# 10

# Informative Versus Persuasive Advertising in a Dynamic Hotelling Monopoly

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## 10.1 Introduction

The analysis of the optimal behaviour of a monopolist in a dynamic model dates back to the pioneering contributions of Evans (1924), Tintner (1937), Eisner and Strotz (1963) and Gould (1968), which mostly focussed on pricing and investment decisions.[1] The building blocks (if not the earliest contributions) of the static approach to monopoly in discrete choice models are Mussa and Rosen (1978) for vertical differentiation and Bonanno (1987) for horizontal differentiation. Both deal with optimal product proliferation, and while Mussa and Rosen (1978) illustrate the well-known problem of downward quality distortion due to the firm's

[1]For an overview of optimal control or dynamic programming approaches to dynamic monopoly, see Lambertini (2018, ch. 2). For more on technical details, see Chiang (1992).

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S. Colombo (ed.), *Spatial Economics Volume I*,
https://doi.org/10.1007/978-3-030-40098-9_10

intention to preserve its profit margin in the highest quality niche, Bonanno (1987) outlines the mechanism of symmetric product proliferation aiming at segmenting the market, as a form of spatial price discrimination.

The relatively scant literature on the dynamic analysis of a monopolistic industry *à la* Hotelling (1929) has investigated the issues of network externalities (Artle and Averous 1973; Dhebar and Oren 1985; Lambertini and Orsini 2004; Rohlfs 1974, *inter alia*), advertising (Lambertini 2005), product development (Lambertini 2007) and productive capacity accumulation (Lambertini 2009).

My aim in the present paper is to offer a view of different types of advertising campaigns in a dynamic Hotelling monopoly, in which neither one can be directly traced back to the classical approaches of Vidale and Wolfe (1957) and Nerlove and Arrow (1962), in particular as far as the formalisation of advertising campaigns is concerned. Here, the modelling approach will alternatively focus on informative versus persuasive advertising efforts, where by informative advertising it is meant that the monopolist aims at increasing the density of consumers at every point along the linear city, while by persuasive advertising it is meant that the advertising effort is devoted to increasing their reservation price. In both cases, partial market coverage is assumed and the magnitude not being targeted is a constant parameter.

The first problem can be solved via the method of dynamic programming, while the second must necessarily be coped with as an optimal control one, since its form does not suggest any plausible shape for the value function. In both cases, however, the existence of a single saddle-point equilibrium is analytically characterised. Then, the steady-state performances of the firm are comparatively evaluated, to find out that the monopolist's preferences about the nature of the advertising campaign crucially depend on the set of initial conditions.

The remainder of the chapter is organised as follows. The features of the two models are laid out in Sect. 10.2. Sections 10.3 and 10.4 illustrate the analysis of the two cases, which are then compared in Sect. 10.5. Concluding remarks are in Sect. 10.6.

## 10.2 The Setup

We model the optimal dynamic behaviour of a dynamic monopolist operating over continuous time $t \in [0, \infty)$ in a Hotelling (1929) linear city under partial market coverage, in which the firm, in addition to the price-quantity pair, may choose between informative and persuasive advertising to expand its demand basin or enhance consumers' willingness to pay for its product. For the time being, the explicit indication of the time argument will be omitted—for a reason that will become evident very soon.

Each consumer at $x \in [0, 1]$ is characterised by a linear-quadratic preference structure

$$U = s - p - (x - 1/2)^2 \tag{10.1}$$

where $s > 0$ is gross surplus (or the reservation price), $p$ is the mill price and $(x - 1/2)^2$ is the disutility of transportation associated with reaching the firm optimally located in the middle of the linear city, along which there are $d$ consumer at each point, so that $d$ measures also the total mass of the population of consumers.

On the basis of the assumption of partial market coverage, the utility of the two marginal consumers symmetrically located to the left and right of 1/2 must be nil, and therefore monopoly price must be equal to

$$p_M = s - (x - 1/2)^2 = s - (2x - 1)^2 /4 \tag{10.2}$$

while demand (or the extent of market coverage) is $q_M = d\,(2x - 1)$, admissible for all $x \in (1/2, 1]$. This amounts to saying that the monopolist chooses the optimal demand to maximise its appropriate objective function by identifying two marginal consumers enjoying zero surplus, that is, by choosing $x$ optimally.

The first scenario deals with informative advertising and relies on the idea that consumer density $d\,(t)$ be treated as a state variable obeying

$$\dot{d} = k\,(t) - \eta d\,(t) \tag{10.3}$$

where $k(t)$ is the firm's instantaneous advertising intensity aimed at attracting more costumers into the market. The presence of a constant decay rate $\eta > 0$ tells that, in the absence of advertising, the population of consumers shrinks as consumers are 'forgetful'.

The second scenario is a slightly modified version of Lambertini (2005). Here, persuasive advertising must convince customers to pay higher prices for the good being supplied, so that the relevant state variable is $s(t)$, obeying

$$\dot{s} = k(t) - \delta s(t) \tag{10.4}$$

in which the decay rate is $\delta$, again time-invariant and positive, but not necessarily equal to $\eta$. In both scenarios, the instantaneous cost of advertising investment is $\Gamma(t) = bk^2(t)$, where $b$ is a positive constant. Marginal cost is constant and, without further loss of generality, is posed equal to zero, in such a way that $\Gamma(t)$ is also the total instantaneous cost function.

In both settings, the firm has two controls and faces a single state. Quite interestingly, we are about to see that the first version of the dynamic problem, based upon (10.3), can be solved using the dynamic programming approach, that is, through the Hamilton-Jacobi-Bellman (HJB) equation by guessing a linear-quadratic value function, while the second version, based upon (10.4), cannot be treated in the same way (because its structure—in particular, the value function—does not lend itself to an intuitive guess, being not linear quadratic) and therefore must be solved as an optimal control problem on the basis of the Hamiltonian function (as in Lambertini 2005).

After the characterisation of the saddle-point equilibria of both models, the resulting steady-state magnitudes (prices, outputs, profits and advertising efforts) are compared in the space of states $(d, s)$ to show that the firm's preferences concerning the nature of the advertising campaign are not univocally defined, as the choice essentially depends upon the initial conditions of both states.

## 10.3 Informative Advertising

Here the monopolist uses advertising to attract additional consumers by increasing density $d(t)$ along the linear city, while the reservation price $s$ of the generic consumer remains constant. Accordingly, the relevant state equation is (10.3), and the firm's instantaneous profit function is

$$\pi(t) = p_M(t)\, q_M(t) - \Gamma(t) = \frac{s - [2x(t) - 1]^2}{4} \cdot d(t)\,[2x(t) - 1] - bk^2(t) \tag{10.5}$$

The firm has to solve the following problem:

$$\max_{x(t),k(t)} \Pi = \int_0^\infty \pi(t)\, e^{-\rho t} dt \tag{10.6}$$

s.t. (10.3), and the initial condition $d_0 = d(0) > 0$. Parameter $\rho > 0$ measures the constant discount rate. The Hamilton-Jacobi-Bellman (HJB) equation is the following:

$$\rho V(d(t)) = \max_{x(t),k(t)} \left\{ \pi(t) + V'(d(t)) \cdot \dot{d} \right\} \tag{10.7}$$

where $V(d(t))$ is the value function and $V'(d(t)) \equiv \partial V(d(t)) / \partial d(t)$ is its partial derivative w.r.t. the state variable.

From (10.7) we obtain the following first-order conditions (FOCs):

$$\begin{gathered} V'(d(t)) - 2bk(t) = 0 \\ d(t)\left[2s - \frac{3[2x(t) - 1]^2}{2}\right] = 0 \end{gathered} \tag{10.8}$$

yielding[2]

$$k^* = \frac{V'(d(t))}{2b}\,;\; x^* = \frac{1}{2} + \sqrt{\frac{s}{3}} \tag{10.9}$$

It is worth noting that the solution determining the extent of market coverage, $x^*$, is indeed static and replicates unmodified forever, while the optimal advertising effort is endogenously determined by the state at all times, through the partial derivative of the value function. Moreover, since $x^* \in (1/2, 1]$, in order to respect the initial assumption of partial market coverage, we have to restrain $s$ to the interval $(0, 3/4]$, outside which all consumers along the linear city would be able to buy the good supplied by the monopolist, irrespective of the level of consumer density.

Now we may stipulate $V(d(t)) = \varepsilon_1 d^2(t) + \varepsilon_2 d(t) + \varepsilon_3$, so that $V'(d(t)) = 2\varepsilon_1 d(t) + \varepsilon_2$. Plugging these and (10.9) into (10.7), the HJB equation can be simplified as follows:[3]

$$\frac{36\varepsilon_1[\varepsilon_1 - b(2\eta+\rho)]d^2 + 4\left[9\varepsilon_1\varepsilon_2 + b\left(4s\sqrt{3s} - 9\varepsilon_2(\eta+\rho)\right)\right]d + 9\left(\varepsilon_2^2 - 4b\rho\varepsilon_3\right)}{36b} = 0 \tag{10.10}$$

which gives rise to the system of Riccati equations:

$$\begin{aligned} 36\varepsilon_1[\varepsilon_1 - b(2\eta+\rho)] &= 0 \\ 9\varepsilon_1\varepsilon_2 + b\left(4s\sqrt{3s} - 9\varepsilon_2(\eta+\rho)\right) &= 0 \\ \varepsilon_2^2 - 4b\rho\varepsilon_3 &= 0 \end{aligned} \tag{10.11}$$

The above system has to be solved w.r.t. the triple of undetermined parameters $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, to obtain

$$\begin{aligned} &\varepsilon_3 = \frac{\varepsilon_2^2}{4b\rho}\,;\; \varepsilon_2 = -\frac{4bs\sqrt{s}}{3\sqrt{3}[\varepsilon_1 - b(\eta+\rho)]} \\ &\varepsilon_{11} = 0\,;\; \varepsilon_{12} = b(2\eta+\rho) \end{aligned} \tag{10.12}$$

[2]The remaining solution of the second FOC, $x = 1/2 - \sqrt{s/3}$, can be disregarded in view of the definition of $q_M$.

[3]Henceforth, the time argument will be omitted throughout the analysis of this case, for the sake of brevity.

Of course, given the linear-quadratic form of the model at hand, we have two solutions for $\varepsilon_1$, which can be alternatively substituted into the expression of the optimal investment effort

$$k^* = \frac{d\varepsilon_1}{b} - \frac{2s\sqrt{s}}{3\sqrt{3}\left[\varepsilon_1 - b\left(\eta + \rho\right)\right]} \tag{10.13}$$

to deliver the pair of linear feedback strategies:

$$k_1^* = \frac{2s\sqrt{s}}{3\sqrt{3}b\left(\eta + \rho\right)}$$
$$k_2^* = d\left(2\eta + \rho\right) - \frac{2s\sqrt{s}}{3\sqrt{3}b\eta} \tag{10.14}$$

The first, $k_1^*$, is the open-loop control which would obtain from the solution of the corresponding optimal control problem based upon the Hamiltonian function (and, as such, it is independent of the state at any time $t$), while the second, $k_2^*$, is a proper feedback strategy defined as a function of the state at all times. Either one can be inserted into (10.3) to impose stationarity and obtain the single steady-state level of the state variable:

$$d^{ss} = \frac{2s\sqrt{s}}{3b\eta\sqrt{3}\left(\eta + \rho\right)} \tag{10.15}$$

where the meaning of superscript $ss$ is intuitive.

The phase diagram drawn in Fig. 10.1 illustrates the stability properties of the state-control system (recall that the market variable has a quasi-static nature) and, given the sign of $\dot{d}$ above and below the steady-state advertising effort $k^{ss} = \eta d$, allows us to deduce that the state-independent open-loop control $k_1^*$ is indeed the stable one.

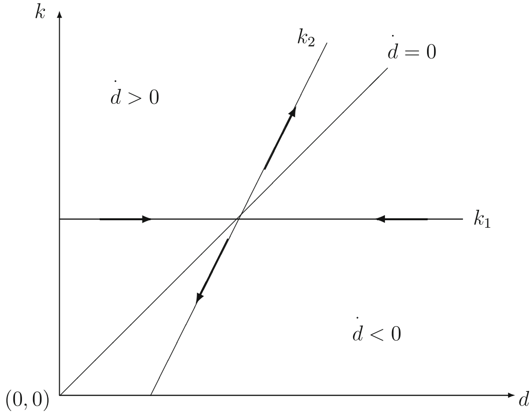


**Fig. 10.1** The phase diagram under informative advertising

The foregoing discussion boils down to the following:

**Proposition 10.1** *Assume* $s \in (0, 3/4]$. *If so, then there exists a unique saddle- point equilibrium at*

$$d^{ss} = \frac{2s\sqrt{s}}{3b\eta\sqrt{3}(\eta + \rho)};\ k^{ss} = \eta d^{ss};\ x^{ss} = \frac{1}{2} + \sqrt{\frac{s^{ss}}{3}}.$$

For later reference, we may also simplify the firm's profit function (10.5) in correspondence of the above steady-state coordinates, to obtain the level of steady-state profits:

$$\pi^{ss}(d) = \frac{4s^3(\eta + 2\rho)}{27b\eta(\eta + \rho)^2} \tag{10.16}$$

## 10.4 Persuasive Advertising

In this case, the state variable is the reservation price $s(t)$; consequently, the relevant state equation is (10.4). The monopolist's instantaneous profit function looks much the same as in the previous section, except that $d$ is an exogenous parameter:

$$\pi(t) = p_M(t)\, q_M(t) - \Gamma(t) = \frac{s(t) - [2x(t) - 1]^2}{4} \cdot d\,[2x(t) - 1] - bk^2(t) \tag{10.17}$$

The firm has to maximise the discounted profit flow

$$\max_{x(t),k(t)} \Pi = \int_0^{\infty} \pi(t)\, e^{-\rho t} dt \tag{10.18}$$

s.t. (10.4) and the initial condition $s_0 = s(0) > 0$.

It is easily ascertained that this problem cannot be treated via the dynamic programming approach, as the model is not defined in a linear-quadratic form and there is no intuitive guess about the shape of the value function appearing in the relevant HJB equation:

$$\rho V(s(t)) = \max_{x(t),k(t)} \left\{ \pi(t) + V'(s(t)) \cdot \dot{s} \right\} \tag{10.19}$$

The FOCs deliver the same expression for the optimal choice of the marginal consumer $x^*$ as in (10.9), except of course for the fact that the reservation price is the relevant state, and $k^* = V'(s(t)) / (2b)$. However, conjecturing a linear-quadratic value function $V(d(t)) = \zeta_1 s^2(t) + \zeta_2 s(t) + \zeta_3$ is not appropriate, as the simplified HJB equation reveals:

$$\frac{36\zeta_1[\zeta_1 - b(2\delta + \rho)]\, s^2 + 16\sqrt{3} bds\sqrt{s} + 36\zeta_2[\zeta_1 - b(\delta + \rho)]\, s + 9\left(\varepsilon_2^2 - 4b\rho\varepsilon_3\right)}{36b} = 0 \tag{10.20}$$

The reason is the presence of $s(t)\sqrt{s(t)}$, as we already know from (10.10). Consequently, one has to solve the optimal control problem relying on the Hamiltonian function:

$$\mathcal{H}(t) = e^{-\rho t} \left\{ \pi(t) + \lambda(t) \cdot \dot{s} \right\} \tag{10.21}$$

in this case written in current value, $\lambda(t) = \mu(t)e^{\rho t}$ being the 'capitalised' costate variable associated with the state dynamics, while $\mu(t)$ is the costate variable.

The resulting FOCs on controls are (the discount factor is omitted)

$$\frac{\partial \mathcal{H}(t)}{\partial x(t)} = \frac{d\left[4s(t) - 3(2x(t) - 1)^2\right]}{2} = 0 \tag{10.22}$$

$$\frac{\partial \mathcal{H}(t)}{\partial k(t)} = \lambda(t) - 2bk(t) = 0 \tag{10.23}$$

while the costate equation is

$$-\frac{\partial \mathcal{H}(t)}{\partial s(t)} = \dot{\lambda}(t) - \rho\lambda(t) \Rightarrow \tag{10.24}$$

$$\dot{\lambda}(t) = (\delta + \rho)\lambda(t) - d[2x(t) - 1]$$

Intuitively, $x^* = 1/2 + \sqrt{s/3}$ solves (10.22) once again. From (10.23), we obtain $\lambda^* = 2bk$ as well as the advertising control kinematics $\dot{k} = \dot{\lambda}/(2b)$ which, on the basis of (10.24) and $\lambda^*$, can be written in its final form as follows:

$$\dot{k} = k(\delta + \rho) - \frac{d}{b} \cdot \sqrt{\frac{s}{3}} \tag{10.25}$$

This, together with (10.4), constitutes the state-control system of the present optimal control problem. Its only solution identifies the steady-state point:

$$s^{ss} = \frac{d^2}{3b^2\delta^2(\delta + \rho)^2};\ k^{ss} = \delta s^{ss} \tag{10.26}$$

and the associated position of the marginal consumer to the r.h.s. of the firm is $x^{ss} = 1/2 + \sqrt{s^{ss}/3}$.

In order to evaluate the stability properties of the steady-state point $(s^{ss}, k^{ss})$, we have to examine the trace and determinant of the $2 \times 2$ Jacobian matrix associated with the state-control system:

$$J = \begin{bmatrix} \frac{\partial \dot{s}}{\partial s} = -\delta & \frac{\partial \dot{s}}{\partial k} = 1 \\ \frac{\partial \dot{k}}{\partial s} = -\frac{d}{2b\sqrt{3s}} & \frac{\partial \dot{k}}{\partial k} = \delta + \rho \end{bmatrix} \tag{10.27}$$

The trace is

$$\mathcal{T}(J) = \frac{\partial \dot{s}}{\partial s} + \frac{\partial \dot{k}}{\partial k} = \rho > 0 \tag{10.28}$$

and the determinant is

$$\Delta(J) = \frac{\partial \dot{s}}{\partial s} \cdot \frac{\partial \dot{k}}{\partial k} - \frac{\partial \dot{s}}{\partial k} \cdot \frac{\partial \dot{k}}{\partial s} = \frac{d}{2b\sqrt{3s}} - \delta(\delta + \rho) \tag{10.29}$$

which, posing $s = s^{ss}$, simplifies as $\Delta(J^{ss}) = -\delta(\delta + \rho)/2$. Consequently, we may formulate:

**Proposition 10.2** *Assume* $s \in (0, 3/4]$. *If so, then the unique steady-state equilibrium at*

$$s^{ss} = \frac{d^2}{3b^2\delta^2(\delta + \rho)^2}\,;\; k^{ss} = \delta s^{ss}\,;\; x^{ss} = \frac{1}{2} + \sqrt{\frac{s^{ss}}{3}}.$$

*is a saddle point.*

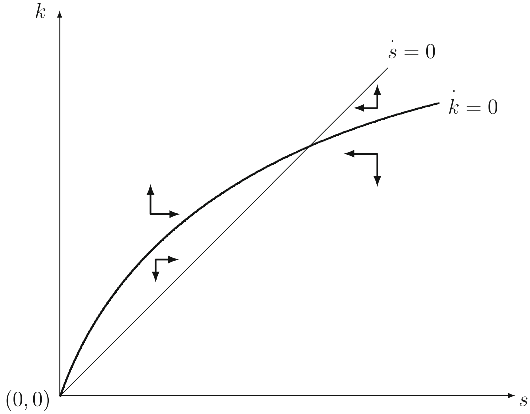


**Fig. 10.2** The phase diagram under persuasive advertising

The saddle-point stability property is illustrated by the arrows appearing in the phase diagram drawn in Fig. 10.2, where the concavity of the locus $\dot{k} = 0$ is also intuitively suggesting the impossibility of using the HJB equation to solve this case. Moreover, the phase diagram also implies that the origin at which $s = k = 0$ is unstable and may therefore be disregarded (in addition to be inadmissible, as $s = 0$ implies that the market does not exist).

The level of steady-state profits at $(s^{ss}, k^{ss})$ amounts to

$$\pi^{ss}(s) = \frac{d^4(\delta + 4\rho)}{27b^3\delta^3(\delta + \rho)^4} \tag{10.30}$$

## 10.5 Comparing Equilibria

Now we are in a position to comparatively assess the equilibrium performance of the firm in the two settings. To begin with, we may take a look at steady-state profits. As it appears from (10.16) and (10.30), $\pi^{ss}(d)$ contains $s$ as a parameter, while $\pi^{ss}(s)$ contains $d$. Hence, one might draw the conclusion that the profit comparison is problematic—to say the least—as the two problems considered in the foregoing analysis take either the consumer reservation price or density as given and endogenise the other magnitude as a state variable.

Yet, there is a sensible way out of this seemingly tricky conundrum which can be envisaged as follows. Since both cases require an exogenously given initial condition on the state, we may suppose that such initial level be also the relevant level of the same magnitude in the alternative scenario where either $d$ or $s$ is taken to be time-invariant, that is, a parameter. Once this standpoint is adopted, the issue of assessing the relative size of profit levels at the steady state becomes relatively easy to tackle.

The difference between profits (10.16) and (10.30) has the following feature:

$$\begin{aligned} & sign\left\{\pi^{ss}(d)-\pi^{ss}(s)\right\} \\ & \quad = sign\left\{4b^2s^3\delta^3(\delta+\rho)^4(\eta+2\rho)-d^4\eta(\delta+4\rho)(\eta+\rho)^2\right\} \end{aligned} \tag{10.31}$$

which involve a quartic polynomial in $d$. However, this can be treated (and easily solved) by posing $D = d^2$, whereby, since

$$\Psi \equiv 4b^2s^3\delta^3(\delta+\rho)^4(\eta+2\rho)-D^2\eta(\delta+4\rho)(\eta+\rho)^2 \tag{10.32}$$

is concave in $D$, the sign of $\pi^{ss}(d)-\pi^{ss}(s)$ is positive for all $D$ inside the interval identified by the roots of $\Psi = 0$, that is,

$$D_{\pm} = \pm\frac{2bs\delta(\delta+\rho)^2\sqrt{s\delta(\eta+2\rho)}}{(\eta+\rho)\sqrt{(\delta+4\rho)\eta}} \tag{10.33}$$

and since the smaller root is negative, $\pi^{ss}(d) > \pi^{ss}(s)$ for all $D \in (0, D_+)$ or, equivalently, for all $d \in \left(0, \sqrt{D_+}\right)$. To complement this result, one may also note that $D_+$ increases monotonically in $s$.

There remains to check whether $\sqrt{D_+}$ is larger or smaller than $d^{ss}$. It turns out that the sign of $\sqrt{D_+} - d^{ss}$ is independent of $s$, the reason being that both are defined as a multiple of $s\sqrt{s}$, in such a way that

$$
\begin{aligned}
& sign\left\{\sqrt{D_+} - d^{ss}\right\} \\
& \quad = sign\left\{9b^2\delta\eta\left(\delta+\rho\right)\sqrt{\delta\left(\eta+2\rho\right)} - \sqrt{3\eta\left(\delta+4\rho\right)}\right\} \qquad (10.34)
\end{aligned}
$$

so that $\sqrt{D_+} > d^{ss}$ for all

$$
b > \frac{\sqrt{3\eta\left(\delta+4\rho\right)}}{9b^2\delta\eta\left(\delta+\rho\right)\sqrt{\delta\left(\eta+2\rho\right)}} \equiv \overline{b} \qquad (10.35)
$$

and conversely. Hence, keeping in mind that a parameter in one setting is taken to coincide with the initial condition in the other setting, we may formulate the following:

**Corollary 10.3** *The relative size of steady-state profits $\pi^{ss}(d)$ and $\pi^{ss}(s)$ depends on the levels of initial conditions, $d_0$ and $s_0$, as well as the steepness of the instantaneous cost of advertising, measured by parameter $b$:*

- *if $b > \overline{b}$, then $\pi^{ss}(d) > \pi^{ss}(s)$ for all $d_0 \in (0, d^{ss})$;*
- *if instead $b \in \left(0, \overline{b}\right)$, then $\pi^{ss}(d) > \pi^{ss}(s)$ for all $d_0 \in \left(0, \sqrt{D_+(s_0)}\right)$ and conversely for all $d_0 \in \left(\sqrt{D_+(s_0)}, d^{ss}\right)$.*
- *Moreover, the threshold below which $\pi^{ss}(d) > \pi^{ss}(s)$ increases as $s_0$ increases, irrespective of its relative position w.r.t. $d^{ss}$.*

The above corollary can be spelt out more intuitively by saying that the richer is the generic consumer along the linear city, the more likely it becomes for the firm to find it preferable to invest in informative rather than persuasive advertising. Additionally, it appears that it is certainly so if the marginal cost of advertising is high enough. A plausible interpretation of this result may be found in a *quantity effect*, because monopoly output $q_M = d(2x-1)$ is linearly increasing in $d$ and $x$, but the equilibrium level of $x$ is concave in $s$, and this fact suggests that, all else equal (in particular, for any given $b$), informative advertising may turn out to be more profitable than persuasive advertising in a larger portion of the parameter constellation.

## 10.6 Concluding Remarks

The foregoing analysis has delved into the details of two alternative forms of advertising (informative or persuasive) in a Hotelling monopoly existing over an infinite horizon, under the assumption of partial market coverage. The stability analysis has analytically proved the existence of a single steady-state equilibrium enjoying the property of saddle-point stability in each of the two settings.

The exercise carried out on comparative profit evaluation at the steady state has shown that the relative performance of the two types of advertising is determined by the relative size of initial conditions on density and reservation price, respectively, with the former resulting relatively more effective than the latter, at least under the specific modelling strategy adopted here.

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