Chapter 12
Stochastic partial differential equations

Second order stochastic partial differential equations are discussed from a rough path point of view. In the linear and finite-dimensional noise case we follow a Feynman–Kac approach which makes good use of concentration of measure results, as those obtained in Section 11.2. Alternatively, one can proceed by flow decomposition and this approach also works in a number of nonlinear situations. Secondly, now motivated by some semilinear SPDEs of Burgers' type with infinite-dimension noise, we study the stochastic heat equation (in space dimension 1) as evolution in Gaussian rough path space *relative to the spatial variable*, in the sense of Chapter 10.

12.1 First order rough partial differential equations

12.1.1 Rough transport equation

As a prototypical linear first order PDE with noise we consider the transport equation, posed (without loss of generality) as a terminal value problem. This is,

$$-\partial_t u(t,x) = \sum_{i=1}^{d} f_i(x) \cdot D_x u(t,x) \dot{W}_t^i \equiv \Gamma u_t(x) \dot{W}_t \;, \quad u(T,\bullet) = g \;, \qquad (12.1)$$

where $u : [0,T] \times \mathbf{R}^n \to \mathbf{R}$, with vector fields $f = (f_1, \ldots, f_d)$ driven by a $\mathcal{C}^1$ driving signal $W = (W^1, \ldots, W^d)$, and we write indifferently $u(t,x) = u_t(x)$. The canonical pairing of $Du = D_x u = (\partial_{x^1} u, \ldots, \partial_{x^n} u)$ with a vector field is indicated by a dot, and we already used the operator / vector notation

$$\Gamma_i = f_i(x) \cdot D_x, \;\; \Gamma = (\Gamma_1, \ldots, \Gamma_d). \qquad (12.2)$$

By the methods of characteristics, the unique (classical) $\mathcal{C}^{1,1}$-solution $u : [0,T] \times \mathbf{R}^n \to \mathbf{R}$, is given explicitly by

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$$u(s,x) = u(s,x;W) := g(X_T^{s,x}) , \tag{12.3}$$

provided $g \in \mathcal{C}^1$ and the vector fields $f_1, \ldots, f_d$ are nice enough ($\mathcal{C}_b^1$ will do) to ensure a $\mathcal{C}^1$ solution flow for the ODE $\dot{X} = \sum_{i=1}^d f_i(X)\dot{W}^i \equiv f(X)\dot{W}$; here $X^{s,x}$ denotes the unique solution started from $X_s = x$.

We start with a rough path stability result for the transport equation, the proof of which is an immediate consequence of our results on flow stability of RDEs.

Proposition 12.1. *Let $g \in \mathcal{C}(\mathbf{R}^m)$ and $W^\varepsilon \in \mathcal{C}^1([0,T],\mathbf{R}^d)$, with geometric rough path limit $\mathbf{W} \in \mathscr{C}_g^{0,\alpha}$, $\alpha > 1/3$. Write $u^\varepsilon(s,x) := u(s,x;W^\varepsilon)$, defined as in (12.3) with W replaced by W^ε. Let $f \in \mathcal{C}_b^3$. Then u^ε converges locally uniformly to*

$$u(s,x;\mathbf{W}) := g(X_T^{s,x}) \tag{12.4}$$

where $X^{s,x}$ denotes the (unique) RDE solution to $dX = f(X)d\mathbf{W}$, started from $X_s = x$. (In particular, the limit depends on $\mathbf{W}$ but not on the approximating sequence.)

It is instructive to consider the case of Brownian motion $B = B(t,\omega)$ with Stratonovich lift as prototypical example of a (random) geometric rough path. The RDE solution X is then equivalently described by a Stratonovich SDE and $u(t,x;\omega) = g(X_T^{t,x}(\omega))$ is $\mathcal{F}_t^T$-measurable. The so-defined random field should then constitute a (backward adapted) solution to the Stratonovich backward stochastic partial differential equation

$$-du_t(x) = \Gamma u_t(x) \circ \overleftarrow{dB}_t \ , \quad u(T,\bullet) = g \ , \tag{12.5}$$

where $\overleftarrow{dB}$ stands for backward Stratonovich integration (cf. Section 5.4) provided g (und then Γu_t) are sufficiently regular to make this Stratonovich integral meaningful. If rewritten in Itô-form, a matrix valued second order $\Gamma^2 = (\Gamma_i \Gamma_j)_{1 \le i,j \le d}$ appears, which of course must not change the hyperbolic nature of the stochastic transport equation. (In classical SPDE theory on has the *stochastic parabolicity* condition, which in the transport case is fully degenerate.)

All this strongly suggests that rough transport noise must be geometric (i.e. $\mathbf{W} \in \mathscr{C}_g^\alpha$). We now prepare the definition of (regular, backward) solution to the rough transport equation. Since we are in the fortunate position to have an explicit solution (candidate) we derive a graded set of rough path estimates that provide a natural generalisation of the classical the transport differential equation. In what follows we abbreviate estimates of the form $|(a) - (b)| \lesssim |t-s|^\gamma$ by writing $(a) \overset{\gamma}{=} (b)$. (Both sides may depend on s,t and the multiplicative constant hidden in $\lesssim$ is assumed uniform over bounded intervals).

Proposition 12.2. *Consider vector fields $f = (f_1, \ldots, f_d) \in \mathcal{C}_b^5$, with associated first order differential operators $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$. There is a unique $\mathcal{C}^3$ solution flow for the RDE $dX = f(X)d\mathbf{W}$ with $\mathbf{W} \in \mathscr{C}_g^\alpha, \alpha > 1/3$. Let $g \in \mathcal{C}^3$ and define $u(s,x;\mathbf{W}) := g(X_T^{s,x})$ as in (12.4). Then $u = u(s,x) \in \mathcal{C}^{\alpha,3}, u_T = g$, and we have the estimates, with Einstein summation,*

$$
\begin{aligned}
u_s(x) &\overset{3\alpha}{=} u_t(x) + \Gamma_i u_t(x) W^i_{s,t} + \Gamma_i \Gamma_j u_t(x) \mathbb{W}^{i,j}_{s,t} \\
\Gamma_i u_s(x) &\overset{2\alpha}{=} \Gamma_i u_t(x) + \Gamma_i \Gamma_j u_t(x) W^j_{s,t}\,, \\
\Gamma_i \Gamma_j u_s(x) &\overset{\alpha}{=} \Gamma_i \Gamma_j u_t(x)\,,
\end{aligned}
$$

with $0 \leqslant s < t \leqslant T$, $i, j = 1, \ldots, d$, *locally uniformly in* x, *and, as consequence,*

$$
u_s(x) - g(x) = u_s(x) - u_T(x) = \int_s^T \Gamma u_t(x)\, d\mathbf{W}_t\ .
$$

Remark 12.3. The first 3α estimate is nothing but Davie's definition of solution for a linear RDE, here of the form $-du = \Gamma u\, d\mathbf{W}$. In finite dimensions, a linear map Γ is necessarily bounded (equivalently: continuous) as linear operator, so that the cascade of lower order $(2\alpha, \alpha)$ estimates are a trivial consequence of the first. This is different in the present situation, where u_t takes values in a function space where each application of Γ amounts to take one derivative. These estimates then have the interpretation that time regularity of u, in the stated ("$k\alpha$") controlled sense, can be traded against space regularity.

Remark 12.4. The rough integral formulation needs explanation. Indeed, while it is clear from $\delta \Xi \overset{3\alpha}{=} \delta u(x) = 0$ that $\Xi_{s,t} = \Gamma_i u_t(x) W^i_{s,t} + \Gamma_i \Gamma_j u_t(x) \mathbb{W}^{i,j}_{s,t}$ has a sewing limit, the right-point evalution requires attention, cf. Proposition 5.12 and the subsequent discussion about the subtleties of "right-point" rough integrals. Fortunately, one checks that $(\Gamma u, -(\Gamma^2 u)^T) \in \mathscr{D}^{2\alpha}_X$ so that, thanks to (5.10), Remark 5.13, this sewing limit, over all partitions of $[0, T]$ say, is exactly identified as

$$
\lim_{|\mathcal{P}| \downarrow 0} \sum_{[s,t] \in \mathcal{P}} \left(\Gamma u_t X_{s,t} - (\Gamma^2 u_t)^T \mathbb{X}_{s,t} \right) = \int_0^T (\Gamma u, -\Gamma^2 u^T) d\mathbf{X}\,,
$$

where we omitted x for better readability. (Since the matrix $\Gamma^2 u_t = (\Gamma_i \Gamma_j u_t)_{1 \le i,j \le d}$ is in general not symmetric, a careful check of the controlledness condition is best spelled out in coordinates.)

Notwithstanding the elegance of the rough integral formulation, additional quantifiers, such as local uniformity in x, are better formulated at the level of the detailed estimates which brings us to

Definition 12.5. Any $\mathcal{C}^{\alpha,3}$-function $u : [0, T] \times \mathbf{R}^n \to \mathbf{R}$, for which the (locally uniform) estimates in Proposition 12.2 hold is called a *regular solution* to the *rough backward transport equation*

$$
-du = \Gamma u d\mathbf{W}.
$$

Proof (Proposition 12.2). Consider a solution $X = X^{s,x}$ to $dX = f(X) d\mathbf{W}$, started from $X_s = x$ so that

$$
X_t \overset{3\alpha}{=} x + f(x) W_{s,t} + f' f(x) \mathbb{W}_{s,t}.
$$

Fix times $s < t < T$. By uniqueness of RDE flow, $X_T^{t,y} = X_T^{s,x}$ whenever $y = X_t^{s,x}$. From $u(s,x) := g(X_T^{s,x})$ and uniqueness of the RDE flow it is clear that, for all such t,

$$u(s,x) = u(t, X_t^{s,x}).$$

Note that $u_t = u(t,\bullet) \in \mathcal{C}^3$ follows from $g \in \mathcal{C}^3$, $f \in \mathcal{C}^5$; the claimed $\mathcal{C}^{\alpha,3}$ regularity is then easy to see. We can expand

$$u_t(X_t^{s,x}) \stackrel{3\alpha}{=} u_t(x) + Du_t(x)(f(x)W_{s,t} + (Df)f(x)\mathbb{W}_{s,t}) + \frac{1}{2}D^2u_t(x)(f(x)W_{s,t})^2$$

where the final term is really the contraction $\partial_{ij}u_t f_k^i f_l^j (\frac{1}{2}W_{s,t} \otimes W_{s,t})^{k,l}$ with summation over all repeated indices. Using geometricity of $\mathbf{X}$ and symmetry of $D^2u_t(x)(f,f)$ the right-hand side becomes

$$u_t(x) + Du_t(x)f(x)W_{s,t} + \{Du_t(x)(Df)f(x) + D^2u_t(x)(f,f)(x)\}\mathbb{W}_{s,t}.$$

(We essentially repeated the proof of Itô's formula here, cf. Section 7.5.) In terms of the first order differential operators Γ_i associated to the vector fields f_i this can be written elegantly as

$$u_s(x) \stackrel{3\alpha}{=} u(t,x) + \Gamma u_t(x)W_{s,t} + \Gamma^2 u_t(x)\mathbb{W}_{s,t}\ .$$

This relation actually implies that with $\Xi_{s,t} := \Gamma_i u_t(x)W_{s,t}^i + \Gamma_i\Gamma_j u_t(x)\mathbb{W}_{s,t}^{i,j}$ we have $|(\delta\Xi)_{r,s,t}| = \mathrm{O}(|t-r|^{3\alpha})$ and hence (after a line of algebra) $(\Gamma_i u_{s,t} - \Gamma_i\Gamma_j u_t W_{s,t}^j)W_{r,s}^i \stackrel{3\alpha}{=} 0$ which strongly suggests validity of the desired 2α-estimate, for all $i = 1,\dots,d$,

$$\Gamma_i u_s(x) \stackrel{2\alpha}{=} \Gamma_i u_t(x) + \Gamma_i\Gamma_j u_t(x)W_{s,t}^j\ .$$

Since no true roughness condition on W is imposed (W could be zero!), one has to check this by hand from $u(s,x) = g(X_T^{s,x})$, left to the reader. Similarly, the previous relation gives $(\Gamma^2 u_t - \Gamma^2 u_s)W_{s,t} \stackrel{2\alpha}{=} 0$ so that the same argument suggests $\Gamma^2 u_s(x) - \Gamma^2 u_t(x) \stackrel{\alpha}{=} 0$. Here again, a direct verification is not hard (and amounts to check α-Hölder regularity of $s \mapsto \Gamma^2 g(X_T^{s,x})$, with $g \in \mathcal{C}^3$.) □

We can now show that solutions in the sense of Definition 12.5 are unique.

Theorem 12.6. *Consider vector fields $f = (f_1,\dots,f_d) \in \mathcal{C}_b^5$, with associated first order differential operators $\Gamma = (\Gamma_1,\dots,\Gamma_d)$ and $\mathbf{W} \in \mathscr{C}_g^\alpha([0,T],\mathbf{R}^d)$ with $\alpha > 1/3$. For $g \in \mathcal{C}^3$, there exists a unique regular solution $u : [0,T] \times \mathbf{R}^n \to \mathbf{R}$ of $\mathcal{C}^{\alpha,3}$ regularity to the rough backward transport equation*

$$-du = \Gamma u d\mathbf{W}\ , \quad u(T,\bullet) = g\ .$$

Proof. Existence is clear, since Proposition 12.2 exactly says that $(s,x) \mapsto g(X_T^{s,x})$ gives a regular solution. Let now u be any solution with $u_T = g$. We show that, whenever X solves $dX = f(X)d\mathbf{W}$,

$$u(t,X_t) - u(s,X_s) \overset{3\alpha}{=} 0.$$

Since $3\alpha > 1$ this entails that $t \mapsto u(t,X_t)$ is constant, and so $u(s,x) = u(T,X_T^{s,x}) = g(X_T^{s,x})$. In fact, we show for $k = 1,2,3$

$$\Gamma^{3-k} u_t(X_t) \overset{k\alpha}{=} \Gamma^{3-k} u_s(X_s).$$

(Case $k = 1$.) Write

$$\Gamma^2 u_t(X_t) - \Gamma^2 u_s(X_s) = \Gamma^2 u_t(X_t) - \Gamma^2 u_s(X_t) + \Gamma^2 u(s,X_t) - \Gamma^2 u(s,X_s).$$

From the (third) defining property of a solution, the first difference on the right-hand side of order α. Since solutions are $\mathcal{C}^3$ in space, hence $\Gamma^2 u(s,\bullet) \in \mathcal{C}^1$, always uniformly in $s \in [0,T]$ the final difference is also of order α, as required.
(Case $k = 2$.) Write

$$\Gamma u_t(X_t) - \Gamma u_s(X_s) = \Gamma u_t(X_t) - \Gamma u_s(X_t) + \Gamma u_s(X_t) - \Gamma u_s(X_s).$$

By the second defining property of a solution, the first difference on the right-hand side equals $-\Gamma^2 u_t(X_t) W_{s,t}$ (up to order 2α). On the other hand, $\Gamma u_s \in \mathcal{C}^2$ so that the final difference can be replaced by

$$D\Gamma u_s(X_s)(X_t - X_s) \overset{2\alpha}{=} D\Gamma u_s(X_s) f(X_s) W_{s,t} = \Gamma^2 u_s(X_s) W_{s,t}.$$

Put together we have $\Gamma u_t(X_t) - \Gamma u_s(X_s) = (\Gamma^2 u_s(X_s) - \Gamma^2 u_t(X_t)) W_{s,t}$. We see that this is of (desired) order 2α, thanks to the case $k = 1$ and $W_{s,t} \overset{\alpha}{=} 0$.
(Case $k = 3$.) We write

$$u(t,X_t) - u(s,X_s) = u(t,X_t) - u(s,X_t) + u(s,X_t) - u(s,X_s).$$

By the (first) defining property of a solution, the the first difference on the right-hand side equals $-\Gamma u_t(X_t) W_{s,t} - \Gamma^2 u_t(X_t) \mathbb{W}_{s,t}$ (up to order 3α). On the other hand, $u(s,\bullet) \in \mathcal{C}^3$ so that the final difference can be replaced, using a second order Taylor expansion, exactly as in the proof of Proposition 7.8, by

$$\begin{aligned} Du_s(X_s)(f(X_s)W_{s,t} + f'f(X_s)\mathbb{W}_{s,t}) + \frac{1}{2} D^2 u_s(f(X_s), f(X_s)) W_{s,t} \otimes W_{s,t} \\ = \Gamma u_s(X_s) W_{s,t} + \Gamma^2 u_s(X_s) \mathbb{W}_{s,t} \end{aligned}$$

Put together (and using the cases $k = 1,2$) gives the desired estimate. □

12.1.2 Continuity equation and analytically weak formulation

Given a finite measure $\varrho \in \mathcal{M}(\mathbf{R}^n)$ and a continuous bounded function $\varphi \in \mathcal{C}_b(\mathbf{R}^n)$, we write $\varrho(\varphi) = \int \varphi(x) \varrho(dx)$ for the natural pairing. We are interested in measure-valued (forward) solutions to the continuity equation

$$\partial_t \varrho = -\sum_{i=1}^{d} \operatorname{div}_x(f_i(x)\varrho_t)\dot{W}_t^i \equiv \Gamma^\star \varrho_t \dot{W}_t$$

when W becomes a (geometric) rough path. As before, $\Gamma_i = f_i(x) \cdot D_x$, with formal adjoint $\Gamma_i^\star = -\operatorname{div}_x(f_i \bullet)$.

Definition 12.7. We say that $\varrho : [0,T] \to \mathcal{M}(\mathbf{R}^n)$ is a measure-valued forward RPDE solution to the rough continuity equation

$$d\varrho_t + \operatorname{div}_x(f(x)\varrho_t)d\mathbf{W}_t = 0 \tag{12.6}$$

if, uniformly over φ bounded in $\mathcal{C}_b^3$,

$$\begin{aligned}
\varrho_t(\varphi) &\overset{3\alpha}{=} \varrho_s(\varphi) + \varrho_s(\Gamma\varphi)W_{s,t} + \varrho_s(\Gamma^2\varphi)\mathbb{W}_{s,t} \\
\varrho_t(\Gamma\varphi) &\overset{2\alpha}{=} \varrho_s(\Gamma\varphi) + \varrho_s(\Gamma^2\varphi)W_{s,t} \\
\varrho_t(\Gamma^2\varphi) &\overset{\alpha}{=} \varrho_s(\Gamma^2\varphi).
\end{aligned}$$

(Note $\Gamma\varphi, \Gamma^2\varphi \in \mathcal{C}_b$ so all pairings are well-defined. Formally, the second and third estimate follow from the first with φ replaces by $\Gamma\varphi$ and $\Gamma^2\varphi$), however doing so would require test functions up to $\Gamma^4\varphi \notin C_b$. Itemizing the estimates allows us to keep track of the correct regularity of φ.)

These estimates imply immediately the following (analytically) weak formulation

$$\forall \varphi \in \mathcal{C}_b^3 : \varrho_t(\varphi) - \varrho_0(\varphi) = \int_0^t (\varrho_s(\Gamma\varphi), \varrho_s(\Gamma^2\varphi))d\mathbf{W}_s \;,$$

but the finer information, as put foward in the definition, is crucial for uniqueness. (Remark 12.9 below comments on time-dependent test functions.)

Theorem 12.8. *Consider vector fields $f = (f_1, \ldots, f_d) \in \mathcal{C}_b^5$, with associated first order differential operators $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$ and $\mathbf{W} \in \mathscr{C}_g^\alpha([0,T], \mathbf{R}^d)$ with $\alpha > 1/3$. For every measure $\nu \in \mathcal{M}(\mathbf{R}^n)$, there exists a unique measure-valued solution to the rough continuity equation*

$$d\varrho_t + \operatorname{div}_x(f(x)\varrho_t)d\mathbf{W}_t \;, \quad \varrho_0 = \nu \;, \tag{12.7}$$

with explicit representation, for $\varphi \in \mathcal{C}_b^3$, given by

$$\varrho_t(\varphi) = \int \varphi(X_t^{0,x})\nu(dx) \;.$$

Proof. (Existence) Let $X = X^{0,x}$ be a solution to the RDE $dX = f(X)d\mathbf{W}$, started at $X_0 = x$. By Proposition 7.8, a form of Itô's formula for controlled rough paths,

$$\varphi(X_t) \overset{3\alpha}{=} \varphi(X_s) + \varphi(X_s)X_s'W_{s,t} + (D\varphi(X_s)X_s'' + D^2\varphi(X_s)(X_s', X_s'))\mathbb{W}_{s,t} \;,$$

uniformly in $\varphi \in \mathcal{C}_b^3$. Taking into account $X' = f(X), X'' = (Df)f$ gives

$$\varphi(X_t) \stackrel{3\alpha}{=} \varphi(X_s) + \Gamma\varphi(X_s)W_{s,t} + (\Gamma^2\varphi)(X_s)\mathbb{W}_{s,t} \ .$$

Combining this with $\varrho_t(\varphi) := \varphi(X_t)$ yields the claimed 3α-estimate. Similar, but now using standard facts on composition of controlled rough paths with regular functions, we obtain

$$\varphi(X_t) \stackrel{2\alpha}{=} \varphi(X_s) + \Gamma\varphi(X_s)W_{s,t},$$

uniformly over φ bounded in $\mathcal{C}_b^2$. At last, the third estimate comes from α-Hölder regularity of $t \mapsto \varrho_t(\Gamma^2\varphi)=\Gamma^2\varphi(X_t)$, itself a manifest consequence of $\Gamma^2\varphi \in \mathcal{C}_b^1$ and α-Hölder regularity of X.

We are not yet done, because until now, we have only handled the case of Dirac initial data $\varrho_0 = \delta_x$. (Since $\varrho_0(\varphi) = \varphi(X_0^{0,x}) = \varphi(x)$.) Fortunately, we are in a linear situation so that, given $\varrho_0 = \nu \in \mathcal{M}$, it suffices to generalise our construction and define

$$\varrho_t(\varphi) := \int \varphi(X_t^{0,x})\nu(dx).$$

It remains to see that such an integration in x respects all graded $3\alpha, 2\alpha, \alpha$ estimates. This is indeed the case, because all required estimates are uniform in $X_0 = x$. (A pleasant consequence of dealing with bounded vector fields so that all quantitative bounds are invariant under shift.)

(Uniqueness) Given any $g = u_T \in \mathcal{C}_b^3$, there exists a regular backward RPDE solution, $u_t = u(t, \bullet) \in \mathcal{C}_b^3$, with

$$u_s - u_t \stackrel{3\alpha}{=} u_t' W_{s,t} + u_t'' \mathbb{W}_{s,t}$$

(and then $u' = \Gamma u \in \mathcal{C}_b^2$ etc). Write $u_{s,t} = u_t - u_s$ and similarly for ϱ. Then

$$\varrho_t(u_t) - \varrho_s(u_s) = \varrho_{s,t}(u_t) + \varrho_s(u_{s,t}) \ .$$

The first summand on the right-hand side expands, using the very definition of weak solution (applied with $\varphi = u_t \in \mathcal{C}_b^3$, uniformly in $t \in [0,T]$),

$$\varrho_{s,t}(u_t) \stackrel{3\alpha}{=} \varrho_s(\Gamma u_t)W_{s,t} + \varrho_s(\Gamma^2 u_t)\mathbb{W}_{s,t} \ .$$

The second summand on the other hand expands, using the defining property of regular backward equation,

$$\varrho_s(u_{s,t}) = -\varrho_s(u_s - u_t) \stackrel{3\alpha}{=} -\varrho_s(\Gamma u_t)W_{s,t} - \varrho_s(\Gamma^2 u_t)\mathbb{W}_{s,t} \ .$$

(Here one needs to argue that the 3α-bound on $u_{s,t}(x)+\Gamma u_t(x)W_{s,t}+\Gamma^2 u_t(x)\mathbb{W}_{s,t}$ is uniform in x, for $u_T \in \mathcal{C}_b^3$, and thus the same 3α-estimate holds after integrating against $\varrho_s(dx)$.) Taken together we see a perfect cancellation so that $\varrho_t(u_t) - \varrho_s(u_s) \stackrel{3\alpha}{=} 0$. By a familiar argument (using $3\alpha > 1$) this implies that $t \mapsto \varrho_t(u_t)$ is constant and thus

$$\varrho_T(g) = \varrho_T(u_T) = \varrho_0(u_0) = \nu(u_0)$$

where u is a regular backward RPDE solution (with terminal data $g = u_T \in \mathcal{C}_b^3$). (Uniqueness of the regular backward RPDE solutions is not used here.) Hence, with given initial data $\varrho = \nu \in \mathcal{M}$ we see that $\varrho_T(g)$ is determined for all $g \in \mathcal{C}_b^3$ and this (uniquely) determines the measure $\varrho_T \in \mathcal{M}$. □

Remark 12.9. The uniqueness part of the proof actually shows that analytically weak solutions to the rough PDE (12.6) can be tested in space-time with test functions $\varphi = \varphi(t,x)$ that have a precise controlled structure, starting with

$$\varphi_s - \varphi_t \stackrel{3\alpha}{=} \varphi_t' W_{s,t} + \varphi_t'' \mathbb{W}_{s,t}$$

(and then 2α, resp. α expansions for φ' and φ''). This space of test functions is tailored to the realisation of the noise $\mathbf{W}$.

12.2 Second order rough partial differential equations

12.2.1 Linear theory: Feynman–Kac

As motivation, consider the second order stochastic partial differential equation with d-dimensional Brownian noise in (backward) Stratonovich form, posed as terminal value problem,

$$-du = L[u]dt + \Gamma[u] \circ \overleftarrow{dB}\,, \qquad u(T,\bullet) = g\,, \tag{12.8}$$

for $u = u(\omega) : [0,T] \times \mathbf{R}^n \to \mathbf{R}$, with differential operators L and $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$ given by

$$L[u] \stackrel{\text{def}}{=} \frac{1}{2}\mathrm{Tr}\big(\sigma(x)\sigma^T(x)D^2u\big) + b(x)\cdot Du + c(x)u\,, \tag{12.9}$$
$$\Gamma_i[u] \stackrel{\text{def}}{=} \beta_i(x)\cdot Du + \gamma_i(x)u\,.$$

The coefficients $\sigma = (\sigma_1, \ldots, \sigma_m)$, b and $\beta = (\beta_1, \ldots, \beta_d)$ are viewed as vector fields on $\mathbf{R}^n$, while $c, \gamma_1, \ldots, \gamma_d$ are scalar functions. For simplicity only, all coefficients are assumed to be bounded with bounded derivatives of all orders (but see Remark 12.12). We assume $g \in \mathcal{BC}(\mathbf{R}^n)$, that is bounded and continuous.[1] As in the previous section, we are interested in replacing W by a genuine (geometric) rough path $\mathbf{W}$, such as to solve the *rough partial differential equation* (RPDE)

$$-du = L[u]dt + \Gamma[u]d\mathbf{W}\,, \qquad u(T,\bullet) = g\,. \tag{12.10}$$

[1] In contrast to the space $\mathcal{C}_b$ we shall equip $\mathcal{BC}$ with the topology of locally uniform convergence.

We have already treated the fully degenerate case $L = 0$, with pure transport noise, $\Gamma_i = \beta_i(x) \cdot D_x$, in Section 12.1.1. Since geometric rough paths are limits of smooth paths, we start with the case when **W** is replaced by $\dot{W} dt$, for $W \in \mathcal{C}^1\big([0,T], \mathbf{R}^d\big)$. It is a basic exercise in Itô calculus that any bounded $\mathcal{C}^{1,2}$ solution to

$$-\partial_t u = L[u] + \sum_{i=1}^{d} \Gamma_i[u] \dot{W}_t^i \;, \qquad u(T, \bullet) = g \;, \tag{12.11}$$

is given by the classical Feynman–Kac formula (and hence also unique),

$$\begin{aligned} u(s,x) \;&=\; \mathbf{E}^{s,x}\left[g(X_T) \exp\left(\int_s^T c(X_t) dt + \int_s^T \gamma(X_t) \dot{W}_t dt \right) \right] & (12.12) \\ &=: \mathcal{S}[W; g](s,x), & (12.13) \end{aligned}$$

where X is the (unique) strong solution to

$$dX_t = \sigma(X_t) dB(\omega) + b(X_t) dt + \beta(X_t) \dot{W}_t dt, \tag{12.14}$$

where B is a m-dimensional standard Brownian motion. When $\sigma \equiv 0$, this is nothing but the method of characteristics, previously encountered for the transport equation in (12.3). (For the moment, we keep $W \in \mathcal{C}^1$, but will soon encounter *rough stochastic characteristics*.)

Remark 12.10. The natural form of the Feynman–Kac formula is the reason for considering terminal value problems here, rather than Cauchy problems of the form $\partial_t u = L[u] + \Gamma[u]\dot{W}$ with given initial data $u(0, \bullet)$. Of course, a change of the time variable $t \mapsto T - t$ allows to switch between these problems.

Clearly, there are situations when solutions cannot be expected to be $\mathcal{C}^{1,2}$, notably when $g \notin \mathcal{C}^2$ and L fails to provide smoothing as is the case, for example, in "transport" equations where L is of first order. In such a case, formula (12.12) is a perfectly good way to define a generalised solution to (12.11). Such a solution need not be $\mathcal{C}^{1,2}$ although it is bounded and continuous on $[0,T] \times \mathbf{R}^n$, as one can see directly from (12.12). As a matter of fact, (12.12) yields a (analytically) weak PDE solution (cf. Exercise 12.1). It is also a stochastic representation of the unique (bounded) viscosity solution [CIL92, FS06] to (12.11) although this will play no role for us in the present section. The main result here is the following rough path stability for linear second order RPDEs.

Theorem 12.11. *Let* $\alpha \in (\frac{1}{3}, \frac{1}{2}]$*. Given a geometric rough path* $\mathbf{W} = (W, \mathbb{W}) \in \mathscr{C}_g^{0,\alpha}([0,T], \mathbf{R}^d)$*, pick* $W^\varepsilon \in \mathcal{C}^1\big([0,T], \mathbf{R}^d\big)$ *so that*

$$(W^\varepsilon, \mathbb{W}^\varepsilon) := \left(W^\varepsilon, \int_0^\cdot W_{0,t}^\varepsilon \otimes dW_t^\varepsilon \right) \to \mathbf{W}$$

in α*-Hölder rough path metric. Then there exists* $u = u(t,x) \in \mathcal{BC}([0,T] \times \mathbf{R}^n)$*, not dependent on the approximating* (W^ε) *but only on* $\mathbf{W} \in \mathscr{C}_g^{0,\alpha}([0,T], \mathbf{R}^d)$*, so* *that, for $g \in \mathcal{BC}(\mathbf{R}^n)$,*

$$u^\varepsilon = \mathcal{S}[W^\varepsilon; g] \to u =: \mathcal{S}[\mathbf{W}; g]$$

as $\varepsilon \to 0$ in the sense of locally uniform convergence. Moreover, the resulting solution map

$$\mathcal{S} : \mathscr{C}_g^{0,\alpha}([0,T], \mathbf{R}^d) \times \mathcal{BC}(\mathbf{R}^n) \to \mathcal{BC}([0,T] \times \mathbf{R}^n)$$

is continuous. We say that u satisfies the RPDE (12.10).

Proof. **Step 1:** Write $X = X^W$ for the solution to (12.14) whenever $W \in \mathcal{C}^1$. The first step is to make sense of the *stochastic RDS*

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt + \beta(X_t)d\mathbf{W}_t. \tag{12.15}$$

This is clearly not an equation that can be solved by Itô theory alone. But is also not immediately well-posed as rough differential equation since for this we would need to understand B and $\mathbf{W} = (W, \mathbb{W})$ jointly as a rough path. In view of the Itô-differential dB in (12.15), we take $\big(B, \mathbb{B}^{\text{Itô}}\big)$, as constructed in Section 3.2), and are basically short of the cross-integrals between B and W. (For simplicity of notation only, pretend over the next few lines W, B to be scalar.) We can define $\int W dB(\omega)$ as Wiener integral (Itô with deterministic integrand), and then $\int B dW = WB - \int W dB$ by imposing integration by parts. We then easily get the estimate

$$\mathbf{E}\bigg(\int_s^t W_{s,r} dB_r\bigg)^2 \lesssim \|W\|_\alpha^2 |t-s|^{2\alpha+1} ,$$

also when switching the roles of W, B, thanks to the integration by parts formula. It follows from Kolmogorov's criterion that $\mathbf{Z}^W(\omega) := \mathbf{Z} = (Z, \mathbb{Z}) \in \mathscr{C}^{\alpha'}$ a.s. for any $\alpha' \in (1/3, \alpha)$ where

$$Z_t = \begin{pmatrix} B_t(\omega) \\ W_t \end{pmatrix}, \qquad \mathbb{Z}_{s,t} = \begin{pmatrix} \mathbb{B}_{s,t}^{\text{Itô}}(\omega) & \int_s^t W_{s,r} \otimes dB_r(\omega) \\ \int_s^t B_{s,r} \otimes dW_r(\omega) & \mathbb{W}_{s,t} \end{pmatrix}$$

where we reverted to tensor notation reflecting the multidimensional nature of B, W. It is easy to deduce from Theorem 3.3 that, for any $q < \infty$,

$$\Big| \varrho_{\alpha'}\Big(\mathbf{Z}^{\mathbf{W}}, \mathbf{Z}^{\tilde{\mathbf{W}}}\Big)\Big|_{L^q} \lesssim \varrho_\alpha\Big(\mathbf{W}, \tilde{\mathbf{W}}\Big). \tag{12.16}$$

We are hence able to say that a solution $X = X(\omega)$ of (12.15) is, by definition, a solution to the genuine (random) rough differential equation

$$dX = (\sigma, \beta)(X) d\mathbf{Z}^{\mathbf{W}}(\omega) + b(X)dt \tag{12.17}$$

driven by the random rough path $\mathbf{Z} = \mathbf{Z}^{\mathbf{W}}(\omega)$. Moreover, as an immediate consequence of (12.16) and continuity of the Itô–Lyons map, we see that X is really the limit, e.g. in probability and uniformly on $[0,T]$, of classical Itô SDE solutions X^ε, obtained by replacing $d\mathbf{W}_t$ by the $\dot{W}^\varepsilon_t dt$ in (12.15).

Step 2: Given (s,x) we have a solution $(X_t : s \le t \le T)$ to the hybrid equation (12.15), started at $X_s = x$. In fact $(X,X') \in \mathscr{D}^{2\alpha'}_Z$ with $X' = (\sigma,\beta)(X)$. In particular, the rough integral

$$\int \gamma(X) d\mathbf{W} := \int (0,\gamma(X)) d\mathbf{Z}$$

is well-defined, as is - with regard to the Feynman–Kac formula (12.12) - the random variable

$$g(X_T)\exp\left(\int_s^T c(X_t)dt + \int_s^T \gamma(X_t)d\mathbf{W}_t\right)(\omega). \tag{12.18}$$

One can see, similar to (11.10), but now also relying on RDE growth estimates as established in Proposition 8.2), with $p = 1/\alpha'$,

$$\left|\int_s^t \gamma(X)d\mathbf{W}\right| \lesssim |||\mathbf{Z}|||_{p\text{-var};[s,t]}$$

whenever $|||\mathbf{Z}|||_{p\text{-var};[s,t]}$ is of order one. An application of the generalised Fernique Theorem 11.7, similar to the proof of Theorem 11.13 but with $\varrho = 1$ in the present context, then shows that the number of consecutive intervals on which $\mathbf{Z}$ accumulates unit p-variation has Gaussian tails; in fact, uniformly in $\varepsilon \in (0,1]$, if $\mathbf{W}$ is replaced by W^ε with limit $\mathbf{W}$.) This implies that (12.18) is integrable (and uniformly integrable with respect to ε when $\mathbf{W}$ is replaced by W^ε). It follows that

$$u(s,x) := \mathbf{E}^{s,x}\left[g(X_T)\exp\left(\int_s^T c(X_t)dt + \int_s^T \gamma(X_t)d\mathbf{W}_t\right)\right] \tag{12.19}$$

is indeed well-defined and the pointwise limit of u^ε (defined in the same way, with $\mathbf{W}$ replaced by W^ε). By an Arzela–Ascoli argument, the limit is locally uniform. At last, the claimed continuity of the solution map follows from the same arguments, essentially by replacing W^ϵ by $\mathbf{W}^\epsilon$ everywhere in the above argument, and of course using (12.19) with $g, \mathbf{W}$ replaced by $g^\varepsilon, \mathbf{W}^\varepsilon$, respectively. □

Remark 12.12. The proof actually shows that our solution $u = u(s,x;\mathbf{W})$ to the linear RDPE (12.10) enjoys a Feynman–Kac type representation, namely (12.19), in terms of the process constructed as solution to the hybrid Itô-rough differential equation (12.15). Assume now W is a Brownian motion, independent of B, and $\mathbf{W}(\omega) = \mathbf{W}^{\text{Strat}} = (W, \mathbb{W}^{\text{Strat}}) \in \mathscr{C}^{0,\alpha}_g$ a.s. It is not difficult to show that $u = u(.,.,\mathbf{W}^{\text{Strat}}(\omega))$ coincides with the Feynman–Kac SPDE solution derived by Pardoux [Par79] or Kunita [Kun82], via conditional expectations given $\sigma(\{W_{u,v} : s \le u \le v \le T\}$, and so provides an identification with classical SPDE theory. In conjunction with continuity of the solution map $\mathcal{S} = \mathcal{S}[\mathbf{W};g]$ one obtains, along the lines of Sections 9.2, SPDE limit theorems of Wong–Zakai type, Stroock–Varadhan type support statements and Freidlin–Wentzell type small noise large deviations.

Remark 12.13. It is easy to quantify the required regularity of the coefficients. The argument essentially relies on solving (12.17) as bona fide rough differential equation. It is then clear that we need to impose $\mathcal{C}_b^3$-regularity for the vector fields σ and β. The drift vector field b may be taken to be Lipschitz and $c \in \mathcal{C}_b$.

Remark 12.14. We have not given meaning to the actual equation (12.10) which we here reproduce equivalently (cf. Remark 12.10) in the form

$$du = L[u]dt + \Gamma[u]d\mathbf{W} \;, \qquad u(0,\bullet) = u_0 \;. \tag{12.20}$$

Indeed, in the absence of ellipticity or Hörmander type conditions on L, the solution may not be any more regular than the initial data g so that in general (for $g \in \mathcal{C}_b$, say) the action of the first order differential operator $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$ on u has no pointwise meaning, let alone its rough integral against $\mathbf{W}$. On the other hand, we can (at least formally) test the equation against $\varphi \in \mathcal{D} = \mathcal{C}_c^\infty(\mathbf{R}^n)$ and so arrive the following "analytically weak" formulation: call $u = u(s,x;\mathbf{X})$ a weak solution to (12.20) if for every $\varphi \in \mathcal{D}$ and $0 \le t \le T$ the following integral formula holds:

$$\langle u_t, \varphi\rangle = \langle u_0, \varphi\rangle + \int_0^t \langle u_s, L^*\varphi\rangle ds + \int_0^t \langle u_s, \Gamma^*\varphi\rangle d\mathbf{W}_s. \tag{12.21}$$

In Exercise 12.1 the reader is invited to check that our Feynman–Kac solution is indeed a weak solution in this sense. In particular, the final integral term is a bona fide rough integral of the controlled rough path $(Y, Y') \in \mathscr{D}_W^{2\alpha}$ against $\mathbf{W}$, where

$$Y_t = \langle u_t, \Gamma^*\varphi\rangle \;, \qquad Y_t' = \langle u_t, \Gamma^*\Gamma^*\varphi\rangle \;. \tag{12.22}$$

It is seen in [DFS17] that a uniqueness result holds for such weak RPDE solutions holds, provided in the definition a suitable uniformity over the test function φ is required. The strategy is a very similar to what was seen in Section 12.1.2: arguing (for convenience) on the terminal value formulation (12.10), we construct a *regular* forward solution and then employ a forward-backward argument to obtain uniqueness. This is essentially the uniqueness argument employed in Theorem 12.8, with switched roles of forward and backward evolution. Alternatively, in [HH18] the unbounded rough driver framework of [DGHT19b] has been adapted to linear second order RPDEs with L in divergence form.

Remark 12.15. Let $u = u(t,x;\mathbf{X})$ be a weak solution in the sense of (12.21), and W be a Brownian motion with Stratonovich rough path lift $\mathbf{W} = \mathbf{W}^{\text{Strat}}(\omega)$. Then, thanks to Theorem 5.14, it follows that $u(t,x;\omega) := u(t,x;\mathbf{W}^{\text{Strat}}(\omega))$ yields an analytically weak SPDE solution in the sense that for every $\varphi \in \mathcal{D}$ and $0 \le t \le T$ one has, with probability one,

$$\langle u_t, \varphi\rangle = \langle u_0, \varphi\rangle + \int_0^t \langle u_s, L^*\varphi\rangle ds + \int_0^t \langle u_s, \Gamma^*\varphi\rangle \circ dW_s \;,$$

where the existence of the Stratonovich integral is implied by Corollary 5.2.

12.2.2 Mild solutions to semilinear RPDEs

We now turn to a class of "abstract" rough evolution problems introduced by Gubinelli–Tindel [GT10], although our exposition is taken from [GH19]. Following a familiar picture in PDE theory, we would like to view an RPDE solution as a controlled path with values in a Hilbert space H which solves an RDE of the form

$$du_t = Lu_t dt + F(u_t) d\mathbf{X}_t \quad \text{and} \quad u_0 = \xi \in H \; . \tag{12.23}$$

Here, $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\gamma([0,T], \mathbf{R}^d), \gamma \in (1/3, 1/2]$, not necessarily geometric. L is a negative definite self-adjoint operator, $F = (F_1, \ldots, F_d)$ are suitable (essentially 0-order) operators. In particular, no transport noise is covered by our setup so that – in contrast to previous sections – there is no restriction here to geometric rough paths.

Remark 12.16. Unlike Section 12.2.1 (Feynman–Kac) and Section 12.2.4 below (maximum principle), the present section is not really restricted to second order equations, even though these constitute the typical examples we have in mind.

To fix ideas, we give an example that will fit into the framework described below.

Example 12.17. Consider the *rough reaction diffusion equation*[2]

$$du_t(x) = \Delta u(x)\, dt + f(u_t(x))\, dt + p(u_t(x))\, d\mathbf{X}_t, \tag{12.24}$$

with $u_t : \mathbf{T}^n \to \mathbf{R}^l$ where $\mathbf{T}^n$ is the n-dimensional torus with Laplace operator Δ, and polynomial nonlinearities f and $p = (p_1, \ldots, p_d)$ on $\mathbf{R}^l$. As as typical in PDE theory, one looks for solutions $u_t \in H^k(\mathbf{T}^n, \mathbf{R}^l) =: H$, where H^k is the L^2-based Sobolev space with k weak derivates in L^2. Of course, Δ is negative definite self-adjoint on H, with dense domain $\mathrm{Dom}(\Delta) = H_1$, where we set (in agreement with a later abstract interpolation space definition) $H_\alpha = H^{k+2\alpha}(\mathbf{T}^n, \mathbf{R}^l)$, and also note that the heat semigroup $(e^{\Delta t})_{t \geq 0}$ acts naturally on this Sobolev scale. The nonlinearity in this example is given by composition with a polynomial. Smoothness of this operation requires H to be an algebra, which, by basic Sobolev theory, requires $k > n/2$. The main theorem below requires each nonlinearity (as operator, here: $u \mapsto p_i \circ u$) to be $\mathcal{C}^3$ in Fréchet sense as map from $H_{-2\gamma} = H^{k-4\gamma}$ into itself. Therefore we have the requirement on k to satisfy $k > n/2 + 4\gamma$. This means that $\gamma = 1/3^+$ is the optimal choice (in a level-2 rough path setting). Of course, this covers the case of Brownian rough paths so that $\mathbf{X}$ above can be replaced by $\mathbf{W}^{\text{Itô}}(\omega)$ or $\mathbf{W}^{\text{Strat}}(\omega)$.

[2] As in the case of RDEs with additional drift vector field, Exercise 8.5, the extra nonlinearity $(f \circ u_t)\, dt$ can be absorbed in the $\mathbf{X}$-term, by working with the space-time extensions of $\mathbf{X}$. Less trivially, a direct analysis allows for more general nonlinearities in (12.23) such as to handle 2D Navier–Stokes with rough noise.

We want to give meaning to the rough partial differential equation (12.23). Similar to (12.21), there is a natural – still formal – analytically weak formulation: for every $h \in \mathrm{Dom}(L) \subset H$ and $0 \le t \le T$ the following integral formula holds (angle brackets denote the inner product in H):

$$\langle u_t, h\rangle = \langle \xi, h\rangle + \int_0^t \langle u_s, Lh\rangle ds + \int_0^t \langle F(u_s), h\rangle d\mathbf{X}_s \,. \tag{12.25}$$

On the other hand, if $(S_t)_{t\ge 0}$ denotes the associated semigroup $S_t = e^{Lt}$ (which is analytic since L is assumed to be selfadjoint) one expects a mild formulation of the form, for all $0 \le t \le T$

$$u_t = S_t \xi + \int_0^t S_{t-s} F(u_s) d\mathbf{X}_s \,, \tag{12.26}$$

where the identity holds between elements in H. The regularity of F will be measured in Fréchet sense, as map from H_α to itself, for a to be specified range of $\alpha \in \mathbf{R}$.[3] Here, for $\alpha \ge 0$, the interpolation space $H_\alpha = \mathrm{Dom}((-L)^\alpha)$ is a Hilbert space when endowed with the norm $\|\bullet\|_{H_\alpha} = \|(-L)^\alpha \bullet\|_H$. Similarly, $H_{-\alpha}$ is defined as the completion of H with respect to the norm $\|\bullet\|_{H_{-\alpha}} = \|(-L)^{-\alpha}\bullet\|_H$. Note that this setting is compatible with that of Exercise 4.16.

The weak formulation requires of course that $s \mapsto \langle F(u_s), h\rangle$ has meaning as a controlled rough path, so that (12.25) is well-defined. In the mild formulation (12.26) on the other hand we recognise the rough convolution integral previously defined in (4.47), provided that $s \mapsto F(u_s)$ is mildly controlled in the sense of (4.46). It can be seen that weak and mild solutions coincide [GH19]. (The proof of this involves a simple variant of the rough Fubini theorem from Exercise 4.11.) In what follows we only consider the mild formulation.

We introduce the following spaces which are a slight strengthening of the spaces $\mathscr{D}^{2\gamma}_{S,X}$ introduced in Exercise 4.17:

$$\mathcal{D}^{2\gamma}_X([0,T], H_\alpha) = \mathscr{D}^{2\gamma}_{S,X}([0,T], H_\alpha) \cap \big(\hat{\mathcal{C}}^\gamma([0,T], H_{\alpha+2\gamma}) \times L^\infty([0,T], H_{\alpha+2\gamma})\big) \,.$$

The basic ingredients, stability of mildly controlled rough paths under rough convolution and composition with regular functions were already established in Exercises 4.17 and 7.3. Taken together, they show that the image of $(Y, Y') \in \mathcal{D}^{2\gamma}_X([0,T], H)$ under the map

$$\mathscr{M}_T(Y, Y')_t := \Big(S_t \xi + \int_0^t S_{t-u} F(Y_u) d\mathbf{X}_u,\, F(Y_t)\Big) \tag{12.27}$$

yields again an element of $\mathcal{D}^{2\gamma}_X([0,T], H)$. We now show that for small enough times this map has a unique fixed point:

[3] This rules out taking any derivatives in F. In particular, the previously considered transport noise, involved $D_x u$, is not accommodated in this setting.

Theorem 12.18 (Rough Evolution Equation). *Let $\xi \in H$, $F_1, \ldots, F_d : H \to H$, bounded in $\mathcal{C}^3(H_\beta, H_\beta)$ on bounded sets for every $\beta \geq -2\gamma$, for some $\gamma \in (1/3, 1/2]$, and $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\gamma(\mathbf{R}_+, \mathbf{R}^d)$. Then there exists $\tau > 0$ and a unique element $(Y, Y') \in \mathcal{D}_X^{2\gamma}([0,\tau), H)$ such that $Y' = F(Y)$ and*

$$Y_t = S_t\xi + \int_0^t S_{t-u} F(Y_u) d\mathbf{X}_u\,, \quad t < \tau. \tag{12.28}$$

Proof. First note $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\gamma \subset \mathscr{C}^\eta$ for $1/3 < \eta < \gamma \leq 1/2$. Fixing $T < 1$, we will find a solution $(Y, Y') \in \mathcal{D}_X^{2\eta}([0,T], H_{2\eta-2\gamma})$ as a fixed point of the map $\mathscr{M}_T$ given by (12.27). In the end we will briefly describe how one can make an improvement and show that one actually has $(Y, Y') \in \mathcal{D}_X^{2\gamma}([0,T], H)$. The proof is analogous to Theorem 8.3, the only difference being that we have two different scales of space regularity for which we need to be able to obtain the bound (7.14), as prepared in Exercise 4.17. We will therefore show only invariance of the solution map (12.27), because proving it already contains all the techniques that are not present in the Theorem 8.3.

Note that if (Y, Y') is such that $(Y_0, Y_0') = (\xi, F(\xi))$ then the same is true for $\mathscr{M}_T(Y, Y')$, so we can view $\mathscr{M}_T$ as a map on the complete metric space

$$\begin{aligned} B_T = \{(Y, Y') \in \mathcal{D}_X^{2\eta}([0,T], H_{2\eta-2\gamma}) :\ & Y_0 = \xi,\ Y_0' = F(\xi), \|(Y,Y')\|^\wedge_{X,2\eta;-2\gamma} \\ & + \|Y - S_\bullet F(\xi) X_{0,\bullet}\|_{\eta;2\eta-2\gamma} + \|Y' - S_\bullet F(\xi)\|_{\infty;2\eta-2\gamma} \leq 1\}\,. \end{aligned}$$

(We use the same notational convention as in Exercise 4.17, namely indices after a semicolon indicate in which one of the H_α norms are taken.) Note that by the triangle inequality for $(Y, Y') \in B_T$ we have

$$\|(Y, Y')\|_{\mathcal{D}_X^{2\eta}} \lesssim (1 + \|\xi\| + \|F(\xi)\|)(1 + \|X\|_\gamma) \lesssim 1.$$

Here and below we write $A \lesssim B$ as a shorthand for $A \leq CB$ for a constant C that may depend on $\gamma, \eta, X, \mathbb{X}, F$ and ξ, but is uniform over $T \in (0,1]$.

It remains to show that for T small enough $\mathscr{M}_T$ leaves B_T invariant and is contracting there, so that the claim follows from the Banach fixed point theorem. We will consider the simpler case when F is $\mathcal{C}_b^3$. For $(Z_t, Z_t') = (F(Y_t), DF(Y_t) \circ Y_t')$ we have by Exercise 7.3

$$\|(Z, Z')\|_{X,2\eta} \lesssim (1 + \|(Y, Y')\|_{\mathcal{D}_X^{2\eta}})^2 \lesssim (1 + \|\xi\| + \|F(\xi)\|)^2 \lesssim 1\,,$$

and from Exercise 4.17

$$\begin{aligned} \|\mathscr{M}_T(Y, Y')\|_{X,2\eta} &= \Big\|\Big(\int_0^\bullet S_{\bullet-u} Z_u dX_u, Z\Big)\Big\|_{X,2\eta} \\ &\lesssim \|Z\|_{\eta,-2\gamma} + (\|Z_0'\|_{H_{-2\gamma}} + \|(Z, Z')\|^\wedge_{X,2\eta;-2\gamma})\varrho_\eta(0, \mathbf{X}) \\ &\lesssim \|Z\|_{\eta,-2\gamma} + (\|Z_0'\|_{H_{-2\gamma}} + \|(Z, Z')\|^\wedge_{X,2\eta;-2\gamma}) T^{\gamma-\eta}. \end{aligned}$$

Since $(Y, Y') \in B_T$, we have the bound $\|Y\|_{\eta,-2\gamma} \le (\|X\|_\gamma + 1)T^{\gamma-\eta}$. One can also show along the same lines as in Exercise 7.3 that

$$\begin{aligned}\|\hat{\delta}Z_{s,t}\|_{H_{-2\gamma}} &\lesssim \|\hat{\delta}Y_{s,t}\|_{H_{-2\gamma}} + \|S_{t-s}Y_s - Y_s\|_{H_{-2\gamma}} + |t-s|^{2\eta}\|F(Y_s)\|_{H_{2\eta-2\gamma}} \\ &\lesssim \big(T^{\gamma-\eta}|t-s|^\eta + |t-s|^{2\eta}\|Y_s\|_{H_{2\eta-2\gamma}} + T^\eta|t-s|^\eta\big) \\ &\lesssim \big(T^{\gamma-\eta} + T^{\gamma+\eta} + T^\eta\big)|t-s|^\eta.\end{aligned}$$

Therefore since $T < 1$ we conclude that $\|Z\|_{\eta,-2\gamma} \lesssim T^{\gamma-\eta}$.

To estimate $\|\mathscr{M}_T(Y) - S_\cdot F(\xi)X_{0,\cdot}\|_{\eta,2\eta-2\gamma}$ we use the identity

$$\hat{\delta}(S_\cdot F(\xi)X_{0,\cdot})_{t,s} = S_t F(\xi) X_{s,t}$$

and since $2\eta < 1$ we can use a better bound from (4.48) to deduce:

$$\begin{aligned}\|\hat{\delta}(\mathscr{M}_T(Y) - S_\cdot F(\xi)X_{0,\cdot})_{t,s}\|_{H_{2\eta-2\gamma}} &= \Big\|\int_s^t S_{t-u}F(Y_u)dX_u - S_tF(\xi)X_{s,t}\Big\|_{H_{2\eta-2\gamma}} \\ &\le (\|F(\xi)\|_H + \|Z\|_{\infty;-2\gamma})\|X\|_\eta|t-s|^\eta + \|Z'\|_{\infty;-2\gamma}\|\mathbb{X}\|_{2\eta}|t-s|^{2\eta} \\ &\quad + C(\|X\|_\eta|R^Z|_{2\eta} + \|\mathbb{X}\|_{2\eta}\|Z'\|_\eta)|t-s|^{3\eta-2\eta} \\ &\lesssim (\|F(\xi)\|_H + \|Z_0'\|_{H_{-2\gamma}} + \|(Z,Z')\|^\wedge_{X,2\eta;-2\gamma})|t-s|^\eta \\ &\lesssim T^{\gamma-\eta}|t-s|^\eta.\end{aligned}$$

Finally we estimate the term $\|\mathscr{M}_T(Y)'_t - S_tF(\xi)\|_{H_{2\eta-2\gamma}}$:

$$\begin{aligned}\|\mathscr{M}_T(Y)'_t - S_tF(\xi)\|_{H_{2\eta-2\gamma}} &= \\ &= \|F(Y_t) - F(S_t\xi) + F(S_t\xi) - F(\xi) + F(\xi) - S_tF(\xi)\|_{H_{2\eta-2\gamma}} \\ &\lesssim \|Y_t - S_t\xi\|_{H_{2\eta-2\gamma}} + \|S_t\xi - \xi\|_{H_{2\eta-2\gamma}} + \|F(\xi) - S_tF(\xi)\|_{H_{2\eta-2\gamma}} \\ &\lesssim \|Y_t - S_t\xi - S_tF(\xi)X_{t,0}\|_{H_{2\eta-2\gamma}} + \|F(\xi)\|_H\|X\|_\gamma T^\gamma \\ &\quad + t^{2\gamma-2\eta}\|\xi\|_H + t^{2\gamma-2\eta}\|F(\xi)\|_H \\ &\lesssim (\|Y - S_\cdot F(\xi)X_{0,\cdot}\|_{\eta,2\eta-2\gamma}T^\eta + T^\gamma + T^{2\gamma-2\eta}) \lesssim T^{\gamma-\eta}.\end{aligned}$$

Putting it all together we obtain the bound

$$\begin{aligned}\|\mathscr{M}_T(Y) - S_\cdot F(\xi)X_{0,\cdot}\|_{\eta;2\eta-2\gamma} + \|\mathscr{M}_T(Y)' - S_\cdot F(\xi)\|_{\infty;2\eta-2\gamma} \\ + \|\mathscr{M}_T(Y,Y')\|^\wedge_{X,2\eta;-2\gamma} \lesssim T^{\gamma-\eta}.\end{aligned}$$

If T is small enough we guarantee that the left-hand side of the above expression is smaller than 1, thus proving that B_T is invariant under $\mathscr{M}_T$. In order to show contractivity of $\mathscr{M}_T$, one can use analogous steps to first show

$$\|\mathscr{M}_T(Y,Y') - \mathscr{M}_T(V,V')\|_{\mathcal{D}_X^{2\eta}} \lesssim \|(Y-V, Y'-V')\|_{\mathcal{D}_X^{2\eta}}T^{\gamma-\eta}.$$

This guarantees contractivity for small enough T, completing the fixed point argument and thus showing the existence of the unique maximal solution to (12.28).

Let now $(Y,Y') \in \mathcal{D}_X^{2\eta}([0,T],H_{2\eta-2\gamma})$ be the solution constructed above, we sketch an argument showing that in fact it belongs to $\mathcal{D}_X^{2\gamma}([0,T],H)$. We know that

$$Y_t = S_t\xi + S_tF(\xi)X_{0,t} + S_tDF(\xi)F(\xi) + R_{0,t}, \tag{12.29}$$
$$Y_t - S_{t-s}Y_s = S_{t-s}F(Y_s)X_{t,s} + S_{t-s}DF(Y_s)F(Y_s)\mathbb{X}_{s,t} + R_{s,t}. \tag{12.30}$$

Here $R_{s,t} = \int_s^t S_{t-r}F(Y_r)dX_r - S_{t-s}F(Y_s)X_{s,t} - S_{t-s}DF(Y_s)F(Y_s)\mathbb{X}_{s,t}$. From the estimate on $R_{0,t}$ using (4.48) and since $\xi \in H$, we see that (12.29) implies $Y \in L^\infty([0,T],H)$. Moreover (12.30) implies $Y \in \hat{\mathcal{C}}^\gamma([0,T],H_{-2\gamma})$ which, together with $Y \in L^\infty([0,T],H)$, implies $F(Y) \in \hat{\mathcal{C}}^\gamma([0,T],H^d_{-2\gamma}) \cap L^\infty([0,T],H^d_{2\eta-2\gamma})$. This itself implies that $(Y,F(Y)) \in \mathscr{D}^{2\gamma}_{S,X}([0,T],H_{-2\gamma})$ (using again (12.30)) and $(F(Y),DF(Y)F(Y)) \in \mathscr{D}^{2\gamma}_{S,X}([0,T],H_{-2\gamma})$ which enables us to get an estimate for every $\beta < 3\gamma$:

$$\|R_{s,t}\|_{H_\beta} \lesssim \|F(Y),DF(Y)F(Y)\|^{\wedge}_{X,2\gamma;-2\gamma}|t-s|^{3\gamma-\beta}.$$

Taking $\beta = 2\gamma$ and using (12.30) again we show that $Y \in \hat{\mathcal{C}}^\gamma([0,T],H)$, which completes the proof that $(Y,Y') \in \mathcal{D}_X^{2\gamma}([0,T],H)$. □

12.2.3 Fully nonlinear equations with semilinear rough noise

We now consider nonlinear rough partial differential equations of the form

$$du = F[u]dt + \sum_{i=1}^d H_i[u] \circ dW_t^i(\omega)\,, \qquad u(0,\cdot) = g\,, \tag{12.31}$$

with fully nonlinear, possibly degenerate, operator

$$F[u] = F(x,u,Du,D^2u),$$

and semilinear

$$H_i[u] = H_i(x,u,Du)\,, \qquad i = 1,\dots,d\,.$$

We essentially rule out nonlinear dependence on Du, hence the terminology "semilinear noise", which makes a (global) flow transformation method work. In a stochastic setting such transformation (at least in the linear case) are attributed to Kunita. As already noted in the context of first order equations, the case $H_i = H_i(x,Du)$ requires a subtle local version of such as transformation and is topic of the pathwise Lions–Souganidis theory of stochastic viscosity theory for fully nonlinear SPDEs; [LS98a, LS98b, LS00b] and [Sou19] for a recent overview.

As in the previous section we aim to replace $\circ dW$ by a "rough" differential $d\mathbf{W}$, for some geometric rough path $\mathbf{W} \in \mathscr{C}_g^{0,\alpha}([0,T],\mathbf{R}^d)$, and show that an RPDE solution arises as the unique limit under approximations $(W^\varepsilon,\mathbb{W}^\varepsilon) \to \mathbf{W}$. Of course, there is little one can say at this level of generality and we have not even clarified in which sense we mean to solve (12.31) when $W \in \mathcal{C}^1$! Let us postpone this discussion and assume momentarily that F and H are sufficiently "nice" so that, for every $W \in \mathcal{C}^1$ and $g \in \mathcal{BC}$, say, there is a classical solution $u = u(t,x)$ for $t > 0$.

With noise of the form $H[u]\dot{W} = \sum_i H_i(x,u,Du)\dot{W}^i$, we shall focus on the following three cases.

a) **Transport noise.** For sufficienly nice vector fields β_i on $\mathbf{R}^n$,

$$H_i[u] = \beta_i(x) \cdot Du \; ;$$

b) **Semilinear noise.** For a sufficiently nice function H_i on $\mathbf{R}^n \times \mathbf{R}$,

$$H_i[u] = H_i(x,u);$$

c) **Linear noise.** With β_i as above and sufficiently nice functions γ_i on $\mathbf{R}^n$

$$H_i[u] = \Gamma_i[u] := \beta_i(x) \cdot Du + \gamma_i(x)u.$$

We now develop the "calculus" for the transformations associated to each of the above cases. All proofs consist of elementary computations and are left to the reader.

Proposition 12.19 (Case a). *Assume that $\psi = \psi^W$ is a $\mathcal{C}^3$ solution flow of diffeomorphisms associated to the ODE $\dot{Y} = -\beta(Y)\dot{W}$, where $W \in \mathcal{C}^1$. (This is the case if $\beta \in \mathcal{C}^3_b$.) Then u is a classical solution to*

$$\partial_t u = F\big(x,u,Du,D^2u\big) + \langle \beta(x), Du \rangle \dot{W}$$

if and only if $v(t,x) = u(t,\psi_t(x))$ is a classical solution to

$$\partial_t v - F^\psi\big(t,x,v,Dv,D^2v\big) = 0$$

where F^ψ is determined from

$$\begin{aligned} &F^\psi(t,\psi_t(x),r,p,X) \\ &\quad \overset{\mathit{def}}{=} F\big(x,r,\langle p, D\psi_t^{-1}\rangle, \langle X, D\psi_t^{-1} \otimes D\psi_t^{-1}\rangle + \langle p, D^2\psi_t^{-1}\rangle\big) \; . \end{aligned}$$

Proposition 12.20 (Case b). *For any fixed $x \in \mathbf{R}^n$, assume that the one-dimensional ODE*

$$\dot{\varphi} = H(x,\varphi)\dot{W} \; , \qquad \varphi(0;x) = r \; ,$$

has a unique solution flow $\varphi = \varphi^W = \varphi(t,r;x)$ which is of class $\mathcal{C}^2$ as a function of both r and x. Then u is a classical solution to

$$\partial_t u = F\big(x,u,Du,D^2u\big) + H(x,u)\dot{W}$$

if and only if $v(t,x) = \varphi^{-1}(t,u(t,x),x)$, or equivalently $\varphi(t,v(t,x),x) = u(t,x)$, is a solution of

$$\partial_t v - {}^{\varphi}F\big(t,x,r,Dv,D^2v\big) = 0\,,$$

with

$$\begin{aligned}{}^{\varphi}F(t,x,r,p,X) \stackrel{\text{def}}{=} \frac{1}{\varphi'}F(t,x,\varphi,D\varphi + \varphi' p,\\ \varphi'' p\otimes p + D\varphi'\otimes p + p\otimes D\varphi' + D^2\varphi + \varphi' X\big)\,,\end{aligned} \tag{12.32}$$

where φ' denotes the derivative of $\varphi = \varphi(t,r,x)$ with respect to r.

Remark 12.21. It is worth noting that the "quadratic gradient" term $\varphi'' p\otimes p$ disappears in (12.32) whenever $\varphi'' = 0$. This happens when $H(x,u)$ is linear in u, i.e. when

$$H_i[u] = \gamma_i(x)u\,, \qquad i = 1,\ldots,d\,.$$

in which case we have

$$\varphi(t,r,x) = r\exp\left(\int_0^t \gamma(x)dW_s\right) = r\exp\left(\sum_{i=1}^d \gamma_i(x)W^i_{0,t}\right)\,. \tag{12.33}$$

Remark 12.22. Note that all dependence on $\dot W$ has disappeared in (12.33), and consequently (12.32). In the SPDE / filtering context this is known as *robustification*: the transformed PDE $(\partial_t - {}^{\varphi}F)v = 0$ can be solved for any $W \in \mathcal{C}([0,T],\mathbf{R}^d)$. This provides a way to solve SPDEs of the form $du = F[u]dt + \sum_{i=1}^d \gamma_i(x)u\circ dW_t$ pathwise, so that u depends continuously on W in uniform topology.

We now turn our attention to case c). The point here is that the "inner" and "outer" transformation seen above, namely

$$v(t,x) = u(t,\psi_t(x))\,, \qquad v(t,x) = \varphi^{-1}(t,u(t,x),x)\,,$$

respectively, can be combined to handle noise coefficients obtained by adding those from cases a) and b), i.e. noise coefficients of the type $\langle\beta_i(x),Du\rangle + H_i(x,u)$. We content ourselves with the linear case

$$H_i[u] = \langle\beta_i(x),Du\rangle + \gamma_i(x)u\,.$$

Proposition 12.23 (Case c). *Let $\psi = \psi^W$ be as in case a) and set $\varphi(t,r,x) = r\exp\left(\int_0^t \gamma(\psi_s(x))dW_s\right)$. Then u is a (classical) solution to*

$$\partial_t u = F\big(x,u,Du,D^2u\big) + \big(\langle\beta(x),Du\rangle + \gamma(x)u\big)\dot W\,,$$

if and only if $v(t,x) = u(t,\psi_t(x))\exp\left(-\int_0^t \gamma(\psi_s(x))dW_s\right)$ is a (classical) solution to

$$\partial_t v - {}^{\varphi}(F^{\psi})\big(t,x,v,Dv,D^2v\big) = 0.$$

Remark 12.24. It is worth noting that the outer transformation $F \to F^{\psi}$ preserves the class of linear operators. That is, if $F[u] = L[u]$ as given in (12.9), then F^{ψ} is again a linear operator. Because of the appearance of quadratic terms in Du, this is not true for the inner transformation $F \to {}^{\varphi}F$ *unless* $\varphi'' = 0$. Fortunately, this happens in the linear case and it follows that the transformation $F \to {}^{\varphi}(F^{\psi})$ used in case c) above does preserve the class of linear operators.

Let us reflect for a moment on what has been achieved. We started with a PDE that involves $\dot{W}$ and in all cases we managed to transform the original problem to a PDE where all dependence on $\dot{W}$ has been isolated in some auxiliary ODEs. In the stochastic context ($\circ dW$ instead of $dW = \dot{W}dt$) this is nothing but the reduction, via stochastic flows, from a *stochastic* PDE to a *random* PDE, to be solved ω-wise. In the same spirit, the rough case is now handled with the aid of flows for RDEs and their stability properties.

Given $\mathbf{W} \in \mathscr{C}_g^{0,\alpha}$, we pick an approximating sequence (W^ε), and transform

$$\partial_t u^\varepsilon = F[u^\varepsilon] + H[u^\varepsilon]\dot{W}^\varepsilon \tag{12.34}$$

to a PDE of the form

$$\partial_t v^\varepsilon = F^\varepsilon[v^\varepsilon], \tag{12.35}$$

e.g. with $F^\varepsilon = F^\psi$ and $\psi = \psi^{W^\epsilon}$ in case a) and accordingly in the other cases. Then

$$F^\varepsilon[w] = F^\varepsilon[t, x, w, Dw, D^2 w]$$

(in abusive notation) and the function F^ε which appears on the right-hand side above converges (e.g. locally uniformly) as $\varepsilon \to 0$, due to stability properties of flows associated to RDEs as discussed in Section 8.10.

All one now needs is a (deterministic) PDE framework with a number of good properties, along the following "wish list".

1. All approximate problems, i.e. with $W^\varepsilon \in \mathcal{C}^1([0,T], \mathbf{R}^d)$

$$\partial_t u^\varepsilon = F[u^\varepsilon] + \sum_{i=1}^d H_i[u^\varepsilon]\dot{W}_t^{\varepsilon,i}\,, \qquad u^\varepsilon(0,\cdot) = g^\varepsilon,$$

   should admit a unique solution, in a suitable class $\mathcal{U}$ of functions on $[0,T] \times \mathbf{R}^n$, for a suitable class of initial conditions in some space $\mathcal{G}$.
2. The change of variable calculus (Propositions 12.19–12.23) should remain valid, so that $u^\varepsilon \in \mathcal{U}$ is a solution to (12.34) if and only if its transformation $v^\varepsilon \in \mathcal{U}$ is a solution to (12.35).
3. There should be a good stability theory, so that $g^\varepsilon \to g^0$ in $\mathcal{G}$ and $F^\varepsilon \to F^0$ (in a suitable sense) allows to obtain convergence in $\mathcal{U}$ of solutions v^ε to (12.35) with intitial data g^ε to the (unique) solution of the limiting problem $\partial_t v^0 = F^0[v^0]$ with initial data g^0.

4. At last, the topology of $\mathcal{U}$ should be weak enough to make sure that $v^\epsilon \to v^0$ implies that the "back-transformed" u^ϵ converges in $\mathcal{U}$, with limit u^0 being v^0 back-transformed.[4]

The final point suggests to *define* a solution to

$$du = F[u]dt + H[u]d\mathbf{W} , \qquad u(0,\bullet) = g , \tag{12.36}$$

as an element in $\mathcal{U}$ which, under the correct flow transformation associated to $\mathbf{W}$ and H, solves the transformed equation $\partial_t v = F^0[v]$, $v(0,\bullet) = g$. To make this more concrete, consider the transport case a). As before, $\psi = \psi^{\mathbf{W}}$ is the flow associated to the RDE $dY = -\beta(Y)d\mathbf{W}$ and u solves the above RPDE (with $H[u] = \langle \beta(x), Du\rangle$) if, by definition, $v(t,x) := u(t,\psi_t(x))$ solves $\partial_t v = F^\psi[v]$, with $v(0,\bullet) = g$. The same logic applies to cases b) and c).

We then have the following (meta-)theorem, subject to a PDE framework with the above properties.

Theorem 12.25. *Let* $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. *Given a geometric rough path* $\mathbf{W} = (W, \mathbb{W}) \in \mathscr{C}_g^{0,\alpha}([0,T], \mathbf{R}^d)$, *pick* $W^\varepsilon \in \mathcal{C}^1\big([0,T], \mathbf{R}^d\big)$ *so that*

$$(W^\varepsilon, \mathbb{W}^\varepsilon) := \left(W^\varepsilon, \int_0^\bullet W^\varepsilon_{0,t} \otimes dW^\varepsilon_t \right) \to \mathbf{W}$$

in α*-Hölder rough path metric. Consider unique solutions* $u^\epsilon \in \mathcal{U}$ *to the PDEs*

$$\begin{cases} \partial_t u^\epsilon = F[u^\epsilon] + H[u^\epsilon]\dot{W}^\epsilon \\ u^\epsilon(0,\bullet) = g \in \mathcal{G}. \end{cases} \tag{12.37}$$

Then there exists $u = u(t,x) \in \mathcal{U}$, *not dependent on the approximating* (W^ε) *but only on* $\mathbf{W} \in \mathscr{C}_g^{0,\alpha}([0,T], \mathbf{R}^d)$, *so that*

$$u^\varepsilon = \mathcal{S}[W^\varepsilon; g] \to u =: \mathcal{S}[\mathbf{W}; g]$$

as $\varepsilon \to 0$ *in* $\mathcal{U}$. *This* u *is the unique solution to the RPDE (12.36) in the sense of the above definition. Moreover, the resulting solution map,*

$$\mathcal{S} : \mathscr{C}_g^{0,\alpha}([0,T], \mathbf{R}^d) \times \mathcal{G} \to \mathcal{U}$$

is continuous.

It remains to identify suitable PDE frameworks, depending on the nonlinearity F. When $\partial_t u = F[u]$ is a scalar conservation law, entropy solutions actually provide a suitable framework to handle additional rough noise, at least of (linear) type c), [FG16b]. On the other hand, when $F = F[u]$ is a fully nonlinear second order operator, say of Hamilton–Jacobi–Bellman (HJB) or Isaacs type, the natural framework is viscosity theory [CIL92, FS06] and the problem of handling additional "rough"

[4] Given the roughness in t of our transformations, typically α-Hölder, it would not be wise to incorporate temporal $\mathcal{C}^1$-regularity in the definition of the space $\mathcal{U}$.

noise, in the sense of $W \notin \mathcal{C}^1$, also with nonlinear $H = H(Du)$, was first raised by Lions–Sougandis [LS98a, LS98b, LS00a, LS00b].

12.2.4 Rough viscosity solutions

Consider a real-valued function $u = u(x)$ with $x \in \mathbf{R}^m$ and assume $u \in \mathcal{C}^2$ is a classical supersolution,

$$-G\big(x, u, Du, D^2u\big) \geq 0,$$

where G is continuous and *degenerate elliptic* in the sense that $G(x,u,p,A) \leq G(x,u,p,A+B)$ whenever $B \geq 0$ in the sense of symmetric matrices. The idea is to consider a (smooth) test function φ which touches u from below at some interior point $\bar{x}$. Basic calculus implies that $Du(\bar{x}) = D\varphi(\bar{x}),\ D^2u(\bar{x}) \geq D^2\varphi(\bar{x})$ and, from degenerate ellipticity,

$$-G\big(\bar{x}, \varphi, D\varphi, D^2\varphi\big) \geq 0. \tag{12.38}$$

This motivates the definition of a *viscosity supersolution* (at the point $\bar{x}$) to $-G = 0$ as a (lower semi-)continuous function u with the property that (12.38) holds for any test function which touches u from below at $\bar{x}$. Similarly, *viscosity subsolutions* are (upper semi-)continuous functions defined via test functions touching u from above and by reversing inequality in (12.38); *viscosity solutions* are both super- and subsolutions. Observe that this definition covers (completely degenerate) first order equations as well as parabolic equations, e.g. by considering $\partial_t - F = 0$ on $[0,T] \times \mathbf{R}^n$ where F is degenerate elliptic. Let us mention a few key results of viscosity theory, with special regard to our "wish list".

1. One has existence and uniqueness results in the class of $\mathcal{BC}$ solutions to the initial value problem $(\partial_t - F)u = 0,\ u(0,\bullet) = g \in \mathcal{BUC}(\mathbf{R}^n)$[5], provided $F = F(t,x,u,Du,D^2u)$ is continuous, degenerate elliptic, there exists $\gamma \in \mathbf{R}$ such that, uniformly in t,x,p,X,

$$\gamma(s-r) \leq F(t,x,r,p,X) - F(t,x,s,p,X) \text{ whenever } r \leq s, \tag{12.39}$$

and some technical conditions hold.[6] Without going into technical details, the conditions are met for $F = L$ as in (12.9) and are robust under taking inf and sup (provided the regularity of the coefficients holds uniformly). As a consequence, HJB and Isaacs type nonlinearities, where F takes the form $\inf_a L_a, \inf_a \sup_{a'} L_{a,a'}$, are also covered.
2. The change of variables "calculus" of Propositions 12.19–12.23 remains valid for (continuous) viscosity solutions. This can be checked directly from the definition of a viscosity solution.

[5] the space of bounded uniformly continuous functions

[6] …the most important of which is [CIL92, (3.14)]. Additional assumptions on F are necessary, however, in particular due to the unboundedness of the domain $\mathbf{R}^n$, and these are not easily found in the literature; see [DFO14]. One can also obtain existence and uniqueness result in $\mathcal{BUC}$.

3. In fact, the technical conditions mentioned in 1. imply a particularly strong form of uniqueness, known as *comparison*: assume u (resp. v) is a subsolution (resp. supersolution) and $u_0 \le v_0$; then $u \le v$ on $[0,T] \times \mathbf{R}^n$. A key feature of viscosity theory is what workers in the field simply call *stability*, a powerful incarnation of which is known as *Barles and Perthame procedure* [FS06, Section VII.3] and relies on comparison for (semicontinuous) sub- and super-solutions. In the form relevant for us, one assumes comparison for $\partial_t - F^0$ and considers viscosity solutions to $(\partial_t - F^\varepsilon)v^\varepsilon = 0$, with $v^\varepsilon(0,\bullet) = g^\varepsilon$, assuming locally uniform boundedness of v^ε and $g^\varepsilon \to g^0$ locally uniformly. Then $v^\varepsilon \to v^0$ locally uniformly where v^0 is the (unique) solution to the limiting problem $\big(\partial_t - F^0\big)v^0 = 0$, with $v^0(0,\bullet) = g^0$.

In the context of RPDEs above, again with focus on the transport case a) for the sake of argument, $F^0 = F^\psi$ where $\psi = \psi^{\mathbf{W}}$, where ψ is a flow of $\mathcal{C}^3$-diffeomorphisms (associated to the RDE $dY = -\beta(Y)d\mathbf{W}$ thereby leading to the assumption $\beta \in \mathcal{C}^5_b$). As a structural condition on F, we may simply assume "ψ-invariant comparison" meaning that comparison holds for $\partial_t - F^\psi$, for any $\mathcal{C}^3$-diffeomorphism with bounded derivatives. Checking this condition turns out to be easy. First, when $F = L$ is linear, we have $F^\psi = L^\psi$ also linear, with similar bounds on the coefficients as L due to the stringent assumptions on the derivatives of ψ. From the above discussion, and in particular from what was said in 1., it is then clear that L satisfies ψ-invariant comparison. In fact, stability of the condition in 1. under taking inf and sup, also implies that HJB and Isaacs type nonlinearities satisfy ψ-invariant comparison.

It is now possible to implement the arguments of the previous Theorem 12.25 in the viscosity framework [CFO11], see also [FO11] for applications to splitting methods. We tacitly assume that all approximate problems of the form (12.40) below have a viscosity solution, for all $W^\varepsilon \in \mathcal{C}^1$ and $g \in \mathcal{BUC}$, but see Remark 12.27.

Theorem 12.26. *Let $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. Given a geometric rough path $\mathbf{W} = (W, \mathbb{W}) \in \mathscr{C}_g^{0,\alpha}([0,T], \mathbf{R}^d)$, pick $W^\varepsilon \in \mathcal{C}^1\big([0,T], \mathbf{R}^d\big)$ so that $(W^\varepsilon, \mathbb{W}^\varepsilon) \to \mathbf{W}$ in α-Hölder rough path metric. Consider unique $\mathcal{BC}$ viscosity solutions u^ϵ to*

$$\begin{cases} \partial_t u^\epsilon = F[u^\epsilon] + \langle \beta(x), Du \rangle \dot{W}^\epsilon \\ u^\epsilon(0,\bullet) = g \in \mathcal{BUC}(\mathbf{R}^n) \end{cases} \tag{12.40}$$

where F satisfies ψ-invariant comparison. Then there exists $u = u(t,x) \in \mathcal{BC}$, not dependent on the approximating (W^ε) but only on $\mathbf{W} \in \mathscr{C}_g^{0,\alpha}([0,T], \mathbf{R}^d)$, so that

$$u^\varepsilon = \mathcal{S}[W^\varepsilon; g] \to u =: \mathcal{S}[\mathbf{W}; g]$$

as $\varepsilon \to 0$ in local uniform sense. This u is the unique solution to the RPDE (12.36) with transport noise $H[u] = \langle \beta(x), Du \rangle$ in the sense of the definition given previous to Theorem 12.25. Moreover, we have continuity of the solution map,

$$\mathcal{S} : \mathscr{C}_g^{0,\alpha}([0,T], \mathbf{R}^d) \times \mathcal{BUC}(\mathbf{R}^n) \to \mathcal{BC}([0,T] \times \mathbf{R}^n)\ .$$

Remark 12.27. In the above theorem, existence of RPDE solutions actually relies on existence of approximate solutions u^ε, which one of course expects from standard viscosity theory. Mild structural conditions on F, satisfied by HJB and Isaacs examples, which imply this existence are reviewed in [DFO14]. One can also establish a modulus of continuity for RPDE solutions, so that $u \in \mathcal{BUC}$ after all.

Remark 12.28. Rough partial differential equations as considered here, $du = F[u]dt + \langle \beta(x), Du \rangle d\mathbf{W}$, with $F = \inf_a L_a$ of HJB form, arise in pathwise stochastic control [LS98b, BM07, DFG17], also in conjunction with filtering [AC19].

Unfortunately, in case b), it turns out the structural assumptions one has to impose on F in order to have the necessary comparison for $\partial_t - F^0 = 0$ is rather restrictive, although semilinear situations are certainly covered. Even in this case, due to the appearance of a quadratic nonlinearity in Du, the argument is involved and requires a careful analysis on consecutive small time intervals, rather than $[0,T]$; see [LS00a, DF12]. A nonlinear Feynman–Kac representation, in terms of *rough backward stochastic differential equations* is given in [DF12].

At last, we return to the fully linear case of Section 12.2.3. That is, we consider the (linear noise) case c) with linear $F = L$. With some care [FO14], the double transformation leading to the transformed equation $\partial_t - {}^{\varphi}(F^{\psi}) = 0$ can be implemented with the aid of coupled flows of rough differential equations. We can then recover Theorem 12.11, but with somewhat different needs concerning the regularity of the coefficients. (For instance, in the aforementioned theorem we really needed $\sigma, \beta \in \mathcal{C}_b^3$ whereas now, using flow decomposition, we need $\beta \in \mathcal{C}_b^5$ but only $\sigma \in \mathcal{C}_b^1$.

Remark 12.29. By either approach, case c) with linear $F = L$ or Theorem 12.11, we obtain a robust view on classes SPDEs which contain the Zakai equation from filtering theory, provided the initial law admits a $\mathcal{BUC}$-density. Robustness is an important issue in filtering theory, see also Exercise 12.3.

12.3 Stochastic heat equation as a rough path

Nonlinear stochastic partial differential equations driven by very singular noise, say space-time white noise, may suffer from the fact that their nonlinearities are ill-posed. For instance, even in space dimension one, there is no obvious way of giving "weak" meaning to Burgers-like stochastic PDEs of the type

$$\partial_t u^i = \partial_x^2 u^i + f(u) + \sum_{j=1}^{n} g_j^i(u) \partial_x u^j + \xi^i \,, \qquad i = 1, \ldots, n \,, \tag{12.41}$$

where $\xi = \left(\xi^i\right)$ denotes space-time white noise (strictly speaking, n independent copies of scalar space-time white noise). Recall that, at least formally, space-time white noise is a Gaussian generalised stochastic process such that

$$\mathbf{E}\xi^i(t,x)\xi^j(s,y) = \delta_{ij}\delta(t-s)\delta(x-y) \;.$$

As a consequence of the lack of regularity of ξ, it turns out that the solution to the stochastic heat equation (i.e. the case $f = g = 0$ in (12.41) above) is only α-Hölder continuous in the spatial variable x for any $\alpha < 1/2$. In other words, one would not expect any solution u to (12.41) to exhibit spatial regularity better than that of a Brownian motion.

As a consequence, even when aiming for a weak solution theory, it is not clear how to define the integral of a spatial test function φ against the nonlinearity. Indeed, this would require us to make sense of expressions of the type

$$\int \varphi(x) g_j^i(u)\partial_x u^j(t,x)\,dx \;,$$

for fixed t. When g happens to be a gradient, such an integral can be defined by postulating that the chain rule holds and integrating by parts. For a general g, as arising in applications from path sampling [HSV07], this approach fails. This suggests to seek an understanding of $u(t,\bullet)$ as a spatial rough path. Indeed, this would solve the problem just explained by allowing us to define the nonlinearity in a weak sense as

$$\int \varphi(x) g_j^i(u)\,d\mathbf{u}^{\mathbf{j}}(t,x) \;,$$

where $\mathbf{u}$ is the rough path associated to u.

In the particular case of (12.41), it is actually sufficient to be able to associate a rough path to the solution ψ to the stochastic heat equation

$$\partial_t \psi = \partial_x^2 \psi + \xi \;.$$

Indeed, writing $u = \psi + v$ and proceeding formally for the moment, we then see that v should solve

$$\partial_t v^i = \partial_x^2 v^i + f(v+\psi) + \sum_{j=1}^n g_j^i(v+\psi)\big(\partial_x \psi^j + \partial_x v^j\big) \;.$$

If we were able to make sense of the term appearing in the right-hand side of this equation, one would expect it to have the same regularity as $\partial_x\psi$ so that, since $\psi(t,\bullet)$ turns out to belong to $\mathcal{C}^\alpha$ for every $\alpha < 1/2$, one would expect $v(t,\bullet)$ to be of regularity $\mathcal{C}^{\alpha+1}$ for every $\alpha < 1/2$. In particular, we would not expect the term involving $\partial_x v^j$ to cause any trouble, so that it only remains to provide a meaning for the term $g_j^i(v+\psi)\partial_x\psi^j$. If we know that $v \in \mathcal{C}^1$ and we have an interpretation of $\psi(t,\bullet)$ as a rough path $\boldsymbol{\psi}$ (in space), then this can be interpreted as the distribution whose action, when tested against a test function φ, is given by

$$\int \varphi(x) g_j^i(\psi + v))\,d\boldsymbol{\psi}^{\mathbf{j}}(t,x) \;.$$

This reasoning can actually be made precise, see the original article [Hai11b]. In this section we limit ourselves to providing the construction of ψ and giving some of its basic properties.

12.3.1 The linear stochastic heat equation

We now study the model problem in this context - the construction of a spatial rough path associated, in essence, to the above SPDE in the case $f = g = 0$. More precisely, we are considering stationary (in time) solution to the stochastic heat equation[7],

$$d\psi_t = -A\psi_t dt + \sigma dW_t, \tag{12.42}$$

where, for fixed $\lambda > 0$

$$Au = -\partial_x^2 u + \lambda u;$$

and W is a cylindrical Wiener process over $L^2(\mathbf{T})$, the L^2-space over the one-dimensional torus $\mathbf{T} = [0, 2\pi]$, endowed with periodic boundary conditions. Let $(e_k : k \in \mathbf{Z})$ denote the standard Fourier-basis of $L^2(\mathbf{T})$

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}} \sin(kx) & \text{for } k > 0 \\ \frac{1}{\sqrt{2\pi}} & \text{for } k = 0 \\ \frac{1}{\sqrt{\pi}} \cos(kx) & \text{for } k < 0 \end{cases}$$

which diagonalises the operator A in the sense that

$$Ae_k = \mu_k e_k , \qquad mu_k = k^2 + \lambda , \qquad k \in \mathbf{Z} .$$

Thanks to the fact that we chose $\lambda > 0$, the stochastic heat equation (12.42) has indeed a stationary solution which, by taking Fourier transforms, may be decomposed as $\psi(x, t; \omega) = \sum_k Y_t^k(\omega) e_k(x)$. The components Y_t^k are then a family of independent stationary one-dimensional Ornstein-Uhlenbeck processes given by

$$dY_t^k = -\mu_k Y_t^k dt + \sigma dB_t^k ,$$

where $(B^k : k \in \mathbf{Z})$ is a family of i.i.d. standard Brownian motions. An explicit calculation yields

$$\mathbf{E}\big(Y_s^k Y_t^k\big) = \frac{\sigma^2}{2\mu_k} \exp\left(-\mu_k |t-s|\right) ,$$

so that in particular, for any fixed time t,

$$\mathbf{E}\big(Y_t^k\big)^2 = \frac{\sigma^2}{2\mu_k} .$$

[7] With $\lambda = 0$, the 0^{th} mode of ψ behaves like a Brownian motion and ψ cannot be stationary in time, unless one identifies functions that only differ by a constant.

Lemma 12.30. *For each fixed t, the spatial covariance of ψ is given by*

$$\mathbf{E}(\psi(x,t)\psi(y,t)) = K(|x-y|)$$

where K is given by

$$K(u) := \frac{1}{4\pi}\sigma^2 \sum_{k\in\mathbf{Z}} \frac{\cos(ku)}{\mu_k} = \frac{\sigma^2}{4\sqrt{\lambda}\sinh\left(\sqrt{\lambda}\pi\right)} \cosh\left(\sqrt{\lambda}(u-\pi)\right).$$

Here, the second equality holds for u restricted to $[0,2\pi]$. In fact, the cosine series is the periodic continuation of the r.h.s. restricted to $[0,2\pi]$.

Proof. From the basic identity $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$,

$$e_{-k}(x)e_{-k}(y) + e_k(x)e_k(y) = \frac{1}{\pi}\cos(k(x-y)),\ \ k\in\mathbf{Z}\,.$$

Inserting the respective expansion in $R(x,y) := \mathbf{E}(\psi(x,t)\psi(y,t))$, and using the independence of the $\left(Y^k : k\in\mathbf{Z}\right)$, gives

$$\begin{aligned} R(x,y) &= \sum_{k\in\mathbf{Z}} e_k(x)e_k(y)\mathbf{E}\big(Y_t^k\big)^2 = \frac{1}{2\pi}\mathbf{E}\big(Y_t^0\big)^2 + \frac{1}{\pi}\sum_{k=1}^{\infty}\cos(k(x-y))\mathbf{E}\big(Y_t^k\big)^2 \\ &= \frac{\sigma^2}{4\pi}\sum_{k\in\mathbf{Z}}\frac{\cos(k(x-y))}{\lambda+k^2}\,, \end{aligned}$$

and then $R(x,y) = K(|x-y|)$ where

$$K(x) = \frac{\sigma^2}{4\pi}\sum_{k\in\mathbf{Z}}\frac{\cos(kx)}{\lambda+k^2}\,.$$

At last, expand the (even) function $\cosh\left(\sqrt{\lambda}(|\cdot|-\pi)\right)$ in its (cosine) Fourier-series to get the claimed equality. □

Proposition 12.31. *Fix $t\geq 0$. Then $\psi_t(x;\omega) = \psi(t,x;\omega)$, indexed by $x\in[0,2\pi]$, is a centred Gaussian process with covariance of finite 1-variation. More precisely,*

$$\left\|R_{\psi(t,\cdot)}\right\|_{1;[x,y]^2} \leq 2\pi\|K\|_{\mathcal{C}^2;[0,2\pi]}|x-y|\,,$$

and so (cf. Theorem 10.4), for each fixed $t\geq 0$, the $\mathbf{R}^d$-valued process

$$[0,2\pi]\ni x\mapsto \left(\psi_t^1(x),\dots,\psi_t^d(x)\right),$$

consisting of d i.i.d. copies of ψ_t, lifts canonically to a Gaussian rough path $\boldsymbol{\psi}_t(\cdot)\in \mathscr{C}_g^{0,\alpha}\left([0,2\pi],\mathbf{R}^d\right)$.

Proof. This follows immediately from Exercise 10.4. □

Remark 12.32. There are ad-hoc ways to construct a (spatial) rough path lift associated to the stochastic heat-equation, for instance be writing $\psi(t,\bullet)$ as Brownian bridge plus a random smooth function. In this way, however, one ignores the large body of results available for general Gaussian rough paths: for instance, rough path convergence of hyper-viscosity or Galerkin approximation, extensions to fractional stochastic heat equations, concentration of measure can all be deduced from general principles.

We now show that solutions to the stochastic heat equation induces a continuous stochastic evolution in rough path space.

Theorem 12.33. *There exists a continuous modification of the map* $t \mapsto \boldsymbol{\psi}_t$ *with values in* $\mathscr{C}_g^\alpha\big([0,2\pi],\mathbf{R}^d\big)$.

Proof. Fix s and t. The proof then proceeds in two steps. First, we will verify the assumptions of Corollary 10.6, namely we will show that

$$|\varrho_\alpha(\boldsymbol{\psi}_s,\boldsymbol{\psi}_t)|_{L^q} \le C \sup_{x,y\in[0,2\pi]} \Big[\mathbf{E}(|\psi_s(x,y)-\psi_t(x,y)|^2)\Big]^\theta ,$$

for some constant C that is independent of s and t. In the second step, we will show that (here we may assume $d=1$), with $\psi_s(x,y) := \psi_s(y)-\psi_s(x)$, one has the bound

$$\sup_{x,y\in[0,2\pi]} \mathbf{E}\Big[|\psi_s(x,y)-\psi_t(x,y)|^2\Big] = \mathrm{O}\big(|t-s|^{1/2}\big) .$$

The existence of a continuous (and even Hölder) modification is then a consequence of the classical Kolmogorov criterion.

For the first step, we write $X = \big(\psi_s^1(\cdot),\ldots,\psi_s^d(\cdot)\big)$ and $Y = \big(\psi_t^1(\cdot),\ldots,\psi_t^d(\cdot)\big)$. Note that one has independence of $\big(X^i,Y^i\big)$ with $\big(X^j,Y^j\big)$ for $i\neq j$. We have to verify finite 1-variation (in the 2D sense) of the covariance of (X,Y). In view of Proposition 12.31, it remains to establish finite 1-variation of

$$\begin{aligned}(x,y)\mapsto R_{(X^1,Y^1)}(x,y) = \mathbf{E}\big[\psi_s^1(x)\psi_t^1(y)\big] &= \sum_{k\in\mathbf{Z}} e_k(x)e_k(y)\mathbf{E}\big(Y_s^kY_t^k\big)\\ &= \frac{\sigma^2}{4\pi}\sum_{k\in\mathbf{Z}} \frac{\cos\big(k(x-y)\big)}{\lambda+k^2} e^{-\left(\lambda+k^2\right)|t-s|}. =: R_\tau(x,y).\end{aligned}$$

For every $\tau>0$, exponential decay of the Fourier-modes implies smoothness of R_τ. We claim

$$\|R_\tau\|_{1\text{-var};[u,v]^2} \le C|v-u| < \infty,$$

uniformly in $\tau\in(0,1]$ and u,v. To see this, write

$$\begin{aligned}\|R_\tau\|_{1\text{-var};[u,v]^2} &= \int_u^v\int_u^v |\partial_{xy}R_\tau|\,dx\,dy\\ &\sim \int_u^v\int_u^v \Big|\sum k^2\frac{e^{ik(x-y)}}{\lambda+k^2}e^{-\left(\lambda+k^2\right)\tau}\Big|\,dx\,dy\end{aligned}$$

$$\begin{aligned}&\sim \int_u^v \int_u^v \Big|\sum e^{ik(x-y)} e^{-k^2\tau}\Big| dx\, dy \\ &= \int_u^v \int_u^v p_\tau(x-y) dy\, dx \le |v-u| \;,\end{aligned}$$

where we used the trivial estimate $\int_u^v p_\tau(x-y)dy \le \int_0^{2\pi} p_\tau(x-y)dy = 1$. In this expression, p denotes the (positive) transition kernel of the heat semigroup on the torus. The step above, between second and third line, where we effectively set $\lambda = 0$ is harmless. The factor $e^{-\lambda\tau}$ may simply be taken out, and

$$\Big|\sum_k \Big(1 - \frac{k^2}{\lambda + k^2}\Big) e^{ik(x-y)} e^{-k^2\tau}\Big| \le \sum_k \Big|1 - \frac{k^2}{\lambda + k^2}\Big| = \sum_k \frac{\lambda}{\lambda + k^2} < \infty \;.$$

After integrating over $[u,v]^2$, we see that the error made above is actually of order $\mathrm{O}\big(|v-u|^2\big)$. This is more than enough to conclude that

$$\big\|R_{(X^1,Y^1)}\big\|_{1\text{-var};[u,v]^2} \le C|v-u| < \infty \;,$$

uniformly in $\tau \in (0,1]$ and u, v.

We now turn to the second step of our proof. We claim that $\mathbf{E}|\psi_s^1(x,y) - \psi_t^1(x,y)|^2 = \mathrm{O}\big(|t-s|^{1/2}\big)$, uniformly in $x, y \in [0,2\pi]$. Since

$$\big|\psi_s^1(x,y) - \psi_t^1(x,y)\big| \le \big|\psi_s^1(x) - \psi_t^1(x)\big| + \big|\psi_s^1(y) - \psi_t^1(y)\big| \;,$$

the question reduces to a similar bound on $\mathbf{E}|\psi_s^1(x) - \psi_t^1(x)|^2$, uniform in $x \in [0,2\pi]$. This quantity is equal to

$$\begin{aligned}&\mathbf{E}\big[\psi_s^1(x)\psi_s^1(x)\big] - 2\mathbf{E}\big[\psi_s^1(x)\psi_t^1(x)\big] + \mathbf{E}\big[\psi_t^1(x)\psi_t^1(x)\big] \\ &\qquad = \frac{\sigma^2}{4\pi} \sum_{k\in\mathbf{Z}} \frac{2\big(1 - e^{-(\lambda+k^2)|t-s|}\big)}{\lambda + k^2}. \\ &\qquad \le \frac{\sigma^2}{4\pi} \sum_{|k|<N} 2|t-s| + 2\frac{\sigma^2}{4\pi} \sum_{k\ge N} \frac{2\big(1 - e^{-(\lambda+k^2)|t-s|}\big)}{\lambda + k^2} \;,\end{aligned}$$

where we used that $1 - e^{-cx} \le cx$ for $c, x > 0$ in the first sum. We then take $N \sim |t-s|^{-1/2}$, so that the first sum is of order $\mathrm{O}\big(|t-s|^{1/2}\big)$. For the second sum, we use the trivial bound $1 - e^{-(\lambda+k^2)|t-s|} \le 1$. It then suffices to note that

$$\sum_{k\ge N} \frac{1}{\lambda + k^2} \le \sum_{k\ge N} \frac{1}{k^2} = \mathrm{O}(1/N) = \mathrm{O}\big(|t-s|^{1/2}\big) \;,$$

which completes the proof. □

Remark 12.34. The final estimate in the above proof, namely

$$\mathbf{E}\big|\psi_s^1(x) - \psi_t^1(x)\big|^2 = \mathrm{O}\big(|t-s|^{1/2}\big)\;,$$

also implies “almost $\frac{1}{4}$-Hölder” temporal regularity of the stochastic heat equation.

12.4 Exercises

Exercise 12.1 (From [DFS17]) *a) Assume $W \in \mathcal{C}^1$. Show that the Feynman–Kac (or equivalently viscosity) solution to (12.11) is an analytically weak solution in the sense of (12.21) with $d\mathbf{W}$ replaced by $\dot{W}dt$.*

b) Assume now $\mathbf{W} = (W, \mathbb{W}) \in \mathscr{C}_g^{0,\alpha}$. Show that $(Y, Y') \in \mathscr{D}_W^{2\alpha}$.

c) Show that the Feynman–Kac solution constructed in Theorem 12.11 is an analytically weak solution in the sense of (12.21).

Exercise 12.2 (From [CDFO13]) *A crucial role in the proof of Theorem 12.11 was played by a hybrid Itô-rough differential equation of the form*

$$dX_t = \sigma(X_t)dB + \beta(X_t)d\mathbf{W}, \tag{12.43}$$

ultimately solved as (random) rough differential equation, subject to $\sigma, \beta \in \mathcal{C}_b^3$. Give an alternative construction to the hybrid equation based on flow decomposition. That is, use the flow associated to the RDE $dY = \beta(Y)d\mathbf{W}$ and transform (12.43) into a bona fide Itô differential equation.

Hint: *When $\mathbf{W}$ is replaced by a $\mathcal{C}^1$ path W^ε this is a straightforward computation. Use the stability of RDE flows, combined with stability results for Itô SDEs to conclude. Specify the regularity requirements on σ, β.*

Exercise 12.3 (Robust filtering, [CDFO13]) *Consider a pair of processes (X, Y) with dynamics*

$$dX_t = V_0(X_t, Y_t)dt + \sum_k Z_k(X_t, Y_t)dW_t^k + \sum_j V_j(X_t, Y_t)dB_t^j, \tag{12.44}$$

$$dY_t = h(X_t, Y_t)dt + dW_t, \tag{12.45}$$

with $X_0 \in L^\infty$ and $Y_0 = 0$. For simplicity, assume coefficients $V_0, V_1, \dots, V_{d_B} : \mathbf{R}^{d_X+d_Y} \to \mathbf{R}^{d_X}$, $Z_1, \dots, Z_{d_Y} : \mathbf{R}^{d_X+d_Y} \to \mathbf{R}^{d_X}$ and $h = (h^1, \dots, h^{d_Y}) : \mathbf{R}^{d_X+d_Y} \to \mathbf{R}^{d_Y}$ to be bounded with bounded derivatives of all orders; W and B are independent Brownian motions of the correct dimension. We now interpret X as a signal and Y as noisy and incomplete observation. The filtering problem consists in computing the conditional distribution of the unobserved component X, given the observation Y. Equivalently, one is interested in computing

$$\pi_t(g) = \mathbf{E}[g(X_t, Y_t)|\mathcal{Y}_t]\;,$$

where $\mathcal{Y}_t$ is the observation filtration and g is a suitably chosen test function. Measure theory tells us that there exists a Borel-measurable map $\theta_t^g : \mathcal{C}([0,t], \mathbf{R}^{d_Y}) \to \mathbf{R}$, such that a.s. $\pi_t(g) = \theta_t^g(Y)$ where we consider $Y = Y(\omega)$ as a $\mathcal{C}([0,t], \mathbf{R}^{d_Y})$-valued random variable. Note that θ_t^g is not uniquely determined (after all, modifications on null sets are always possible). On the other hand, there is obvious interest to have a robust filter, in the sense of having a continuous *version of θ_t^g, so that close observations lead to nearby conclusions about the signal.*

a) Give an example showing that, in general, θ_t^g does not admit a continuous version.
b) Let $\alpha \in (1/2, 1/3)$. Show that there exists a continuous map on rough path space

$$\Theta_t^g : \mathscr{C}_g^{0,\alpha}([0,t], \mathbf{R}^{d_Y}) \to \mathbf{R}\,,$$

such that a.s.

$$\pi_t(g) = \Theta_t^g(\mathbf{Y})\,, \tag{12.46}$$

where $\mathbf{Y}$ is the random geometric rough path obtained from Y by iterated Stratonovich integration.

Hint: *You may use the "Kallianpur–Striebel formula", a standard result in filtering theory which asserts that*

$$\pi_t(g) = \frac{p_t(g)}{p_t(1)}, \quad p_t(g) := \mathbf{E}_0[g(X_t, Y_t)v_t | \mathcal{Y}_t]$$

where

$$\left.\frac{\mathrm{d}\mathbf{P}_0}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_t} = \exp\left(-\sum_i \int_0^t h^i(X_s, Y_s) dW_s^i - \frac{1}{2}\int_0^t ||h(X_s, Y_s)||^2 ds\right)$$

and $v = \{v_t, t > 0\}$ is defined as the right-hand side above with $-W$ replaced by Y.

Exercise 12.4 *Show almost sure "$(\frac{1}{4} - \varepsilon)$-Hölder" temporal regularity of $\psi = \psi_t(x;\omega)$, solution to the stochastic heat equation. Show that, for fixed x, $\psi_t(x;\omega)$ is not a semimartingale.*

Exercise 12.5 (Spatial Itô–Stratonovich correction [HM12]) *Writing $\mathbf{T}$ for the interval $[0, 2\pi]$ with periodic boundary, let us say that*

$$u = u(t, x; \omega) : [0, T] \times \mathbf{T} \times \Omega \to \mathbf{R}$$

is a (analytically) weak solution to

$$\partial_t u = \partial_{xx} u - u + \frac{1}{2}\partial_x\big(u^2\big) + \xi\,, \tag{$\star$}$$

if and only if $u = v + \psi$ where ψ is the stationary solution to $\partial_t \psi = \partial_{xx}\psi - \psi + \xi$ and, for all test functions $\varphi \in \mathcal{C}^\infty(\mathbf{T})$,

$$\partial_t \langle v, \varphi \rangle = \langle v, \partial_{xx}\varphi \rangle - \langle v, \varphi \rangle - \left\langle \frac{1}{2}u^2, \partial_x \varphi \right\rangle .$$

a) *Replace* $\frac{1}{2}\partial_x(u^2)$ *in* ($\star$) *by a (spatially right) finite-difference approximation,*

$$\frac{1}{2}\frac{u(.+\varepsilon)^2 - u^2}{\varepsilon};$$

write u^ε *for a solution to the resulting equation. Assume* $u^\varepsilon \to u$ *locally uniformly in probability. Show that* u *is a solution to* ($\star$).

b) *At least formally,* $\partial_x\left(\frac{1}{2}u^2\right) = u\partial_x u$ *in* ($\star$)*, which suggests an alternative finite difference approximation, namely,*

$$u\frac{(u(.+\varepsilon) - u)}{\varepsilon};$$

Assume $v^\varepsilon = u^\varepsilon - \psi \to v := u - \psi$ *and its first (spatial) derivatives converge locally uniformly in probability. Show that* u *is an analytically weak solution to the perturbed equation*

$$\partial_t u = \partial_{xx} u + \frac{1}{2}\partial_x\left(u^2\right) + C + \xi$$

with $C \neq 0$*. Determine the value of* C*.* **Hint:** *Use Exercise 10.6.*

Solution. a) By switching to suitable subsequences, we may assume $u^\varepsilon \to u$ locally uniformly with probability one. Write $D_{\varepsilon,l}, D_{\varepsilon,r}$ for a discrete (left, right) finite difference approximation. Note

$$\left\langle D_{\varepsilon,r}\left(\frac{1}{2}u^2\right), \varphi \right\rangle = -\left\langle \frac{1}{2}u^2, D_{\varepsilon,l}\varphi \right\rangle \to -\left\langle \frac{1}{2}u^2, \partial_x\varphi \right\rangle.$$

Given that $v^\varepsilon = u^\varepsilon - \psi \to v := u - \psi$ locally uniform it then suffices to pass to the limit in the (integral formulation) of

$$\partial_t \langle v^\varepsilon, \varphi \rangle = \langle v^\varepsilon, \partial_{xx}\varphi \rangle - \langle v^\varepsilon, \varphi \rangle + \left\langle \frac{1}{2}u^2, D_{\varepsilon,l}\varphi \right\rangle.$$

b) We note

$$\begin{aligned} D_{\varepsilon,r}\left(\frac{1}{2}u^2\right) &= \frac{1}{2}\frac{u(.+\varepsilon)^2 - u^2}{\varepsilon} = \frac{(u(.+\varepsilon) + u)}{2}\frac{(u(.+\varepsilon) - u)}{\varepsilon} \\ &= u\frac{(u(.+\varepsilon) - u)}{\varepsilon} + \frac{1}{2\varepsilon}(u(.+\varepsilon) - u)^2 . \end{aligned}$$

It follows that

$$\begin{aligned}\partial_t\langle v^\varepsilon,\varphi\rangle &= \langle v^\varepsilon,\partial_{xx}\varphi\rangle - \langle v^\varepsilon,\varphi\rangle + \left\langle u^\varepsilon\frac{(u^\varepsilon(.+\varepsilon)-u^\varepsilon)}{\varepsilon},\varphi\right\rangle.\\ &= \langle v^\varepsilon,\partial_{xx}\varphi\rangle - \langle v^\varepsilon,\varphi\rangle\\ &\quad - \left\langle \frac{1}{2}(u^\varepsilon)^2, D_{\varepsilon,l}\varphi\right\rangle - \left\langle \frac{1}{2\varepsilon}(u^\varepsilon(.+\varepsilon)-u^\varepsilon)^2,\varphi\right\rangle .\end{aligned}$$

In order to pass to the $\varepsilon \to 0$ limit, we must understand the final "quadratic variation" term. By assumption v^ε are of class $\mathcal{C}^1$, uniformly in ε. Hence

$$\begin{aligned}[u^\varepsilon(.+\varepsilon)-u^\varepsilon] &= \psi(.+\varepsilon)-\psi+v^\varepsilon(.+\varepsilon)-v^\varepsilon\\ &= \psi(.+\varepsilon)-\psi+\mathrm{O}(\varepsilon)\end{aligned}$$

and so, with osc $(\psi;\varepsilon)\mathrm{O}(1)+\mathrm{O}(\varepsilon)=\mathrm{o}(1)$ as $\varepsilon\to 0$,

$$\frac{1}{2\varepsilon}(u^\varepsilon(.+\varepsilon)-u^\varepsilon)^2 = \frac{1}{2\varepsilon}(\psi(.+\varepsilon)-\psi)^2+\mathrm{o}(1)$$

we have

$$\left\langle \frac{1}{2\varepsilon}(u^\varepsilon(.+\varepsilon)-u^\varepsilon)^2,\varphi\right\rangle = \left\langle \frac{1}{2\varepsilon}(\psi(.+\varepsilon)-\psi)^2,\varphi\right\rangle + \mathrm{o}(1)\ .$$

From Lemma 12.30 we know that

$$\mathbf{E}[\psi^2_{x,x+\varepsilon}] = 2(K(0)-K(\varepsilon)) = -2K'(0)\varepsilon+\mathrm{o}(\varepsilon) = C\varepsilon+\mathrm{o}(\varepsilon)\ .$$

Since $K(u) = \frac{\cosh(u-\pi)}{4\sinh(\pi)}$, we have $C=-2K'(0)=\frac{1}{2}$, and it follows from Exercise 10.6 that

$$\begin{aligned}\left\langle \frac{1}{2\varepsilon}(\psi(.+\varepsilon)-\psi)^2,\varphi\right\rangle &= \frac{1}{2}\int \varphi(x)\frac{\psi^{\,2}_{x,x+\varepsilon}}{\varepsilon}dx\\ &\to \frac{1}{2}\int\varphi(x)C\,dx = \left\langle\frac{1}{4},\varphi\right\rangle,\end{aligned}$$

where the convergence takes place in probability. It follows that u is a solution (in the above analytically weak sense) of

$$\partial_t u = \partial_{xx}u - u + \frac{1}{2}\partial_x\big(u^2\big) + \frac{1}{4} + \xi\ .$$

12.5 Comments

Section 12.1: The explicit solution of the rough transport equation in Section 12.1.1 is a (geometric) rough-pathification of the classical method of characteristics and Kunita's (Stratonovich) stochastic version thereof [Kun84], first pointed out in [CF09].

Our intrinsic definition of (regular vs. weak / measure-valued) RPDE solution is essentially taken from Diehl et al. [DFS17] and Bellingeri et al. [BDFT20], which also treats the low regularity case. Bailleul–Gubinelli [BG17] suggest an abstract framework of (unbounded) rough drivers in which $(\Gamma[\bullet]W_{s,t}, \Gamma^2[\bullet]\mathbb{W}_{s,t})$, with Γ as in (12.2), are viewed as (s,t)-indexed familiy of *unbounded operators*

$$\mathbf{A}_{s,t} = (A_{s,t}, \mathbb{A}_{s,t})$$

on a suitable scale of Banach spaces, which satisfy an operator Chen relation and then the (operator) geometricity condition $A_{s,t}^2/2 = \mathbb{A}_{s,t}$. The rough transport equation, say $du_t = \Gamma u_t d\mathbf{W}$ if written as initial value problem, then fits into an abstract rough linear equation of the form

$$du_t = \mathbf{A}(dt)u_t \ .$$

An analytically weak formulation (somewhat similar to our Section 12.1.2, but now formulated via Banach duals) then allows them to obtain existence and uniqueness under $\mathcal{C}_b^3$ assumptions on the vector fields, at the price of a doubling of variables argument related in the spirit to Di Perna–Lions [DL89].

Entropy solutions to scalar conservation laws with rough forcing are studied by Friz–Gess [FG16b]; in [HNS20] Hocquet et al. study a generalized Burgers equation with rough transport noise. A different class of *rough scalar conservation laws*, closely related to rough transport, is given by

$$du + \mathrm{div}_x(A(x,u))d\mathbf{W} = 0 \ , \quad u = u_0 \ , \tag{12.47}$$

where $u : [0,T] \times \mathbf{R}^n \to \mathbf{R}$, with $A = (A_j^i : 1 \le i \le n, 1 \le j \le d)$ sufficiently smooth, matrix valued functions and $\mathbf{W}$ a geometric Hölder rough path over $\mathbf{R}^d$. (The case of linear $A(x,u) = f(x)u$ is precisely the rough continuity equation treated in Section 12.1.2.)

Such equations were studied from a "pathwise" point of view (essentially possible when $A = A(u)$ has no x-dependence or when $d = 1$) in Lions, Perthame and Souganidis [LPS13] and [LPS14], followed by Gess–Souganidis [GS15] who treat the general case (12.47) and then Hofmanová [Hof16]. When $d\mathbf{W} = \dot{W}dt$, this falls into the well established theories of entropy solutions and kinetic solutions. The latter formulation related to rough transport as follows. With

$$\chi(x,\xi,t) := \chi(u(x,t),\xi) := \begin{cases} +1 \ \text{ if } \ 0 \le \xi \le u(x,t), \\ -1 \ \text{ if } \ u(x,t) \le \xi \le 0, \\ \ 0 \ \text{ otherwise}, \end{cases} \tag{12.48}$$

one can rewrite (12.47) in its (formal) kinetic form: for $T > 0$ fixed,

$$d_t\chi + \big(\partial_u A(x,\xi) \cdot D_x\chi - \mathrm{div}_x\, A(x,\xi)\partial_\xi\chi\big)d\mathbf{W} = (\partial_\xi m)dt \ , \tag{12.49}$$

on $\mathbf{R}^n \times \mathbf{R} \times (0,T]$ with initial data $\chi(\bullet,*,0) = \chi(u_0(\bullet),*)$ where $\mathrm{div}_x A = (\mathrm{div}_x A_1, \dots, \mathrm{div}_x A_d)$ and m is a bounded nonnegative measure on $\mathbf{R}^n \times \mathbf{R} \times [0,T]$, known as *defect measure*, which is part of the solution. The definition of rough kinetic solution [GS15] is then given as analytically weak solution of (12.49), with test functions obtained as (spatially) regular solutions to an auxilary rough transport equation, similar in spirit to Section 12.1.2. See also Gess et al. [GPS16] for a semi-discretisation. The idea of test functions with (here: temporal) structure tailor-made to a realisation of the noise (a.k.a. rough path) is central to RPDEs. A well-posedness result for rough kinetic solutions was also obtained by Deya et al. [DGHT19b], in an extended setting of RPDEs with (unbounded) rough drivers, of the form

$$du_t = \mu(dt) + \mathbf{A}(dt)u_t \ ,$$

where the abstract assumptions on the drift term μ are seen to accommodate the defect measure. *Rough Hamilton–Jacobi equations* are of the form

$$du + H(Du, x)d\mathbf{W} = 0 \ , \qquad u(0, \bullet) = u_0 \ , \tag{12.50}$$

on $(0,T] \times \mathbf{R}^n$, with Hamiltonians $H = (H_1, \ldots, H_d)$. When $d\mathbf{W} = \dot{W}dt$, this falls into the well established theory of viscosity solutions, with intrinsic notion of sub (resp. super) solutions via "touching" test functions $\varphi = \varphi(t,x) \in \mathcal{C}^{1,1}$. Short-time regular solutions via the method of "rough" characteristics then supply the correct class of test functions (depending on the noise realisation modelled by $\mathbf{W}$): when inserted in the equation, they at least formally "eliminate" the rough part, this is basically a local change of the unknown. (A global change of coordinates is sometimes possible, notably in the case of transport noise when $H(p,x)$ is linear in p, cf. Section 12.2.3 below.) These ideas form the basis of Lions–Souganidis' stochastic viscosity theory [LS98a, LS98b, LS00b] which predates most works on rough paths, the resulting "pathwise" theory essentially requires $H = H(p)$ with no x-dependence, or $d = 1$; see also [FGLS17] (x-dependent quadratic Hamiltonian) and [GGLS20] (speed of propagation). In spatial dimension $n = 1$, there is a noteworthy connection with rough conservation laws: if v solves the rough HJ equation $dv + A(\partial_x v, x)d\mathbf{W} = 0$, then, at least formally, $u = \partial_x v$ satisfies the rough conservation law $du + \partial_x(A(u,x))d\mathbf{W} = 0$.

Section 12.2: Linear stochastic partial differential equations go back at least to Krylov–Rozovskii [KR77] and play an important problem in filtering theory (Zakai equation). A Feynman–Kac representation appears in Pardoux [Par79] and Kunita [Kun82]. Kunita also has flow decompositions of SPDE solutions. Caruana–Friz [CF09] implement this in the rough path setting in a framework of classical PDE solutions. The construction of hybrid stochastic / rough differential equations which underlies the "rough" Feynman–Kac approach, Theorem 12.11, is taken from [DOR15] (see also [FHL20]). Diehl et al. [DFS17] establish existence and uniqueness, based on an intrinsic definition for (linear) RPDEs, numerical algorithms are given by Bayer et al. [BBR$^+$18]. Hofmanova–Hocquet [HH18] study (linear) RPDEs from a variational perspective and unbounded rough driver perspective, as does Hofmanová et al. [HLN19] for the Navier–Stokes equation perturbed by rough transport noise.

An extension of Lions, Perthame and Souganidis [LPS13, LPS14] to rough, scalar, degenerate parabolic-hyperbolic equation is given in [GS17].

In the context of Crandall–Ishii–Lions viscosity setting, by nature a theory for second order equations with a maximum principle, *stochastic (pathwise) viscosity solutions* for fully non-linear equations were introduced by Lions–Souganidis [LS98a, LS98b, LS00a, LS00b]. Caruana, Friz and Oberhauser [CFO11] introduce *rough viscosity solutions* by a limiting procedure for classes of nonlinear SPDEs with transport noise; an intrinsic definition (via global transformaion) is given e.g. in [DFO14]. An adaption of the original intrinsic definition of (pathwise) viscosity solutions to fully non-linear equations [LS98a] is given in Seeger [See18b]. Extensions to different noise situations are due to Diehl–Friz, [DF12] and then [FO14]. Nonlinear noise, x-dependent and quadratic in Du is considered by Friz, Gassiat, Lions and Souganidis [FGLS17]. Approximation schemes for (pathwise) viscosity solutions of fully nonlinear problems are studied [See18a].

A nonlinear Feynman–Kac representation (with relations to "rough BSDEs") is given in [DF12]. In a filtering context, a (rough path) robustified Kalianpur–Striebel formula (cf. Exercise 12.3) was given by Crisan, Diehl, Friz and Oberhauser [CDFO13], which is also the first source of hybrid differential equations. At last, we refer to Gubinelli–Tindel, Deya et al. and Teichmann [GT10, DGT12, Tei11] for some other rough path approaches to SPDEs. Theorem 12.18 is essentially due to [GH19], but very closely related to the earlier results of [GT10]. Compared to the latter, we restrict ourselves to finite-dimensional drivers, but allow for a more natural class of nonlinearities thanks to a slightly different use of the various interpolation spaces.

Section 12.3: The construction of a spatial rough path associated to the stochastic heat equation is due to Hairer [Hai11b] and allows to deal with otherwise ill-posed SPDEs of stochastic Burgers type, see also Hairer–Weber [HW13] and Friz, Gess, Gulisashvili, Riedel [FGGR16] for various extensions (including multiplicative noise, and fractional Laplacian / non-periodic boundary respectively). This construction is also an ingredient in one construction for solutions to the KPZ equation, see Hairer [Hai13] and Chapter 15. Exercise 12.5, in the spirit of Föllmer – rather than rough path – integration, is taken from Hairer–Maas [HM12]. Similar results are available for rough SPDEs of type (12.41), see Hairer, Maas and Weber [HMW14], but this is beyond the scope of these notes. Bellingeri [Bel20] uses regularity structures to establish an Itô formula for the stochastic heat equation.