Chapter 7
Uncertainty-Based Multidisciplinary Design Optimization (UMDO)

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7.1 Introduction

This chapter is devoted to the description of the MDO formulations in the presence of uncertainty. In Chapter 1, deterministic MDO formulations have been introduced, highlighting the interest of such methodologies to solve complex and multidisciplinary design problems. Uncertainty-based multidisciplinary design optimization (UMDO) deals with the presence of uncertainty in MDO problems. The understanding of the importance of UMDO is spreading among academia and industry quickly. Nevertheless, in comparison with the deterministic MDO approaches, the UMDO methodologies are still in the early stages of development and numerous challenges have still to be solved. In the last decades, important improvements have been made in this field of research and are presented in this chapter. UMDO problems combine the challenges of deterministic MDO (organization of the design process, control of interdisciplinary couplings, etc.) and the difficulties involved by uncertainty propagation for multi-physics problems. Most of the existing UMDO formulations are built on the uncertainty propagation techniques dedicated to multidisciplinary problems presented in Chapter 6. The algorithms used to solve these UMDO problems are not discussed in this chapter, but all the presented optimization techniques in Chapter 5 may be used to solve UMDO problems.

The objective of this chapter is to describe the existing UMDO formulations and to understand their characteristics and limitations with illustrations on toy cases. To understand the existing UMDO formulations, this chapter is organized as follows. In Section 7.2, the differences between deterministic MDO and UMDO are highlighted, distinction between robustness-based UMDO and reliability-based UMDO is discussed, and useful notations are introduced. Section 7.3 is focused on the existing coupled UMDO formulations relying on MDA and the use of uncertainty propagation techniques introduced in Chapter 6 in the case of mul-

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tidisciplinary systems. Section 7.4 presents the existing single-level procedures for UMDO problems that separate the uncertainty propagation phase from the optimization. Section 7.5 is focused on the distributed UMDO formulations and the interdisciplinary coupling satisfaction in the presence of uncertainty. Alternative hybrid UMDO formulations are introduced in Section 7.6. These formulations use MDA to ensure the interdisciplinary couplings while allowing the uncertainty propagation though a decoupled system. In Section 7.7, UMDO strategies to deal with both aleatory and epistemic uncertainties are briefly reviewed.

For each approach, the principle of the formulation is exposed, then the mathematical formulation with an explanatory scheme is presented, and finally the advantages and drawbacks are outlined. Moreover, these approaches are illustrated over a toy case.

7.2 Differences Between MDO and UMDO

Concerning the handling of interdisciplinary couplings, several hypotheses are made on the considered UMDO problem in order to simplify the notations and the understanding of the UMDO formulations:

- besides the interdisciplinary coupling equation, only inequality constraints are considered. Indeed, in aerospace vehicle design problems, tolerances are often considered on equality constraints (such as orbit injection constraints) to account for the presence of uncertainty and therefore equality constraints may be transformed into inequality constraints involving tolerances.
- without loss of generality, the state variables $\mathbf{x}$ and the state equation residuals $\mathbf{r}(\cdot)$ are handled at the discipline level and therefore do not appear in the UMDO formulations.

The presence of uncertainty in MDO problems requires tools and techniques coming from the fields of:

- uncertainty characterization and modeling (see Chapter 2), of sensitivity analysis (see Chapter 3),
- reliability analysis (see Chapter 4),
- uncertainty propagation (see Chapters 3 and 6),
- adapted optimization algorithms (see Chapter 5).

Moreover, in order to efficiently assess the performance and the reliability of the system, an efficient organization of the design process is essential. Important differences exist between the UMDO formulation and the deterministic MDO formulation described in Chapter 1. The UMDO formulations require uncertainty modeling and measures, uncertainty propagation, and optimization algorithms under uncertainty.

The introduction of uncertainty in a MDO problem leads to a new general UMDO problem (Yao et al. 2011) formulated as follows:

$$\begin{aligned} \min \quad & \Xi\left[f(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U})\right] && (7.1) \\ \text{w.r.t.} \quad & \mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}} \\ \text{s.t.} \quad & \mathbb{K}_k\left[g_k(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U})\right] \leq 0 \quad \text{for } k \in \{1, \ldots, m\} && (7.2) \\ & \forall i \neq j \in [1, N]^2, \forall \mathbf{u} \in \mathbb{R}^d, \; \mathbf{y}_{ij}(\boldsymbol{\theta}_{\boldsymbol{Y}ij}, \mathbf{u}_i) = \mathbf{c}_{ij}(\mathbf{z}_i, \mathbf{y}_{.i}(\boldsymbol{\theta}_{\boldsymbol{Y}.i}, \mathbf{u}_{.i}), \mathbf{u}_i) && (7.3) \\ & \mathbf{z}_{\min} \leq \mathbf{z} \leq \mathbf{z}_{\max} && (7.4) \end{aligned}$$

The notations employed in this generic UMDO problem formulation and the difference with respect to deterministic MDO are explained in the following paragraphs.

- First, $\mathbf{U}$ is the uncertain vector (assumed to be real-valued random vector,$\mathbf{U} \in \mathbb{R}^d$). We note $\mathbf{U}_i$ the input uncertain vector of the discipline i and $\mathbf{U} = \bigcup_{i=1}^{N} \mathbf{U}_i$ without duplication. In this chapter (except in Section 7.7), it is assumed that the uncertain variables are modeled with the probability theory, and that the joint input variable probability distributions are known. To simplify the notations in the following of the chapter, for all the uncertain variables, the realization $U(w)$ is noted u. The kth sample generated for instance by CMC of the random vector $\mathbf{U}_i$ is noted $\mathbf{u}_{i(k)}$. The pth coordinate of the kth sample of the random vector $\mathbf{U}_i$ is noted $u_{i(k)}^{(p)}$. As highlighted in Chapter 2, other uncertainty modelings exist such as evidence theory (Dempster 1967), possibility theory (Negoita et al. 1978), or interval analysis (Moore et al. 2009), and their combination with UMDO is discussed in Section 7.7. Moreover, it is assumed that the design variables $\mathbf{z}$ are deterministic variables, and all the uncertainties are represented by $\mathbf{U}$. We note $(\boldsymbol{\Omega}, \boldsymbol{\sigma}_{\boldsymbol{\Omega}}, \mathbb{P})$ a probability space with $\boldsymbol{\Omega}$ a sample space, $\boldsymbol{\sigma}_{\boldsymbol{\Omega}}$ a sigma-algebra, and $\mathbb{P}[\cdot]$ a probability measure. We note $\phi(\cdot)$ the joint probability density function (PDF) of the uncertain vector $\mathbf{U}$. If a design variable is considered as uncertain, then, its contribution is composed of two parts: one deterministic that is controlled by the optimizer and one aleatory that is propagated through the system. For instance, if the propellant mass m is considered as an uncertain design variable, therefore, the expected value of the propellant mass μ_m is the deterministic design variable controlled by the optimizer (it can be considered as a system-level specification) and the propellant mass uncertainty around the expected value is propagated through the system according to the propellant mass PDF.
- Secondly, $\Xi[\cdot]$ denotes the objective function uncertainty measure. The measure $\Xi[\cdot]$ quantifies the uncertainty in the objective function to be optimized. Within the probability formalism, the expected value $\mathbb{E}\left[f(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U})\right]$ or an aggregation of the expected value and the standard deviation are commonly used to quantify the uncertainty in the objective function (Baudoui 2012) (see Chapter 5 for more details on the uncertainty measures for optimization).
- Eventually, because of the presence of the uncertain vector $\mathbf{U}$, the coupling variable vector $\mathbf{Y}$ is also an uncertain vector. In the decoupled formulations (as in deterministic MDO), the input coupling variables have to be controlled

by the optimizer. However in the presence of uncertainty, the optimizer cannot directly control the uncertain coupling variables. Indeed, as the input coupling variables are functions of $\mathbf{U}$, in order to avoid infinite dimension optimization problem, the optimizer does not directly control the uncertain coupling variables but rather deterministic parameters $\boldsymbol{\theta}_{\boldsymbol{Y}}$ modeling the uncertain input coupling vector $\mathbf{Y}$. These parameters may be some realizations of the uncertain variables, the statistical moments, the parameters of the PDF, parameters of a functional representation of $\mathbf{Y}$, etc.

Concerning the UMDO constraints using the probability theory, two classical measures of uncertainty exist. In the solving of UMDO problems, the largest part of the computational effort is generally devoted to the constraint evaluations (Du et al. 2008). Similarly to the single discipline optimization under uncertainty (see Chapter 5), depending on the choice of the constraint measures, two UMDO problem formulations may be distinguished, the robustness-based UMDO and the reliability-based UMDO (Yao et al. 2011). Different definitions have been proposed, we consider in this chapter that the difference between the two approaches results from the constraint uncertainty measures as illustrated in the following paragraphs.

Robustness-Based UMDO Formulation In the robustness-based formulation (also called robust formulation), the constraint of the UMDO problem (Equation 7.2) can be rewritten as:

$$\mathbb{K}_k\left[g_k(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U})\right] = \mathbb{E}\left[g_k(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U})\right] + \eta_k \sigma\left[g_k(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U})\right],$$

where $\mathbb{E}[g_k(\cdot)]$ and $\sigma[g_k(\cdot)]$ are the expected value and the standard deviation of the kth coordinate of the constraint function vector $\mathbf{g}(\cdot)$. The robust formulation is based on the statistical moments of the inequality function vector to ensure that despite the uncertainty, the system will stay feasible. $\eta_k \in \mathbb{R}^+$indicates the restriction of the feasible region to η_k standard deviations away from the mean value of the constraint function vector. The robust UMDO formulation may be written such as:

$$\begin{aligned}
\min \quad & \Xi\left[f(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U})\right] && (7.5)\\
\text{w.r.t.} \quad & \mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}} \\
\text{s.t.} \quad & \mathbb{E}\left[g_k(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U})\right] + \eta_k \sigma\left[g_k(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U})\right] \leq 0 \ \text{ for } k \in \{1, \ldots, m\} && (7.6)\\
& \forall i \neq j, \forall \mathbf{u} \in \mathbb{R}^d, \ \mathbf{y}_{ij}(\boldsymbol{\theta}_{\boldsymbol{Y}ij}, \mathbf{u}_i) = \mathbf{c}_{ij}(\mathbf{z}_i, \mathbf{y}_{.i}(\boldsymbol{\theta}_{\boldsymbol{Y}.i}, \mathbf{u}_{.i}), \mathbf{u}_i) && (7.7)\\
& \mathbf{z}_{\min} \leq \mathbf{z} \leq \mathbf{z}_{\max} && (7.8)
\end{aligned}$$

For instance, the approach called multidisciplinary optimization and robust design approaches applied to concurrent engineering (MORDACE) (Giassi et al. 2004) proposes to solve UMDO problems with a robust formulation through concurrent design of subsystems ensuring effective design work distribution. The method relies on surface response methods of each discipline in order to concurrently optimize them. Then, a compromise strategy (based on a Pareto frontier analysis) is performed in order to identify the potential optimal candidates of the different possible combinations of the subsystem optimization results.

Reliability-Based UMDO Formalism In the reliability-based formulation, the uncertainty measure of the constraint (Equation 7.2) is written as:

$$\mathbb{K}_k\left[g_k(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U})\right] = \Lambda_k\left[g_k(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U}) > 0\right] - \Lambda_{t_k},$$

where $\Lambda_k[g_k(\cdot)]$ stands for the measure vector of uncertainty for the inequality constraint function vector. The uncertainty measure of the constraint has to be at most equal to a given threshold Λ_{t_k} (Agarwal et al. 2004b). The computation of such a constraint involves reliability analysis methods such as the ones presented in Chapters 4 and 6. It reflects the requirement for the optimized system to lie in the feasible region with a given reliability despite the uncertainty. As the uncertain variables are modeled within the probability theory, we have for the kth coordinate of the vector of the measures of uncertainty:

$$\mathbb{K}_k\left[g_k(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U})\right] = \mathbb{P}\left[g_k(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U}) > 0\right] - P_{t_k} = \int_{\mathscr{I}_k} \phi(\mathbf{u})\mathrm{d}\mathbf{u} - P_{t_k} \tag{7.9}$$

with $g_k(\cdot)$ the kth component of the inequality constraint vector, $\mathscr{I}_k = \{\mathbf{u} \in \mathbb{R}^d | g_k(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{u}) > 0\}$, and P_{t_k} the maximal allowed failure probability. In reliability-based UMDO, the formulation may be rewritten such as:

$$\begin{aligned}
\min \quad & \Xi\left[f(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U})\right] & (7.10)\\
\text{w.r.t.} \quad & \mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}} & \\
\text{s.t.} \quad & \mathbb{P}\left[g_k(\mathbf{z}, \boldsymbol{\theta}_{\boldsymbol{Y}}, \mathbf{U}) > 0\right] - P_{t_k} \leq 0 \ \text{ for } k \in \{1, \ldots, m\} & (7.11)\\
& \forall i \neq j, \forall \mathbf{u} \in \mathbb{R}^d,\ \mathbf{y}_{ij}(\boldsymbol{\theta}_{\boldsymbol{Y}ij}, \mathbf{u}_i) = \mathbf{c}_{ij}(\mathbf{z}_i, \mathbf{y}_{.i}(\boldsymbol{\theta}_{\boldsymbol{Y}.i}, \mathbf{u}_{.i}), \mathbf{u}_i) & (7.12)\\
& \mathbf{z}_{\min} \leq \mathbf{z} \leq \mathbf{z}_{\max} & (7.13)
\end{aligned}$$

As in deterministic MDO, in UMDO three types of formulations may be distinguished for the coupling handing:

- Interdisciplinary coupling satisfaction handled with using a coupled approach (with multidisciplinary analysis using Fixed Point Iteration),
- Interdisciplinary coupling satisfaction handled with using a decoupled approach,
- Interdisciplinary coupling satisfaction through hybrid strategy combining both coupled and decoupled techniques.

In this chapter, the coupled, hybrid, and decoupled existing UMDO formulations are presented in the point of view of the interdisciplinary coupling satisfaction in the presence of uncertainty. Firstly, the existing coupled approaches are introduced in Section 7.3. Then, in Sections 7.4 and 7.5 the single-level procedures and distributed

UMDO formulations are reviewed. Finally in Section 7.6 a brief overview of hybrid UMDO formulations is performed.

7.3 Coupled UMDO Formulations

7.3.1 MultiDisciplinary Feasible (MDF) Formulation Under Uncertainty

Classical MDF Under Uncertainty

Principle As in deterministic MDO, MDF under uncertainty (Koch et al. 2002) is the most used UMDO formulation. MDF under uncertainty is a single-level coupled formulation. It takes advantages of the simplicity of the deterministic version of MDF and derives it in the presence of uncertainty. The most straightforward approach to ensure the coupling satisfaction in UMDO is to use Crude Monte Carlo simulations (CMC) to propagate uncertainty while solving the system of interdisciplinary equations by MDA for each realization of the CMC sample set (Oakley et al. 1998; Koch et al. 2002; Jaeger et al. 2013) (Figure 7.1). The MDF under uncertainty formulation is given by

$$\min \quad \Xi\left[f(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right] \tag{7.14}$$

$$\text{w.r.t.} \quad \mathbf{z}$$

$$\text{s.t.} \quad \mathbb{K}_k\left[g_k(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right] \leq 0 \ \text{ for } k \in \{1, \ldots, m\} \tag{7.15}$$

$$\mathbf{z}_{\min} \leq \mathbf{z} \leq \mathbf{z}_{\max} \tag{7.16}$$

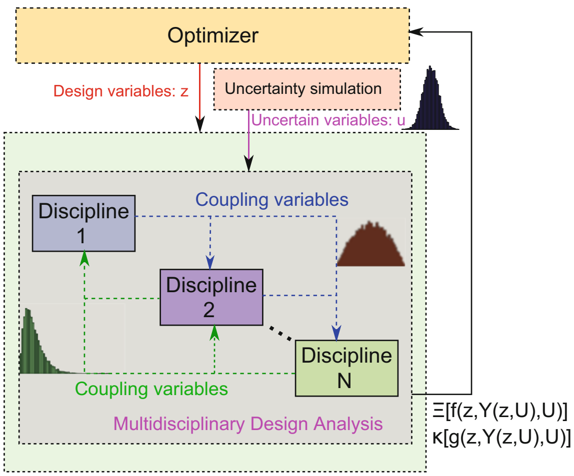


Fig. 7.1 Multidiscipline feasible (MDF) under uncertainty

For a given design variable vector $\mathbf{z}$, to evaluate the objective function and the constraint functions, it is necessary to propagate the uncertainty in the entire system (through the different disciplines). In the coupled formulations, $\mathbf{Y}$ (which some readers might prefer to read as $\mathbf{y}(\mathbf{z}, \mathbf{U})$ but, for readability, the uppercase notation denoting random variables to $\mathbf{Y}$ is carried except to refer to realizations) is the coupling variable vector satisfying the following system of interdisciplinary equations:

$$\forall\, \mathbf{u} \in \mathbb{R}^d,\ \forall (i, j) \in \{1, \ldots, N\}^2\ i \neq j,\ \mathbf{y}_{ij} = \mathbf{c}_{ij}(\mathbf{z}_i, \mathbf{y}_{.i}, \mathbf{u}) \tag{7.17}$$

It is assumed that for a given realization of the uncertain vector $\mathbf{U}$, there exists a unique set of coupling variable values such that the coupling variables satisfy

$$\forall\, \mathbf{u} \in \mathbb{R}^d,\ \forall (i, j) \in \{1, \ldots, N\}^2\ i \neq j, \exists!(\mathbf{y}_{ij}, \mathbf{y}_{ji}) |\ \mathbf{y}_{ij} = \mathbf{c}_{ij}(\mathbf{z}_i, \mathbf{y}_{.i}, \mathbf{u}) \tag{7.18}$$

To compute the uncertainty measure of the performance $\Xi[f(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U})]$ and the constraints $\mathbb{K}_k\,[g_k(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U})]$, repeated MDAs are carried out for a set of uncertain variable realizations sampled by CMC. Classical techniques introduced for deterministic MDO in Chapter 1 such as Fixed Point Iteration or auxiliary optimization process may be used to solve the system of interdisciplinary couplings for each CMC sample.

MDF under uncertainty allows one to ensure the interdisciplinary coupling satisfaction for all the realizations of the uncertain variables guarantying an appropriate estimation of the system performance and reliability. Indeed, at each iteration of the system-level optimizer in $\mathbf{z}$, for each realization of the uncertain variables, the system of interdisciplinary equations (7.17) is solved with using the MDA. This approach ensures the multidisciplinary feasibility of the optimal design system and also for all the design points evaluated during the optimization process.

This formulation is computationally expensive due to the repeated evaluations of the disciplines. The computational cost of MDA under uncertainty with CMC corresponds to one MDA multiplied by the number of CMC samples (Haldar and Mahadevan 2000). Therefore, the computational cost of MDF under uncertainty is increased by the propagation of uncertainty and becomes intractable for the design of complex systems (Du et al. 2008). MDF under uncertainty is considered as the reference UMDO formulation due to its intrinsic interdisciplinary coupling satisfaction.

Nonparametric approaches have been proposed in Cho et al. (2016) considering limited available information on data samples for the uncertain variables. A FPI-based approach is used to propagate uncertainty characterized by a set of limited samples. Then, a Kolmogorov–Smirnov test is used to calculate the maximum difference of the empirical cumulative distribution functions between the previous and the current iterations to check the FPI convergence. Then, a nonparametric uncertainty analysis based on the Akaike information criterion (AIC) is proposed (Akaike 1973) to select the best fitted distribution from several potential candidate distributions. The interdisciplinary coupling satisfaction is ensured for the probability distribution of the coupling variables but not for all the realizations of the uncertain variables as in the reference MDF formulation.

Application to Toy Case In order to illustrate the different UMDO methodologies described in this chapter, the following toy case problem is considered.

The robustness-based MDF formulation is given by

$$\begin{aligned} \text{min} \quad & \mathbb{E}\left[f(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right] && (7.19)\\ \text{w.r.t.} \quad & \mathbf{z} = [z_{sh}, z_1, z_2] \\ \text{s.t.} \quad & \mathbb{E}\left[g(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right] + 3 \times \sigma\left[g(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right] \leq 0 && (7.20)\\ & \mathbf{0} \leq \mathbf{z} \leq \mathbf{5} && (7.21) \end{aligned}$$

with the two disciplines defined by

$$\begin{aligned} \text{—Discipline 1: } y_{12} &= -z_{sh}^{0.2} + u_{sh} + 0.25 \times u_1^{0.2} + z_1 + y_{21}^{0.58} \\ &\quad + u_1^{0.4} \times y_{21}^{0.47} && (7.22)\\ \text{—Discipline 2: } y_{21} &= -z_{sh} + u_{sh}^{0.1} - z_2^{0.1} + 3 \times y_{12}^{0.47} + u_2^{0.33} \\ &\quad + y_{12}^{0.16} \times u_2^{0.05} + y_{12}^{0.6} \times u_2^{0.13} + 100 && (7.23) \end{aligned}$$

and the objective and constraint functions:

$$\begin{aligned} \text{—Objective: } f &= \left(\frac{1}{5}\right) \times \Big[(z_{sh} - 4)^2 + (z_1 - 3)^2 + (z_2 - 2)^2 \\ &\quad + (y_{21} + z_1 \times z_2)^{0.6} + (u_{sh} + 0.9)^2\Big] && (7.24)\\ \text{—Constraint: } g &= \exp(-0.01 \times u_1^2) \times (z_{sh} - 1) \times z_1 - 0.02 \times u_2^5 \times z_2^3 \\ &\quad + 0.01 \times y_{12}^{2.5} \times z_2 \times \exp(-0.1 \times u_{sh}) && (7.25) \end{aligned}$$

$\mathbf{U} = [U_{sh}, U_1, U_2]$ is the vector of uncertain variables. These latter are distributed according to the following probability distributions:

- U_{sh} is distributed according to a normal distribution: $U_{sh} \sim \mathcal{N}(2, 0.3)$
- U_1 is distributed according to a uniform distribution: $U_1 \sim \mathcal{U}(0, 1)$
- U_2 is distributed according to a normal distribution: $U_2 \sim \mathcal{N}(4, 0.5)$

In the following, all the figures and discussions are presented with design variable values normalized in [0,1]. To illustrate the variation of the objective function and the constraint, the histogram for these two functions is represented in Figures 7.2 and 7.3 for three different values of the design variable vector. The distributions for $f(\cdot)$ and $g(\cdot)$ are obtained with a FPI between the two disciplines and the propagation of 10^4 CMC samples generated according to the PDF of the uncertain variables (see Chapter 6 for more details on the uncertainty propagation

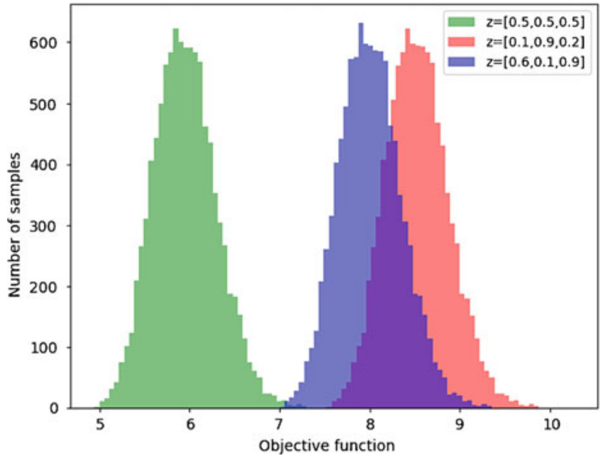
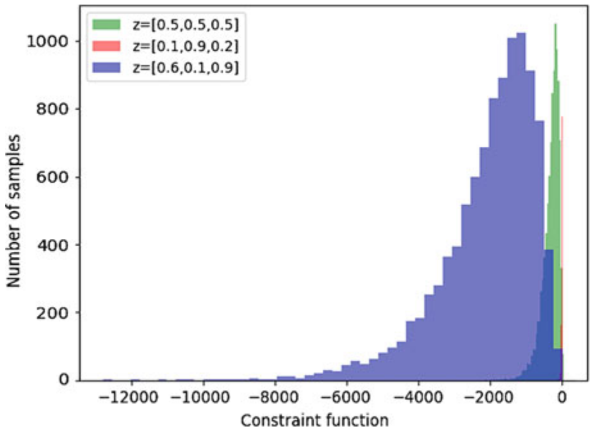


Fig. 7.2 Histogram of the objective function for three different values of the design variable vector

Fig. 7.3 Histogram of the constraint function for three different values of the design variable vector



for multidisciplinary systems). The value of the decision variable vector $\mathbf{z}$ has an important impact both on the objective and the constraint function distributions. For instance, for $\mathbf{z} = [0.5, 0.5, 0.5]$, the mean value of the objective function is around $\mu_f \simeq 6.0$, whereas for $\mathbf{z} = [0.6, 0.1, 0.9]$ its value is around $\mu_f \simeq 8.0$. Similarly, the impact on the constraints and the feasibility of the three design candidates is illustrated in Figure 7.3.

In Figure 7.4, the iso-surface of the limit state $g(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U}) = 0$ for $\mathbf{z} = [0.5, 0.5, 0.5]$ is represented with the colormap corresponding to the distance to the origin in the uncertain space. This iso-surface separates the safe and failed states.

In Figure 7.5, the histograms for two different values of the design variable vector are represented. The number of samples satisfying the constraint $g(\cdot) \leq 0$ is displayed in green, while the failed samples are represented in red. The value of

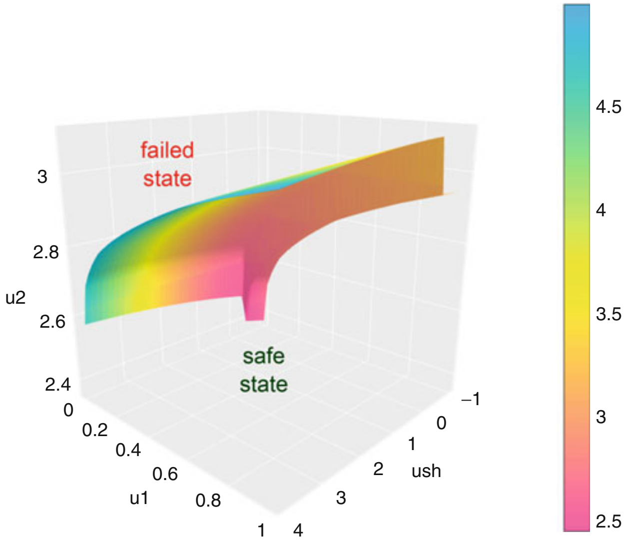


Fig. 7.4 Iso-surface of the limit state $g(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U}) = 0$ for $\mathbf{z} = [0.5, 0.5, 0.5]$, colormap corresponds to the distance to origin

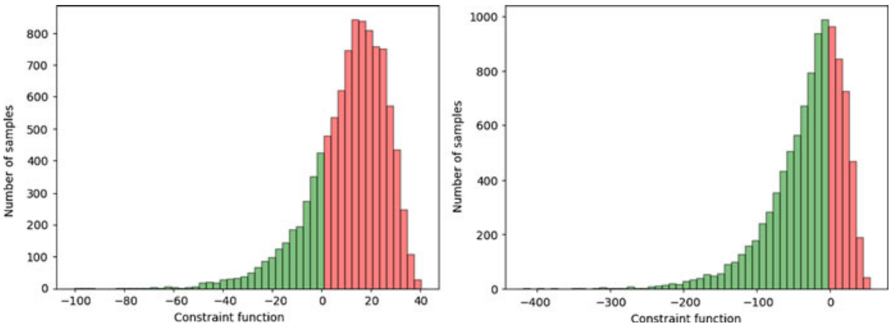


Fig. 7.5 Histogram of constraint function for $z = [0.5, 0.5, 0.2]$ (left) and $z = [0.5, 0.5, 0.3]$ (right), safe (green), and failure (red) samples

the decision variable vector influences the probability of failure which is larger on the graph of the left side than on the right side.

Using a coupled MDF formulation, the robust optimization problem is solved with CMA-ES (Hansen and Ostermeier 1996) (see Chapter 5 for more details on CMA-ES algorithm) constituted of a population of 10 individuals and a penalization strategy for the constraint handling in order to find the optimal value of the design vector $\mathbf{z}$ corresponding to the problem defined by Equations (7.19–7.21). At each iteration of CMA-ES, a propagation of uncertainty by CMC (using 10^4 samples) is carried out to estimate the objective function and the constraint. The optimization is stopped after 150 iterations. Figure 7.6 presents the convergence plots for the robust

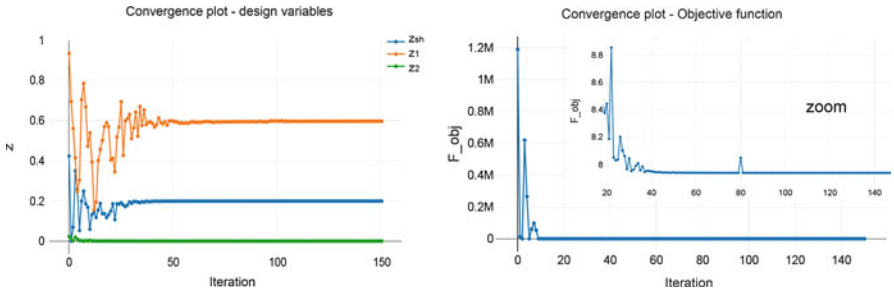


Fig. 7.6 Convergence plots for the robust-based MDF formulation, design variables (left), objective function (right)

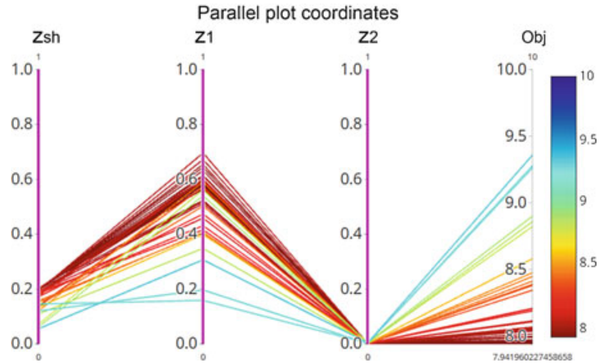


Fig. 7.7 Parallel plot for the robust-based MDF formulation using CMA-ES optimization algorithm

MDF formulation for the normalized design variables (left graph) and the objective function (right graph).

Figure 7.7 illustrates the parallel plot coordinates for the robust-based MDF formulation using CMA-ES and represents with broken lines the best individual variable values at each iteration. The algorithm succeeds to find an optimal value for the design variables $\mathbf{z}^* = [0.20, 0.60, 0.00]$ with an objective function value of $\mu_f = 7.94$ and satisfying the constraint $\mathbb{E}\left[g(\mathbf{z}^*, \mathbf{Y}(\mathbf{z}^*, \mathbf{U}), \mathbf{U})\right] + 3 \times \sigma\left[g(\mathbf{z}^*, \mathbf{Y}(\mathbf{z}^*, \mathbf{U}), \mathbf{U})\right] \leq 0$. CMA-ES stabilizes after 50 iterations.

At each iteration of CMA-ES, for each of the 10 individuals, the FPI process converges in 4 iterations and 10^4 CMC samples are propagated through FPI leading to a total number of discipline evaluations of $N_{call} = 4 \times 10 \times 10^4 = 4 \times 10^5$. At the convergence of the optimization, the two disciplines have been evaluated 6×10^7 times, which is possible here due to the simplicity of the disciplines; however, such an approach might not be possible for complex system design using computationally intensive disciplines.

Considering the same disciplines, objective function, and constraint equations, a reliability-based MDF formulation is given by

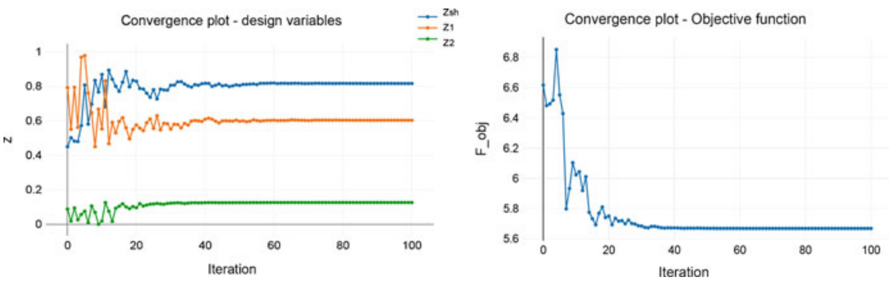
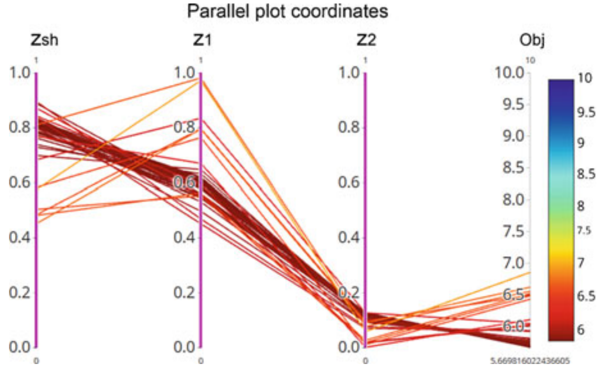


Fig. 7.8 Convergence plot for the reliability-based MDF, design variables (left), objective function (right)

Fig. 7.9 Parallel plot for reliability-based MDF using CMA-ES optimization algorithm



$$\min \quad \mathbb{E}\left[f(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right] \tag{7.26}$$

$$\text{w.r.t.} \quad \mathbf{z} = [z_{sh}, z_1, z_2]$$

$$\text{s.t.} \quad \mathbb{P}\left[g(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U}) > 0\right] \leq 10^{-3} \tag{7.27}$$

$$\mathbf{0} \leq \mathbf{z} \leq \mathbf{5} \tag{7.28}$$

The same process as for the robustness-based MDF formulation is employed for the uncertainty propagation and the estimation of the probability of failure is carried out by CMC. Figure 7.8 presents the convergence plot for the reliability MDF formulation for the normalized design variables (left graph) and objective function (right graph).

Figure 7.9 illustrates the parallel plot coordinates for the robust MDF formulation using CMA-ES and representing with a broken line the best individual variable values at each iteration. The algorithm succeeds to find an optimal value for the design variables $\mathbf{z}^* = [0.81, 0.60, 0.13]$ with an objective function value of $\mu_f = 5.67$ and satisfying the constraint $\mathbb{P}[g(\cdot) > 0] \leq 10^{-3}$. CMA-ES stabilizes after 50 iterations. The solution found is different from the robustness-based UMDO formulation highlighting the importance of problem formulation in the presence of uncertainty and choice uncertainty measures as discussed in Chapter 5.

The histogram of the objective and the constraint functions at the optimum $\mathbf{z}^*$ for the reliability-based MDF formulation is displayed in Figures 7.10 and 7.11. For the histogram of the constraint function, it can be seen that among the 10^4 samples less than 10 samples do not satisfy the constraint (in red) leading to a probability of failure under the target.

To reduce the computational cost, derivations of MDF under uncertainty based on surrogate models have been proposed and are detailed in the following section.

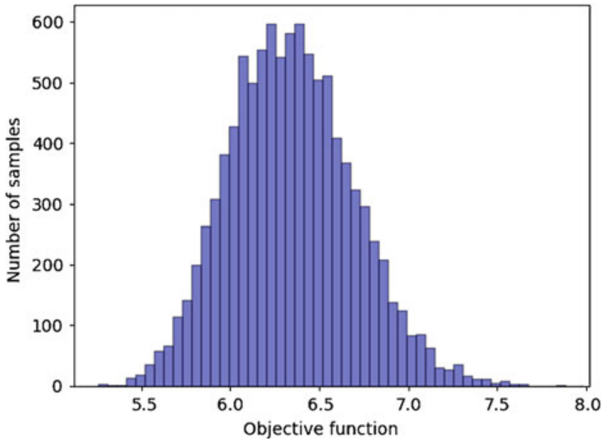
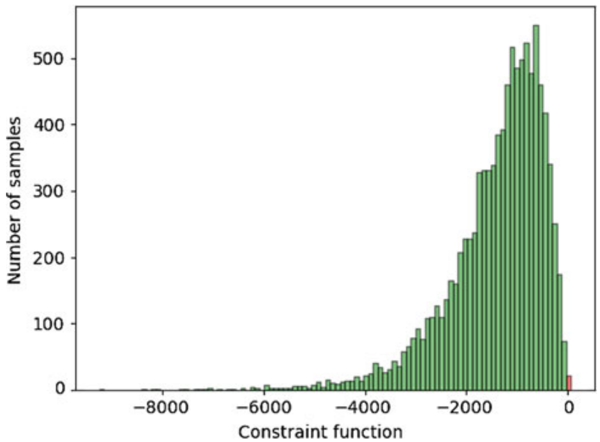


Fig. 7.10 Histogram of the objective function—reliability-based MDF

Fig. 7.11 Histogram of the constraint—reliability-based MDF—safe (green) and failure (red) samples



Surrogate Model-Based MDF Under Uncertainty

Several alternatives to the classical MDF under uncertainty approach have been proposed to reduce the computational cost introduced by repetitive calls to the MDA. A first class of alternatives is the use of surrogate models of the disciplines or even of the entire MDA allowing to repetitively evaluate the disciplines and MDA without an explosion of the computational cost. Several surrogate models have been used for that purpose such as Taylor series expansion, Kriging, polynomial chaos expansion, neural networks, etc. Naive approaches consider the creation of the surrogate models offline and use them for the uncertainty propagation and/or the optimization without refinement and appropriate control of surrogate model uncertainty. Alternative more advanced approaches propose adaptive surrogate models for the disciplines to ensure the convergence of MDA and the satisfaction of the interdisciplinary coupling consistency (Dubreuil et al. 2016).

Leotardi et al. (2016) proposed a derivation of MDF using surrogate models (thin plate spline metamodel (Duchon 1977) which is a special case of radial basis function) with varying the accuracy of the surrogate models. The variability is present in the training set size and in the definition domain of the DoE for the metamodel training, in the accuracy (sample size) for the uncertainty propagation, and in the discipline coupling tolerance in the MDA (introducing a variable precision for the multidisciplinary consistency). The surrogate model is used to replace the objective function in the robust-based optimization (using Particle Swarm Optimization) and speeding-up the uncertainty propagation by using quasi-Monte Carlo (quasi-MC). At the first step of the proposed process, the training points for the surrogate model are distributed in the entire definition domain and the corresponding objective function values are obtained considering both a low level of accuracy in the uncertainty propagation (sample size of the quasi-MC) and a weak coupling between disciplines. After the first optimization step, a refined subdomain is defined centered on the current optimum and a new training set is used, with the corresponding objective function values obtained increasing both the accuracy of quasi-MC and the FPI tolerance (Gauss–Seidl convergence tolerance). The procedure is iterated until the convergence of the process.

System Uncertainty Analysis (SUA) and Concurrent SubSystem Uncertainty Analysis (CSSUA)

Principle To overcome the computational burden introduced by the repetitive MDAs in the MDF formulation under uncertainty, Du and Chen (2002) and Du (2002) proposed a formulation called System Uncertainty Analysis (SUA) derived from MDF in which the uncertainty propagation by CMC on MDA is replaced by an approximation of the first two statistical moments of the interdisciplinary couplings (Figure 7.12). The proposed approach is a robust MDO formulation:

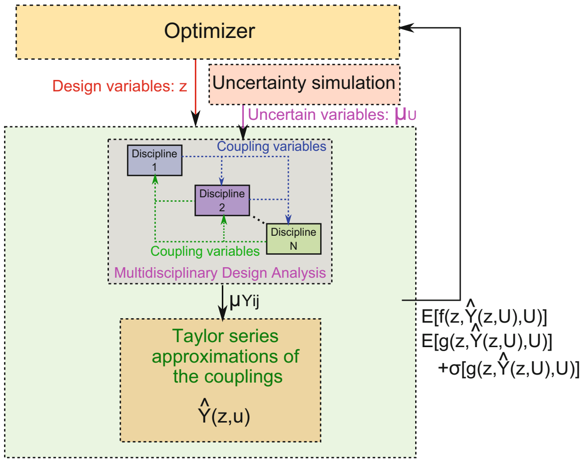


Fig. 7.12 System uncertainty analysis (SUA)

$$\begin{aligned}
\min \quad & \Xi\left[\mathbb{E}\left[\mathbf{f}(\mathbf{z}, \hat{\mathbf{Y}}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right], \sigma\left[\mathbf{f}(\mathbf{z}, \hat{\mathbf{Y}}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right]\right] && (7.29)\\
\text{w.r.t.} \quad & \mathbf{z} \\
\text{s.t.} \quad & \mathbb{E}\left[g_k(\mathbf{z}, \hat{\mathbf{Y}}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right] + \eta\sigma\left[g_k(\mathbf{z}, \hat{\mathbf{Y}}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right] \leq 0 \\
& \quad \text{for } k \in \{1, \ldots, m\} && (7.30)\\
& \mathbf{z}_{\min} \leq \mathbf{z} \leq \mathbf{z}_{\max} && (7.31)
\end{aligned}$$

with $\Xi\left[\mathbb{E}\left[\mathbf{f}(\mathbf{z}, \hat{\mathbf{Y}}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right], \sigma\left[\mathbf{f}(\mathbf{z}, \hat{\mathbf{Y}}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right]\right]$ a combination of the expected value and the standard deviation of the objective function. In practice, the surrogate model of the coupling relations $\hat{\mathbf{Y}}(\mathbf{z}, \mathbf{U})$ is obtained by a first-order Taylor series expansion and is only used to estimate the first two statistical moments of the coupling variables. The local approximation is made around $\boldsymbol{\mu}_{\mathbf{U}}$ and the current design vector value $\mathbf{z}$ and is given by

$$\begin{aligned}
\hat{\mathbf{Y}}_{ij}\left(\mathbf{z}, \mathbf{Y}_{.i}, \mathbf{u}_{sh}, \mathbf{u}_i\right) = \boldsymbol{\mu}_{\mathbf{Y}_{ij}} + \left.\frac{\partial \mathbf{c}_{ij}}{\partial \mathbf{u}_{sh}}\right|_{\mathbf{u}=\boldsymbol{\mu}_{\mathbf{u}}} (\mathbf{u}_{sh} - \boldsymbol{\mu}_{\mathbf{u}_{sh}}) + \left.\frac{\partial \mathbf{c}_{ij}}{\partial \mathbf{u}_i}\right|_{\mathbf{u}=\boldsymbol{\mu}_{\mathbf{u}}} (\mathbf{u}_i - \boldsymbol{\mu}_{\mathbf{u}_i}) \\
+ \left.\frac{\partial \mathbf{c}_{ij}}{\partial \mathbf{Y}_{.i}}\right|_{\mathbf{u}=\boldsymbol{\mu}_{\mathbf{u}}} (\mathbf{Y}_{.i} - \boldsymbol{\mu}_{\mathbf{Y}_{.i}}) \qquad (7.32)
\end{aligned}$$

See Chapter 6 for more details on SUA, notably the calculation of $\boldsymbol{\mu}_{\mathbf{Y}_{.i}}$. A first-order Taylor series expansion of the objective function $f(\cdot)$ and the constraints $g_k(\cdot)$ for $k \in \{1, \ldots, m\}$ is used to propagate the uncertainties and to estimate their first two statistical moments. It is possible therefore to estimate the expected value $\mathbb{E}[f(\cdot)]$ and $\mathbb{E}[g_k(\cdot)] + \eta_k\sigma[g_k(\cdot)]$. The first-order Taylor series expansion provides a functional relationship of the coupling dependence with respect to the uncertain variables. The method enables to find the optimal design while ensuring the interdisciplinary couplings for all the uncertain variable realizations. However, as detailed in Chapter 6, this method has several limits: the first-order Taylor approximation is only valid for functions that can be locally approximated as linear functions and the method requires to perform a MDA to locally build the surrogate model. This process is repeated for each value of the design variables provided by the system-level optimizer.

In order to further improve SUA, Du and Chen (2002) and Du (2002) proposed an amelioration named Concurrent SubSystem Uncertainty Analysis (CSSUA) to avoid the FPI to locally build the surrogate models. An optimization problem replaces the required FPI in SUA to find the expected value of the coupling variables $\boldsymbol{\mu}_{\mathbf{Y}_{ij}}$ formulated as:

$$\begin{aligned} \min \quad & \sum_{i=1}^{N} \left|\left| \boldsymbol{\mu}_{\mathbf{Y}_{i.}} - \mathbf{c}_{i.}\left(\mathbf{z}_i, \boldsymbol{\mu}_{\mathbf{Y}_{.i}}, \boldsymbol{\mu}_{\mathbf{U}}\right) \right|\right|_2^2 \\ \text{w.r.t.} \quad & \boldsymbol{\mu}_{\mathbf{Y}} \end{aligned} \tag{7.33}$$

As in SUA, a linear approximation of the disciplines is made by assuming that

$$\forall (i, j) \in \{1, \ldots, N\}^2 \; i \neq j, \;\; \boldsymbol{\mu}_{\mathbf{Y}_{ij}} = \mathbf{y}_{ij}\left(\mathbf{z}_i, \boldsymbol{\mu}_{\mathbf{U}}\right). \tag{7.34}$$

This optimization problem allows one to call the disciplines in parallel, potentially reducing the computational cost compared to FPI. Once the expected value of the coupling variables is found, the same uncertainty propagation as in SUA is performed. CSSUA suffers from the same drawbacks as SUA. In aerospace vehicle design, some disciplines such as the aerodynamics and the trajectory involve highly non-linear dynamics and the linearity hypothesis would introduce high errors compared to the classical MDF under uncertainty approach.

Application to Toy Case As presented in Chapter 6, SUA is valid for MDO problems which involve discipline models that can be approximated by first-order linear models. The toy case presented in Section 7.3.1 is slightly modified in order for SUA to be applicable. The distribution for U_1 is set to $U_1 \sim \mathcal{N}(0.5, 0.1)$, otherwise as illustrated in Chapter 6, SUA is not able to properly approximate the coupling variables. This UMDO problem is solved with SUA where the uncertainty is propagated thanks to first-order Taylor series expansion. The following formulation is adopted:

$$\min \quad \mathbb{E}\left[f(\mathbf{z}, \hat{\mathbf{Y}}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right] \tag{7.35}$$

$$\text{w.r.t.} \quad \mathbf{z} = [z_{sh}, z_1, z_2]$$

$$\text{s.t.} \quad \mathbb{E}\left[g(\mathbf{z}, \hat{\mathbf{Y}}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right] + 3 \times \sigma\left[g(\mathbf{z}, \hat{\mathbf{Y}}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right] \leq 0 \tag{7.36}$$

$$\mathbf{0} \leq \mathbf{z} \leq \mathbf{5} \tag{7.37}$$

At each iteration of the UMDO problem, taking into account the values of the design variables, a local model $\hat{\mathbf{Y}}(\mathbf{z}, \mathbf{U}), \mathbf{U})$ of the coupling variables is built. The local approximation is performed around the current value of the decision variable vector $\mathbf{z}$ and $\boldsymbol{\mu}_{\mathbf{u}}$.

$$\hat{\mathbf{Y}}(\mathbf{z}, \mathbf{u}) = \begin{bmatrix} \hat{y}_{12}(\mathbf{z}, \mathbf{u}) \\ \hat{y}_{21}(\mathbf{z}, \mathbf{u}) \end{bmatrix} = \mathbf{A}^{-1} \times \mathbf{B} \times \left(\mathbf{u}_{sh} - \boldsymbol{\mu}_{\mathbf{u}_{sh}}\right) + \mathbf{A}^{-1} \times \mathbf{C} \times \left(\mathbf{u}_i - \boldsymbol{\mu}_{\mathbf{u}_i}\right) \tag{7.38}$$

where: $\mathbf{A} = \begin{bmatrix} 1 & -\left.\frac{\partial c_{12}}{\partial y_{21}}\right|_{\mathbf{u}=\boldsymbol{\mu}_{\mathbf{u}}} \\ -\left.\frac{\partial c_{21}}{\partial y_{12}}\right|_{\mathbf{u}=\boldsymbol{\mu}_{\mathbf{u}}} & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -\left.\frac{\partial c_{12}}{\partial z_{sh}}\right|_{\mathbf{u}=\boldsymbol{\mu}_{\mathbf{u}}} \\ -\left.\frac{\partial c_{21}}{\partial z_{sh}}\right|_{\mathbf{u}=\boldsymbol{\mu}_{\mathbf{u}}} \end{bmatrix}$ and

$$\mathbf{C} = \begin{bmatrix} -\left.\frac{\partial c_{12}}{\partial z_1}\right|_{\mathbf{u}=\boldsymbol{\mu}_{\mathbf{u}}} & 0 \\ 0 & -\left.\frac{\partial c_{21}}{\partial z_2}\right|_{\mathbf{u}=\boldsymbol{\mu}_{\mathbf{u}}} \end{bmatrix}$$

This model is used to propagate the uncertainty and to estimate the objective function and the constraint. In order to avoid additional simplifications, the objective and the constraint functions are not linearized and the CMC propagation is performed using only the approximated models of the coupling relationships. CMA-ES is used to optimize this problem. Figure 7.13 presents the convergence plots for SUA in terms of normalized design variables and objective function. The convergence is reached after 40 iterations. Figure 7.14 illustrates the parallel plot for the CMA-ES run. The objective function converges to a value of 7.95 that corresponds to $\mathbf{z}^* = [0.60, 0.20, 0.00]$. Compared to a reference robustness-based MDF (using FPI and CMC), SUA converges to the exact same values; therefore, the linear approximation is well suited in such a toy case. Moreover, it enables to reduce the number of discipline evaluations by a factor 10^3. Indeed, at each iteration, only 10 exact discipline evaluations (in the FPI) are necessary to estimate the objective function and the constraints, whereas 10^4 are required in the classical robustness-based MDF formulation. Figure 7.15 compares the distributions of the coupling

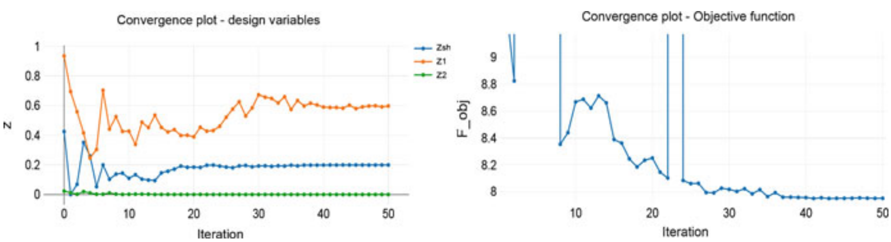


Fig. 7.13 Convergence plot for robust-based SUA, design variables (left), objective function (right)

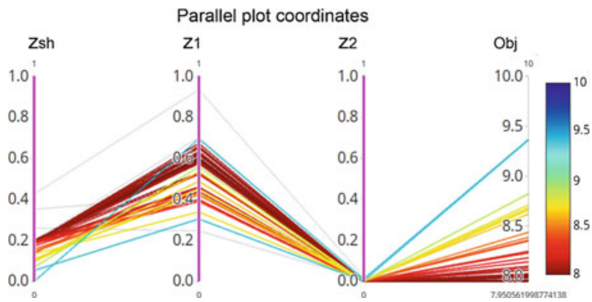


Fig. 7.14 Parallel plot for robust-based SUA using CMA-ES, each line represents the best CMA-ES individual at each algorithm iteration

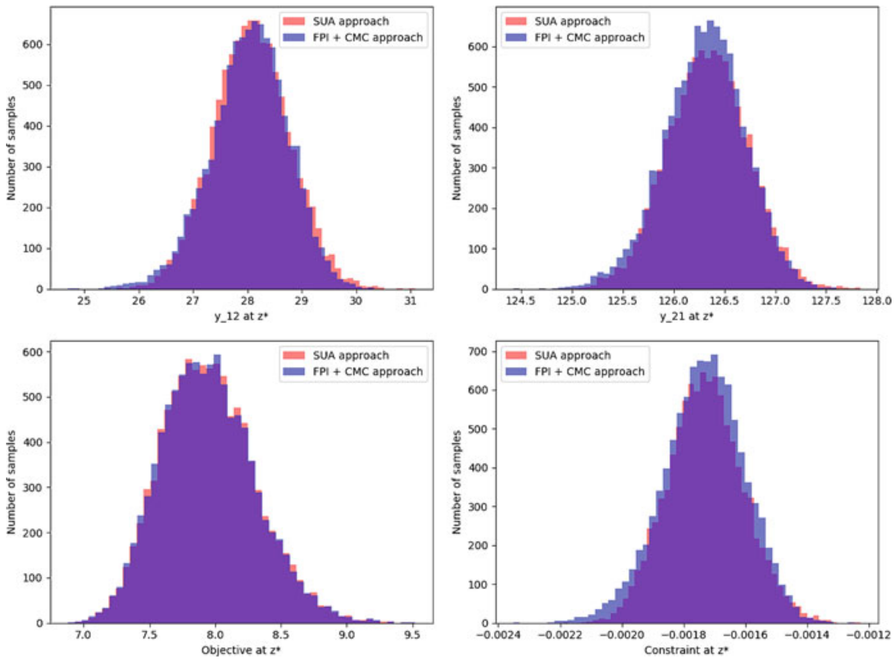


Fig. 7.15 Comparison of SUA results with robust-based MDF for the interdisciplinary couplings, the objective and the constraint functions

variables, the objective function, and the constraint at the respective optimum designs for SUA and the robustness-based MDF formulation. The distributions are very similar, the discrepancies might be explained by the linear approximation in SUA, but in this problem the two coupling variables are appropriately approximated by linear functions.

The classical double-loop approach (one loop to propagate uncertainty and one loop for MDA) may become computationally inefficient for disciplinary models of medium to high-fidelity due to the computational cost. The computational cost associated with MDF under uncertainty using CMC is $N_{opt} \times M_{CMC} \times N_{MDA}$, where N_{opt} is the number of calls to the MDA by the system-level optimizer, M_{CMC} is the number of samples used to propagate the uncertainty, and N_{MDA} is the number of calls to the disciplines required by the MDA to converge. To further improve the computational cost, several works (Rhodes and Sues 1994; Koch et al. 2000) proposed to use parallel computing tools to perform UMDO and reduce computational time. For instance, the DoE design for the MDA approximation modeling or the CMC uncertainty propagation can be executed in parallel.

An alternative strategy to full uncertainty propagation in the case of robust-based UMDO has been proposed by Baudoui et al. (2012). This strategy is suited for problems in which the disciplines are black boxes that cannot be modified and adding new variables into an existing framework is too difficult. It allows not to modify in depth the existing framework and to provide the designer with a preliminary step towards robust-based UMDO, before integrating full uncertainty propagation into a dedicated framework. The main idea consists in using a criterion to estimate the importance of the disciplines affected by uncertainties over the objective function variations. If this applicability criterion is verified for the problem at hand, it enables to propagate uncertainty locally (through disciplines but not the whole system) using the LOcal Uncertainty Processing (LOUP) methodology. This approach is only valid if the outputs of the disciplines have a significant effect on the objective function. In this case, local uncertainty computations may be carried out instead of a complete uncertainty propagation over the MDA, making the uncertainty handling easier and less computationally intensive.

Alternatives to coupled UMDO formulations have been proposed to transform the organization of the UMDO process, including MDA, disciplinary analyses, and uncertainty propagation. Two families may be distinguished: single-level procedures and distributed procedures. Single-level approaches decouple the uncertainty propagation from the optimization, for instance, using a sequential procedure. The main interest in this category of approaches is that it enables to directly use existing deterministic MDO formulations. Distributed procedures apply a process for UMDO problems similar to decoupled formulations in deterministic MDO by removing MDA. Hence, the existing decomposed formulations for deterministic MDO, such as IDF, CSSO, CO, ATC, etc., can be employed to split the integrated optimization and uncertainty analysis problem into several disciplinary or subsystem-level uncertainty optimization problems, leading to manageable subproblems.

7.4 Single-Level Procedures

7.4.1 Unilevel Method for UMDO

In order to remove the nested loop imposed by the UMDO process, Agarwal et al. (2004a) proposed to transform the reliability analysis carried out by FORM, which is an optimization problem, into its corresponding first-order necessary Karush–Kuhn–Tucker (KKT) optimality conditions, and to impose these KKT conditions on the upper optimization loop. This formulation enables to remove the inner loop at each iteration of the outer loop while ensuring the reliability requirements. In this approach, the optimal design variables $\mathbf{z}^*$ and the corresponding MPP $\mathbf{x}^*$ (corresponding to the transformation of $\mathbf{u}^*$ into the standard normal space) are the decision variables of the single-level procedure.

$$\begin{aligned}
\text{min} \quad & \Xi\left[f(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right] && (7.39)\\
\text{w.r.t.} \quad & \mathbf{z}, \mathbf{x}_1, \ldots, \mathbf{x}_m && \\
\text{s.t.} \quad & G_k(\mathbf{x}_k) \leq 0, \quad k = 1, \ldots, m && (7.40)\\
& \| \mathbf{x}_k \| \| \nabla_{\mathbf{x}} G_k(\mathbf{x}_k) \| + \mathbf{x}_k^{\mathrm{T}} \nabla_{\mathbf{x}} G_k(\mathbf{x}_k) = 0, \quad k = 1, \ldots, m && (7.41)\\
& \| \mathbf{x}_k \| - \beta_k = 0, \quad k = 1, \ldots, m && (7.42)\\
& \mathbf{z}_{\min} \leq \mathbf{z} \leq \mathbf{z}_{\max}, && (7.43)
\end{aligned}$$

where β is the reliability index and $G_k(\cdot)$ the kth limit state in the standard normal space (m is the number of reliability constraints). For the multidisciplinary coupled problem, GSE (*Global Sensitivity Equation*) is used to implicitly estimate the gradient of the limit-state functions. This approach is mathematically equivalent to the initial nested loop UMDO problem given that the KKT optimality conditions for the reliability analysis are satisfied. It should be noted that only the disciplinary reliability constraints and no system-level reliability constraint are considered (therefore in the proposed approach $G_k(\cdot)$ does not depend on $\mathbf{z}$ and $\mathbf{Y}$, limiting its applicability to UMDO problems). Furthermore, enforcing equality constraints on the upper optimization loop may lead to poor numerical convergence behavior. Besides, the KKT conditions are derived from FORM and the accuracy can be questioned for highly non-linear uncertainty problems.

Chen et al. (1997) proposed another methodology to approximately identify the MPP location of each active reliability constraint with using the gradients of the limit-state function and the desired safety factor. Therefore, the FORM analysis performed at the lower loop can be removed and the approximation of the MPP can be directly embedded in the outer optimization loop with equivalent deterministic constraints.

These approaches enable to remove the nested loop of UMDO reducing the computational cost but are limited to problems with no system-level reliability constraints.

7.4.2 Sequential Optimization and Reliability Assessment (SORA)

Principle

The other family of single-level procedure consists in decoupling the inner uncertainty propagation phase and the outer optimization phase with sequential cycles of uncertainty analysis and deterministic MDO. Indeed, the reliability analysis is computationally expensive especially on multidisciplinary systems. To avoid to perform this reliability analysis at each system-level iteration to compute the constraints, a sequential approach has been proposed. At each iteration of the sequential cycle, the reliability constraints are transformed into equivalent deterministic constraints and then integrated in the next deterministic MDO problem solving to guide the optimum search towards the feasible region. The key challenge in these approaches is how to transform the reliability constraints into the equivalent deterministic ones.

Sues et al. proposed a method (Sues and Cesare 2000) to search for the MPP at the initial design during the reliability analysis phase and then to approximate each limit-state function at its MPP with first-order linearized models. These models are used as equivalent deterministic constraints in the following deterministic MDO phase. Once the optimum is reached in the deterministic MDO problem, a reliability analysis is carried out to determine the new MPP for each reliability constraint. The sequential cycle is stopped when a convergence criterion is reached on the design variable values and the MPP. This approach is easy to implement but is limited due to the linear approximation of the limit-state functions.

Instead of decoupling the disciplines to avoid computationally intensive MDA under uncertainty, an alternative is to decouple the optimization of the design variables and the uncertainty propagation to estimate the probability of failure in the reliability-based approach. Du et al. (2008) proposed the sequential optimization and reliability assessment (SORA) for UMDO problems. The main idea is to separate the optimization and the reliability analysis. The UMDO problem is divided into a sequence of deterministic MDO problems and reliability analyses. SORA replaces the probabilistic reliability constraints by deterministic approximation of the reliability constraints evaluated at the Most Probable Point (MPP). Reliability analysis is performed by FORM (Rackwitz 2001) to find the MPP (noted $\mathbf{u}^*$). It is assumed here that the uncertain variables are given in the standard normal space. If it is not the case, different statistical transformations may be applied on the input distributions (such as Nataf (1962) or Rosenblatt (1952) transformations).

In SORA (Du et al. 2008), four steps are distinguished (Figure 7.16):

- *Step 1*: at the kth SORA iteration, the deterministic MDO problem is solved with the uncertain variables fixed at their MPP $\mathbf{u}^{*k-1}$ found at the $[k-1]$th iteration for the constraints and fixed at their mean values $\bar{\mathbf{u}}$ for the objective function. The coupling variable values are in concordance with the uncertain variable values: coupling variables at the MPP for the constraints and mean values of the coupling variables for the objective function.

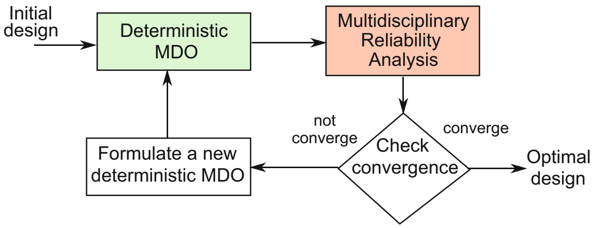


Fig. 7.16 SORA procedure for UMDO (Du et al. 2008)

- *Step 2*: a reliability analysis is performed to identify the MPP $\mathbf{u}^{*k}$ of all the inequality constraints by using an inverse FORM with the design variables fixed at the optimal design $\mathbf{z}^{*k}$ found in step 1 and given a reliability index β.
- *Step 3*: the convergence is checked. If the inequality constraints ($\mathbb{P}\left[g_i(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U}) > 0\right] - \mathbb{P}_{t_i} \leq 0$) transformed into the MPP problem are verified and the objective function becomes stable (Du et al. 2008), the solution is found.
- *Step 4*: if the convergence is not reached, or the inequality constraints are not satisfied, a new deterministic MDO problem is formulated for $\mathbf{u} = \mathbf{u}^{*k}$, back to Step 1.

Deterministic MDO: Step 1

The deterministic MDO problem of step 1 may be solved with the classical decoupled MDO methods (IDF, AAO, BLISS, ATC, etc). With the IDF formulation (Du et al. 2008), the deterministic MDO problem at the SORA kth-cycle, ($k \geq 2$), is formulated as follows:

$$\text{given} \quad \mathbf{u}^{*k-1}, \bar{\mathbf{u}} \tag{7.44}$$

$$\min \quad f(\mathbf{z}, \bar{\mathbf{y}}, \bar{\mathbf{u}}) \tag{7.45}$$

$$\text{w.r.t.} \quad \mathbf{z}, \mathbf{y}^*, \bar{\mathbf{y}}$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{z}, \mathbf{y}^*, \mathbf{u}^{*k-1}\right) \leq 0 \tag{7.46}$$

$$\forall (i, j) \in \{1, \ldots, N\}^2 \; i \neq j, \;\; \bar{\mathbf{y}}_{ij} = \mathbf{c}_{ij}\left(\mathbf{z}_i, \bar{\mathbf{y}}_{.i}, \bar{\mathbf{u}}_i\right) \tag{7.47}$$

$$\forall (i, j) \in \{1, \ldots, N\}^2 \; i \neq j, \;\; \mathbf{y}^*_{ij} = \mathbf{c}_{ij}\left(\mathbf{z}_i, \mathbf{y}^*_{.i}, \mathbf{u}_i^{*k-1}\right) \tag{7.48}$$

$$\mathbf{z}_{\min} \leq \mathbf{z} \leq \mathbf{z}_{\max} \tag{7.49}$$

The interdisciplinary couplings (Equations 7.47–7.48) are ensured for two particular realizations of the uncertain variables corresponding to the MPP $\mathbf{u}^{*k-1}$ and the mean value $\bar{\mathbf{u}}$.

Reliability Analysis: Step 2

The reliability analysis is performed for the design variables fixed at $\mathbf{z}^{*k}$ based on an inverse FORM (Chiralaksanakul and Mahadevan 2007; Du et al. 2008). The percentile value of the performance function is calculated based on a reliability index target β:

$$\text{given} \quad \mathbf{z}^{*k} \tag{7.50}$$

$$\max \quad \mathbf{g}\left(\mathbf{z}^{*k}, \mathbf{y}, \mathbf{u}\right) \tag{7.51}$$

$$\text{w.r.t.} \quad \mathbf{u}, \mathbf{y}$$

$$\text{s.t.} \quad \| \mathbf{u} \| = \beta \tag{7.52}$$

$$\forall (i, j) \in \{1, \ldots, N\}^2 \; i \neq j, \; \mathbf{y}_{ij} = \mathbf{c}_{ij}\left(\mathbf{z}^{*k}, \mathbf{y}_{.i}, \mathbf{u}_i\right) \tag{7.53}$$

This optimization provides the MPP value $\mathbf{u}^{*k}$ for the uncertain variables at the SORA kth-cycle. The reliability analysis is performed on a decoupled multidisciplinary system and the interdisciplinary couplings are satisfied at the MPP in (Equation 7.53). By decoupling the reliability analysis from the deterministic MDO, SORA tends to decrease the number of calls to the disciplinary functions compared to MDF under uncertainty (Du et al. 2008). SORA has been implemented with various MDO formulations such as MDF (Chiralaksanakul and Mahadevan 2007), IDF (Chiralaksanakul and Mahadevan 2007), CO (Li et al. 2010; Zhang and Zhang 2013a), CSSO (Li et al. 2014; Zhang and Zhang 2013b), or BLISS (Ahn and Kwon 2006) but the coupling satisfaction relies on the same approach: satisfaction at the MPP of the coupling variables and at their mean value.

The interdisciplinary coupling satisfaction within SORA presents several advantages:

- Possibility to perform the disciplinary analyses in parallel,
- Satisfaction of the interdisciplinary couplings at the MPP value and mean value for the coupling variables at the optimum,
- Reduction of the computational cost compared to MDF under uncertainty with CMC.

However, SORA has also several limitations. The reliability analysis is performed by FORM which locally linearizes the inequality constraints and may lead to inaccurate estimation of the probability of failure. FORM also assumes the uniqueness of the MPP which might be a limiting hypothesis in practical applications (Dubourg et al. 2013). Furthermore, in terms of interdisciplinary coupling satisfaction, the couplings are ensured only at the mean value and the MPP which is the most likely failure point to happen but another failure less likely may occur. Moreover, during the deterministic MDO problem solving, the objective function and the constraints are not evaluated for the same realizations of the uncertain variables. The objective function is evaluated at the mean values of the uncertain variables while the constraints are evaluated at the MPP.

Several derived SORA approaches have been proposed to improve the reliability analysis. Meng et al. (2015) developed an approach based on subset simulation to accurately estimate the probability of failure in SORA. From subset simulation, the estimation of the MPPs is obtained by listing all the MCMC samples in ascending order according to their performance values $\text{MPP} = \text{argmin}\left[g\left(\mathbf{z}^*, \mathbf{y}, \mathbf{u}_{(1)}\right), \ldots, g\left(\mathbf{z}^*, \mathbf{y}, \mathbf{u}_{(M)}\right)\right]$. From these values, a shifting vector $\mathbf{S}^* = \mu_{\mathbf{u}} - \mathbf{u}^*$ is constructed for the next deterministic MDO problem with a deterministic constraint: $g(\mathbf{z}, \mathbf{y}, \mu_{\mathbf{u}} - \mathbf{S}^*)$. The design optimization problem considered in the paper involves low strength couplings and the CO formulation is implemented for SORA step deterministic MDO.

Application to Toy Case

The test case presented in Section 7.3.1 is solved using SORA. At the iteration k of SORA, the deterministic MDO phase is solved using a MDF formulation as follows:

$$
\begin{aligned}
\text{given: } & \mathbf{u}^{*k-1}, \bar{\mathbf{u}} \\
\min\ & f\left(\mathbf{z}, \mathbf{Y}\left(\mathbf{z}, \bar{\mathbf{u}}\right), \bar{\mathbf{u}}\right) && (7.54)\\
\text{w.r.t. } & \mathbf{z} = [z_{sh}, z_1, z_2] \\
\text{s.t. } & g\left(\mathbf{z}, \mathbf{Y}\left(\mathbf{z}, \mathbf{u}^{*k-1}\right), \mathbf{u}^{*k-1}\right) \leq 0 && (7.55)\\
& \mathbf{0} \leq \mathbf{z} \leq \mathbf{5} && (7.56)
\end{aligned}
$$

CMA-ES is used to solve the deterministic MDF formulation.

The reliability analysis phase (with an inverse reliability technique and FORM) is carried out using FPI to estimate the couplings $\mathbf{Y}\left(\mathbf{z}^{*k}, \mathbf{u}\right)$ as follows:

$$
\begin{aligned}
\text{given } & \mathbf{z}^{*k} && (7.57)\\
\max\ & \mathbf{g}\left(\mathbf{z}^{*k}, \mathbf{Y}\left(\mathbf{z}^{*k}, \mathbf{u}\right), \mathbf{u}\right) && (7.58)\\
\text{w.r.t. } & \mathbf{u} = [u_{sh}, u_1, u_2] \\
\text{s.t. } & \parallel \mathbf{u} \parallel = \beta && (7.59)\\
& && (7.60)
\end{aligned}
$$

with $\beta = 3.09$ leading to the probability of failure $\mathbb{P}\left[g(\mathbf{z}, \mathbf{Y}(\mathbf{z}, \mathbf{U}), \mathbf{U}) > 0\right] = 10^{-3}$. A SQP optimization algorithm is used to solve the inverse reliability analysis.

In Figure 7.17(left side), the convergence of the design variables at each iteration of SORA is presented. The design variable vector $\mathbf{z}$ converges to $\mathbf{z}^*_{\text{SORA}} =$

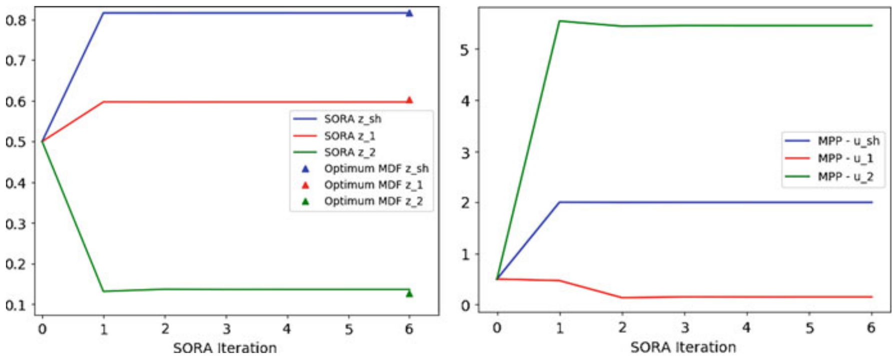
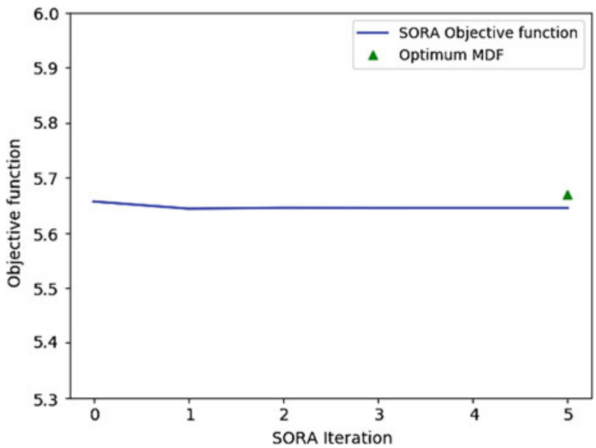


Fig. 7.17 Convergence plot for SORA: design variables and MPP values

Fig. 7.18 Convergence plot for SORA, objective function



[0.82, 0.560, 0.13] which is similar to the value obtained in Section 7.3.1 with the coupled MDF formulation with FPI and CMC $\mathbf{z}^*_{\text{MDF}} = [0.81, 0.60, 0.13]$.

Figure 7.17(right side) depicts the MPP values obtained at each iteration of SORA that converge to the optimal MPP $\mathbf{u}^* = [1.99, 0.14, 5.45]$. Figure 7.19 displays the iso-surface of the limit state $\mathbf{g}\left(\mathbf{z}^{*k}, \mathbf{y}, \mathbf{u}^*\right) = 4.73$ which is the maximum obtained with the reliability analysis. The MPP for SORA with a reliability index fixed at $\beta = 3.09$ is also represented. The colormap corresponds to the distance to the origin. The convergence plot in terms of objective function is represented in Figure 7.18. The difference between SORA and MDF under uncertainty is due to the approximation of the objective function at the mean value of the uncertain variables.

Compared to MDF under uncertainty (see Section 7.3.1), SORA requires less evaluations of the exact disciplines. The deterministic MDF solving of SORA carried out by CMA-ES converges in the order of 2.8×10^3 calls to the disciplines and the reliability analysis phase using SQP requires only 1.2×10^3 discipline

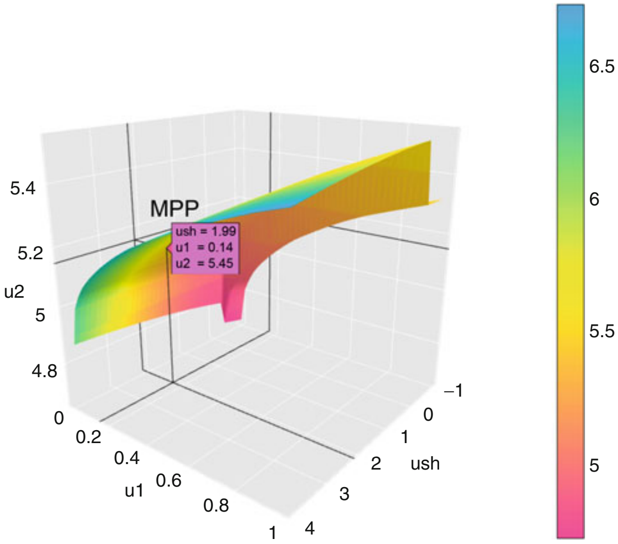


Fig. 7.19 Iso-surface of the limit-state function $\mathbf{g}\left(\mathbf{z}^{*k}, \mathbf{y}, \mathbf{u}\right) = 4.73$ and SORA MPP for reliability index $\beta = 3.09$. Colormap corresponds to the distance to the origin

evaluations. In the end, SORA requires approximatively 10^4 calls to the disciplines where MDF under uncertainty requires about 10^7 discipline evaluations. The reduction in terms of number of discipline evaluations is significant. SORA might provide less accurate results for more complex problems with important non-linearities and the presence of multiple MPPs. However, it can indicate a region of interest to reduce the design space for a more suited methodology to the considered problem.

7.5 Distributed UMDO Approaches

One of the main concluding remarks of the previous sections concerns the challenges of the UMDO process organization and the handling of the interdisciplinary couplings in the presence of uncertainty. The main difficulty results in ensuring the system consistency whatever the unexpected event realization. The distributed strategies under uncertainty could benefit from the same advantages as in the deterministic case; however, it must not be to the detriment of the multidisciplinary feasibility when impacted by uncertainty. In order to maintain the mathematical equivalence between the classical coupled formulations and the decoupled formulations, the interdisciplinary couplings should be satisfied for all the realizations of the uncertain variables.

To avoid the repeated calls to the MDA used in MDF under uncertainty, the decoupled approaches aim at propagating the uncertainty on decoupled disciplines allowing one to evaluate them in parallel and to ensure the coupling satisfaction by introducing equality constraints in the UMDO problem formulation. However, two main challenges are faced to decouple the design process:

- The uncertain input coupling variable vector **Y** has to be controlled by the system-level optimizer. The uncertain coupling variables are functions and infinite-dimensional problems are complex to solve. Dedicated methods have to be used.
- Equality constraints between the input coupling variables **Y** and the output coupling variables computed by $c(\cdot)$, which are two uncertain variables, have to be imposed. Equality between two uncertain variables corresponds to an equality between two functions which is not obvious to formulate and implement.

In order to understand these two challenges, a quick focus on decoupled deterministic MDO formulations is necessary.

Let us consider two disciplines i and j, one scalar feedforward coupling y_{ij}, and one scalar feedback coupling y_{ji} as illustrated in Figure 7.20. In a deterministic decoupled MDO approach, to remove the feedforward coupling, there is only one equality constraint that has to be imposed at the system level in the optimization formulation between the input coupling variable y_{ij} and the output coupling variable $c_{ij}(\mathbf{z}_i, y_{ji})$:

$$y_{ij} = c_{ij}(\mathbf{z}_i, y_{ji}) \tag{7.61}$$

However, in the presence of uncertainty, the coupling satisfaction involves an equality constraint between two uncertain variables. An uncertain variable is a function (see Chapter 2 on probability theory for more details). Two uncertain

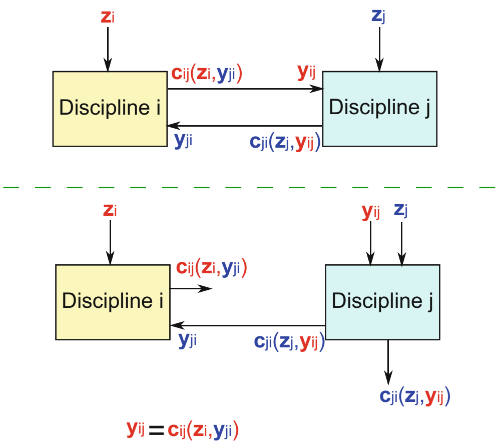


Fig. 7.20 Illustration of couplings between two disciplines

variables are equal if and only if the two corresponding functions have the same initial and final sets and the same mappings. To ensure the coupling satisfaction *in realizations*, an infinite number of equality constraints (Equation 7.62) have to be imposed, one for each realization of the uncertain variables used to compute the objective and constraint functions:

$$\forall \mathbf{u} \in \mathbb{R}^d, \quad y_{ij} = c_{ij}(\mathbf{z}_i, y_{ji}, \mathbf{u}_i), \tag{7.62}$$

where $c_{ij}(\cdot)$ is a function of the uncertain variable realizations $\mathbf{u}$. However, it is important to notice that even if the coupling variables are random variables, for one realization $\mathbf{u}_{(0)}$ there is in general only one converged coupling value that satisfies $y_{ij_{(0)}} = c_{ij}\left(\mathbf{z}_i, y_{ji_{(0)}}, \mathbf{u}_{(0)}\right)$ ensuring the multidisciplinary feasibility. Indeed, the disciplines are considered here as deterministic functions, all the uncertainties arise in the discipline inputs.

To tackle this issue, a first category of approaches proposes to reduce the statistical description of the interdisciplinary coupling variables to their statistical moments and to transform the UMDO problem into a classical deterministic MDO. These methods are described in the next section.

7.5.1 Statistical Moment Matching Formulations

A Hierarchical Approach to Collaborative Multidisciplinary Robust Design

Principle In order to replace the MDA, decoupled approaches inspired from CO (Braun et al. 1997) have been proposed (Du and Chen 2001; McAllister and Simpson 2003; Gu et al. 2006; Liu et al. 2006; Ghosh et al. 2014; Xiong et al. 2014). The idea is to extend the CO framework to robust design. In these methods, the uncertain input coupling vector is replaced by its statistical moments. Therefore, the system-level optimizer only controls deterministic parameters. For instance, Du and Chen (2001) proposed the hierarchical approach to collaborative multidisciplinary robust design method in which the input coupling variables are represented by their expected values $\boldsymbol{\mu}_Y$ and standard deviations $\boldsymbol{\sigma}_Y$. In this formulation, the system-level optimizer controls the design variable vector $\mathbf{z}$, the input coupling variable expected values $\boldsymbol{\mu}_Y$, and standard deviations $\boldsymbol{\sigma}_Y$.

As in SUA and CSSUA, the disciplines, the objective function and the constraints are approximated by a first-order Taylor series expansion to estimate the first- and second-order statistical moments. The aim of the subsystem level is to determine its local design variables $\mathbf{z}_i^*$ in order to find an agreement with the other subsystems with respect to the coupling variable values. The subsystem-level objective is a measure of the relative errors between the discipline outputs and the system-level targets.

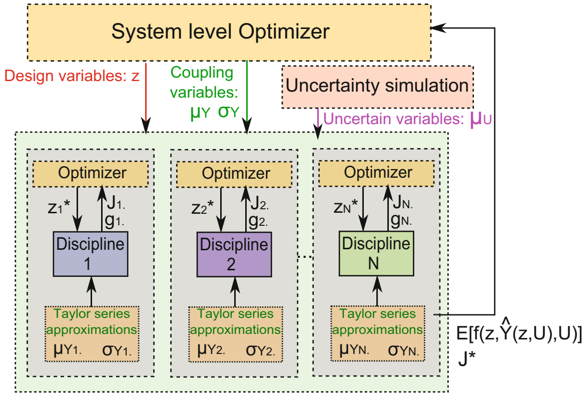


Fig. 7.21 Hierarchical approach to collaborative multidisciplinary robust design

The formulation proposed by Du and Chen (2001) (Figure 7.21) is

$$\min \quad \mathbb{E}\left[f(\mathbf{z}, \hat{\mathbf{Y}}(\mathbf{z}, \mathbf{U}), \mathbf{U})\right] \tag{7.63}$$

$$\text{w.r.t.} \quad \mathbf{z}_{sh}, \boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\sigma}_{\mathbf{Y}}$$

$$\text{s.t.} \quad \mathbf{J}^*_{i.}(\mathbf{z}_{sh}, \mathbf{z}^*_i, \boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\sigma}_{\mathbf{Y}}) = 0, \forall i \in \{1, \dots, N\} \tag{7.64}$$

$$\mathbf{z}_{\min} \le \mathbf{z} \le \mathbf{z}_{\max} \tag{7.65}$$

with $\mathbf{J}^*_{i.}$ the optimized objective function of the ith discipline and $\mathbf{z}^*_i$ the decision variables found by the subsystem optimizer. The ith subsystem optimization problem is given by

$$\min \quad \mathbf{J}_{i.} = \left|\left|\mathbf{z}^*_{sh} - \mathbf{z}_{sh}\right|\right|_2^2 + \left|\left|\boldsymbol{\mu}_{\mathbf{Y}_{i.}} - \mathbf{c}_{i.}\left(\mathbf{z}^*_{sh}, \mathbf{z}^*_i, \boldsymbol{\mu}_{\mathbf{Y}_{.i}}, \boldsymbol{\mu}_{\mathbf{U}}\right)\right|\right|_2^2$$
$$+ \left|\left|\boldsymbol{\sigma}^2_{\mathbf{Y}_{i.}} - \sum_{l=1}^{d}\left(\left.\frac{\partial c_{i.}}{\partial u^{(l)}}\right|_{\mathbf{u}=\boldsymbol{\mu}_{\mathbf{U}}}\right)^2 \sigma^2_{U^{(l)}}\right|\right|_2^2 \tag{7.66}$$

$$\text{w.r.t.} \quad \mathbf{z}^*_{sh}, \quad \mathbf{z}^*_i$$

$$\text{s.t.} \quad \mathbb{E}\left[g_k(\mathbf{z}, \hat{\mathbf{Y}}(\mathbf{z}^*, \mathbf{U}), \mathbf{U})\right] + \eta_k \sigma\left[g_k(\mathbf{z}, \hat{\mathbf{Y}}(\mathbf{z}^*, \mathbf{U}), \mathbf{U})\right] \le 0$$
$$\text{for } k \in \{1, \dots, m\} \tag{7.67}$$

$$\mathbf{z}_{\min} \le \mathbf{z}^* \le \mathbf{z}_{\max}, \tag{7.68}$$

where $\mathbf{z}_{sh}$ is the target value given by the system-level optimizer. The shared design variables between the disciplines $\mathbf{z}_{sh}$ are controlled at the system level and the decision variables specific to each discipline and local copies of the shared design variables are controlled at the lower-level. This formulation relies on CSSUA to estimate the statistical moments of the coupling variables and does not involve any MDA offering the possibility for parallel discipline optimizations. The interdisciplinary coupling constraints at the system level ensure that the input coupling variables and the output coupling variables have the same expected values and the same standard deviations. The first two statistical moments of the coupling variables are matched between the different disciplines to ensure the multidisciplinary feasibility.

It should be noted that the proposed formulation by Du et al. and most of the alternative moment matching approaches described in the next section are originally not suited to handle problems involving system-level constraints $\mathbf{g}(\cdot)$ depending on $\mathbf{z}_{sh}$, $\mathbf{z}_i$, and $\mathbf{y}_{ij}$. Indeed, this type of constraints is not specific to a single discipline (e.g. local); therefore, it has to be evaluated at the system level. However, it depends on design variables $\mathbf{z}_i$ that are determined by the lower-level optimization problems and therefore their values do not take into account a possible violation of the global constraints. In order to handle such constraints, a target variable $\mathbf{z}_i$ has to be controlled at the system level and should match the copy handled at the subsystem level (Xiong et al. 2014).

Alternative Moment Matching Approaches Alternative formulations have to be adopted for such a general type of problems. Xiong et al. (2014) proposed a modified robust-CO formulation to avoid the first-order Taylor series approximation. CMC replaces the Taylor series expansion to propagate the uncertainty and to estimate the coupling variable statistical moments. However, the proposed approach is only compatible with parametrical PDF for the interdisciplinary coupling variables (often considered distributed according to a Gaussian distribution). The statistical moments of the distributions are controlled at the system level by the optimizer. In order to offer more flexibility to the subsystem optimization problems, a copy of the statistical moments of the coupling variable distributions is controlled at the subsystem level and a matching constraint is added at the system level.

To further improve the method, Ghosh et al. (2014) proposed to capture the statistical dependence of the coupling variables by introducing the covariance matrix to model the correlations between them. The coefficients of the covariance matrix in addition to the expected values are controlled by the system-level optimizer. It increases the number of decision variables controlled by the system-level optimizer but it enhances the fidelity of the moment matching. In this approach, the ranges of uncertainty are assumed to be small and the coupling variables follow a multivariate Gaussian distribution which is not necessary the case for non-linear disciplines. Moreover, this approach has been extended to reliability-based UMDO (Huang et al. 2010) to enable a reliability analysis on the constraints instead of statistical moment estimations.

Furthermore, moment matching approaches have been adapted in other UMDO formulations such as probabilistic ATC (Kokkolaras et al. 2004; Liu et al. 2006; Xiong et al. 2010) which presents similarities with CO. In these formulations, the advanced mean value method is used to generate CDF of each subsystem response and the mean and standard deviation of each subsystem are passed upwards to its parents. Therefore, between the levels, the statistical moments of the coupling variables are matched between the target provided by the upper level and the resulting uncertainty propagation at the lower one. The subsystem-level optimization problems are solved upwards level by level up to the top of the hierarchical decomposition. Once the higher-level problem is solved, the new targets start to be cascaded level by level to the lower ones. With the updated parameters, the subsystem optimization is solved again from the bottom to the top. This iterative process is stopped when the convergence criterion is reached. Probabilistic ATC generalizes the proposed approach in CO under uncertainty to a higher number of levels. Xiong et al. (2010) extended this probabilistic ATC approach to match the covariance matrix of the coupling variables between the different levels.

The moment matching formulations are interesting since they preserve some disciplinary autonomy *via* parallel sybsystem-level uncertainty propagations and optimizations. However, the interdisciplinary couplings are satisfied only in terms of statistical moments (expected value, standard deviation, or covariance matrix) of the coupling variables and, most of the time, Gaussian distributions are assumed for the coupling variables.

Application to Toy Case The methodology proposed by Xiong et al. (2014) is applied on the toy case described at the beginning of this chapter. The following robust-CO formulation is adopted for the problem solving:

The optimization problem at the system level is given by

$$\min \quad \mathbb{E}\left[f\left(\mathbf{z}, \hat{\mathbf{Y}}(\mathbf{z}, \mathbf{U}), \mathbf{U}\right)\right] \tag{7.69}$$

$$\text{w.r.t.} \quad z_{sh}, z_1, z_2, \mu_{y_{12}}, \sigma_{y_{12}}, \mu_{y_{21}}, \sigma_{y_{21}}$$

$$\text{s.t.} \quad J_{12}^*\left(z_{sh}^*, z_1^*, \mu_{y_{12}}, \sigma_{y_{12}}, \mu_{y_{21}}, \sigma_{y_{21}}\right) = 0 \tag{7.70}$$

$$J_{21}^*\left(z_{sh}^*, z_2^*, \mu_{y_{12}}, \sigma_{y_{12}}, \mu_{y_{21}}, \sigma_{y_{21}}\right) = 0 \tag{7.71}$$

$$\mathbf{0} \leq \mathbf{z} \leq \mathbf{5} \tag{7.72}$$

and the two subsystem problems are given by

$$\min \quad J_{12} = \left|\left|z_{sh}^* - z_{sh}\right|\right|_2^2 + \left|\left|z_1^* - z_1\right|\right|_2^2 + \left|\left|\mu_{y_{12}} - c_{12}\left(z_{sh}^*, z_1^*, \mu_{y_{21}}, \sigma_{y_{21}}, \boldsymbol{\mu}_{\mathbf{U}}\right)\right|\right|_2^2$$

$$+ \left|\left|\sigma_{y_{12}}^2 - \sum_{l=1}^{d}\left(\left.\frac{\partial c_{12}}{\partial u^{(l)}}\right|_{\mathbf{u}=\boldsymbol{\mu}_{\mathbf{U}}}\right)^2 \sigma_{U^{(l)}}^2\right|\right|_2^2 \tag{7.73}$$

$$\text{w.r.t.} \quad z_{sh}^*, \quad z_1^*$$

$$\mathbf{0} \leq z_{sh}^*, z_1^* \leq \mathbf{5} \tag{7.74}$$

$$\min \quad J_{21} = \left|\left|z_{sh}^* - z_{sh}\right|\right|_2^2 + \left|\left|z_2^* - z_2\right|\right|_2^2 + \left|\left|\mu_{y_{21}} - c_{21}\left(z_{sh}^*, z_2^*, \mu_{y_{12}}, \sigma_{y_{12}}, \boldsymbol{\mu}_{\mathbf{U}}\right)\right|\right|_2^2$$

$$+\left\|\sigma^2_{y_{21}} - \sum_{l=1}^{d}\left(\frac{\partial c_{21}}{\partial u^{(l)}}\bigg|_{\mathbf{u}=\boldsymbol{\mu}_{\mathbf{U}}}\right)^2 \sigma^2_{U^{(l)}}\right\|_2^2 \tag{7.75}$$

$$\text{w.r.t.} \quad z^*_{sh}, \quad z^*_2$$

$$\text{s.t.} \quad \mathbb{E}\left[g_2(\mathbf{z}, \hat{y}_{21}, \mathbf{u})\right] + \eta\sigma\left[g_2(\mathbf{z}, \hat{y}_{21}, \mathbf{u})\right] \leq 0 \tag{7.76}$$

$$\mathbf{0} \leq z^*_{sh}, z^*_2 \leq \mathbf{5} \tag{7.77}$$

with $g_2 = \exp(-0.05 * u_2^2) * z_2 - 0.2 * z_2^4 * u_2^2 + 10^{-4} * y_{21}^2$.

In order to optimize the subsystem problems, CMA-ES is used due to the non-convex objective and constraint functions and the presence of numerous minima. Indeed, by using a classical SQP algorithm with different initializations, this algorithm converges to different local minima leading to a non-robust convergence of the lower-level (Figure 7.22). Figure 7.22 represents a boxplot of 20 repetitions of the subsystem one solving with different initializations of the SQP algorithm for a given system-level design variable vector. The range of optimum values for the objective function varies between 1.2 and 30. for J_{12}. In this case, SQP is clearly non-robust to the initialization.

In contrast, by exploring the design space, CMA-ES succeeds to find a better optimum than the gradient-based approach and is more robust with respect to the initialization (that is unknown and set in the center of the subsystem design space) than the gradient-based optimization. CMA-ES is parallelized in order to evaluate each candidate solution with a multiprocessing approach to increase the computational efficiency of the subsystem problem solving. A penalization approach is carried out to control the optimization problem constraints at the subproblem level.

Figure 7.23 illustrates the distribution of the coupling variables for robustness-based CO formulation and robustness-based MDF formulation. Due to the assumption of Gaussian distribution for the interdisciplinary coupling variables, it can

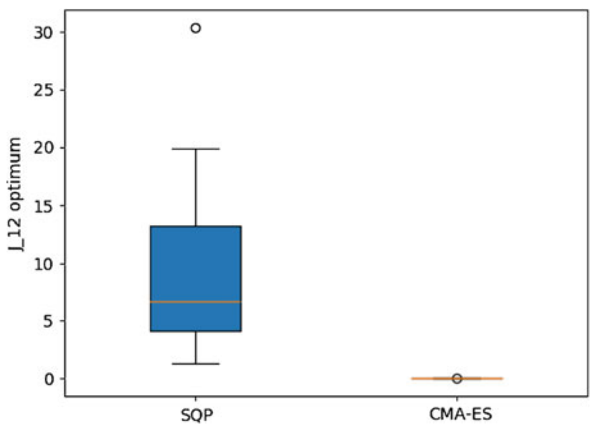


Fig. 7.22 Boxplot of J_{12} optimum for subproblem 1 using SQP and CMA-ES—20 repetitions with random initializations

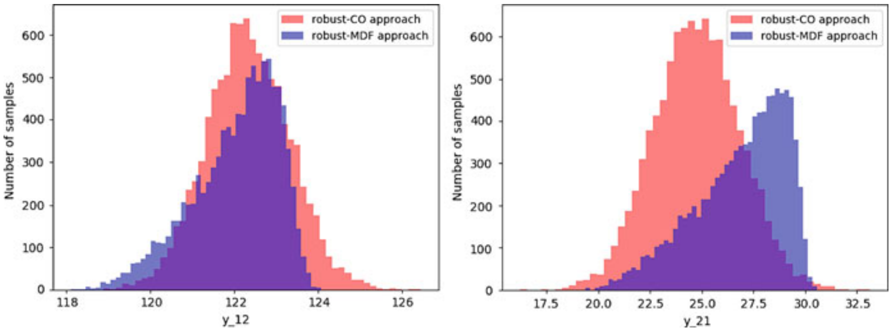
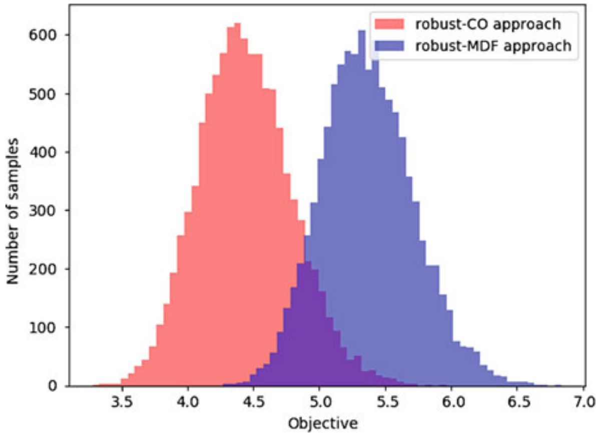


Fig. 7.23 Comparison between robust-based CO and robustness-based MDF for the interdisciplinary couplings

Fig. 7.24 Comparison between robust-based CO and robustness-based MDF for the objective function



be seen that the PDFs do not match perfectly between the two approaches. The distribution obtained for the coupling variables with robustness-based MDF is non-Gaussian especially for y_{21}. These disparities have a direct impact on the objective function, explaining the differences observed in Figure 7.24. Moreover, the fact that the final objective function in robustness-based CO formulation is lower than robustness-based MDF is explained by the fact that J_{12} and J_{21} are not yet equal to 0 and there still exist discrepancies between the system-level targets and lower-level responses. In terms of moment matching, it can be seen that between the system-level targets provided by the system-level optimizer and the subsystem-level discipline outputs, there is an accurate moment matching between the two distributions for the interdisciplinary couplings (Figure 7.25).

The convergence of the robust-CO formulation is more difficult to achieve than with robustness-based MDF (Figure 7.26). Indeed, the two nested levels of optimization converge slowly and require an important number of discipline evaluations, mainly due to the difficulty to solve the optimization problem at the lower level (Figure 7.22). At the subsystem level, using CMC to propagate

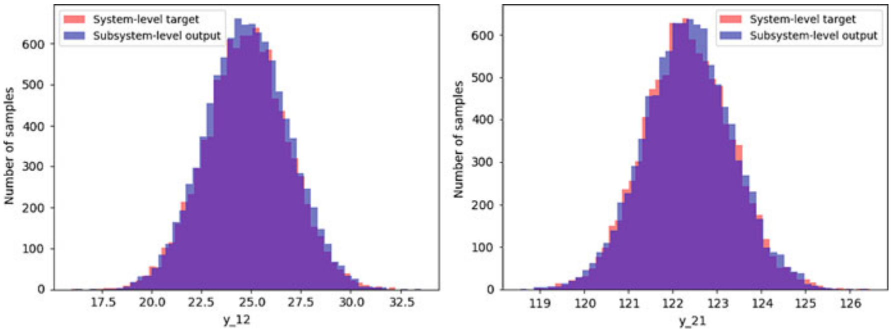


Fig. 7.25 Matching statistical moments, comparison between the distributions of the couplings for the system-level target and the subsystem-level output

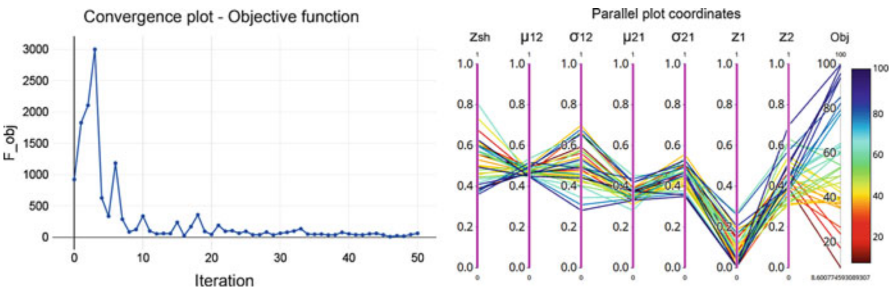


Fig. 7.26 Convergence plots at the system level (CO)

uncertainty with 10^4 samples and with 500 evaluations of the CMA-ES algorithm, at each iteration of the system level, there are $5 \times 10^2 \times 10^4 = 5 \times 10^6$ discipline evaluations. The system-level optimizer is stopped after 400 subsystem evaluations leading to a total number of discipline simulations in the order of 10^9. This is significantly higher than the 6×10^7 evaluations required by the robustness-based MDF. Therefore, for this particular problem, robustness-based CO seems not to provide an advantage compared to robustness-based MDF.

7.5.2 Individual Discipline Feasible—Polynomial Chaos Expansion (IDF-PCE) and Multi-level Hierarchical Optimization Under Uncertainty (MHOU)

Principle

As described in the previous section, to ensure interdisciplinary coupling satisfaction for all the realizations of the uncertain variables, it is necessary to introduce an infinite number of constraints at the system level. Solving an optimization problem with an infinite number of constraints is a challenging task. To bypass this issue, considering a UMDO problem involving N disciplines, in Brevault et al. (2015a,b) and Brevault (2015) proposed a new integral form for the interdisciplinary coupling constraint:

$$\forall(i, j) \in \{1, \ldots, N\}^2,\ i \neq j,\ \mathbf{J}_{ij} = \int_{\mathbb{R}^d} \left[\mathbf{c}_{ij}(\mathbf{z}_i, \mathbf{y}_{.i}, \mathbf{u}_i) - \mathbf{y}_{ij}\right]^2 \phi(\mathbf{u})\mathrm{d}\mathbf{u} = 0 \tag{7.78}$$

In (Equation 7.78), in order to have the integrals equal to zero, the input coupling variables must be equal to the output coupling variables for each realization of the uncertain variables almost surely (except maybe over null measure sets). The interdisciplinary coupling constraints $\mathbf{J}_{ij}$ represent the integration over the entire uncertain space of a loss function (the difference between the input and the output coupling variables). If these constraints (Equation 7.78) are satisfied, a mathematical equivalence is conserved with coupled formulations because, similarly with MDA using FPI, the following system of equations is satisfied for the coupling variables:

$$\forall\, \mathbf{u} \in \mathbb{R}^d,\ \forall(i, j) \in \{1, \ldots, N\}^2,\ i \neq j,\ \mathbf{y}_{ij} = \mathbf{c}_{ij}(\mathbf{z}_i, \mathbf{y}_{.i}, \mathbf{u}_i) \tag{7.79}$$

However, in order to decompose the disciplines, the uncertain input coupling variables $\mathbf{Y}$ must be controlled by the optimizer along with the design variables. Uncertain variables are measurable functions (mapping between the uncertainty space and the set of real numbers) and finding a function that is a solution to an infinite-dimensional optimization problem is a challenge. Several approaches have been proposed for this type of problems such as optimal control (Zhou et al. 1996), calculus of variations (Noton 2013), and shape optimization (Sokolowski and Zolesio 1992). To avoid to directly solve an infinite-dimensional problem, the function is often discretized and the discretization points are controlled by the optimizer (Devolder et al. 2010). The discretization strategy must be carried out in concordance with the optimization problem. Brevault et al. proposed to replace the scalar coupling variable y_{ij} by a surrogate model mimicking the coupling functional relations:

$$y_{ij} \rightarrow \hat{y}_{ij}\left(\mathbf{u}, \boldsymbol{\alpha}^{(ij)}\right) \tag{7.80}$$

The metamodel, noted $\hat{y}_{ij}\left(\mathbf{u}, \boldsymbol{\alpha}^{(ij)}\right)$, gives a functional representation of the dependency between the uncertain variables $\mathbf{U}$ and the input coupling variables with $\boldsymbol{\alpha}^{(ij)}$ the surrogate model hyperparameters. This approach allows to ensure that the functional dependency between the uncertain variables and the coupling variables is taken into account. In the proposed formulations, each coupling is replaced by a surrogate model. This latter is also a function, represented by hyperparameters that may be used to decompose the UMDO problem and let the system-level optimizer controlling the surrogate model coefficients. The infinite-dimensional optimization problem is converted into a q-dimensional optimization one with q the number of coefficients needed to model all the removed coupling variables.

To model the coupling functional relations, Brevault et al. proposed to use polynomial chaos expansion (PCE) as this metamodel is particularly suited for uncertainty analysis and propagation (Eldred 2009). PCE is adapted to represent the input coupling variables as it is dedicated to model functions that take as input uncertain variables as illustrated in Chapter 3. The scalar coupling y_{ij} is represented by

$$\hat{y}_{ij}\left(\mathbf{u}, \boldsymbol{\alpha}^{(ij)}\right) = \sum_{k=1}^{d_{\text{PCE}}} \alpha_{(k)}^{(ij)} \Psi_k(\mathbf{u}) \tag{7.81}$$

with d_{PCE} the PCE decomposition degree and Ψ_k orthogonal polynomials basis defined in accordance with the uncertain variable PDF distributions.

In order not to have too complex surrogate model $\hat{y}_{ij}(\cdot)$, the dependency between $\hat{y}_{ij}(\cdot)$ and $\mathbf{z}$ is not present here. Indeed, $\hat{y}_{ij}(\cdot)$ is not an explicit function of $\mathbf{z}$, it is learnt for the unknown specific $\mathbf{z}^*$ where the optimization converges. The interdisciplinary coupling satisfaction that is ensured for all the realizations of the uncertain variables enables to guaranty that the system is multidisciplinary feasible. This transformation of the complex original infinite-dimensional problem into a finite-dimensional one enables to solve it in practice while ensuring the mathematical equivalence between coupled and decoupled formulations in terms of coupling compatibility.

This methodology relies on an iterative construction of the PCEs along with the system-level UMDO. At the optimum, the surrogate models of the coupling functional relations simulate these mappings as would MDA under uncertainty do (Figure 7.27). Moreover, this approach does not require any MDA, allowing to fully decompose the process and to reduce the number of calls to the disciplines.

IDF-PCE (Individual Discipline Feasible—polynomial chaos expansion) is a single-level decoupled UMDO formulation relying on a functional representation of the coupling variables. IDF-PCE is formulated as follows:

$$\min \quad \Xi\left[f(\mathbf{z}, \boldsymbol{\alpha}, \mathbf{U})\right] \tag{7.82}$$

$$\text{w.r.t.} \quad \mathbf{z}, \boldsymbol{\alpha}$$

$$\text{s.t.} \quad \mathbb{K}_k\left[g_k(\mathbf{z}, \boldsymbol{\alpha}, \mathbf{U})\right] \leq 0 \;\; \text{for } k \in \{1, \ldots, m\} \tag{7.83}$$

$$\forall (i, j) \in \{1, \ldots, N\}^2, \; i \neq j,$$

$$\mathbf{J}_{ij} = \int_{\mathbb{R}^d} \left[\mathbf{c}_{ij}\left(\mathbf{z}_i, \hat{\mathbf{y}}_{.i}\left(\mathbf{u}, \boldsymbol{\alpha}^{(.i)}\right), \mathbf{u}_i\right) - \hat{\mathbf{y}}_{ij}\left(\mathbf{u}, \boldsymbol{\alpha}^{(ij)}\right)\right]^2 \phi(\mathbf{u})\mathrm{d}\mathbf{u} = \mathbf{0} \tag{7.84}$$

$$\mathbf{z}_{\min} \leq \mathbf{z} \leq \mathbf{z}_{\max}, \tag{7.85}$$

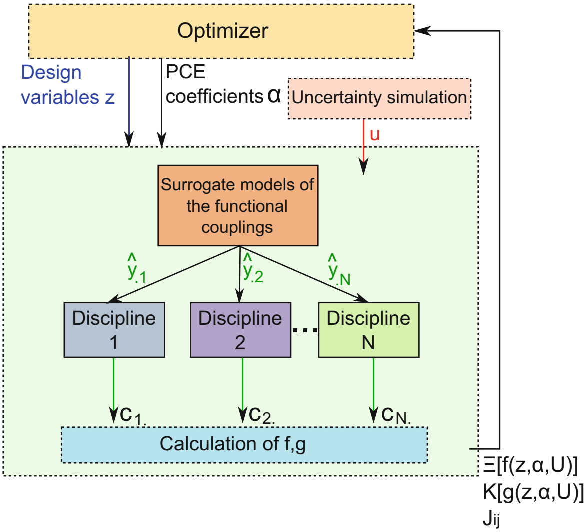


Fig. 7.27 IDF-PCE with the surrogate models of the coupling functional relations

where $\mathbf{J}_{ij}$ is the interdisciplinary constraint vector for the couplings from the discipline i to the discipline j and $\hat{\mathbf{y}}_{.i}\left(\mathbf{u}, \boldsymbol{\alpha}^{(.i)}\right)$ are the PCEs of all the input coupling variables of the discipline i.

In this formulation, the system-level optimizer controls the design variables $\mathbf{z}$ and the PCE hyperparameters of the coupling variables $\boldsymbol{\alpha}$. The handling of the PCE coefficients at the system level enables to decouple the disciplines and to simulate them in parallel (Figure 7.27). In comparison with the coupled formulations, the dimension of the design space is therefore increased with the number of hyperparameters $\boldsymbol{\alpha}$. To ensure the multidisciplinary compatibility at the optimum, equality constraints derived from the generalization error are added (Equation 7.84). These constraints take the input coupling variables modeled by PCE and the output coupling variables resulting from the discipline evaluations. The integral form for the constraints allows to ensure the coupling satisfaction for all the possible realizations of the uncertain variables. If this equation is verified: $\forall (i, j) \in \{1, \ldots, N\}^2\ \forall i \neq j,\ \mathbf{J}_{ij} = 0$, then the couplings are satisfied for all the realizations $\mathbf{u} \in \mathbb{R}^d$ almost surely.

The distances with respect to the MDA coupling satisfaction are represented by $\mathbf{J}(\cdot)$. Indeed, using MDA (FPI and CMC), $\mathbf{J}(\mathbf{z}) = 0, \forall \mathbf{z} \in [\mathbf{z}_{\min}, \mathbf{z}_{\max}]$. In IDF-PCE, $\mathbf{J}(\mathbf{z}) = 0$ has to be verified only at the UMDO optimum $\mathbf{z} = \mathbf{z}^*$. The interdisciplinary compatibility is not ensured all along the optimization, as in the deterministic IDF formulation.

In IDF-PCE, the robustness-based UMDO and the reliability-based UMDO problem formulation may be considered. Different measures of uncertainty for the inequality constraints (Equation 7.83) may be used, such that for instance:

$$\mathbb{K}\left[g_k(\mathbf{z}, \boldsymbol{\alpha}, \mathbf{U})\right] = \mathbb{E}\left[g_k(\mathbf{z}, \boldsymbol{\alpha}, \mathbf{U})\right] + \eta_k \sigma\left[g_k(\mathbf{z}, \boldsymbol{\alpha}, \mathbf{U})\right] \leq 0 \tag{7.86}$$

$$\mathbb{K}\left[\mathbf{g}(\mathbf{z}, \boldsymbol{\alpha}, \mathbf{U})\right] = \mathbb{P}_{g_k(\mathbf{z},\boldsymbol{\alpha},\mathbf{U})\leq 0} - P_{t_k} \leq 0 \quad (7.87)$$

with P_{t_k} the admissible maximal probability of failure. The first measure, (Equation 7.86), is based on the statistical moments of the inequality constraint functions and the second (Equation 7.87) is based on the probability of failure. In practice, the multidimensional integrals associated to the statistical moments (expectations, standard deviations), to the coupling constraints $\mathbf{J}$, or to the probability of failure require uncertainty propagation. Three techniques to estimate the statistical moments and the coupling constraints have been considered in Brevault et al. (2015a,b) and Brevault (2015): CMC, quadrature rules, and decomposition of the output coupling variables over a PCE. In order to estimate the probability of failure a technique combining subset sampling with support vector machines has been proposed. Depending on the approach carried out to propagate uncertainty, this lead to three variants of IDF-PCE.

The different steps of the IDF-PCE strategy are summarized in Figure 7.28. In the proposed formulation, the design variables are considered as deterministic (such as specifications) and all the uncertainty is assumed to be represented by uncertain parameters $\mathbf{U}$. However, an extension of these UMDO formulations to uncertain design variables by letting the optimizer controlling the expected value of the design variables is possible as, for instance, done in Liu et al. (2006) and Lin and Gea (2013). Furthermore, PCE is employed to characterize the functional relations between the coupling variables at the UMDO problem convergence as PCE are well suited to model functions which take as input uncertain variables. The general idea

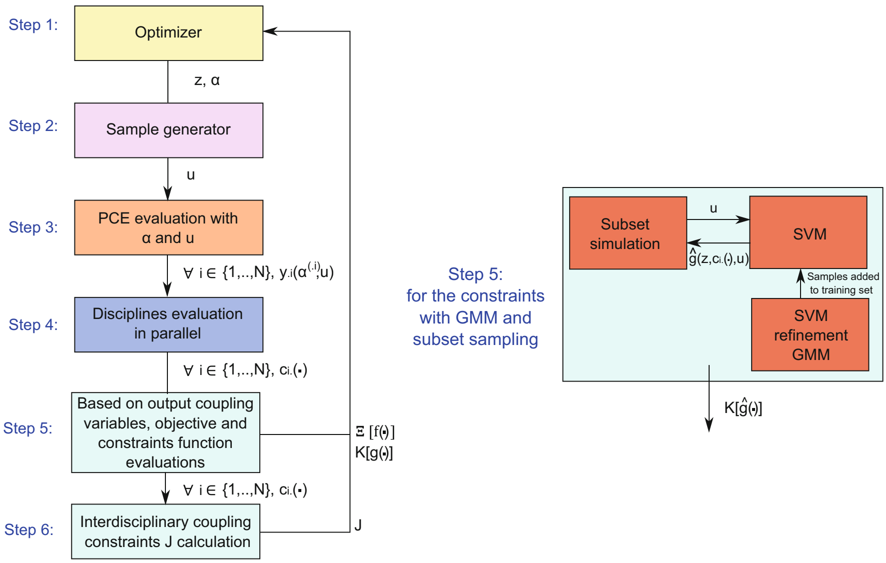


Fig. 7.28 Steps of the IDF-PCE algorithm

to control interdisciplinary couplings at the system level could be extended to any parametric surrogate model. These hyperparameters would have to be controlled by the system-level optimizer in addition with the design variables.

IDF-PCE solves a finite-dimensional problem which is manageable compared to infinite-dimensional problems which are difficult to implement in practice. One of the principal advantages is that this formulation ensures the coupling satisfaction at the UMDO optimum for each uncertain variable realization almost surely. For that purpose, it models the functional dependency between the uncertain variables, the design variables, and the coupling variables, which may be useful beyond optimization, for instance, for a post-optimality sensitivity analysis. IDF-PCE is a decomposed single-level UMDO formulation allowing to decouple the UMDO problem and to concurrently evaluate the disciplines. Furthermore, it does not require any full MDA evaluation. This decomposition of the UMDO process allows to reduce the management tasks compared to coupled formulations because each discipline just converses with the system level and no more interdisciplinary information exchange is required during the subsystem level uncertainty propagation.

However, this formulation has some drawbacks similar to deterministic IDF. Indeed, the number of variables controlled by the system-level optimizer (design variables and PCE coefficients) is increased. This makes the system-level optimization task more complex. Moreover, the PCE decomposition order has to be chosen a priori based on the coupling linearity knowledge. The increase of the PCE decomposition order may highly increase the size of the design space, a curse of dimensionality is observed due to the PCE construction. Alternative approaches such as sparse PCE may be investigated to tackle this issue. Finally, IDF-PCE increases the number of equality constraints at the system level which also makes the optimization problem solving more complex.

Multi-level Hierarchical Optimization Under Uncertainty (MHOU)

The aim of MHOU (Brevault et al. 2015b) is to ease the system-level optimization by introducing a subsystem-level optimization (Figure 7.29). It is derived from IDF-PCE. MHOU is a semi-decomposed hierarchical method that removes all the feedback interdisciplinary couplings in order to avoid the expensive disciplinary loops through MDA. Due to the curse of dimensionality of the surrogate model-based decoupling technique proposed in IDF-PCE, only the feedback couplings are removed in MHOU. It enables a hierarchical process without any loop between the subsystems. This type of decomposition has been proposed in the context of launch vehicle design (Brevault et al. 2015b), but it may be generalized to a set of problems. Indeed, the formulation assumes that the system-level objective $\Xi[f(\cdot)]$ can be decomposed into a sum of subsystem contributions $\Xi[f(\cdot)] = \sum_{k=1}^{N} \Xi[f_k(\cdot)]$ with $\Xi[f_k(\cdot)]$ the kth subsystem objective function. Most of the systems may be decomposed according to the contribution of the subsystems (contributions of the subsystem masses, of the subsystem costs, etc.).

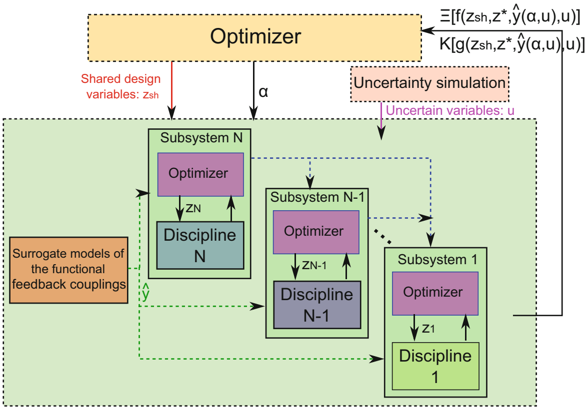


Fig. 7.29 Multi-level hierarchical optimization under uncertainty (MHOU)

In MHOU, the system-level and subsystem-level formulations are postulated as follows:

- System level:

$$\min \quad \sum_{k=1}^{N} \Xi\left[f_k\left(\mathbf{z}_{sh}, \mathbf{z}_k^*, \boldsymbol{\alpha}, \mathbf{U}\right)\right] \tag{7.88}$$

$$\text{w.r.t.} \quad \mathbf{z}_{sh}, \boldsymbol{\alpha}$$

$$\text{s.t.} \quad \mathbb{K}_k\left[g_k(\mathbf{z}_{sh}, \mathbf{z}_k^*, \boldsymbol{\alpha}, \mathbf{U})\right] \leq 0 \quad \text{for } k \in \{1, \ldots, m\} \tag{7.89}$$

$$\forall (k, j) \in \{1, \ldots, N\}^2, \quad j \neq k \ \mathbf{J}_{kj}\left(\mathbf{z}_{sh}, \mathbf{z}_k^*, \boldsymbol{\alpha}\right) = 0 \tag{7.90}$$

$$\forall s \in \{1, \ldots, N\}, \ \mathbb{K}\left[\mathbf{g}_s\left(\mathbf{z}_{sh}, \mathbf{z}_k^*, \boldsymbol{\alpha}, \mathbf{U}\right)\right] \leq 0 \tag{7.91}$$

$$\mathbf{z}_{sh_{\min}} \leq \mathbf{z}_{sh} \leq \mathbf{z}_{sh_{\max}} \tag{7.92}$$

- Subsystem level:
  $k = N$

$$\text{While} \quad k > 0$$

$$\text{Given} \quad \mathbf{y}_{Nk}, \ldots, \mathbf{y}_{(k+1)k}$$

For the kth subsystem

$$\min \quad \Xi\left[f_k(\mathbf{z}_{sh}, \mathbf{z}_k, \boldsymbol{\alpha}, \mathbf{U})\right] \tag{7.93}$$

$$\text{w.r.t.} \quad \mathbf{z}_k$$

$$\text{s.t.} \quad \mathbb{K}_s\left[g_s(\mathbf{z}_{sh}, \mathbf{z}_k, \boldsymbol{\alpha}, \mathbf{U})\right] \leq 0 \quad \text{for } s \in \{1, \ldots, m_s\} \tag{7.94}$$

$$\forall j \in \{1, \ldots, N\}, \quad j \neq k \ \mathbf{J}_{kj} =$$

$$\int_{\mathbb{R}^d} \left[\mathbf{c}_{kj}\left(\mathbf{z}_{sh}, \mathbf{z}_k, \hat{\mathbf{y}}_{.k}\left(\mathbf{u}, \boldsymbol{\alpha}^{(.k)}\right), \mathbf{u}_k\right) - \hat{\mathbf{y}}_{kj}\left(\mathbf{u}, \boldsymbol{\alpha}^{(kj)}\right)\right]^2 \phi(\mathbf{u})\mathrm{d}\mathbf{u} = \mathbf{0} \tag{7.95}$$

$$\mathbf{z}_{k_{\min}} \leq \mathbf{z}_k \leq \mathbf{z}_{k_{\max}} \tag{7.96}$$

$$k \leftarrow k-1,$$

where $\mathbf{z}_k$ is the local design variable vector of discipline k and $\mathbf{z}_{sh}$ is the shared design variable vector between several disciplines. $\mathbf{z}_k^*$ corresponds to the optimal design variables found by the subsystem-level optimizer. This formulation enables to optimize each subsystem independently in a hierarchical process. The system-level optimizer controls $\mathbf{z}_{sh}$ and the PCE hyperparameters $\boldsymbol{\alpha}$ of the feedback coupling variables. The handling of the PCE coefficients at the system level allows one to remove the feedback couplings and to optimize the subsystems in sequence. The surrogate models of the functional feedback couplings provide the required input couplings to the different subsystems. The kth subsystem-level optimizer controls $\mathbf{z}_k$ and the corresponding problem aims at minimizing the subsystem contribution to the global system objective while ensuring the subsystem-level constraints $\mathbb{K}_s\left[g_s(\cdot)\right]$. The interdisciplinary coupling constraint (Equation 7.95) guarantees the coupling consistency whatever the realization of the uncertain variables. In MHOU (Equation 7.95) is only considered for $k \neq N$. This formulation is particularly suited for launch vehicle design as it is a natural way to decompose the design process into the different stage optimizations. The decreasing order of the discipline optimization from N to 1 is more convenient for a launch vehicle (the last stage is optimized first, then the intermediate stages, and the first one is optimized last). For a general problem any order may be adopted, but in practice the disciplines are organized to have the minimal number of feedbacks in order to decrease the number of coupling variables controlled at the system level and then the complexity of the optimization problem. This formulation has been applied for launch vehicle design problems, using Stage-Wise Optimal Rocket Decomposition (SWORD) (Balesdent et al. 2013) and MHOU (Brevault et al. 2015b). In this case, the feedforward coupling variables are the masses of the different stages (the mass is passed from stage i to stage $i-1$) and the feedback couplings are the separation conditions (e.g., altitude, velocity, flight path angle) and trajectory loads.

Application to Toy Case

Numerical comparisons between MDF under uncertainty (using FPI and CMC) and IDF-PCE are carried out for the toy case. The UMDO problem presented in Figure 7.30 is a constrained optimization problem composed of

- Discipline 1: $y_{12} = c_{12}(\mathbf{z}, \mathbf{u}) = -z_{sh}^{0.2} + u_{sh} + 0.25 \times u_1^{0.2} + z_1 + y_{21}^{0.58} + u_1^{0.4} \times y_{21}^{0.47}$

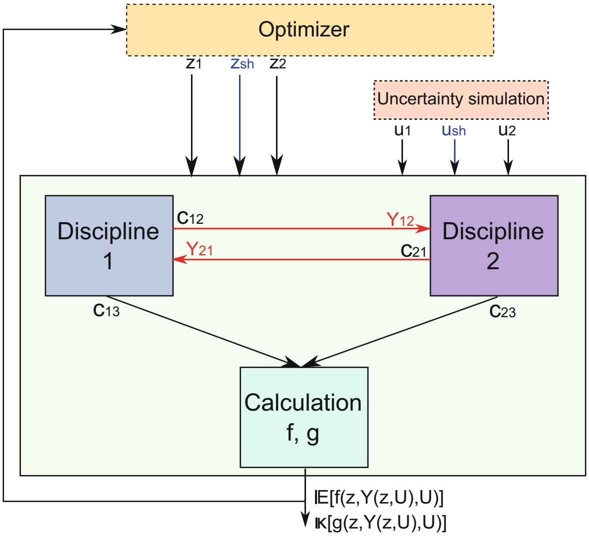


Fig. 7.30 Analytical test case of a multidisciplinary coupled system

- Discipline 2: $y_{21} = c_{21}(\mathbf{z}, \mathbf{u}) = -z_{sh} + u_{sh}^{0.1} - z_2^{0.1} + 3 \times y_{12}^{0.47} + u_2^{0.33} + y_{12}^{0.16} \times u_2^{0.05} + y_{12}^{0.6} \times u_2^{0.13} + 100$
- Calculation of $f = \frac{1}{5}\left[(z_{sh} - 5)^2 + (z_1 - 3)^2 + (z_2 - 7)^2 + (y_{21} + z_1 \times z_2)^{0.6} + (u_{sh} + 9)^2\right]$
- Calculation of $g = 150 + \exp(-0.01 \times u_1^2) \times z_{sh} \times z_1 - 0.02 \times z_2^3 \times u_2^5 + 0.01 \times y_{12}^{2.5} \times z_2 \times \exp(-0.1 u_{sh})$
- Objective function: $\Xi[f(\mathbf{z}, \mathbf{Y}, \mathbf{U})] = \mathbb{E}[f(\mathbf{z}, \mathbf{Y}, \mathbf{U})]$,
- Constraint function: $\mathbb{K}[g(\mathbf{z}, \mathbf{Y}, \mathbf{U})] = \mathbb{E}[g(\mathbf{z}, \mathbf{Y}, \mathbf{U}))] + 3\sigma[g(\mathbf{z}, \mathbf{Y}, \mathbf{U}))] \leq 0$
  The problem has 3 design variables: $z_1 \in [0, 1]$, $z_2 \in [0, 1]$ and the shared variable $z_{sh} \in [0, 1]$, and 3 uncertain variables: $U_1 = \mathscr{U}(-1, 1)$, $U_2 = \mathscr{N}(0, 1)$ and the shared uncertain variable $U_{sh} = \mathscr{N}(0, 1)$.

This toy case involves non-linear disciplines and results in non-Gaussian coupling variable distributions in order to illustrate the challenge of dealing with highly non-linear couplings. Moreover, it has been numerically verified for all $\mathbf{z}$ and $\mathbf{u}$ values tried that this problem is such that the MDA converges (it is a contraction mapping by FPI) and it converges to a unique coupling value.

MDF Under Uncertainty Uncertainties are propagated with CMC on MDA (using FPI) for each realization of the uncertain variables with a sample size $M = 150{,}000$ in order to have an error lower than 10^{-3} on the estimation of the statistical moments. As the first two statistical moments are estimated with CMC, the objective function is noisy; therefore, gradient-based optimizers are not appropriate for this test case. Stochastic gradient optimizer is not suited due to the multiple local minima. An Ant Colony optimizer (ACOmi) (Hirmajer et al. 2010) is used. Optimizations are stopped if there is no improvement in 50 consecutive objective function evaluations with a tolerance of 10^{-3} on the objective function and the constraint. The MDA convergence criterion for the Fixed Point Iteration has been set to 10^{-4} as it corresponds to a variation in the objective and constraint functions smaller than 10^{-3}. Based on the numerical experimentation, five iterations are in general required to converge under the tolerance with the FPI methods. $\epsilon_g = -0.004$ is a conservative tolerance due to the estimation of the mean and the standard deviation of the constraint by CMC to ensure that the constraint is inferior or equal to 0.

IDF-PCE Formulation IDF-PCE formulation (Figure 7.31) is given by

$$\min \quad \mathbb{E}\left[f\left(\mathbf{z}, \boldsymbol{\alpha}, \mathbf{U}\right)\right] \tag{7.97}$$

$$\text{w.r.t.} \quad \mathbf{z}, \boldsymbol{\alpha}^{(12)}, \boldsymbol{\alpha}^{(21)}$$

$$\text{s.t.} \quad \mathbb{E}[g(\mathbf{z}, \boldsymbol{\alpha}, \mathbf{U})] + 3\sigma[g(\mathbf{z}, \boldsymbol{\alpha}, \mathbf{U})] \leq \epsilon_g \tag{7.98}$$

$$J_{12} = \int_{\mathbb{R}^3} \left[c_{12}\left(z_{sh}, z_1, u_{sh}, u_1, \hat{y}_{21}\left(\mathbf{u}, \boldsymbol{\alpha}^{(21)}\right)\right) - \hat{y}_{12}\left(\mathbf{u}, \boldsymbol{\alpha}^{(12)}\right)\right]^2 \phi(\mathbf{u})\mathrm{d}\mathbf{u} \leq \epsilon \tag{7.99}$$

$$J_{21} = \int_{\mathbb{R}^3} \left[c_{21}\left(z_{sh}, z_2, u_{sh}, u_2, \hat{y}_{12}\left(\mathbf{u}, \boldsymbol{\alpha}^{(12)}\right)\right) - \hat{y}_{21}\left(\mathbf{u}, \boldsymbol{\alpha}^{(21)}\right)\right]^2 \times \phi(\mathbf{u})\mathrm{d}\mathbf{u} \leq \epsilon \tag{7.100}$$

$$\mathbf{z} \in [0, 1]^3 \tag{7.101}$$

The system-level coupling variables are decomposed according to: $\hat{y}_{ij}(\mathbf{U}, \boldsymbol{\alpha}^{(ij)}) = \sum_{k=0}^{19} \alpha_k^{(ij)} \Psi_k(\mathbf{U})$, with $\Psi_k(\cdot)$ the product of Legendre and Hermite polynomials with a total expansion order of degree 3 in order to take into account the non-linearity of the problem. These polynomial bases are orthogonal to the input density distributions (Gaussian and uniform). As there are three uncertain variables for the decomposition, $\dim\left(\boldsymbol{\alpha}^{(12)}\right) = \dim\left(\boldsymbol{\alpha}^{(21)}\right) = \frac{(3+3)!}{3!3!} = 20$. The design space is of dimension 43. The methods to compute the multivariate integrals are detailed in the next paragraphs.

IDF-PCE (CMC) Formulation In IDF-PCE with CMC, the interdisciplinary constraints are computed with:

$$J_{ij} \simeq \frac{1}{M} \sum_{k=1}^{M} \left[c_{ij}\left(z_{sh}, z_1, u_{sh(k)}, u_{1(k)}, \hat{y}_{.i}\left(\mathbf{u}_{(k)}, \boldsymbol{\alpha}^{(.i)}\right)\right) - \hat{y}_{ij}\left(\mathbf{u}_{(k)}, \boldsymbol{\alpha}^{(ij)}\right)\right]^2 \tag{7.102}$$

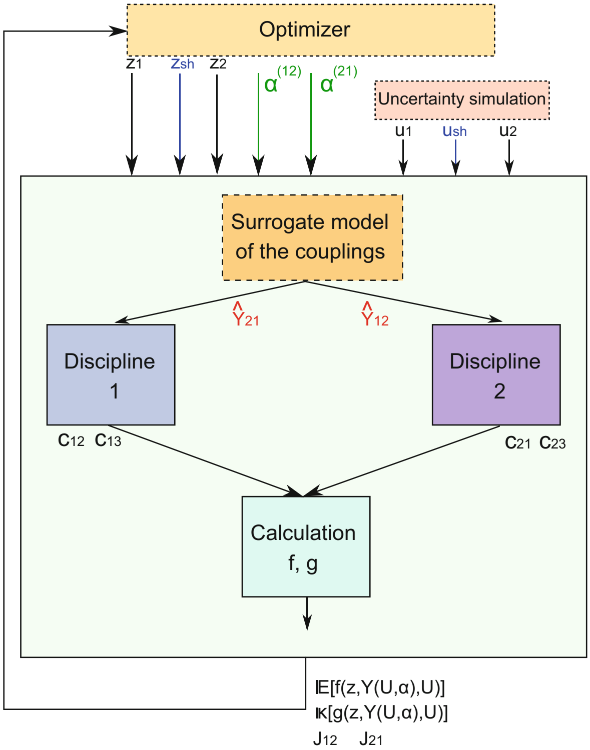


Fig. 7.31 IDF-PCE design process

The mean of the objective function, and the mean and the standard deviation of the constraint g are computed by CMC. The uncertainties are propagated with CMC on the decoupled system with a sample size of $M = 150{,}000$. The interdisciplinary constraints on the couplings are such that $\epsilon = 10^{-3}$ in order to have on average a coupling error under ϵ. Another alternative to avoid a fixed ϵ value set by the designer is to convert the equality constraint of $J_{ij} = 0$ into an inequality constraint $J_{ij} \leq \epsilon$, where ϵ is a small positive real number, and use an additional dynamic slack variable to carry out the optimization process in order to minimize the value of ϵ to be as close as possible to 0.

IDF-PCE (Quadrature) Formulation In IDF-PCE with quadrature rules, the coupling constraints are computed as follows:

$$J_{ij} = \sum_{k=1}^{M_{sh}} \sum_{l=1}^{M_1} \sum_{m=1}^{M_2} (w_{sh(k)} \otimes w_{1(l)} \otimes w_{2(m)}) \Big[c_{ij} \Big(z_{sh}, z_1, u_{sh(k)}, u_{1(l)}, \hat{y}_{.i} \Big(u_{sh(k)}, u_{1(l)}, u_{2(m)}, \boldsymbol{\alpha}^{(.i)} \Big) \Big) - \hat{y}_{ij} \Big(u_{sh(k)}, u_{1(l)}, u_{2(m)}, \boldsymbol{\alpha}^{(ij)} \Big) \Big]^2 \quad (7.103)$$

The expected value of $f(\cdot)$ and the mean and the standard deviation of the constraint $g(\cdot)$ are computed as explained in the paragraph describing the quadrature rules. The quadrature rules used to compute the multidimensional integrals correspond to the tensor product of the one-dimensional Gauss–Hermite and Gauss–Legendre quadratures. The number of sampling points in each direction is: $M_{sh} = M_2 = 8$ and $M_1 = 10$, resulting in a tensor product of 640 discipline evaluations to compute the multivariate integrals. This number of quadrature points is selected in order to have an error less than 10^{-3} compared to a CMC computation of the integrals with 10^6 points. The decomposition of the coupling variables is the same as in IDF-PCE with CMC formulation.

IDF-PCE (PCE) Formulation In this approach, the output PCE coefficients $\tilde{\boldsymbol{\alpha}}^{(ij)}$ are computed by orthogonal spectral projection in which the multivariate integrals are estimated by quadrature rules. The interdisciplinary constraints J_{12} and J_{21} are replaced by:

$$\| \boldsymbol{\alpha}^{(12)} - \tilde{\boldsymbol{\alpha}}^{(12)} \|^2 \leq \epsilon_\alpha \tag{7.104}$$

$$\| \boldsymbol{\alpha}^{(21)} - \tilde{\boldsymbol{\alpha}}^{(21)} \|^2 \leq \epsilon_\alpha \tag{7.105}$$

To compute the output PCE coefficients, the same quadrature rules as in IDF-PCE (quadrature) formulation are used: $M_{sh} = M_2 = 8$ and $M_1 = 10$. The constraints on the couplings are such that $\epsilon_\alpha = 0.5$ as it generates an error on average smaller than 10^{-3} compared to CMC approximation of the integral. The main difference with the IDF-PCE quadrature formulation is in the coupling constraints to ensure the interdisciplinary couplings in realizations. In IDF-PCE (PCE) the quadratic constraints only involve the PCE coefficients and could facilitate the optimizer convergence.

Results Due to the presence of uncertainty and the use of a population-based optimizer, each problem solving is repeated 10 times and the results given in Table 7.1 are the averages over the 10 repetitions. The ratio of the standard deviation over the expected value of the results is added in parenthesis in order to quantify the robustness of the results.

To insist on the importance of incorporating uncertainty in MDO problems, a deterministic MDF with the uncertain variables set to their mean values is carried out. The found optimal objective value is 0.466, the set of optimal design variables is: $z_{sh} = 0.504$, $z_1 = 0.452$, $z_2 = 0.682$ and the constraint is active. A propagation of uncertainty based on the found deterministic optimum $\mathbf{z}^*$ results in a violation of the constraint $\mathbb{E}[g(\mathbf{z}^*, \mathbf{Y}(\mathbf{z}^*, \mathbf{U}), \mathbf{U}))] + 3\sigma[g(\mathbf{z}^*, \mathbf{Y}(\mathbf{z}^*, \mathbf{U}), \mathbf{U})] = 2.73 > 0$ and the performance is decreased compared to the deterministic value $\mathbb{E}[f(\mathbf{z}^*, \mathbf{Y}(\mathbf{z}^*, \mathbf{U}), \mathbf{U}))] = 0.831$. For this toy case, the presence of uncertainty modifies both the optimal objective and the set of optimal design variables, but it results in a non-robust deterministic solution.

Table 7.1 Analytical test case results for the different proposed IDF-PCE formulations. In parenthesis, standard deviation over average of each result

Results	MDF (ref)	IDF-PCE (CMC)	IDF-PCE (quadrature)	IDF-PCE (PCE)
Objective	$\mu_f = 0.928$ (0.64%)	$\mu_f = 0.926$ (0.65%)	$\mu_f = 0.926$ (0.70%)	$\mu_f = 0.914$ (0.49%)
Design	$z_{sh} = 0.520$ (0.63%)	$z_{sh} = 0.511$ (0.86%)	$z_{sh} = 0.514$ (1.34%)	$z_{sh} = 0.523$ (1.03%)
Variables	$z_1 = 0.340$ (1.13%)	$z_1 = 0.339$ (1.11%)	$z_1 = 0.340$ (1.27%)	$z_1 = 0.349$ (1.13%)
	$z_2 = 0.658$ (1.55%)	$z_2 = 0.661$ (1.30%)	$z_2 = 0.661$ (1.68%)	$z_2 = 0.649$ (0.95%)
Coupling	$\lvert c_{12} - y_{12}\rvert^2 \leq 0.0001$	$J_{12} = 0.00067$ (1.23%)	$J_{12} = 0.00054$ (1.08%)	$\parallel \boldsymbol{\alpha}^{(21)} - \tilde{\boldsymbol{\alpha}}^{(21)} \parallel^2 = 0.48$ (1.56%)
Constraints	$\lvert c_{21} - y_{21}\rvert^2 \leq 0.0001$	$J_{21} = 0.00057$ (1.15%)	$J_{21} = 0.00074$ (1.12)%	$\parallel \boldsymbol{\alpha}^{(21)} - \tilde{\boldsymbol{\alpha}}^{(21)} \parallel^2 = 0.3$ (2.13%)
Constraint $\mathbb{K}$ value	−0.001 (1.87%)	−0.002 (1.43%)	−0.001 (2.04%)	−0.002 (2.53%)
Design space dimension	3	43	43	43
Number of optimization iterations	$N_i = 2016$ (5.34%)	$N_i = 5608$ (14.5%)	$N_i = 5501$ (9.56%)	$N_i = 5262$ (8.10%)
Calls to each discipline	$N_d = 1512 \times 10^6$	$N_d = 841.2 \times 10^6$	$N_d = 3.52 \times 10^6$	$N_d = 3.37 \times 10^6$
Number of calls reduction factor	1 (Ref)	1.80	429.55	448.66

The MDF under uncertainty formulation is considered as the reference for interdisciplinary coupling satisfaction (couplings are satisfied for each uncertain variable realization). MDF and the IDF-PCE formulations converge to the same design variable values $\mathbf{z}$ with errors inferior to 2.04% in terms of distance to MDF results. IDF-PCE (quadrature) is the closest to MDF. The higher error stems from IDF-PCE (PCE) due to the approximations introduced by the output coupling PCEs $\tilde{\mathbf{c}}(\cdot)$. In terms of objective values, the relative error compared to MDF is of 0.21% for IDF-PCE (CMC) and IDF-PCE (quadrature) and 1.5% for IDF-PCE (PCE). There is one constraint in MDF, whereas in the proposed formulations three constraints are present. The design space dimension is three in the MDF approach, whereas it is 43 in the proposed formulations. The increase in number of design variables and constraints requires more iterations for the optimization process to converge (but less calls to the disciplines). The number of calls to each discipline is 1512×10^6 for MDF. It is divided by 1.80 for IDF-PCE (CMC) formulation. Compared to MDF, the number of calls to each discipline decreases by a factor of 430 in IDF-PCE (quadrature) and by 449 in IDF-PCE (PCE). The reduction of the number of calls to the disciplines is due to the absence of complete MDA and the uncertainty propagation technique (quadrature and PCE) instead of CMC. While the absence of MDA decreases the number of calls to the disciplines, it generates higher errors in the interdisciplinary coupling satisfaction: the couplings are satisfied with a precision of 10^{-4} in MDF for all the realizations of the uncertain variables and with a precision of 6.7×10^{-4} on average in IDF-PCE (CMC). The replacement of CMC in IDF-PCE enables to decrease the number of calls to the disciplines while ensuring coupling satisfaction with a precision of 7.4×10^{-4} on average. The PDF of the performance values given by MDF and by the proposed approaches are similar (Figures 7.32 and 7.33). The distributions of the coupling variable Y_{12} have the same shape for MDF and the decoupled approaches. Moreover, the proposed approaches succeed to handle the multimodal probability distribution for the coupling variables. Differences exist in the distribution tails (Figures 7.32 and 7.33) due to the error introduced by the PCE to represent the coupling relations. All the distributions of the coupling errors $\mathbf{J}$ for the proposed formulation are given in Figure 7.34.

Similarly to hybrid uncertainty propagation techniques for multidisciplinary systems presented in Chapter 6, several UMDO formulations have been proposed to combine the advantages of coupled and decoupled strategies to solve UMDO problems. A brief overview of several techniques is discussed in the next section.

7.6 Hybrid UMDO Approaches

Deterministic Concurrent Subspace Optimization (CSSO) framework has been derived to solve UMDO problems. One such method has been proposed by Padmanabhan and Batill (2002) in which the CSSO architecture has been used to perform reliability-based optimization for UMDO problems. The proposed procedure is a sequential process: starting from a design variable vector $\mathbf{z}^k$ a system-

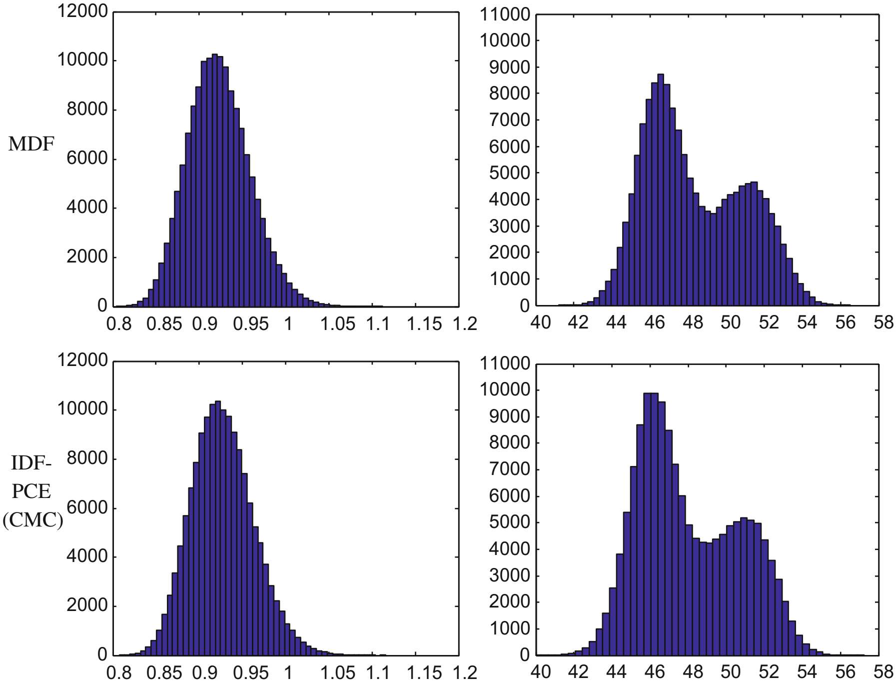


Fig. 7.32 Distribution of the performance $f(\cdot)$ (left column) and the coupling variable Y_{12} (right column) estimated from 150,000 **U** samples with MDF and IDF-PCE (CMC)

level reliability analysis is carried out to obtain outputs of the objective function, the coupling variables, the reliability constraints and their sensitivities with respect to the design variables, and the uncertain parameters at the current design point $\mathbf{z}^k$. Then, these information are used with the first-order Taylor series approximation models to build surrogate models of the non-local state variables and reliability constraints. These metamodels are employed in the subspace optimization enabling to concurrently carry out local subspace optimization and providing the next design point $\mathbf{z}^{k+1}$. The sequential process is stopped when the convergence of the design point is reached.

Hu et al. (2016) proposed an approach relying on active subspace identification to reduce the uncertainty dimension and applies a one-dimensional regression surface with least squares as the surrogate model in order to reduce the computational cost associated with the uncertainty propagation. To estimate the measure of uncertainty on the objective and constraint functions while avoiding repeated calls to MDA, a partial first-order second moment (p-FOSM) is carried out. It consists in estimating the first two statistical moments of the coupling variables using a first-order approximation with a limited number of MDA evaluations. Then, assuming Gaussian distribution for the coupling variables, the estimation of the statistical moments of the objective and the constraint functions can be performed by

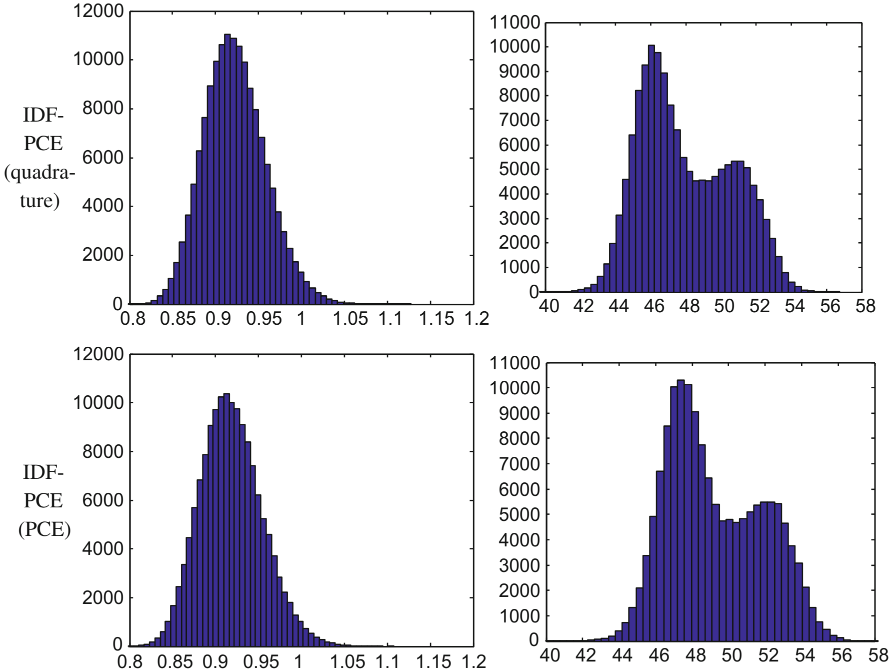


Fig. 7.33 Distribution of the performance $f(\cdot)$ (left column) and the coupling variable Y_{12} (right column) estimated from 150,000 **U** samples with IDF-PCE (quadrature) and IDF-PCE (PCE)

evaluating the disciplines in parallel with the corresponding input coupling variable distributions.

Ahn et al. proposed an extension of BLISS formulation (see Chapter 1) named ProBLISS for reliability-based MDO problems (Ahn and Kwon 2006). ProBLISS uses a single-level reliability-based MDO in which the reliability analysis and the optimization are carried out sequentially by approximating the limit-state functions. As the main challenge with BLISS concerns the accuracy of the discipline approximations, the convergence of the strategy is ensured by employing a trust region-sequential quadratic programming framework, in order to validate the approximation models within the trust region radius. In the proposed approach, the interdisciplinary coupling satisfaction is partially ensured with multidisciplinary analysis (for instance, with FPI) performed at the mean value of the uncertain variables. Moreover, the determination of the most probable failure point is assessed using also MDA. The approximation models are built locally around the solutions found with MDA.

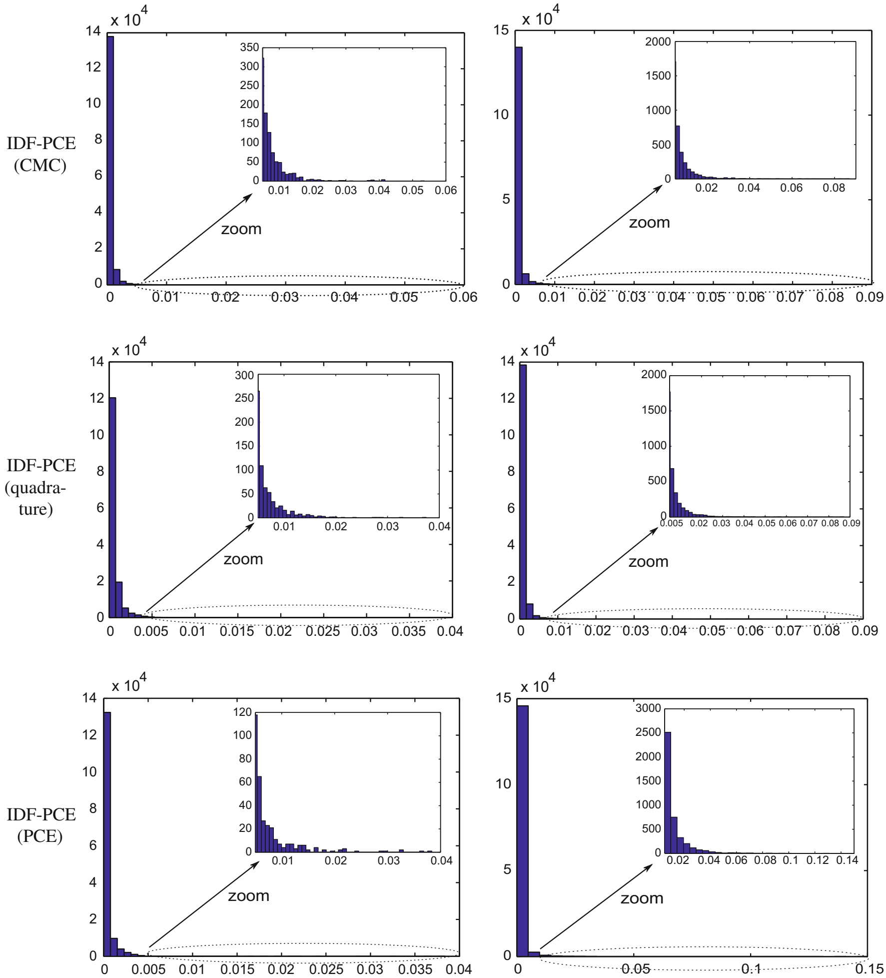


Fig. 7.34 Distribution of the coupling errors **J**. Left column, estimations of J_{12} , right column, estimations of J_{21}, from 150,000 **U** samples

7.7 UMDO with Mixed Aleatory and Epistemic Uncertainties

UMDO process has been adapted to the presence of both aleatory and epistemic uncertainties. These types of formulations require to propagate the uncertainty described with different mathematical formalisms and to manage them in the optimization process.

A first type of mixed uncertainty MDO problems combining interval and probability formalisms has been treated in the literature. Xia et al. (2016) proposed an adaptation of the sequential-MDO (S-MDO) formulation with interval uncertainty.

It is a decomposed bi-level MDO formulation. In the first step of S-MDO, a concurrent optimization of each discipline is performed without considering the other disciplines. It supposes that each discipline (or subsystem) may present objective functions. In this step, the interdisciplinary coupling variables are considered as local variables and optimized along with the local design variables. After these multiple local optimizations, even though different solutions for the different subsystems are found, the obtained local design variables and interdisciplinary couplings are used to form an hyper-interval from which a potential Pareto points could be determined. Based on the obtained potential Pareto points, a combination of global variables and coupling variables is dispatched into subsystems. Then sequentially, each subsystem carries out optimization under consistency constraints to ensure multidisciplinary feasibility. S-MDO formulation has been extended to account for interval uncertainty. During the first step of concurrent optimization, a fixed range (by expert opinion, for instance) is associated with the coupling variables. Then, a robust optimization under interval uncertainty is solved for each subsystem. A double-loop approach using genetic algorithm is employed to propagate interval uncertainty (inner loop) and to solve the full autonomy robust optimization problems. In the second step, in order to ensure interdisciplinary feasibility, a design point is said to be consistently robust if the maximum variation between the output coupling variable and the input coupling variable is within a specified tolerance range (interval). The advantage of such an approach is that it does not require to assume an hypothetical distribution for the uncertain variables. However, tolerance ranges for the coupling variables and acceptable variation range of objectives should be determined by the decision maker. These tolerance ranges have an important impact on the ability for the problem solution to converge to an optimum.

Hu et al. (2013) proposed an approximation assisted multi-objective collaborative robust optimization under interval uncertainty by using a single-objective optimization problem to coordinate all system and subsystem multi-objective optimization problems with a CO framework. The developed formulation converts the consistency constraints of CO into penalty terms which are integrated into the system and subsystem objective functions. The upper-level problem coordinates the shared and coupling variables and guides the lower-level problems, while a system problem in the lower-level attempts to find the optimum design solutions. Uncertainties using interval formalism are considered and an interdisciplinary uncertainty propagation technique is used to quantify the impact on the objective and constraints. The robustness evaluation is assessed at the lower-level using a worst-case robustness evaluation approach.

Xu et al. (2017) developed a non-probabilistic CO formulation to deal with bounded correlated uncertain variables (e.g., using interval formalism). The authors proposed a model of correlated uncertainties based on the ellipsoidal model (Ball et al. 1997) and interval theory. The proposed approach consists in constructing the correlated uncertainty models and incorporate their effects in MDO through the addition of constraints in the CO framework.

Yang et al. (2018) developed an UMDO approach relying on Gaussian processes to account for mixed aleatory and interval uncertainty variables. Gaussian processes are built using MDA (solved with FPI) to replace the computationally intensive constraints. MDF under uncertainty and robustness-based CO are carried out to solve such a problem. CMC using Gaussian processes is used to propagate the uncertainty. An optimization algorithm is used to determine the bounds of the reliability index using a derivation of FORM to account for interval uncertain variables.

Wang et al. (2018) derived a sequential multidisciplinary design optimization with uncertainty described using intervals. To propagate the uncertainty, a dimension-by-dimension method (DDM) is adopted (Wang et al. 2018). DDM relies on orthogonal polynomial to fit the function between the system outputs and the uncertain variables dimension by dimension. SORA is carried out to decouple the reliability analysis and the deterministic MDO. In the deterministic MDO, a shifting distance is used to shift the deterministic constraint to take into account the uncertainty. The shifting distance is determined based on interval analysis. The deterministic MDO problem is solved with MDF.

Zaman and Mahadevan (2017) proposed MDO formulations in the presence of both aleatory and epistemic uncertainty (expressed through the interval formalism). The proposed formulations deal with data uncertainty for random variables arising from interval data. Both types of uncertainties are treated with a unified probabilistic formalism. The use of a four-parameter flexible Johnson family distributions to represent the interval uncertainty enables the data to be fitted with different distribution function shapes and removes fixed distribution type assumption.

Several UMDO approaches focus on evidence theory to model epistemic uncertainty. Agarwal et al. (2004b) proposed an evidence theory-based approach to multidisciplinary RBDO in the presence of only epistemic uncertainty using continuous response surfaces for belief and plausibility functions and a sequential approximate optimization approach. Evidence theory is used to model the uncertainty arising due to incomplete information or the lack of knowledge. As the belief functions are discontinuous, a sequential approximate optimization strategy is carried out using simplified trust region to drive the design optimization. Multidisciplinary system is handled using a coupled approach.

Yao et al. (2013) proposed a MDO formulation under both aleatory and epistemic uncertainties using probability and evidence theory formalisms. The methodology consists of an extension of the deterministic multi-level MDO procedure MDF-CSSO (Yao et al. 2012), where a sequential optimization and mixed uncertainty analysis (SOMUA) algorithm is used to decompose the traditional double-level reliability-based optimization problem into separate deterministic optimization and mixed uncertainty analysis sub-problems, which are iteratively solved until the convergence is achieved. SOMUA generalizes SORA by relying on FORM-UUA (FORM-Unified Uncertainty Analysis, see Chapter 4 for more details on FORM-UUA), a generalization of FORM to account for epistemic uncertainty described with evidence theory. In MDF-CSSO, in a first step, surrogate models of the exact disciplinary models are built independently. Then in a second step, based on these surrogate models and an initial baseline, MDF is carried out to quickly identify the promising region and roughly identify the optimum. In a third step, CSSO is used to organize the concurrent disciplinary optimization and the system coordination. The discipline surrogate models are improved during CSSO by adding the feasible candidates into the DoE. Then, based on the CSSO results, a MDA with the exact disciplines is performed and defines the new baseline for the next MDF. This process is repeated until convergence. The efficiency of this formulation depends on the accuracy of surrogate models employed in CSSO. Moreover, the proposed approach transforms the problem into a deterministic one by adding a constraint for each focal element, thus making the approach computationally expensive.

Guoqiang et al. (2018) proposed a derivation of the SORA UMDO strategy to carry out multidisciplinary reliability analysis taking into account both probability and evidence theory formalisms. Using on Bayes theorem and the maximum entropy principle, the probability distribution is assumed as piece-wise uniform in focal elements of evidence theory, and then the mean, variance, PDF, and CDF of epistemic uncertainty variables are defined. For the reliability analysis, a MPP-based method is modified to account for the presence of evidence theory. MDF and BLISS are used to solve the SORA step which involves deterministic MDO problem.

Zhang and Huang (2010) developed a UMDO formulation to deal with mixed aleatory and fuzzy uncertainties. The proposed approach extends SORA to enable epistemic uncertainty description using the possibility theory. At each iteration of SORA, a probability/possibility analysis and a deterministic MDO are carried out. After solving the deterministic MDO, the maximal grade point of each fuzzy design variable and the expected value of each random variable may be determined. Then, probability/possibility analysis is applied to estimate the feasibility of each probability/possibility constraint at the current deterministic MDO design point. To improve the feasibility of constraints which violate the probability/possibility requirements, constraints in deterministic MDO are shifted with the MPPs determined in probability/possibility analysis of the previous cycle.

Within the framework of mixed possibility and probability theories, Li et al. (2013) proposed also an extension of SORA where the reliability analysis and the deterministic MDO use single-level and bi-level MDO strategies. The multi-level MDO process relies on the work performed around the combination of BLISS and CO formulations for deterministic problems (Zhao and Cui 2011) but adapted for the presence of possibility theory. Based on the fuzzy random variables theory and the use of $a-$cut set (see Chapter 2 on fuzzy sets), the initial fuzzy UMDO problem is transformed into a standard probability UMDO problem at the optimal $a-$cut set.

7.8 Summary

The importance of taking into account uncertainty in MDO problems has spread among academia and industry in the last decades; however, UMDO is still in the early stage of development compared to deterministic MDO. In addition to the

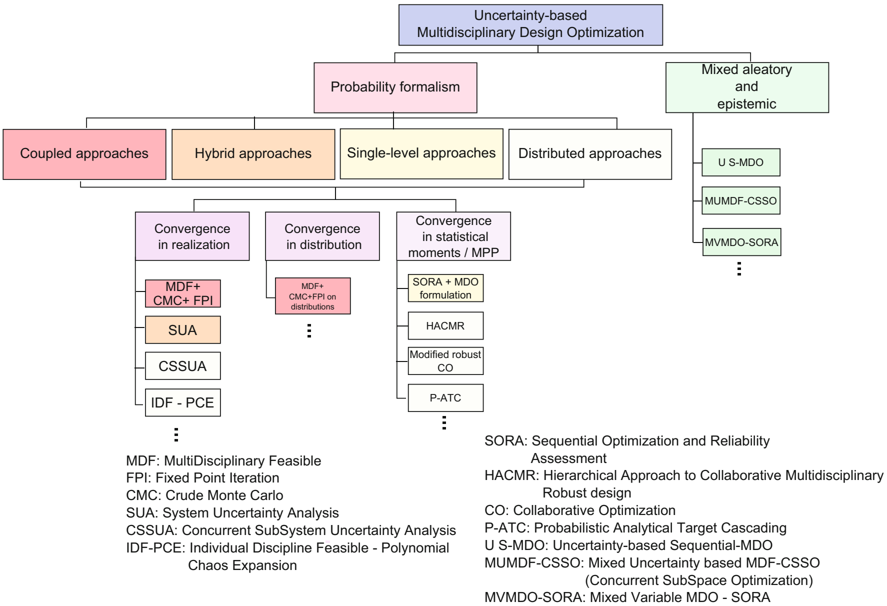


Fig. 7.35 Classification of uncertainty-based multidisciplinary design optimization

challenges of deterministic MDO, the presence of uncertainty in UMDO makes the UMDO problem solving more complex. In this chapter, based on the uncertainty propagation methodologies for multidisciplinary systems presented in Chapter 6, UMDO formulations have been described. The specificity of MDO relies on the management of interdisciplinary couplings which become uncertain variables in the presence of uncertainty. Different strategies for UMDO problem solving have been proposed in the literature and in summary the classification presented in Figure 7.35 may be adopted.

A toy case problem has been used to illustrate the consequences of the presence of uncertainty in MDO and to highlight the advantages and drawbacks of each technique described in the chapter.

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