Vertex-Edge Domination in Unit Disk Graphs

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Abstract. Let $G = (V, E)$ be a simple graph. A set $D \in V$ is called a vertex-edge dominating set of G if for each edge $e = (u, v) \in E$, either u or v is in D or one vertex from their neighbor is in D. Simply, a vertex $v \in V$, vertex-edge dominates every edge (u, v), as well as every edge adjacent to these edges. The vertex-edge dominating problem is to find a minimum vertex-edge dominating set of G. Herein, we study the vertex-edge dominating set problem in unit disk graphs and prove that this problem is NP-hard in that class of graphs. We also show that the problem admits a polynomial time approximation scheme (PTAS) in unit disk graphs.

Keywords: Dominating set · Vertex-edge dominating set · Unit disk graph · Approximation algorithm · Approximation scheme

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. The *open neighbourhood* of a vertex $v \in V$ in G is the set $N_G(v) = \{u \in V \mid (u, v) \in E\}$ whereas the *closed neighbourhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. A *dominating set* D of G is a subset of V such that every vertex in V is in D or adjacent to at least one vertex in D. A vertex $v \in D$ *dominates* all its neighbors and itself. The *dominating set problem* is to find a minimum cardinality subset $D \subseteq V$ such that D dominates all the vertices of G.

A *vertex-edge dominating set* (VEDs) of a simple undirected graph $G = (V, E)$ is a set $D \subseteq V$ of G such that every edge of G is incident with a vertex of D or a vertex adjacent to a vertex of D. The *VEDs problem* asks to find a VEDs of minimum size in a given graph. A set $D \subseteq V$ is a *double* vertex-edge dominating set if every edge $e \in E$ is vertex-edge dominated by at least two vertices in D. A set $D \subseteq V$ is called a *total* vertex-edge dominating set if every edge $e \in E$ is vertex-edge dominated by D and the graph induced by D has no isolated vertices.

2 Releated Work

The vertex-edge dominating set problem was introduced by Peters [18] and then studied further by different researchers. In particular, bounds on the vertex-edge

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domination number in several graph classes were studied in [3,14–16,20], vertex-edge degrees and vertex-edge domination polynomials of different graphs were discussed in [4,9,23,24], whereas the relations between some vertex-edge domination parameters were discussed in [3,5,12,15,16], several algorithmic aspects were discussed in [15]. Some variants of vertex-edge domination problem were studied in [2,6,11,13,19].

The minimum cardinality of a vertex-edge dominating set (double vertex-edge dominating set, respectively) of G is termed the *vertex-edge domination number* and denoted by $\gamma_{ve}(G)$ (the *double vertex-edge domination number*, $\gamma_{dve}(G)$, respectively). Krishnakumari et al. [14] proved that for every tree T of order $n \geq 3$ with ℓ leaves and s support vertices, we have $\frac{(n-\ell-s+3)}{4} \leq \gamma_{ve}(T) \leq \frac{n}{3}$. In [11], Krishnakumari et al. showed that determining $\gamma_{dve}(G)$ for bipartite graphs is NP-hard, whereas for every non-trivial connected graphs G, $\gamma_{dve}(G) \geq \gamma_{ve}(G)+1$, and for every tree T, we have $\gamma_{dve}(T) = \gamma_{ve}(T)+2$. They also provided two lower bounds on the double vertex-edge domination number of trees and unicycle graphs in terms of order n, the number of leaves and support vertices, respectively.

Boutrig et al. [3] presented a new relationship between the vertex-edge domination and some other domination parameters, answering the four open questions posed by Lewis [15]. Then, for every non-trivial connected $K_{1,k}$-free graph, with $k \geq 3$, they provided an upper bound for the independent vertex-edge domination number in terms of the vertex-edge domination number and showed that for every non-trivial tree the independent vertex-edge domination number can be bounded by the domination number. For connected C_5-free graphs, they also established an upper bound on the vertex-edge domination number. Next Boutrig and Chellali [2] studied the total vertex-edge domination. The minimum cardinality of a total vertex-edge dominating set of graph G called the *total vertex-edge domination number* and denoted by $\gamma_{ve}^{t}(G)$. They showed that determining $\gamma_{ve}^{t}(G)$ for bipartite graphs is NP-hard, and in case of tree T different from a star having order n, with ℓ leaves and s support vertices, respectively, we have $\gamma_{ve}^{t}(G^T) \leq \frac{(n-\ell+s)}{2}$. In the same article, they established a necessary condition for a graph G such to satisfy $\gamma_{ve}^{t}(G) = 2\gamma_{ve}(G)$ and for a tree T, $\gamma_{ve}^{t}(T) = 2\gamma_{ve}(T)$.

Later Venkatakrishnan and Kumar [22] proved that the minimum double vertex-edge dominating set problem is NP-hard for chordal graphs and APX-hard for bipartite graphs with maximum degree 5. They also proposed a linear-time algorithm for finding a minimum double vertex-edge dominating set in proper interval graphs. In addition, showed that the minimum double vertex-edge dominating set problem can not be approximated the factor $(1-\epsilon)\ln|V|$ for any $\epsilon \geq 0$ unless $NP \subset DTIME(|V|^{O(\ln\ln|V|)})$. Finally, influence of edge removal, edge addition and edge subdivision on the double vertex-edge domination number of a graph was investigated by Krishnakumari and Venkatakrishnan [12]. Next, Horoldagva et al. [9] obtained some results on the regularity and irregularity of vertex-edge and edge-vertex degrees in graphs. Recently, Żyliński [25] proved that for any connected graph G of order $n \geq 6$, $\gamma_{ve}(G) \leq \lfloor \frac{n}{3} \rfloor$.

3 Our Contribution

We study the VEDs problem in unit disk graphs. A *unit disk graph* (UDG) is the intersection graph of equal-radii disks in the plane. Given a set $S = \{d_1, d_2, \ldots, d_n\}$ of n circular disks in the plane, each having diameter 1, the corresponding UDG $G = (V, E)$ is defined as follows: each vertex $v_i \in V$ corresponds to the disk $d_i \in S$, and there is an edge between two vertices if and only if the Euclidean distance between the relevant disk centers is at most 1.

We show that the decision version of the VEDs problem is NP-complete in unit disk graphs (Sect. 4). We also propose a polynomial-time approximation scheme for the problem (Sect. 5).

4 NP-Hardness

In this section, we show a polynomial-time reduction from the NP-hard *vertex cover* problem in planar graphs [7] to the VEDs problem to prove that the latter one is also NP-hard. The decision versions of both these problems are defined below.

The VEDs problem on UDGs (VEDS-UDG)
Instance: A unit disk graph $G = (V, E)$ and a positive integer k.
Question: Does there exist a vertex-edge dominating set D of G such that $|D| \leq k$?

The vertex cover problem on planar graphs (VC-PLA)
Instance: A planar graph $G = (V, E)$ having maximum degree 3 and a positive integer k.
Question: Does there exist a vertex cover C of G such that $|C| \leq k$?

Lemma 1 ([21]). *A planar graph $G = (V, E)$ with maximum degree 4 can be embedded in the plane using $O(|V|^2)$ area in such a way that its vertices are at integer coordinates and its edges are drawn so that they are made up of line segments of the form $x = i$ or $y = j$, for some i and j.*

This embedding is known as the *orthogonal drawing* of a graph. There is a linear-time algorithm given by Biedl and Kant [1] that gives an orthogonal drawing of a given graph with at most 2 bends along each edge (see Fig. 1).

Corollary 1. *A planar graph $G = (V, E)$ with maximum degree 3 and $|V| \geq 3$ can be embedded in the plane with its vertices are at $(4i, 4j)$ and its edges are drawn as a sequence of consecutive line segments on the lines $x = 4i$ or $y = 4j$, for some i and j.*

Lemma 2. *Let $G = (V, E)$ be an instance of* VC-PLA *with $|E| \geq 2$. An instance $G' = (V', E')$ of* VEDS-UDG *can be constructed from G in polynomial-time.*

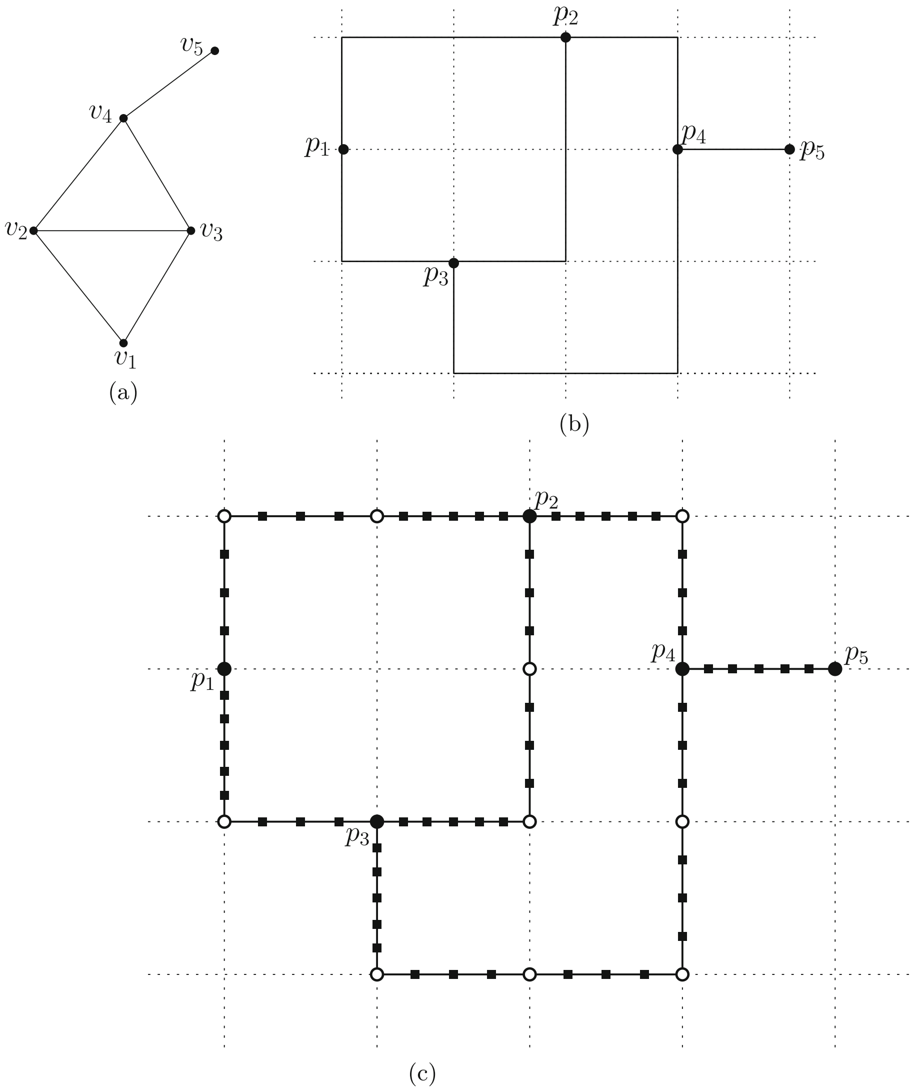

Fig. 1. (a) A planar graph G, (b) its embedding on a grid, and (c) a UDG construction from the embedding.

Proof. Our construction of G' from G is done in three steps.

Step 1: Embedding graph G into a grid of size $4n \times 4n$. G can be embedded in the plane using one of the algorithms [8,10] (see Lemma 1 and Corollary 1) with each of its edges as a sequence of connected line segment(s) of length four units. Let the total number of line segments used in the embedding is ℓ. The points $\{p_1, p_2, \ldots, p_n\}$ are termed *node points* in the embedding correspond to the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ (see Fig. 1(a) and (b)).

Step 2: Adding extra points. For each edge (p_i, p_j) having length 4 units, (i) we add two points α and β on the edge (p_i, p_j) such that α is 0.8 unit apart from p_i and β is 0.8 unit appart from p_j, and (ii) add another three points between α and β with distance 0.6 unit from each other, respectively (thus adding five points in total, see edge (p_4, p_5) in Fig. 1(c)). For each edge of length greater than 4 units, we also add points as follows: (i) add a point in the joining point (grid point) of each line segments other than the *node points* and name it as a *joint point* (see empty circular points in Fig. 1(c)), (ii) for one of the two line segments whose one endpoint is associated with node point, we add five points at distances 0.8, 1.4, 2, 2.6 and 3.2 units from the node point, and for other line segments we add three points at distance 1 units from each other excluding the *joint points* (see the edge (p_3, p_4) in Fig. 1(c)). We name the points added in this step as *added points*.

Step 3: Construction step. For convenience, denote the set of node points by N and set of added points by A, respectively, that is, $N = \{p_i \mid v_i \in V\}$ and $A = \{q_1, q_2, \ldots, q_{4\ell+|E|}\}$. We construct a UDG $G' = (V', E')$, where $V' = N \cup A$ and there is an edge between two points in V' if and only if the Euclidean distance between the points is at most 1 (see Fig. 1(c)). Observe that $|N| = |V|(= n)$ and $|A| = 4\ell + |E|$, where ℓ is the total number of line segments in the embedding and $|E|$ is the total number of edges in G. Since G is planar, $|E| = O(n)$. It also follows from Lemma 1 that $\ell = O(n^2)$. Therefore both $|V'|$ and $|E'|$ are bounded by $O(n^2)$, and hence G' can be constructed in polynomial-time. □

Theorem 1. VEDS-UDG *is NP-complete.*

Proof. For any given set $D \subseteq V$ and a positive integer k, we can verify in polynomial-time whether D is a vertex-edge dominating set of size at most k by checking whether each edge in E is vertex-edge dominated by a vertex in D or not. Hence, VEDS-UDG $\in$ NP.

Now, we need to prove VEDS-UDG $\in$ NP-hard. For the hardness proof, we show a polynomial time reduction from VC-PLA to VEDS-UDG. Let $G = (V, E)$ be an instance of VC-PLA. Construct the instance $G' = (V', E')$ of VEDS-UDG as discussed in Lemma 2. We have the following claim.

Claim. *G has a vertex cover of size at most k if and only if G' has a vertex-edge dominating set of size at most $k + \ell$.*

Necessity. Let $C \subseteq V$ be a vertex cover of G such that $|C| \leq k$. Let $N' = \{p_i \in N \mid v_i \in C\}$, i.e., N' is the set of vertices in G' that correspond to the vertices in C. The idea is to choose one vertex from each segment in the embedding such that the chosen vertex set $A'(\subseteq A)$ together with N', i.e., $N' \cup A'$ will form a VEDs of cardinality $k + \ell$ in G'. As C is a vertex cover in G, every edge in G has at least one of its endpoints in C. Let (v_i, v_j) be an edge in G and assume $v_i \in C$ (the same argument works for $v_j \in C$ or if both v_i and $v_j \in C$). It follows from the construction of G' that the edge (p_i, p_j) is represented as a sequence of line segments in the graph G', where p_i and p_j are nodes in G' corresponding to vertices v_i and v_j in G. Start traversing the segments from p_i, and add each fourth vertex to A' encountered from p_i to p_j in the traversal (see Fig. 2 for an illustration, where both big circles and squares belong to A' while traversing from p_1 to p_2, p_2 to p_3, p_1 to p_3, p_4 to p_2, p_4 to p_5 and p_4 to p_3, respectively).

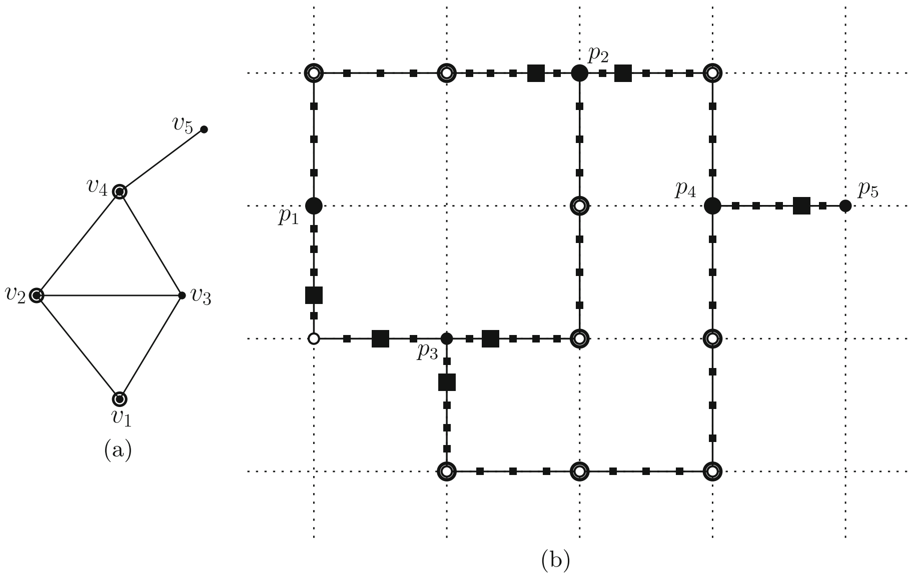

Fig. 2. (a) A vertex cover $\{v_1, v_2, v_4\}$ of G, and (b) the construction of A' in G'

Apply the same process to each chain of line segments in G' corresponding to each edge in G. Observe that the cardinality of A' is ℓ as we have chosen one vertex from each segment in the embedding. Let $D = N' \cup A'$. Now, observe that D is a vertex-edge dominating set in G' as each edge in G' is vertex-edge dominated by at least one vertex in D and $|D| = |N'| + |A'| \leq k + \ell$ as required.

Sufficiency. Let $D \subseteq V'$ be a VEDs of size at most $k + \ell$. We argue that G has a vertex cover of size at most k based upon the following claim: (i) at least one vertex on each segment in the embedding must belongs to D and hence $|A \cap D| \geq \ell$, where ℓ is the total number of segments in the embedding. We shall show that, by removing and/or replacing some vertices in D, a set of at most k vertices from N can be chosen such that the corresponding vertices in G is a vertex cover. Let $C = \{v_i \in V \mid p_i \in D \cap N\}$. If any edge (v_i, v_j) in G has none of its end vertices in C, then consider the points p_i and p_j corresponding to v_i and v_j respectively.

Case (i): If p_i is the only vertex that is connected with p_j in G', then the chain of segments (say ℓ') in the path $p_i \rightsquigarrow p_j$ in G' has at least $\ell' + 1$ vertices in D (see Fig. 3(a) for example). In this case, we delete one point from the segment containing two points in D and introduce p_i in D.

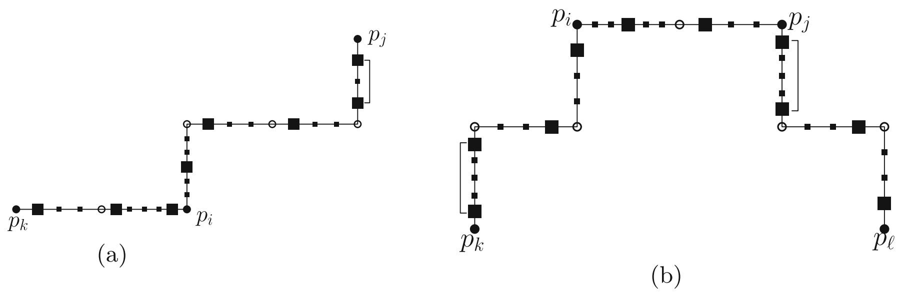

Fig. 3. (a) p_i is connected with only p_j, (b) p_i is connected with p_k and p_j is connected with p_ℓ.

Case (ii): If both p_i and p_j are connected with some points p_k and p_ℓ respectively in G', then either the chain of segments (say ℓ') in the path $p_i \rightsquigarrow p_j$ in G' has at least $\ell' + 1$ vertices in D (see Case (i)) or the chain of segments (say ℓ') in both the path $p_i \rightsquigarrow p_k$ and ($p_j \rightsquigarrow p_\ell$) in G' has at least $\ell' + 1$ vertices in D (see Fig. 3(b) for example). In this case, we choose the segment having two points in D and remove one point of the segment from D and introduce p_j in D if $p_k \in D$ or p_ℓ is only connected with p_j, otherwise introduce p_i in D. Update C and repeat the process till every edge has at least one of its end vertices in C. Due to Claim (i), C is a vertex cover in G with $|C| \leq k$. Therefore, VEDS-UDG is NP-hard.

As VEDS-UDG is in NP as well as NP-hard, VEDS-UDG is NP-complete. □

5 Approximation Scheme

In this section, we propose a PTAS for the VEDs problem in UDGs. Let $G = (V, E)$ be a given UDG. Our PTAS is based on the concept of m-separated collection of subsets of V for some integer m. Given a graph G, let $d(u, v)$ denote the number of edges on a shortest path between u and v. For $V_1, V_2 \subseteq V$, $d(V_1, V_2)$ is defined as $d(V_1, V_2) = min_{u \in V_1, v \in V_2}\{d(u, v)\}$. We use notations $VED(A)$ and $VED_{opt}(A)$ to denote a vertex-edge dominating set of A ($\subseteq V$) in G and an optimal vertex-edge dominating set of A in G. We also define the closed neighborhood of a set $A \subseteq V$ as $N_G[A] = \bigcup_{v \in A} N_G[v]$ and the r-th neighborhood of a vertex v as $N_G^r[v] = \{u \in V \mid d(u, v) \leq r\}$ in G.

Let $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k\}$ be a collection of disjoint vertex subsets in G such that each $\mathcal{S}_i \subset V$ for $i = 1, 2, \ldots, k$. $\mathcal{S}$ is refered as a m-separated collection of vertices if $d(\mathcal{S}_i, \mathcal{S}_j) > m$, for $1 \leq i, j \leq k$ and $i \neq j$ (see Fig. 4 for a 4-separated collection). Nieberg and Hurink [17] considered 2-separated collection to propose a PTAS for the minimum dominating set problem on unit disk graphs.

Lemma 3. *If* $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k\}$ *is a* m*-separated collection in a graph* $G = (V, E)$*, then* $\sum_{i=1}^{k} |VED_{opt}(S_i)| \leq |VED_{opt}(V)|$ *for each* $m \geq 4$.

Proof. For each $\mathcal{S}_i \in \mathcal{S}$, consider $P_i = \{u \in V \mid v \in S_i \text{ and } d(u,v) \leq 2\}$, for $i = 1, 2, \ldots, k$. Since $m \geq 4$, $P_i \cap P_j = \emptyset$ as $d(S_i, S_j) > m$ for $i \neq j$. Observe that, for each $i = 1, 2, \ldots, k$, $\mathcal{S}_i \subseteq P_i$ and $P_i \cap VED_{opt}(V)$ is a vertex-edge dominating set of $\mathcal{S}_i$. Therefore, $(P_i \cap VED_{opt}(V)) \cap (P_j \cap VED_{opt}(V)) = \emptyset$, and hence, we have $\sum_{i=1}^{k} |(P_i \cap VED_{opt}(V))| \leq |VED_{opt}(V)|$. As $P_i \cap VED_{opt}(V)$ is a vertex-edge dominating set of $\mathcal{S}_i$, for $i = 1, 2, \ldots, k$, and $VED_{opt}(V)$ is a minimum vertex-edge dominating set of the graph G, we obtain $\sum_{i=1}^{k} |VED_{opt}(\mathcal{S}_i)| \leq \sum_{i=1}^{k} |(P_i \cap VED_{opt}(V))| \leq |VED_{opt}(V)|$. □

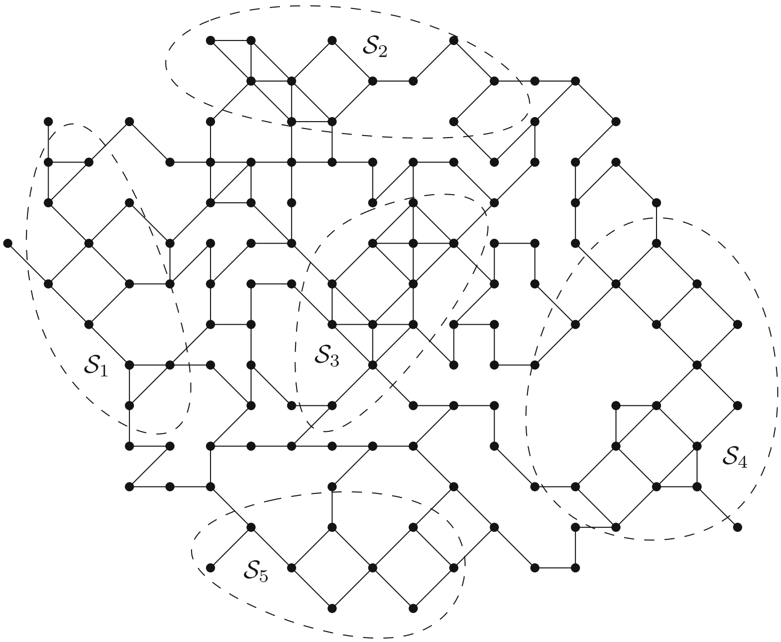

Fig. 4. A 4-separated collection $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5\}$

Lemma 4. *Let $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k\}$ be an m-separated collection in a graph $G = (V, E)$, $m \geq 4$, and let $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$ be subsets of V with $\mathcal{S}_i \subseteq \mathcal{R}_i$ for all $i = 1, 2, \ldots, k$. If there exists $\rho \geq 1$ such that $|VED_{opt}(\mathcal{R}_i)| \leq \rho |VED_{opt}(\mathcal{S}_i)|$ holds for all $i = 1, 2, \ldots, k$, and if $\bigcup_{i=1}^{k} VED_{opt}(\mathcal{R}_i)$ is a vertex-edge dominating set in G, then $\sum_{i=1}^{k} |VED_{opt}(\mathcal{R}_i)|$ is at most ρ times the size of a minimum vertex-edge dominating set in G.*

Proof. $\sum_{i=1}^{k} |VED_{opt}(\mathcal{S}_i)| \leq |VED_{opt}(V)|$ (from Lemma 3).

Hence, $\sum_{i=1}^{k} |VED_{opt}(\mathcal{R}_i)| \leq \rho \sum_{i=1}^{k} |VED_{opt}(\mathcal{S}_i)| \leq \rho |VED_{opt}(V)|$. □

5.1 Construction of Subsets

In this section, we discuss the process of constructing the desired 4-separated collection of subsets $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k\}$ and the corresponding subsets $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$ of V such that $\mathcal{S}_i \subseteq \mathcal{R}_i$ for all $i = 1, 2, \ldots, k$. The algorithm proceeds in an iterative manner. The basic idea of the algorithm is as follows: start with an arbitrary vertex $v \in V_i$, where V_i is the vertex set in the i-th iteration of the algorithm. Note that in the first iteration $V_1 = V$ and the algorithm computes $\mathcal{S}_1$ and $\mathcal{R}_1$. More specifically, for $r = 1, 2, \ldots$, we find the vertex-edge dominating set of the graphs induced by the r-th neighborhood as well as the $(r+4)$-th neighborhood of the vertex v until $|VED(N_G^{r+4}[v])| > \rho |VED(N_G^r[v])|$ holds. Here, $VED(N_G^{r+4}[v])$ and $VED(N_G^r[v])$ are vertex-edge dominating sets of $N_G^{r+4}[v]$ and $N_G^r[v]$, respectively, and $\rho = 1 + \epsilon$ $(\epsilon > 0)$. Let $\hat{r}$ be the smallest r violating the above condition. Set $\mathcal{S}_i = N_G^{\hat{r}}[v]$, $\mathcal{R}_i = N_G^{\hat{r}+4}[v]$ and $V_i' = V_i \setminus N_G^{\hat{r}+3}[v]$. Note that removing $N_G^{\hat{r}+4}[v]$ from V_i implies removing the relevant edges connecting $N_G^{\hat{r}+4}[v]$ to $V_i \setminus N_G^{\hat{r}+4}[v]$ for which vertex-edge domination may not be maintained. Hence, removing $N_G^{\hat{r}+3}[v]$ from V_i removes the edges for which $VED(N_G^{\hat{r}+4}[v])$ is a vertex-edge dominating set. Let T_i be the set of vertices consisting of all singleton vertices after removing $N_G^{\hat{r}+3}[v]$ vertices from V_i in the i-th iteration of the algorithm. Set $V_{i+1} = V_i' \setminus T_i$. The process stops while $V_{i+1} = \emptyset$ and returns the sets $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k\}$ and $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k\}$. The collection of the sets $\mathcal{S}$ is a 4-separated collection.

We compute the vertex-edge dominating set of the r-th neighborhood of a vertex v, $VED(N_G^r[v])$ with respect to G as follows. Find a maximal independent set I for the graph induced by the vertices of $N_G^r[v]$. Observe that if we choose each vertex $v_i \in I$ in $VED(N_G^r[v])$, then it forms a vertex-edge dominating set for $N_G^r[v]$ (see Lemma 5).

Lemma 5. *$VED(N_G^r[v])$ is a VEDs of $N_G^r[v]$ in G.*

Proof. Suppose to the contrary, assume that $VED(N_G^r[v])$ is not a VEDs of the graph $G' = (V', E')$ induced by $N_G^r[v]$. That means, there exist an edge $(u, v) \in E'$ such that $N_{G'}[u] \notin VED(N_G^r[v])$ and $N_{G'}[v] \notin VED(N_G^r[v])$. It contradicts the fact that I is a maximal independent set in G'. Thus, the lemma. □

Lemma 6. *The worst case size of a vertex-edge dominating set of the r-th neighborhood of a vertex v is bounded by $(r+2)^2$, i.e.,$|VED(N_G^r[v])| \leq (r+2)^2$.*

Proof. We compute a maximal independent set I before computing a vertex-edge dominating set in the graph $G' = (V', E')$ induced by $N_G^r[v]$. The cardinality of a maximal independent set in the UDG G' is bounded by the number of non-intersecting unit disks packed in a disk of radius $r+2$ centered at v. So, $|I| \leq \frac{\pi(r+2)^2}{\pi(1)^2} = (r+2)^2$. From Lemma 5, in any graph, the cardinality of a minimum vertex-edge dominating set is bounded by the cardinality of maximal independent set. Therefore, $|VED(N_G^r[v])| \leq (r+2)^2$. □

Lemma 7. *For $\rho = 1+\epsilon$, there always exists an r violating the condition $|(VED(N_G^{r+4}[v])| > \rho|VED(N_G^r[v])|$.*

Proof. Suppose to the contrary that there exists a vertex $v \in V$ such that $|(VED(N_G^{r+4}[v])| > \rho|VED(N_G^r[v])|$ for all $r = 1, 2, \ldots$.

From Lemma 6, we have $|(VED(N_G^{r+4}[v])| \leq (r+6)^2$.

Therefore, if r is even,
$(r+6)^2 \geq |(VED(N_G^{r+4}[v])| > \rho|VED(N_G^r[v])| > \cdots > \rho^{\frac{r}{2}}|VED(N_G^2[v])| \geq \rho^{\frac{r}{2}}$,
and if r is odd,
$(r+6)^2 \geq |(VED(N_G^{r+4}[v])| > \rho|VED(N_G^r[v])| > \cdots > \rho^{\frac{r-1}{2}}|VED(N_G^1[v])| \geq \rho^{\frac{r-1}{2}}$.

Hence,

$$(r+6) > \begin{cases} (\sqrt{\rho})^r, & \text{if r is even} \\ (\sqrt{\rho})^{r-1}, & \text{if r is odd} \end{cases} \quad (1)$$

Observe that in the inequality (1), the right side is an exponential function where as the left side is a polynomial function in r, which results in a contradiction. □

Lemma 8. *The smallest r violating inequality (1) is bounded by $O(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$.*

Proof. Let $\hat{r}$ be the smallest r violating the inequalities (1). We prove $\hat{r} \leq O(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$ by using the inequality $\log(1+\epsilon) > \frac{\epsilon}{2}$ for $0 < \epsilon < 1$. For a fixed $\epsilon > 0$, consider the inequality $(1+\epsilon)^x < x^2$. Let $x = \frac{c}{\epsilon}\log\frac{1}{\epsilon}$, for some constant $c > 0$. By taking logarithm on the both sides of the inequality, we get $\log c + \log\log\frac{1}{\epsilon} > 0$. Note that we can always find ϵ' such that $\log c + \log\log\frac{1}{\epsilon} > 0$ for $0 < \epsilon < \epsilon'$. Therefore, $(1+\epsilon)^x < x^2 < (x+6)^2$ holds for sufficiently smaller ϵ values and hence, $\hat{r} \leq O(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$. □

Lemma 9. *For a given $v \in V$, minimum vertex-edge dominating set $VED_{opt}(\mathcal{R}_i)$ of $\mathcal{R}_i$ can be computed in polynomial time.*

Proof. Let $G' = (V', E')$ be a graph induced by $\mathcal{R}_i \subseteq N_G^{r+4}[v]$. From Lemma 6, the size of $N_G^{r+4}[v]$ is bounded by $O(r^2)$, so we take every possible tuple of size at most $O(r^2)$ and check whether the selected tuple is a vertex-edge dominating set of the graph G'. This process takes $O(\binom{n}{r^2}) = O(n^{r^2})$ time. Since $r = O(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$ by Lemma 8, $VED_{opt}(\mathcal{R}_i)$ can be computed in polynomial time. □

Lemma 10. *For the collection of subsets $\{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k\}$, $\mathcal{D} = \bigcup_{i=1}^{k} VED(\mathcal{R}_i)$ is a vertex-edge dominating set in $G = (V, E)$.*

Proof. To prove $\mathcal{D}$ is a vertex-edge dominating set of the graph G, we need to prove for every edge $(v_i, v_j) \in E$, there exists at least one vertex from $N_G[v_i]$ or $N_G[v_j]$ in $\mathcal{D}$. It follows from our construction of the subsets $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$ (Sect. 5.1) that each edge (v_i, v_j) belongs to a particular subset $\mathcal{R}_i$ and $VED(\mathcal{R}_i)$ is a vertex-edge dominating set of the graph induced by the vertices of $\mathcal{R}_i$. Thus the lemma. □

Corollary 2. $\mathcal{D}^* = \bigcup_{i=1}^{k} VED_{opt}(\mathcal{R}_i)$ *is a vertex-edge dominating set in* G*, for the collection* $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k\}$.

Theorem 2. *For a given UDG,* $G = (V, E)$*, and an* $\epsilon > 0$*, we can design a* $(1 + \epsilon)$*-factor approximation algorithm to find a VEDs in* G *with running time* $n^{O(c^2)}$*, where* $c = O(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$.

Proof. The proof of the theorem follows from Lemmas 4, 7, 9 and Corollary 2.□

6 Conclusion

In this article, we studied the minimum vertex-edge dominating set problem (VEDs) on unit disk graphs, and showed that the VEDs problem is NP-complete. We also proposed a PTAS for the VEDs problem.

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