Self-centeredness of Generalized Petersen Graphs

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Abstract. A connected graph is said to be self-centered if all its vertices have the same eccentricity. The family of generalized Petersen graphs $P(n,k)$, introduced by Coxeter [6] and named by Watkins [18], is a family of cubic graphs of order $2n$ defined by positive integral parameters n and k, $n \geq 2k$. Not all generalized Petersen graphs are self-centered. In this paper, we prove self-centeredness of $P(n,k)$ whenever k divides n and $k < \frac{n}{2}$, except the case when n is odd and k is even. We also prove non-self-centeredness of generalized Petersen graphs $P(n,k)$ when n even with $k = \frac{n}{2}$; $n = 4m+2$ with $k = \frac{n}{2} - 1$ for some positive integer $m \geq 3$; $n \geq 9$ is odd and $k = 2$ or $k = \frac{n-1}{2}$; and $n = m(4m+1) \pm (m+1)$ with $k = 4m+1$ for any positive integer $m \geq 2$. Finally, we make an exhaustive computer search and get all possible values of n and k for which $P(n,k)$ is non-self-centered.

Keywords: Eccentricity · Center of graph · Self-centered graph · Generalized Petersen graphs

1 Introduction

Graph centrality plays a significant importance in facility location problem, and has a great role in designing a communication network. In a locality, for the efficient use of resources, we place them at central nodes. Because of this, self-centered graphs are ideal as the facility can be placed (located) at any node or vertex of the locality. In the paper, by a *graph* $G = (V(G), E(G))$ (or simply G) we mean a simple finite graph with the vertex set $V(G)$ and the edge set $E(G)$. The length of a shortest u–v path in a graph G gives the *distance* between vertices u and v, which is denoted by $d_G(u,v)$ (or $d(u,v)$). The maximum of distances from a vertex v to all other vertices in a graph G is known as the *eccentricity* (denoted by $e(v)$) of the vertex v. The *radius* of G, denoted by $rad(G)$, is the minimum eccentricity of vertices in G. Similarly, the *diameter* of G, denoted by $diam(G)$, is the maximum eccentricity of vertices. Vertices with minimum eccentricity are called *central* vertices and the subgraph induced on these vertices is called the *center* $C(G)$ of the graph G. A graph G is known as a *self-centered* graph if $C(G) = G$. In other words, for a self-centered graph

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G, $rad(G)$ is equal to $diam(G)$. Further, if eccentricity of every vertex in a self-centered graph is d then the graph is known as *d-self-centered graph*.

As a generalization of the well-known Petersen graph, the generalized Petersen graph has attracted the attention of several researchers. For each positive integers n and k with $n \geq 2k$, the *generalized Petersen graph* $P(n,k)$ is a graph with vertex set $V(P(n,k)) = \{u_0, u_1, u_2, ..., u_{n-1}, v_0, v_1, v_2, \ldots, v_{n-1}\}$ and the edge set $E(P(n,k)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : 0 \leq i \leq n-1\}$, where subscripts are addition modulo n. Throughout the paper, we refer this notation for vertex set and edge set of $P(n,k)$. For $n = 5$ and $k = 2$, $P(5,2)$ is the well known Petersen graph.

The generalized Petersen graphs, named by Watkins [18] were defined by Coxeter [6] but not with this name. The essence of the Petersen graph is a remarkable configuration that serves as a counterexample to many optimistic predictions and conjectures about what might be true for graphs in general. The generalized Petersen graphs have been studied by several authors; for instance, Tait coloring of generalized Petersen graphs have been studied and analysed in [4], generalization of generalized Petersen graphs on the basis of symmetry properties have been discussed in [13]. A result on maximum number of vertices in a generalized Petersen graph was given by authors in [1], where number of vertices is treated as a function of diameter. A formula for number of isomorphism classes of generalized Petersen graphs was presented by Steimle and Staton [17]. For works related to domination number in generalized Petersen graphs, one can refer to [5,7], and [9]. However, there were no significant work done related to self-centeredness of generalized Petersen graphs because of the complex structure of these graphs. This motivated us to work on the self-centeredness property of generalized Petersen graphs.

The theorem below gives a criteria for generalized Petersen graphs to be isomorphic.

Theorem 1. *[17] Let $n > 3$ and k, l relatively prime to n with $kl \equiv 1 \pmod{n}$. Then $P(n,k) \cong P(n,l)$.*

The theorem stated below is useful in proving the self-centeredness of generalized Petersen graph $P(n,1)$.

Theorem 2. *[16] Let $G = G_1 \square G_2$ be the Cartesian product of graphs G_1 and G_2. If G_1 and G_2 are l- and m-self centered graphs, respectively, then G is $(l+m)$-self centered graph.*

We note that $P(n,1)$ is the Cartesian product of the cycle C_n and the complete graph K_2. Since C_n is $\lfloor \frac{n}{2} \rfloor$-self-centered graph and K_2 is 1-self-centered graph, by Theorem 2 we get the result below.

Theorem 3. *For $n \geq 3$, the generalized Petersen graph $P(n,1)$ is a d-self-centered graph, where $d = \lfloor \frac{n}{2} \rfloor + 1$.*

Vertex transitive graphs are self-centered. In [8], the authors have proved that $P(n,k)$ is vertex transitive if and only if $k^2 \equiv \pm 1 (\mathrm{mod}\ n)$, or $n = 10$ and $k = 2$. So, one get the following result.

Theorem 4. *For $n \geq 3$, generalized Petersen graph $P(n,k)$ is self-centered for $k^2 \equiv \pm 1 \pmod n$, or $n = 10$ and $k = 2$. Moreover, $P(10,2)$ is 5-self-centered and the other $P(n,k)$ are d-self-centered, where*

$$d = \begin{cases} k+1, & \text{if } n = k^2 - 1, \\ k+1, & \text{if } n = k^2 + 1 \text{ and } k \text{ is even, } k \neq 2, \\ k+2, & \text{if } n = k^2 + 1 \text{ and } k \text{ is odd.} \end{cases}$$

Self-centered graphs were studied and surveyed by many authors in the last few decades. For the same, we refer the articles [2], [3], and [10–12]. Self-centeredness of different types of graph products are studied by the authors in [14,15].

In the remaining of the paper, we assume $k \geq 2$. The main technique followed in this paper for verification of self-centeredness of $P(n,k)$ is the determination of eccentricities of u_0 and v_0, because $P(n,k)$ is symmetric on outer vertices $u_0, u_1, u_2, \ldots, u_{n-1}$ and also symmetric on inner vertices $v_0, v_1, v_2, \ldots, v_{n-1}$. If $e(u_0) = e(v_0)$ then the generalized Petersen graph is self-centered, otherwise not.

The rest of the paper is organized as follows. In Sect. 2, we prove the non-self-centeredness of generalized Petersen graphs $P(n,k)$ when n even, $k = \frac{n}{2}$ or $n = 4m+2$ with $k = \frac{n}{2} - 1$ for some positive integer $m \geq 3$. Then we prove self-centeredness of $P(n,k)$ whenever n is even, $k < \frac{n}{2}$ and k divides n. In Sect. 3, we study self-centeredness of generalized Petersen graphs $P(n,k)$ for odd n. We prove that $P(n,k)$ is not self-centered for odd $n \geq 9$ with $k = 2$ or $k = \frac{n-1}{2}$. Then we prove the self-centeredness of $P(n,k)$ for odd n and odd k, $k < \frac{n}{2}$, and $k|n$. Also, we prove non-self-centeredness of $P(n,k)$ when $n = m(4m+1) \pm (m+1)$ with $k = 4m+1$ for any positive integer $m \geq 2$. Finally, we make an exhaustive computer search and get all possible values of n and k for which $P(n,k)$ is non-self-centered.

2 Self-centeredness of $P(n,k)$ for an Even n

In the following result, we investigate the self-centeredness of $P(n,k)$ for an even n and $k = \frac{n}{2}$.

Theorem 5. *Let $P(n,k)$ be a generalized Petersen graph such that $n \geq 4$ is even and $k = \frac{n}{2}$. Then $P(n,k)$ is not a self-centered graph.*

Proof. To prove the result, it is sufficient to show that eccentricity of two vertices in $P(n,k)$ are not equal. We note that $C : v_0, u_0, u_1, u_2, \ldots, u_k, v_k, v_0$ induces a cycle of length $k+3$, see Fig. 1, where the cycle C is highlighted by thick lines. We observe that $d(u_0, u_i) = d(u_0, u_{n-i})$, for $i \in \{1, 2, 3, ..., \frac{n}{2}\}$. Depending on the parity of k, we distinguish following two cases.

Case 1. The integer k is even.

Since $k = \frac{n}{2}$, in this case n will be a multiple of four. Consider vertices u_0 and v_0. Since u_0 lies on C and C is of length $k+3$,

$$\max\{d(u_0, u_i) : 1 \leq i \leq \frac{n}{2}\} = \lfloor \frac{k+3}{2} \rfloor = \frac{k+2}{2} = \frac{n+4}{4} = \frac{n}{4} + 1. \quad (1)$$

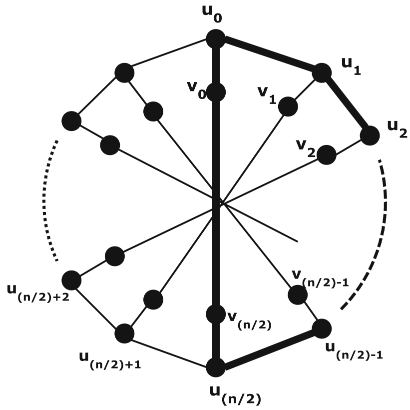

Fig. 1. Generalized Petersen graph $P(n, k)$ with n even and $k = \frac{n}{2}$

Next, we find $d(u_0, v_i)$ for $1 \leq i \leq \frac{n}{2}$. First, let $1 \leq i \leq \frac{n}{4}$. For these values of i, we get that $d(u_0, v_i) = d(u_0, u_i) + 1$. Now, $\max\{d(u_0, u_i) : 1 \leq i \leq \frac{n}{4}\} = \frac{n}{4}$ and so

$$\max\{d(u_0, v_i) : 1 \leq i \leq \frac{n}{4}\} = \max\{d(u_0, u_i) + 1 : 1 \leq i \leq \frac{n}{4}\} = \frac{n}{4} + 1. \quad (2)$$

For $\frac{n}{4} < i \leq \frac{n}{2}$, a shortest v_i–u_0 path is given by $P_i : v_i, v_{i+k}, u_{i+k}, u_{i+(k+1)}, u_{i+(k+2)}, \ldots, u_0$, where $l(P_i) = n - i - k + 2 = \frac{n}{2} + 2 - i$ (since $k = \frac{n}{2}$). The maximum length of P_i is for $i = \frac{n}{4} + 1$, and hence

$$\max\{l(P_i) : \frac{n}{4} < i \leq \frac{n}{2}\} = \frac{n}{4} + 1. \quad (3)$$

From Eqs. (1)–(3), we get that $e(u_0) = \frac{n}{4} + 1$.
Since v_0 lies on C and n is a multiple of four,

$$\max\{d(v_0, u_i) : 1 \leq i \leq \frac{n}{2}\} = \lfloor \frac{k+3}{2} \rfloor = \frac{n}{4} + 1. \quad (4)$$

Next for $\frac{n}{4} \leq i \leq \frac{n}{2}$, we obtain $d(v_0, v_i) = d(u_0, v_i) + 1$, and this gives

$$\max\{d(v_0, v_i) : \frac{n}{4} \leq i \leq \frac{n}{2}\} = \frac{n}{4} + 1. \quad (5)$$

Finally, for $1 \leq i \leq \frac{n}{4}$, a shortest v_i–v_0 path is given by $P'_i : v_i, v_{i+k}, u_{i+k}, u_{i+(k+1)}, \ldots, u_0$, where $l(P'_i) = n - i - k + 3 = \frac{n}{2} + 3 - i$, and the maximum length of P'_i is for $i = \frac{n}{4} + 1$, i.e.,

$$\max\{l(P'_i) : 1 \leq i \leq \frac{n}{4}\} = \frac{n}{2} + 3 - \frac{n}{4} - 1 = \frac{n}{4} + 2. \quad (6)$$

Hence $e(v_0) = \frac{n}{4} + 2$. Thus, we get that $e(u_0) \neq e(v_0)$ and $P(n, k)$ is not a self-centered graph in this case.

Case 2. The integer k is odd.

In this case n is not a multiple of four but cycle C is of an even length. Since u_0 and v_0 lie on C, for $1 \leq i \leq \frac{n}{2}$, we have

$$\max\{d(u_0, u_i) : 1 \leq i \leq \frac{n}{2}\} = \frac{k+3}{2} \quad \text{and,} \tag{7}$$

$$\max\{d(v_0, u_i) : 1 \leq i \leq \frac{n}{2}\} = \frac{k+3}{2}. \tag{8}$$

For $1 \leq i \leq \lfloor \frac{n}{4} \rfloor + 1$, a shortest u_0–v_i path is given by $Q_i : u_0, u_1, u_2, \ldots, u_i, v_i$, where $l(Q_i) = i + 1$ and maximum length of Q_i is for $i = \lfloor \frac{n}{4} \rfloor + 1$, i.e.,

$$\max\{l(Q_i) : 1 \leq i \leq \lfloor \frac{n}{4} \rfloor + 1\} = \lfloor \frac{n}{4} \rfloor + 1 + 1 = \lfloor \frac{k+3}{2} \rfloor. \tag{9}$$

Again, for $\lfloor \frac{n}{4} \rfloor + 2 \leq i \leq \frac{n}{2}$, a shortest u_0–v_i path is given by $Q'_i : v_i, v_{i+k}, u_{i+k}, u_{i+(k+1)}, \ldots, u_0$ and the length of Q'_i is $\frac{n}{2} + 2 - i$. The path Q'_i has a maximum length for $i = \lfloor \frac{n}{4} \rfloor + 2$, i.e.,

$$\max\{l(Q'_i) : \lfloor \frac{n}{4} \rfloor + 2 \leq i \leq \frac{n}{2}\} = \frac{n}{2} - \lfloor \frac{n}{4} \rfloor = \frac{k+1}{2}. \tag{10}$$

From Eqs. (7), (9), and (10), we have $e(u_0) = \frac{k+3}{2}$.

Next, we consider the vertex v_0. For $1 \leq i \leq \lfloor \frac{n}{4} \rfloor + 1$, a shortest v_0–v_i path is given by $T_i : v_0, u_0, u_1, u_2, \ldots, u_i, v_i$. The length of the path T_i is $i + 2$. The maximum length of T_i is for $i = \lfloor \frac{n}{4} \rfloor + 1$, i.e.,

$$\max\{l(T_i) : 1 \leq i \leq \lfloor \frac{n}{4} \rfloor + 1\} = \lfloor \frac{n}{4} \rfloor + 1 + 2 = \frac{k+5}{2}. \tag{11}$$

Finally, for $\lfloor \frac{n}{4} \rfloor + 2 \leq i \leq \frac{n}{2}$, a shortest v_0–v_i path is given by $T'_i : v_i, v_{i+k}, u_{i+k}, u_{i+(k+1)}, \ldots, u_n = u_0, v_0$, where $l(T'_i) = \frac{n}{2} + 3 - i$. We get the maximum length of T'_i for $\lfloor \frac{n}{4} \rfloor + 2$, i.e.,

$$\max\{l(T'_i) : \lfloor \frac{n}{4} \rfloor + 2 \leq i \leq \lfloor \frac{n}{2} \rfloor\} = \frac{k}{2} + 1. \tag{12}$$

From Eqs. (8), (11), and (12), we have $e(v_0) = \frac{k+5}{2}$. This proves that $e(u_0) \neq e(v_0)$. Hence, $P(n, k)$ is not a self-centered graph in this case also. □

In the following theorem we get some non-self-centered generalized Petersen graphs $P(n, k)$, where n is even but not divisible by k.

Theorem 6. *Let $P(n, k)$ be a generalized Petersen graph such that $n = 4m + 2$ for some positive integer $m \geq 3$ and $k = \frac{n}{2} - 1$. Then $P(n, k)$ is not a self-centered graph.*

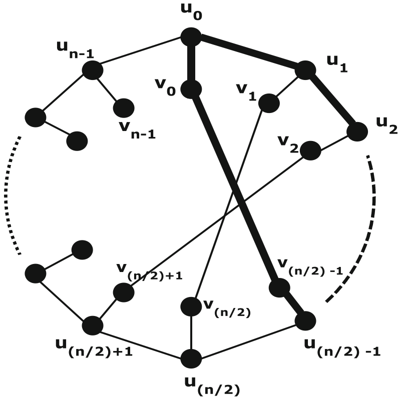

Fig. 2. Generalized Petersen graph $P(n,k)$ with $n = 4m+2$ and $k = \frac{n}{2} - 1$

Proof. Here we obtain a cycle $C: u_0, u_1, u_2, \ldots, u_{\frac{n}{2}-1}, v_{\frac{n}{2}-1}, v_0, u_0$ of length $\frac{n+4}{2}$ i.e. of length $2m+3$, see Fig. 2, where the cycle C is highlighted by thick lines. Since u_0 and v_0 lie on C, we have

$$\max\{d(u_0, u_i) : 1 \leq i \leq \frac{n}{2} - 1\} = \max\{d(v_0, u_i) : 1 \leq i \leq \frac{n}{2} - 1\} = \lfloor \frac{2m+3}{2} \rfloor = m+1. \tag{13}$$

We have $d(u_0, u_{m+1}) = d(u_0, u_{m+2}) = m+1$ and $d(v_0, u_m) = d(v_0, u_{m+1}) = m+1$. Next, for every $j \in \{1, 2, \ldots, m\} \cup \{m+3, m+4, \ldots, 2m+1\}$, we get $d(u_0, v_j) \leq m+1$. Similarly, for every $l \in \{1, 2, \ldots, m-1\} \cup \{m+2, m+4, \ldots, 2m+1\}$, $d(v_0, v_l) \leq m+1$. Now for $l = m+1$ and $m+2$, a shortest u_0–v_i paths P_1 and P_2 are given below.

$$P_1 : u_n = u_0, u_{n-1}, u_{n-2}, \ldots, u_{n-(m-1)}, v_{n-(m-1)}, v_{m+1} = v_l, \quad \text{and}$$
$$P_2 : u_n = u_0, u_{n-1}, u_{n-2}, \ldots, u_{n-(m-2)}, v_{n-(m-2)}, v_{m+2} = v_l.$$

Also,

$$l(P_1) = m+1 \text{ and } l(P_2) = m. \tag{14}$$

Further for $l = m$ and $m+1$, smallest v_0–v_i paths are as given below.

$P_3 : v_0, u_0, u_1, u_2, \ldots, u_m, v_l = v_m$, and

$P_4 : v_0, u_0, u_{n-1}, u_{n-2}, \ldots, u_{n-(m-1)}, v_{n-(m-1)}, v_l = v_{m+1}$. We notice that

$$l(P_3) = m+2 \text{ and } l(P_4) = m+2. \tag{15}$$

From the above three equations, we conclude that $e(u_0) = m+1$ and $e(v_0) = m+2$. Thus, $e(u_0) \neq e(v_0)$, and hence $P(n,k)$ is not a self-centered graph. □

Theorem 7. *Let $P(n,k)$ be a generalized Petersen graph with n even, $k < \frac{n}{2}$ and n be divisible by k. If $n = kq$ then $P(n,k)$ is a d-self-centered graph, where*

$$d = \begin{cases} \frac{q}{2} + \lfloor \frac{k+3}{2} \rfloor, & \text{if } k \text{ divides } \frac{n}{2}, \\ \lfloor \frac{q+1}{2} \rfloor + \lfloor \frac{k}{2} \rfloor + 1, & \text{otherwise.} \end{cases}$$

Proof. To prove the result, we consider the following sets of vertices in $P(n,k)$.

- $R_1 : \{u_0, v_0, v_k, u_k, u_{k-1}, u_{k-2}, \ldots, u_3, u_2, u_1\}$ and $R'_1 : \{v_1, v_2, \ldots, v_{k-1}\}$
- $R_2 : \{u_k, v_k, v_{2k}, u_{2k}, u_{2k-1}, u_{2k-2}, \ldots, u_{k+1}\}$ and $R'_2 : \{v_{k+1}, v_{k+2}, \ldots, v_{2k-1}\}$

  $\vdots$
- $R_q : \{u_{(q-1)k}, v_{(q-1)k}, v_{qk}, u_{qk}, u_{qk-1}, u_{qk-2}, \ldots, u_{qk-k+1}\}$ and

  $R'_q : \{v_{k(\frac{q}{2}-1)}, v_{k(\frac{q}{2}-1)+1}, \ldots, v_{\frac{kq}{2}-1}\}$

We see that vertices of each R_i, $i \in \{1, 2, \ldots, q\}$, induces a cycle of length $k+3$, vertices of each R'_i induces independent set and $\bigcup_{i=1}^{q}(R_i \cup R'_i) = V(P(n,k))$. Next, we find distances from u_0 and v_0 to other vertices of $P(n,k)$. It is sufficient to find the distances from u_0 and v_0 to vertices $u_1, u_2, \ldots, u_{\frac{n}{2}}$, $v_0, u_0, v_1, v_2, \ldots, v_{\frac{n}{2}}$. Here, we have following two cases.

Case 1. When k divides $\frac{n}{2}$.

Since vertices of R_1 induces a cycle $u_0, v_0, v_k, u_k, \ldots, u_2, u_1, u_0$ of length $k+3$, we have

$$\max\{d(u_0,x) : x \in R_1\} = \max\{d(v_0,x) : x \in R_1\} = \lfloor \frac{k+3}{2} \rfloor. \tag{16}$$

$$\max\{d(u_0,x) : x \in R'_1\} = \max\{d(v_0,x) : x \in R'_1\} = \lfloor \frac{k+3}{2} \rfloor + 1. \tag{17}$$

We see that for any vertex $x \in R_2$, $d(u_0,x) = d(u_0,v_0) + d(v_0,v_k) + d(v_k,x)$ and thus, we have

$$\max\{d(u_0,x) : x \in R_2\} = \lfloor \frac{k+3}{2} \rfloor + 2, \quad \text{and} \tag{18}$$

$$\max\{d(u_0,x) : x \in R'_2\} = \lfloor \frac{k+3}{2} \rfloor + 3, \tag{19}$$

and,

$$\max\{d(v_0,x) : x \in R_2\} = \lfloor \frac{k+3}{2} \rfloor + 1, \quad \text{and} \tag{20}$$

$$\max\{d(v_0,x) : x \in R'_2\} = \lfloor \frac{k+3}{2} \rfloor + 2. \tag{21}$$

Similarly, we get

$$\max\{d(u_0,x) : x \in R_q\} = \lfloor \frac{k+3}{2} \rfloor + \frac{q}{2}, \quad \text{and} \tag{22}$$

for the vertices $v_j \in R'_q$ for $j \neq \frac{k(q-1)}{2}$ and $\frac{k(q-1)}{2}+1$ when k is even, then

$$\max\{d(u_0,x) : x \in R'_q, x \neq v_{\frac{k(q-1)}{2}}, v_{\frac{k(q-1)}{2}+1}\} = \lfloor \frac{k+3}{2} \rfloor + \frac{q}{2}. \tag{23}$$

For $j = \frac{k(q-1)}{2}$, a shortest u_0–v_j path is given by $P : u_0, u_1, \ldots, u_{\frac{k}{2}}, v_{\frac{k}{2}}, v_{\frac{k}{2}+k}, v_{\frac{k}{2}+2k}, \ldots, v_{\frac{k}{2}+\frac{k(q-1)}{2}}$ and thus $d(u_0,u_j) = \frac{k+q}{2}$. Due to symmetric structure of the graph, the same can be obtained for $j = \frac{k(q-1)}{2}+1$.

Finally, consider vertices $v_j \in R'_q$ for $j \neq \frac{q(k-1)}{2}$ when k is odd. Then

$$\max\{d(u_0,x) : x \in R'_q, x \neq v_{\frac{q(k-1)}{2}}\} = \lfloor \frac{k+3}{2} \rfloor + \frac{q}{2}, \tag{24}$$

and for $j = \frac{q(k-1)}{2}$ a shortest u_0–v_j path is given by $P' : u_0, u_1, \ldots, u_{\frac{k+1}{2}}, v_{\frac{k+1}{2}}, v_{\frac{k+1}{2}+k}, v_{\frac{k+1}{2}+2k}, \ldots, v_{\frac{k+1}{2}+k(\frac{q}{2}-1)}$ of length $\frac{k+q+1}{2}$ and thus $d(u_0,u_j) = \frac{k+q+1}{2}$.

Now, for the vertex v_0, we obtain

$$\max\{d(v_0,x) : x \in R_q\} = \lfloor \frac{k+3}{2} \rfloor + \frac{q}{2} - 1. \tag{25}$$

$$\max\{d(v_0,x) : x \in R'_q\} = \lfloor \frac{k+3}{2} \rfloor + \frac{q}{2}. \tag{26}$$

From Eqs. (16)–(26), we conclude that $e(u_0) = e(v_0) = \frac{q}{2} + \lfloor \frac{k+3}{2} \rfloor$.

Case 2. When k does not divide $\frac{n}{2}$.

Given that $n = kq$ and k does not divide $\frac{n}{2}$, so q is an odd integer. This means k must be even. In this case, the distance between the vertex u_0 and a vertex in $R_1 \cup \ldots \cup R_{\frac{q-1}{2}}$ is the same as obtained in the Case 1. That is, the maximum distance between u_0 and any vertex from $R_{\frac{q-1}{2}}$ is $\lfloor \frac{q-1}{2} \rfloor + \lfloor \frac{k+3}{2} \rfloor$. Next consider the vertices from the region $R_{\frac{q+1}{2}}$. Now, because of the symmetry of $P(n,k)$ for an even n, the vertex farthest from u_0 (v_0) lie in the region $R_{\frac{q+1}{2}}$. The vertex farthest from u_0 and v_0 are the vertices $u_{\frac{n}{2}}$ and $v_{\frac{n}{2}}$, respectively, at a distance $\lfloor \frac{q+1}{2} \rfloor + \lfloor \frac{k}{2} \rfloor + 1$, and hence the result. □

Theorem 8. *The generalized Petersen graph $P(n,k)$ is not self-centered for $n = 4m(4m+1)$ and $k = 2m(4m-1)$ for some positive integer $m \geq 1$.*

Proof. In this case, we find that $v_{\frac{n}{2}}$ is the farthest vertex from both u_0 and v_0 and have obtained that $d(u_0, v_{\frac{n}{2}}) = 4m+2$ and $d(v_0, v_{\frac{n}{2}}) = 4m+1$. So, $e(u_0) = 4m+2$ and $e(v_0) = 4m+1$ and hence the given generalized Petersen graphs are not self-centered in this case. □

3 Self-centeredness of $P(n,k)$ for Odd n

In this section, we first investigate self-centeredness of $P(n,k)$ for $k = 2$. First of all, we prove that $P(5,2)$ and $P(7,2)$ are self-centered graphs.

Theorem 9. *The generalized Petersen graph $P(n,2)$ is 2- or 3-self-centered graphs for $n = 5$ or 7, respectively.*

Proof. For $n = 5$, the graph $P(n,2)$ is the well known Petersen graph and we know that radius and diameter of $P(5,2)$ is two. Thus, $P(5,2)$ is 2-self-centered graph.

Let us consider the vertices u_0 and v_0 in $P(7,2)$. The shortest path from u_0 to u_1, u_2, or u_3 is through the edges in cycle $C : u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_0$. So we get that $d(u_0,u_1), d(u_0,u_2)$, and $d(u_0,u_3)$ are equal to $1, 2$, and 3, respectively. A shortest u_0–v_i path for $i = 0, 1, 2$, and 3 is (u_0, v_0), (u_0, u_1, v_1), (u_0, v_0, v_2), and (u_0, u_1, v_1, v_3) with lengths $1, 2, 2$, and 3, respectively. Similarly, a shortest $v_0 - u_i$ path for $i = 0, 1, 2$, and 3 is (v_0, u_0), (v_0, u_0, u_1), (v_0, v_2, u_2), (v_0, v_2, u_2, u_3) with lengths 1, 2, 2, and 3, respectively. Further, a shortest v_0–v_i path for $i = 1, 2$, and 3 is (v_0, u_0, u_1, v_1), (v_0, v_2), and (v_0, v_5, v_3) with lengths 3, 1, and 2 respectively. From this we can say that $e(u_0) = e(v_0) = 3$. Hence $P(7,2)$ is a 3-self-centered graph. □

Theorem 10. *The generalized Petersen graph $P(n,2)$ is not self-centered for odd integers $n \geq 9$.*

Proof. We take $n = 4m+1$ or $4m+3$ for some positive integer $m \geq 2$. We shall find $d(u_0,u_i)$, $d(u_0,v_i)$, $d(v_0,u_i)$, and $d(v_0,v_i)$ for $i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. First, we consider the vertex u_0. We note that $d(u_0,u_i) = i$ for $i = 1, 2, 3$, and 4.

For an even index i, $6 \leq i \leq 2m$, a shortest u_0–u_i and u_0–v_i path is given by $P_i : u_0, v_0, v_2, v_4, \ldots, v_i, u_i$ and $P'_i : u_0, v_0, v_2, \ldots, v_i$, where $l(P_i) = \frac{i+4}{2}$ and $l(P'_i) = \frac{i+2}{2}$. Now,

$$\max\{l(P_i) : 6 \leq i \leq 2m, i \text{ even}\} = m + 2. \tag{27}$$

$$\max\{l(P'_i) : 6 \leq i \leq 2m, i \text{ even}\} = m + 1. \tag{28}$$

For an odd index i, a shortest u_0–u_i and u_0–v_i path is given by $Q_i : u_0, v_0, v_2, v_4, \ldots, v_{i-1}, u_{i-1}, u_i$ and $Q'_i : u_0, u_1, v_1, v_3, \ldots, v_i$, where $l(Q_i) = \frac{i+5}{2}$ and $l(Q'_i) = \frac{i+3}{2}$. If $n = 4m+1$ then

$$\max\{l(Q_i) : 5 \leq i \leq 2m-1, i \text{ odd}\} = m + 2 \tag{29}$$

$$\max\{l(Q'_i) : 5 \leq i \leq 2m-1, i \text{ odd}\} = m + 1, \tag{30}$$

and for $n = 4m+3$, we get

$$\max\{l(Q_i) : 5 \leq i \leq 2m+1, i \text{ odd}\} = m + 3 \tag{31}$$

$$\max\{l(Q'_i) : 5 \leq i \leq 2m+1, i \text{ odd}\} = m + 2. \tag{32}$$

Next, we consider the vertex v_0. For an even index i, $6 \leq i \leq 2m$, a shortest v_0–u_i and v_0–v_i path is given by $L_i : v_0, v_2, v_4, \ldots, v_i, u_i$ and $L'_i : v_0, v_2, v_4, \ldots, v_i$, where $l(L_i) = \frac{i+2}{2}$ and $l(L'_i) = \frac{i}{2}$.

$$\max\{l(L_i) : 6 \leq i \leq 2m, i \text{ even}\} = m + 1. \tag{33}$$

$$\max\{l(L'_i) : 6 \leq i \leq 2m, i \text{ even}\} = m. \tag{34}$$

Next let i be an odd index. When $n = 4m + 1$, for $5 \leq i < 2m - 1$, a shortest v_0–v_i path is given by $M_i : v_0, v_2, v_4, \ldots, v_{i-1}, u_{i-1}, u_i, v_i$ and the length of the path M_i is $\frac{i+5}{2}$, and for $i = 2m - 1$, a shortest v_0–v_i path is $M'_i : v_0, v_{4m-1}, v_{4m-3}, \ldots, v_{4m-(2m+1)}$ with length $m + 1$. When $n = 4m + 3$, for $5 \leq i < 2m + 1$, a shortest v_0–v_i path is given by $N_i : v_0, v_2, v_4, \ldots, v_{i-1}, u_{i-1}, u_i, v_i$ and the length of the path N_i is $\frac{i+5}{2}$, and for $i = 2m + 1$, a shortest v_0–v_i path is $N'_i : v_0, v_{4m+1}, v_{4m-1}, v_{4m-3}, \ldots, v_{4m-(2m-1)}$ with length $m + 1$. Now, we get the following.

$$\max\{l(M_i) : 5 \leq i \leq 2m - 3, i \text{ odd}\} = m + 1. \tag{35}$$

$$\max\{l(N_i) : 5 \leq i \leq 2m - 1, i \text{ odd}\} = m + 2. \tag{36}$$

From the Eqs. (27)–(36), we have

$$e(u_0) = \begin{cases} m + 2, & \text{for } n = 4m + 1 \\ m + 3, & \text{for } n = 4m + 3, \end{cases}$$

and

$$e(v_0) = \begin{cases} m + 1, & \text{for } n = 4m + 1 \\ m + 2, & \text{for } n = 4m + 3 \end{cases}$$

Thus, $e(u_0) \neq e(v_0)$ and hence $P(n, 2)$ is not self-centered graph. □

Corollary 1. *The generalized Petersen graph $P(n, k)$ is not self-centered for odd integers $n \geq 9$ and $k = \frac{n-1}{2}$.*

Proof. By the structure of generalized Petersen graph, for an odd integer n we get that $P(n, \frac{n+1}{2})$ and $P(n, \frac{n-1}{2})$ are isomorphic. Since $\frac{n+1}{2}$ is relatively prime with n, and n is odd, by Theorem 1 we get $P(n, 2)$ and $P(n, \frac{n+1}{2})$ are isomorphic and thus $P(n, 2)$ and $P(n, \frac{n-1}{2})$ are isomorphic. Since, $P(n, 2)$ is not a self-centered graph for $n \geq 9$ with odd n, $P(n, \frac{n-1}{2})$ is also not a self-centered graph and hence the result. □

In the next theorem we prove that the generalized Petersen graph is a self-centered graph when both n and k are odd, and n is divisible by k.

Theorem 11. *Let $P(n, k)$ be a generalized Petersen graph, where n and k are both odd and k divides n. Then $P(n, k)$ is a d-self-centered graph, where $d = \frac{q+k}{2} + 1$ and $n = kq$.*

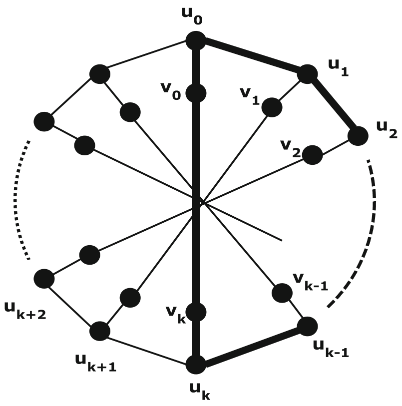

Fig. 3. Generalized Petersen graph $P(n, k)$ with both n and k odd, k divides n

Proof. Due to symmetric structure of $P(n, k)$ we have $d(u_0, u_i) = d(u_0, u_{n-i})$ for $i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. We consider the cycle $C : u_0, u_1, u_2, \ldots, u_k, v_k, v_0, u_0$ of length $k + 3$, see Fig. 3, where C is highlighted by thick lines. Since $u_0, v_0 \in C$, we have

$$\max\{d(u_0, u_i) : u_i \in C\} = \max\{d(v_0, u_i) : u_i \in C\} = \frac{k+3}{2}. \tag{37}$$

Next, we determine $d(u_0, u_i)$, $d(u_0, v_i)$, $d(v_0, u_i)$, and $d(v_0, v_i)$, where u_i, $v_i \notin C$. Let $m = 1, 2, \ldots, \frac{q-1}{2} - 1, \frac{q-1}{2}$. We have following cases depending on the values of i. In the first three cases, we consider $m = 1, 2, \ldots, \frac{q-1}{2} - 1$, and in the fourth case we take $m = \frac{q-1}{2}$.

Case 1. $mk \le i < mk + \frac{k+1}{2}$, $m = 1, 2, \ldots, \frac{q-1}{2} - 1$.

Since $mk \le i$, we can write $i = mk + j$ for $j = 0, 1, 2, \ldots, \frac{k-1}{2}$. Now, a shortest u_0–u_i and v_0–u_i paths are given by the paths P_1 and P_2 respectively, where

$$P_1 : u_0, v_0, v_k, v_{2k}, \ldots, v_{mk}, u_{mk}, u_{mk+1}, \ldots, u_{mk+j}$$
$$P_2 : v_0, v_k, v_{2k}, \ldots, v_{mk}, u_{mk}, u_{mk+1}, \ldots, u_{mk+j}.$$

Moreover, $l(P_1) = m + j + 2$ and $l(P_2) = m + j + 1$. Both P_1 and P_2 obtain their maximum length for $m = \frac{q-1}{2} - 1$ and $j = \frac{k-1}{2}$, i.e.

$$\max\{l(P_1) : i < mk + \frac{k+1}{2}\} = \frac{q+k}{2}. \tag{38}$$

$$\max\{l(P_2) : i < mk + \frac{k+1}{2}\} = \frac{q+k}{2} - 1. \tag{39}$$

Similarly, a shortest u_0–v_i and v_0–v_i paths are given by P_3 and P_4, respectively, where

$$P_3 : v_i, v_{i-k}, v_{i-2k}, v_{i-3k}, \ldots, v_{i-mk}, u_{i-mk}, u_{i-mk-1}, \ldots, u_0$$
$$P_4 : v_0, v_k, v_{2k}, \ldots, v_{mk}, u_{mk}, u_{mk+1}, \ldots, u_{mk+j}, v_{mk+j}$$

Now, $l(P_3) = 1 + i - (k-1)m$ and $l(P_4) = m + j + 2$. Length of path P_3 is maximum for $m = \frac{q-1}{2} - 1$ and corresponding value of i i.e. $i = mk + \frac{k+1}{2} - 1$. This gives

$$\max\{l(P_3)\} = \frac{q+k}{2} - 1. \tag{40}$$

Also, the path P_4 obtains its optimum length for $m = \frac{q-1}{2} - 1$ and $j = \frac{k-1}{2}$, i.e.

$$\max\{l(P_4)\} = \frac{q+k}{2}. \tag{41}$$

Case 2. $mk + \frac{k+1}{2} < i < mk + k$, $m = 1, 2, \ldots, \frac{q-1}{2} - 1$.

Since $mk + \frac{k+1}{2} < i < mk + k$, we write $i = mk + (k - x)$ for $x = 1, 2, \ldots, \frac{k-3}{2}$. A shortest u_0–u_i and v_0–u_i paths are given below.

$$P_5 : u_0, v_0, v_k, v_{2k}, \ldots, v_{mk}, v_{mk+k}, u_{mk+k}, u_{mk+(k-1)}, u_{mk+(k-2)}, \ldots, u_{mk+(k-x)}$$
$$P_6 : v_0, v_k, v_{2k}, \ldots, v_{mk}, v_{mk+k}, u_{mk+k}, u_{mk+(k-1)}, u_{mk+(k-2)}, \ldots, u_{mk+(k-x)}$$

We see that $l(P_5) = 3 + m + x$ and $l(P_6) = 2 + m + x$. Now, maximum length of the paths P_5 and P_6 are obtained when $m = \frac{q-1}{2} - 1$ and $x = \frac{k-3}{2}$, respectively. Hence,

$$\max\{l(P_5)\} = \frac{q+k}{2} + 1, \text{and } \max\{l(P_6)\} = \frac{q+k}{2} - 1. \tag{42}$$

Further, shortest u_0–v_i and v_0–v_i paths are given by P_7 and P_8, respectively, where

$$P_7 : v_i, v_{i-k}, v_{i-2k}, \ldots, v_{i-mk}, v_{n+i-mk-k}, u_{n+i-mk-k}, u_{n+i-mk-k-1}, u_{n+i-mk-k-2}, \ldots, u_0$$
$$P_8 : v_0, v_k, v_{2k}, v_{3k}, \ldots, v_{mk}, v_{mk+k}, u_{mk+k}, u_{mk+(k-1)}, u_{mk+(k-2)}, \ldots, u_{mk+(k-x)}, v_{mk+(k-x)}$$

Moreover, $l(P_7) = 2 + m + (m+1)k - i$ and $l(P_8) = 3 + m + x$, and so

$$\max\{l(P_7)\} = \frac{q+k}{2} - 1, \text{and } \max\{l(P_8)\} = \frac{q+k}{2}. \tag{43}$$

Case 3. $i = mk + \frac{k+1}{2}$, $m = 1, 2, \ldots, \frac{q-1}{2} - 1$.

In this case, a shortest u_0–u_i, v_0–u_i, u_0–v_i, and v_0–v_i paths are given by the following paths P_9, P_{10}, P_{11}, and P_{12}, respectively, where

$$P_9 : u_0, v_0, v_k, v_{2k}, \ldots, v_{mk}, u_{mk}, u_{mk+1}, u_{mk+2}, \ldots, u_{mk+\frac{k+1}{2}}$$
$$P_{10} : v_0, v_k, v_{2k}, \ldots, v_{mk}, u_{mk}, u_{mk+1}, u_{mk+2}, \ldots, u_{mk+\frac{k+1}{2}}$$
$$P_{11} : v_i, v_{i-k}, v_{i-2k}, \ldots, v_{i-mk}, u_{i-mk}, u_{i-mk-1}, u_{i-mk-2}, \ldots, u_0$$
$$P_{12} : v_0, v_k, v_{2k}, \ldots, v_{mk}, u_{mk}, u_{mk+1}, u_{mk+2}, \ldots, u_{mk+\frac{k+1}{2}}, v_{mk+\frac{k+1}{2}}.$$

Moreover, $l(P_9) = 2+m+\frac{k+1}{2}$, $l(P_{10}) = 1+m+\frac{k+1}{2}$, $l(P_{11}) = 1+i-(k-1)m$, and $l(P_{12}) = 2+m+\frac{k+1}{2}$. These paths attain their maximum length for $m = \frac{q-1}{2} - 1$, and hence we get

$$\max\{l(P_9)\} = \frac{q+k}{2} + 1, \max\{l(P_{10})\} = \frac{q+k}{2}, \max\{l(P_{11})\} = \frac{q+k}{2},$$
$$\text{and } \max\{l(P_{12})\} = \frac{q+k}{2} + 1 \quad (44)$$

Case 4. $i = mk + l$ and $m = \frac{q-1}{2}$, where $l = 0, 1, 2, \ldots, \lfloor \frac{k}{2} \rfloor$

Shortest u_0–u_i, v_0–u_i, and v_0–v_i paths are given by:

$P_{13} : u_0, v_0, v_k, v_{2k}, \ldots, v_{(m+1)k}, u_{(m+1)k}, u_{(m+1)k+1}, \ldots, u_{(m+1)k+l}$

$P_{14} : v_0, v_k, v_{2k}, \ldots, v_{(m+1)k}, u_{(m+1)k}, u_{(m+1)k+1}, \ldots, u_{(m+1)k+l},$

$P_{15} : v_0, v_k, v_{2k}, \ldots, v_{(m+1)k}, u_{(m+1)k}, u_{(m+1)k+1}, \ldots, u_{(m+1)k+l}, v_{(m+1)k+l}.$

We have $l(P_{13}) = 3 + m + l$, $l(P_{14}) = 2 + m + l$, and $l(P_{15}) = 3 + m + l$, respectively. Finally, the path $P_{16} : v_i, v_{i-k}, v_{i-2k}, \ldots, v_{i-mk}, u_{i-mk}, u_{i-mk-1}, u_{i-mk-2}, \ldots, u_0$ gives a shortest u_0–v_i path and the length of the path P_{16} is $1 + m + i - mk$. The path P_{16} attains its maximum length for $m = \frac{q-1}{2}$ and $i = mk + \lfloor \frac{k}{2} \rfloor$, i.e.

$$\max\{l(P_{16})\} = \frac{q+k}{2}. \quad (45)$$

From the Eqs. (38)–(45), we conclude that $e(u_0) = e(v_0) = \frac{q+k}{2} + 1$. Thus, generalized petersen graph is a d-self-centered graph, where $d = \frac{q+k}{2} + 1$. □

In the theorem given below, we investigate self-centeredness of $P(n, k)$ for $n = m(4m + 1) \pm (m + 1)$ and $k = 4m + 1$.

Theorem 12. *For $n = m(4m + 1) \pm (m + 1)$, $k = 4m + 1$, and any positive integer $m \geq 2$, the generalized Petersen graph is not a self-centered graph.*

Proof. We have following two cases here.

Case 1. $k = 4m + 1$ and $n = m(4m + 1) + (m + 1) = 4m^2 + 2m + 1$ for any $m \geq 2$.

In this case, $v_{\frac{k-1}{2}}$ and $v_{n-(\frac{k-1}{2})}$ are equivalently most distant vertices from u_0 as well as v_0 and their path lengths are $\frac{k-1}{2}$ and $\frac{k-1}{2} + 1$, respectively. This implies that, $e(u_0) = \frac{k-1}{2}$ and $e(v_0) = \frac{k-1}{2} + 1$ and they differ by one.

Case 2. $k = 4m + 1$ and $n = m(4m + 1) - (m + 1) = 4m^2 - 1$ for any $m \geq 2$.

In this case, $v_{\frac{k+1}{2}}$ and $v_{n-(\frac{k+1}{2})}$ are equivalently most distant vertices from u_0 as well as v_0 and their path lengths are $\frac{k+1}{2}$ and $\frac{k+1}{2} + 1$, respectively. This implies that, $e(u_0) = \frac{k+1}{2}$ and $e(v_0) = \frac{k+1}{2} + 1$ and the eccentricities differ by one.

In both the above cases we get that $e(u_0) \neq e(v_0)$. Hence, the generalized Petersen graph is not a self-centered. □

4 Computer Search and Concluding Remarks

We made an exhaustive computer search and found all possible values of n and k for which $P(n,k)$ are non self-centered. In this search, we have discarded isomorphs of these generalized Petersen graphs using Theorem 1. We list all non-isomorphic generalized Petersen graphs in Tables 1 and 2 and address their theoretical proofs obtained in this paper.

Table 1. Non-self-centeredness of $P(n,k)$, n odd

n	k	Theoretical support
$n \geq 9$	$k = 2$	Theorem 9
$n = m(4m+1) \pm (m+1)$, $m \geq 2$	$k = 4m+1$	Theorem 11

Table 2. Non-self-centeredness of $P(n,k)$, n even

n	k	Theoretical support
$n \geq 4$	$k = \frac{n}{2}$	Theorem 5
$n = 4m+2$, m is a positive integer	$k = \frac{n}{2} - 1$	Theorem 6
$n = 4m(4m+1)$, m is a positive integer	$k = 2m(4m-1)$	Theorem 8

For complete characterization, one has to prove theoretically that all generalized Petersen graphs $P(n,k)$ other than those in Tables 1 and 2, and their isomorphs are self-centered. Hence, we make the following conjecture.

Conjecture: The generalized Petersen graphs $P(n,k)$ other than those in Tables 1 and 2 and their isomorphs are self-centered.

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