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On the Geodetic and Hull Numbers of Shadow Graphs

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Abstract. Given two vertices u, v in a graph G, a shortest (u, v)-path in G is called an (u, v)-geodesic. Let $I_G[u, v]$ denote the set of all vertices in G lying on some (u, v)-geodesic. Given a set $T \subseteq V(G)$, let $I_G[T] = \cup_{u,v \in T} I_G[u, v]$. If $I_G[T] = T$, we call T a convex set. The convex hull, denoted by $\langle T \rangle_G$, is the smallest convex set containing T. A subset T of vertices of a graph G is a hull set if $\langle T \rangle_G = V(G)$. Moreover, T is a geodetic if $I_G[T] = V(G)$. The hull number $h(G)$ of a graph G is the minimum size of a hull set. The geodetic number $g(G)$ of G is the minimum size of a geodetic set. The shadow graph, denoted by $S(G)$, of a graph G is the graph obtained from G by adding a new vertex v' for each vertex v of G and joining v' to the neighbors of v in G. In this paper, we study the geodetic and hull numbers of shadow graphs. Bounds for the geodetic and hull numbers of shadow graphs are obtained and for several classes exact values are determined. Graphs G for which $g(S(G)) \in \{2, 3\}$ are characterized.

Keywords: Convex set · Geodetic number · Hull number · Simplicial vertex · Shadow graphs

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1 Introduction

Convexities in graphs are extensively studied due to their prominent role in graph theory as well as their contributions to axiomatic convexity theory. Given

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a finite set X, a family $\mathcal{C}$ of subsets of X is a *convexity* on X if $\varnothing \in \mathcal{C}, X \in \mathcal{C}$, and $\mathcal{C}$ is closed under intersections [12,13,21]. A set $T \subseteq X$ is said to be $\mathcal{C}$-*convex* if $T \in \mathcal{C}$. The $\mathcal{C}$-*convex hull* of $T \subseteq X$, $\langle T \rangle_{\mathcal{C}}$, is the minimum $\mathcal{C}$-convex set containing T. The cardinality of minimum set whose convex hull is X is the *hull number of* $\mathcal{C}$.

The most studied graph convexities are convexities defined by a family of paths $\mathcal{P}$, in a way that a set T of vertices of G is convex if and only if each vertex that lies on an (u, v)-path of $\mathcal{P}$ belongs to T. In this paper, we consider the geodetic convexity in graphs. In this convexity, $\mathcal{P}$ is the family of geodesics (shortest paths) of the graph.

One of the most studied numbers associated with graphs is the chromatic number. The chromatic number $\chi(G)$ is the minimum number of colors that can be assigned to the vertices of G so that adjacent vertices are colored differently. It is clear that $\chi(G) \geq \omega(G)$, where $\omega(G)$ is the size of a largest clique in G. However, a graph G may have arbitrarily large chromatic number without triangles ($\omega(G) = 2$). In 1955 Jan Mycielski used a fascinating construction called the Mycielskian or Mycielski graph [9,16]. His construction preserves the property of being triangle-free but increases the chromatic number. Applying the construction repeatedly to a triangle-free starting graph, we obtain a triangle-free graph with arbitrarily large chromatic number. A graph closely related to this construction is called the shadow graph. The *shadow graph* $S(G)$ of a graph G is the graph obtained from G by adding a new vertex v' for each vertex v of G and joining v' to the neighbors of v in G (the vertex v' is called the *shadow vertex* of v). The *star shadow graph* of a graph G is the graph obtained from the shadow graph $S(G)$ of G by adding a new vertex s^*(*star vertex*) and joining s^* to all shadow vertices. The Mycielski's construction consists of repeatedly finding the star shadow of the previous one beginning with the cycle C_5. Particularly, $s^*(C_5)$ is called the *Grötzsch graph* (a triangle-free graph) with chromatic number four. The term shadow graph was coined in [9,11].

In this paper, we continue our investigation of hull and geodetic numbers on shadow graphs. In Sect. 2, we fix the notation, terminologies and discuss some preliminary results of the geodetic and hull numbers already available in the literature.

2 Preliminaries

Let G be a connected graph and $u, v \in V(G)$. The *distance* $d_G(u, v)$ between u and v is the minimum number of edges on a (u, v)-path. The maximum distance between all pairs of vertices of G is the *diameter* $diam(G)$ of G. A (u, v)-path of length $d_G(u, v)$ is called an (u, v)-*geodesic*. Then, the *geodetic interval* $I_G[u, v]$ between vertices u and v of a graph G is the set of vertices x such that there exists a (u, v)-geodesic which contains x. For $T \subseteq V(G)$ we set $I_G[T] = \bigcup_{u,v \in T} I_G[u, v]$. The set T is a geodetic set if $I_G[T] = V(G)$. The geodetic number, denoted by $g(G)$, is the size of a minimum geodetic set. To simplify the writing, we may omit the index G in the above notation provided that G is clear from the context. The geodetic number of a graph was introduced in [14] and [1,2,6–8,17,18] contain numerous results and references concerning geodetic sets and the geodetic number.

The set T is *convex* in G if $I_G[T] = T$. The *convex hull* $\langle T\rangle_G$ of T is the smallest convex set that contains T, and T is a *hull set* of G if $\langle T\rangle_G$ is the whole vertex set of G. A smallest hull set is a *minimum hull set* of G, its cardinality is the *hull number* $h(G)$ of G. The convex hull $\langle T\rangle_G$ can also be formed from the sequence $\{I_G^k[T]\}$, $k \geq 0$, where $I_G^0[T] = T$, $I_G^1[T] = I_G[T]$ and $I_G^k[T] = I_G[I_G^{k-1}[T]]$ for $k \geq 2$. From some term on, this sequence must be constant. Let p be the smallest number such that $I_G^p[T] = I_G^{p+1}[T]$. Then $I_G^p[T]$ is the convex hull $\langle T\rangle_G$. The hull number of a graph was introduced by Everett and Seidman in [15]. See [2,5,10,17] for recent developments on the hull sets and the hull number of a graph. The hull number of composition, cartesian product, and strong product of graphs were studied in [3,4] and [19], respectively. A vertex v is called a simplicial vertex G if the subgraph induced by the neighbors of v is complete. The set of all simplicial vertices in a graph G is denoted by $simp(G)$ and $sp(G) = |simp(G)|$. A graph is *chordal* if it contains no induced cycle of length greater than three. A graph G is an *extreme hull graph* if the set of all simplicial vertices forms a hull set. In this paper, we make use of the following result.

Lemma 1. [2,10] *In a connected graph G, each simplicial vertex belongs to every hull set of G.*

3 Hull Number of Shadow Graphs

In this section, we estimate the upper and lower bounds of the hull number of shadow graphs and simplify the exact values for the shadows of complete graphs, hyper-cubes, grids, cycles and complete bipartite graphs. We prove a formula for the hull number of the shadow graph of a tree.

Lemma 2. *For any non-trivial connected graph G, a vertex v in $S(G)$ is a simplicial vertex of $S(G)$ if and only if v is a shadow of a simplicial vertex in G.*

Proof. Since $N_{S(G)}(v') = N_G(v)$, it follows that v' is a simplicial vertex in $S(G)$ for any simplicial vertex v in G. On the otherhand, let x be any simplicial vertex of $S(G)$. Then observe that x must be a shadow vertex, say $x = v'$, shadow of v. Now, if v is non-simplicial, then there exist non-adjacent neighbors, say s and t of v in G. This shows that s and t are also non-adjacent neighbors of v' in $S(G)$, impossible. Thus v must be simplicial in G. ■

Lemma 3. (i) *Let x and y be non-adjacent vertices in G. Then*
(1) $d_{S(G)}(x, y) = d_G(x, y)$.
(2) $d_{S(G)}(x, y') = d_G(x, y)$.
(3) $d_{S(G)}(x', y') = d_G(x, y)$.
(ii) *Let x and y be adjacent vertices in G. Then*
(1) $d_{S(G)}(x, y) = d_G(x, y) = 1$.

(2) $d_{S(G)}(x, y') = d_{S(G)}(x, y) = 1$.

(3) $d_{S(G)}(x', y') = \begin{cases} 2 & \textit{if } xy \textit{ lies on an induced } K_3 \\ 3 & \textit{otherwise} \end{cases}$

Proof. (i) First consider the case x and y are non-adjacent in G. Let $d_G(x, y) = d \geq 2$ and let $P : x = x_0, x_1, \ldots, x_d = y$ be a (x, y)-geodesic in G. This shows that x' is adjacent with x_1 and y' is adjacent with x_{d-1} in $S(G)$. Hence $x', x_1, x_2, \ldots, x_{d-1}, y'$ is an (x', y')-path of length d in $S(G)$ and so $d_{S(G)}(x', y') \leq d$. Now, suppose that $d_{S(G)}(x', y') = k < d$. Let $Q : x' = y_0, y_1, \ldots, y_k = y'$ be an (x', y')-geodesic of length k. Then it follows from the definition of $S(G)$ that $y_1, y_{k-1} \in V(G)$ and $xy_1, yy_{k-1} \in E(G)$. Now, suppose that the (y_1, y_{k-1})-subpath Q_1 of Q in $S(G)$ contains a shadow vertex u' of u, say $y_i = u'$. Then $2 \leq i \leq k-2$ and $y_{i-1}, y_{i+1} \in V(G)$. Then it follows from the definition of $S(G)$, the vertex u is adjacent to both y_{i-1} and y_{i+1} in G. This shows that (y_1, y_{i-1})-subpath of Q_1 together with the path y_{i-1}, u, y_{i+1} and the (y_{i+1}, y_{k-1})-subpath of Q_1 is a (y_1, y_{k-1})-path which has the same length of Q_1. Hence for each shadow vertex in Q_1, can be replaced with the corresponding vertex without changing the length of Q_1. Hence without loss of generality, we may assume that Q_1 has no shadow vertices and so Q_1 is a path in G. Then Q_1 together with the edges xy_1 and $y_{k-1}y$ is an (x, y)-walk of length k in G. Then $d_G(x, y) \leq k < d$, a contradiction. Thus $d_{S(G)}(x', y') = d_G(x, y)$. Proof for the remaining cases are similar. ■

Lemma 4. *For every graph G, it holds $h(S(G)) \leq h(G) + sp(G)$.*

Proof. Let T be a hull set of G and T' be the set of all vertices of $S(G)$ formed by the shadow vertices of T. We claim that $T' \cup simp(G)$ is a hull set of $S(G)$.

Observe from Lemma 3 that $V(G) \backslash T$ is contained in the convex hull of T'. Next, let $v \in T \backslash simp(G)$ and let $u, w \in N_G(v)$ such that $uw \notin E(G)$. If $u \in T$, then $u' \in T'$, otherwise, u belongs to the convex hull of T'. Therefore, $V(G)$ is contained in the convex hull of $T' \cup simp(G)$.

Now, by Lemma 2, every shadow vertex not in T' is not a simplicial vertex of $S(G)$, and then it has two non-adjacent neighbors in $V(G)$. ■

Theorem 1. *For any non-trivial connected graph G of order n,*

$$\max\{2, sp(G)\} \leq h(S(G)) \leq \min\{n, h(G) + sp(G)\}.$$

Proof. The left inequality is an immediate consequence of Lemmas 1 and 2. By Lemma 4, it remains to prove that $h(S(G)) \leq n$. Now, let V' be the set of shadow vertices of $V(G)$ in $S(G)$. We claim that V' is a hull set in $S(G)$. For, let v be any vertex in G. First suppose that $deg_G(v) \geq 2$. Let u and w be two distinct neighbors of v in G. Then the shadow vertices u' and w' of u and w, respectively are adjacent to v in $S(G)$. This shows that $v \in I_{S(G)}[u', w'] \subseteq \langle V' \rangle_{S(G)}$. So, assume that v is a pendent vertex in G. Let u be the unique neighbor of v in G and let v' and u' be the corresponding shadow vertices of v and u respectively. Then it follows from Lemma 3 that $d_{S(G)}(u', v') = 3$ and $v \in I_{S(G)}[u', v'] \subseteq \langle V' \rangle_{S(G)}$.

This shows that $\langle V' \rangle_{S(G)} = V(S(G))$ and so V' is a hull set of $S(G)$. Hence $h(S(G)) \leq |V'| = n$. ■

Now, the following formulas that can be easily deduced from the Theorem 1. The k-cube Q_k has the vertex set $\{0,1\}^k$, two vertices being adjacent if they differ in precisely one coordinate.

- $h(S(K_n)) = h(K_n) = n$, where $n \geq 2$.
- $h(S(K_{m,n})) = h(K_{m,n}) = 2$, where $m, n \geq 2$.
- $h(S(Q_n)) = h(Q_n) = 2$, where $n \geq 2$.
- $h(S(C_n)) = h(C_n) = \begin{cases} 2 \text{ if } n \text{ is even} \\ 3 \text{ if } n \text{ is odd} \end{cases}$
- $h(S(G_{n,m})) = h(G_{n,m}) = 2$, where $G_{n,m} (n, m \geq 2)$ is the 2 dimensional grid of order nm.

In view of Theorem 1, we also have the following result.

Theorem 2. *Let G be a connected graph of order n. If $h(S(G)) = n$, then G is a chordal graph of diameter at most three.*

Proof. First suppose that $h(S(G)) = n$. Let V' denotes the set of all shadow vertices of $V(G)$ in $S(G)$. We first claim that $diam(G) \leq 3$. Assume the contrary that there is a shortest path, say $u = u_0, u_1, u_2, u_3, u_4 = v$ of length four in G. For each i in the interval $0 \leq i \leq 4$, let u_i' denote the shadow vertices of u_i. Then by Lemma 3, $d_{S(G)}(u_0', u_4') = d_G(u_0, u_4) = 4$. We prove that $V' \backslash u_2'$ is a hull set of $S(G)$. Since V' is a hull set of $S(G)$, it is enough to prove that $u_2' \in I_{S(G)}[V' \backslash u_2']$. Now, since $P : u_0', u_1, u_2', u_3, u_4'$ is a path of length four and $d_{S(G)}(u_0', u_4') = 4$, we know that P is a (u_0', u_4')-geodesic containing the vertex u_2'. Hence $u_2' \in I_{S(G)}[u_0', u_4'] \subseteq I[V' \backslash u_2']$ and so $V' \backslash u_2'$ is a hull set of $S(G)$. This shows that $h(S(G)) \leq |V' \backslash u_2'| = n - 1$, a contradiction. Thus $diam(G) \leq 3$. Now, suppose that G contains an induced cycle, say $C : u_1, u_2, \ldots, u_n, u_1$ of length $n \geq 4$. As above, we claim that $V' \backslash u_1'$ is a hull set of $S(G)$. Since $\langle V' \rangle_{S(G)} = V(S(G))$, it is enough to prove that $u_1' \in I_{S(G)}[V' \backslash u_1']$. Now, since C is chordless, it follows from Lemma 3 that $u_1 \in I_{S(G)}[u_2', u_n'], u_3 \in I_{S(G)}[u_2', u_4']$ and $u_{n-1} \in I_{S(G)}[u_{n-2}', u_n']$. Hence $u_2, u_n \in I^2_{S(G)}[V' \backslash u_1']$. Thus $u_1' \in I_{S(G)}[u_n, u_2] \subseteq I^3_{S(G)}[V' \backslash u_1']$. This shows that $u_1' \in \langle V' \backslash u_1' \rangle_{S(G)}$ and so $V' \subseteq \langle V' \backslash u_1' \rangle_{S(G)}$. Hence $\langle V' \backslash u_1' \rangle_{S(G)} = \langle V' \rangle_{S(G)} = V(S(G))$. This leads to the fact that $h(S(G)) \leq |V' \backslash u_1'| = n - 1$, a contradiction. This proves that G is a chordal graph. ■

The converse of Theorem 2 need not be true. Consider the chordal graph of diameter 2 shown in the Fig. 1. The set $T' = \{v_1', v_4', v_5'\}$ is the set of all simplicial vertices of the shadow graph of G. Now, $I_{S(G)}[T'] = T' \cup \{v_2, v_3\}$ and $I^2_{S(G)}[T'] = I_{S(G)}[T']$ and so T' is not a hull set of $S(G)$.

On the otherhand, since the set $T' = \{v_1', v_4', v_5', v_3'\}$ is a hull set of $S(G)$. It follows that $h(S(G)) = 4 < 5 = n$. This example also shows that the inequalities in Theorem 1 can be strict.

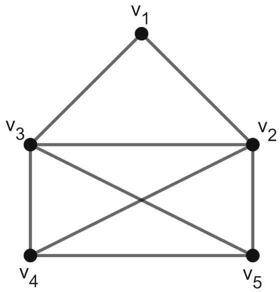

Fig. 1. G

In the following, we determine a formula for the hull number of shadow graph of a tree. A vertex v in a tree T is a *support vertex* if it is adjacent to a pendant vertex in T. A support vertex v is said to be a *first order support vertex* of T, if $T - v$ has atmost one non-trivial component.

Theorem 3. *Let T be a tree with k end vertices and l first order support vertices. Then $h(S(T)) = k + l$.*

Proof. Let $v_1, v_2, \ldots, v_k$ be the end vertices of T. Since each end vertex is a simplicial vertex, it follows from Lemma 1 that $h(S(T)) \geq k$. Let R be a hull set of $S(T)$. Then $v'_1, v'_2, \ldots, v'_k \in R$. Let v be a first order support vertex of T. Then $T - v$ has at most one non-trivial component. This shows that if v lies on a (u, w)-geodesic in T, then atleast one end, say u, must be an end vertex of T. Now, suppose that the shadow vertex v' of v lies only on a (x, y)-geodesic in $S(T)$, say $P : x = x_0, x_1, \ldots, x_i = v', \ldots, x_n = y$. Then $Q : x = x_0, x_1, \ldots, x_i = v, \ldots, x_n = y$ is also an (x, y)-geodesic in $S(T)$ containing the vertex v. Now, if Q contains any shadow vertex, then we can replace each shadow vertex by the corresponding vertex in T. Hence we can assume without loss of generality that Q is (x, y)-geodesic in T containing v. Thus $x = u$ or $y = u$, say $x = u$. Also note that the vertex u lies internally only on (v, v')-geodesic in $S(T)$. This shows that either $u \in R$ or $v' \in R$ and so $|R| \geq k + l$. Now , on the otherhand consider the set $R' = \{v'_1, v'_2, \ldots, v'_k, w'_1, w'_2, \ldots, w'_l\}$, where $w_1, w_2, \ldots, w_l$ are the first order support vertices of T. Then $I_{S(G)}[R'] = V(S(T)) - \{v_i\}$. Then $I^2_{S(G)}[R'] = V(S(T))$, implies that R' is a hull set of $S(T)$. Therefore, $h(S(T)) = |R'| = k + l$. ■

The hull number of the shadow graph of a graph can be significantly small. For instance, in the class of wheels $W_{1,n} (n \geq 5)$, one can observe that $h(W_{1,n}) = \left\lfloor \frac{n}{2} \right\rfloor$. Whereas, one can easily check that the set $T = \{u, u'\}$(u is the vertex with largest degree in $W_{1,n}$) is a hull set of the shadow graph of $W_{1,n}$. Thus $h(S(W_{1,n})) = 2$. In general, if a graph G of order n has a vertex of degree $n - 1$, then $h(S(G)) = 2$. In view of this observation, we leave the following problems as open.

Problem 1. Characterize graphs G for which $h(S(G)) = 2$.

Problem 2. Characterize graphs G for which $h(S(G)) = 3$.

4 Geodetic Number of Shadow Graphs

In this section, we estimate the upper and lower bounds of the geodetic number of any shadow graph and for several classes of shadow graphs the exact values are determined. We prove that K_2 is the only connected graph in which $g(S(G)) = 2$; and the graphs K_3 and P_3 are the only connected graphs in which $g(S(G)) = 3$.

Definition 1. *For any set $T' \subseteq V(S(G))$, the set $\pi(T')$ is defined as $\pi(T') = \{v \in V(G) : v \in T'$ or $v' \in T'\}$.*

Lemma 5. *If T' is a geodetic set of $S(G)$, then $\pi(T')$ is a geodetic set of G.*

Proof. Let x be any vertex in G such that $x \notin \pi(T')$. Then $x' \notin T'$. Since T' is a geodetic set in $S(G)$, there exist vertices $u, v \in T'$ such that $x' \in I_{S(G)}[u, v]$. Let $P : u = u_0, u_1, \ldots, u_i = x', \ldots, u_n = v$ be a (u, v)-geodesic in $S(G)$ containing the vertex x'. Then $u_{i-1}, u_{i+1} \in V(G)$.
Case 1: Both $u, v \in V(G)$. Since $d_G(u, v) \geq 2$, by Lemma 3, $d_G(u, v) = d_{S(G)}(u, v)$. Now, the path $Q : u = u_0, \ldots, u_{i-1}, x, u_{i+1}, \ldots, u_n = v$ is a (u, v)-geodesic in $S(G)$ containing the vertex x. Now, if Q contains any shadow vertex y' then we can replace y' by the corresponding vertex y. Hence without loss of generality, we may assume that Q has no shadow vertices and so Q is a (u, v)-geodesic in G containing the vertex x. Thus $x \in I_G[u, v] \subseteq I_G[\pi(T')]$.
Case 2: Both u and v are shadow vertices, say $u = r'$ and $v = t'$. Since $x' \in I_{S(G)}[r', t']$, it is clear that $d_{S(G)}(r', t') \geq 4$ and so $d_G(r, t) = d_{S(G)}(r', t')$. This shows that the path $Q : r, u_1, \ldots, u_{i-1}, x, u_{i+1}, \ldots, u_{n-1}, t$ is an (r, t)-geodesic in $S(G)$ containing the vertex x. Again without loss of generality, we may assume that Q has no shadow vertices and so Q is a (r, t)-geodesic in G containing the vertex x. Then $x \in I_G[r, t] \subseteq I_G[\pi(T')]$.
Case 3: $u = r'$ and $v \in V(G)$. Again, since $x' \in I_{S(G)}[r', v]$, we have that $v \neq r$ and $d_{S(G)}(r', v) \geq 3$. Hence similar to the above cases, we have that the path $Q : r, u_1, \ldots, u_{i-1}, x, u_{i+1}, \ldots, u_n = v$ is an (r, v)-geodesic in G containing the vertex x. Again, we may assume that Q has no shadow vertex and so Q is a (r, v)-geodesic in G containing the vertex x. Hence $x \in I_G[r, v] \subseteq I_G[\pi(T')]$. Thus in all cases $\pi(T')$ is a geodetic set in G. ■

A set T of vertices in a graph G is an *open geodetic set* if for each vertex v in G, either (1) v is a simplicial vertex of G and $v \in T$ or (2) v is an internal vertex of an (x, y)-geodesic for some $x, y \in T$. The minimum size of an open geodetic set is the open geodetic number $og(G)$ of G. In the following, we obtain an upper bound for the geodetic number of $S(G)$ in terms of the open geodetic number of G. The open geodetic number of a graph was introduced and studied in [20].

Theorem 4. *Let G be any connected graph of order n. Then*

$$g(G) \leq g(S(G)) \leq \min\{n, og(G) + sp(G)\}.$$

Proof. Let T' be a minimum geodetic set of $S(G)$. Then by Lemma 5, $\pi(T')$ is a geodetic set of G. Thus $g(G) \leq |\pi(T')| \leq |T'| = g(S(G))$. On the other hand, let R be a minimum open geodetic set in G. We claim that $T = R \cup simp(S(G))$ is a geodetic set of $S(G)$. Observe that $V(G) \subseteq I_G[R] \subseteq I_{S(G)}[T]$. Now let v' be any shadow vertex in $S(G)$ corresponding to the vertex v of G. If v is simplicial, then $v' \in T$. So, assume that v is non-simplicial. Since R is an open geodetic set, there exist $x, y \in R$ such that $v \in I_G[x, y]$ with $v \neq x$ and $v \neq y$. Let $P : x = u_0, u_1, \ldots, u_i = v, u_{i+1}, \ldots, v_n = y$ with $1 \leq i \leq n-1$ be a (x, y)-geodesic in G containing the vertex v internally. Then by Lemma 3, the path $P : x = u_0, u_1, \ldots, v_{i-1}, v', v_{i+1}, \ldots, v_n = y$ is an (x, y)-geodesic in $S(G)$ containing the vertex v'. Hence $v' \in I_{S(G)}[T]$. This shows that T is a geodetic set of $S(G)$ and so $g(S(G)) \leq |T| = og(G) + sp(G)$. Now, as in the case of Theorem 1, one can easily verify that the set of all shadow vertices of $V(G)$ is a geodetic set of $S(G)$. Thus $g(S(G)) \leq n$. Hence the result follows. ■

The following two observations are used to characterize graphs G in which $g(S(G)) \in \{2, 3\}$.

Observation 5. *Let u and v be two distinct vertices in G. Then $u \in I_{S(G)}[u', v']$ if and only if u and v are adjacent vertices in G having no common neighbors.*

Observation 6. *Let u be any vertex in G. Then $u' \notin I_{S(G)}[u, x]$ for any $x \in V(S(G))$ distinct from u'.*

Theorem 7. *Let G be any connected graph. Then $g(S(G)) = 2$ if and only if $G = P_2$.*

Proof. Let $T = \{x, y\}$ be a geodetic set in $S(G)$. If $x \in V(G)$, then by Observation 6, $x' \notin I_{S(G)}[x, y]$. Hence x must be a shadow vertex, say $x = v'$ for some $v \in V(G)$. Now, since T is a geodetic set of $S(G)$, we have that $v \in I_{S(G)}[x, y]$. But, in this case, one can easily observe that $v \in I_{S(G)}[x, y]$ if and only if $y = u'$ where u is adjacent to v in G. This is possible only when $G = P_2$. Hence the result follows. ■

Theorem 8. *Let G be any connected graph. Then $g(S(G)) = 3$ if and only if G is either K_3 or P_3.*

Proof. First, suppose that $G = K_3$ or $G = P_3$, then one can easily verify that $g(S(G)) = 3$. Conversely, assume that $g(S(G)) = 3$. If G has only three vertices, then $G = K_3$ or $G = P_3$. So, assume that G contains at least four vertices. Let T be a geodetic set in $S(G)$ of size three. We consider the following cases.
Case 1: $T = \{x', y', z\}$, where x' and y' are shadow vertices of x and y in G and $z \in V(G)$. First suppose that $z = x$ or $z = y$, say $z = x$. Choose $w \in V(G)$ be such that $w \notin \{x, y, z\}$ (This is possible because G has at least

four vertices). Now, since $d_{S(G)}(x', x) = 2$, it follows that $w' \in I_{S(G)}[x', y']$. This shows that $d_{S(G)}(x', y') \geq 4$ and hence by Lemma 3, x and y are non-adjacent in G. This shows that $y \notin I_{S(G)}[x, x']$ and so $y \in I_{S(G)}[y', x']$, a contradiction to Observation 5. Hence $z \neq x$ and $z \neq y$. Now, it follows from Observation 6 that $z' \in I_{S(G)}[x', y']$. Thus $d_{S(G)}(x', y') \geq 4$. Let $x' = x_0, x_1, \ldots, x_{i-1}, x_i = z', x_{i+1}, \ldots, x_n = y'$ be an (x', y')-geodesic in $S(G)$ containing the vertex z'. Then $x_{i-1}, x_{i+1} \in V(G)$ and so the path $P : x' = x_0, x_1, x_{i-1}, z, x_{i+1}, \ldots, x_n = y'$ is also an (x', y')-geodesic containing the vertex z. This shows that $z \in I_{S(G)}[x', y']$ and so $I_{S(G)}[x', z] \subseteq I_{S(G)}[x', y']$ and $I_{S(G)}[z, y'] \subseteq I_{S(G)}[x', y']$. This shows that the set $U = \{x', y'\}$ is a geodetic set in $S(G)$, a contradiction to the fact that $g(S(G)) = 3$.

Case 2: $T = \{x', y, z\}$, where x' is a shadow vertex of x in G and $y, z \in V(G)$. Now, if $x = y$ or $x = z$, say $x = y$, then it follows from Observation 6 that $z' \in I_{S(G)}[y, y']$. This leads to the fact that $d_{S(G)}(y, y') \geq 3$, a contradiction. Hence $x \neq y$ and $x \neq z$. Again by Observation 6, $z' \in I_{S(G)}[x', y]$. Hence as in the previous case, we can show that $z \in I_{S(G)}[x', y]$. Thus $I_{S(G)}[x', z] \subseteq I_{S(G)}[x', y]$ and $I_{S(G)}[z, y] \subseteq I_{S(G)}[x', y]$. This leads to the fact that the set $U = \{x', y\}$ is a geodetic set in $S(G)$, a contradiction.

Case 3: $T = \{x, y, z\} \subseteq V(G)$. By Observation 6, we have that $y' \in I_{S(G)}[x, z]$. Hence as in the previous cases, $y \in I_{S(G)}[x, z]$ and hence $U = \{x, z\}$ is a geodetic set in $S(G)$, a contradiction.

Case 4: All the three vertices of T are shadow vertices, say $T = \{x', y', z'\}$. Choose $w \in V(G)$ be such that $w \notin \{x, y, z\}$. Since T is a geodetic set, we may assume that $w' \in I_{S(G)}[x', y']$. This shows that $d_{S(G)}(x', y') \geq 4$. Hence by Lemma 3, the vertices x and y are non-adjacent in G. Moreover, $d_G(x, y) = d_{S(G)}(x', y') \geq 4$. Now, by Observation 5, it follows that both $x, y \notin I_{S(G)}[x', y']$. Now, suppose that $x \in I_{S(G)}[x', z']$ and $y \in I_{S(G)}[y', z']$, it follows from Observation 5 that both x and y must be adjacent with z in G. Then $d_{S(G)}(x', y') = d_G(x, y) = 2$, a contradiction. Hence either $x \notin I_{S(G)}[x', z']$ or $y \notin I_{S(G)}[y', z']$, say $x \notin I_{S(G)}[x', z']$. Now, since T is a geodetic set of $S(G)$, we have that $x \in I_{S(G)}[y', z']$. Moreover, recall that $d_G(x, y) \geq 4$. This shows that $d_{S(G)}(y', z') \geq 4$ and so by Lemma 3, $d_G(y, z) = d_{S(G)}(y', z') \geq 4$. Now, let $P : y' = u_0, u_1, \ldots, u_{i-1}, u_i = x, u_{i+1}, \ldots, u_n = z'$ be a (y', z')-geodesic in $S(G)$ containing the vertex x. Note that $u_{i-1} \neq y'$ and $u_{i+1} \neq z'$. Now without loss of generality, we may assume that both u_{i-1} and u_{i+1} are vertices in G. Otherwise, if they are shadow vertices, we can replace these vertices by the corresponding vertices in G. This shows that the path $P' : y' = u_0, u_1, \ldots, u_{i-1}, x', u_{i+1}, \ldots, u_n = z'$ is a (y', z')-geodesic containing x' and so $x' \in I_{S(G)}[y', z']$. Hence as the previous cases, the set $U = \{y', z'\}$ must be a geodetic set in $S(G)$, a contradiction. Hence the result follows. ■

The following result is an immediate consequence of Theorems 4, 7 and 8.

- $g(S(K_n)) = g(K_n) = n$, where $n \geq 2$.
- $g(S(K_{m,n})) = og(K_{m,n}) = 4$, where $m, n \geq 2$.
- $g(S(Q_n)) = og(Q_n) = 4$, where $n \geq 2$.
- $g(S(G_{n,m})) = og(G_{n,m}) = 4$, where $n, m \geq 2$.

Theorem 9. *Let T be a tree with k end vertices and m support vertices. Then $g(S(T)) = k + m$.*

Proof. Let $v_1, v_2, \ldots, v_k$ be end vertices of T, let $u_1, u_2, \ldots, u_m$ be the corresponding support vertices and let M' be a geodetic set of $S(T)$. Then by Lemma 2, $v_i' \in M'$ for all $i = 1, 2, \ldots, k$. For each $i = 1, 2, \ldots, k$, if the vertex v_i in $S(G)$ lies internally on an (x, y)-geodesic in $S(T)$, then $x = u_i'$ or $y = u_i'$, where u_i is the corresponding support vertices of v_i. This shows that either $u_i \in M'$ or $v_i \in M'$. Hence $|M'| \geq k + m$. On the other hand, consider the set $R' = \{v_1, v_2, \ldots, v_k\} \cup \{u_1', u_2', \ldots, u_m'\}$. Let x be any vertex of $S(T)$ such that $x \notin R'$.
Case 1: $x \in V(G)$. Then x lies on a (v_i, v_j)-geodesic in T, say $v_i = y_0, y_1, \ldots, y_r = x, y_{r+1}, \ldots, y_k = v_j$. Then by Lemma 3, the path $y_0', y_1, \ldots, y_r, x, y_{r+1}, \ldots, y_{k-1}, y_k'$ is an (y_0', y_k')-geodesic in $S(T)$ containing the vertex x and so $x \in I_{S(G)}[R']$.
Case 2: $x = u'$, a shadow vertex of u in T. If u is a support vertex, then $u' \in R'$. So, assume that u is not a support vertex. This shows that u lies on a (v_i, v_j)-geodesic in R', say $v_i = y_0, y_1, \ldots, y_r = u, y_{r+1}, \ldots, y_l = v_j$. Since u is not a support vertex of T, we have that $2 \leq r \leq l - 2$.

This shows that by Lemma 3, the path $v_i' = y_0, y_1, \ldots, y_{r-1}, u', y_{r+1}, \ldots, y_l = v_j'$ is a (v_i', v_j')- geodesic in $S(T)$ containing the vertex u'. Hence R' is a geodetic set of $S(T)$ and so $g(S(T)) = k + m$. ■

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