1

Classic Spatial Models

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1.1 Introduction

In this chapter, we illustrate some classic spatial models. Starting from the seminal paper of Hotelling (1929), space has become a crucial variable in the economic analysis of oligopolistic models.[1] The aim of this chapter is mainly pedagogical: we aim to provide a useful collection of the principal classic spatial models, by illustrating their characteristics and the main results. Indeed, classic spatial models are a flexible tool which adopts the

[1]Obviously, the importance of the spatial dimension has been well recognized even before Hotelling. For example, Thunen (1826), Launhardt (1885), Marshall (1890), and Weber (1909) developed relevant frameworks to understand the implications of space for consumers and firms' behavior. However, none of these models has been used for plenty of applications as the Hotelling one and its epigones.

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space dimension to model a wide array of economic questions, including industrial organization, regional science, and marketing. This is not an exhaustive review of spatial models. Indeed, hundreds and hundreds of spatial models have been developed by scholars; we have selectively chosen those contributions we believe represent the cornerstone of modern spatial economy, in order to provide a toolkit for those who are approaching this field for the first time.

Before starting, we briefly put forward the common characteristic of classic spatial models: classic spatial models do not want to represent "stylized facts", but, rather, to highlight and describe the forces that determine the choices of the firms or the consumers. Indeed, classic spatial models are often too simple to provide a good description for what happens in the real word, but they are sufficiently simple to capture which incentives are at work when the firms or the consumers take their decision. This is the main purpose of classic spatial models.

The rest of the chapter proceeds as follows. In Sect. 1.2 we introduce and describe the linear model. In Sect. 1.3, we consider the circular model. In Sect. 1.4 we describe some spatial models adopting price discrimination, whereas in Sect. 1.5 we introduce elastic demand. In Sect. 1.6 we discuss the "barbell" model. In Sect. 1.7 we consider a spatial model of vertical differentiation. Section 1.8 concludes.

1.2 The Linear Model

In this section, we describe the linear model, which is based on the work of Hotelling (1929). The aim of the model consists in providing a simple framework to describe product differentiation, that is, a situation where a slight decrease of the price of one firm does not determine an abrupt increase of the demand of that firm, but rather a gradual shift of demand. In fact, Hotelling was rather skeptical about Bertrand's (1883) criticism of Cournot (1838) equilibrium. Indeed, "in all [Bertrand's] illustrations of competition one merchant can take away his rival's entire business by undercutting his price ever so slightly. This discontinuities appear, though a discontinuity, like a vacuum, is abhorred by nature. More typical of real situations is the case in which the quantity sold by each merchant is a continuous function of two variables. His own price and his competitor's. Quite commonly a tiny increase in price by one seller will send only a few customers to the other" (Hotelling 1929, p. 44). Therefore, the linear model originates in order to show that price competition does not necessarily lead to the perfect competition outcome (the so-called Bertrand paradox). Intuitively, this happens because, once space is introduced, product differentiation arises, and this allows avoiding the Bertrand paradox. Nowadays, the linear (Hotelling) model is the most widely used model to describe oligopolistic competition between firms selling non-homogenous products.

Suppose there is a continuum of consumers located along a segment of length 1, from 0 to 1. The segment might have a "spatial" interpretation or a "product differentiation" interpretation. In the first case, the location of a consumer or a firm in the segment refers to the location in a strict physical sense (i.e., the consumer or the firm is really located at a certain point). In the second case, the segment is a metaphor of the product characteristic space. In this case, the location of a consumer represents the product's variety which is preferred by that consumer, whereas the location of a firm represents the product variety produced by that firm.

Consumers are uniformly distributed along the segment. Let $x \in [0, 1]$ indicate the location of each consumer on the segment. Each consumer buys just one or zero unit of good. That is, there is unit demand function. Suppose there are two firms, Firm A and Firm B, whose location is a and b, respectively. It is assumed, without loss of generality, that $0 \leq a \leq b \leq 1$ (in other words, Firm A is the firm which is located at the left). Furthermore, the firms cannot be located outside the segment. There are no production costs.

For the moment, we suppose that the locations of the firms are exogenous. In particular, we assume that $a = 0$ and $b = 1$.[2] The utility function of a consumer which is located at x and buys from Firms A and B is the following, respectively, $U_A = v - p_A - tx$ and $U_B = v - p_B - t(1 - x)$, where v is the reservation price of consumers,[3] p_A and p_B is the price set by Firms A and B, respectively, and $t > 0$

[2] It should be observed that the main purpose of spatial models is to derive *endogenously* the "locations" of firms. However, it might be useful to start with the case of exogenous locations.

[3] v is assumed to be sufficiently high so that the market is always covered in equilibrium.

is the unit transport cost sustained by the consumer when he goes to the firm's location to pick up the good (note that in the case of the product differentiation interpretation, this can be interpreted as a "disutility cost" deriving from purchasing a less-than-preferred product variety). It is important to note that the transport costs are linear in the distance. In what follows we derive the equilibrium prices. Suppose that the firms set simultaneously the price. First, observe that for any possible couple of prices (p_A, p_B), it is possible to determine a consumer whose location, say $\hat{x}$, is such that $U_A\left(p_A, p_B, \hat{x}\right) = U_B\left(p_A, p_B, \hat{x}\right)$. Therefore, consumer $\hat{x}$ is indifferent between buying from Firm A and from Firm B. In addition, all consumers located at the left of $\hat{x}$ buy from Firm A, and all consumers located at the right of $\hat{x}$ buy from Firm B (formally, if $x < (>)\ \hat{x}$, then $U_A\left(p_A, p_B, \hat{x}\right) > (<)\ U_B\left(p_A, p_B, \hat{x}\right)$). Therefore, the demand of Firm A is $\hat{x}$, and the demand of Firm B is $1 - \hat{x}$. By solving $U_A\left(p_A, p_B, \hat{x}\right) = U_B\left(p_A, p_B, \hat{x}\right)$, we get $\hat{x} = \frac{p_B - p_A}{2t} + \frac{1}{2}$. The profit functions are therefore $\pi_A = p_A\hat{x}$ and $\pi_B = p_B\left(1 - \hat{x}\right)$. By maximizing the profit functions, we have the following best-reply functions, $p_i\left(p_j\right) = \frac{p_j + t}{2}$, with $i, j = A, B$. By solving the system of best-reply functions, the equilibrium prices and profits follow: $p_i{}^* = t$ and $\pi_i * = \frac{t}{2}$. Therefore, when products are differentiated, the firms avoid the Bertrand paradox, that is, the prices do not fall to the marginal cost level, and profits are positive. Intuitively, space introduces product differentiation. Indeed, consumers do not perceive the products as homogenous: even if the two products are identical, each consumer, all else being equal, prefers the closer firm to save on transport costs. In this sense, t is a measure of product differentiation, and the higher is t, the higher are the equilibrium prices and profits.

In what follows, we discuss what happens when firms decide where to locate before setting prices. That is, we look for the locations emerging endogenously in the model. We assume the following two-stage game[4]:

1. Stage 1. The firms choose simultaneously where to locate.
2. Stage 2. The firms choose simultaneously the price.

[4]This is not the only possible timing. For example, one might consider a simultaneous choice of location and price. However, the sequential timing is more reasonable when one considers that it is often more difficult to modify the location/product characteristic rather than the price.

Due to the dynamic structure of the game, the appropriate solution concept is the Subgame Perfect Nash Equilibrium (Selten 1975). Therefore, we solve the model by proceeding by backward induction. In other words, first we find the equilibrium prices for *any* possible pair of locations. Then, by anticipating the second-stage equilibrium prices, we find the first-stage equilibrium locations. Unfortunately, when Hotelling wrote his contribution, game theory has not appeared yet. Therefore, it is not surprising that the main conclusion in terms of expected locations is not correct. In particular, Hotelling (1929) claims that the two firms are expected to engage in a fierce competition in order to obtain greater demand, so that they will end up choosing the same central location (1/2) ("they crowd together as closely as possible", p. 53). This conclusion, even if not correct (see later), is well known as the Minimum Differentiation Principle.

As mentioned above, the Minimum Differentiation Principle does not hold in the original framework of Hotelling (1929), as shown in the famous contribution of D'Aspremont et al. (1979), whose model is based on a simple variation of the Hotelling model (quadratic transportation costs rather than linear transportation costs). In particular, D'Aspremont et al. (1979) show that the Minimum Differentiation principle is invalid at the Hotelling conditions. The main intuition is based on the following argument: in the Hotelling model there are no Subgame Perfect Nash Equilibria; therefore it cannot be said that firms decide to locate in the middle of the segment.

To understand this non-existence result, consider Fig. 1.1, where the total cost of purchase (i.e., price plus the transport costs) of each consumer is represented, given the locations a and b. Note that the demand might be discontinuous. Indeed, suppose that p_A reduces so much that $p_A = p_B - t(b - a)$, that is, the consumer located at b is indifferent between the two firms (i.e., $\hat{x} = b$). Note that also all consumers such that $x > b$ are indifferent between the two firms. Therefore, if p_A reduces a bit further, there is a jump in the demand of Firm A, because now Firm A serves all consumers. As a consequence, the profits of Firm A are illustrated in Fig. 1.2. They are continuous in p_A until $p_A = p_B - t(b - a)$,

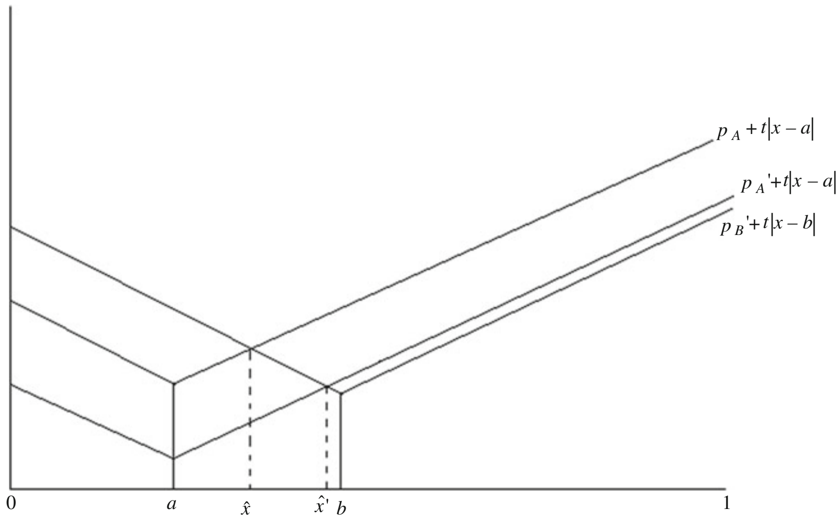

Fig. 1.1 Total cost of purchase

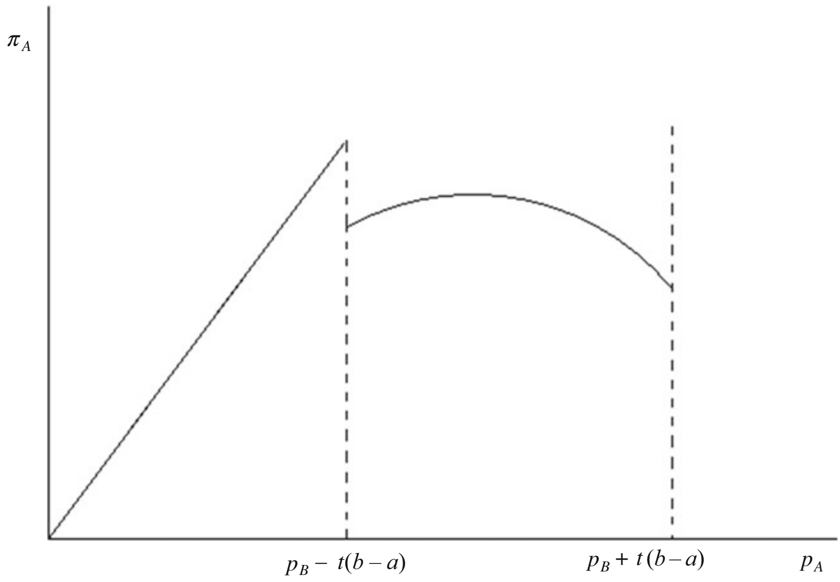

Fig. 1.2 The profits of Firm A

they are concave in p_A until $p_A = p_B + t(b - a)$, and then they are zero. Therefore, Firm A's profits are not *everywhere* continuous.

D'Aspremont et al. (1979) show that it is possible to find second-stage equilibrium prices only when the locations of firms satisfy certain conditions. In particular, the firms must be sufficiently distant from each other (alternatively, they must be located in the same point): if the firms are rather near to each other (but not located in the same point), there is no equilibrium in the second stage of the game. Intuitively, this happens for the following reason. If the two firms are quite near, each firm has a strong incentive to undercut the rival's price. Indeed, from Fig. 1.2 it can be observed that there are two local maxima, the first at $\tilde{p}_A = p_B - t(b - a) - \varepsilon$ and the second, say $\hat{p}_A(p_B)$, where the profit function is concave. In the first maximum, Firm A serves the whole market, and Firm B has no demand. So here there is no equilibrium, because Firm B would decrease the price to get a positive demand. Therefore, an equilibrium is possible if and only if $p_A* = \hat{p}_A(p_B*)$ and $p_B* = \hat{p}_B(p_A*)$. But this implies that $\hat{p}_i\left(p_j\right)$ must be a global maximum, and not just a local maximum, which in turn requires that the two firms are not too close to each other (intuitively, if the two firms are sufficiently distant, a focal firm should significantly reduce the price to serve the whole market, and thus this strategy is not profitable, that is, $\hat{p}_i\left(p_j\right)$ is a global maximum). On the other side, case $a = b$ is obvious: the two firms are undifferentiated, so the standard Bertrand argument applies and the prices are equal to the marginal costs.

Even more importantly, D'Aspremont et al. (1979) show that there are no Subgame Perfect Nash Equilibria in the original Hotelling framework. Indeed, if a and b satisfy the conditions for the existence of the price equilibrium in the second stage, the profits of Firm A (B) are increasing (decreasing) in a (b). Therefore, the two firms would like to move toward the center, but in this way, the locations end up not satisfying the conditions for the existence of the equilibrium prices in the second stage. At the same time, the pair $a = b$ cannot be an equilibrium, because each firm has the incentive to separate from the rival in order to get positive profits.

Therefore, the Minimum Differentiation Principle is invalid under the assumptions in Hotelling (1929). In order to explore the properties of the location-than-price equilibrium, D'Aspremont et al. (1979) propose to modify the original Hotelling model by adopting quadratic transportation costs rather than unit transportation costs. Therefore, the relevant utility functions become $U_A = v - p_A - t(a - x)^2$ and $U_B = v - p_B - t(b - x)^2$. By equating U_A and U_B, we get the indifferent consumer, $\hat{x}\,(a, b, p_A, p_B) = a + \frac{b-a}{2} + \frac{p_B - p_A}{2t(b-a)}$. To interpret this equation, note that, at equal prices, Firm A controls its own turf (the first term in the equation) and receives half of the consumers located between the two firms (the second term in the equation). The last term in the equation expresses the sensitivity of the demand to the price differential. It can be shown that the second-stage price equilibrium always exists, and it is given by $p_A\,(a, b) = \frac{t(b-a)(2+a+b)}{3}$ and $p_B\,(a, b) = \frac{t(b-a)(4-a-b)}{3}$. Now we consider the first-period choice of locations. Firm A maximizes (similarly for Firm B): $\pi_A\,(a, b) = p_A\,(a, b)\,\hat{x}\,(a, b, p_A\,(a, b)\,,\,p_B\,(a, b))$. The profits of Firm A are strictly decreasing in a (symmetrically, the profits of Firm B are strictly increasing in b). Therefore, the two firms separate as much as possible: this result is known as the Maximum Differentiation Principle. The maximum differentiation principle is the result of two contrasting forces (Tirole 1988). On the one hand, there is a *demand effect*, which captures the incentive of each firm to move toward to the center of the segment in order to increase the demand. On the other hand, there is also a *strategic effect* that describes the fact that, when A moves closer to B, the two firms are more similar and then competition is fiercer (indeed, the equilibrium price is lower when the two firms are located closer). Therefore, the strategic effect induces each firm to move toward the endpoints. Under the assumptions of D'Aspremont et al. (1979) model, the strategic effect always dominates, and therefore the unique equilibrium is characterized by maximum differentiation of the firms.

The linear model with quadratic transportation costs (D'Aspremont et al. 1979) has been proven to be particularly useful, as it allows a full characterization of the location-price equilibrium. For example, it can be used to discuss welfare implications. Suppose that a social planner wants to maximize the overall welfare. Clearly, due to the unit demand function, prices are simply a transfer from consumers to firms, and they do not affect welfare. Welfare depends (negatively) only on transportation costs. Therefore, welfare is maximized when the overall transportation costs are minimized, which occurs when the two firms are located at 1/4 and 3/4, respectively. Given that in equilibrium the two firms maximally differentiate, we can conclude that there is too differentiation in equilibrium. Intuitively, when choosing the location, each firm does not take into account the increase in the consumers' transportation costs, but just aims to avoid disruptive competition with the rival.

The linear model has been extended in many directions. Here we focus on some extensions which are particularly relevant. Economides (1986) considers a more general class of transportation costs. In particular, the transportation costs are assumed to be equal to $t|a - x|^{\alpha}$ and $t|b - x|^{\alpha}$ when buying from Firm A and Firm B, respectively, and with $\alpha \in [1, 2]$. Therefore, α measures the convexity of the transportation costs. Economides (1986) shows that when the transportation costs are sufficiently convex (i.e., $\alpha \in [5/3, 2]$), the Maximum Differentiation Principle holds, as the two firms choose to locate at the endpoints of the segment. However, when the degree of the convexity of the transportation costs is intermediate (i.e., $\alpha \in [63/50, 5/3]$), the location equilibrium is characterized by interior solutions, ranging from 0 to 0.3 for Firm A and from 1 to 0.7 for Firm B. Finally, when the transportation costs are almost linear (i.e., $\alpha \in [1, 63/50]$), there is no equilibrium. Therefore, on the one hand, Economides (1986) confirms that the Minimum Differentiation Principle does not hold even for more general transportation costs. On the other hand, he shows that the Maximum Differentiation Principle is valid only when the transportation costs are sufficiently convex. Another relevant extension concerns the assumption of the uniform distribution of consumers over the linear market. This assumption is mainly motivated by the need to find closed-form solutions. However, it is reasonable to imagine that in many situations consumers are not uniformly distributed. For example, suppose that the distribution of the consumers is symmetric around 1/2, but there is increasing density of consumers toward the center. In such a framework, Neven (1986) considers a location-price game. He shows that when the consumers are rather dispersed, the unique equilibrium consists in maximal differentiation of firms. However, if consumers are quite concentered around the center, partial differentiation of firms emerges in equilibrium. Indeed, when there are more consumers in the center of the market, the *demand effect* is rather strong, thus inducing the firms to move inner. Finally, both the Hotelling model (1929) and the D'Aspremont et al. (1979) model assume that firms are constrained to locate between the endpoints. However, in many situations, firms are free to locate outside the "city boundaries", that is, the firms can locate in points of the space where there are no consumers. Lambertini (1994) considers the D'Aspremont et al. (1979) model and explores the characteristics of the location-price equilibrium by removing the assumption $0 \leq a \leq b \leq 1$ and just assuming $a \leq b$. It is found that there is a unique equilibrium, where Firm A and Firm B locates at $-1/4$ and $5/4$, respectively, that is, the two firms locate outside the endpoints of the segment. Therefore, the firms maximally differentiate only if they are constrained to locate between 0 and 1; otherwise, the equilibrium differentiation is finite. Intuitively, the larger is the distance between the firms, the stronger is the demand effect and the weaker is the strategic effect: at the equilibrium locations $-1/4$ and $5/4$, the two effects compensate. Finally, it is worth mentioning the two-dimensional extension of the Hotelling linear market, which has been introduced by Tabuchi (1994). In particular, it is shown that, in a location-price game, in equilibrium the two firms maximize their distance in one dimension, but minimize their distance in the other dimension.

1.3 The Circular Model

The linear model (Hotelling 1929) has received relevant attention by economists. However, the existence of the boundaries often makes the model intractable when the firms are more than two. For example, Brenner (2005) finds analytically the location-price equilibrium in the case of three firms under quadratic transportation costs, and he also numerically characterizes the equilibrium up to nine firms. However, when the number of firms is larger than nine, a solution is hard to find, both analytically and numerically. The main problem with the linear market is that firms are intrinsically asymmetric. Indeed, the most-to-the-left and the most-to-the-right firms compete with just another firm; at the opposite, any other firm competes with two rivals.

The "classic" model that solves this kind of "asymmetry" in the Hotelling line is the circular model, which has been introduced by Vickrey (1964) and Salop (1979). The basic idea is very simple: instead of assuming that the consumers are distributed along a segment, they are distributed along a circle (of length 1). Now, no point is better than another. In what follows, we illustrate the main characteristics of the circular model.

As before, let $x \in [0, 1]$ indicate the location of consumers. Instead of considering just two firms, we consider a large number of identical potential firms. Firms are also located in the circle, and they can locate in just one position. Consumers wish to buy one unit of the good, and sustain linear transportation costs to move to the firm. The only cost sustained by a firm is the fixed cost f in the case of entry. Suppose the following two-stage entry-price game. In the first stage of the game, potential entrants simultaneously decide whether or not to enter. Let us indicate by n the number of firms that enter in the market. We assume that firms do not choose their locations: in particular, the firms are assumed to be automatically located equidistant from one another in the circle. In other words, maximal differentiation is assumed. It follows that the circle can be divided in n segments: the length of each of them is $1/n$. In the second stage of the game, the firms that are entered set simultaneously the price.

Since there are many identical firms, the number of firms in equilibrium is determined by the zero-profit condition (up to the integer problem).[5] We solve the game by backward induction. Consider the second stage. Assume that the number of firms that entered in the market is sufficiently high, so that there is competition between the existing firms (in other words, there are no local monopolies in the circle): intuitively, this amounts requiring that f is not too large. Let us focus on the focal firm, say Firm i. Since the firms are identical, we can assume that all

[5]That is, the number of firms must be an integer.

the other firms are setting the same price, say p. Note that Firm i has just two real competitors, that is, the two firms that surround it. For example, suppose Firm i is located at point 0 (or 1): its two competitors are the firms located at $1/n$ and $-1/n$. Consider a consumer located at $x \in [0, 1/n]$. This consumer is indifferent between buying from Firm i and its closest (to the right) competitor, if the following condition is verified: $v - p_i - tx = v - p - t\left(\frac{1}{n} - x\right)$, that is, $\hat{x} = \frac{p+t/n-p_i}{t}$. It follows that the demand of Firm i is $2\hat{x}$. The profit function is $2p_i\hat{x}$. By maximizing the profit function with respect to p_i, and then setting $p_i = p$, we get the equilibrium price: $p* = \frac{t}{n}$. Not surprisingly, the price increases with the level of product differentiation (t) and decreases with the number of competing firms (n). Due to symmetry, the demand of each firm in equilibrium is $1/n$. Let us consider now the first period. Since there is free-entry net of the entry costs, the equilibrium number of firms is determined by the zero-profit condition, which yields $n* = \sqrt{\frac{t}{f}}$. It is interesting to note that, in equilibrium, the price is higher than the marginal costs. However, the profits are zero, due to the free-entry conditions. This result is similar to the monopolistic competition of Chamberlin (1933) (Fig. 1.3).

The Salop model is useful also to derive implications on welfare. More specifically, is the equilibrium number of firms too high or too low from the point of view of welfare? Suppose that a social planner wants to maximize the overall welfare. Welfare is only determined by the equilibrium transportation costs. Note that, in equilibrium, the consumer's average transportation cost is $2n\int_0^{\frac{1}{2n*}} tx\,dx = \frac{\sqrt{tf}}{4}$. The social planner chooses n in such a way to minimize the sum of the average transportation costs and the overall cost of entry. Therefore, the optimal number of firms, n^o, is given by $n^o \in \arg\min\left[2n\int_0^{\frac{1}{2n}} tx\,dx + nf\right]$, that is, $n^o = \frac{n*}{2}$. We can conclude that the market generates too many firms in equilibrium. Excess of entry is due to fixed costs of entry. In particular, the private and the social incentives of entrance do not coincide: entrance is socially justified only if the savings in the transportation costs compensate for the entry costs, whereas the private incentive to entry is linked to stealing the business of other firms. However, it should be noted that

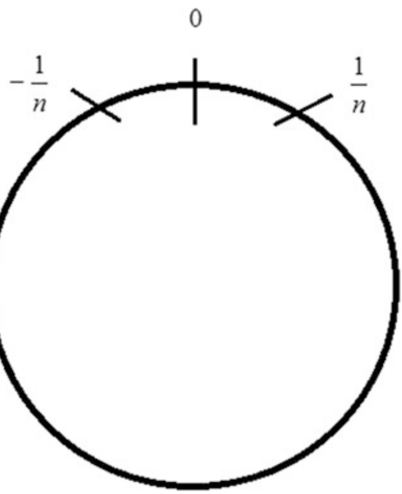

Fig. 1.3 The Salop model

this "excess of entry" result is not general. For example, Gu and Wenzel (2009) remove the assumption of unit demand function and introduce a demand function with constant elasticity. They show that the number of firms entering into the market in equilibrium decreases with the demand elasticity: when the demand elasticity is sufficiently large, there is insufficient entry from the welfare's point of view.

In general, the linear and the circular markets yield different outcomes, due to the asymmetric nature of the former and symmetric nature of the latter. Interestingly, the literature has developed a more general model aiming to encompass the basic models and limit cases. This model is known as the "quasi-linear" city and it has been introduced first by Takahashi and de Palma (1993). We briefly describe it. There is a unit length circular city with a caveat at point 0. When passing through this point, there is an additional cost equal to β (e.g., this cost can be interpreted as a barrier such as a congested bridge, a mountain, or a river). When this cost is nil, we have the standard circular model; when it is extremely high, no consumer passes through this point, and therefore we are back to the standard linear model. When β is intermediate, we have a mixture of the linear and the circular model.

1.4 Spatial Price Discrimination

One crucial assumption in Hotelling and Salop models is that the firms set the same price for all consumers. This kind of pricing technique is also known as FOB (free-on-board) or mill or uniform pricing. However, it is also possible that a firm sets a price which depends on the location of the consumer which is served. This pricing technique is known as "delivered" pricing. Therefore, delivered pricing is a peculiar form of price discrimination, where discrimination is based on location (Greenhut and Greenhut 1975). Alternatively, if we assume a "product differentiation" interpretation of the spatial models, delivered pricing can be interpreted as follows. The firm might produce a single standardized variety and sets a price for it: this is equivalent to FOB pricing. In contrast, the firm might decide to offer a basic product with a series of options with different prices: this is equivalent to delivered pricing.

Thisse and Vives (1988) introduce the analysis of price discrimination into the basic Hotelling linear model. They consider the following situation. There are two firms which are exogenously located at the endpoints of the Hotelling line (i.e., $a = 0$ and $b = 1$). The consumers sustain linear transportation costs. Before setting the price, each firm has to decide its own pricing policy, which might be uniform or discriminatory. If the firm chooses uniform price, it is constrained to set the same price for all consumers; if the firm chooses price discrimination, it is free to set a different price for each consumers. Note that spatial price discrimination is assumed to be "perfect", that is, there is a price for each possible location. The game is two-stage: in the first stage of the game, each firm commits to a pricing policy (U and D) which will be followed in the second stage. In the second stage of the game, the two firms set the price or the price schedule, depending on the first-period choice. The game is solved by backward induction, by starting from the last stage of the game and discussing each subgame in turn (indeed, there are four possible subgames).

Subgame UU: both firms have chosen uniform pricing in the first stage. The analysis is the same as in the case of the standard Hotelling model with linear transportation costs and maximum differentiation.

Subgame DD: both firms have chosen discriminatory pricing in the first stage. Consider a consumer located in x. Define with $p_i(x)$ the price charged by Firm $i = A, B$ to consumer x. The utility of that consumer when he buys from Firm A (B) is $u_A(x) = v - p_A(x) - tx$ ($u_B(x) = v - p_B(x) - t(1 - x)$). The consumer buys from the firm which gives the higher utility; if the utility is the same, it is assumed that he buys from the nearer firm. Suppose that consumer x is nearer to Firm i. Given the price set by Firm $j \neq i$, the best thing Firm i can do is to set a price that gives the consumer the same utility he receives from Firm j: this is the highest possible price that guarantees that consumer x buys from i. Given that the price is never lower than the marginal costs, the equilibrium price schedules are $p_A(x)* = \begin{cases} t(1-x) - tx & if \;\; x \leq 1/2 \\ 0 & if \; x \geq 1/2 \end{cases}$ and $p_B(x)* = \begin{cases} 0 & if \; x \leq 1/2 \\ tx - t(1-x) & if \; x \geq 1/2 \end{cases}$. The firms' profits are $\pi_A^{DD}* = \int_0^{\frac{1}{2}} p_A(x) * dx = \frac{t}{4}$ and $\pi_B^{DD}* = \int_{\frac{1}{2}}^1 p_B(x) * dx = \frac{t}{4}$.

Subgame UD: only Firm B has chosen discriminatory pricing in the first stage. If the utility of the consumer is the same, he buys from the discriminating firm. The firm setting a uniform price moves first. Consider a generic consumer x. The best-reply function of Firm B consists in setting $p_B(x) = p_A + tx - t(1 - x)$. If Firm A sets $p_A > t(1 - x) - tx$, Firm B can always serve consumer x by undercutting the uniform price set by Firm A: therefore consumer x will always buy from Firm B. In order to have a positive demand, Firm A must set a uniform price such that $p_A \leq t(1 - x) - tx$, which cannot be undercut by Firm B. Therefore, the highest uniform price is $p_A = t(1 - x) - tx$. Solving for x, we obtain the most at the right consumer served by Firm A: $x_*{}' = (t - p_A)/2t$. Therefore, the demand of Firm A is $x_*{}'$. Maximizing the profits of Firm A with respect to p_A, we get $p_A{}^* = t/2$. Substituting p_A into the best-reply function of Firm B, we get the equilibrium price schedule: $p_B(x)* = \begin{cases} 2tx - t/2 & if \; x \geq 1/4 \\ 0 & if \; x \leq 1/4 \end{cases}$. The profits are $\pi_A^{UD}* = p_A * x *' = \frac{t}{8}$ and $\pi_B^{UD}* = \int_{x*'}^1 p_B(x) * dx = \frac{9t}{16}$.

Subgame DU: symmetric to subgame UD.

Table 1.1 Equilibrium profits in the Thisse and Vives model

π^A	π^B	
	U	D
U	$t/2$; $t/2$	$t/8$; $9t/16$
D	$9t/16$; $t/8$	$t/4$; $t/4$

We consider now the first stage of the game. In Stage 1, the two firms decide simultaneously whether to price discriminate (D) or not (U), by anticipating the equilibrium profits in Stage 2. Consider Table 1.1.

We observe that there is a unique pricing policy equilibrium, DD, and that the profits in DD are lower than in UU (Prisoner Dilemma). The intuition is the following. For any given pricing policy strategy of the rival, each firm would like to be as flexible as possible in setting prices. Therefore, each firm chooses D, which is the dominant strategy. However, when both firms price discriminate, competition is very fierce, as each firm can reduce the price in one location without fearing to reduce the price elsewhere. Therefore, the firms would be better off in UU, but they fail to coordinate on that equilibrium. The crucial difference between uniform pricing and discriminatory pricing has been well described by Hoover (1948): "The difference between market competition under FOB pricing [. . .] and discriminatory delivered pricing is something like the difference between trench warfare and guerrilla warfare. In the former case all the fighting takes place along a definite battle line; in the second case the opposing forces are intermingled over a broad area" (p.57).

Lederer and Hurter (1986) also consider spatial price discrimination. However, they are not interested in the resulting pricing policy equilibrium. In contrast, they consider a location-price game in a highly general spatial model and find an important relation between equilibrium locations and optimal locations. Consider a two-dimensional compact market region denoted as S (i.e., there is no specific assumption about the shape of the space). Let the locations of Firm A and Firm B be indicated by $z_A = (x_A, y_A)$ and $z_B = (x_B, y_B)$, respectively. The marginal costs of production of Firm A and Firm B are c_A and c_B, respectively. Therefore, the two firms are not restricted to be symmetric. Let the location of a consumer in the space be indicated by $z \in \Re^2$. The consumers' distribution over S is a generic distribution $\rho(z)$. Let the cost of transporting the good from the plant to the consumer be given by $f_A(z_A, z)$ for Firm A and by $f_B(z_B, z)$ for Firm B (therefore, the transportation costs are sustained by the firms). The consumer buys from the cheapest source. If the two sources are equal, we assume that the consumer is served by the firm with the least total marginal costs (production plus transport costs). The firms are assumed to perfectly price discriminate. That is, they can set a delivered price schedule where the price depends on the location of the consumer which is served. The game is two-stage: in the first stage of the game, the firms choose simultaneously the locations, and, in the second stage, they choose simultaneously the price schedule. In the second stage, the equilibrium price schedule is the following: $p_*(z_A, z_B, z) = \max\,[f_A(z_A, z) + c_A, f_B(z_B, z) + c_B]$. Intuitively, the proof is the following. If the low-cost firm does not serve the demand, it could undercut the low-price firm. The current low-price firm must be pricing above or at its marginal cost: thus by cutting its price, the low-cost firm can raise its profits. Furthermore, in equilibrium the low-cost firm must price at the marginal cost of the next efficient firm at each market point and that firm must price at its marginal cost. If the next most efficient firm priced above this amount, the low-cost firm would price at this price and would serve the demand. This would induce the next most efficient firm to cut its price.[6]

Now, we consider the first stage of the game, where the firms choose the locations by anticipating the equilibrium prices in the second stage of the game. Denote the social cost as the total cost incurred by the firms to supply demand to customers in S in a cost-minimizing manner. Therefore, it is $K(z_A, z_B) = \int\int_S \min\,[f_A(z_A, z) + c_A, f_B(z_B, z) + c_B]\rho(z)dz$. Note that the profits of Firm $i = A, B$ under the equilibrium prices can be written as follows: $\pi_i(z_A, z_B, p_*) = \int\int_S [f_j(z_j, z) + c_j]\rho(z)dz - K(z_A, z_B)$ with

[6]Note that there is no contradiction with the equilibrium prices in Thisse and Vives (1988) in the DD subgame. In that case the transportation costs were sustained by the consumers. Here the transportation costs are sustained by the firm. Therefore, the profit margin is the same.

Fig. 1.4 Liu and Serfes model

$j \neq i$.[7] Therefore, the location equilibrium (z_{A*}, z_{B*}) minimizes the social costs, that is,[8] $K(z_{A*}, z_{B*}) \leq K(z_A, z_{B*})$ and $K(z_{A*}, z_{B*}) \leq K(z_{A*}, z_B)$.

The spatial models considered until now represent quite extreme situations. In Hotelling (1929), D'Aspremont et al. (1979) and Salop (1979) models assume that the firms set a uniform price for all consumers; in Lederer and Hurter (1986) and Thisse and Vives (1988), the firms set a different price for any possible location, thus implying "perfect" price discrimination. However, in many real-world situations, the firms are able to "imperfectly" discriminate, that is, they are able to set different prices for different "groups" of consumers, but they are able not to distinguish within each group.

The analysis of imperfect spatial price discrimination has been developed first by Liu and Serfes (2004). As in Hotelling (1929), the consumers are uniformly distributed on a linear segment of length 1 and sustain linear transportation costs. There is an information technology which allows the firms to partition the consumers into different groups: the linear market is partitioned into n sub-segments indexed by m, with $m = 1, \ldots, n$. Each sub-segment is of equal length, $1/n$. It follows that sub-segment m can be expressed as the interval $\left[\frac{m-1}{n}; \frac{m}{n}\right]$ (Fig. 1.4). A firm can price discriminate between consumers belonging to different sub-segments, but not between the consumers belonging to the same sub-segment. The cost of using the technology is zero. Denote with p_i^m the price set by Firm $i = A,B$ on consumers belonging to sub-segment m. Assume that $n = 2^k$, with

[7] Indeed,

$$\begin{aligned} \pi_i(z_A, z_B, p*) &= \int\int_S \left[f_j(z_j, z) + c_j - f_i(z_i, z) - c_i\right] \rho(z) dz \\ &= \int\int_S \left[f_j(z_j, z) + c_j\right] \rho(z) dz - \int\int_S \min\left[f_j(z_j, z) + c_j, f_i(z_i, z) + c_i\right] \rho(z) dz. \\ &= \int\int_S \left[f_j(z_j, z) + c_j\right] \rho(z) dz - K(z_A, z_B) \end{aligned}$$

[8] The continuity of function K on S also guarantees that the location equilibrium exists.

$k = 1, 2, 3, 4 \ldots$. Therefore, the higher is n, the higher is the information precision. When $n \to \infty$, we have the perfect price discrimination model of Thisse and Vives (1988); at the opposite, when $n \to 2$ the model is close as possible to the uniform pricing case of Hotelling (1929).

Liu and Serfes (2004) consider the case of firms which are exogenously located at the endpoints of the segment (i.e., $a = 0$ and $b = 1$).[9] As in Thisse and Vives (1988) in the first stage of the game, the firms simultaneously decide between D and U and in the second stage set the prices.

Subgame UU: both firms have chosen uniform pricing in the first stage.

The analysis is the same as in the case of the standard Hotelling model with linear transportation costs and maximum differentiation.

Subgame DD: both firms have chosen discriminatory pricing in the first stage. Consider segment m. Define x^{m*} as the consumer on segment m which is indifferent between buying from Firm A and from Firm B for a given couple of discriminatory prices, p_A^m and p_B^m. Equating the utility in the two cases and solving for x, we get $x^m* = \frac{1}{2} + \frac{p_B^m - p_A^m}{2t}$. Therefore, the demand of Firm A and Firm B on segment m is, respectively, $d_A^m = \frac{1}{2} + \frac{p_B^m - p_A^m}{2t} - \frac{m-1}{n}$ and $d_B^m = \frac{m}{n} - \frac{1}{2} - \frac{p_B^m - p_A^m}{2t}$. Therefore, the profits of Firm i on segment m are $\pi_i^m = p_i^m d_i^m$. Define $m_A \equiv \frac{n}{2} - 1$ and $m_B \equiv \frac{n}{2} + 2$, with $m_B > m_A$. The equilibrium price schedules in DD are as follows: if $m_A < m < m_B$, then $p_A^m* = \frac{t(4-2m+n)}{3n}$ and $p_B^m* = \frac{t(2+2m-n)}{3n}$; if $m \leq m_A$, then $p_A^m* = t\left(1 - \frac{2m}{n}\right)$ and $p_B^m* = 0$; and if $m \geq m_B$, then $p_A^m* = 0$ and $p_B^m* = t\left(\frac{2m-2-n}{n}\right)$. Intuitively, Firm A is a constrained monopolist in all segments $m \leq m_A$, whereas Firm B is a constrained monopolist in all segments $m \geq m_B$; the two firms compete in the remaining segments. Therefore, in each segment where a firm is a constrained monopolist,

[9] Colombo (2011) extends to the case of endogenous locations.

the firm sets the highest price that allows serving the whole sub-segment without being undercut by the rival. The firms' profits are therefore

$$\Pi_A^{DD}* = \sum_{m=1}^{m_A} \frac{t}{n}\left(1-\frac{2m}{n}\right) + \sum_{m=m_A+1}^{m_B-1} \frac{t(4-2m+n)}{3n}\left(\frac{2-m}{3n}+\frac{1}{6}\right)$$
$$= \frac{t\left(9n^2-18n+40\right)}{36n^2}$$

$$\Pi_B^{DD}* = \sum_{m=m_A+1}^{m_B-1} \frac{t(2+2m-n)}{3n}\left(\frac{m+1}{3n}-\frac{1}{6}\right)$$
$$+ \sum_{m=m_B}^{n} \frac{t}{n}\left(\frac{2m-2-n}{n}\right)$$
$$= \frac{t\left(9n^2-18n+40\right)}{36n^2}.$$

By comparing the profits in the case UU and the profits in the case DD, it can be observed that the profits in the case of imperfect price discrimination are always lower than the profits in the case of uniform pricing. However, the profits in DD are U-shaped in the precision of segmentation, n. Indeed, there are two contrasting forces at work, the *intensified competition* effect and the *surplus extraction* effect. The first refers to the fact that, when both firms sell positive quantities in a given segment of consumers, an information refinement intensifies competition. The second refers to the fact that some segments are monopolized by a firm, and on these segments, the firm extracts the consumer surplus. When n is low and it increases, the number of competitive segments increases: the intensified competition effect dominates, so the profits decrease. For further increases of n, the number of competitive segments is constant, but the number of monopolized segments increases: the surplus extraction effect dominates, so the profits increase.

Subgame UD: only Firm B has chosen discriminatory pricing in the first stage. Denote $m\hat{} = \frac{n+7}{4}$. The equilibrium prices are $p_A* = \frac{t(n+1)}{2n}$ and $p_B^m* = \begin{cases} \frac{t}{n} & if\ m = m\hat{} - 1 \\ \frac{t(4m-3-n)}{2n} & if\ m \geq m\hat{} \end{cases}$. The firms' profits are therefore $\Pi_A^{UD}* = \frac{t\left(n^2+2n+1\right)}{8n^2}$ and $\Pi_B^{UD}* = \frac{t\left(9n^2-6n+5\right)}{16n^2}$.

Subgame DU: only Firm A has chosen discriminatory pricing in the first stage. This case is symmetric to case UD.

Consider now the first stage of the game. By comparing the profits, it can be shown that if n is low, the dominant strategy is U and there is no Prisoner Dilemma, whereas if n is high, the dominant strategy is D and there is a Prisoner Dilemma. Therefore, we can conclude that the adoption of spatial price discrimination (and, consequently, the existence of a Prisoner Dilemma) emerges if and only if the information about the consumers' location is precise enough.

1.5 Spatial Models with Elastic Demand Functions

Classic models typically assume that consumers have unit demand functions (i.e., each consumer buys one or zero unit of good). However, it might be reasonable to assume that consumers might have elastic rather than unit demand functions. Introducing elastic demand function within a spatial model with uniform pricing is difficult. Indeed, as shown by Rath and Zhao (2001), equilibrium prices and equilibrium locations can only be defined implicitly (when transportation costs are quadratic). In general, introducing elastic demand functions into a spatial model with uniform pricing does not allow getting easily interpretable solutions (Peitz, 2002).

On the other hand, introducing elastic demand in a spatial model with (spatial) price discrimination is more fruitful, as shown by Hamilton et al. (1989). The Hamilton et al. (1989) model maintains the same assumptions of Hotelling (1929) with the only difference that each consumer has a linear demand function of this type: $p_x = 1 - (q_{A,x} + q_{B,x})$, where $q_{A,x}$ ($q_{B,x}$) is the quantity produced by Firm A (B) at location x. Therefore, as in Thisse and Vives (1988), the firms can spatially price discriminate, as they can deliver different quantities at different locations in the space, thus making the price different at any location. The firms pay linear transportation costs to ship the good from the plant to consumers. Therefore, the profits of

Firm A (B) at point x are $\pi_{A,x} = (1 - q_{A,x} - q_{B,x} - t|x - a|)q_{A,x}$ ($\pi_{B,x} = (1 - q_{A,x} - q_{B,x} - t|x - b|)q_{B,x}$). Overall profits of Firm A (B) are $\Pi_A = \int_0^1 \pi_{A,x} dx$ ($\Pi_B = \int_0^1 \pi_{B,x} dx$). Provided that the transportation costs are not too high, no point in the space is monopolized by one firm. The game is a two-stage location-than-quantity game. Consider the second stage. Each location x can be treated as a separated market. Indeed, due to spatial price discrimination, a firm's quantity decision at a particular location has no effect on other locations. As a result, at each location, the Cournot equilibrium is $q_{A,x}(a, b) = (1 - 2t|a - x| + t|b - x|)/3$ and $q_{B,x}(a, b) = (1 - 2t|b - x| + t|a - x|)/3$. By anticipating the second-stage equilibrium quantity schedules, in the first stage, each firm chooses the location that maximizes its own profits. There is a unique equilibrium, that is, $a* = b* = \frac{1}{2}$. Therefore, with spatial discrimination and quantity competition, agglomeration occurs. It should be mentioned that agglomeration in the case of spatial Cournot competition is a quite general result. For example, agglomeration arises in the case of different production costs throughout the city (Mayer 2000), in the case of product differentiation (Shimizu 2002), and in the case of different transportation costs (Colombo 2013).[10] Furthermore, since each firm's sales are distributed symmetrically around the market center, each firm is located so as to minimize the transportation costs associated with its sales pattern.

Hamilton et al. (1989) also consider the case where the firms set price rather than quantity in the second stage (Bertrand competition). The relevant demand function is now $q_x = 1 - p_x$: the consumer located at x buys from the firm charging the lower delivered price. When the delivered prices are equal, the firm with lower transport costs provides the good to the consumers. As under quantity competition, the price problem can be solved at each location separately. Following a standard Bertrand argument, the equilibrium price at location x is $p_{A,x}(a, b) = p_{B,x}(a, b) = \max\, [t|a - x|, t|b - x|]$. Note that, differently from the Cournot case, each location x is served by only one firm. Let

[10] However, it does not emerge in the case of hyperbolic demand function (Colombo 2016).

us consider now the first stage of the game. The equilibrium locations are $a* = 1 - b* = \frac{10t-8+\sqrt{(10t-8)^2+24t(4-3t)}}{24t}$. Therefore, with spatial price discrimination and price competition, agglomeration never occurs. Indeed, the firms do not locate in the same point to avoid zero profits. The equilibrium locations are such that the two firms locate between the first and third quartiles and very close to them.

The intuition can be summarized as follow. Under both Cournot and Bertrand, the firms select the locations that minimize the transportation costs, given the expected second-stage quantity/price schedules. Since under Cournot there is complete overlapping, the transport costs are minimized when each firm locates in the middle of the segment. In contrast, in Bertrand the market areas are completely disjointed: therefore, the two firms locate "close" to the first and the third quartiles in order to minimize the transport costs. Note that they do not locate at the first and third quartiles: as the price decreases uniformly from the boundary to the center, the firms sell more in the in-the-between region than in the hinterlands. Therefore, in order to reduce the transportation costs, the firms locate closer to the center (i.e., between the first and third quartiles). As the optimal locations are at the first and third quartiles, we can conclude that the unit demand assumption is a necessary condition for the equilibrium locations to be transport cost minimizing (Lederer and Hurter 1986). Furthermore, the dispersed locations in Bertrand make the total transport costs lower under Bertrand than under Cournot. Since the equilibrium prices are lower in Bertrand than in Cournot, we can conclude that welfare is higher under Bertrand.

1.6 The "Barbell" Model

The Hotelling model assumes uniform distribution of consumers. Non-uniform distribution of consumers makes it difficult to obtain closed-form solutions (see Sect. 1.2). However, one particular case of non-uniform distribution of consumers has received considerable attention due to its tractability. It is the case of consumers located at endpoints of the linear segment. This is the "barbell" model introduced by Hwang and

Mai (1990). This model is particularly appealing in a geographical/spatial perspective.

Suppose a segment from 0 to 1. Consumers are located at the two endpoints, the "cities". Denote by 1 (2) the city located at the left (right) endpoint. A monopolist has to decide the location and the price. Denote by $a \in [0, 1]$ the location of the monopolist. First, we consider the case where the monopolist cannot price discriminate, and then we will consider the case of price discrimination. The demand function in City 1 (2) is $q_1 = 1 - cp$ ($q_2 = 1 - dp$). Therefore, the higher is c and d, the flatter is the corresponding demand curve (so, c and d are positively related to demand elasticity). The monopolist sustains linear transportation costs to carry the good to the cities. The profits of the monopolist are $\pi = (1 - cp)(p - ta) + (1 - dp)(p - t(1 - a))$. By maximizing with respect to price, we get $p = \frac{2+tca+td(1-a)}{2(c+d)}$. Note that $\frac{\partial^2 \pi}{\partial a^2} = \frac{t^2(c-d)^2}{2(c+d)} \geq 0$. Therefore, the profits are convex in the location, a. It follows that the optimal location is either $a = 0$ or $a = 1$. By comparing $\pi(a = 0)$ with $\pi(a = 1)$, we get $a* = \begin{cases} 0 \\ 1 \end{cases} a* = \begin{cases} 0 & \text{if } c \geq d \\ 1 & \text{if } c \leq d \end{cases}$. That is, the firm locates where the demand curve is flatter. Indeed, at equal prices, the demand is larger when the demand curve is flatter. Therefore, in order to minimize the transportation costs, the monopolist locates where the demand curve is flatter (as here the quantity sold is larger). We consider now the case of spatial price discrimination. The profit function now is $\pi = (1 - cp_1)(p_1 - ta) + (1 - dp_2)(p_2 - t(1 - a))$. By maximizing, we get $p_1 = \frac{1+tca}{2c}$ and $p_2 = \frac{1+td(1-a)}{2d}$. Note that $\frac{\partial^2 \pi}{\partial a^2} = \frac{t^2(c+d)}{2} \geq 0$. Therefore, the profits are convex in a. By comparing $\pi(a = 0)$ with $\pi(a = 1)$, we get $a* = \begin{cases} 0 & \text{if } c \leq d \\ 1 & \text{if } c \geq d \end{cases}$. That is, the firm locates where the demand curve is steeper. Note that this result is the opposite with respect to uniform pricing. Indeed, under price discrimination, all else being equal, even if the firm sets a lower price where the elasticity is higher, in equilibrium the demand is lower in the market characterized by a flatter demand curve.[11] Therefore, the firm locates where the demand curve is steeper in order to minimize the transportation costs.

1.7 Vertical Differentiation

All the models we have considered until now assume that, at equal prices, some consumers prefer the product of Firm A, whereas others prefer the product of Firm B. That is, these models describe horizontal product differentiation. However, there are many situations where, at equal prices, all consumers prefer the product of, say, Firm A to the product of Firm B (e.g., because the quality of Firm A is higher). In this case, we refer to vertical product differentiation. Spatial models are also useful to analyze vertical product differentiation. In this case their correct interpretation is the product characteristic one.

In what follows, we discuss one of the most famous spatial models of vertical differentiation, which dates back to Shaked and Sutton (1982). Suppose that each consumer buys one or zero unit of the good. The preferences of the consumer, if he buys the good, are expressed by the following utility function, $U = \vartheta s - p$, where s is a quality index of the good. If the consumer does not buy the good, the utility is zero. Parameter ϑ is a taste parameter: all consumers prefer high quality to low quality, for a given price; however, a consumer with a high ϑ is more willing to pay to obtain a higher quality.[12] Suppose the following (uniform) distribution of tastes across the population: $\vartheta \in \left[\underline{\vartheta}, \overline{\vartheta}\right]$, where $\underline{\vartheta} > 0$. Furthermore, we assume for the moment that $\overline{\vartheta} \geq 2\underline{\vartheta}$ (i.e., there is "sufficient" heterogeneity). Suppose there are two firms, Firm A and Firm B. Firm A (B) produces a good of quality s_A (s_B), with $s_A \geq s_B$: that is, Firm A (B) produces the high (low)-quality good.

[11] In other words, the lower price is not sufficient to compensate for the higher sensitivity to the price of consumers.

[12] It can be shown that ϑ is the inverse of the marginal rate of substitution between income and quality. That is, consumers have different incomes, and wealthier consumers have a lower marginal utility of income and a higher ϑ.

Denote $\Delta \equiv s_A - s_B$. Consider a two-stage game. In the first stage, the two firms choose simultaneously the quality; in the second they choose simultaneously the price. Consider the second stage. Suppose the market is covered.[13] Denote by $\hat{\vartheta}$ the consumer which is indifferent between buying from Firm A and from Firm B. Solving $\vartheta s_A - p_A = \vartheta s_B - p_B$, we get $\hat{\vartheta} = \frac{p_A - p_B}{\Delta}$. Clearly, high-$\vartheta$ consumers buy the high-quality good, whereas low-ϑ consumers buy the low-quality good. Therefore, the demand functions are $D_A = \overline{\vartheta} - \hat{\vartheta}$ and $D_B = \hat{\vartheta} - \underline{\vartheta}$, and the profits are $\pi_A = p_A D_A$ and $\pi_B = p_B D_B$. By maximizing with respect to price, we get $p_A* = \frac{\Delta(2\overline{\vartheta} - \underline{\vartheta})}{3}$ and $p_B* = \frac{\Delta(\overline{\vartheta} - 2\underline{\vartheta})}{3}$. The equilibrium profits (for given qualities) are $\pi_A(p_A*, p_B*) = \frac{\Delta(2\overline{\vartheta} - \underline{\vartheta})^2}{9}$ and $\pi_B(p_A*, p_B*) = \frac{\Delta(\overline{\vartheta} - 2\underline{\vartheta})^2}{9}$. Therefore, the high-quality firm sets a higher price and gets higher profits. Furthermore, note that the prices increase with consumers' heterogeneity.

Consider now the first stage. Suppose that the quality choice is without cost. From the profit functions above, it is immediate to see that the two firms will maximally differentiate. In particular, suppose that s must belong to $\left[\underline{s}, \overline{s}\right]$. If we assume that $s_A \geq s_B$, then the equilibrium qualities would be $s_A* = \overline{s}$ and $s_B* = \underline{s}$ (maximal differentiation). The intuition is the same as for spatial models of horizontal differentiation: the firms differentiate in order to reduce price competition. In particular, the strategic effect dominates, so that, even if producing a high-quality good is costless, the low-quality firm reduces the quality of its good as much as possible, in order to soften price competition.[14] Clearly, if the two firms enter sequentially, the firm that enters first chooses $\overline{s}$, whereas the other chooses $\underline{s}$.

Suppose now that $\overline{\vartheta} < 2\underline{\vartheta}$ (low consumer heterogeneity). In this case, in the price equilibrium, Firm B has no demand. Therefore, it sets a price equal to zero, whereas Firm A sets a price equal to $\overline{\vartheta}\Delta/2$

[13] Under some appropriate restrictions on the parameters, this conjecture is correct in equilibrium.

[14] However, this conclusion is not always true: if the lowest level of quality is particularly low so that the market is uncovered, the low-quality firm would end up with zero demand. In this case, there is less than maximal differentiation in equilibrium.

and gets positive profits. Therefore, even if there are constant return to scale and no entrance costs, there is only one firm in the market. This is in contrast with the locational models under horizontal product differentiation, where there is an infinite number of firms when there are no entrance costs (see the Salop model when f tends to zero). In the case of vertical differentiation, when consumer heterogeneity is low, more intense price competition drives the low-quality firm out of the market. Indeed, if the lower quality is very "low", the low-quality firm cannot resist to the competition of the high-quality firm. More generally, the following "finiteness result" can be stated: provided that the marginal cost of quality does not increase too quickly with quality, there can be at most a finite number of firms with a positive market share in the industry regardless of entry costs.

Before concluding, it is worth stressing the existence of spatial models combining the "horizontal" and the "vertical" dimension: Gabszewicz and Thisse (1986) introduce vertical differentiation by allowing firms to be asymmetrically located outside the linear market, Dos Santos Ferreira and Thisse (1996) use asymmetric transport costs to generate vertical differentiation in the Hotelling set-up, and Gabszewicz and Wauthy (2012) nest horizontal and vertical differentiation by means of a measure of the "natural market" of each firm (i.e., when the "natural market" of a firm is the whole market, there is pure vertical differentiation; when the "natural market" of both firms is of equal size, there is pure horizontal differentiation).

1.8 Conclusions

This chapter illustrates some prominent classic spatial models. In particular, we consider some cornerstones of classic spatial economics, including the linear model, the circular model, and the vertical differentiation model, and some extensions to them. The aim of the chapter is mainly pedagogical: we want to discuss the main spatial models and their implications. Of course, many other relevant models are not discussed, even if they contribute to our comprehension of the role of space in shaping economic phenomena.

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