Chapter 13
Introduction to regularity structures

We give a short introduction to the main concepts of the general theory of regularity structures. This theory unifies the theory of (controlled) rough paths with the usual theory of Taylor expansions and allows to treat situations where the underlying space is multidimensional.

13.1 Introduction

While a full exposition of the theory of regularity structures is well beyond the scope of this book, we aim to give a concise overview to most of its concepts and to show how the theory of controlled rough paths fits into it. In most cases, we will only state results in a rather informal way and give some ideas as to how the proofs work, focusing on conceptual rather than technical issues. The only exception is the "reconstruction theorem", Theorem 13.12 below, which is one of the linchpins of the whole theory. Since its proof (or rather a slightly simplified version of it) is relatively concise, we provide a fully self-contained version. For precise statements and complete proofs of most of the results exposed here, we refer to the original article [Hai14b]. See also the review articles [Hai15, Hai14a] for shorter expositions that complement the one given here.

It should be clear by now that a controlled rough path $(Y, Y') \in \mathscr{D}_W^{2\alpha}$ bears a strong resemblance to a differentiable function, with the Gubinelli derivative Y' describing the coefficient in front of a "first-order Taylor expansion" of the type

$$Y_t = Y_s + Y'_s W_{s,t} + \mathrm{O}(|t - s|^{2\alpha}) . \tag{13.1}$$

Compare this to the fact that a function $f : \mathbf{R} \to \mathbf{R}$ is of class $\mathcal{C}^\gamma$ with $\gamma \in (k, k+1)$ if for every $s \in \mathbf{R}$ there exist coefficients $f_s^{(1)}, \ldots, f_s^{(k)}$ such that

$$f_t = f_s + \sum_{\ell=1}^{k} f_s^{(\ell)} (t - s)^\ell + \mathrm{O}(|t - s|^\gamma) . \tag{13.2}$$

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Of course, $f_s^{(\ell)}$ is nothing but the ℓth derivative of f at the point s, divided by $\ell!$. In this sense, one should really think of a controlled rough path $(Y, Y') \in \mathscr{D}_W^{2\alpha}$ as a 2α-Hölder continuous function, but with respect to a "model" given by W, rather than the usual Taylor polynomials. This formal analogy between controlled rough paths and Taylor expansions suggests that it might be fruitful to systematically investigate what are the "right" objects that could possibly take the place of Taylor polynomials, while still retaining many of their nice properties.

13.2 Definition of a regularity structure and first examples

The first step in such an endeavour is to set up an algebraic structure reflecting the properties of Taylor expansions. First of all, such a structure should contain a vector space T that will contain the coefficients of our expansion. It is natural to assume that T has a graded structure: $T = \bigoplus_{\alpha \in A} T_\alpha$, for some set A of possible "homogeneities". For example, in the case of the usual Taylor expansion (13.2), it is natural to take for A the set of natural numbers and to have T_ℓ contain the coefficients corresponding to the derivatives of order ℓ. In the case of controlled rough paths however, it is natural to take $A = \{0, \alpha\}$, to have again T_0 contain the value of the function Y at any time s, and to have T_α contain the Gubinelli derivative Y'_s. This reflects the fact that the "monomial" $t \mapsto X_{s,t}$ only vanishes at order α near $t = s$, while the usual monomials $t \mapsto (t-s)^\ell$ vanish at integer order ℓ.

This however isn't the full algebraic structure describing Taylor-like expansions. Indeed, one of the characteristics of Taylor expansions is that an expansion around some point x_0 can be re-expanded around any other point x_1 by writing

$$(x - x_0)^m = \sum_{k+\ell=m} \frac{m!}{k!\ell!} (x_1 - x_0)^k \cdot (x - x_1)^\ell \;. \tag{13.3}$$

(In the case when $x \in \mathbf{R}^d$, k, ℓ and m denote multi-indices and $k! = k_1! \dots k_d!$.) Somewhat similarly, in the case of controlled rough paths, we have the (rather trivial) identity

$$W_{s_0,t} = W_{s_0,s_1} \cdot 1 + 1 \cdot W_{s_1,t} \;. \tag{13.4}$$

What is a natural abstraction of this fact? In terms of the coefficients of a "Taylor expansion", the operation of reexpanding around a different point is ultimately just a linear operation from $\Gamma \colon T \to T$, where the precise value of the map Γ depends on the starting point x_0, the endpoint x_1, and possibly also on the details of the particular "model" that we are considering. In view of the above examples, it is natural to impose furthermore that Γ has the property that if $\tau \in T_\alpha$, then $\Gamma\tau - \tau \in \bigoplus_{\beta<\alpha} T_\beta$. In other words, when reexpanding a homogeneous monomial around a different point, the leading order coefficient remains the same, but lower order monomials may appear.

These heuristic considerations can be summarised in the following definition of an abstract object we call a *regularity structure*:

Definition 13.1. A *regularity structure* $\mathscr{T} = (T, G)$ consists of the following elements:

- A *structure space* given as graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$ where each T_α is a Banach space, with *index set* $A \subset \mathbf{R}$ bounded from below and locally finite.[1] Elements of T_α are said to have *degree* α and we write $\deg \tau = \alpha$ for $\tau \in T_\alpha$. Given $\tau \in T$, we will write $\|\tau\|_\alpha$ for the norm of its component in T_α.
- A *structure group* G of continuous linear operators acting on T such that, for every $\Gamma \in G$, every $\alpha \in A$, and every $\tau_\alpha \in T_\alpha$, one has

$$\Gamma \tau_\alpha - \tau_\alpha \in T_{<\alpha} \stackrel{\text{def}}{=} \bigoplus_{\beta < \alpha} T_\beta \;. \tag{13.5}$$

A *sector* V of $\mathscr{T}$ is a linear subspace $V = \bigoplus_{\alpha \in A} V_\alpha \subset T$, with closed linear subspaces $V_\alpha \subset T_\alpha$, invariant under G, such that $(V, G|_V)$ is a regularity structure in its own right.

Remark 13.2. In principle, the index set A can be infinite. By analogy with the polynomials, it is then natural to interpret T as the set of all formal series of the form $\sum_{\alpha \in A} \tau_\alpha$, where only finitely many of the τ_α's are non-zero. This also dovetails nicely with the particular form of elements in G. In practice however we will only ever work with finite subsets of A so that the precise topology on T does not matter as long as each of the T_α is finite-dimensional, which is the case in all of the examples we will consider here.

The space T should be thought of as consisting of "abstract" Taylor expansions (or "jets") , where each element of T_α would correspond to a "homogeneous polynomial of degree α" (this will be made in combination with the definition of a model in Definition 13.5 below). To avoid confusion between "abstract" elements of T and "concrete" associated functions (or distributions), we will use colour to denote elements of T, e.g. τ. Typically, T will be generated (as a free vector space) by a set of "basis symbols", so that T consists of all formal (finite) linear combination obtained from regarding these symbols as basis vectors. Given basis symbols / vectors $\tau_1, \tau_2, \ldots$ we indicate this by

$$T = \langle \tau_1, \tau_2, \ldots \rangle. \tag{13.6}$$

Important convention: basis symbols will always by listed in order of increasing homogeneities. That is, $\tau_i \in T_{\alpha_i}$ with $\alpha_1 \leq \alpha_2 \leq \ldots$ in (13.6). We now turn to some first examples of regularity structures.

[1] In [Hai14b], T was called *model space*, somewhat in clash with the space of *models*.

13.2.1 The polynomial structure

We start with two simple special cases followed by the general polynomial structure. Fix $\gamma \in (0, 1)$ and consider a real-valued function belonging to the Hölder space of exponent γ, say $f \in \mathcal{C}^\gamma$. In other words, $f : \mathbf{R} \to \mathbf{R}$, and $|f_x - f_y| \lesssim |y - x|^\gamma$ uniformly for x, y on compacts. The trivial regularity structure

$$T = T_0 = \langle \mathbf{1} \rangle \cong \mathbf{R} \;, \qquad G = \{\mathrm{Id}\} \;,$$

allows us to interpret the function f as a T-valued map

$$x \mapsto f(x) := f_x \mathbf{1}.$$

Consider next a real-valued function $f : \mathbf{R} \to \mathbf{R}$ of class $\mathcal{C}^{2+\gamma}$, with $\gamma \in (0, 1)$. By this we mean that continuous derivatives Df and $D^2 f$ exist, with $D^2 f$ locally γ-Hölder continuous. The minimal regularity structure allowing to capture the fact that $f \in \mathcal{C}^{2+\gamma}$ is

$$T = T_0 \oplus T_1 \oplus T_2 = \langle \mathbf{1}, X, X^2 \rangle \cong \mathbf{R}^3 \;,$$

with structure group $G = \{\Gamma_h \in \mathcal{L}(T, T) : h \in (\mathbf{R}, +)\}$ where Γ_h is given, with respect to the ordered basis $\mathbf{1}, X, X^2$, by the matrix

$$\Gamma_h \cong \begin{pmatrix} 1 & h & h^2 \\ 0 & 1 & 2h \\ 0 & 0 & 1 \end{pmatrix}.$$

In other words,

$$\Gamma_h \mathbf{1} = \mathbf{1} \;, \quad \Gamma_h X = X + h\mathbf{1} \;, \quad \Gamma_h X^2 = (X + h\mathbf{1})^2 \;,$$

with the obvious abuse of notation in the last expression.

Note that $\Gamma_g \circ \Gamma_h = \Gamma_{g+h}$, so that G inherits its group structure from $(\mathbf{R}, +)$. Moreover, the triangular form, with ones on the diagonal, expresses exactly the requirement (13.5). This structure allows to represent the function f and its first two derivatives as a truncated Taylor series, namely as the T-valued map

$$x \mapsto f(x) := f_x \mathbf{1} + Df_x X + \frac{1}{2} D^2 f_x X^2.$$

It is now an easy matter to generalise the above considerations to general Hölder maps of several variables, say $f : \mathbf{R}^d \to \mathbf{R}$ in the Hölder space $\mathcal{C}^{n+\gamma}$, which is defined by the obvious generalisation of (13.2) to functions on $\mathbf{R}^d$. In this case, we would take T to be the space of polynomials of degree at most n in d commuting indeterminates $X_1, \ldots, X_d$. This motivates the following definition.

Definition 13.3. The *polynomial regularity structure* on $\mathbf{R}^d$ is given by

- $T = \mathbf{R}[X_1, \ldots, X_d]$ is the space of real polynomials in d commuting indeterminates and T_α is given by the homogeneous polynomials of degree $\alpha \in \mathbf{N}$.
- The structure group $G \sim (\mathbf{R}^d, +)$ acts on T via

$$\Gamma_h P(X) = P(X + h\mathbf{1}) , \qquad h \in \mathbf{R}^d ,$$

for any polynomial P.

Given an arbitrary multi-index $k = (k_1, \ldots, k_d)$, we write X^k as a shorthand for $X_1^{k_1} \cdots X_d^{k_d}$, and we write $|k| = k_1 + \cdots + k_d$. With this notation, for any $\alpha \in A = \mathbf{N}$,

$$T_\alpha = \langle X^k : |k| = \alpha \rangle. \tag{13.7}$$

Note that $T_{\le\alpha} = T_0 \oplus T_1 \oplus \cdots \oplus T_\alpha$, i.e. the space of polynomials of degree at most α, any $\alpha \in A = \mathbf{N}$, is a sector of the polynomial regularity structure.

13.2.2 The rough path structure

We start again from simple examples. What structure would be appropriate for Young integration? Fix $\alpha \in (0,1)$ and consider the problem of integrating a (continuous) path Y against a scalar $W \in \mathcal{C}^\alpha$. In the case of smooth W, the indefinite integral $Z = \int Y dW$exists in Riemann–Stieltjes' sense and one has $\dot{Z} = Y\dot{W}$. In general, $\dot{W}$ only exists as a distribution, more precisely an element of the negative Hölder space $\mathcal{C}^{\alpha-1}$. A regularity structure allowing to describe this situation is given by

$$T = T_{\alpha-1} \oplus T_0 = \langle \dot{W} \rangle \oplus \langle \mathbf{1} \rangle \cong \mathbf{R}^2 , \qquad G = \{\mathrm{Id}\} . \tag{13.8}$$

The potentially ill-defined product $\dot{Z} = Y\dot{W}$ can now be replaced by the perfectly well-defined T-valued map

$$s \mapsto \dot{Z}(s) := Y_s \dot{W} .$$

We shall see later how $\dot{Z}$ gives rise to $\dot{Z}$, the distributional derivative of the indefinite Young integral $\int Y dW$, provided of that Y is sufficiently regular, namely $Y \in \mathcal{C}^\beta$ with $\alpha + \beta > 1$.

Let us next consider the "task" of representing a controlled rough path in a suitable regularity structure. More precisely, consider $\alpha \in (1/3, 1/2]$, a path $W \in \mathcal{C}^\alpha$ with values in $\mathbf{R}$, say, and $(Y, Y') \in \mathscr{D}_W^{2\alpha}$ so that

$$Y_t \approx Y_s + Y'_s W_{s,t} . \tag{13.9}$$

The right-hand side above is some sort of Taylor expansion, based on $W \in \mathcal{C}^\alpha$, which describes Y well near the (time) point s. We want to formalise this by attaching to each time s the "jet"

$$Y(s) := Y_s \mathbf{1} + Y'_s W .$$

Performing the substitution $\mathbf{1} \mapsto 1$, $W \mapsto W_{s,\cdot}$ gets us back to the right-hand side of (13.9). This suggests to define the following regularity structure

$$T = T_0 \oplus T_\alpha = \langle \mathbf{1} \rangle \oplus \langle W \rangle \cong \mathbf{R}^2 ,$$

with structure group $G = \{\Gamma_h \in \mathcal{L}(T, T) : h \in (\mathbf{R}, +)\}$ where Γ_h acts as

$$\Gamma_h \mathbf{1} = \mathbf{1} , \qquad \Gamma_h W = W + h\mathbf{1} .$$

The regularity structure relevant for rough integration is essentially a combination of the two previous ones. Let $\mathbf{W} = (W, \mathbb{W}) \in \mathscr{C}^\alpha$ and $(Y, Y') \in \mathscr{D}_W^{2\alpha}$ and consider the rough integral $Z := \int Y d\mathbf{W}$. Since, for $s \approx t$, we have

$$Z_{s,t} = \int_s^t Y d\mathbf{W} \approx Y_s W_{s,t} + Y_s' \mathbb{W}_{s,t} ,$$

this suggests (rather informally at this stage), that in the vicinity of any fixed time s, the distributional derivative of Z should have an expansion of the type

$$\dot{Z} \approx Y_s \dot{W} + Y_s' \dot{\mathbb{W}}_s , \tag{13.10}$$

where $\dot{W} := \partial_t W_t$ and $\dot{\mathbb{W}}_s := \partial_t \mathbb{W}_{s,t}$ are distributional derivatives. This suggests to attach the following "jet" at each point s,

$$\dot{Z}(s) := Y_s \dot{W} + Y_s' \dot{\mathbb{W}} . \tag{13.11}$$

The case of multi-component rough paths just needs more basis vectors $\dot{W}^i$, $\dot{\mathbb{W}}^{j,k}$, W^l (with $1 \le i, j, k, l \le e$). This suggests the following definition.

Definition 13.4. Let $\alpha \in (1/3, 1/2]$. The *regularity structure for α-Hölder rough paths (over $\mathbf{R}^e$)* is given by $T = T_{\alpha-1} \oplus T_{2\alpha-1} \oplus T_0 \oplus T_\alpha \cong \mathbf{R}^{e+e^2+1+e}$ with

$$\begin{aligned} T_0 &= \langle \mathbf{1} \rangle , & T_\alpha &= \langle W^1, \ldots, W^e \rangle , \\ T_{\alpha-1} &= \langle \dot{W}^1, \ldots, \dot{W}^e \rangle , & T_{2\alpha-1} &= \langle \dot{\mathbb{W}}^{ij} : 1 \le i, j \le e \rangle , \end{aligned}$$

and structure group $G \sim (\mathbf{R}^e, +)$ acting on T by

$$\begin{aligned} \Gamma_h \mathbf{1} &= \mathbf{1} , & \Gamma_h W^i &= W^i + h^i \mathbf{1} , \\ \Gamma_h \dot{W}^i &= \dot{W}^i , & \Gamma_h \dot{\mathbb{W}}^{ij} &= \dot{\mathbb{W}}^{ij} + h^i \dot{W}^j . \end{aligned} \tag{13.12}$$

It will be seen later in Proposition 13.21 that in this framework the function $\dot{Z}$ defined in (13.11) does indeed give rise naturally to $\dot{Z}$, the distributional derivative of the indefinite rough integral $\int Y d\mathbf{W}$.

In a Brownian (rough path) context, one has Hölder regularity with exponent $\alpha = 1/2 - \kappa$, for arbitrarily small $\kappa > 0$. The above index set A, relevant for a "regularity structure view" on stochastic integration, then becomes $A = \{ -\frac{1}{2} - \kappa, -2\kappa, 0, \frac{1}{2} - \kappa\}$, which, in abusive but convenient notation, we write as

$$A = \Big\{ -\frac{1}{2}^-, 0^-, 0, \frac{1}{2}^- \Big\} .$$

Index sets of this form ("half-integers$^-$") will also be typical in later SPDE situations driven by spatial or space-time white noise.

13.3 Definition of a model and first examples

At this stage, a regularity structure is a completely abstract object. It only becomes useful when endowed with a *model*, which is a concrete way of associating to any $\tau \in T$ and $x \in \mathbf{R}^d$, the actual "Taylor polynomial based at x" represented by τ. Furthermore, we want elements $\tau \in T_\alpha$ to represent functions (or possibly distributions!) that "vanish at order α" around the given point x, thereby justifying our terminology of calling α a degree.

Since we would like to allow A to contain negative values and therefore allow elements in T to represent actual distributions, we need a suitable notion of "vanishing at order α". We achieve this by considering the size of our distributions, when tested against test functions that are localised around the given point x_0. Given a test function φ on $\mathbf{R}^d$, we write φ_x^λ as a shorthand for

$$\varphi_x^\lambda(y) = \lambda^{-d}\varphi\big(\lambda^{-1}(y-x)\big) .$$

Given $r \in \mathbf{N}$, we also denote by $\mathcal{B}_r$ the set of all *smooth* test functions $\varphi\colon \mathbf{R}^d \to \mathbf{R}$ such that $\varphi \in \mathcal{C}^r$ with $\|\varphi\|_{\mathcal{C}^r} \le 1$ that are furthermore supported in the unit ball around the origin; clearly $\mathcal{B}_r \subset \mathcal{D}(\mathbf{R}^d)$, the test function space for $\mathcal{D}'(\mathbf{R}^d)$, the space of distributions on $\mathbf{R}^d$. With these notations, our definition of a model for a given regularity structure $\mathscr{T}$ is as follows.

Definition 13.5. Given a regularity structure $\mathscr{T} = (T, G)$ and an integer $d \ge 1$, a *model* $\mathrm{M} = (\Pi, \Gamma)$ for $\mathscr{T}$ on $\mathbf{R}^d$ consists of maps

$$\begin{aligned} \Pi\colon \mathbf{R}^d &\to \mathcal{L}\big(T, \mathcal{D}'(\mathbf{R}^d)\big) & \qquad \Gamma\colon \mathbf{R}^d \times \mathbf{R}^d &\to G \\ x &\mapsto \Pi_x & (x,y) &\mapsto \Gamma_{xy} \end{aligned}$$

such that $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ and $\Pi_x \Gamma_{xy} = \Pi_y$. Write r for the smallest integer such that $r > |\min A| \ge 0$ and impose that for every compact set $\mathfrak{K} \subset \mathbf{R}^d$ and every $\gamma > 0$, there exists a constant $C = C(\mathfrak{K}, \gamma)$ such that the bounds

$$\big|\big(\Pi_x \tau\big)(\varphi_x^\lambda)\big| \le C\lambda^\alpha \|\tau\|_\alpha , \qquad \|\Gamma_{xy}\tau\|_\beta \le C|x-y|^{\alpha-\beta}\|\tau\|_\alpha , \tag{13.13}$$

hold uniformly over $x, y \in \mathfrak{K}$, $\lambda \in (0,1]$, $\varphi \in \mathcal{B}_r$, $\tau \in T_\alpha$ with $\alpha \le \gamma$ and $\beta < \alpha$.

We then call Π the *realisation map*, since $\Pi_x\tau$ *realises* an element $\tau \in T$ as a distribution, and Γ the *reexpansion map*.

One very important remark is that the space $\mathscr{M}$ of all models for a given regularity structure is *not* a linear space. However, it can be viewed as a closed subset (determined by the nonlinear constraints $\Gamma_{xy} \in G$, $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$, and $\Pi_y = \Pi_x \Gamma_{xy}$) of the linear space with seminorms (indexed by the compact set $\mathfrak{K}$ and the upper bound γ) given by the smallest constant C in (13.13). In particular, there is a natural collection of "distances" between models (Π, Γ) and $(\bar{\Pi}, \bar{\Gamma})$ given by the smallest constant C in (13.13), when replacing Π_x by $\Pi_x - \bar{\Pi}_x$ and Γ_{xy} by $\Gamma_{xy} - \bar{\Gamma}_{xy}$. Since this collection is essentially countable (consider for example the sequence of pseudometrics d_n corresponding to the choices $(\mathfrak{K}_n, \gamma_n)$ with $\mathfrak{K}_n$ the centred ball of radius n and $\gamma_n = n$), it determines a metrisable topology (take for example $d = \sum_{n \geq 1} 2^{-n}(d_n \wedge 1)$).

Remark 13.6. The precise choice of r in Definition 13.5 is not very important, as one can see that any other choice $r > |\min A| \geq 0$ leads to the same definition. See Lemma 14.13 for a similar statement in the context of Hölder spaces.

Remark 13.7. The test functions appearing in (13.13) are smooth. It turns out that if these bounds hold for smooth elements of $\mathcal{B}_r$, then $\Pi_x \tau$ can be extended canonically to allow any $\mathcal{C}^r$ test function with compact support.

Remark 13.8. The identity $\Pi_x \Gamma_{xy} = \Pi_y$ reflects the fact that Γ_{xy} is the linear map that takes an expansion around y and turns it into an expansion around x. The first bound in (13.13) states what we mean precisely when we say that $\tau \in T_\alpha$ represents a term that vanishes at order α. (See Exercise 13.2; note that α can be negative, so that this may actually not vanish at all!) The second bound in (13.13) is very natural in view of both (13.3) and (13.4). It states that when expanding a monomial of order α around a new point at distance h from the old one, the coefficient appearing in front of lower-order monomials of order β is of order at most $h^{\alpha-\beta}$.

Remark 13.9. In many cases of interest, it is natural to scale the different directions of $\mathbf{R}^d$ in a different way. This is the case for example when using the theory of regularity structures to build solution theories for parabolic stochastic PDEs, in which case the time direction "counts as" two space directions. This "parabolic scaling" can be formalised by the integer vector $(2, 1, \ldots, 1)$. More generally, one can introduce a scaling $\mathfrak{s}$ of $\mathbf{R}^d$, which is just a collection of d scalars $\mathfrak{s}_i \in [1, \infty)$ and to define φ_x^λ in such a way that the ith direction is scaled by $\lambda^{\mathfrak{s}_i}$. The polynomial structure introduced earlier, in particular (13.7), should be changed accordingly by postulating that the degree of X^k is given by $|k|_\mathfrak{s} = \sum_{i=1}^d \mathfrak{s}_i k_i$. In this case, the Euclidean distance between two points $x, y \in \mathbf{R}^d$ should be replaced everywhere by the corresponding scaled distance $|x-y|_\mathfrak{s} = \sum_i |x_i - y_i|^{1/\mathfrak{s}_i}$. See [Hai14b] for more details.

With these definitions at hand, it is then natural to define an analogue in this context of the space of γ-Hölder continuous functions in the following way.

Definition 13.10. Given a regularity structure $\mathscr{T}$ equipped with a model $\mathrm{M} = (\Pi, \Gamma)$ over $\mathbf{R}^d$, the space $\mathscr{D}_\mathrm{M}^\gamma$ is given by the set of functions $f : \mathbf{R}^d \to T_{<\gamma}$ such that, for every compact set $\mathfrak{K}$ and every $\alpha < \gamma$, there exists a constant C with

$$\|f(x) - \Gamma_{xy} f(y)\|_\alpha \le C|x-y|^{\gamma-\alpha} \tag{13.14}$$

uniformly over $x, y \in \mathfrak{K}$. Such functions f are called *modelled distributions*. For fixed $\mathfrak{K}$, a seminorm $\|f\|_{\mathrm{M},\gamma;\mathfrak{K}}$ is defined as the smallest constant C in the bound (13.14). The space $\mathscr{D}^\gamma_{\mathrm{M}}$ endowed with this family of seminorms is then a Fréchet space.

It is furthermore convenient to be able to compare two modelled distributions defined over two different models. In this case, a natural way of comparing them is to take as a "metric" the smallest constant C in the bound

$$\|f(x) - \Gamma_{xy} f(y) - \bar{f}(x) + \bar{\Gamma}_{xy} \bar{f}(y)\|_\alpha \le C|x-y|^{\gamma-\alpha} .$$

Remark 13.11. (Compare with Remark 4.8 in the rough path context.) It is important to note that while the space of models $\mathscr{M}$ is not a linear space, the space $\mathscr{D}^\gamma_{\mathrm{M}}$ is a linear (in fact: Fréchet) space given a model $\mathrm{M} \in \mathscr{M}$. The twist of course is that the space in question depends in a crucial way on the choice of M. The total space then is the disjoint union

$$\mathscr{M} \ltimes \mathscr{D}^\gamma \stackrel{\text{def}}{=} \bigsqcup_{\mathrm{M} \in \mathscr{M}} \{\mathrm{M}\} \times \mathscr{D}^\gamma_{\mathrm{M}},$$

with base space $\mathscr{M}$ and "fibres" $\mathscr{D}^\gamma_{\mathrm{M}}$.

The most fundamental result in the theory of regularity structures then states that given $f \in \mathscr{D}^\gamma$ with $\gamma > 0$, there exists a *unique* distribution $\mathcal{R} f$ on $\mathbf{R}^d$ such that, for every $x \in \mathbf{R}^d$, $\mathcal{R} f$ "looks like $\Pi_x f(x)$ near x". More precisely, one has

Theorem 13.12 (Reconstruction). *Let* $\mathrm{M} = (\Pi, \Gamma)$ *be a model for a regularity structure* $\mathscr{T}$ *on* $\mathbf{R}^d$. *Assume* $f \in \mathscr{D}^\gamma_{\mathrm{M}}$ *with* $\gamma > 0$. *Then, there exists a unique linear map*

$$\mathcal{R} = \mathcal{R}_{\mathrm{M}} \colon \mathscr{D}^\gamma_{\mathrm{M}} \to \mathcal{D}'(\mathbf{R}^d)$$

such that

$$\big|\big(\mathcal{R} f - \Pi_x f(x)\big)(\varphi^\lambda_x)\big| \lesssim \lambda^\gamma , \tag{13.15}$$

uniformly over $\varphi \in \mathcal{B}_r$ *and* λ *as before, and locally uniformly in* x. *For* $\gamma < 0$, *everything remains valid but uniqueness of* $\mathcal{R}$.

Remark 13.13. With a look to Remark 13.11, and $\mathrm{M} = (\Pi, \Gamma) \in \mathscr{M}$, one should really view $\mathcal{R} = \mathcal{R}_{\mathrm{M}} f$ as a map from $\mathscr{M} \ltimes \mathscr{D}^\gamma$ into $\mathcal{D}'$. Since the space $\mathscr{M} \ltimes \mathscr{D}^\gamma$ is *not* a linear space, this shows that the map $\mathcal{R}$ isn't actually linear, despite appearances. However, the map $(\Pi, \Gamma, f) \mapsto \mathcal{R} f$ turns out to be locally Lipschitz continuous provided that the distance between (Π, Γ, f) and $(\bar{\Pi}, \bar{\Gamma}, \bar{f})$ is given by the smallest constant C such that

$$\begin{aligned} \|f(x) - \bar{f}(x) - \Gamma_{xy} f(y) + \bar{\Gamma}_{xy} \bar{f}(y)\|_\alpha &\le C|x-y|^{\gamma-\alpha} , \\ \big|\big(\Pi_x \tau - \bar{\Pi}_x \tau\big)(\varphi^\lambda_x)\big| &\le C \lambda^\alpha \|\tau\| , \\ \|\Gamma_{xy} \tau - \bar{\Gamma}_{xy} \tau\|_\beta &\le C|x-y|^{\alpha-\beta} \|\tau\| . \end{aligned}$$

Here, in order to obtain bounds on $\big(\mathcal{R}f - \bar{\mathcal{R}}\bar{f}\big)(\psi)$ for some smooth compactly supported test function ψ, the above bounds should hold uniformly for x and y in a neighbourhood of the support of ψ. The proof that this stronger continuity property also holds is actually crucial when showing that sequences of solutions to mollified equations all converge to the same limiting object. However, its proof is somewhat more involved which is why we chose not to give it here but refer instead to [Hai14b, Thm 3.10].

Remark 13.14. There are obvious analogies between the construction of the reconstruction operator $\mathcal{R}$ and that of the "rough integral" in Section 4. As a matter of fact, there exists a slightly more abstract formulation of the reconstruction theorem which can be interpreted as a multidimensional analogue to the sewing lemma, Lemma 4.2, see [Hai14b, Prop. 3.25].

Remark 13.15. The reconstruction theorem with $\gamma < 0$ allows one to recover the Lyons–Victoir extension theorem previously obtained in Exercise 2.14, see also Exercise 13.6. Note that the reconstruction theorem does *not* hold for $\gamma = 0$ (even if we forego uniqueness of $\mathcal{R}$), for the same reason that the Lyons–Victoir extension theorem fails for $\alpha = \frac{1}{2}$ (and more generally when $1/\alpha \in \mathbf{N}$).

In the particular case where $\Pi_x\tau$ happens to be a continuous function for every $\tau \in T$ (and every $x \in \mathbf{R}^d$), we will see in Remark 13.27 that $\mathcal{R}f$ is also a continuous function and $\mathcal{R}$ is given by the somewhat trivial explicit formula

$$\big(\mathcal{R}f\big)(x) = \big(\Pi_x f(x)\big)(x) \ .$$

We postpone the proof of the reconstruction theorem to Section 13.4 and turn instead to our previous list of regularity structures, now adding the relevant models and indicating the interest of the reconstruction map.

13.3.1 The polynomial model

Recall the polynomial regularity structure in d variables defined in Section 13.2.1. In this context, the polynomial model P is given by

$$\big(\Pi_x X^k\big) = \big(y \mapsto (y-x)^k\big) \ , \qquad \Gamma_{xy} = \Gamma_h \,\big|_{h=x-y} \ .$$

We leave it as an exercise to the reader to verify that this does indeed satisfy the bounds and relations of Definition 13.5.

In the sense of the following proposition, modelled distributions in the context of the polynomial model are nothing but classical Hölder functions.

Proposition 13.16. *Let $\beta = n + \gamma$ with $n \in \mathbf{N}$ and $\gamma \in (0,1)$. If f belongs to the Hölder space $\mathcal{C}^\beta$, then $f \in \mathcal{D}_{\mathrm{P}}^\beta$ with*

$$f(x) = f(x)\mathbf{1} + \sum_{1\le|k|\le n} \frac{f^{(k)}(x)}{k!} X^k \,.$$

Conversely, if $\hat f \in \mathscr{D}_{\rm P}^\beta$ *then* $f := \langle \hat f, \mathbf{1}\rangle$ *is in* $\mathcal{C}^\beta$ *and necessarily* $\hat f = f$. □

This proposition is essentially a consequence of the (well-known) fact that $f \in \mathcal{C}^\beta$ if and only if for every $x \in \mathbf{R}^d$, there exists a polynomial $P_x = P_x(y)$ of degree n, such that, locally uniformly in x, y, one has $|f(y) - P_x(y)| \lesssim |y-x|^\beta$. Necessarily then, such a function f is n times continuously differentiable, and P_x is its Taylor polynomial of degree n. This characterisation and the above proposition remain valid for integer values of β with the caveat that in this context $\mathcal{C}^\beta$ means $\beta - 1$ times continuously differentiable with the highest order derivatives locally Lipschitz continuous.

It will be convenient for the sequel to introduce a suitable notion of "negative" Hölder spaces. In fact, the definition of a model (see also Exercise 13.2) suggests that a very natural space of distributions is obtained in the following way. Given $\alpha > 0$, we denote by $\mathcal{C}^{-\alpha}$ the space of all distributions η such that, with r the smallest integer such that $r > \alpha$,

$$\left|\eta(\varphi_x^\lambda)\right| \lesssim \lambda^{-\alpha}\,,$$

uniformly over all $\varphi \in \mathcal{B}_r$ and $\lambda \in (0,1]$, and locally uniformly in x. Given any compact set $\mathfrak{K}$, the best possible constant such that the above bound holds uniformly over $x \in \mathfrak{K}$ yields a seminorm. The collection of these seminorms endows $\mathcal{C}^{-\alpha}$ with a Fréchet space structure.

Remark 13.17. In terms of the scale of classical Besov spaces, the space $\mathcal{C}^{-\alpha}$ is a local version of $\mathcal{B}^{-\alpha}_{\infty,\infty}$. It is in some sense the largest space of distributions that is invariant under the scaling $\varphi(\cdot) \mapsto \lambda^{-\alpha}\varphi(\lambda^{-1}\cdot)$, see for example [BP08].

Let us now give a very simple application of the reconstruction theorem. It is a classical result in the "folklore" of harmonic analysis (see for example [BCD11, Thm 2.52] for a very similar statement) that the product extends naturally to $\mathcal{C}^\beta \times \mathcal{C}^{-\alpha}$ into $\mathcal{D}'(\mathbf{R}^d)$ if and only if $\beta > \alpha$, which can also be seen as higher-dimensional version of the Young integral, cf. Exercise 13.1. We illustrate how to use the reconstruction theorem in order to obtain a straightforward proof of the "if" part of this result:

Theorem 13.18. *For* $\beta > \alpha > 0$*, there is a continuous bilinear map*

$$B\colon \mathcal{C}^\beta \times \mathcal{C}^{-\alpha} \to \mathcal{D}'(\mathbf{R}^d)$$

such that $B(f,g) = fg$ *for any two continuous functions* f *and* g.

Proof. Assume from now on that $g = \xi \in \mathcal{C}^{-\alpha}$ for some $\alpha > 0$ and that $f \in \mathcal{C}^\beta$ for some $\beta > \alpha$. We then build a regularity structure $\mathscr{T}$ in the following way. For the index set A, we take $A = \mathbf{N} \cup (\mathbf{N} - \alpha)$ and for T, we set $T = V \oplus W$, where each one of the spaces V and W is a copy of the polynomial regularity structure (in

d commuting variables). We also choose Γ as in the polynomial case above, acting simultaneously and identically on each of the two instances.

As before, we denote by X^k the canonical basis vectors in V. We also use the suggestive notation "ΞX^k" for the corresponding basis vector in W, but we postulate that $\Xi X^k \in T_{|k|-\alpha}$ rather than $\Xi X^k \in T_{|k|}$. Given any distribution $\xi \in \mathcal{C}^{-\alpha}$, we then define a model (Π^ξ, Γ), where Γ is as in the canonical model, while Π^ξ acts as

$$\big(\Pi^\xi_x X^k\big)(y) = (y-x)^k \;, \qquad \big(\Pi^\xi_x \Xi X^k\big)(y) = (y-x)^k \xi(y) \;,$$

with the obvious abuse of notation in the second expression. It is then straightforward to verify that $\Pi_y = \Pi_x \circ \Gamma_{xy}$ and that the relevant analytical bounds are satisfied, so that this is indeed a model.

Denote now by $\mathcal{R}^\xi$ the reconstruction map associated to the model (Π^ξ, Γ) and, for $f \in \mathcal{C}^\beta$, denote by f the element in $\mathscr{D}^\beta$ given by the local Taylor expansion of f of order β at each point. Note that even though the space $\mathscr{D}^\beta$ does in principle depend on the choice of model, in our situation $f \in \mathscr{D}^\beta$ for any choice of ξ. It follows immediately from the definitions that the map $x \mapsto \Xi f(x)$ belongs to $\mathscr{D}^{\beta-\alpha}$ so that, provided that $\beta > \alpha$, one can apply the reconstruction operator to it. This suggests that the multiplication operator we are looking for can be defined as

$$B(f,\xi) = \mathcal{R}^\xi\big(\Xi f\big) \;.$$

By Theorem 13.12, this is a jointly continuous map from $\mathcal{C}^\beta \times \mathcal{C}^{-\alpha}$ into $\mathcal{D}'(\mathbf{R}^d)$, provided that $\beta > \alpha$. If ξ happens to be a smooth function, then it follows immediately from the remark after Theorem 13.12 that $B(f,\xi) = f(x)\xi(x)$, so that B is indeed the requested continuous extension of the usual product. $\square$

Remark 13.19. In the context of this theorem, one can actually show that $B(f,g) \in \mathcal{C}^{-\alpha}$. More generally, denoting by $-\alpha$ the smallest degree arising in a given regularity structure $\mathscr{T}$, i.e. $\alpha = -\min A$, it is possible to show that the reconstruction operator $\mathcal{R}$ takes values in $\mathcal{C}^{-\alpha}$.

The reader may notice that one can also work with a finite-dimensional regularity structure, based on index set $\tilde N \cup (\tilde N - \alpha)$, with $\tilde N = \{0,1,\ldots,n\}$ and $\beta = n + \gamma$. In particular, if $n = 0$, the regularity structure used here is exacty the one already encountered in (13.8).

13.3.2 The rough path model

Let us see now how some of the results of Section 4 can be reinterpreted in the light of this theory. Fix $\alpha \in (1/3, 1/2]$ and let $\mathscr{T}$ be the rough path regularity structure put forward in Definition 13.4. Recall that this means that $T_0 = \langle \mathbf{1} \rangle$, T_α and $T_{\alpha-1}$ are copies of $\mathbf{R}^e$ with respective basis vectors W^j and $\dot W^j$, and $T_{2\alpha-1}$ is a copy of $\mathbf{R}^{e\times e}$ with basis vectors $\dot{\mathbb{W}}^{ij}$. The structure group G is isomorphic to $\mathbf{R}^e$ and, for $h \in \mathbf{R}^e$, acts on T via

$$\Gamma_h \mathbf{1} = \mathbf{1}\,, \quad \Gamma_h \dot W^i = \dot W^i\,, \quad \Gamma_h W^i = W^i + h^i \mathbf{1}\,, \quad \Gamma_h \dot{\mathbb{W}}^{ij} = \dot{\mathbb{W}}^{ij} + h^i \dot W^j\,. \tag{13.16}$$

Let now $\mathbf{W} = (W, \mathbb{W})$ be an α-Hölder continuous rough path over $\mathbf{R}^e$. It turns out that this defines a model for $\mathscr{T}$ in the following way:

Lemma 13.20. *Given an α-Hölder continuous rough path $\mathbf{W}$, one can define a model $\mathrm{M} = \mathrm{M}_{\mathbf{W}}$ for $\mathscr{T}$ on $\mathbf{R}$ by setting $\Gamma_{t,s} = \Gamma_{W_{s,t}}$ and*

$$\big(\Pi_s \mathbf{1}\big)(t) = 1\,, \qquad \big(\Pi_s W^j\big)(t) = W^j_{s,t}$$
$$\big(\Pi_s \dot W^j\big)(\psi) = \int \psi(t)\, dW^j_t\,, \qquad \big(\Pi_s \dot{\mathbb{W}}^{ij}\big)(\psi) = \int \psi(t)\, d\mathbb{W}^{ij}_{s,t}\,.$$

Here, both integrals are perfectly well-defined Riemann integrals, with the differential in the second case taken with respect to the variable t. Given a controlled rough path $(Y, Y') \in \mathscr{D}^{2\alpha}_W$, this then defines an element $Y \in \mathscr{D}^{2\alpha}_{\mathrm{M}}$ by

$$Y(s) = Y(s)\,\mathbf{1} + Y'_i(s)\, W^i\,,$$

with summation over i implied.

Proof. We first check that the algebraic properties of Definition 13.5 are satisfied. It is clear that $\Gamma_{s,u}\Gamma_{u,t} = \Gamma_{s,t}$ and that $\Pi_s \Gamma_{s,u} \tau = \Pi_u \tau$ for $\tau \in \{\mathbf{1}, W^j, \dot W^j\}$. Regarding $\dot{\mathbb{W}}^{ij}$, we differentiate Chen's relations (2.1) which yields the identity

$$d\mathbb{W}^{i,j}_{s,t} = d\mathbb{W}^{i,j}_{u,t} + W^i_{s,u}\, dW^j_t\,.$$

The last missing algebraic relation then follows at once. The required analytic bounds follow immediately (exercise!) from the definition of the rough path space $\mathscr{C}^\alpha$.

Regarding the function Y defined in the statement, we have

$$\|Y(s) - \Gamma_{s,u} Y(u)\|_0 = |Y(s) - Y(u) + Y'_i(u) W^i_{s,u}|\,,$$
$$\|Y(s) - \Gamma_{s,u} Y(u)\|_\alpha = |Y'(s) - Y'(u)|\,,$$

so that the condition (13.14) with $\gamma = 2\alpha$ does indeed coincide with the definition of a controlled rough path. □

Theorems 4.4 and 4.10 can then be recovered as a particular case of the reconstruction theorem in the following way.

Proposition 13.21. *In the same context as above, let $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, and consider the modelled distribution $Y \in \mathscr{D}^{2\alpha}_{\mathrm{M_W}}$ built as above from a controlled rough path $(Y, Y') \in \mathscr{D}^{2\alpha}_W$. Then, the map $Y \dot W^j$ given by*

$$\big(Y \dot W^j\big)(s) := Y(s)\, \dot W^j + Y'_i(s)\, \dot{\mathbb{W}}^{ij}$$

belongs to $\mathscr{D}^{3\alpha-1}$. Furthermore, there exists a function Z, unique up to addition of constants, such that

$$\big(\mathcal{R} Y\dot{W}^j\big)(\psi) = \int \psi(t)\, dZ(t)\;,$$

and such that $Z_{s,t} = Y(s)\, W^j_{s,t} + Y_i'(s)\, \mathbb{W}^{i,j}_{s,t} + \mathcal{O}(|t-s|^{3\alpha})$.

Proof. The fact that $Y\dot{W}^j \in \mathscr{D}^{3\alpha-1}$ is an immediate consequence of the definitions. Since $\alpha > \frac{1}{3}$ by assumption, we can apply the reconstruction theorem to it, from which it follows that there exists a unique distribution η such that, if ψ is a smooth compactly supported test function, one has

$$\eta(\psi_s^\lambda) = \int \psi_s^\lambda(t) Y(s)\, dW^j_t + \int \psi_s^\lambda(t) Y_i'(s)\, d\mathbb{W}^{i,j}_{s,t} + \mathcal{O}(\lambda^{3\alpha-1})\;.$$

By a simple approximation argument, see Exercise 13.10, one can take for ψ the indicator function of the interval $[0,1]$, so that

$$\eta(\mathbf{1}_{[s,t]}) = Y(s)\, W^j_{s,t} + Y_i'(s)\, \mathbb{W}^{i,j}_{s,t} + \mathcal{O}(|t-s|^{3\alpha})\;.$$

Here, the reason why one obtains an exponent 3α rather than $3\alpha - 1$ is that it is really $|t-s|^{-1}\mathbf{1}_{[s,t]}$ that scales like an approximate δ-distribution as $t \to s$. □

Remark 13.22. Using the formula (13.26), it is straightforward to verify that if W happens to be a smooth function and $\mathbb{W}$ is defined from W via (2.2), but this time viewing it as a definition for the right-hand side, with the left-hand side given by a usual Riemann integral, then the function Z constructed in Proposition 13.21 coincides with the usual Riemann integral of Y against W^j.

Remark 13.23. The theory of (controlled) rough paths of lower regularity already hinted at in Section 2.4 can be recovered from the reconstruction operator and a suitable choice of regularity structure (essentially two copies of the truncated tensor algebra) in virtually the same way.

13.4 Proof of the reconstruction theorem

The proof of the reconstruction theorem originally given in [Hai14b] relied on wavelet analysis, in particular on the existence of compactly supported wavelets of arbitrary regularity [Dau88]. More recently, Otto and Weber [OSSW18] and then Moinat and Weber [MW18] obtained a version of the reconstruction theorem that bypasses this theory and is completely self-contained. The version of the proof given here is inspired by their work and has the advantage of being purely local: although we state the result for models and modelled distributions that are assumed to be defined on all of $\mathbf{R}^d$, the proof generalises immediately to arbitrary domains. The proof given here also generalises immediately to non-Euclidean scalings, even in situations where the ratios between scaling exponents are irrational.

A crucial ingredient is the following remark. Fix $\alpha > 0$ and let $\varrho\colon \mathbf{R}^d \to \mathbf{R}$ be even, smooth, compactly supported in the ball of radius 1, such that

$$\int x^k \varrho(x)\,dx = \delta_{k,0}\;, \qquad 0 < |k| \le \alpha\;, \tag{13.17}$$

where k denotes a d-dimensional multi-index and δ denotes Kronecker's delta. Note that such a function necessarily exists, since otherwise one would be able to find a polynomial P of degree at most α such that $\int P(x)\varphi(x)\,dx = 0$ for *every* smooth and compactly supported φ, which is clearly absurd. (See also Exercise 13.8 for a constructive proof.)

Given such a function ϱ, we define $\varrho^{(n)}(x) = 2^{nd}\varrho(2^n x)$, as well as

$$\varrho^{(n,m)} = \varrho^{(n)} * \varrho^{(n+1)} * \cdots * \varrho^{(m)}\;, \tag{13.18}$$

where $*$ denotes convolution. We also set $\varphi^{(n)} = \lim_{m\to\infty}\varrho^{(n,m)}$, so that in particular $\varphi^{(n)} = \varrho^{(n)} * \varphi^{(n+1)}$ and we write $\varrho_x^{(n)}(y) = \varrho^{(n)}(y-x)$ and similarly for $\varphi_x^{(n)}$; see Exercise 13.7 to see that the limit $\varphi^{(n)}$ exists and belongs to $\mathcal{C}_c^\infty$. We then have the following preliminary lemma.

Lemma 13.24. *Let $\alpha > 0$, let ϱ be as above and let $\xi_n \colon \mathbf{R}^d \to \mathbf{R}$ be a sequence of functions such that for every compact $\mathfrak{K}$ there exists $C_\mathfrak{K}$ such that $\sup_{x\in\mathfrak{K}}|\xi_n(x)| \le C_\mathfrak{K} 2^{\alpha n}$, and such that furthermore $\xi_n = \varrho^{(n)} * \xi_{n+1}$. Then, the sequence ξ_n is Cauchy in $\mathcal{C}^{-\beta}$ for every $\beta > \alpha$ and its limit ξ satisfies $\xi_n = \varphi^{(n)} * \xi$.*

If furthermore, for some $x \in \mathbf{R}^d$ and $\gamma > -\alpha$ one has the bound $|\xi_n(y)| \le 2^{\alpha n}\big(|x-y|^{\gamma+\alpha} + 2^{-(\gamma+\alpha)n}\big)$, uniformly over $n \ge 0$ and $|y-x| \le 1$, then $|\xi(\psi_x^\lambda)| \lesssim \lambda^\gamma$ for $\lambda \le 1$.

Proof. Let $\lambda \in (0,1]$ and let ψ_λ be a test function that is supported in the ball of radius λ and such that $|D^k\psi| \le \lambda^{-d-|k|}$ for all $|k| \le \alpha + 1$. In order to show that ξ_n is Cauchy in $\mathcal{C}^{-\beta}$ it then suffices to exhibit a bound of the type

$$|\psi_\lambda * (\xi_n - \xi_{n+1})| \lesssim \lambda^{-\beta} 2^{(\alpha-\beta)n}\;, \tag{13.19}$$

locally uniformly in x, for a proportionality constant independent of ψ_λ. Since there exists $\bar C > 0$ such that $\int |\psi_\lambda(x)|\,dx \le \bar C$, uniformly over λ and ψ_λ, it follows from the assumption $|\xi_n(x)| \le C 2^{\alpha n}$ that the left-hand side of (13.19) is bounded by $(1+2^\alpha)C\bar C 2^{\alpha n}$, so that the bound (13.19) holds whenever $\lambda \le 2^{-n}$.

To deal with the converse case $2^{-n} \le \lambda$, we rewrite the left-hand side of (13.19) as $|(\psi_\lambda * \varrho^{(n)} - \psi_\lambda) * \xi_{n+1}|$ and we note that, by Taylor's remainder theorem,

$$\big|\psi_\lambda(y) - T_x^{(\alpha)}(y)\big| \stackrel{\text{def}}{=} \Big|\psi_\lambda(y) - \sum_{|k|\le\alpha} \frac{D^k\psi_\lambda(x)}{k!}(y-x)^k\Big| \lesssim \lambda^{-N-d}|y-x|^N\;, \tag{13.20}$$

where $N = \lceil\alpha\rceil$. Since, by (13.17), one has $\varrho^{(n)} * T_x^{(\alpha)} = T_x^{(\alpha)}$ and since $T_x^{(\alpha)}(x) = \psi_\lambda(x)$, one has

$$\big(\psi_\lambda * \varrho^{(n)} - \psi_\lambda\big)(x) = \big(\varrho^{(n)} * (\psi_\lambda - T_x^{(\alpha)})\big)(x)\;,$$

which is bounded by $\lambda^{-N-d}2^{-nN}$ as an immediate consequence of (13.20). Since furthermore the support of this function has diameter at most 2λ, it follows that its integral is at most $\lambda^{-N}2^{-nN}$ so that, combining this with the a priori bound $|\xi_{n+1}| \lesssim 2^{\alpha n}$, we conclude that

$$|\psi_\lambda * (\xi_n - \xi_{n+1})| \lesssim \lambda^{-N}2^{(\alpha-N)n} .$$

Since $N \geq \alpha$, the bound (13.19) then follows for $2^{-n} \leq \lambda$ as required.

Since we have just shown that the sequence ξ_n is Cauchy, it has a limit $\xi \in \mathcal{C}^{-\beta}$. Given a test function ψ, we have

$$\xi_n(\psi) = \xi_{n+1}(\varrho^{(n)} * \psi) = \xi_m(\varrho^{(n,m)} * \psi) = \xi(\varphi^{(n)} * \psi) ,$$

showing that $\xi_n = \varphi^{(n)} * \xi$ as required. (Here we use the fact that the convergence $\varrho^{(n,m)} \to \varphi^{(n)}$ takes place in $\mathcal{C}^r$ for $r = r_\beta$ by Exercise 13.7.)

The proof of the second claim follows the same lines. We write

$$\xi(\psi_x^\lambda) = \xi_n(\psi_x^\lambda) + \sum_{k\geq n}(\xi_{k+1} - \xi_k)(\psi_x^\lambda) ,$$

where n is chosen in such a way that $\lambda \in [2^{-(n+1)}, 2^{-n}]$. As a consequence of this choice and of our assumption on ξ_n, one has the bound

$$\begin{aligned}|\xi_n(\psi_x^\lambda)| &\lesssim \lambda^{-d}\int_{B_x(\lambda)} 2^{\alpha n}\bigl(|x-y|^{\gamma+\alpha} + 2^{-(\gamma+\alpha)n}\bigr)\,dy \\ &\lesssim \lambda^{\gamma+\alpha}2^{\alpha n} + 2^{-\gamma n} \lesssim \lambda^\gamma .\end{aligned}$$

To bound $(\xi_{k+1} - \xi_k)(\psi_x^\lambda)$ we proceed as above so that

$$\begin{aligned}\bigl|(\xi_{k+1} - \xi_k)(\psi_x^\lambda)\bigr| &\lesssim \lambda^{-N-d}2^{-nN}\int_{B_x(2\lambda)} |\xi_{n+1}(y)|\,dy \\ &\lesssim \lambda^{\gamma+\alpha-N}2^{(\alpha-N)n} + \lambda^{-N}2^{-(\gamma+N)n} .\end{aligned}$$

Since $N > \alpha$ and $N > -\gamma$, this is summable and its sum is again of order λ^γ, thus concluding the proof. □

Remark 13.25. Note the strong similarity of this setting with that of multiresolution analysis [Mey92]: the image of the convolution operator with $\varphi^{(n)}$ plays the role of V_n and convolution with $\varrho^{(n)}$ plays the role of the projection $V_{n+1} \to V_n$.

Let us now restate the reconstruction theorem for the reader's convenience. (We only consider the case $\gamma > 0$ here.)

Theorem 13.26. *Let $\mathscr{T}$ be a regularity structure as above and let (Π, Γ) a model for $\mathscr{T}$ on $\mathbf{R}^d$. Then, for $\gamma > 0$, there exists a unique linear map $\mathcal{R}\colon \mathscr{D}^\gamma \to \mathcal{D}'(\mathbf{R}^d)$ such that*

$$\bigl|\bigl(\mathcal{R}f - \Pi_x f(x)\bigr)(\psi_x^\lambda)\bigr| \lesssim \lambda^\gamma ,$$

uniformly over $\psi \in \mathcal{B}_r$ *and* $\lambda \in (0,1]$, *and locally uniformly in* x. *The statement still holds for* $\gamma < 0$, *except that uniqueness fails.*

Proof. We first define operators $\mathcal{R}^{(m,m)}$ by

$$\big(\mathcal{R}^{(m,m)} f\big)(y) = (\varphi^{(m)} * \Pi_y f(y))(y) = (\Pi_y f(y))(\varphi_y^{(m)}) . \tag{13.21}$$

The idea then is to obtain $\mathcal{R}$ as the limit of $\mathcal{R}^{(m,m)}$ as $m \to \infty$. This however turns out not to be that easy to obtain directly. Instead, we try to make use of Lemma 13.24 and define, for $m > n$,

$$\mathcal{R}^{(n,m)} f = \varrho^{(n,m-1)} * \mathcal{R}^{(m,m)} f ,$$

so that, as a consequence of the identity $\Pi_z = \Pi_y \Gamma_{yz}$,

$$\big(\mathcal{R}^{(n,m)} f - \mathcal{R}^{(n,m+1)} f\big)(x) = \int \varrho_x^{(n,m-1)}(y) \int \varrho_z^{(m)}(y)\big(\Pi_y\big(f(y) - \Gamma_{yz} f(z)\big)\big)(\varphi_z^{(m+1)})\, dz\, dy .$$

At this stage we note that, as a consequence of the analytical bounds (13.13) imposed in the definition of a model, the quantity $\big(\Pi_y \tau\big)(\varphi_z^{(m+1)})$ is bounded by $C 2^{-\alpha m}\|\tau\|_\alpha$, uniformly over $|y - z| \lesssim 2^{-m}$ and $\tau \in T_\alpha$. On the other hand, the definition of the spaces $\mathscr{D}^\gamma$ guarantees that the component of $f(y) - \Gamma_{yz} f(z)$ in T_α is bounded by $2^{(\alpha-\gamma)m}$, again uniformly over $|y - z| \lesssim 2^{-m}$. Since $\int |\varrho_x^{(n,m-1)}(y)|\, dy \lesssim 1$, uniformly over m and n, we conclude that

$$\big\|\big(\mathcal{R}^{(n,m)} - \mathcal{R}^{(n,m+1)}\big) f\big\|_{L^\infty} \lesssim 2^{-\gamma m} , \tag{13.22}$$

uniformly over $n \geq 0$ and $m \geq n$. Furthermore, it is straightforward to check that

$$\big\|\mathcal{R}^{(n,n)} f\big\|_{L^\infty} \lesssim 2^{-\underline{\alpha} n} , \tag{13.23}$$

where $\underline{\alpha}$ denotes the smallest degree in the ambient regularity structure. It follows that $\mathcal{R}^{(n)} f = \lim_{m\to\infty} \mathcal{R}^{(n,m)} f$ is well-defined and also satisfies the bound (13.23). Since the identity

$$\mathcal{R}^{(n,m)} f = \varrho^{(n)} * \mathcal{R}^{(n+1,m)} f$$

holds for every $m \geq n+1$, it follows that $\mathcal{R}^{(n)} f = \varrho^{(n)} * \mathcal{R}^{(n+1)} f$, so that $\mathcal{R} f = \lim_{n\to\infty} \mathcal{R}^{(n)} f$ exists in $\mathcal{C}^\alpha$ for every $\alpha < \underline{\alpha}$ by Lemma 13.24.

It remains to show that one has the bound

$$\big|\big(\mathcal{R} f - \Pi_x f(x)\big)(\psi_x^\lambda)\big| \lesssim \lambda^\gamma . \tag{13.24}$$

For this, we note first that if we define $f_x \in \mathscr{D}^\gamma$ by $f_x(y) = \Gamma_{yx} f(x)$, then one has $\mathcal{R}^{(n)} f_x = \varphi^{(n)} * \Pi_x f(x)$, so that (13.24) can be written as

$$\big|\mathcal{R}\big(f - f_x\big)(\psi_x^\lambda)\big| \lesssim \lambda^\gamma . \tag{13.25}$$

Since $\|(f - f_x)(y)\|_\alpha \lesssim |y - x|^{\gamma-\alpha}$, it follows from the definition (13.14) of $\mathscr{D}^\gamma$ that

$$\big|\big(\mathcal{R}^{(n,n)}\big(f - f_x\big)\big)(y)\big| = \big|\big(\Pi_y\big(f - f_x\big)\big)(\varphi_y^{(n)})\big| \lesssim \sum_{\alpha<\gamma} 2^{-\alpha n}|y - x|^{\gamma-\alpha}$$
$$\lesssim 2^{-\underline{\alpha} n}\big(|y - x|^{\gamma-\underline{\alpha}} + 2^{(\underline{\alpha}-\gamma)n}\big) .$$

By (13.22) the same bound also holds for $\mathcal{R}^{(n)}$, so that the claim follows from the second part of Lemma 13.24.

The case $\gamma < 0$ works in a similar way, but this time we explicitly define

$$\mathcal{R}f = \mathcal{R}^{(0,0)}f + \sum_n \big(\varrho^{(n)} - \delta\big) * \mathcal{R}^{(n,n)}f ,$$

where δ denotes the Dirac delta-distribution. We leave it as an exercise for the reader to verify that this sum does indeed converge in $\mathcal{C}^\alpha$ for every $\alpha < \underline{\alpha}$ and that the limit satisfies the required bound. □

Remark 13.27. In the particular case where $\Pi_x\tau$ happens to be a continuous function for every $\tau \in T$ (and every $x \in \mathbf{R}^d$), $\mathcal{R}f$ is also a continuous function and one has the identity

$$\big(\mathcal{R}f\big)(x) = \big(\Pi_x f(x)\big)(x) . \tag{13.26}$$

We leave it as an exercise to show that this is the case, taking (13.21) as a starting point.

13.5 Exercises

Exercise 13.1 *a) Relate Theorem 13.18, in case $d = 1$, with the Young integral.*

b) Draw inspiration from Weierstrass's construction of a continuous nowhere differentiable function to construct examples demonstrating the "only if" part of Theorem 13.18.

Exercise 13.2 (Hölder spaces) *For $k \in \mathbf{N}$ and $\alpha \in (0,1)$, it is customary to define $\mathcal{C}^{k+\alpha}$ as the space of k times continuously differentiable functions $f : \mathbf{R}^d \to \mathbf{R}$ such that their derivatives of order k are α-Hölder continuous. Show that this agrees with the obvious extension to $\mathbf{R}^d$ of the definition given earlier in (13.2).*

Exercise 13.3 *Show that in general, the function Z from Proposition 13.21 coincides, up to an additive constant, with the rough integral $\int_0^t Y(s)\,dX_s^j$, in the sense of Remark 4.12.*

♯ **Exercise 13.4** *Let $\bar\gamma \geq \gamma > 0$ and let $f \in \mathcal{C}(\mathbf{R}^d, T_{<\bar\gamma})$ such the "modelled distribution" bound (13.14) holds for every $\alpha < \gamma$.*

$$\|f\|_{\mathscr{D}^\gamma} < \infty .$$

Show that the projection of f on $T_{<\gamma}$ belongs to $\mathscr{D}^\gamma$.

Exercise 13.5 *Let (Π, Γ) be a model for the "rough path" regularity structure given in Definition 13.4 with the additional property that $\Pi_s \dot{W}^i$ is the distributional derivative of $\Pi_s W^i$ for every s. Show that it is then necessarily of the form $\mathrm{M}_{\mathbf{W}}$ for some α-Hölder rough path $\mathbf{W}$ as in Lemma 13.20.*

Exercise 13.6 *Using the regularity structure defined in Section 13.3.2, give a proof of the Lyons–Victoir extension theorem using the case $\gamma < 0$ of the reconstruction theorem. **Hint:** A useful fact is that, for any symbol τ of degree α and any model (Π, Γ), the function $y \mapsto f_x^\tau(y) = \Gamma_{yx}\tau - \tau$ belongs to $\mathscr{D}^\alpha$.*

$*$ **Exercise 13.7** *Show that the limit $\varphi^{(n)} = \lim_{m\to\infty} \varrho^{(n,m)}$ with $\varrho^{(n,m)}$ as in (13.18) exists and belongs to $\mathcal{C}_c^\infty$, with the limit being taken in $\mathcal{C}^r$ for any $r > 0$. Show furthermore that, despite the fact that one necessarily has $\int |\varrho(x)|\,dx > 1$ (why?), there exists a constant C such that $\int |\varrho^{(n,m)}(x)|\,dx < C$, uniformly over $n, m \in \mathbf{N}$. **Hint:** Work in Fourier space to show existence and smoothness of the limit and in direct space to show that it has compact support.*

Exercise 13.8 *Show that it is possible to find a smooth compactly supported function ϱ such that (13.17) holds. **Hint:** Note first that for any ψ integrating to 1 one can find a differential operator $\mathcal{L}$ of order α with constant coefficients and without constant term such that $\int \psi(x)P(x)\,dx = \big((\mathrm{Id} - \mathcal{L})P\big)(0)$ for all polynomials P of degree α. Show that then $\varrho = \sum_{k\le\alpha} (\mathcal{L}^*)^k\psi$ does the trick, where $\mathcal{L}^*$ denotes the formal adjoint of $\mathcal{L}$.*

Exercise 13.9 *Show that the construction of Section 2.4 determines a regularity structure with $T = T^{(p)}(\mathbf{R}^d)$, structure group $G^{(p)}(\mathbf{R}^d)$, and such that $\deg e_w = \alpha|w|$. Show also that every rough path $\mathbf{X}$ determines a model for this regularity structure and that the definition of a controlled path given in Definition 4.18 coincides with the definition of the space $\mathscr{D}^{p\alpha}$ for the model associated to the rough path $\mathbf{X}$.*

Exercise 13.10 *Show that one can indeed take $\varphi = \mathbf{1}_{[0,1]}$ in the last step of the proof of Proposition 13.21. **Hint:** show first that one can write*

$$\mathbf{1}_{[0,1]} = \sum_{n\ge 0} (\varphi_n + \psi_n)\;,$$

where φ_n is supported on $[0, 2^{-n}]$, ψ_n is supported on $[1 - 2^{-n}, 1]$, all of these functions are smooth, and $\|D^k\varphi_n\|_\infty + \|D^k\psi_n\|_\infty \le C2^{kn}$ for some $C > 0$, uniformly over $n \ge 0$ and $k \in [0, r]$.

Exercise 13.11 *Given a fixed regularity structure and model, given $\gamma > 0$, $\tau \in T_\gamma$ and $x \in \mathbf{R}^d$, define a function $f_{x,\tau}\colon \mathbf{R}^d \to T_{<\gamma}$ by*

$$f_{x,\tau}(y) = \Gamma_{yx}\tau - \tau\;.$$

Show that $f_{x,\tau} \in \mathscr{D}^\gamma$ and that one has $\mathcal{R} f_{x,\tau} = \Pi_x\tau$. Use this to give another proof of Lyons' extension theorem (Exercise 4.6).

13.6 Comments

All basic definitions (regularity structure, model, modelled distribution, ...) are taken from [Hai14b]. An alternative theory to the theory of regularity structures was introduced more or less simultaneously in Gubinelli–Imkeller–Perkowski [GIP15]. Instead of the reconstruction theorem, that theory builds on properties of Bony's paraproduct [Bon81, BMN10, BCD11] and it introduces a notion of "paracontrolled distribution" which replaces the notion of "modelled distribution". This theory is also able to deal with stochastic PDEs like the KPZ equation or the dynamical Φ^4_3 equation, see Catellier–Chouk [CC18b], but its scope is not as wide as that of the theory of regularity structures. For example, as it stands it does not appear to be able to deal with classical one-dimensional parabolic SPDEs driven by space-time white noise with a diffusion coefficient depending on the solution or the type of equation arising as natural evolutions on the space of loops with values in a manifold [Hai16, BGHZ19]. This is however evolving rapidly as a number of recent results show that paracontrolled calculus can alternatively be used as the foundation for the analytical aspects of the theory of regularity structures. We refer to [BB19, BH18, MP18, BH19, BM19] for more details.

One advantage of the paraproduct-based theory is that one generally deals with globally defined objects rather than the "jets" used in the theory of regularity structures. It also uses some already well-studied objects, so that it can rely on a substantial body of existing literature. On the flip side, it usually achieves a less clean break between the analytical and the algebraic aspects of a given problem. Furthermore, while the probabilistic aspects of the theory are expected to be equivalent to some extent, it is not completely clear how an analogue of the results [CH16] would even be formulated in the paracontrolled setting, although the results mentioned above may provide a hint. A third approach, closer in spirit to Wilson's renormalisation group ideas, was developed by Kupiainen [Kup16] who used it to give an alternative construction of the solutions to the dynamical Φ^4_3 equation.

The regularity structure view on rough paths, Sections 13.2.2 and 13.3.2, is further explored in [BCFP19]; see also [Hai14b, Sec. 4.4]. As already mentioned, the original proof of the reconstruction theorem given in [Hai14b] (also reproduced in the first edition of this book) relies on wavelet analysis, in particular on the existence of compactly supported wavelets of arbitrary regularity [Dau88]. The new proof in Section 13.4 was inspired by [OSSW18, MW18] and has the advantage of being entirely self-contained. One additional advantage is that the current proof immediately generalises to scalings $\mathfrak{s}$ that are not necessarily rational. (Rationality of $\mathfrak{s}$ was required in the original articles in order to be able to build a suitable wavelet basis by tensorisation of one-dimensional wavelet bases.)

One advantage of the proof using wavelets is that it implies that a model is uniquely determined by the actions of Π_x and Γ_{xy} on countably many translates and scalings of a finite number of functions and for a countable number of values of x, y. It also makes it very easy to prove a Kolmogorov-type criterion for models, see [Hai14b, Prop. 3.32 & Thm. 10.7].