






Partitioning Cographs into Two Forests and One Independent Set

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Abstract. We consider a variation of arboricity, where a graph is partitioned into p forests and q independent sets. These problems are NP-complete in general, but polynomial-time solvable in the class of cographs; in fact, for each p and q there are only finitely many minimal non-partitionable cographs. In previous investigations it was revealed that when $p = 0$ or $p = 1$, these minimal non-partitionable cographs can be uniformly described as one family of obstructions valid for all values of q . We investigate the next case, when $p = 2$; we provide the complete family of minimal obstructions for $p = 2, q = 1$, and find that they include more than just the natural extensions of the previously described obstructions for $p = 2, q = 0$. Thus a uniform description for all q seems unlikely already in the case $p = 2$.

Our result gives a concrete forbidden induced subgraph characterization of cographs that can be partitioned into two forests and one independent set. Since our proof is algorithmic, we can apply our characterization to complement the recognition algorithm for partitionable cographs by an algorithm to certify non-partitionable cographs by finding a forbidden induced subgraph.

Keywords: Vertex arboricity · Independent vertex feedback set · Cograph · Forbidden subgraph characterization · Colouring · Partition

1 Introduction and Motivation

The *vertex-arboricity* of a graph G is the minimum integer p such that the vertices of G can be partitioned into p parts each of which induces a *forest*. It is, in general, NP-complete to decide if a graph G has arboricity less than or equal to a fixed p , $p \geq 2$ [9]. This is a situation analogous to deciding if a graph G has chromatic number less than or equal a fixed q , $q \geq 3$ [6]. Both problems can be

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efficiently solved on the class of cographs, and in [8], the authors have studied, for cographs, a blended problem, whereby a graph is partitioned into p parts inducing forests and q parts that are independent sets. Each of these problems can be efficiently solved in the class of cographs, and in fact characterized by a finite number of minimal cograph obstructions. This parallels the situation for a similar blended problem studied earlier, where a cograph G is to be partitioned into k independent sets and ℓ cliques [2, 10].

Cographs are one of the most popular and intensively studied classes of perfect graphs. We say that G is a *cograph* if it has no induced subgraph isomorphic to P_4 , the path on 4 vertices. Equivalently [1], cographs can be recursively defined as follows: (i) The graph on single vertex is a cograph; (ii) If G_1, G_2, \dots, G_k are cographs then so is their union, $G_1 \cup G_2 \cup \dots \cup G_k$; and (iii) If G is a cograph, then so is its complement \overline{G} . Since cographs are perfect, many intractable problems can be solved in polynomial time on the class of cographs [7]. Moreover, the recursive description of cographs corresponds to a natural data structure (called a *co-tree* [1]), and partition problems like the chromatic number or arboricity can be solved in linear time directly on the co-tree. This is explicitly done for the chromatic number in [1], and can be done in a very similar fashion for vertex-arboricity. In fact, in [8], the authors similarly solve, for cographs, the blended problem of partition into p forests and q independent sets (and for even more general partitions). Furthermore, it follows from [3] that each of these problems has a characterization by a finite set of minimal cograph obstructions. Here a *minimal cograph obstruction* is a cograph G that does not admit a required partition, but each proper induced subgraph of G does admit such a partition. Thus a cograph admits a required partition if and only if it does not contain an induced subgraph isomorphic to a minimal cograph obstruction.

Minimal cograph obstructions for partition into k independent sets and ℓ cliques were described in [2, 4, 5, 10]; they have $(k+1)(\ell+1)$ vertices, and admit a partition into $k+1$ independent sets of size $\ell+1$ as well as a partition into $\ell+1$ cliques of size $k+1$. In particular, the unique minimal cograph obstruction for partition into k independent sets is K_{k+1} , and the minimal cograph obstruction for partition into ℓ cliques is $\overline{K_{\ell+1}}$ (as is required for perfect graphs).

Minimal cograph obstructions for partition into p forests and q independent sets were investigated in [8]. Consider first the special case of $q = 0$, that is partitions into forests (arboricity). Since cographs are perfect, there are two minimal cograph obstructions for being a forest, i.e., admitting a partition with $p = 1$: these are the cycles C_3 and C_4 . For partitions into $p = 2$ forests, there turn out to be exactly 7 minimal cograph obstructions, forming the family \mathcal{A}_2 depicted in Fig. 1.

Each of these obstructions has a natural generalization to minimal cograph obstruction for partition into p forests. For example, K_5 generalizes to K_{2p+1} , $\overline{3K_3}$ generalizes to $\overline{(p+1)K_{p+1}}$, and so on. These 7 generalizations form a family \mathcal{A}_p , given by an explicit uniform description in [8]. They are all minimal cograph obstructions to partition into p forests. Nevertheless, it turns out that there are in general many additional minimal cograph obstructions, and in fact the number

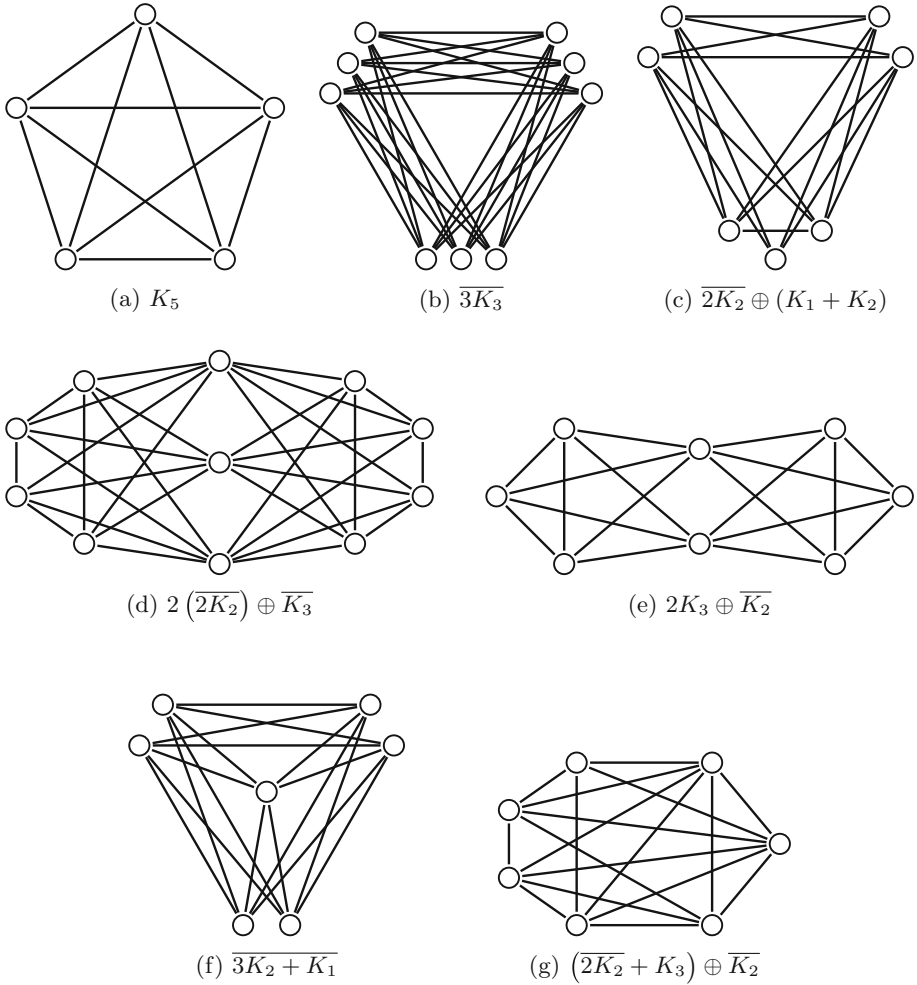


Fig. 1. The family \mathcal{A}_2 .

of minimal cograph obstructions for partition to p forests grows exponentially with p [8].

There is, however, a class of partition problems in which minimal cograph obstructions can be uniformly described. This is the class of problems generalizing the problem of independent vertex feedback set [8]. A q -colourable vertex feedback set of a graph G is a set V of vertices such that $G \setminus V$ admits a q -colouring. Thus a graph admits a q -colourable vertex-feedback set if and only if it has a partition into $p = 1$ forest and q independent sets. It is shown in [8] that there are precisely two minimal cograph obstructions for such a partition,

namely K_{q+3} and $\overline{(q+2)K_2}$. (Note that for $q = 0$ we again obtain C_3 and C_4 as the minimal cograph obstructions to being a forest.) This family describes all minimal cograph obstructions for partitions into $p = 1$ forest and q of independent sets, uniformly for all values of q . As mentioned above, there is only one minimal cograph obstruction for partitions into ($p = 0$ forests and) an arbitrary number q of independent sets, namely K_{q+1} , which is again a family uniformly described for all values of q . This motivates the natural question whether there are other values of p for which such uniformity is possible.

In this paper we investigate the first open case of $p = 2$. In order to address the question of possible uniform description, we explicitly describe all minimal cograph obstructions for partition into $p = 2$ forests and $q = 1$ independent set. Each member of the family \mathcal{A}_2 again has a natural generalization as an obstruction for such a partition. For example, K_5 generalizes to K_6 , because an independent set will take only one vertex and the remaining K_5 cannot be partitioned into 2 forests. Similarly, $\overline{3K_3} = K_{3,3,3}$ generalizes to $K_{3,3,3,3}$, $\overline{3K_2 + K_1} = K_{2,2,2} \oplus K_1$ generalizes to $K_{2,2,2,2} \oplus K_1$, $\overline{2K_2} \oplus (K_1 + K_2) = K_{2,2} \oplus \overline{K_{1,2}}$ generalizes to $K_{2,2,2} \oplus \overline{K_{1,2}}$, and so on. Below, we present a complete set \mathcal{F} of minimal cograph obstructions for partition into $p = 2$ forests and $q = 1$ independent set. The family \mathcal{F} contains 9 cographs, and while most of them can be interpreted as generalizations of members of \mathcal{A}_2 , some appear to be definitely new. In particular, the last member, 9, of the family \mathcal{F} , does not seem to arise from any member of the family \mathcal{A}_2 in any obvious fashion. Thus the evidence suggests that a uniform description of minimal cograph obstructions for all $(2, q)$ -partitions seems unlikely.

The reference [8] presents a linear-time dynamic programming algorithm to decide whether an input cograph G admits a $(2, 1)$ -partition (or any other (p, q) -partition. As an application of our result we explain how to certify a negative outcome of the algorithm (i.e., a non-partitionable cograph G) by finding an actual forbidden induced subgraph.

2 The List of Minimal Obstructions

For brevity, we call a partition of a graph G into p forests and q independent sets, a (p, q) -partition of G . Thus, in the remainder of the paper, we describe all minimal cograph obstructions for $(2, 1)$ -partition.

We introduce the family of cographs \mathcal{F} . The members of the family are:

1. K_6
2. $K_{3,3,3,3}$
3. $K_{2,2,2} \oplus \overline{K_{1,2}}$
4. $K_{2,2,2,2} \oplus K_1$
5. $2K_3 \oplus K_{2,2}$
6. $(K_{2,2} + K_3) \oplus K_{2,2}$
7. $2K_{2,2} \oplus K_{3,3}$
8. $2K_{2,2} \oplus 2K_{2,2}$
9. $(K_4 + K_{3,3,3}) \oplus \overline{K_2}$

Lemma 1. *Each graph in \mathcal{F} is a minimal cograph obstruction to $(2, 1)$ -partition.*

Proof. It is clear from their descriptions that each graph in the \mathcal{F} family is a cograph. We claim that each of these graphs is a minimal obstruction for $(2, 1)$ -partition.

Consider first K_6 : it does not have a $(2, 1)$ -partition, because any forest in K_6 can have at most two vertices, and hence two forests can have at most four vertices. This leaves at least two vertices, but no two vertices in K_6 form an independent set. Moreover, when a vertex is removed we have K_5 , which has an obvious $(2, 1)$ -partition where each forest is one edge and the independent set is a single vertex. Therefore K_6 is a minimal obstruction.

For the graph $G = K_{3,3,3,3}$, we observe that any induced forest in G has at most four vertices, and this happens only when the forest is a tree. Thus two forests can cover at most eight vertices, and since $K_{3,3,3,3}$ has no independent set of size four, it does not have a $(2, 1)$ -partition. When a vertex is removed, we obtain $K_{2,3,3,3}$, where we can take one independent set consisting of a part with three vertices, and cover the vertices of the remaining two parts of size three by stars centered at the remaining two vertices. Hence $K_{3,3,3,3}$ is also a minimal obstruction.

The proof for most of the remaining obstructions follows a similar approach, and we skip the details (which are included in the last section). We do include the proof for the last two obstructions on our list, which are more interesting.

Consider the graph $2K_{2,2} \oplus 2K_{2,2}$ from 8. Any independent set must be on one side of the join, and include at most four vertices. The remaining vertices contain an induced $2(2\overline{K_2}) \oplus \overline{K_3}$, which is one of the obstructions for $(2, 0)$ -partition from Fig. 1. When a vertex is deleted, we obtain the graph $((K_{1,2} + K_{2,2}) \oplus 2K_{2,2})$, which has the following $(2, 1)$ -partition: one independent set of four vertices on the bigger side of the join, one forest consisting of $2K_{1,2}$ on the smaller side of the join, and one forest which is a star on five vertices. Thus $2K_{2,2} \oplus K_{2,2}$ is indeed a minimal cograph obstruction for $(2, 1)$ -partition.

Finally, we prove that the graph $(K_4 + K_{3,3,3}) \oplus \overline{K_2}$ is a minimal cograph obstruction for $(2, 1)$ -partition. We consider what an independent set S must contain in order for none of the minimal cograph obstructions for $(2, 0)$ -partition (from Fig. 1) to remain after S is removed. Note that our graph contains $K_{2,3,3,3}$, while in Fig. 1 there is both a $K_{3,3,3}$ and a $K_{1,2,2,2}$. Moreover, when S is removed there must not remain a copy of K_5 . To satisfy just these restrictions, S must contain one vertex of the K_4 , and three vertices of one entire part of the $K_{3,3,3}$. Since this is a maximal independent set, S must be this set; but then its removal results in a graph containing an induced $(2\overline{K_2} + K_3) \oplus \overline{K_2}$ (the last graph in Fig. 1). It remains to partition the graphs resulting from deleting a vertex from $(K_4 + K_{3,3,3}) \oplus \overline{K_2}$. If a vertex in the K_4 is deleted, then we obtain a $(2, 1)$ -partition by taking the independent set S as above, and two stars centered at the two vertices of the $\overline{K_2}$, each involving one 3-vertex part of the $K_{3,3,3}$ and one vertex of the K_4 . If a vertex of the $\overline{K_2}$ is deleted, we can take again the independent set S , one forest consisting of an edge from the $\overline{K_4}$ and one part of the $K_{3,3,3}$, and one star centered at the other vertex of the $\overline{K_2}$. If a vertex v in

the $K_{3,3,3}$ is deleted, we can take for the independent set the vertices in the $\overline{K_2}$, and partition the remaining vertices into two forests each consisting of one edge of the K_4 and one star on four vertices.

3 The Completeness of the List

We now prove that the list of minimal cograph obstructions for $(2, 1)$ -partition given in Lemma 1 is complete.

Theorem 1. *A cograph has a $(2, 1)$ -partition if and only if it does not contain an induced subgraph from \mathcal{F} .*

Proof. Let G be a cograph. It is easy to see that a disconnected cograph G admits a $(2, 1)$ -partition if and only if each connected component of G admits a $(2, 1)$ -partition. Thus we may assume G is a connected cograph which does not contain an induced subgraph from \mathcal{F} , and proceed to prove it has a $(2, 1)$ -partition.

For brevity, we shall say that a graph is F -free if it does not contain F as an induced subgraph, and \mathcal{F} -free, if it doesn't contain any member of the family \mathcal{F} as an induced subgraph.

Since G is connected, there exist cographs G_1 and G_2 such that $G = G_1 \oplus G_2$. If G_1 and G_2 are forests, then G trivially has a $(2, 1)$ -partition. So, at least one of G_1, G_2 must contain an induced cycle. Without loss of generality, assume that at least G_1 has an induced cycle; since G_1 is a cograph, the only cycles possible are C_3 or C_4 .

1. **Assume G_1 is C_3 -free.** In this case G_1 has an induced C_4 ; moreover, G_1 is a bipartite graph. We will take a concrete bipartition and refer to (X, Y) as the parts. If G_2 is a forest, then we have a trivial $(2, 1)$ -partition with two independent sets and a forest. Thus we may assume that G_2 also has a cycle. We have the following two subcases:

(a) **Both G_1 and G_2 are C_3 free.** This implies that both cographs G_1 and G_2 are bipartite, and each has an induced C_4 . Both G_1 and G_2 cannot have more than one connected component with C_4 because G is $2K_{2,2} \oplus 2K_{2,2}$ -free. Hence without loss of generality we may assume that G_2 has exactly one component, say A , with a C_4 , and the other components are trees. Note that A must be a complete bipartite graph since G_2 has no induced P_4 . The graph G_1 must also contain at least one connected component, say B , which is a complete bipartite graph. If G_1 has other components with an induced C_4 , then one of the parts of A in G_2 has exactly two vertices, because G is $2K_{2,2} \oplus K_{3,3}$ -free. If the other connected components of G_1 are trees, then one of the subgraphs A or B has a bipartition with one of the parts having exactly two vertices, since G is $K_{3,3,3,3}$ -free. In either case, we can obtain a $(2, 1)$ -partition of G as follows. Suppose the connected component A of the graph G_2 has a bipartition (X, Y) , where X has exactly two vertices. The first forest is obtained by taking one vertex from X , the entire other part Y , and the remaining tree components of G_2 . Since graph G_1 is also bipartite, another forest can be obtained by taking one of the parts of

G_1 and the remaining vertex in X . The remaining vertices form an independent set in G_1 .

(b) G_1 is C_3 free but G_2 contains a C_3 . Since G_1 contains a $K_{2,2}$ and since G is $2K_3 \oplus K_{2,2}$ -free and $(K_{2,2} + K_3) \oplus K_{2,2}$ -free, there is exactly one component of G_2 with a C_3 , and the other components of G_2 are forests. Let the set (v_1, v_2, v_3) induce a C_3 in G_2 , and let B be the component of G_2 containing it. Since B is a connected cograph, we have $B = B_1 \oplus B_2$ for cographs B_1, B_2 . The component B cannot contain an induced K_4 and hence none of the graphs B_1 and B_2 have a C_3 . So, we assume without loss of generality that $v_1, v_2 \in V(B_1)$, and $v_3 \in V(B_2)$; moreover, B_2 must be an independent set since G is K_6 -free. If B_2 has at least two elements, then B_1 must be a K_2 , since G is $K_{2,2,2} \oplus \overline{K_{1,2}}$ -free and $K_{2,2,2,2} \oplus K_1$ -free. Hence either B_1 is a K_2 or B_2 is a K_1 . We construct a $(2, 1)$ -partition in both the cases.

When $B_1 = K_2$, then taking one of the parts of the bipartite graph G_1 along with one vertex in B_1 we obtain one first forest in our partition. To construct the second forest we include B_2 along with the remaining vertex in B_1 and the remaining tree components of G_2 . The remainder in G_1 is the independent set in the partition.

When B_2 consists of a single vertex, then taking this vertex with one of the parts of the bipartite graph G_1 yields the first forest in the partition. The remaining parts of G_2 form a forest which becomes the second forest in the partition. The remaining part of G_1 is our independent set in the $(2, 1)$ -partition.

This concludes the first case.

2. Assume that G_1 contains C_3 . Without loss of generality we can assume that G_2 is a forest as otherwise we have a K_6 , or a situation symmetric to the case 1(b). We consider several possible cases, noting that in all the cases where G_2 has at least one edge, G_1 does not contain K_4 , since G is K_6 -free.

(a) **Suppose first that G_2 has at least three vertices and at least one edge.** Consider a copy of C_3 on v_1, v_2, v_3 in G_1 , and the component B of G_1 containing it. Since B is a connected cograph, we have $B = B_1 \oplus B_2$ for cographs B_1, B_2 . Since B does not contain a K_4 , neither B_1 nor B_2 can contain a C_3 . So, we assume without loss of generality that $v_1, v_2 \in V(B_1)$, and $v_3 \in V(B_2)$; moreover, B_2 is an independent set. If B_1 has an induced C_4 , then B_2 will be just a single vertex because G is $K_{2,2,2} \oplus \overline{K_{1,2}}$ -free and $K_{2,2,2,2} \oplus K_1$ -free. (Note that G_2 contains a copy of $\overline{K_{1,2}}$ or $K_{1,2}$.) In conclusion, each component $B = B_1 \oplus B_2$ of G_1 which contains a C_3 either has a single vertex in B_2 and a bipartite B_1 , or an independent set B_2 and a forest B_1 . Each component of G_1 without a C_3 is bipartite.

We find a $(2, 1)$ -partition of the graph G as follows. One forest will be formed by the vertices in G_2 . We partition G_1 into a forest and an independent set; it suffices to partition each component B of G_1 separately. A component $B = B_1 \oplus B_2$ with C_3 which has a single vertex v in B_2 yields a star centered at v and using one part of the bipartition of B_1 , with the other part of the bipartition yielding an independent set. In a component $B = B_1 \oplus B_2$ with C_3 where B_2 is an independent set and B_1 is a forest, we trivially have a desired partition.

Finally, each remaining component B is bipartite and one part can be taken as a forest and the other part as an independent set.

(b) **Assume G_2 has exactly two vertices, which are adjacent.** Since G_1 does not contain K_4 , it is three colourable. One of the colour classes along with one vertex of G_2 forms one forest of the partition. Another colour class with the other vertex of G_2 yields another forest. The remainder is a single colour class which forms the independent set of the partition.

(c) **Assume G_2 has exactly two vertices, which are not adjacent.** If G_1 does not contain an induced K_4 , we obtain a partition of G as in case 2(b); so we assume that G_1 has a K_4 . Note that we may take G_2 for the independent set of a $(2, 1)$ -partition, and it remains to find a partition of G_1 into two forests (a $(2, 0)$ -partition). Clearly, it suffices to find such a partition for each component B of G_1 separately.

Note that while at least one component of G_1 has a K_4 , there could be other components B of G_1 without a K_4 . Such components B must have a $(2, 0)$ -partition because otherwise G_1 contains a minimal cograph obstruction for $(2, 0)$ -partition from the family \mathcal{A}_2 , and adding the independent set G_2 would yield a member of \mathcal{F} . (This can be easily seen by comparing the two families.)

Now we consider components $B = B_1 \oplus B_2$ of G_1 which do contain a K_4 .

Suppose first that both B_1, B_2 are bipartite. Note that both B_1 and B_2 cannot contain an induced C_4 since G is $K_{2,2,2,2} \oplus K_1$ -free. If both B_1 and B_2 are forests, then we have a trivial partition of B into two forests. Hence, say, B_1 has a C_4 and B_2 is a forest. In fact, B_2 must be just an edge, say uv , because G is $K_{2,2,2,2} \oplus K_1$ -free and $K_{2,2,2} \oplus \overline{K_{1,2}}$ -free. In this case a $(2, 0)$ -partition of B is formed by taking one star centered at u with one part of the bipartition of B_1 , and one star centered at v with the other part of B_1 .

Thus we may assume that one of B_1, B_2 , say B_1 , contains a C_3 . Since G is K_6 -free, B is K_5 -free, and so B_2 must be an independent set. Now we further consider each component $D = D_1 \oplus D_2$ of B_1 . At least one such component D' must contain a C_3 , but there could also be bipartite components D ; all must be K_4 -free.

If B_2 has at least two vertices, then exactly one component, namely D' , of B_1 has a cycle (specifically a C_3). Bipartite components D cannot have a cycle (i.e., a C_4), because G is $(K_3 + K_{2,2}) \oplus K_{2,2}$ -free. Moreover no other component $D \neq D'$ can have a C_3 , because G is $2K_3 \oplus K_{2,2}$ -free. Hence if B_2 has at least two vertices then all the components D of B_1 , other than D' , are forests.

Suppose that v_1, v_2, v_3 form a C_3 in D . Since $D = D_1 \oplus D_2$ is K_4 -free, neither of the graphs D_1, D_2 has a C_3 . So we may assume $v_1, v_2 \in V(D_1)$ and $v_3 \in V(D_2)$; moreover we may assume D_1 is a bipartite graph and D_2 is an independent set.

If the bipartite graph D_1 contains a C_4 , then both D_2 and B_2 must consist of a single vertex because G is $K_{2,2,2,2} \oplus K_1$ -free.

If D_1 is a forest with more than the two vertices v_1, v_2 , then it contains an induced $K_{1,2}$ or $\overline{K_{1,2}}$. Therefore, at least one of D_2, B_2 must be a single vertex, since G is $K_{2,2,2,2} \oplus K_1$ -free and $K_{2,2,2} \oplus \overline{K_{1,2}}$ -free.

Otherwise D_1 is just the edge v_1v_2 .

Finally, if there is no C_3 in D , i.e., D is bipartite, then D_1 is an independent set.

We now describe a $(2, 0)$ -partition of $B = B_1 \cup B_2$. Recall that B_2 is an independent set, and B_1 consists of components $D = D_1 \oplus D_2$ where each D_2 is an independent set and each D_1 is bipartite, with the following four possibilities: (i) D_1 contains a C_4 , in which case D_2 , as well as B_2 , has a single vertex; (ii) D_1 is a forest of more than two vertices, in which case D_2 or B_2 has a single vertex; (iii) D_1 is an edge v_1v_2 ; or (iv) D_1 is an independent set. Moreover, in cases (ii–iv), if B_2 has more than one vertex, then all but one component D of B_1 are forests.

We first describe a $(2, 0)$ -partition of $B = B_1 \cup B_2$ when B_2 has at least two vertices. In this case, there is one component $D' = D'_1 \oplus D'_2$ of B_1 with D'_1 a forest with one or more vertices (cases (ii, iii)), and all other components D of B are forests themselves. We obtain a $(2, 0)$ -partition of G_1 as follows. If D'_1 is just an edge, say xy , the first forest consists of a star centred at the vertex x covering the independent set D'_2 , along with the rest of the forest components of B'_1 . The second forest is a star centred at the remaining vertex y covering the independent set B_2 . If D'_1 has at least two vertices, then D'_2 is a single vertex u , and we can take D'_1 together with all other components D as one forest; the other forest will be the star centered at u and covering B_2 .

Now consider a component $B = B_1 \cup B_2$ of G_1 when B_2 has a single vertex, say v . We put together one forest for a $(2, 0)$ -partition of B from the following forests in the various components $D = D_1 \oplus D_2$ of B . From components D of type (i) we take the star centered at the single vertex of D_2 and covering one part of the bipartition of D_1 ; from components D of type (ii–iv) we take the forests D_1 . The other forest for a $(2, 0)$ -partition of B will be formed by a star centered at v and covering all the remaining vertices. (These are the other parts of all D_1 for components of type (i), as well as all D_2 for components of type (ii–iv); note that this is an independent set of vertices.)

(d) **Finally, we assume that G_2 is just a single vertex, say v .** The proof here is similar to the case 2(c), except that in the case (i), when D_1 contains an induced C_4 , we can only claim that B_2 or D_2 is a single vertex, and in the case (ii), when D_1 is a forest with more than two vertices, we cannot claim anything about the size of B_2 or D_2 (Fig. 2).

Nevertheless, there is a $(2, 1)$ -partition of the entire G . (Since G_2 is a single vertex v , we may use v to form a star for the forests of the partition, and we no longer use G_2 as the independent set.) Before describing the partition, recall that G consists of a vertex v adjacent to all other vertices, and $G \setminus v$ has components $B = B_1 \oplus B_2$ of two kinds, either B_2 is a single vertex, or B_2 is an independent set with at least two vertices. For components $B' = B'_1 \oplus B'_2$ of the first kind (where B'_2 is a single vertex w), we only note that B'_1 consists of bipartite components

D. For the components B of the second kind (where B_2 is a larger independent set), we distinguish components $D' = D'_1 \oplus D'_2$ in which D'_2 consists of a single vertex z , and other components $D = D_1 \oplus D_2$ where D_2 is a larger independent set and D_1 is a forest. We now describe the first forest of a $(2, 1)$ -partition of G . It is a star centered at v and covering the sets D_2 of all components D of B'_1 for the components B' of the first kind (where B'_2 is a single vertex), as well as the sets B_2 of all components B of the second kind. The second forest of the partition contains, for each component B' of the first kind (where B'_2 is a single vertex w), a star centered at w and covering all first parts of the bipartitions of all D_1 of the components D of B'_1 . It also contains, for each component B of the second kind, and each component D' in which D'_2 consists of a single vertex z , a star centered at z and covering the first part of the bipartition of D'_1 , and containing D_1 for each component D in which D_1 is a forest. The remaining vertices are easily seen to form an independent set which we take for the desired $(2, 1)$ -partition of G .

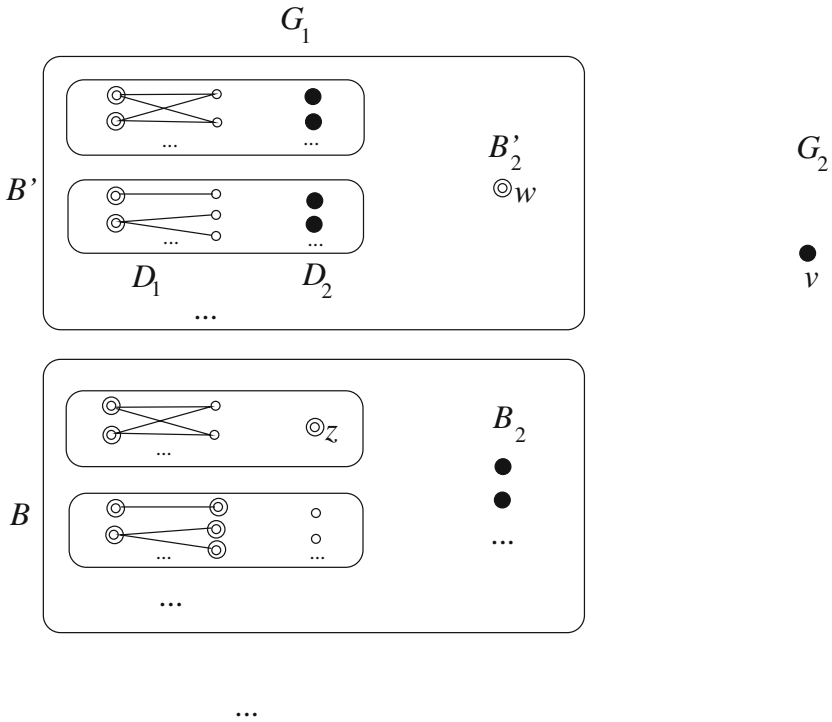


Fig. 2. An illustration of the case 2(d): one forest is indicated by large filled circles, the other forest by double circles, the remainder is independent

4 Conclusions

Theorem 1 implies that any cograph that is not $(2, 1)$ -partitionable must have a member of \mathcal{F} as an induced subgraph. In [8] there is a linear-time dynamic programming algorithm to recognize $(2, 1)$ -partitionable cographs. It computes for each cograph G the set of all triples (p, q, r) such that $p + q + r \leq 3, p \leq 2, q \leq 1$, and G has a set R with r vertices such that $G \setminus R$ is (p, q) -partitionable. The cograph G is $(2, 1)$ -partitionable if and only if $(2, 1, 0)$ is included in this set of triples. To compute these triples is trivial when G has one vertex, and explicit formulas are given for computing them when $G = G_1 + G_2$ and when $G = G_1 \oplus G_2$. Thus a bottom-up calculation on the cotree of G (a data structure that captures the description of the recursive construction of G) allows us to compute the triples for G . If the triple $(2, 1, 0)$ is not present, we can apply a top-down process to actually recover a forbidden induced subgraph from the family \mathcal{F} . We can identify in the cotree a cograph G' , subgraph of G , which does not have a $(2, 1)$ -partition, but all of whose descendants have a $(2, 1)$ -partition. Clearly, this means that G' was obtained by a join operation, and we may assume $G' = G_1 \oplus G_2$. Reading the proof of Theorem 1, we see that the relevant information we need includes whether or not G_1, G_2 contain an induced C_3 and where (or whether they are bipartite), and similarly for C_4 , or K_4 ; we would also like to keep track of how many vertices they contain. Moreover, if G_1 or G_2 are themselves obtained by a disjoint union operation, we need similar information about their descendants. It is easy to see all this information can be computed during the construction of the cotree, so we assume we have it available for G' and its descendants. Then the proof of Theorem 1 directly specifies how to find a forbidden induced subgraph, since G' is known to not be $(2, 1)$ -partitionable. For example, if G_1 and G_2 are both bipartite and both are joins of two bipartite cographs each of which contains a C_4 , then case 1(a) of the proof explains we can identify (from those C_4 's) an induced copy of $2K_{2,2} \oplus 2K_{2,2}$. The remaining cases are similar, with the exception of components B without K_4 in case 2(c). In this case we rely on the corresponding results of [8], which identify forbidden induced subgraphs for a $(2, 0)$ -partition, to which we need to add one or both vertices of G_2 . It is not difficult to implement all this in linear time. A detailed implementation will be presented in the third author's M.Sc. thesis.

5 The Remaining Proofs for Lemma 1

For the graph $G = K_{2,2,2} \oplus \overline{K_{1,2}}$, we note that any subgraph of G on at least four vertices contains an induced cycle. Hence, one forest in the partition can cover at most three vertices and two forest can cover at most six vertices, and since G has no independent set of size three, $K_{2,2,2} \oplus \overline{K_{1,2}}$ is an obstruction. For $H = K_{1,2,2} \oplus \overline{K_{1,2}}$, one of the two forests will be $\overline{K_{1,2}}$, and removing another forest on three vertices, the remainder is an independent set on two vertices, yielding a required $(2, 1)$ -partition. For $H = K_{2,2,2} \oplus K_2$, take one of the vertices

of K_2 with one of the parts in $K_{2,2,2}$ to obtain one forest. We obtain the other forest in similar way and the remainder is just an independent set of size two. Hence, $K_{2,2,2} \oplus \overline{K_{1,2}}$ is a minimal obstruction.

To prove that $G = K_{2,2,2,2} \oplus K_1$, is an obstruction, note that any forest must be a tree and hence can have at most three vertices. Two forests can cover at most six vertices, and the remaining three vertices will contain an edge, and hence not be independent. To prove that G is indeed minimal, note that both $H_1 = K_{2,2,2,2}$ and $H_2 = K_{1,2,2,2} \oplus K_1$ have a $(2, 1)$ -partition in which each forest is a tree on three vertices and the independent set has two vertices.

For $G = 2K_3 \oplus K_{2,2}$, one of the two forests can cover at most four vertices and the other forest can cover at most three vertices. Hence, two forests can cover at most seven vertices and there is no independent set of size at least three in G . Hence, G does not have a $(2, 1)$ -partition. Now we will show that $H_1 = (K_2 + K_3) + \oplus K_{2,2}$ and $2K_3 \oplus K_{1,2}$ have a $(2, 1)$ -partition. For H_1 , the partition consists of one forest that is $2\overline{K_2}$ and has four vertices. The other forest is a tree on three vertices and the remainder is just an independent set on two vertices. In H_2 , the first forest consists of the middle vertex in $K_{1,2}$ along with one vertex in the each of the K_3 . The other forest has four vertices consisting of $2K_2$ each. The remainder is an independent set on two vertices. Thus $G = 2K_3 \oplus K_{2,2}$ is indeed minimal.

Similarly, for $G = (K_{2,2} + K_3) \oplus K_{2,2}$, one of the two forests in G can have at most five vertices, and then the other forest can have at most three vertices. The remainder will have at least 3 vertices. Since G does not have an independent set of size three, G does not have a $(2, 1)$ -partition. We will prove that all the graphs obtained from deleting one vertex from G have a $(2, 1)$ -partition. That is, $H_1 = (K_{1,2} + K_3) \oplus K_{2,2}$, $H_2 = (K_{2,2} + K_2) \oplus K_{2,2}$ and $H_3 = (K_{2,2} + K_3) \oplus K_{1,2}$, each have a $(2, 1)$ -partition. For the graph H_1 , the partition has one forest on five vertices consisting of $K_{1,2}$ and K_2 , and the other forest is just a $K_{1,2}$, leaving an independent set on two vertices.

For $G = 2K_{2,2} \oplus K_{3,3}$, two forests can cover at most ten vertices. Either there is only one forest on six vertices and the other forest then can have at most four vertices, or one can obtain two forest on five vertices each. There is no independent set on four vertices., so G does not admit a $(2, 1)$ -partition. To see that G is indeed minimal, note that both $H_1 = (K_{2,2} + K_{1,2}) \oplus K_{3,3}$ and $H_2 = 2K_{2,2} \oplus K_{2,3}$ have a $(2, 1)$ -partition. For H_1 one such partition has one forest consisting of two copies of $K_{1,2}$ and other forest is a $K_{1,3}$, leaving an independent set of three vertices. For H_2 , a partition can be obtained with two forests which are stars on five vertices each, and the remainder is just an independent set on three vertices.

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