpha \in (1/3,1/2]$ *and assume that* $(Y,Y') = (Y,f(Y)) \in \mathscr{D}_X^{2\alpha}$ *is an RDE* *solution to* $dY = f(Y)\, d\mathbf{X}$ *started at* $Y_0 = \xi \in W$. *That is, for all* $t \in [0,T]$,

$$Y_t = \xi + \int_0^t f(Y_s)\, d\mathbf{X}_s \,, \tag{8.2}$$

with integral interpreted in the sense of Theorem 4.10 and $(f(Y), f(Y)') \in \mathscr{D}_X^{2\alpha}$ *built from* Y *by Lemma 7.3. (Thanks to* $\mathcal{C}_b^2$*-regularity of* f *and Lemma 7.3 the above rough integral equation (8.2) is well-defined.*[1]*)*

Then the following (a priori) estimate holds true

$$\|Y\|_\alpha \le C\Big[\Big(\|f\|_{\mathcal{C}_b^2}\|\mathbf{X}\|_\alpha\Big) \vee \Big(\|f\|_{\mathcal{C}_b^2}\|\mathbf{X}\|_\alpha\Big)^{1/\alpha}\Big]$$

where $C = C(\alpha)$ *is a suitable constant.*

Proof. Consider an interval $I := [s,t]$ so that, using basic estimates for rough integrals (cf. Theorem 4.10),

$$\begin{aligned}
\big|R^Y_{s,t}\big| &= |Y_{s,t} - f(Y_s)X_{s,t}| \\
&\le \left|\int_s^t f(Y)dX - f(Y_s)X_{s,t} - Df(Y_s)f(Y_s)\mathbb{X}_{s,t}\right| + |Df(Y_s)f(Y_s)\mathbb{X}_{s,t}| \\
&\lesssim \Big(\|X\|_{\alpha;I}\big\|R^{f(Y)}\big\|_{2\alpha;I} + \|\mathbb{X}\|_{2\alpha;I}\|f(Y)\|_{\alpha;I}\Big)|t-s|^{3\alpha} \\
&\qquad + \|\mathbb{X}\|_{2\alpha;I}|t-s|^{2\alpha}.
\end{aligned} \tag{8.3}$$

Recall that $\|\cdot\|_\alpha$ is the usual Hölder seminorm over $[0,T]$, while $\|\cdot\|_{\alpha;I}$ denotes the same norm, but over $I \subset [0,T]$, so that trivially $\|X\|_{\alpha;I} \le \|X\|_\alpha$. Whenever notationally convenient, multiplicative constants depending on α and f are absorbed in $\lesssim$, at the very end we can use scaling to make the f dependence reappear. We will also write $\|\cdot\|_{\alpha;h}$ for the supremum of $\|\cdot\|_{\alpha;I}$ over all intervals $I \subset [0,T]$ with length $|I| \le h$. Again, one trivially has $\|X\|_{\alpha;I} \le \|X\|_{\alpha;h}$ whenever $|I| \le h$. Using this notation, we conclude from (8.3) that

$$\big\|R^Y\big\|_{2\alpha;h} \lesssim \|\mathbb{X}\|_{2\alpha;h} + \Big(\|X\|_{\alpha;h}\big\|R^{f(Y)}\big\|_{2\alpha;h} + \|\mathbb{X}\|_{2\alpha;h}\|f(Y)\|_{\alpha;h}\Big)h^\alpha.$$

We would now like to relate $R^{f(Y)}$ to R^Y. As in the proof of Lemma 7.3, we obtain the bound

$$\begin{aligned}
R^{f(Y)}_{s,t} &= f(Y_t) - f(Y_s) - Df(Y_s)Y'_sX_{s,t} \\
&= f(Y_t) - f(Y_s) - Df(Y_s)Y_{s,t} + Df(Y_s)R^Y_{s,t}
\end{aligned}$$

so that,

$$\big\|R^{f(Y)}\big\|_{2\alpha;h} \le \frac{1}{2}\big|D^2f\big|_\infty\|Y\|^2_{\alpha;h} + |Df|_\infty\big\|R^Y\big\|_{2\alpha;h}$$

[1] Later we will establish existence and uniqueness under $\mathcal{C}_b^3$-regularity.

$$\lesssim \|Y\|_{\alpha;h}^2 + \|R^Y\|_{2\alpha;h}.$$

Hence, also using $\|f(Y)\|_{\alpha;h} \lesssim \|Y\|_{\alpha;h}$, there exists $c_1 > 0$, not dependent on **X** or Y, such that

$$\begin{aligned} \|R^Y\|_{2\alpha;h} \le c_1 \|\mathbb{X}\|_{2\alpha;h} + c_1 \|X\|_{\alpha;h} h^\alpha \|Y\|_{\alpha;h}^2 \\ + c_1 \|X\|_{\alpha;h} h^\alpha \|R^Y\|_{2\alpha;h} + c_1 \|\mathbb{X}\|_{2\alpha;h} h^\alpha \|Y\|_{\alpha;h} . \end{aligned} \tag{8.4}$$

We now restrict ourselves to h small enough so that $\|\mathbf{X}\|_\alpha h^\alpha \ll 1$. More precisely, we choose it such that

$$c_1 \|X\|_\alpha h^\alpha \le \frac{1}{2} , \qquad c_1 \|\mathbb{X}\|_{2\alpha}^{1/2} h^\alpha \le \frac{1}{2} .$$

Inserting this bound into (8.4), we conclude that

$$\|R^Y\|_{2\alpha;h} \le c_1 \|\mathbb{X}\|_{2\alpha;h} + \frac{1}{2}\|Y\|_{\alpha;h}^2 + \frac{1}{2}\|R^Y\|_{2\alpha;h} + \|\mathbb{X}\|_{2\alpha;h}^{1/2} \|Y\|_{\alpha;h} .$$

This in turn yields the bound

$$\begin{aligned} \|R^Y\|_{2\alpha;h} &\le 2c_1 \|\mathbb{X}\|_{2\alpha;h} + \|Y\|_{\alpha;h}^2 + 2\|\mathbb{X}\|_{2\alpha;h}^{1/2} \|Y\|_{\alpha;h} \\ &\le c_2 \|\mathbb{X}\|_{2\alpha;h} + 2\|Y\|_{\alpha;h}^2 , \end{aligned} \tag{8.5}$$

with $c_2 = (2c_1 + 1)$. On the other hand, since $Y_{s,t} = f(Y_s)X_{s,t} - R_{s,t}^Y$ and f is bounded, we have the bound

$$\|Y\|_{\alpha;h} \lesssim \|X\|_\alpha + \|R^Y\|_{2\alpha;h} h^\alpha .$$

Combining this bound with (8.5) yields

$$\begin{aligned} \|Y\|_{\alpha;h} &\le c_3 \|X\|_\alpha + c_3 \|\mathbb{X}\|_{2\alpha;h} h^\alpha + c_3 \|Y\|_{\alpha;h}^2 h^\alpha \\ &\le c_3 \|X\|_\alpha + c_4 \|\mathbb{X}\|_{2\alpha;h}^{1/2} + c_3 \|Y\|_{\alpha;h}^2 h^\alpha , \end{aligned}$$

for some constants c_3 and c_4. Multiplication with $c_3 h^\alpha$ then yields, with $\psi_h := c_3 \|Y\|_{\alpha;h} h^\alpha$ and $\lambda_h := c_5 \|\mathbf{X}\|_\alpha h^\alpha \to 0$ as $h \to 0$,

$$\psi_h \le \lambda_h + \psi_h^2.$$

Clearly, for all h small enough depending on Y (so that $\psi_h \le 1/2$) $\psi_h \le \lambda_h + \psi_h/2$ implies $\psi_h \le 2\lambda_h$ and so

$$\|Y\|_{\alpha;h} \le c_6 \|\mathbf{X}\|_\alpha.$$

To see that this is true for all h small enough without dependence on Y, pick h_0 small enough so that $\lambda_{h_0} < 1/4$. It then follows that for each $h \le h_0$, one of the following two estimates must hold true

$$\psi_h \geq \psi_+ \equiv \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_h} \geq \frac{1}{2}$$
$$\psi_h \leq \psi_- \equiv \frac{1}{2} - \sqrt{\frac{1}{4} - \lambda_h} = \frac{1}{2}\Big(1 - \sqrt{1 - 4\lambda_h}\Big) \sim \lambda_h \text{ as } h \downarrow 0.$$

(In fact, for reasons that will become apparent shortly, we may decrease h_0 further to guarantee that for $h < h_0$ we have not only $\psi_h < 1/2$ but $\psi_h < 1/6$.) We already know that we are in the regime of the second estimate above as $h \downarrow 0$. Noting that $\psi_h (< 1/6) < 1/2$ in the second regime, the only reason that could prevent us from being in the second regime for all $h < h_0$ is an (upwards) jump of the (increasing) function $(0, h_0] \ni h \mapsto \psi_h$. But $\psi_h \leq 3 \lim_{g \uparrow h} \psi_g$, as seen from

$$\|Y\|_{\alpha;h} \leq 3\|Y\|_{\alpha;h/3} \leq 3 \lim_{g\uparrow h} \|Y\|_{\alpha;g} ,$$

(and similarly: $\lim_{g\downarrow h} \psi_g \leq 3\psi_h$) which rules out any jumps of relative jump size greater than 3. However, given that $\psi_h \geq 1/2$ in the first regime and $\psi_h < 1/6$ in the second, we can never jump from the second into the first regime, as h increases (from zero). And so, we indeed must be in the second regime for all $h \leq h_0$. Elementary estimates on ψ_-, as function of λ_h then show that

$$\|Y\|_{\alpha;h} \leq c_6 \|\mathbf{X}\|_\alpha ,$$

for all $h \leq h_0 \sim \|\mathbf{X}\|^{-1/\alpha}$. We conclude with Exercise 4.5, arguing exactly as in the Young case, Proposition 8.1. □

8.5 Rough differential equations

The aim of this section is to show that if f is regular enough and $(X, \mathbb{X}) \in \mathscr{C}^\beta$ with $\beta > \frac{1}{3}$, then we can solve differential equations driven by the rough path $\mathbf{X} = (X, \mathbb{X})$ of the type

$$dY = f(Y)\, d\mathbf{X} .$$

Such an equation will yield solutions in $\mathscr{D}_X^{2\alpha}$ and will be interpreted in the corresponding integral formulation, where the integral of $f(Y)$ against X is defined using Lemma 7.3 and Theorem 4.10. More precisely, one has the following local existence and uniqueness result. (The construction of a *maximal solution* is left as Exercise 8.4.)

Theorem 8.3. *Given* $\xi \in W$, $f \in \mathcal{C}^3(W, \mathcal{L}(V, W))$ *and a rough path* $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\beta([0, T], V)$ *with* $\beta \in (\frac{1}{3}, \frac{1}{2})$, *there exists* $0 < T_0 \leq T$ *and a unique element* $(Y, Y') \in \mathscr{D}_X^{2\beta}([0, T_0], W)$, *with* $Y' = f(Y)$, *such that, for all* $0 \leq t \leq T_0$,

$$Y_t = \xi + \int_0^t f(Y_s)\, d\mathbf{X}_s . \tag{8.6}$$

Here, the integral is interpreted in the sense of Theorem 4.10 and $f(Y) \in \mathscr{D}_X^{2\beta}$ is built from Y by Lemma 7.3. Moreover, if f is linear or $f \in \mathcal{C}_b^3$, we may take $T_0 = T$, and thus global existence holds on $[0,T]$.

Remark 8.4. The condition $Y' = f(Y)$ (and then $f(Y)' = Df(Y)Y'$ by Lemma 7.3) is crucial for uniqueness. To see what can happen, consider the canonical lift of $X \in \mathcal{C}^1$ to $\mathbf{X} = (X, \int X \otimes dX)$, in which case *any* choice of $f(Y)' \in \mathcal{C}^\beta$ yields a pair $(f(Y), f(Y)') \in \mathscr{D}_X^{2\beta}$. (Indeed, thanks to $|X_{s,t}| \lesssim |t-s|$, the term $f(Y)'_s X_{s,t}$ can always be absorbed in the 2β-remainder.) On the other hand, regardless of the choice of Y', or $f(Y)'$, the rough integral in (8.6) here always agrees with the Riemann-Stieltjes integral $\int f(Y)dX$, so that (8.6) is satisfied whenever Y solves the ODE $\dot{Y} = f(Y)\dot{X}$, with $Y_0 = \xi$.

Proof. With $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\beta \subset \mathscr{C}^\alpha$, $\frac{1}{3} < \alpha < \beta$ and $(Y, Y') \in \mathscr{D}_X^{2\alpha}$ we know from Lemma 7.3 that

$$(\Xi, \Xi') := \big(f(Y), f(Y)'\big) := (f(Y), Df(Y)Y') \in \mathscr{D}_X^{2\alpha} .$$

Restricting from $[0,1]$ to $[0,T]$, any $T \le 1$, Theorem 4.10 allows to define the map

$$\mathcal{M}_T(Y, Y') \stackrel{\text{def}}{=} \left(\xi + \int_0^\cdot \Xi_s d\mathbf{X}_s, \Xi\right) \in \mathscr{D}_X^{2\alpha} .$$

The RDE solution on $[0,T]$ we are looking for is a fixed point of this map. Strictly speaking, this would only yield a solution (Y, Y') in $\mathscr{D}_X^{2\alpha}$. But since $\mathbf{X} \in \mathscr{C}^\beta$, it turns out that this solution is automatically an element of $\mathscr{D}_X^{2\beta}$. Indeed, $|Y_{s,t}| \le |Y'|_\infty |X_{s,t}| + \left\|R^Y\right\|_{2\alpha} |t-s|^{2\alpha}$, so that $Y \in \mathcal{C}^\beta$. From the fixed point property it then follows that $Y' = f(Y) \in \mathcal{C}^\beta$ and also $R^Y \in \mathcal{C}_2^{2\beta}$, since $\mathbb{X} \in \mathcal{C}_2^{2\beta}$ and

$$\begin{aligned} \left|R^Y_{s,t}\right| = \left|Y_{s,t} - Y'_s X_{s,t}\right| &= \left|\int_s^t (f(Y_r) - f(Y_s))d\mathbf{X}_t\right| \\ &\le |Y'|_\infty |\mathbb{X}_{s,t}| + \mathrm{O}\big(|t-s|^{3\alpha}\big) . \end{aligned}$$

Note that if (Y, Y') is such that $(Y_0, Y'_0) = (\xi, f(\xi))$, then the same is true for $\mathcal{M}_T(Y, Y')$. Therefore, $\mathcal{M}_T$ can be viewed as map on the space of controlled paths started at $(\xi, f(\xi))$, i.e.

$$\big\{(Y, Y') \in \mathscr{D}_X^{2\alpha}([0,T], W) : Y_0 = \xi, Y'_0 = f(\xi)\big\} .$$

Since $\mathscr{D}_X^{2\alpha}$ is a Banach space (under the norm $(Y, Y') \mapsto |Y_0| + |Y'_0| + \|Y, Y'\|_{X,2\alpha}$) the above (affine) subspace is a complete metric space under the induced metric. This is also true for the (closed) unit ball $\mathcal{B}_T$ centred at, say

$$t \mapsto (\xi + f(\xi)X_{0,t}, f(\xi)).$$

(Note here that the apparently simpler choice $t \mapsto \big(\xi, f(\xi)\big)$ does in general not belong to $\mathscr{D}_X^{2\alpha}$.) In other words, $\mathcal{B}_T$ is the set of all $(Y, Y') \in \mathscr{D}_X^{2\alpha}([0,T], W)$:

$Y_0 = \xi, Y_0' = f(\xi)$ and

$$
\begin{aligned}
|Y_0 - \xi| + |Y_0' - f(\xi)| + \|(Y - (\xi + f(\xi)X_{0,\cdot}), Y_\cdot' - f(\xi))\|_{X,2\alpha} \\
= \|(Y - f(\xi)X_{0,\cdot}, Y_\cdot' - f(\xi))\|_{X,2\alpha} \le 1.
\end{aligned}
$$

In fact, $\|(Y - f(\xi)X_{0,\cdot}, Y_\cdot' - f(\xi))\|_{X,2\alpha} = \|Y, Y_\cdot'\|_{X,2\alpha}$ as a consequence of the triangle inequality and $\|(f(\xi)X_{0,\cdot}, f(\xi))\|_{X,2\alpha} = \|f(\xi)\|_\alpha + \|0\|_{2\alpha} = 0$, so that

$$
\mathcal{B}_T = \Big\{(Y, Y') \in \mathscr{D}_X^{2\alpha}([0,T], W) : Y_0 = \xi, Y_0' = f(\xi) : \|(Y, Y_\cdot')\|_{X,2\alpha} \le 1\Big\}.
$$

Let us also note that, for all $(Y, Y') \in \mathcal{B}_T$, one has the bound

$$
|Y_0'| + \|(Y, Y')\|_{X,2\alpha} \le |f|_\infty + 1 =: M \in [1, \infty). \tag{8.7}
$$

We now show that, for T small enough, $\mathcal{M}_T$ leaves $\mathcal{B}_T$ invariant and in fact is contracting. Constants below are denoted by C, may change from line to line and may depend on $\alpha, \beta, X, \mathbb{X}$ without special indication. They are, however, uniform in $T \in (0,1]$ and we prefer to be explicit (enough) with respect to f such as to see where $\mathcal{C}_b^3$-regularity is used. With these conventions, we recall the following estimates, direct consequences from Lemma 7.3 and Theorem 4.10 , respectively,

$$
\begin{aligned}
\|\Xi, \Xi'\|_{X,2\alpha} &\le CM\|f\|_{\mathcal{C}_b^2}\big(|Y_0'| + \|Y, Y'\|_{X,2\alpha}\big) \\
\left\|\int_0^\cdot \Xi_s d\mathbf{X}_s, \Xi\right\|_{X,2\alpha} &\le \|\Xi\|_\alpha + \|\Xi'\|_\infty \|\mathbb{X}\|_{2\alpha} \\
&\quad + C\big(\|X\|_\alpha \big\|R^\Xi\big\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha}\|\Xi'\|_\alpha\big) \\
&\le \|\Xi\|_\alpha + C\big(|\Xi_0'| + \|\Xi, \Xi'\|_{X,2\alpha}\big)(\|X\|_\alpha + \|\mathbb{X}\|_{2\alpha}) \\
&\le \|\Xi\|_\alpha + C\big(|\Xi_0'| + \|\Xi, \Xi'\|_{X,2\alpha}\big)T^{\beta-\alpha}.
\end{aligned}
$$

Invariance: For $(Y, Y') \in \mathcal{B}_T$, noting that $\|\Xi\|_\alpha = \|f(Y)\|_\alpha \le \|f\|_{\mathcal{C}_b^1}\|Y\|_\alpha$ and that $|\Xi_0'| = |Df(Y_0)Y_0'| \le \|f\|_{\mathcal{C}_b^1}^2$, we obtain the bound

$$
\begin{aligned}
\big\|\mathcal{M}_T(Y, Y')\big\|_{X,2\alpha} &= \left\|\int_0^\cdot \Xi_s d\mathbf{X}_s, \Xi\right\|_{X,2\alpha} \\
&\le \|\Xi\|_\alpha + C\big(|\Xi_0'| + \|\Xi, \Xi'\|_{X,2\alpha}\big)T^{\beta-\alpha} \\
&\le \|f\|_{\mathcal{C}_b^1}\|Y\|_\alpha + C\Big(\|f\|_{\mathcal{C}_b^1}^2 + CM\|f\|_{\mathcal{C}_b^2}\big(|Y_0'| + \|Y, Y'\|_{X,2\alpha}\big)\Big)T^{\beta-\alpha} \\
&\le \|f\|_{\mathcal{C}_b^1}(\|f\|_\infty + 1)T^{\beta-\alpha} + CM\Big(\|f\|_{\mathcal{C}_b^1}^2 + \|f\|_{\mathcal{C}_b^2}(\|f\|_\infty + 1)\Big)T^{\beta-\alpha},
\end{aligned}
$$

where in the last step we used (8.7) and also $\|Y\|_{\alpha;[0,T]} \le C_f T^{\beta-\alpha}$, seen from

$$
|Y_{s,t}| \le |Y'|_\infty |X_{s,t}| + \big\|R^Y\big\|_{2\alpha}|t-s|^{2\alpha}
$$

$$\leq (|Y_0'| + \|Y'\|_\alpha)\|X\|_\beta |t-s|^\beta + \big\|R^Y\big\|_{2\alpha}|t-s|^{2\alpha}\ .$$

Then, using $T^\alpha \leq T^{\beta-\alpha}$ and $\big\|R^Y\big\|_{2\alpha} \leq \|Y,Y'\|_{X,2\alpha} \leq 1$, we obtain the bound

$$\begin{aligned}\|Y\|_{\alpha;[0,T]} &\leq \big(|Y_0'| + \|Y,Y'\|_{X,2\alpha}\big)\|X\|_\beta T^{\beta-\alpha} + \big\|R^Y\big\|_{2\alpha}T^{\beta-\alpha} \\ &\leq \big((\|f\|_\infty + 1)\|X\|_\beta + 1\big)T^{\beta-\alpha}\ .\end{aligned} \tag{8.8}$$

In other words, $\|\mathcal{M}_T(Y,Y')\|_{X,2\alpha} = \|\mathcal{M}_T(Y,Y')\|_{X,2\alpha;[0,T]} = \mathrm{O}\big(T^{\beta-\alpha}\big)$ with constant only depending on α, β, $\mathbf{X}$ and $f \in \mathcal{C}_b^2$. By choosing $T = T_0$ small enough, we obtain the bound $\|\mathcal{M}_{T_0}(Y,Y')\|_{X,2\alpha;[0,T_0]} \leq 1$ so that $\mathcal{M}_{T_0}$ leaves $\mathcal{B}_{T_0}$ invariant, as desired.

Contraction: Setting $\Delta_s = f(Y_s) - f\big(\tilde{Y}_s\big)$ as a shorthand, we have the bound

$$\begin{aligned}\big\|\mathcal{M}_T\big(Y,Y'\big) - \mathcal{M}_T\big(\tilde{Y},\tilde{Y}'\big)\big\|_{X,2\alpha} &= \left\|\int_0^\cdot \Delta_s d\mathbf{X}_s, \Delta\right\|_{X,2\alpha} \\ &\leq \|\Delta\|_\alpha + C\big(|\Delta_0'| + \|\Delta,\Delta'\|_{X,2\alpha}\big)T^{\beta-\alpha} \\ &\leq C\|f\|_{\mathcal{C}_b^2}\big\|Y-\tilde{Y}\big\|_\alpha + C\|\Delta,\Delta'\|_{X,2\alpha}T^{\beta-\alpha}\ .\end{aligned}$$

The contraction property is obvious, provided that we can establish the following two estimates:

$$\big\|Y-\tilde{Y}\big\|_\alpha \leq CT^{\beta-\alpha}\big\|Y-\tilde{Y},Y'-\tilde{Y}'\big\|_{X,2\alpha}\ , \tag{8.9}$$

$$\big\|\Delta,\Delta'\big\|_{X,2\alpha} \leq C\big\|Y-\tilde{Y},Y'-\tilde{Y}'\big\|_{X,2\alpha}\ . \tag{8.10}$$

To obtain (8.9), replace Y by $Y-\tilde{Y}$ in (8.8), noting $Y_0' - \tilde{Y}_0' = 0$, and this shows

$$\begin{aligned}\big\|Y-\tilde{Y}\big\|_\alpha &\leq \big\|Y'-\tilde{Y}'\big\|_\alpha\|X\|_\beta T^{\beta-\alpha} + \big\|R^Y - R^{\tilde{Y}}\big\|_{2\alpha}T^{\beta-\alpha} \\ &\leq CT^{\beta-\alpha}\big\|Y-\tilde{Y},Y'-\tilde{Y}'\big\|_{X,2\alpha}.\end{aligned}$$

We now turn to (8.10). Similar to the proof of Lemma 7.5, $f \in \mathcal{C}^3$ allows to write $\Delta_s = G_s H_s$ where

$$G_s := g\big(Y_s,\tilde{Y}_s\big)\ ,\quad H_s := Y_s - \tilde{Y}_s\ ,$$

and $g \in \mathcal{C}_b^2$ with $\|g\|_{\mathcal{C}_b^2} \leq C\|f\|_{\mathcal{C}_b^3}$. Lemma 7.3 tells us that $(G,G') \in \mathscr{D}_X^{2\alpha}$ (with $G' = (D_Y g)Y' + (D_{\tilde{Y}} g)\tilde{Y}'$) and in fact immediately yields an estimate of the form

$$\|G,G'\|_{X,2\alpha} \leq C\|f\|_{\mathcal{C}_b^3}\ ,$$

uniformly over $\big(Y,Y'\big), \big(\tilde{Y},\tilde{Y}'\big) \in \mathcal{B}_T$ and $T \leq 1$. On the other hand, $\mathscr{D}_X^{2\alpha}$ is an algebra in the sense that $(GH,(GH)') \in \mathscr{D}_X^{2\alpha}$ with $(GH)' = G'H + GH'$. In fact, we leave it as easy exercise to the reader to check that

$$\|GH, (GH)'\|_{X,2\alpha} \lesssim \big(|G_0| + |G_0'| + \|G, G'\|_{X,2\alpha}\big) \\ \times \big(|H_0| + |H_0'| + \|H, H'\|_{X,2\alpha}\big) \ .$$

In our situation, $H_0 = Y_0 - \tilde{Y}_0 = \xi - \xi = 0$, and similarly $H_0' = 0$, so that, for all $\big(Y, Y'\big), \big(\tilde{Y}, \tilde{Y}'\big) \in \mathcal{B}_T$, we have

$$\begin{aligned} \big\|\Delta, \Delta'\big\|_{X,2\alpha} &\lesssim \big(|G_0| + |G_0'| + \|G, G'\|_{X,2\alpha}\big)\|H, H'\|_{X,2\alpha} \\ &\lesssim \big(\|g\|_\infty + \|g\|_{\mathcal{C}_b^1}\big(\big|Y_0'\big| + \big|\tilde{Y}_0'\big|\big) + C\|f\|_{\mathcal{C}_b^3}\big)\big\|Y - \tilde{Y}, Y' - \tilde{Y}'\big\|_{X,2\alpha} \\ &\lesssim \big\|Y - \tilde{Y}, Y' - \tilde{Y}'\big\|_{X,2\alpha} \ , \end{aligned}$$

where we made use of $\|g\|_\infty, \|g\|_{\mathcal{C}_b^1} \lesssim \|f\|_{\mathcal{C}_b^3}$ and $|Y_0'| = \big|\tilde{Y}_0'\big| = |f(\xi)| \le |f|_\infty$.

The argument from here on is identical to the Young case: the previous estimates allow for a small enough $T_0 \le 1$ such that $\mathcal{M}_{T_0}(\mathcal{B}_{T_0}) \subset \mathcal{B}_{T_0}$ and for all $\big(Y, Y'\big), \big(\tilde{Y}, \tilde{Y}'\big) \in \mathcal{B}_{T_0}$:

$$\big\|\mathcal{M}_{T_0}\big(Y, Y'\big) - \mathcal{M}_{T_0}\big(\tilde{Y}, \tilde{Y}'\big)\big\|_{X,2\alpha} \le \frac{1}{2}\big\|Y - \tilde{Y}, Y' - \tilde{Y}'\big\|_{X,2\alpha}$$

and so $\mathcal{M}_{T_0}(\cdot)$ admits a unique fixed point $(Y, Y') \in \mathcal{B}_{T_0}$, which is then the unique solution Y to (8.1) on the (possibly rather small) interval $[0, T_0]$. Noting that the choice of T_0 can again be done uniformly in the starting point, the solution on $[0, 1]$ is then constructed iteratively as before. □

In many situations, one is interested in solutions to an equation of the type

$$dY = f_0(Y, t)\, dt + f(Y, t)\, d\mathbf{X}_t \ , \tag{8.11}$$

instead of (8.6). On the one hand, it is possible to recast (8.11) in the form (8.6) by writing it as an RDE for $\hat{Y}_t = (Y_t, t)$ driven by $\hat{\mathbf{X}}_t = (\hat{X}, \hat{\mathbb{X}})$ where $\hat{X} = (X_t, t)$ and $\hat{\mathbb{X}}$ is given by $\mathbb{X}$ and the "remaining cross integrals" of X_t and t, given by usual Riemann-Stieltjes integration. However, it is possible to exploit the structure of (8.11) to obtain somewhat better bounds on the solutions. See [FV10b, Ch. 12].

8.6 Stability III: Continuity of the Itô–Lyons map

We now obtain continuity of solutions to rough differential equations as function of their (rough) driving signals.

Theorem 8.5 (Rough path stability of the Itô–Lyons map). *Let $f \in \mathcal{C}_b^3$ and, for $\alpha \in \big(\frac{1}{3}, \frac{1}{2}\big]$, let $(Y, f(Y)) \in \mathscr{D}_X^{2\alpha}$ be the unique RDE solution given by Theorem 8.3 to*

$$dY = f(Y)\, d\mathbf{X}, \quad Y_0 = \xi \in W \ .$$

Similarly, let $(\tilde{Y}, f(\tilde{Y}))$ *be the RDE solution driven by* $\tilde{\mathbf{X}}$ *and started at* $\tilde{\xi}$ *where* $\mathbf{X}, \tilde{\mathbf{X}} \in \mathscr{C}^\alpha$. *Assuming*

$$\|\mathbf{X}\|_\alpha, \|\tilde{\mathbf{X}}\|_\alpha \le M < \infty$$

we have the local Lipschitz estimates

$$d_{X,\tilde{X},2\alpha}\big(Y, f(Y); \tilde{Y}, f(\tilde{Y})\big) \le C_M\big(|\xi - \tilde{\xi}| + \varrho_\alpha(\mathbf{X}, \tilde{\mathbf{X}})\big) ,$$

and also

$$\big\|Y - \tilde{Y}\big\|_\alpha \le C_M\big(|\xi - \tilde{\xi}| + \varrho_\alpha(\mathbf{X}, \tilde{\mathbf{X}})\big) ,$$

where $C_M = C(M, \alpha, f)$ *is a suitable constant.*

Remark 8.6. The proof only uses the a priori information that RDE solutions remain bounded if the driving rough paths do, combined with basic stability properties of rough integration and composition.

Proof. Recall that, for given $\mathbf{X} \in \mathscr{C}^\alpha$, the RDE solution $(Y, f(Y)) \in \mathscr{D}_X^{2\alpha}$ is constructed as the unique fixed point of

$$\mathcal{M}_T(Y, Y') := (Z, Z') := \left(\xi + \int_0^\cdot f(Y_s) d\mathbf{X}_s, f(Y_\cdot)\right) \in \mathscr{D}_X^{2\alpha} ,$$

and similarly for $\tilde{\mathcal{M}}_T\big(\tilde{Y}, f\big(\tilde{Y}\big)\big) \in \mathcal{C}_{\tilde{X}}^\alpha$. Then, thanks to the fixed point property

$$(Y, f(Y)) = (Y, Y') = (Z, Z') = (Z, f(Y)) ,$$

(similarly with tilde) and the local Lipschitz estimate for rough integration, Theorem 4.17, and writing $(\Xi, \Xi') := \big(f(Y), f(Y)'\big)$ for the integrand, we obtain the bound

$$\begin{aligned} d_{X,\tilde{X},2\alpha}\big(Y, Y'; \tilde{Y}, \tilde{Y}'\big) &= d_{X,\tilde{X},2\alpha}\big(Z, Z'; \tilde{Z}, \tilde{Z}'\big) \\ &\lesssim \varrho_\alpha\big(\mathbf{X}, \tilde{\mathbf{X}}\big) + \big|\xi - \tilde{\xi}\big| + T^\alpha d_{X,\tilde{X},2\alpha}\big(\Xi, \Xi'; \tilde{\Xi}, \tilde{\Xi}'\big) , \end{aligned}$$

Thanks to the local Lipschitz estimate for composition, Theorem 7.6, uniform in $T \le 1$,

$$d_{X,\tilde{X},2\alpha}\big(\Xi, \Xi'; \tilde{\Xi}, \tilde{\Xi}'\big) \lesssim \varrho_\alpha\big(\mathbf{X}, \tilde{\mathbf{X}}\big) + \big|\xi - \tilde{\xi}\big| + d_{X,\tilde{X},2\alpha}\big(Y, f(Y); \tilde{Y}, f\big(\tilde{Y}\big)\big) .$$

In summary, for some constant $C = C(\alpha, f, M)$, we have the bound

$$\begin{aligned} d_{X,\tilde{X},2\alpha}\big(Y, f(Y); \tilde{Y}, f\big(\tilde{Y}\big)\big) \le C\big(&\varrho_\alpha\big(\mathbf{X}, \tilde{\mathbf{X}}\big) + \big|\xi - \tilde{\xi}\big| \\ &+ T^\alpha d_{X,\tilde{X},2\alpha}\big(Y, f(Y); \tilde{Y}, f\big(\tilde{Y}\big)\big)\big) . \end{aligned}$$

By taking $T = T_0(M, \alpha, f)$ smaller, if necessary, we may assume that $CT^\alpha \le 1/2$, from which it follows that

$$d_{X,\tilde{X},2\alpha}\big(Y, f(Y); \tilde{Y}, f\big(\tilde{Y}\big)\big) \le 2C\big(\varrho_\alpha\big(\mathbf{X}, \tilde{\mathbf{X}}\big) + \big|\xi - \tilde{\xi}\big|\big) ,$$

which is precisely the required bound. The bound on $\|Y - \tilde{Y}\|_\alpha$ then follows as in (4.32), and these bounds can be iterated to cover a time interval of arbitrary (fixed) length. □

8.7 Davie's definition and numerical schemes

Fix $f \in \mathcal{C}_b^2(W, \mathcal{L}(V,W))$ and $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\beta([0,T], V)$ with $\beta > \frac{1}{3}$. Under these assumptions, the rough differential equation $dY = f(Y)d\mathbf{X}$ makes sense as well-defined integral equation. (In Theorem 8.3 we used additional regularity, namely $\mathcal{C}_b^3$, to establish existence of a *unique* solution on $[0,T]$.) By the very definition of an RDE solution, unique or not, $(Y, f(Y)) \in \mathscr{D}_X^{2\beta}$, i.e.

$$Y_{s,t} = f(Y_s)X_{s,t} + \mathrm{O}\big(|t-s|^{2\beta}\big) ,$$

and we recognise a step of *first-order Euler* approximation, $Y_{s,t} \approx f(Y_s)X_{s,t}$, started from Y_s. Clearly $\mathrm{O}\big(|t-s|^{2\beta}\big) = \mathrm{o}(|t-s|)$ if and only if $\beta > 1/2$ and one can show that iteration of such steps along a partition $\mathcal{P}$ of $[0,T]$ yields a convergent "Euler" scheme as $|\mathcal{P}| \downarrow 0$, see [Dav08] or [FV10b].

In the case $\beta \in \big(\frac{1}{3}, \frac{1}{2}\big]$ we have to exploit that we know more than just $(Y, f(Y)) \in \mathscr{D}_X^{2\beta}$. Indeed, since $Y_{s,t} = \int_s^t f(Y)dX$, estimate (4.22) for rough integrals tells us that, for all pairs s,t

$$Y_{s,t} = f(Y_s)X_{s,t} + (f(Y))'_s \mathbb{X}_{s,t} + \mathrm{O}\big(|t-s|^{3\beta}\big) . \tag{8.12}$$

Using the identity $f(Y)' = Df(Y)Y' = Df(Y)f(Y)$, this can be spelled out further to

$$Y_{s,t} = f(Y_s)X_{s,t} + Df(Y_s)f(Y_s)\mathbb{X}_{s,t} + \mathrm{o}(|t-s|) \tag{8.13}$$

and, omitting the small remainder term, we recognise a step of a *second-order Euler* or *Milstein* approximation. Again, one can show that iteration of such steps along a partition $\mathcal{P}$ of $[0,T]$ yields a convergent "Euler" scheme as $|\mathcal{P}| \downarrow 0$; see [Dav08] or [FV10b].

Remark 8.7. This schemes can be understood from simple Taylor expansions based on the differential equation $dY = f(Y)dX$, at least when X is smooth (enough), or via Itô's formula in a semimartingale setting. With focus on the smooth case, the Euler approximation is obtained by a "left-point freezing" approximation $f(Y_\cdot) \approx f(Y_s)$ over $[s,t]$ in the integral equation,

$$Y_{s,t} = \int_s^t f(Y_r)dX_r \approx f(Y_s)X_{s,t}$$

whereas the Milstein scheme, with $\mathbb{X}_{s,t} = \int_s^t X_{s,r}dX_r$ for smooth paths, is obtained from the next-best approximation

$$f(Y_r) \approx f(Y_s) + Df(Y_s)Y_{s,r}$$
$$\approx f(Y_s) + Df(Y_s)f(Y_s)X_{s,r} \; .$$

It turns out that the description (8.13) is actually a formulation that is equivalent to the RDE solution built previously in the following sense.

Proposition 8.8. *The following two statements are equivalent*

i) $(Y, f(Y))$ *is a RDE solution to (8.6), as constructed in Theorem 8.3.*
ii) $Y \in \mathcal{C}([0,T], W)$ *is an "RDE solution in the sense of Davie", i.e. in the sense of (8.13).*

Proof. We already discussed how (8.13) is obtained from an RDE solution to (8.6). Conversely, (8.13) implies immediately $Y_{s,t} = f(Y_s)X_{s,t} + \mathrm{O}\big(|t-s|^{2\beta}\big)$ which shows that $Y \in \mathcal{C}^\beta$ and also $Y' := f(Y) \in \mathcal{C}^\beta$, thanks to $f \in \mathcal{C}_b^2$, so that $(Y, f(Y)) \in \mathscr{D}_X^{2\beta}$. It remains to see, in the notation of the proof of Theorem 4.10, that $Y_{s,t} = (\mathcal{I}\Xi)_{s,t}$ with

$$\Xi_{s,t} = f(Y_s)X_{s,t} + (f(Y))'_s \mathbb{X}_{s,t} = f(Y_s)X_{s,t} + Df(Y_s)f(Y_s)\mathbb{X}_{s,t} \; .$$

To see this, we note that trivially $Y_{s,t} = (\mathcal{I}\tilde{\Xi})_{s,t}$ with $\tilde{\Xi}_{s,t} := Y_{s,t}$. But $\tilde{\Xi}_{s,t} = \Xi_{s,t} + \mathrm{o}(|t-s|)$ and one sees as in Remark 4.13 that $\mathcal{I}\tilde{\Xi} = \mathcal{I}\Xi$. □

8.8 Lyons' original definition

A slightly different notion of solution was originally introduced in [Lyo98] by Lyons.[2] This notion only uses the spaces $\mathscr{C}^\alpha$, without ever requiring the use of the spaces $\mathscr{D}_X^{2\alpha}$ of "controlled rough paths". Indeed, for $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha([0,T], V)$ and $F \in \mathcal{C}_b^2(V, \mathcal{L}(V, W))$ we can define an element $\mathbf{Z} = (Z, \mathbb{Z}) = I_F(X) \in \mathscr{C}^\alpha([0,T], W)$ directly by

$$Z_t \stackrel{\text{def}}{=} \big(\mathcal{I}\Xi\big)_{0,t} \; , \qquad \Xi_{s,t} = F(X_s)\, X_{s,t} + DF(X_s)\mathbb{X}_{s,t} \; ,$$
$$\mathbb{Z}_{s,t} \stackrel{\text{def}}{=} \big(\mathcal{I}\bar{\Xi}^s\big)_{s,t} \; , \quad \bar{\Xi}^s_{u,v} = Z_{s,u}\, Z_{u,v} + \big(F(X_u) \otimes F(X_u)\big)\mathbb{X}_{u,v} \; .$$

It is possible to check that $\bar{\Xi}^s \in \mathcal{C}_2^{\alpha,3\alpha}$ for every fixed s (see the proof of Theorem 4.10) so that the second line makes sense. It is also straightforward to check that $(Z, \mathbb{Z})$ satisfies (2.1), so that it does indeed belong to $\mathscr{C}^\alpha$. Actually, one can see that

$$Z_t = \int_0^t F(X_s)\, d\mathbf{X}_s \; , \qquad \mathbb{Z}_{s,t} = \int_s^t Z_{s,r} \otimes dZ_r \; ,$$

[2] As always, we only consider the step-2 α-Hölder case, i.e. $\alpha > \frac{1}{3}$, whereas Lyons' theory is valid for every Hölder-exponent $\alpha \in (0,1]$ (or: variation parameter $p \geq 1$) at the complication of heaving to deal with $\lfloor p \rfloor$ levels.

where the integrals are defined as in the previous sections, where $F(X) \in \mathscr{D}_X^{2\alpha}$ as in Section 7.3.

We can now define solutions to (8.6) in the following way.

Definition 8.9. A rough path $\mathbf{Y} = (Y, \mathbb{Y}) \in \mathscr{C}^\alpha([0,T], W)$ is a solution in the sense of Lyons to (8.6) if there exists $\mathbf{Z} = (Z, \mathbb{Z}) \in \mathscr{C}^\alpha(V \oplus W)$ such that the projection of $(Z, \mathbb{Z})$ onto $\mathscr{C}^\alpha(V)$ is equal to $(X, \mathbb{X})$, the projection onto $\mathscr{C}^\alpha(W)$ is equal to $(Y, \mathbb{Y})$, and $Z = I_F(Z)$ where

$$F(x,y) = \begin{pmatrix} I & 0 \\ f(y) & 0 \end{pmatrix} .$$

It is straightforward to see that if $(Y, Y') \in \mathscr{D}_X^{2\alpha}(W)$ is a solution to (8.6) in the sense of the previous section, then the path $Z = (X, Y) \in V \oplus W$ is controlled by X. As seen in Section 7.1, it can therefore be interpreted as an element of $\mathscr{C}^\alpha$. It follows immediately from the definitions that it is then also a solution in the sense of Lyons. Conversely, if $(Y, \mathbb{Y})$ is a solution in the sense of Lyons, then one can check that one necessarily has $(Y, f(Y)) \in \mathscr{D}_X^{2\alpha}(W)$ and that this is a solution in the sense of the previous section. We leave the verification of this fact as an exercise to the reader.

8.9 Linear rough differential equations

Let $X \in \mathcal{C}^1([0,1], V)$, $A \in \mathcal{L}(W, \mathcal{L}(V,W))$ with finite operator norm $\|A\|_{\text{op}} = a \in [0,\infty)$, and consider the linear differential equation $dY = AY dX$, with initial data $Y_0 \in W$, written in integral form as

$$Y_t = Y_0 + \int_0^t AY_s dX_s .$$

Clearly $|Y_t| \leq |Y_0| + a \int_0^t |Y_s| d|X|_s$ in terms of the Lipschitz path $|X|_t := \int_0^t |\dot{X}_s| ds$, and the classical Gronwall lemma gives

$$\|Y\|_{\infty;[0,1]} \leq |Y_0| \exp(a \|X\|_{1;[0,1]}) ,$$

with $\|X\|_{1;[0,1]} = \sup_{0 \leq s < t \leq 1} \frac{|X_{s,t}|}{|t-s|} = \sup_{0 \leq s \leq 1} |\dot{X}_s|$. Alternatively, one can extract from the integral formulation the estimate, valid for all $0 \leq s < t \leq 1$,

$$|Y_{s,t}| \leq a \|X\|_{1;[0,1]} \|Y\|_{\infty;[s,t]} |t-s| .$$

The following lemma, applied with $\alpha = 1$, then leads to a similar conclusion. More importantly, it will be seen to be applicable in rough situations with $\alpha < 1$.

Lemma 8.10. *(Rough Gronwall) Assume* $Y \in \mathcal{C}([0,1])$, $\alpha \in (0,1]$, *and*

$$|Y_{s,t}| \le M \|Y\|_{\infty;[s,t]} |t-s|^{\alpha}$$

whenever $0 \le s < t \le 1$. *Then there exists* $c = c_\alpha < \infty$ *such that*

$$\|Y\|_{\infty;[0,1]} \le c \exp(cM^{1/\alpha}) |Y_0|.$$

Remark 8.11. Since $|Y_{s,t}| \le 2\|Y\|_{\infty;[s,t]}$ the assumption is trivially satisfied for "distant" times s,t such that $M|t-s|^{\alpha} \ge 2$. It then suffices to check the assumption for "nearby" times with $M|t-s|^{\alpha} \le \theta$ with $\theta = 2$, and in fact any $\theta > 0$, at the price of replacing M by $\frac{2M}{\theta \wedge 2}$.

Proof. For any $\xi \in [s,t]$ have $|Y_\xi| \le |Y_s| + |Y_{s,\xi}| \le |Y_s| + M\|Y\|_{\infty;[s,t]}|t-s|^{\alpha}$, and so

$$\|Y\|_{\infty;[s,t]}(1 - M|t-s|^{\alpha}) \le |Y_s| \; .$$

Since $e^{-2x} \le 1 - x$ for $x \in [0,1/2]$, we have, for $M|t-s|^{\alpha} \in [0,1/2]$,

$$\|Y\|_{\infty;[s,t]} \le |Y_s| e^{2M|t-s|^{\alpha}} \le e|Y_s| \; .$$

This induces a greedy partition of $[0,1]$, of mesh-size $(2M)^{-1/\alpha}$ and hence no more than $(2M)^{1/\alpha} + 1$ intervals. The final estimate is then

$$\|Y\|_{\infty;[0,1]} \le e^{1+(2M)^{1/\alpha}} |Y_0| \; ,$$

so that the claimed estimate holds with $c = e \vee 2^{1/\alpha}$. □

We now apply this to linear (Young and rough) differential equations, without loss of generality posed on $[0,1]$. By general theory, Theorem 8.3, we have a (non-explosive) solution.

Proposition 8.12. *Let* Y *solve the linear Young differential equation* $dY = AY\,dX$, *started from* Y_0 *and driven by* $X \in \mathcal{C}^{\alpha}([0,1])$, $\alpha > 1/2$, *with* A *of finite operator norm* a. *Then there exists* $c = c(\alpha) \in (0,\infty)$ *so that*

$$\|Y\|_{\infty;[0,1]} \le c \exp\left(c (a \|X\|_{\alpha;[0,1]})^{1/\alpha} \right) |Y_0|.$$

Proof. By scaling A, we can and will assume $\|X\|_{\alpha;[0,1]} = 1$. Young's inequality gives, with $a = |A|$ and $c = c(\alpha)$,

$$|Y_{s,t}| \le |AY_s X_{s,t}| + \left| \int_s^t A(Y_r - Y_s) dX_r \right| \le a|Y_s||t-s|^{\alpha} + ca\|Y\|_{\alpha;[s,t]}|t-s|^{2\alpha}$$

and so $\frac{1}{2}\|Y\|_{\alpha;[s,t]} \le a|Y|_{\infty;[s,t]}$ whenever $ca|t-s|^{\alpha} \le 1/2$. Re-insert the estimate on $\|Y\|_{\alpha;[s,t]}$ (and also use $ca|t-s|^{\alpha} \le 1/2$) above to obtain precisely

$$|Y_{s,t}| \le a|Y_s||t-s|^{\alpha} + a|Y|_{\infty;[s,t]}|t-s|^{\alpha} \le 2a\|Y\|_{\infty;[s,t]}|t-s|^{\alpha}.$$

This holds whenever $ca|t-s|^\alpha \le 1/2$ and so we can conclude with the rough Gronwall lemma (and the remark after it). The constant c is allowed to change of course, but remains $c = c(\alpha)$. □

A similar result also holds in the rough case.

Proposition 8.13. *Let Y solve the linear rough differential equation $dY = AY d\mathbf{X}$, started from Y_0 and driven by $\mathbf{X} \in \mathscr{C}^\alpha([0,1])$, $\alpha > 1/3$, with A of finite operator norm a. Then there exists $c = c(\alpha) \in (0,\infty)$ so that*

$$\|Y\|_{\infty;[0,1]} \le c \exp\Big(c(a|\!|\!|\mathbf{X}|\!|\!|_{\alpha;[0,1]})^{1/\alpha}\Big)|Y_0|.$$

Proof. By scaling A, we can again assume unit (homogeneous) rough path norm for $\mathbf{X}$. By a basic estimate for rough integrals it then holds, with $c = c(\alpha) \in [1,\infty)$ and $a = |A|$,

$$\begin{aligned} |Y^\natural_{s,t}| &\le c\|AY, A^2Y\|_{2\alpha,X}|t-s|^{3\alpha} \le ca\|Y, AY\|_{2\alpha,X}|t-s|^{3\alpha} \\ &= ca(\|AY\|_\alpha + \|Y^\#\|_{2\alpha})|t-s|^{3\alpha}, \end{aligned}$$

using musical notation $Y_{s,t} \equiv AY_sX_{s,t} + Y^\#_{s,t} \equiv AY_sX_{s,t} + A^2Y_s\mathbb{X}_{s,t} + Y^\natural_{s,t}$. This entails

$$|Y^\#_{s,t}| \le |A^2Y_s\mathbb{X}_{s,t}| + |Y^\natural_{s,t}| \le a^2|Y_s||t-s|^{2\alpha} + (a\|Y\|_\alpha + \|Y^\#\|_{2\alpha})ca|t-s|^{3\alpha}$$

and so for all $s < t$ with $ca|t-s|^\alpha \le 1/2$ we obtain

$$\frac{1}{2}\|Y^\#\|_{2\alpha;[s,t]} \le a^2\|Y\|_{\infty;[s,t]} + \frac{a}{2}\|Y\|_\alpha.$$

Similarly, $|Y_{s,t}| \le |AY_sX_{s,t}| + |Y^\#_{s,t}| \le a|Y_s||t-s|^\alpha + (2a^2\|Y\|_{\infty;[s,t]} + a\|Y\|_\alpha)|t-s|^{2\alpha}$ and so

$$\frac{1}{2}\|Y\|_{\alpha;[s,t]} \le a\|Y\|_{\infty;[s,t]} + 2a^2\|Y\|_{\infty;[s,t]}|t-s|^\alpha \le 3a\|Y\|_{\infty;[s,t]}.$$

for all $s < t$ with $a|t-s|^\alpha \le 1/2$. Re-inserting this and the bound for $\|Y\|_\alpha = \|Y\|_{\alpha;[s,t]}$ in the above estimate for $|Y_{s,t}|$, we obtain

$$|Y_{s,t}| \le a|Y_s||t-s|^\alpha + 8a^2\|Y\|_{\infty;[s,t]}|t-s|^{2\alpha} \le 5a\|Y\|_{\infty;[s,t]}|t-s|^\alpha.$$

We conclude with the rough Gronwall lemma, just as in the Young case. □

Remark 8.14. All this can be vector-valued. Assuming X takes values in some space V and Y takes values in W, we should view A as a linear map $A\colon W \otimes V \to W$. The operator $A^2\colon W \otimes V \otimes V \to W$ should then be interpreted as $A \circ (A \otimes \mathrm{Id})$.

8.10 Stability IV: Flows

We briefly state, without proof, a result concerning regularity of flows associated to rough differential equations, as well as local Lipschitz estimates of the Itô–Lyons maps on the level of such flows. More precisely, given a geometric rough path $\mathbf{X} \in \mathscr{C}_g^\alpha([0,T],\mathbf{R}^d)$, we saw in Theorem 8.3 that, for $\mathcal{C}_b^3$ vector fields $f = (f_1,\ldots,f_d)$ on $\mathbf{R}^e$, there is a unique global solution to the rough integral equation

$$Y_t = y + \int_0^t f(Y_s)\, d\mathbf{X}_s\ , \qquad t \geq 0\ . \tag{8.14}$$

Write $\pi_{(f)}(0,y;\mathbf{X}) = Y$ for this solution. Note that the inverse flow exists trivially, by following the RDE driven by $\mathbf{X}(t-.)$,

$$\pi_{(f)}(0,\bullet;\mathbf{X})_t^{-1} = \pi_{(f)}(0,\bullet;\mathbf{X}(t-.))_t.$$

We call the map $y \mapsto \pi_{(f)}(0,y;\mathbf{X})$ the flow associated to the above RDE. Moreover, if X^ϵ is a smooth approximation to $\mathbf{X}$ (in rough path metric), then the corresponding ODE solution Y^ϵ is close to Y, with a local Lipschitz estimate as given in Section 8.6.

It is natural to ask if the flow depends smoothly on y. Given a multi-index $k = (k_1,\ldots,k_e) \in \mathbf{N}^e$, write D^k for the partial derivative with respect to $y^1,\ldots,y^e$. The proof of the following statement is an easy consequence of [FV10b, Chapter 12].

Theorem 8.15. *Let $\alpha \in (1/3,1/2]$ and $\mathbf{X},\tilde{\mathbf{X}} \in \mathscr{C}_g^\alpha$. Assume $f \in \mathcal{C}_b^{3+n}$ for some integer n. Then the associated flow is of regularity $\mathcal{C}^{n+1}$ in y, as is its inverse flow. The resulting family of partial derivatives, $\{D^k\pi_{(f)}(0,\xi;\mathbf{X}), |k| \leq n\}$ satisfies the RDE obtained by formally differentiating $dY = f(Y)d\mathbf{X}$.*

At last, for every $M > 0$ there exist C, K depending on M and the norm of f such that, whenever $\|\!|\mathbf{X}|\!\|_\alpha, \|\!|\tilde{\mathbf{X}}|\!\|_\alpha \leq M < \infty$ and $|k| \leq n$,

$$\begin{aligned}
\sup_{\xi\in\mathbf{R}^e} \big|D^k\pi_{(f)}(0,\xi;\mathbf{X}) - D^k\pi_{(f)}(0,\xi;\tilde{\mathbf{X}})\big|_{\alpha;[0,t]} &\leq C\varrho_\alpha(\mathbf{X},\tilde{\mathbf{X}}),\\
\sup_{\xi\in\mathbf{R}^e} \big|D^k\pi_{(f)}(0,\xi;\mathbf{X})^{-1} - D^k\pi_{(f)}(0,\xi;\tilde{\mathbf{X}})^{-1}\big|_{\alpha;[0,t]} &\leq C\varrho_\alpha(\mathbf{X},\tilde{\mathbf{X}}),\\
\sup_{\xi\in\mathbf{R}^e} \big|D^k\pi_{(f)}(0,\xi;\mathbf{X})\big|_{\alpha;[0,t]} &\leq K,\\
\sup_{\xi\in\mathbf{R}^e} \big|D^k\pi_{(f)}(0,\xi;\mathbf{X})^{-1}\big|_{\alpha;[0,t]} &\leq K.
\end{aligned}$$

8.11 Exercises

Exercise 8.1 *a) Consider the case of a smooth, one-dimensional driving signal $X : [0,T] \to \mathbf{R}$. Show that the solution map to the (ordinary) differential equation $dY = f(Y)dX$, for sufficiently nice f (say bounded with bounded derivatives)* *and started at some fixed point* $Y_0 = \xi$, *is locally Lipschitz continuous with respect to the driving signal in the supremum norm on* $[0,T]$. *Conclude that it admits a unique continuous extension to every continuous driving signal* X.

b) *Show by an example that no such continuous extension is possible, in general, in a multi-dimensional situation, with vector fields* $f = (f_1, \ldots, f_d)$ *driven by a* d*-dimensional signal* $X : [0,T] \to \mathbf{R}^d$, *with* $d > 2$.

♯ c) *Show that a continuous extension is possible for commuting vector fields, in the sense that all Lie bracket* $[f_i, f_j]$, $1 \le i,j \le d$, *vanish or, equivalently, their flows commute.*

Exercise 8.2 (Explicit solution, Chen–Strichartz formula) *View*

$$f = (f_1, \ldots, f_d) \in \mathcal{C}_b^\infty\big(\mathbf{R}^e, \mathcal{L}\big(\mathbf{R}^d, \mathbf{R}^e\big)\big) ,$$

as a collection of d *(smooth, bounded with bounded derivatives of all orders) vector fields on* $\mathbf{R}^e$. *Assume that* f *is step-2 nilpotent in the sense that* $[f_i, [f_j, f_k]] \equiv 0$ *for all* $i,j,k \in \{1, \ldots, d\}$. *Here,* $[\cdot,\cdot]$ *denotes the Lie bracket between two vector fields. Let* $(Y, f(Y))$ *be the RDE solution to* $dY = f(Y)d\mathbf{X}$ *started at some* $\xi \in \mathbf{R}^e$ *and assume that the rough path* $\mathbf{X}$ *is geometric. Give an explicit formula of the type* $Y_t = \exp(\ldots)\xi$ *where* exp *denotes the unit time solution flow along a vector field* $(\ldots)$ *which you should write down explicitly.*

$*$ **Exercise 8.3 (Explosion along linear-growth vector fields)** *Give an example of smooth* f *with linear growth, and* $\mathbf{X} \in \mathscr{C}^\alpha$ *so that* $dY = f(Y)d\mathbf{X}$ *started at some* ξ *fails to have a global solution.*

♯ **Exercise 8.4 (Maximal RDE solution)** *We are in the setting of the local existence and uniqueness Theorem 8.3, with* $\mathcal{C}^3$*-regular coefficients,* $f \in \mathcal{C}^3(W, \mathcal{L}(V,W))$, *and local solution* Y *to (8.6) with values in the Banach space* W.

a) *Show that* Y *can either be extended to a global solution on the whole interval* $[0,T]$ *or only on a subinterval* $[0,\tau)$ *which is maximal with respect to extension of solutions.*

b) *Show that* $\tau = \tau(\mathbf{X})$ *is a lower semicontinuous function of the driving rough path, i.e.* $\underline{\lim}_{n\to\infty} \tau(\mathbf{X}^n) \ge \tau(\mathbf{X})$ *whenever* $\mathbf{X}^n \to \mathbf{X} \in \mathscr{C}^\alpha$.

c) *Assume* f *is* $\mathcal{C}^3$*-bounded on bounded sets. (This is always the case for* $f \in \mathcal{C}^3$ *with* W, V *finite-dimensional.) If a solution only exists on* $[0,\tau)$, *then* $\overline{\lim}_{t\uparrow\tau} |Y_t| = +\infty$ *and we call* $\tau \in (0,T]$ explosion time.

Remark: *In infinite dimensions, there are examples of Banach-valued ODEs with smooth coefficients, where global existence fails but the solution does not explode. In essence, this is possible because a smooth vector field need not map bounded sets into bounded sets.*

Exercise 8.5 *Let* $T > 0, \alpha \in (1/3, 1/2]$ *and* $\mathbf{X}, \tilde{\mathbf{X}} \in \mathscr{C}^\alpha([0,T], \mathbf{R}^d$. *Establish existence, continuity and stability for rough differential equations with drift (cf. (8.6)),*

$$dY_t = f_0(Y_t)\,dt + f(Y_t)\,d\mathbf{X}_t \,. \tag{8.15}$$

a) *First assume f_0 to have the same regularity as f, in which case you may solve $dY = \bar{f}(Y)\bar{\mathbf{X}}$ with $\bar{f} = (f, f_0)$ and $\bar{\mathbf{X}}$ as (canonical) space-time rough path extension of $\mathbf{X}$. (The missing integrals $\int X^i dt, \int t dX^i, i = 1, \ldots, d$ are canonically defined as Riemann–Stieltjes integrals.)*
b) *Give a direct analysis for $f_0 \in \mathcal{C}_b^1$ (or in fact f_0 Lipschitz continuous, without boundedness assumption).*

Exercise 8.6 *Let $f \in \mathcal{C}_b^2$ and assume $(Y, f(Y))$ is a RDE solution to (8.6), as constructed in Theorem 8.3. Show that the* o*-term in Davie's definition, (8.13), can be bounded uniformly over $(X, \mathbb{X}) \in B_R$, any $R < \infty$, where*

$$B_R := \left\{ (X, \mathbb{X}) \in \mathscr{C}^\beta : \|X\|_\beta + \|\mathbb{X}\|_{2\beta} \leq R \right\}, \text{ any } R < \infty.$$

Show also that RDE solutions are β-Hölder, uniformly over $(X, \mathbb{X}) \in B_R$, any $R < \infty$.

Exercise 8.7 *Show that $\|Y, f(Y); Y^n, f(Y^n)\|_{X, X^n, 2\alpha} \to 0$, together with $\mathbf{X} \to \mathbf{X}^n$ in $\mathscr{C}^\beta$ implies that also $(Y^n, \mathbb{Y}^n) \to (Y, \mathbb{Y})$ in $\mathscr{C}^\alpha$. Since, at the price of replacing f by F, cf. Definition 8.9, there is no loss of generality in solving for the controlled rough path $Z = (X, Y)$, conclude that continuity of the RDE solution map (Itô–Lyons map) also holds with Lyons' definition of a solution.*

Exercise 8.8 *Show that $\|Y, f(Y); Y^n, f(Y^n)\|_{X, X^n, 2\alpha} \to 0$, together with $\mathbf{X} \to \mathbf{X}^n$ in $\mathscr{C}^\beta$ implies that also $(Y^n, \mathbb{Y}^n) \to (Y, \mathbb{Y})$ in $\mathscr{C}^\alpha$. Since, at the price of replacing f by F, cf. Definition 8.9, there is no loss of generality in solving for the controlled rough path $Z = (X, Y)$, conclude that continuity of the RDE solution map (Itô–Lyons map) also holds with Lyons' definition of a solution.*

Exercise 8.9 (Lyons extension theorem revisited) *Let $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ and consider $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha([0, T], V)$. Show that $\bar{\mathbf{X}} = (1, \mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \ldots, \mathbf{X}^{(N)})$, the (level-$N$) Lyons lift of $\mathbf{X}$ from Exercise 4.6, solves a linear RDE. Use this and a scaling argument for another proof of the estimate, $0 \leq s < t \leq T$, $n = 1, \ldots, N$,*

$$|\mathbf{X}_{s,t}^{(n)}|^{\frac{1}{n}} \lesssim \|\mathbf{X}\|_\alpha |t - s|^\alpha .$$

8.12 Comments

ODEs driven by not too rough paths, i.e. paths that are α-Hölder continuous for some $\alpha > 1/2$ or of finite p-variation with $p < 2$, understood in the (Young) integral sense were first studied by Lyons in [Lyo94]; nonetheless, the terminology Young-ODEs is now widely used. Existence and uniqueness for such equations via Picard iterations is by now classical, our discussion in Section 8.3 is a mild variation of [LCL07, p.22] where also the *division property* (cf. proof of Lemma 7.5) is emphasised. Existence and uniqueness of solutions to RDEs via Picard iteration in the (Banach!) space of controlled rough paths originates in [Gub04] for regularity $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. This approach also allows to treat arbitrary regularities, [Gub10, HK15]. In case of driving rough paths with jumps, one has to distinguish between forward (think Itô or branched) and geometric (think Marcus canonical) sense, this was started in [Wil01], and the general forward resp. geometric case completed by Chevyrev, Friz and Zhang in [FZ18, CF19], see also comment Section 9.6.

The continuity result of Theorem 8.5 is due to T. Lyons; proofs of uniform continuity on bounded sets were given in [Lyo98, LQ02, LCL07]. Local Lipschitz estimates were pointed out subsequently and in different settings by various authors including Lyons–Qian [LQ02], Gubinelli [Gub04], Friz–Victoir [FV10b], Inahama [Ina10], Deya et al. [DNT12]; Bailleul [Bai15a, Bai14] and Bailleul–Riedel [BR19] take a flow perspective, initially studied in [LQ98]. Smoothness of the Lyons–Itô map is discussed in [LL06, FV10b, Bai15b, CL18], see also comment Section 11.5.

The name *universal limit theorem* was suggested by P. Malliavin, meaning continuity of the Itô–Lyons map in rough path metrics. As we tried to emphasise, the stability in rough path metrics is seen at all levels of the theory.

Lyons' original argument (for arbitrary regularity) also involves a Picard iteration, see e.g. [LCL07, p.88]. In his p-variation setting, vector fields are assumed $\mathrm{Lip}^\gamma, \gamma > p$, which agrees with our $\mathcal{C}_b^\gamma$ in finite dimensions, cf. Sections 1.4 and 1.5, with the usual disclaimer $\gamma \notin \mathbf{N}$ (Lipschitz vs continuously. differentiable). In finite dimensions, existence results are given for $\gamma > p - 1$, see [Dav08, FV10b] for $p < 3$ and general p respectively. In infinite dimensions, due to lack of compactness, extra assumptions on the vector fields are necessary; a *Peano existence theorem*, as in the case of Banach valued RDEs is shown by Caruana [Car10]. On the other hand, under local $\mathcal{C}^\gamma$ regularity one has a unique (in infinite dimensions: not necessarily exploding) maximal solution, cf. Exercise 8.4. In finite dimensions, global existence is guaranteed by non-explosion, discussed in [Dav08, FV10b, Lej12, RS17].

For regularity $1/p = \alpha > 1/3$, Davie [Dav08] establishes existence and uniqueness for Young resp. rough differential equations via discrete Euler resp. Milstein approximations. Step-N Euler schemes. with $\lfloor p \rfloor \le N$, are studied in [FV08b] via sub-Riemannian geodesics in $G^{(N)}(\mathbf{R}^d)$, Boutaib et al. [BGLY14] establish similar estimates in the Banach setting, Boedihardjo, Lyons and Yang [BLY15] study $N \to \infty$.

Our regularity assumption as stated in Theorem 8.3, namely $\mathcal{C}^3$ for a unique (local) solution is not sharp; it is straightforward to push this to $\mathcal{C}^\gamma$ any $\gamma > 1/\alpha$ for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$ (due to our level-2 exposition) in agreement with [Lyo98, Dav08]. It is less straightforward [Dav08, FV10b] to show that uniqueness also holds for $\gamma = 1/\alpha$ and this is optimal, with counter-examples constructed in [Dav08]. Local existence results on the other hand are available for $\gamma > (1/\alpha) - 1$. Setting $\alpha = 1$, this is consistent with the theory of ODEs where it is well known that, at least modulo possible logarithmic divergencies and in finite dimensions, Lipschitz continuity of the coefficients is required for the uniqueness of local solutions, but continuity is sufficient for their existence.

Theorem 8.3 gives global existence for $f \in \mathcal{C}_b^3$ or (affine) linear f. Linear rough differential equations are important (Jacobian of the flow, equations for Malliavin type derivatives, etc) and studied e.g. in [Lyo98, FV10b, CL14], see also [HH10] for related analysis. Solutions can be estimated by the *rough Gronwall lemma* [DGHT19b, Hof18], in a sense a real-analysis abstraction of previously used arguments for linear RDE solutions, [HN07, FV10b].

The existence and uniqueness results for rough differential equations have seen many variations over recents years. Gubinelli, Imkeller and Perkowski apply their theory of paracontrolled distributions to (level-2) RDEs with Hölder drivers [GIP15, Sec.3], extended to Besov drivers by Prömel–Trabs [PT16], revisited with "classical" rough path tools in [FP18].

Rough/stochastic Volterra equations are discussed from a rough path point of view in [DT09, HT19, Com19], from a paracontrolled point of view in [PT18] and in a regularity structure context in [BFG$^+$19, Sec.5]. Bailleul–Diehl then study the inverse problem for rough differential equations [BD15]. For a "joint development" of RDEs and SDEs with stochastic sewing, by a fixed point argument in a space of stochastic controlled rough paths, see [FHL20]. Rough partial differential equations are discussed in Chapter 12.

Last not least, we note that the point of view to construct RDE solutions by fixed point arguments in the (linear) space of controlled rough paths, where the rough path figures as parameter of the fixed point problem, extends naturally to the framework of regularity structures developed in [Hai14b], cf. Chapter 13 onwards. In that context, solutions (to singular SPDEs, say) are found by similar fixed point arguments in a linear space of "modelled distributions"), with enhanced noise ("the model") again as parameter of the fixed point problem. (The question of renormalisation is a priori disconnected from the construction of a solution and only concerns the model / rough path. However, one would like to understand the equation driven by renormalised noise, at least when the latter is smooth. In the setting of rough differential equations such effects have been observed in [FO09], a systematic study in case of branched RDE is found in Bruned et al. [BCFP19], see also [BCEF20].)