A Second Order Time Accurate Finite Volume Scheme for the Time-Fractional Diffusion Wave Equation on General Nonconforming Meshes

Fayssal Benkhaldoun[1,2] and Abdallah Bradji[3(✉)]

[1] LAGA, University of Paris 13, Paris, France
fayssal@math.univ-paris13.fr
[2] UM6P, Benguerir, Morocco
Fayssal.BENKHALDOUN@um6p.ma
[3] LMA (Laboratoire de Mathématiques Appliquées) Faculty of Sciences, University of Annaba, Annaba, Algeria
abdallah.bradji@gmail.com, abdallah.bradji@etu.univ-amu.fr, bradji@math.univ-paris13.fr
https://www.math.univ-paris13.fr/~fayssal/
https://www.i2m.univ-amu.fr/~bradji/

Abstract. SUSHI (Scheme Using Stabilization and Hybrid Interfaces) is a finite volume method has been developed at the first time to approximate heterogeneous and anisotropic diffusion problems. It has been applied later to approximate several types of partial differential equations. The main feature of SUSHI is that the control volumes can only be assumed to be polyhedral. Further, a consistent and stable Discrete Gradient is developed.

In this note, we establish a second order time accurate implicit scheme for the TFDWE (Time Fractional Diffusion-Wave Equation). The space discretization is based on the use of SUSHI whereas the time discretization is performed using a uniform mesh. The scheme is based on the use of an equivalent system of two low order equations. We sketch the proof of the convergence of the stated scheme. The convergence is unconditional. This work is an improvement of [3] in which a first order scheme, whose convergence is conditional, is established.

Keywords: Finite volume · Time Fractional Diffusion Wave Equation · System · Unconditional convergence · Second order time accurate

1 Problem to Be Solved and Aim of This Paper

Let us consider the following time-fractional equation:

$$\partial_t^\alpha u(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t),\ (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{1}$$

I. Lirkov and S. Margenov (Eds.): LSSC 2019, LNCS 11958, pp. 95–104, 2020.
https://doi.org/10.1007/978-3-030-41032-2_10

where Ω is an open bounded connected subset of $\mathbb{R}^d$ ($d \in \mathbb{N}^\star$), $T > 0$, and f is a given function. The operator ∂_t^α denotes the Caputo derivative of order α whose general formula is given by, for $m-1 < \rho < m$ with $m \in \mathbb{N}^\star$

$$\partial_t^\rho u(t) = \frac{1}{\Gamma(m-\rho)} \int_0^t (t-s)^{m-1-\rho} u^{(m)}(s) ds. \tag{2}$$

We are concerned with the case of TFDWE in which α, which appears in the first term on the left hand side of (1), is satisfying

$$1 < \alpha < 2.$$

In this case the operator ∂_t^α is given by

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} u''(s) ds. \tag{3}$$

Initial conditions are given by, for all $\boldsymbol{x} \in \Omega$:

$$u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}) \quad \text{and} \quad u_t(\boldsymbol{x}, 0) = u^1(\boldsymbol{x}), \tag{4}$$

where u^0 and u^1 are given functions defined on Ω.

Homogeneous Dirichlet boundary conditions are given by

$$u(\boldsymbol{x}, t) = 0, \ (\boldsymbol{x}, t) \in \partial\Omega \times (0, T). \tag{5}$$

The TFDWE arises in several applications and several numerical methods have been devoted to approximate such equation, see [7,8] and references therein. In this note, we establish a second order time accurate implicit finite volume scheme using SUSHI [5] for TFDWE in any space dimension along with a brief study for its convergence analysis in several discrete norms. The scheme is based on an equivalent system of low order equations for (1). This work improves [3] which deal with a first order scheme with a conditional convergence.

2 Definition of a Consistent and Stable Discrete Gradient

We consider as discretization in space the mesh of [5]. In brief, such mesh is defined as the triplet $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ where $\mathcal{M}$ is the set of cells, $\mathcal{E}$ is the set of edges, and $\mathcal{P}$ is a set of points $\boldsymbol{x}_K$ in each cell K. We assume that, for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma \in \mathcal{E}_K} \overline{\sigma}$. For any $\sigma \in \mathcal{E}$, we denote by $\mathcal{M}_\sigma = \{K, \sigma \in \mathcal{E}_K\}$. We then assume that, for any $\sigma \in \mathcal{E}$, either $\mathcal{M}_\sigma$ has exactly one element and then $\sigma \subset \partial\Omega$ (the set of these interfaces, called boundary interfaces, denoted by $\mathcal{E}_{\text{ext}}$) or $\mathcal{M}_\sigma$ has exactly two elements (the set of these interfaces, called interior interfaces, denoted by $\mathcal{E}_{\text{int}}$). For all $\sigma \in \mathcal{E}$, we denote by $\boldsymbol{x}_\sigma$ the barycentre of σ. For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$, we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to σ outward to K. Denoting by $d_{K,\sigma}$ the Euclidean distance between $\boldsymbol{x}_K$ and the hyperplane including σ, one assumes that $d_{K,\sigma} > 0$. We then denote by $\mathcal{D}_{K,\sigma}$ the cone with vertex $\boldsymbol{x}_K$ and basis σ. Also, h_K is used to denote the diameter of K. For more details on the mesh, we refer to [5, Definition 2.1, Page 1012].

We define the discrete space $\mathcal{X}_{\mathcal{D},0}$ as the set of all $v = \left((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}\right)$, where $v_K, v_\sigma \in \mathbb{R}$ and $v_\sigma = 0$ for all $\sigma \in \mathcal{E}_{\text{ext}}$. Let $H_{\mathcal{M}}(\Omega) \subset L^2(\Omega)$ be the space of functions which are constant on each control volume K of the mesh $\mathcal{M}$. For all $v \in \mathcal{X}_{\mathcal{D}}$, we denote by $\Pi_{\mathcal{M}}v \in H_{\mathcal{M}}(\Omega)$ the function defined by $\Pi_{\mathcal{M}}v(\boldsymbol{x}) = v_K$, for a.e. $\boldsymbol{x} \in K$, for all $K \in \mathcal{M}$. In order to analyze the convergence, we need to consider the size of the discretization $\mathcal{D}$ defined by $h_{\mathcal{D}} = \sup\{\text{diam(K)}, \ K \in \mathcal{M}\}$ and the regularity of the mesh given by $\theta_{\mathcal{D}} = \max\left(\max_{\sigma\in\mathcal{E}_{\text{int}},K,L\in\mathcal{M}} \frac{d_{K,\sigma}}{d_{L,\sigma}}, \max_{K\in\mathcal{M},\sigma\in\mathcal{E}_K} \frac{h_K}{d_{K,\sigma}}\right)$.

The formulation of the scheme we want to consider involves the discrete gradient, denoted by $\nabla_{\mathcal{D}}$, developed in [5]. The value of $\nabla_{\mathcal{D}}u$, where $u \in \mathcal{X}_{\mathcal{D},0}$, is defined by, for all $K \in \mathcal{M}$, for a.e. $\boldsymbol{x} \in \mathcal{D}_{K,\sigma}$

$$\nabla_{\mathcal{D}}u(\boldsymbol{x}) = \frac{1}{\mathrm{m}(K)} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\,(u_\sigma - u_K)\,\mathbf{n}_{K,\sigma} + \left(\frac{\sqrt{d}}{d_{K,\sigma}}\,(u_\sigma - u_K - \nabla_K u \cdot (\boldsymbol{x}_\sigma - \boldsymbol{x}_K))\right)\mathbf{n}_{K,\sigma}. \tag{6}$$

We define now the inner product defined on $\mathcal{X}_{\mathcal{D},0} \times \mathcal{X}_{\mathcal{D},0}$ and given by

$$\langle u, v\rangle_F = \int_\Omega \nabla_{\mathcal{D}}u(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}}v(\boldsymbol{x})d\boldsymbol{x}. \tag{7}$$

The time discretization is performed with a constant time step $k = \dfrac{T}{N+1}$, where $N \in \mathbb{N}^\star$, and we shall denote $t_n = nk$, for $n \in [\![0, N+1]\!]$. We denote by ∂^1 the discrete first time derivative given by $\partial^1 v^{j+1} = \dfrac{v^{j+1} - v^j}{k}$.

Throughout this paper, the letter C stands for a positive constant independent of the parameters of the space and time discretizations.

3 A High Order Approximation for the Caputo Derivative and Its Properties

For the sake of completeness, we recall in this section some results concerning a second order approximation for the time fractional derivative $\partial_t^\beta \Phi$ with $0 < \beta < 1$ and Φ is smooth, i.e. $\Phi \in \mathcal{C}^3[0,T]$, and its properties. This approximation is given by the so-called $L2-1_\sigma$ formula developed in [1,6]. Such approximation will help to derive a second order time accurate scheme for the considered problem (1)–(5). To construct this high order approximation for the Caputo derivative, we consider the "fractional mesh points" $t_{n+\sigma} = (n+\sigma)k$, for $n \in [\![0, N]\!]$, where

$$\sigma = 1 - \frac{\beta}{2}. \tag{8}$$

Using (2) when $\rho = \beta \in (0, 1)$ and consequently $m = 1$, the value $\partial_t^\beta \Phi(t_{n+\sigma})$ is given by

$$\frac{1}{\Gamma(1-\beta)} \left(\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (t_{n+\sigma} - s)^{-\beta} \Phi_s(s) ds + \int_{t_n}^{t_{n+\sigma}} (t_{n+\sigma} - s)^{-\beta} \Phi_s(s) \right) ds. \quad (9)$$

For each $j \in [\![1, N]\!]$, let $\Pi_{2,j}\Phi$ be the quadratic interpolation defined on (t_{j-1}, t_j) on the points t_{j-1}, t_j, t_{j+1} of Φ. An explicit expansion for $\Pi_{2,j}\Phi$ yields:

$$\begin{aligned}(\Pi_{2,j}\Phi(s))' &= \partial^1 \Phi(t_{j+1}) + \partial^2 \Phi(t_{j+1}) \left(s - t_{j+\frac{1}{2}} \right) \\ &= \partial^1 \Phi(t_j) + \partial^2 \Phi(t_{j+1}) \left(s - t_{j-\frac{1}{2}} \right).\end{aligned}$$

When approximating the terms of the sum (resp. the last term) using quadratic interpolations (resp. a linear interpolation) in (9) of $\partial_t^\beta \Phi(t_{n+\sigma})$, we have to compute the following integrals:

1. **First set of integrals**:

$$\int_{t_{j-1}}^{t_j} \left(s - t_{j-\frac{1}{2}} \right) (t_{n+\sigma} - s)^{-\beta} ds = \frac{k^{2-\beta}}{1-\alpha} b_{n-j}^\sigma,$$

   where
   $b_l^\sigma = \frac{1}{2-\beta} \left((l+\sigma+1)^{2-\beta} - (l+\sigma)^{2-\beta} \right) - \frac{1}{2} \left((l+\sigma+1)^{1-\beta} + (l+\sigma)^{1-\beta} \right).$
2. **Second set of integrals**:

$$\int_{t_{j-1}}^{t_j} (t_{n+\sigma} - s)^{-\beta} ds = \frac{k^{1-\beta}}{1-\beta} d_{n+\sigma-j,\beta}, \quad (10)$$

   with, for all $s > 0$, $d_{s,\beta}$ is given by $d_{s,\beta} = (s+1)^{1-\beta} - s^{1-\beta}$.
3. **Third set of integrals**:

$$\int_{t_n}^{t_{n+\sigma}} (t_{n+\sigma} - s)^{-\beta} ds = \frac{k^{1-\beta}}{1-\beta} \sigma^{1-\beta}. \quad (11)$$

We then obtained approximation for the fractional derivative $\partial_t^\beta \Phi(t_{n+\sigma})$ using (9)–(11)

$$\begin{aligned}&\frac{1}{\Gamma(1-\beta)} \left(\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (t_{n+\sigma} - s)^{-\beta} \left(\Pi_{2,j}\Phi(s) \right)' ds + \frac{k^{1-\beta}}{1-\beta} \sigma^{1-\beta} \partial^1 \Phi(t_{n+1}) \right) \\ &= \frac{k^{1-\beta}}{\Gamma(2-\beta)} \left(\sum_{j=1}^{n} \partial^1 \Phi(t_j) d_{n+\sigma-j,\beta} + \partial^2 \Phi(t_{j+1}) k b_{n-j}^\sigma + \sigma^{1-\beta} \partial^1 \Phi(t_{n+1}) \right).\end{aligned}$$

This gives, after re-ordering the sum (see [1, (27)-(28), Page 429])

$$\partial_t^\beta \Phi(t_{n+\sigma}) \approx \frac{k^{1-\beta}}{\Gamma(2-\beta)} \sum_{j=0}^{n} c_{n-j}^{\sigma,n} \partial^1 \Phi(t_{j+1}), \tag{12}$$

where $c_0^{\sigma,0} = \sigma^{1-\beta}$ and for all $n \geq 1$

$$\begin{aligned} &c_0^{\sigma,n} = \sigma^{1-\beta} + b_0^\sigma, \quad c_j^{\sigma,n} = d_{j+\sigma-1,\beta} + b_j^\sigma - b_{j-1}^\sigma, \quad \forall j \in [\![1, n-1]\!], \\ &c_n^{\sigma,n} = d_{n+\sigma-1,\beta} - b_{n-1}^\sigma. \end{aligned} \tag{13}$$

Let us denote

$$\lambda_j^{n+1} = \frac{c_{n-j}^{\sigma,n}}{k^\beta \Gamma(2-\beta)}. \tag{14}$$

The following lemma summarizes some properties of the approximation given by (12). Some of these results are proved in [1] whereas the other ones can be justified using the explicit form (14).

Lemma 1 (Some results concerning the time discretization, cf. [1,6]**).** *Let $\beta \in (0,1)$ be given and λ_j^{n+1} be defined by (14). Then the following results hold:*

1. Properties of the coefficients λ_j^{n+1}**, cf.** [1, Lemma 4, Page 431]**.**

$$\lambda_n^{n+1} > \lambda_{n-1}^{n+1} > \ldots > \lambda_0^{n+1} > \lambda_0 = \frac{1}{2T^\alpha \Gamma(1-\beta)}, \tag{15}$$

$$\sum_{j=0}^{n} k\lambda_j^{n+1} \leq \frac{T^{1-\beta}}{\Gamma(2-\beta)}, \qquad k\sum_{n=1}^{N} \lambda_1^{n+1} \leq \frac{(4-\beta)T^{1-\beta}}{\Gamma(3-\beta)}, \tag{16}$$

and for all $j \in [\![0, n]\!]$ and $i \in [\![0, m]\!]$ such that $n - j = m - i$, $i \neq 0$, and $j \neq 0$, we have $\lambda_j^{n+1} = \lambda_i^{m+1}$.

2. Stability result, cf. [1, Corollary 1, Page 427]**.** *For all $\left(\beta^j\right)_{j=0}^{N+1} \in \mathbb{R}^{N+2}$, for any $n \in [\![0, N+1]\!]$:*

$$\left(\sigma\beta^{n+1} + (1-\sigma)\beta^n\right) \sum_{j=0}^{n} \lambda_j^{n+1}(\beta^{j+1} - \beta^j) \geq \frac{1}{2} \sum_{j=0}^{n} \lambda_j^{n+1} \left((\beta^{j+1})^2 - (\beta^j)^2\right). \tag{17}$$

3. Consistency result, cf. [1, Lemma 2, Page 429]**.** *For any $\Phi \in \mathcal{C}^3([0,T])$:*

$$\left| \partial^\beta \Phi(t_{n+\sigma}) - \sum_{j=0}^{n} k\lambda_j^{n+1} \partial^1 \Phi(t_{j+1}) \right| \leq Ck^{3-\beta} \left| \Phi^{(3)} \right|_{\mathcal{C}([0,T])}. \tag{18}$$

4 Principles of the Scheme

The principles of the scheme we want to present are:
1. First step. Taking the "fractional mesh point" $t = t_{n+\sigma}$, with σ is given by (8) and $\beta = \alpha - 1$, in (1) and using the general formula (2) implies that (which gives $\partial_t^\alpha u(t) = \partial_t^{\alpha-1}(u_t)$)

$$\partial_t^{\alpha-1}\overline{u}(t_{n+\sigma}) - \Delta u(t_{n+\sigma}) = f(t_{n+\sigma}) \quad \text{and} \quad \overline{u} = u_t. \tag{19}$$

2. Second step. Approximation of $\partial_t^{\alpha-1}\overline{u}(t_{n+\sigma})$, which is the first term in (19), can be deduced from (18) by choosing $\beta = \alpha - 1$ and $\Phi(t) = \overline{u}(t) = u_t$:

$$\partial_t^{\alpha-1}\overline{u}(t_{n+\sigma}) = \sum_{j=0}^{n} k\lambda_j^{n+1}\partial^1\overline{u}(t_{j+1}) + \mathbb{T}_1^{n+1}(\overline{u}), \tag{20}$$

where $|\mathbb{T}_1^{n+1}(\overline{u})| \le Ck^{4-\alpha}\left|u^{(4)}\right|_{\mathcal{C}([0,T])}$.
3. Third step. Approximation of the first equation of (19). We have, thanks to a convenient Taylor expansion

$$u^{n+\sigma} = \sigma u(t_{n+1}) + (1-\sigma)u(t_n) = u(t_{n+\sigma}) + \mathbb{T}_2^{n+1}, \tag{21}$$

where $|\mathbb{T}_2^{n+1}| \le \dfrac{k^2}{2}\|u\|_{\mathcal{C}^2([0,T])}$. From (19)–(21), we deduce that

$$\sum_{j=0}^{n} k\lambda_j^{n+1}\partial^1\overline{u}(t_{j+1}) - \Delta u^{n+\sigma} = f(t_{n+\sigma}) + \mathbb{T}_3^{n+1}, \tag{22}$$

where $|\mathbb{T}_3^{n+1}| \le Ck^2\|u\|_{\mathcal{C}^4([0,T])}$.
4. Fourth step. Approximation of the second equation of (19) when $n \ge 1$. The derivative $u_t(t_{n+\sigma})$ is approximated (to get the stability property) by $\dfrac{(2\sigma+1)u(t_{n+1}) - 4\sigma u(t_n) + (2\sigma-1)u(t_{n-1})}{2k}$. Using a suitable Taylor expansion yields

$$\frac{(2\sigma+1)u(t_{n+1}) - 4\sigma u(t_n) + (2\sigma-1)u(t_{n-1})}{2k} = \overline{u}(t_{n+\sigma}) + \mathbb{T}_4^{n+1}, \tag{23}$$

where $|\mathbb{T}_4^{n+1}| \le Ck^2\|u\|_{\mathcal{C}^3([0,T])}$.

The system (19) is used for instance in [8] to establish a finite difference scheme in one space dimension. Such scheme is based on a Crank-Nicolson method.

5 Formulation of a Second Order Time Accurate Finite Volume Scheme and Statement of Its Convergence

After having explained the principles of the finite volume scheme for problem (1)–(5), we are able now to set its definition. We will denote by $(\cdot,\cdot)_{L^2(\Omega)}$ the $L^2(\Omega)$-inner product and by $v^{n+\sigma}$ the barycentric element given by $v^{n+\sigma} = \sigma v^{n+1} + (1-\sigma)v^n$.

Definition 1 (Definition of a finite volume scheme for (1)–(5)). *Let $\langle \cdot, \cdot \rangle$ be the inner product given by (7):*
1. Discretization of initial conditions (4)*: Find $u_{\mathcal{D}}^0, \overline{u}_{\mathcal{D}}^0 \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$*

$$\langle u_{\mathcal{D}}^0, v \rangle_F = -\left(\Delta u^0, \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)} \quad \text{and} \quad \langle \overline{u}_{\mathcal{D}}^0, v \rangle_F = -\left(\Delta u^1, \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)}. \tag{24}$$

2. Discretization of Eq. (22)*: For any $n \in [\![0, N]\!]$, find $u_{\mathcal{D}}^{n+1}, \overline{u}_{\mathcal{D}}^{n+1} \in \mathcal{X}_{\mathcal{D},0}$ such that*

$$\sum_{j=0}^{n} \lambda_j^{n+1} \left(\Pi_{\mathcal{M}}(\overline{u}_{\mathcal{D}}^{j+1} - \overline{u}_{\mathcal{D}}^{j}), \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)} + \langle u_{\mathcal{D}}^{n+\sigma}, v \rangle_F$$
$$= \left(f(t_{n+\sigma}), \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)}, \quad \forall \in \mathcal{X}_{\mathcal{D},0}. \tag{25}$$

3. Discretization of the second equation of (19)*:*

$$\overline{u}_{\mathcal{D}}^{\frac{1}{2}} = \partial^1 u_{\mathcal{D}}^1 \quad \text{and} \tag{26}$$
$$\overline{u}_{\mathcal{D}}^{n+\sigma} = \frac{1}{2k}\left((2\sigma+1)u_{\mathcal{D}}^{n+1} - 4\sigma u_{\mathcal{D}}^{n} + (2\sigma-1)u_{\mathcal{D}}^{n-1}\right), \quad \forall n \in [\![1, N]\!].$$

We now state one of the main results of this note, that is the convergence of scheme (24)–(26).

Theorem 1 (Error estimates for scheme (24)–(26)). *Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N} \setminus \{0\}$. Assume that the solution of (1)–(5) satisfies $u \in \mathcal{C}^4([0,T]; \mathcal{C}^2(\overline{\Omega}))$ and $\theta_{\mathcal{D}}$ satisfies $\theta \geq \theta_{\mathcal{D}}$. Let $\nabla_{\mathcal{D}}$ be the discrete gradient defined as in (6) and $\langle \cdot, \cdot \rangle$ be the inner product given by (7). Let $k = \frac{T}{N+1}$, with $N \in \mathbb{N}^\star$, and denote by $t_n = nk$, for $n \in [\![0, N+1]\!]$. Let σ be defined by (8) with $\beta = \alpha - 1$. For any $n \in [\![0, N]\!]$, for any $j \in [\![0, n]\!]$, we define the coefficients λ_j^{n+1} as in (14).*
Then there exists a unique solution $\left(\overline{u}_{\mathcal{D}}^n\right)_{n=0}^{N+1}, \left(u_{\mathcal{D}}^n\right)_{n=0}^{N+1} \in \mathcal{X}_{\mathcal{D},0}^{N+2}$ for scheme (24)–(26) and the following error estimates in $L^\infty(H^1)$ and $H^1(L^2)$ discrete semi-norms hold:

$$\max_{n=0}^{N+1} \|\nabla_{\mathcal{D}} u_{\mathcal{D}}^n - \nabla u(t_n)\|_{L^2(\Omega)^d} + \left(\sum_{n=0}^{N+1} k \|\Pi_{\mathcal{M}} \overline{u}_{\mathcal{D}}^n - u_t(t_n)\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}$$
$$\leq C(h_{\mathcal{D}} + k^2)\|u\|_{\mathcal{C}^4(0,T;\mathcal{C}^2(\overline{\Omega}))}. \tag{27}$$

The proof of Theorem 1 is based on the following new a priori estimate result:

Theorem 2 (A priori estimate for the discrete problem) *Under the same hypotheses of Theorem 1, assume that there exists $\left(\overline{\eta}^n\right)_{n=0}^{N+1}, \left(\eta^n\right)_{n=0}^{N+1} \in \left(\mathcal{X}_{\mathcal{D},0}\right)^{N+2}$ such that $\eta_{\mathcal{D}}^0 = \overline{\eta}^0 = 0$ and for all $n \in [\![0, N]\!]$*

$$\sum_{j=0}^{n} \lambda_j^{n+1} \left(\Pi_{\mathcal{M}}(\overline{\eta}_{\mathcal{D}}^{j+1} - \overline{\eta}_{\mathcal{D}}^{j}), \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)} + \langle \eta_{\mathcal{D}}^{n+\sigma}, v \rangle_F = \left(\mathcal{S}^{n+1}, \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)}, \tag{28}$$

where $\mathcal{S}^{n+1} \in L^2(\Omega)$, *for all* $n \in [\![0, N]\!]$. *Then, the following estimate holds:*

$$\max_{n=0}^{N+1} \|\nabla_{\mathcal{D}} \eta_{\mathcal{D}}^n\|_{L^2(\Omega)^d} + \left(\sum_{n=0}^{N+1} k\|\Pi_{\mathcal{M}} \overline{\eta}_{\mathcal{D}}^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \le C\overline{\mathcal{S}}, \tag{29}$$

where

$$\overline{\mathcal{S}} = \max_{n=1}^{N} \left\| \nabla_{\mathcal{D}} \left(\overline{\eta}_{\mathcal{D}}^{n+\sigma} - \frac{(2\sigma+1)\eta_{\mathcal{D}}^{n+1} - 4\sigma\eta^n + (2\sigma-1)\eta_{\mathcal{D}}^{n-1}}{2k} \right) \right\|_{L^2(\Omega)^d} + \max_{n=0}^{N} \|\mathcal{S}^{n+1}\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}}(\partial^1 \eta_{\mathcal{D}}^1 - \overline{\eta}_{\mathcal{D}}^{\frac{1}{2}})\|_{L^2(\Omega)^d} \tag{30}$$

An Overview on the Proof of Theorem 2: In addition to Lemma 1, the proof of Theorem 2 is based on the following inequality:

$$\langle \sigma\eta_{\mathcal{D}}^{n+1} + (1-\sigma)\eta_{\mathcal{D}}^n, (2\sigma+1)\eta_{\mathcal{D}}^{n+1} - 4\sigma\eta_{\mathcal{D}}^n + (2\sigma-1)\eta_{\mathcal{D}}^{n-1}\rangle_F \ge \mathbb{E}^{n+1} - \mathbb{E}^n, \tag{31}$$

where

$$\mathbb{E}^{n+1} = \frac{2\sigma^2+\sigma-1}{2}\|\nabla_{\mathcal{D}}(\eta_{\mathcal{D}}^{n+1} - \eta_{\mathcal{D}}^n)\|_{L^2(\Omega)^d}^2 + \frac{2\sigma+1}{2}\|\nabla_{\mathcal{D}}\eta_{\mathcal{D}}^{n+1}\|_{L^2(\Omega)^d}^2 - \frac{2\sigma-1}{2}\|\nabla_{\mathcal{D}}\eta_{\mathcal{D}}^n\|_{L^2(\Omega)^d}^2.$$

In addition to this $\mathbb{E}^{n+1} \ge \dfrac{1}{2\sigma}\|\nabla_{\mathcal{D}}\eta_{\mathcal{D}}^{n+1}\|_{L^2(\Omega)^d}^2$. However, Theorem 2 demands a rather longer proof. We will detail this in a future paper. □

Sketch of the proof of Theorem 1

The uniqueness for schemes (24) stems from the fact that $\|\nabla_{\mathcal{D}} \cdot \|_{L^2(\Omega)^d}$ is a norm on $\mathcal{X}_{\mathcal{D},0}$. This uniqueness implies the existence since the matrix involved is square. The uniqueness for scheme (25) with (26) can be justified using a priori estimate (29). This uniqueness implies the existence since the matrix involved in (25) is square. To prove estimate (27), we compare (24)–(26) with the following auxiliary schemes: For any $n \in [\![0, N+1]\!]$, find $\Xi_{\mathcal{D}}^n, \Upsilon_{\mathcal{D}}^n \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\langle \Xi_{\mathcal{D}}^n, v\rangle_F = -\left(\Delta u(t_n), \Pi_{\mathcal{D}} v\right)_{L^2(\Omega)} \quad \text{and} \quad \langle \Upsilon_{\mathcal{D}}^n, v\rangle_F = -\left(\Delta u_t(t_n), \Pi_{\mathcal{D}} v\right)_{L^2(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{32}$$

Taking $n = 0$ in (32), using the fact that $u(0) = u^0$ and $u_t(0) = u^1$ (subject of (4)), and comparing with scheme (24), we get $\overline{\eta}_{\mathcal{D}}^0 = \eta_{\mathcal{D}}^0 = 0$, where, for all $n \in [\![0, N+1]\!]$, $\overline{\eta}_{\mathcal{D}}^n, \eta_{\mathcal{D}}^n \in \mathcal{X}_{\mathcal{D},0}$ are given by $\eta_{\mathcal{D}}^n = u_{\mathcal{D}}^n - \Xi_{\mathcal{D}}^n$ and $\overline{\eta}_{\mathcal{D}}^n = \overline{u}_{\mathcal{D}}^n - \Upsilon_{\mathcal{D}}^n$.

First step: Comparison between (u, u_t) **and** $(\Xi_{\mathcal{D}}^n, \Upsilon_{\mathcal{D}}^n)$**.** We have (see [4,5])

$$\|\partial^1 u(t_n) - \Pi_{\mathcal{M}}\partial^1 \Xi_{\mathcal{D}}^n\|_{L^2(\Omega)} + \|\nabla u(t_n) - \nabla_{\mathcal{D}}\Xi_{\mathcal{D}}^n\|_{L^2(\Omega)^d} \le Ch_{\mathcal{D}}\|u\|_{\mathcal{C}^1(0,T;\mathcal{C}^2(\overline{\Omega}))}. \tag{33}$$

and

$$\|u_t(t_n) - \Pi_{\mathcal{M}}\Upsilon_{\mathcal{D}}^n\|_{L^2(\Omega)} + \|\partial^1\left(u_t(t_n) - \Pi_{\mathcal{M}}\Upsilon_{\mathcal{D}}^n\right)\|_{L^2(\Omega)} \le Ch_{\mathcal{D}}\|u\|_{\mathcal{C}^2(0,T;\,\mathcal{C}^2(\overline{\Omega}))}. \tag{34}$$

Second step: Comparison between $(\Xi_{\mathcal{D}}^n, \Upsilon_{\mathcal{D}}^n)$ **and** $(u_{\mathcal{D}}^n, \overline{u}_{\mathcal{D}}^n)$**.** From (25), (22), and (32), we deduce that

$$\sum_{j=0}^{n}\lambda_j^{n+1}\left(\Pi_{\mathcal{M}}(\overline{\eta}_{\mathcal{D}}^{j+1} - \overline{\eta}_{\mathcal{D}}^{j}), \Pi_{\mathcal{M}}v\right)_{L^2(\Omega)} + \left(\nabla_{\mathcal{D}}\eta_{\mathcal{D}}^{n+\sigma}, \nabla_{\mathcal{D}}v\right)_{L^2(\Omega)^d}$$
$$= \left(\mathcal{S}^{n+1}, \Pi_{\mathcal{M}}v\right)_{L^2(\Omega)}, \tag{35}$$

where $\mathcal{S}^{n+1} = \sum_{j=0}^{n} k\lambda_j^{n+1}\partial^1\left(u_t(t_{j+1}) - \Pi_{\mathcal{M}}\Upsilon_{\mathcal{D}}^{j+1}\right) - \mathbb{T}_3^{n+1}$. Using the a priori estimate (29) yields

$$\max_{n=0}^{N+1}\|\nabla_{\mathcal{D}}\eta_{\mathcal{D}}^n\|_{L^2(\Omega)^d} + \left(\sum_{n=0}^{N+1} k\|\Pi_{\mathcal{M}}\overline{\eta}_{\mathcal{D}}^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \le C\overline{\mathcal{S}}, \tag{36}$$

where

$$\overline{\mathcal{S}} = \max_{n=1}^{N}\left\|\nabla_{\mathcal{D}}\left(\overline{\eta}_{\mathcal{D}}^{n+\sigma} - \frac{(2\sigma+1)\eta_{\mathcal{D}}^{n+1} - 4\sigma\eta^n + (2\sigma-1)\eta_{\mathcal{D}}^{n-1}}{2k}\right)\right\|_{L^2(\Omega)^d}$$
$$+ \max_{n=0}^{N}\|\mathcal{S}^{n+1}\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}}(\partial^1\eta_{\mathcal{D}}^1 - \overline{\eta}_{\mathcal{D}}^{\frac{1}{2}})\|_{L^2(\Omega)^d}. \tag{37}$$

Using the triangle inequality, (16), and (34) yields (recall that $\mathbb{T}_3^{n+1}$ is of order two, see (22)) $\mathcal{S} \le C(h_{\mathcal{D}} + k^2)\|u\|_{\mathcal{C}^4(0,T;\,\mathcal{C}^2(\overline{\Omega}))}$. Using (26) implies that $\overline{\eta}_{\mathcal{D}}^{\frac{1}{2}} - \partial^1\eta_{\mathcal{D}}^1 = -\Upsilon_{\mathcal{D}}^{\frac{1}{2}} + \partial^1\Xi_{\mathcal{D}}^1$. On the other hand, using (32) implies that $\left(\nabla_{\mathcal{D}}\left(\partial^1\Xi_{\mathcal{D}}^1 - \Upsilon_{\mathcal{D}}^{\frac{1}{2}}\right), \nabla_{\mathcal{D}}v\right)_{L^2(\Omega)^d} = -\left(\Delta\left(\partial^1 u(t_1) - u_t^{\frac{1}{2}}\right), \Pi_{\mathcal{M}}v\right)_{L^2(\Omega)}$. By taking $v = \partial^1\Xi_{\mathcal{D}}^1 - \Upsilon_{\mathcal{D}}^{\frac{1}{2}}$ in this equation and using the Cauchy Schwarz inequality and the Poincaré inequality [5, Lemma 5.4] imply that $\left\|\nabla_{\mathcal{D}}\left(\overline{\eta}_{\mathcal{D}}^{\frac{1}{2}} - \partial^1\eta_{\mathcal{D}}^1\right)\right\|_{L^2(\Omega)^d} \le Ck^2\|u\|_{\mathcal{C}^3(0,T;\,\mathcal{C}^2(\overline{\Omega}))}$. In the same manner, we justify that the first term in (37) is bounded above by $Ck^2\|u\|_{\mathcal{C}^3(0,T;\,\mathcal{C}^2(\overline{\Omega}))}$. These estimates for the terms of (37) and estimate (36) imply that $\max_{n=0}^{N+1}\|\nabla_{\mathcal{D}}\eta_{\mathcal{D}}^n\|_{L^2(\Omega)^d}$ and $\left(\sum_{n=0}^{N+1} k\|\Pi_{\mathcal{M}}\overline{\eta}_{\mathcal{D}}^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}$ are bounded above by $C(h_{\mathcal{D}}+k^2)\|u\|_{\mathcal{C}^4(0,T;\,\mathcal{C}^2(\overline{\Omega}))}$. This with the triangle inequality and estimates (33)–(34) imply the required estimate (27). This completes the proof of Theorem 1. ◻

6 Some Concluding Remarks and Perspectives

Using an equivalent system of low order equations for TFDWE, we established a second order time accurate finite volume scheme using the discrete gradient of [5]. The time discretization uses the approximation of Caputo derivative of order $0 < \beta < 1$ developed in [1,6]. We *sketched* a proof for the convergence under the *strong regularity assumption* $\mathcal{C}^4(\mathcal{C}^2)$. This regularity can be weakened to $\mathcal{C}^3(H^2)$ in the particular cases when $d = 2$ or $d = 3$, see [5, Remark 4.9, Pages 1033–1034]. Strong regularity assumptions are usually needed when we would like to improve the convergence, see for instance the regularity $\mathcal{C}^3(\mathcal{C}^4)$ in [1, Lemma 5]. The convergence stated in this note includes a convergence in $L^\infty(H^1)$ and $H^1(L^2)$ discrete semi-norms. As mentioned in Abstract and Introduction, the present notes improve [3]. We plan in the near future to detail these notes and to address for instance the technique of graded meshes to get high order approximations.

References

1. Alikhanov, A.A.: A new difference scheme for the fractional diffusion equation. J. Comput. Phys. **280**, 424–438 (2015)
2. Bradji, A.: Convergence order of gradient schemes for time-fractional partial differential equations. C. R. Math. Acad. Sci. Paris **356**(4), 439–448 (2018)
3. Bradji, A.: Some convergence results of a multi-dimensional finite volume scheme for a time-fractional diffusion-wave equation. In: Cancès, C., Omnes, P. (eds.) FVCA 2017. Springer Proceedings in Mathematics & Statistics, vol. 199, pp. 391–399. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-57397-7_32
4. Bradji, A.: An analysis of a second-order time accurate scheme for a finite volume method for parabolic equations on general nonconforming multidimensional spatial meshes. Appl. Math. Comput. **219**(11), 6354–6371 (2013)
5. Eymard, R., Gallouët, T., Herbin, R.: Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. IMA J. Numer. Anal. **30**(4), 1009–1043 (2010)
6. Gao, G.-H., Sun, Z.-Z., Zhang, H.-W.: A new fractional numerical differentiation formula to approximate the Caputo fractional derivative and its applications. J. Comput. Phys. **259**, 33–50 (2014)
7. Jin, B., Lazarov, R., Zhou, Z.: Two fully discrete schemes for fractional diffusion and diffusion-wave equations with nonsmooth data. SIAM J. Sci. Comput. **38**, A146–A170 (2016)
8. Sun, Z.-Z., Wu, X.: A fully discrete difference scheme for a diffusion-wave system. Appl. Numer. Math. **56**, 193–209 (2006)