




The Lexicographic Method for the Threshold Cover Problem

Mathew C. Francis^(✉)  and Dalu Jacob

Indian Statistical Institute, Chennai Centre, Chennai, India
{mathew,dalujacob}@isichennai.res.in

Abstract. The lexicographic method is a technique that was introduced by Hell and Huang [*Journal of Graph Theory*, 20(3): 361–374, 1995] as a way to simplify the problems of recognizing and obtaining representations of comparability graphs, proper circular-arc graphs and proper interval graphs. This method gives rise to conceptually simple recognition algorithms and leads to much simpler proofs for some characterization theorems for these classes. Threshold graphs are a class of graphs that have many equivalent definitions and have applications in integer programming and set packing problems. A graph is said to have a threshold cover of size k if its edges can be covered using k threshold graphs. Chvátal and Hammer conjectured in 1977 that given a graph G , a suitably constructed auxiliary graph G' has chromatic number equal to the minimum size of a threshold cover of G . Although this conjecture was shown to be false in the general case by Cozzens and Leibowitz, it was shown to be true for graphs having a threshold cover of size 2 by Raschle and Simon [*Proceedings of the Twenty-seventh Annual ACM Symposium on Theory of Computing*, STOC '95, pages 650–661, 1995]. That is, a graph G has a threshold cover of size 2 if and only if G' is bipartite—this is the only known forbidden structure characterization of graphs having a threshold cover of size 2. We show how the lexicographic method can be used to obtain a completely new and much simpler proof for this result. This method also gives rise to a simple new LexBFS-based algorithm for recognizing graphs having a threshold cover of size 2. Although this algorithm is not the fastest known, it is a certifying algorithm that matches the time complexity of the fastest known certifying algorithm for this problem. The algorithm can also be easily adapted to give a certifying recognition algorithm for bipartite graphs that can be covered by two chain subgraphs.

Keywords: Lexicographic method · Threshold cover · Chain graph cover

1 Introduction

We consider only simple, undirected and finite graphs. A graph G is said to be a *threshold graph* if it does not contain a pair of edges ab, cd such that $ad, bc \notin E$.

$E(G)$; or equivalently, G is $(2K_2, P_4, C_4)$ -free [1]. A graph $G = (V, E)$ is said to be *covered* by the graphs H_1, H_2, \dots, H_k if $E(G) = E(H_1) \cup E(H_2) \cup \dots \cup E(H_k)$. A graph G is said to have a *threshold cover* of size k if it can be covered by k threshold graphs. The *threshold dimension* of a graph G is defined to be the smallest integer k such that G has a threshold cover of size k . Mahadev and Peled [12] give a comprehensive survey of threshold graphs and their applications.

Chvátal and Hammer [1] showed that the fact that a graph has a threshold cover of size k is equivalent to the following: there exist k linear inequalities on $|V(G)|$ variables such that the characteristic vector of a set $S \subseteq V(G)$ satisfies all the inequalities if and only if S is an independent set of G (see [13] for details). They further defined the auxiliary graph G' (defined in Sect. 2) corresponding to a graph G and showed that any threshold cover of G must have size at least $\chi(G')$. This gave rise to the question of whether there exist any graph G that does not have a threshold cover of size $\chi(G')$. Cozzens and Leibowitz [4] showed the existence of such graphs. In particular, they showed that for every $k \geq 4$, there exists a graph G such that $\chi(G') = k$ but G has no threshold cover of size k . The question of whether such graphs exist for $k = 2$ seems to have been intensely studied but remained open for a decade (see [11]). Ibaraki and Peled [7] showed that if G is a split graph or if G' contains at most two non-trivial components, then $\chi(G') = 2$ if and only if G has a threshold cover of size 2. They further conjectured that every graph G satisfying $\chi(G') \leq 2$ has a threshold cover of size 2. Cozzens and Halsey [3] studied some properties of graphs having a threshold cover of size 2 and show that it can be decided in polynomial time whether the complement of a bipartite graph has a threshold cover of size 2. Finally, in 1995, Raschle and Simon [13] settled the question by extending the methods of Ibaraki and Peled: they showed that every graph G whose auxiliary graph G' is bipartite has a threshold cover of size 2. This proof is very technical and involves the use of a number of complicated reductions and previously known results. In particular, they construct a set of edges that have a “threshold completion” by finding a 2-colouring of G' that is so-called “ AC_{2l} -free”, where $l \geq 2$ (a colouring of G' is AC_{2l} -free, if for each colour class S , there is no cyclical sequence $v_1, v_2, \dots, v_{2l}, v_1$ of vertices in G such that $v_i v_{i+1} \in S$ if and only if i is odd). It is then shown that this reduces to finding a 2-colouring of G' which is AC_6 -free. This further reduces to finding a so-called “ AP_6 -free” 2-colouring of G' which further reduces to finding a so-called “double AP_6 -free 2-colouring” of G' . The most intricate part is the proof of correctness of an algorithm that computes this particular kind of 2-colouring of G' .

The paper of Raschle and Simon also gives an $\mathcal{O}(|E(G)|^2)$ algorithm that checks whether a graph G has a threshold cover of size 2 and outputs two threshold graphs that cover G in case it has. If the input graph G does not have a threshold cover of size 2, the algorithm detects an odd cycle in the auxiliary graph G' . This odd cycle gives edges e_1, e_2, \dots, e_k in G , where k is odd, such that the edges e_i, e_{i+1} , for $1 \leq i < k$, and the edges e_k, e_1 , can never both belong to any threshold subgraph of G (because their endpoints induce a $2K_2, P_4$ or C_4 in G). In this way, the algorithm provides an easily verifiable “certificate” for

the fact that there does not exist two threshold graphs that cover G . If G does have a threshold cover of size 2, then the two threshold graphs returned by the algorithm that cover G form an easily verifiable certificate for that fact. Such algorithms are called *certifying algorithms* [8].

Since as noted above, an odd cycle in the auxiliary graph G' corresponds to a structure present in G that serves as an “obstruction” to it having a threshold cover of size 2, the result of Raschle and Simon can also be seen as a “forbidden structure characterization” of graphs having a threshold cover of size 2. That is, a graph G has a threshold cover of size 2 if and only if the said obstruction is not present in G . Such characterizations are known for many different classes of graphs—for example, interval graphs [9] and circular-arc graphs [5].

In this paper, we propose a completely different and self-contained proof for the theorem of Raschle and Simon that a graph G can be covered by two threshold graphs if and only if G' is bipartite. Our proof is short and direct, and also gives rise to a simpler (although having the same asymptotic worst case running time of $\mathcal{O}(|E(G)|^2)$) certifying recognition algorithm for graphs having a threshold cover of size 2.

Note that faster algorithms for determining if a graph has a threshold cover of size 2 are known. After the algorithm of Raschle and Simon [13], Sterbini and Raschle [15] used some observations of Ma [10] to construct an $\mathcal{O}(|V(G)|^3)$ algorithm for the problem. But this algorithm is not a certifying algorithm in the sense that if the input graph G does not have a threshold cover of size 2, it does not produce an obstruction in G that prevents it from having a threshold cover of size 2. Note that there is an obvious way to make this algorithm a certifying algorithm: if the algorithm answers that the input graph G does not have a threshold cover of size 2, run a secondary algorithm that constructs G' and finds an odd cycle in it (this odd cycle can serve as a certificate). But a naive implementation of the secondary algorithm will take time $\Omega(|E(G)|^2)$, and it is not clear if there is a way to run it in time $o(|E(G)|^2)$.

In the current work, we show that a graph G has a threshold cover of size 2 if and only if its auxiliary graph G' is bipartite using a technique called the *lexicographic method* which was introduced by Hell and Huang [6]. Hell and Huang demonstrated how this method can lead to shorter proofs and simpler recognition algorithms for certain problems that can be viewed as orienting the edges of a graph satisfying certain conditions—for example, they showed how this method can lead to simpler characterization proofs and recognition algorithms for comparability graphs, proper interval graphs and proper circular-arc graphs. The method starts by taking an arbitrary ordering of the vertices of the graph. It then prescribes choosing the lexicographically smallest (with respect to the given vertex ordering) edge to orient and then orienting it in one way or the other, along with all the edges whose orientations are forced by it. Hell and Huang showed that the lexicographic approach makes it easy to ensure that the orientation so produced satisfies the necessary conditions, if such an orientation exists. We adapt this technique to the problem of generating two threshold graphs that cover a given graph, if two such graphs exist. This shows that the

applicability of the lexicographic method may not be limited to only problems involving orientation of edges. However, it should be noted that in our proof, we start with a *Lex-BFS ordering* of the vertices of the graph instead of an arbitrary ordering. It is an ordering of the vertices that gives the order in which a Lex-BFS, or *Lexicographic Breadth First Search*, a graph searching algorithm that was introduced by Rose, Tarjan and Lueker [14], may visit the vertices of the graph. A Lex-BFS ordering always gives an order in which a breadth-first search can visit the vertices of the graph, but has some additional properties. Lex-BFS can be implemented to run in time linear in the size of the input graph and Rose, Tarjan and Lueker originally used this algorithm to construct a linear-time algorithm for recognizing chordal graphs. Later, Lex-BFS based algorithms were discovered for the recognition of many different graph classes (see [2] for a survey).

2 Preliminaries

Let $G = (V, E)$ be any graph. Two edges ab, cd are said to form a pair of *cross edges* in G if $ad, bc \notin E(G)$. If ab, cd form a pair of cross edges in G , we say that the set $\{a, b, c, d\}$ is a *crossing set* in G (such a set is called an AC_4 in [13]). It is easy to see that threshold graphs are exactly the graphs that contain no pairs of cross edges, or equivalently no crossing set.

For a graph G , the auxiliary graph G' is defined to be the graph with $V(G') = E(G)$ and $E(G') = \{e_1e_2 : e_1, e_2 \text{ form a pair of cross edges in } G\}$. We shall refer to the vertex of G' corresponding to an edge $ab \in E(G)$ alternatively as $\{a, b\}$ or ab , depending upon the context. The following lemma is just a special case of the observation of Chvátal and Hammer [1] that a graph G cannot have a threshold cover of size less than $\chi(G')$.

Lemma 1. *If a graph $G = (V, E)$ has a threshold cover of size two then G' is bipartite.*

Proof. Let G be covered by two threshold graphs H_1 and H_2 . By the definition of G' , if $\{ab, cd\} \in E(G')$ then $ad, bc \notin E(G)$. The fact that H_1 and H_2 are threshold subgraphs of G then implies that neither H_1 nor H_2 can contain both the edges ab and cd . We therefore conclude that the sets $E(H_1)$ and $E(H_2)$ are both independent sets in G' . Since G is covered by H_1 and H_2 , we have that $V(G') = E(H_1) \cup E(H_2)$. Thus, $\{E(H_1), E(H_2) \setminus E(H_1)\}$ forms a bipartition of G' into two independent sets. This completes the proof. \square

Our goal is to provide a new proof for the following theorem of Raschle and Simon [13].

Theorem 1. *A graph G can be covered by two threshold graphs if and only if G' is bipartite.*

By Lemma 1, it is enough to prove that if G' is bipartite, then G can be covered by two threshold graphs. In order to prove this, we find a specific 2-coloring of

the non-trivial components of G' using the lexicographic method of Hell and Huang [6].

Let $<$ be an ordering of the vertices of G . Given two k -element subsets $S = \{s_1, s_2, \dots, s_k\}$ and $T = \{t_1, t_2, \dots, t_k\}$ of $V(G)$, where $s_1 < s_2 < \dots < s_k$ and $t_1 < t_2 < \dots < t_k$, S is said to be *lexicographically smaller* than T , denoted by $S < T$, if $s_j < t_j$ for some $j \in \{1, 2, \dots, k\}$, and $s_i = t_i$ for all $1 \leq i < j \leq k$. In the usual way, we let $S \leq T$ denote the fact that either $S < T$ or $S = T$. For a set $S \subseteq V(G)$, we abbreviate $\min_{<} S$ to just $\min S$. Note that the relation $<$ (“is lexicographically smaller than”) that we have defined on k -element subsets of $V(G)$ is a total order. Therefore, given a collection of k -element subsets of $V(G)$, the lexicographically smallest one among them is well-defined.

3 Proof of Theorem 1

Assume that G' is bipartite. Let $<$ denote a Lex-BFS ordering of the vertices of G . The following observation states a well-known property of Lex-BFS orderings [2].

Observation 1. *For $a, b, c \in V(G)$, if $a < b < c$, $ab \notin E(G)$ and $ac \in E(G)$, then there exists $x \in V(G)$ such that $x < a < b < c$, $xb \in E(G)$ and $xc \notin E(G)$.*

We shall now construct a partial 2-coloring of the vertices of G' using the colors $\{1, 2\}$. Notice that choosing a color for any vertex in a component of G' fixes the colors of all the other vertices in that component. Recall that every vertex of G' is a two-element subset of $V(G)$. For every non-trivial component C of G' , perform the following operation: Choose the lexicographically smallest vertex in C (with respect to the ordering $<$) and assign the color 1 to it. This fixes the colors of all the other vertices in C . Note that after this procedure, every vertex of G' that is in a non-trivial component has been colored either 1 or 2. For $i \in \{1, 2\}$, let $F_i = \{e \in V(G') : e \text{ is colored } i\}$. Further, let F_0 denote the set of all isolated vertices in G' . Clearly, F_0 is exactly the set of uncolored vertices of G' and we have $V(G') = F_0 \cup F_1 \cup F_2$. Consider the subgraphs $H_1 = (V, F_1 \cup F_0)$ and $H_2 = (V, F_2 \cup F_0)$ of G . We claim that H_1 and H_2 are two threshold graphs that cover G . Clearly $E(G) = E(H_1) \cup E(H_2)$; so it only remains to be proven that both H_1 and H_2 are threshold graphs. Note that for any edge $ab \in E(G)$, $ab \notin E(H_1) \Rightarrow ab \in F_2$ and $ab \notin E(H_2) \Rightarrow ab \in F_1$.

Observation 2. *If ab, cd form a pair of cross edges in G , then exactly one of the following is true:*

1. $ab \in F_1$ and $cd \in F_2$, or
2. $ab \in F_2$ and $cd \in F_1$.

Therefore, ab and cd cannot be present together in either H_1 or H_2 .

Proof. As ab, cd form a pair of cross edges in G , the vertices ab and cd are adjacent in G' . Therefore one of them will be colored 1 and the other 2 in the

partial 2-coloring of G' . This implies that one of ab, cd belongs to F_1 and the other to F_2 . Since $F_1 = E(H_1) \setminus E(H_2)$ and $F_2 = E(H_2) \setminus E(H_1)$, ab and cd cannot be both present in either $E(H_1)$ or $E(H_2)$. \square

For $i \in \{1, 2\}$, let $\mathcal{P}_i = \{\{x, y, z, w\} : xy, zw \in E(H_i), xw \notin E(G) \text{ and } yz \in E(G) \setminus E(H_i)\}$ and $\mathcal{C}_i = \{\{x, y, z, w\} : xy, zw \in E(H_i), xw, yz \in E(G) \setminus E(H_i)\}$. By Observation 2, it can be seen that the crossing sets in H_i are exactly the elements of $\mathcal{P}_i \cup \mathcal{C}_i$. Define $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Notice that in order to show that both H_1 and H_2 are threshold graphs, we only need to prove that $\mathcal{P} \cup \mathcal{C} = \emptyset$. We shall first show that $\mathcal{P} = \emptyset$. Suppose not. Let $\{a, b, c, d\}$ be the lexicographically smallest element in \mathcal{P} .

Lemma 2. $\{a, b, c, d\} \notin \mathcal{P}_1$.

Proof. Suppose for the sake of contradiction that $\{a, b, c, d\} \in \mathcal{P}_1$. By definition of \mathcal{P}_1 , we can assume without loss of generality that $ab, cd \in E(H_1)$, $ad \notin E(G)$ and $bc \in E(G) \setminus E(H_1)$. Since $E(G) = E(H_1) \cup E(H_2)$, we have that $bc \in E(H_2) \setminus E(H_1)$, which implies that $bc \in F_2$. By the definition of F_2 , we have that bc belongs to a non-trivial component C of G' and has been colored 2. Therefore $\{b, c\}$ is not the lexicographically smallest vertex in C . Let $\{b_k, c_k\}$ be the lexicographically smallest vertex in C (k is defined below). Then we have $\{b_k, c_k\} < \{b, c\}$ and by our construction the vertex $\{b_k, c_k\}$ must have received color 1. Let $bc = b_0c_0, b_1c_1, \dots, b_{k-1}c_{k-1}, b_kc_k$ be a path in C between $\{b, c\}$ and $\{b_k, c_k\}$, where for $0 \leq i < k$, $b_i c_{i+1}, b_{i+1} c_i \notin E(G)$. Note that k is odd, $b_i c_i \in F_2$ for each even i and $b_i c_i \in F_1$ for each odd i , where $0 \leq i \leq k$.

We claim that $ab_i, c_i d \in E(H_1)$ for each even i and $ab_i, c_i d \in E(H_2)$ for each odd i , where $0 \leq i \leq k$. We prove this by induction on i . The case where $i = 0$ is trivial as $b_0 = b$ and $c_0 = c$. So let us assume that $i > 0$. Consider the case where i is odd. As $i - 1$ is even, by the induction hypothesis we have, $ab_{i-1}, c_{i-1} d \in E(H_1)$. As $ab_{i-1}, b_i c_i \in E(H_1)$ and $b_{i-1} c_i \notin E(G)$, by Observation 2, we have that $ab_i \in E(G)$. Now as $ab_i, c_{i-1} d$ form a pair of cross edges in G and $c_{i-1} d \in E(H_1)$ the same observation then implies that $ab_i \in E(H_2)$. Similarly, as $c_{i-1} d, b_i c_i \in E(H_1)$ and $b_i c_{i-1} \notin E(G)$, we have $c_i d \in E(G)$. Again, as $c_i d, ab_{i-1}$ form a pair of cross edges in G and $ab_{i-1} \in E(H_1)$ we have $c_i d \in E(H_2)$. The case where i is even is also similar and hence the claim.

By the above claim, $ab_k, c_k d \in E(H_2)$. Since $b_k c_k \in F_1$, $b_k c_k \notin E(H_2)$. Recalling that $ad \notin E(G)$, we now have that $\{a, b_k, c_k, d\} \in \mathcal{P}_2$. Since $\{b_k, c_k\} < \{b, c\}$, we have that $\{a, b_k, c_k, d\} < \{a, b, c, d\}$, which is a contradiction. \square

Lemma 3. $\{a, b, c, d\} \notin \mathcal{P}_2$.

Proof. Suppose for the sake of contradiction that $\{a, b, c, d\} \in \mathcal{P}_2$. By definition of \mathcal{P}_2 , we can assume without loss of generality that $ab, cd \in E(H_2)$, $ad \notin E(G)$ and $bc \notin E(H_2)$. Recall that $bc \notin E(H_2) \Rightarrow bc \in F_1$. As $bc \in F_1$, the vertex bc belongs to a non-trivial component of G' . Then there exists a neighbor $b'c'$ of bc in G' such that $bc', b'c \notin E(G)$. By Observation 2, $bc \in F_1$ implies $b'c' \in F_2$. Further, $ab, b'c' \in E(H_2)$ and $bc' \notin E(G)$ implies that $ab' \in E(G)$. Now ab', cd

form a pair of cross edges in G . Since $cd \in E(H_2)$, we now have by Observation 2 that $cd \in F_2$ and $ab' \in F_1$. This implies that cd is in a non-trivial component C_1 of G' . Similarly, as $cd, b'c' \in E(H_2)$ and $b'c \notin E(G)$ we have that $c'd \in E(G)$. Now $c'd, ab$ form a pair of cross edges in G . Since $ab \in E(H_2)$, we have by Observation 2 that $ab \in F_2$ and $c'd \in F_1$. This implies that ab is in a non-trivial component C_2 of G' .

We now prove two claims using the fact that $<$ is a Lex-BFS ordering of $V(G)$.

Claim 1. $d < a < c$ is not possible.

Suppose not. Note that $da \notin E(G)$ and $dc \in E(G)$. Then by Observation 1, there exists $x \in V(G)$ such that $x < d < a < c$, $xa \in E(G)$ and $xc \notin E(G)$. Now cd, xa form a pair of cross edges in G . By Observation 2, $cd \in F_2$ implies that $xa \in F_1$. As $bc \in F_1$, $xc \notin E(G)$ and $ab \notin E(H_1)$ (recall that $ab \in F_2$) we then have that $\{x, a, b, c\} \in \mathcal{P}_1$. Further, $x < d$ implies that $\{x, a, b, c\} < \{a, b, c, d\}$ which is a contradiction.

The next claim is symmetric to the claim above, but we give a proof for the sake of completeness.

Claim 2. $a < d < b$ is not possible.

Suppose not. Note that $ad \notin E(G)$ and $ab \in E(G)$. Then by Observation 1, there exists $x \in V(G)$ such that $x < a < d < b$, $xd \in E(G)$ and $xb \notin E(G)$. Now ab, xd form a pair of cross edges in G . By Observation 2, $ab \in F_2$ implies that $xd \in F_1$. As $bc \in F_1$, $xb \notin E(G)$ and $cd \notin E(H_1)$ (recall that $cd \in F_2$) we then have that $\{x, d, c, b\} \in \mathcal{P}_1$. Further, $x < a$ implies that $\{x, d, c, b\} < \{a, b, c, d\}$ which is a contradiction.

As $cd \in F_2$, cd must have received color 2 in the partial 2-coloring of G' . This means that cd is not the lexicographically smallest vertex in the component C_1 . Let $\{c_k, d_k\}$ be the lexicographically smallest vertex in C_1 . Then we have $\{c_k, d_k\} < \{c, d\}$ and by our construction, the vertex $\{c_k, d_k\}$ must have received color 1. Let $cd = c_0d_0, c_1d_1, \dots, c_{k-1}d_{k-1}, c_kd_k$ be a path in C_1 between cd and c_kd_k , where for $0 \leq i < k$, $c_id_{i+1}, c_{i+1}d_i \notin E(G)$. Note that k is odd, $c_id_i \in F_2$ for each even i and $c_id_i \in F_1$ for each odd i , where $0 \leq i \leq k$.

Claim 3. $c_ib, c'd_i \in F_1$ for each even i and $c_ib, c'd_i \in F_2$ for each odd i , where $0 \leq i \leq k$.

We prove this by induction on i . The case $i = 0$ is trivial as $c_0 = c$ and $d_0 = d$. So let us assume that $i > 0$. Consider the case where i is odd. As $i - 1$ is even, we have by the induction hypothesis that $c_{i-1}b, c'd_{i-1} \in F_1$. As $c_id_i, c_{i-1}b \in F_1$ and $c_{i-1}d_i \notin E(G)$, by Observation 2, we have $c_ib \in E(G)$. Now since $c_ib, c'd_{i-1}$ form a pair of cross edges in G (recall that $bc' \notin E(G)$) and $c'd_{i-1} \in F_1$, the same observation then implies that $c_ib \in F_2$. Similarly, as $c_id_i, c'd_{i-1} \in F_1$ and $c_id_{i-1} \notin E(G)$, we have that $c'd_i \in E(G)$. Now since $c'd_i, c_{i-1}b$ form a pair of cross edges in G and $c_{i-1}b \in F_1$, we can deduce as before that $c'd_i \in F_2$. The case where i is even can be proved in the same way. Hence the claim.

Recall that $ab \in F_2$, ab is in a non-trivial component C_2 of G' , and it has color 2 in the partial 2-coloring of G' . Therefore, there exists a lexicographically smallest vertex $\{a_k, b_k\}$ in C_2 which has been colored 1. Clearly, $\{a_k, b_k\} < \{a, b\}$. Let $ab = a_0b_0, a_1b_1, \dots, a_{k-1}b_{k-1}, a_kb_k$, be a path in C_2 between $\{a_k, b_k\}$ and $\{a, b\}$, where for $0 \leq i < k$, $a_ib_{i+1}, a_{i+1}b_i \notin E(G)$. Note that k is odd, $a_ib_i \in F_2$ for each even i and $a_ib_i \in F_1$ for each odd i , where $0 \leq i \leq k$. The following claim is symmetric to Claim 3, but we give a proof for the sake of completeness.

Claim 4. $a_ib', cb_i \in F_1$ for each even i and $a_ib', cb_i \in F_2$ for each odd i , where $0 \leq i \leq k$.

We prove this by induction on i . The case $i = 0$ is trivial as $a_0 = a$ and $b_0 = b$. Consider the case where i is odd. As $i - 1$ is even we have $a_{i-1}b', cb_{i-1} \in F_1$. Now as $a_ib_i, a_{i-1}b' \in F_1$ and $a_{i-1}b_i \notin E(G)$, by Observation 2, we have that $a_ib' \in E(G)$. Now a_ib', cb_{i-1} form a pair of cross edges (recall that $b'c \notin E(G)$) and $cb_{i-1} \in F_1$ the same observation then implies that $a_ib' \in F_2$. Similarly, as $a_ib_i, cb_{i-1} \in F_1$ and $a_ib_{i-1} \notin E(G)$, we have that $cb_i \in E(G)$. Now $cb_i, a_{i-1}b'$ form a pair of cross edges and $a_{i-1}b' \in F_1$, implying that $cb_i \in F_2$. The case where i is even can be proved in the same way. Hence the claim.

Recall that $c_kd_k \in F_1$. By Claim 3, $c_{k-1}b \in F_1$ and $c_kb \in F_2$, implying that $c_kb \notin E(H_1)$. As $c_{k-1}d_k \notin E(G)$ we then have $\{c_{k-1}, b, c_k, d_k\} \in \mathcal{P}_1$. Similarly, as $a_kb_k \in F_1$, $a_kb_{k-1} \notin E(G)$, and by Claim 4, we have $cb_{k-1} \in F_1$ and $cb_k \notin E(H_1)$ (as $cb_k \in F_2$), we have $\{a_k, b_k, c, b_{k-1}\} \in \mathcal{P}_1$. We get the final contradiction from the following claim.

Claim 5. Either $\{a_k, b_k, c, b_{k-1}\} < \{a, b, c, d\}$ or $\{c_{k-1}, b, c_k, d_k\} < \{a, b, c, d\}$.

Suppose $d > a$. By Claim 2, we then have $d > b$. Now, since $\{a_k, b_k\} < \{a, b\}$, we have $\{a_k, b_k, c, b_{k-1}\} < \{a, b, c, d\}$, and we are done. So we shall assume that $d < a$. By Claim 1, we now have that $c < a$, implying that $a > \max\{c, d\}$. If $\min\{c_k, d_k\} < \min\{c, d\}$, then we have $\min\{c_k, d_k\} < a, c, d$, which implies that $\{c_{k-1}, b, c_k, d_k\} < \{a, b, c, d\}$, proving the claim. So we shall assume that $\min\{c_k, d_k\} \geq \min\{c, d\}$. Therefore, since $\{c_k, d_k\} < \{c, d\}$, we have $\min\{c_k, d_k\} = \min\{c, d\}$ and $\max\{c_k, d_k\} < \max\{c, d\}$. Thus we have $a > \max\{c_k, d_k\}$, implying that $\{c_{k-1}, b, c_k, d_k\} < \{a, b, c, d\}$. \square

From Lemmas 2 and 3, it follows that $\mathcal{P} = \emptyset$.

Lemma 4. $\mathcal{C} = \emptyset$.

Proof. Suppose for the sake of contradiction that $\mathcal{C} \neq \emptyset$. Then there exists $i \in \{1, 2\}$ such that $\mathcal{C}_i \neq \emptyset$. Consider an element $\{a, b, c, d\} \in \mathcal{C}_i$. We can assume without loss of generality that $ab, cd \in E(H_i)$, $ad, bc \in E(G) \setminus E(H_i)$. As $ad \in E(G) \setminus E(H_i)$, it belongs to a non-trivial component of G' . Therefore there exists a neighbor $a'd'$ of ad in G' such that $ad', a'd \notin E(G)$. Therefore by Observation 2, we have that $a'd' \in E(H_i)$. As $ab, a'd' \in E(H_i)$, where $ad' \notin E(G)$, by the same observation we then have $a'b \in E(G)$. Now if $a'b \in E(H_i)$, then the fact that $cd \in E(H_i)$, $bc \in E(G) \setminus E(H_i)$ and $a'd \notin E(G)$ implies that $\{a', b, c, d\} \in \mathcal{P}_i$ which is a contradiction to our earlier observation that $\mathcal{P} = \emptyset$. Therefore

$a'b \in E(G) \setminus E(H_i)$. As $ab, a'd' \in E(H_i)$ and $ad' \notin E(G)$, it then follows that $\{a, b, a', d'\} \in \mathcal{P}_i$ which again contradicts the fact that $\mathcal{P} = \emptyset$. This completes the proof. \square

We have now shown that $\mathcal{P} \cup \mathcal{C} = \emptyset$, or in other words, there is no crossing set in either H_1 or H_2 . Thus H_1 and H_2 are two threshold graphs that cover G . We have thus shown that if G' is bipartite then G has a threshold cover of size two. As we already have Lemma 1, this completes the proof of Theorem 1.

4 A Certifying Algorithm

Our proof of Theorem 1 gives an algorithm which when given a graph G as input, either constructs two threshold graphs that cover G , or produces an odd cycle in G' as a certificate that G cannot be covered by two threshold graphs.

Algorithm 2-Threshold-Cover

Input: A graph G .

Output: If G has a threshold cover of size 2, two threshold graphs H_1, H_2 such that they cover G , otherwise the auxiliary graph G' and an odd cycle in it.

1. Run the Lex-BFS algorithm on G (starting from an arbitrarily chosen vertex) to produce a Lex-BFS ordering $<$ of $V(G)$.
2. Construct the auxiliary graph G' .
3. Initialize $V(H_1) = V(H_2) = V(G)$ and $E(H_1) = E(H_2) = \{e \in V(G') : e \text{ belongs to a trivial component of } G'\}$.
4. **While** there exist uncolored vertices in a non-trivial component C of G' , **do**
 - (i) Choose the lexicographically smallest vertex uv in C and assign the color 1 to it.
 - (ii) Complete the 2-coloring of C by doing a BFS starting from the vertex uv . If an odd cycle is detected, return the cycle and exit. Otherwise update $E(H_1) = E(H_1) \cup \{e \in V(C) : e \text{ is colored 1 in } G'\}$, $E(H_2) = E(H_2) \cup \{e \in V(C) : e \text{ is colored 2 in } G'\}$.
5. Output H_1 and H_2 .

Correctness of the algorithm follows from the proof of Theorem 1. The Lex-BFS on G can be done in $\mathcal{O}(|V(G)| + |E(G)|)$ time and the remaining steps in $\mathcal{O}(|V(G')| + |E(G')|)$ time. As G' contains at most $|E(G)|$ vertices and at most $|E(G)|^2$ edges, the running time of this algorithm is $\mathcal{O}(|E(G)|^2)$.

5 The Chain Subgraph Cover Problem

A bipartite graph $G = (A, B, E)$ is called a *chain graph* if it does not contain a pair of edges whose endpoints induce a $2K_2$ in G . A collection of chain graphs $\{H_1, H_2, \dots, H_k\}$ is said to be a *k-chain subgraph cover* of a bipartite graph G if it is covered by H_1, H_2, \dots, H_k . The problem of deciding whether a bipartite

graph G can be covered by k chain graphs, i.e. whether G has a k -chain subgraph cover, is known as the k -chain subgraph cover (k -CSC) problem. He showed that 3-CSC is NP-complete and pointed out that using the results of Ibaraki and Peled [7], the 2-CSC problem can be solved in polynomial time as it can be reduced to the problem of determining whether a split graph can be covered by two threshold graphs. Ma and Spinrad [11] note that a direct implementation of this approach to the 2-CSC problem only gives an $\mathcal{O}(|V(G)|^4)$ algorithm and instead propose an $\mathcal{O}(|V(G)|^2)$ algorithm for the problem. This algorithm works by reducing the 2-CSC problem to the problem of deciding whether a partial order has Dushnik-Miller dimension at most 2. Note that this algorithm does not produce a directly verifiable certificate, such as a forbidden structure in the graph, in case the input graph does not have a 2-chain subgraph cover. Our algorithm can be easily modified to make it an $\mathcal{O}(|E(G)|^2)$ certifying algorithm for deciding if an input bipartite graph G has a 2-chain subgraph cover as explained below. In fact, the only modification that is needed is to change the definition of G' so that two edges of G are adjacent in G' if and only if they induce $2K_2$ in G . As shown below, we can start with an arbitrary ordering of vertices in this case, i.e. we do not need to run the Lex-BFS algorithm to produce a Lex-BFS ordering of the input graph as the first step.

Let $G = (A, B, E)$ be a bipartite graph. We now redefine the meaning of the term ‘‘cross edges’’. Two edges $ab, cd \in E(G)$ are now said to be cross edges if and only if $a, c \in A$, $b, d \in B$ and $ad, bc \notin E(G)$. Note that the meaning of the auxiliary graph G' now changes, but our proof that $\chi(G') \leq 2$ if and only if there exists two graphs H_1, H_2 , each containing no cross edges, such that $E(G) = E(H_1) \cup E(H_2)$ still works verbatim. We could, however, let the ordering $<$ on $V(G)$ be any arbitrary ordering. In that case, we cannot use Observation 1 and any argument that uses it. Note that Observation 1 is used only in the proof of Lemma 3. We show how this proof can be modified so that Observation 1 is no longer needed. Observation 1 is used only in Claims 1 and 2, which in turn are used only in Claim 5. Remove Claims 1 and 2 and replace the proof of Claim 5 with the following proof.

Claim 5. Either $\{a_k, b_k, c, b_{k-1}\} < \{a, b, c, d\}$ or $\{c_{k-1}, b, c_k, d_k\} < \{a, b, c, d\}$.

As $c_{k-1}b, c'd \in F_1$ and $c'b \notin E(G)$, we have $c_{k-1}d \in E(G)$. From Claim 3, we have $c'd_{k-1} \in F_1$. Then, $c'd_{k-1}, cb \in F_1$ where $c'b \notin E(G)$ implies that $cd_{k-1} \in E(G)$. As $c_k d_{k-1}, c_{k-1} d_k \notin E(G)$, we can conclude that $c \neq c_k$ and $d \neq d_k$. Since $c, c_k \in A$ and $d, d_k \in B$, we further have that $c \neq d_k$ and $d \neq c_k$. Therefore we get,

$$c, d > \min\{c_k, d_k\} \quad (\text{as } c_k d_k < cd) \quad (1)$$

From Claim 4, we have $a_{k-1}b' \in F_1$. Now $a_{k-1}b', cb \in F_1$ where $cb' \notin E(G)$ implies that $a_{k-1}b \in E(G)$. Since $cb_{k-1}, ab' \in F_1$ and $cb' \notin E(G)$, we have $ab_{k-1} \in E(G)$. As $a_k b_{k-1}, a_{k-1} b_k \notin E(G)$ we can conclude that $a \neq a_k$ and

$b \neq b_k$. Since $a, a_k \in A$ and $b, b_k \in B$, we further have that $a \neq b_k$ and $b \neq a_k$. Therefore we get,

$$a, b > \min\{a_k, b_k\} \quad (\text{as } a_k b_k < ab) \quad (2)$$

If $a \leq \min\{c_k, d_k\}$ and $d \leq \min\{a_k, b_k\}$, we get by (1) and (2) that $a \leq \min\{c_k, d_k\} < d \leq \min\{a_k, b_k\} < a$, which is a contradiction. Therefore, either $a > \min\{c_k, d_k\}$ or $d > \min\{a_k, b_k\}$. If $a > \min\{c_k, d_k\}$, then by (1), we get $\{c_{k-1}, b, c_k, d_k\} < \{a, b, c, d\}$, and we are done. Similarly, if $d > \min\{a_k, b_k\}$, then by (2), we have $\{a_k, b_k, c, b_{k-1}\} < \{a, b, c, d\}$, again we are done. This proves the claim.

Thus Algorithm 2-Threshold-Cover can be modified into a certifying recognition algorithm for deciding if a bipartite graph has a 2-chain subgraph cover by just changing the definition of G' . Moreover, this algorithm can choose any arbitrary ordering of the vertices of the input graph to start with and hence does not require the implementation of the Lex-BFS algorithm. Note that we do not know the answer to the following question: Would Algorithm 2-Threshold-Cover correctly decide whether the input graph G has a threshold cover of size 2 even if it lets $<$ be an arbitrary ordering of $V(G)$?

6 Conclusion

Chvátal and Hammer [1] showed that the problem of deciding whether an input graph has a threshold cover of size at most k is NP-complete, when k is part of the input. Yannakakis [16] observes that a bipartite graph $G = (A, B, E)$ has a k -chain subgraph cover if and only if the split graph H obtained from G by making every pair of vertices in A adjacent to each other has a threshold cover of size k . He notes that therefore, his proof of the NP-completeness of the 3-CSC problem implies that the problem of deciding if an input graph has a threshold cover of size at most 3 is also NP-complete.

We believe that our result demonstrates once again the power of the lexicographic method in yielding short and elegant proofs for certain kinds of problems that otherwise seem to need more complicated proofs. Further research could establish the applicability of the method to a wider range of problems.

References

1. Chvátal, V., Hammer, P.L.: Aggregations of inequalities. In: Studies in Integer Programming. Annals of Discrete Mathematics, vol. 1, pp. 145–162 (1977)
2. Cornil, D.G.: Lexicographic breadth first search – a survey. In: Hromkovič, J., Nagl, M., Westfechtel, B. (eds.) WG 2004. LNCS, vol. 3353, pp. 1–19. Springer, Heidelberg (2004). https://doi.org/10.1007/978-3-540-30559-0_1
3. Cozzens, M.B., Halsey, M.D.: The relationship between the threshold dimension of split graphs and various dimensional parameters. Discrete Appl. Math. **30**(2), 125–135 (1991)

4. Cozzens, M.B., Leibowitz, R.: Threshold dimension of graphs. *SIAM J. Algebr. Discrete Methods* **5**(4), 579–595 (1984)
5. Francis, M., Hell, P., Stacho, J.: Forbidden structure characterization of circular-arc graphs and a certifying recognition algorithm. In: *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015*, pp. 1708–1727 (2015)
6. Hell, P., Huang, J.: Lexicographic orientation and representation algorithms for comparability graphs, proper circular arc graphs, and proper interval graphs. *J. Graph Theory* **20**(3), 361–374 (1995)
7. Ibaraki, T., Peled, U.N.: Sufficient conditions for graphs to have threshold number 2. In: Hansen, P. (ed.) *Annals of Discrete Mathematics* (11). North-Holland Mathematics Studies, vol. 59, pp. 241–268. North-Holland, Amsterdam (1981)
8. Kratsch, D., McConnell, R., Mehlhorn, K., Spinrad, J.: Certifying algorithms for recognizing interval graphs and permutation graphs. *SIAM J. Comput.* **36**(2), 326–353 (2006)
9. Lekkerkerker, C.G., Boland, J.C.: Representation of a finite graph by a set of intervals on the real line. *Fundamenta Mathematicae* **51**, 45–64 (1962)
10. Ma, T.H.: On the threshold dimension 2 graphs. Technical report. Institute of Information Science, Academia Sinica, Nankang, Taipei, Republic of China (1993)
11. Ma, T.H., Spinrad, J.P.: On the 2-chain subgraph cover and related problems. *J. Algorithms* **17**(2), 251–268 (1994)
12. Mahadev, N.V.R., Peled, U.N.: *Threshold Graphs and Related Topics*, vol. 56. Elsevier, Amsterdam (1995)
13. Raschle, T., Simon, K.: Recognition of graphs with threshold dimension two. In: *Proceedings of the Twenty-Seventh Annual ACM Symposium on Theory of Computing, STOC 1995*, pp. 650–661 (1995)
14. Rose, D.J., Tarjan, R.E., Lueker, G.S.: Algorithmic aspects of vertex elimination on graphs. *SIAM J. Comput.* **5**(2), 266–283 (1976)
15. Sterbini, A., Raschle, T.: An $O(n^3)$ time algorithm for recognizing threshold dimension 2 graphs. *Inf. Process. Lett.* **67**(5), 255–259 (1998)
16. Yannakakis, M.: The complexity of the partial order dimension problem. *SIAM J. Algebr. Discrete Methods* **3**(3), 351–358 (1982)