The Sequence of Carboncettus Octagons

Fabio Caldarola[1(✉)], Gianfranco d'Atri[1], Mario Maiolo[2], and Giuseppe Pirillo[3]

[1] Department of Mathematics and Computer Science, Università della Calabria, Cubo 31/A, 87036 Arcavacata di Rende, CS, Italy
{caldarola,datri}@mat.unical.it

[2] Department of Environmental and Chemical Engineering (DIATIC), Università della Calabria, Cubo 42/B, 87036 Arcavacata di Rende, CS, Italy
mario.maiolo@unical.it

[3] Department of Mathematics and Computer Science U. Dini, University of Florence, viale Morgagni 67/A, 50134 Firenze, Italy
pirillo@math.unifi.it

Abstract. Considering the classic Fibonacci sequence, we present in this paper a geometric sequence attached to it, where the word "geometric" must be understood in a literal sense: for every Fibonacci number F_n we will in fact construct an octagon C_n that we will call the *n-th Carboncettus octagon*, and in this way we obtain a new sequence $\{C_n\}_n$ consisting not of numbers but of geometric objects. The idea of this sequence draws inspiration from far away, and in particular from a portal visible today in the Cathedral of Prato, supposed work of *Carboncettus marmorarius*, and even dating back to the century before that of the writing of the *Liber Abaci* by *Leonardo Pisano* called Fibonacci (AD 1202). It is also very important to note that, if other future evidences will be found in support to the historical effectiveness of a Carboncettus-like construction, this would mean that Fibonacci numbers were known and used well before 1202. After the presentation of the sequence $\{C_n\}_n$, we will give some numerical examples about the metric characteristics of the first few Carboncettus octagons, and we will also begin to discuss some general and peculiar properties of the new sequence.

Keywords: Fibonacci numbers · Golden ratio · Irrational numbers · Isogonal polygons · Plane geometric constructions

1 Introduction

The names here proposed of "n-th Carboncettus octagon" and "Carboncettus sequence/family of octagons", or better, the inspiration for these names, comes from far away, sinking its roots in the early centuries of the late Middle Ages. They are in fact connected to the cathedral of Prato, a jewel of Italian Romanesque architecture, which underwent a profound restructuring in the 11th century, followed by many others afterwards. The side portal shown in

Y. D. Sergeyev and D. E. Kvasov (Eds.): NUMTA 2019, LNCS 11973, pp. 373–380, 2020.
https://doi.org/10.1007/978-3-030-39081-5_32

Fig. 1 (which we will later call simply the *portal*) at the time of its construction seems to have been the main portal of the cathedral. The marble inlays on its sides and the figures represented have aroused many discussions among scholars for many years and in particular have always aroused the attention and interest of G. Pirillo, an interest that he recently transmitted also to the other authors. Pirillo studied the figures of the portal for a long time and traced a fascinating symbolism, typical of medieval culture (see for example [11]). According to these studies, the right part of the portal, for instance, through a series of very regular and symmetrical figures, would recall the divine perfection, while the left part, through figures that approximate the regular ones but are not themselves regular, the imperfection and the limits of human nature. The very interesting fact is that the artist/architect who created the work (which is thought to be a certain *Carboncettus Marmoriarius*, very active at that time and in those places, [11]) seems to have been in part used the mathematical language to express these concepts and ideas, and this thing, if confirmed, would assume enormous importance, because before the 12th century we (and many experts of the field) have no knowledge of similar examples. The construction of the Carboncettus octagon (or better, of the Carboncettus octagons, since they are infinitely many) originates from Fibonacci numbers and yields a sequence not of numbers but of geometrical figures: we will explain the details starting from Sect. 2.

From the historical point of view we cannot avoid to note an interesting, particular coincidence: probably, the most known and most important octagonal monument existing in Calabria dates back to the same period as the construction of the portal of the *Duomo* of Prato, and it is the octagonal tower of the Norman-Swabian Castle in Cosenza. But it is important to specify, for the benefit of the reader, that, in Cosenza, on the site of the actual Norman-Swabian Castle, a fortification had existed from immemorial time, which underwent considerable changes over the years: first a Bruttuan fortress, then Roman, Norman and Swabian, when it had the most important restructuring due to Frederick II of Swabia. In particular, it is Frederick who wanted the octagonal tower visible today, his preferred geometric shape: remember, for example, the octagonal plan of the famous *Castel del Monte* near Bari, in Apulia.

With regard to Fibonacci numbers, we would like to point out to the reader for completeness of information, a recent thesis by G. Pirillo often and many times discussed within this group of authors. In [10,12–14] Pirillo presented the audacious thesis that the first mathematicians who discovered Fibonacci numbers were some members of the Pythagorean School, well documented and active in Crotone in the 6th, 5th and 4th centuries B.C., hence about 1,700 years before that *Leonardo Pisano*, known as *"Fibonacci"*, wrote his famous *Liber Abaci* in 1202. Such a thesis is mainly supported by computational evidences arising from pentagon and pentagram about the well-known Pythagorean discovery of the existence of incommensurable numbers. The interested reader can find further information and references on the Pythagorean School, incommensurable lengths, Fibonacci numbers and some recent developments in [6,8,10,14–17].

Fig. 1. The side portal of the cathedral of Prato. The two topmost figures have octagonal shape: the one on the right is based on a regular octagon, while the one on the left seems to allude to a very particular construction that inspires thus paper and the now called *Carboncettus octagons*.

Similarly to the above thesis note that, since the portal in Prato is dating back to the 12th century, if other future evidences will support the employ of Fibonacci numbers in its geometries, this would mean that they were known before 1202 as well, even if only a few decades.

A final remark on notations: we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$. A sequence of numbers or other mathematical objects is denoted by $\{a_n\}_{n\in\mathbb{N}}$, $\{a_n\}_n$, or simply $\{a_n\}$. If, moreover, A, B, C are three points of the plane, AB denotes the line segment with endpoints A and B, $|AB|$ its length, and $\measuredangle ABC$ the measure of the angle with vertex in B.

2 The Carboncettus Family of Octagons

If r is any positive real number, we denote by Γ_r the circumference of radius r centered in the origin. As usual, let F_n be the n-th Fibonacci number for all $n \in \mathbb{N}_0$, i.e.,

$$F_0 = 0, \quad F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3, \quad F_5 = 5, \quad \text{etc.}$$

If $n \in \mathbb{N}$ we consider a couple of concentric circumferences having radii of length F_n and F_{n+2}, respectively. If $n = 1$ they are represented in green in Fig. 2, were the radius of the inner circumference is 1 and that of the outer one is 2, i.e. F_3. Then we draw two couples of parallel tangents, orthogonal between them, to the inner circumference and we consider the eight intersection points A, B, C, D, E, F, G, L with the outer circumference $\Gamma_{F_{n+2}}$, as in Fig. 2. The octagon obtained by drawing the polygonal through the points $A, B, C, D, E, F, G, L, A$, in red in Fig. 2, is called the *n-th Carboncettus octagon* and is denoted by C_n. Therefore, the red octagon in Fig. 2, is the *first Carboncettus octagon* C_1.

From a geometrical point of view, the Carboncettus octagon C_n is more than a cyclic polygon; it is in fact an *isogonal octagon* for all $n \in \mathbb{N}$, that is, an equiangular octagon with two alternating edge lengths.[1] More recently it is also used to say a *vertex-transitive* octagon: all the vertices are equivalent under the symmetry group of the figure and, in the case of C_n, for every couple of vertices, the symmetry which send the first in the second is unique. The symmetry group of C_n is in fact isomorphic to the one of the square, the dihedral group D_4.[2]

An interesting property of the Carboncettus sequence $\{C_n\}_{n\in\mathbb{N}}$ is the fact that, with the exception of the first three elements C_1, C_2, C_3 (or, at most, also C_4), all the subsequent ones are completely indistinguishable from a regular octagon (see, for example, Fig. 3 representing C_2: it is yet relatively close to a regular octagon). Due to the lack of space, we will deepen these and other important aspects mentioned in the following, in a subsequent paper in preparation.

[1] In this view, a recent result established that a cyclic polygon is equiangular if and only if is isogonal (see [7]). Of course, an equiangular octagon is not cyclic in general, while it is true for 3- and 4-gons (see [2]).

[2] Note, for didactic purposes, how the multiplication table of D_4 emerges much more clearly to the mind of a student thinking to C_1 than thinking to a square.

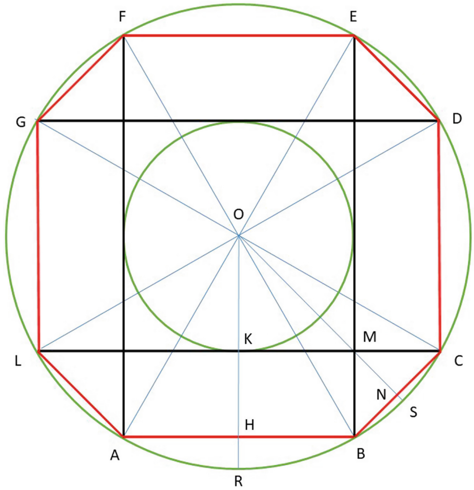

Fig. 2. The construction of the Carboncettus octagon. In the picture, in particular, it is shown in red the octagon C_1. (Color figure online)

3 The First Four Octagons of the Carboncettus Sequence: Geometric Properties and Metric Data

In this section we will give some numerical examples looking closely at the first elements of the sequence $\{C_n\}_{n\in\mathbb{N}}$.

Example 1 (The octagon C_1). The first Carboncettus octagon C_1 is built starting from the circumferences Γ_1 and Γ_2, as said in Sect. 2. In this case we obtain a very particular isogonal octagon: drawing the eight radii

$$OA,\ OB,\ OC,\ OD,\ OE,\ OF,\ OG,\ OL \tag{1}$$

of the circumference Γ_2 as in Fig. 2, the resulting shape has commensurable angle measures, in fact all them are integer multiples of $\pi/12 = 15°$. Not only; in this way C_1 results formed by 4 equilateral triangles (congruent to ABO, see Fig. 2) and 4 isosceles triangles (congruent to BCO). The lengths of their sides and heights are

$$\begin{aligned} |AB| &= |OA| = 2, & |OH| &= |KC| = \sqrt{3}, \\ |BC| &= \sqrt{6} - \sqrt{2}, & |ON| &= \tfrac{\sqrt{6}+\sqrt{2}}{2}, \end{aligned} \tag{2}$$

which, for example, are all incommensurable in pairs. Instead, for the widths of the angles we trivially have

$$\begin{gathered} \measuredangle AOB = \measuredangle OBA = \pi/3 = 60°, \qquad \measuredangle BOC = \measuredangle HOB = \pi/6 = 30°, \\ \measuredangle OBC = 5\pi/12 = 75°. \end{gathered} \tag{3}$$

Discussing the commensurability of the angles for all the sequence $\{C_n\}_n$ is interesting, but we are forced to postpone this elsewhere. The same, as well, considering the commensurability, along all the sequence, of some of the side lengths made explicit in (2). Note lastly that perimeter and area are

$$\text{Per}(C_1) = 8 + 4\sqrt{6} - 4\sqrt{2}, \qquad \text{Area}(C_1) = 4 + 4\sqrt{3}.$$

The second Carboncettus octagon C_2 originates from the circumferences Γ_1 and Γ_3, with radii $F_2 = 1$ and $F_4 = 3$, respectively, and the result is the black octagon in Fig. 3, compared with a red regular one inscribed in the circumference Γ_3 itself. Using the letters disposition of Fig. 2, the lengths of the correspondent sides and heights considered in (2), the angle widths, perimeter and area, are those listed in the second column of Table 1.

Table 1. Some metric data relative to the first three elements of the Carboncettus sequence, after C_1. The letters are displayed in the construction as in Fig. 2.

	C_2	C_3	C_4
$\|OK\|$	1	2	3
$\|OA\|$	3	5	8
$\|AB\|$	2	4	6
$\|BC\|$	$4 - \sqrt{2}$	$\sqrt{42} - 2\sqrt{2}$	$\sqrt{110} - 3\sqrt{2}$
$\|OH\|$	$2\sqrt{2}$	$\sqrt{21}$	$\sqrt{55}$
$\|ON\|$	$2 + \sqrt{2}/2$	$\sqrt{2} + \sqrt{42}/2$	$(3\sqrt{2} + \sqrt{110})/2$
$\measuredangle AOB$	$\approx 38.942°$	$\approx 47.156°$	$\approx 44.049°$
$\measuredangle BOC$	$\approx 51.058°$	$\approx 42.844°$	$\approx 45.951°$
$\measuredangle OAB$	$\approx 70.529°$	$\approx 66.421°$	$\approx 67.976°$
Perim.	$24 - 4\sqrt{2}$	$16 + 4\sqrt{42} - 8\sqrt{2}$	$24 + 4\sqrt{110} - 12\sqrt{2}$
Area	$14 + 8\sqrt{2}$	$34 + 8\sqrt{21}$	$92 + 12\sqrt{55}$

4 The "limit octagon" and Future Researches

Many aspects of the new sequence $\{C_n\}_n$ are interesting to investigate. For example, scaling the octagon C_n by a factor equal to the n-th Fibonacci number F_n, the sequence will converge to a limit octagon C_∞^N (where the top N stands for "normalized") that can be drawn through the "Carboncettus construction" described at the beginning of Sect. 2, by starting from the circumferences with radii given by the following limit ratios

$$\lim_{n\to\infty} \frac{F_n}{F_n} = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{F_{n+2}}{F_n}, \tag{4}$$

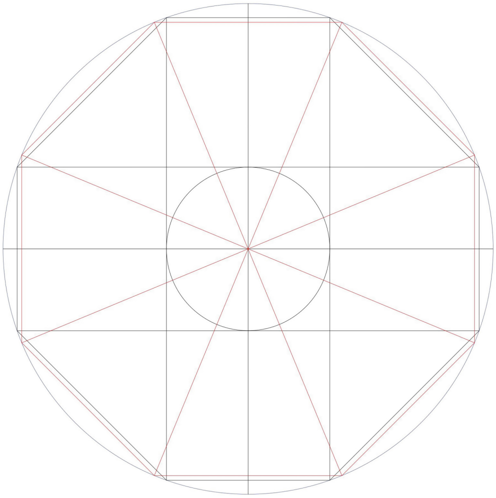

Fig. 3. The second element of the Carboncettus sequence, the octagon C_2, is drawn in black. A regular octagon inscribed in the same circumference $\varGamma_3$, is also represented in red. (Color figure online)

respectively. It is well known that the limit of the ratio of two consecutive Fibonacci numbers F_{n+1}/F_n converges to the *golden ratio*

$$\phi := (1 + \sqrt{5})/2 \approx 1.618033987, \tag{5}$$

hence, the second limit in (4) is simple to compute as follows[3]

$$\lim_{n\to\infty} \frac{F_{n+2}}{F_n} = \lim_{n\to\infty} \frac{F_{n+2}}{F_{n+1}} \cdot \frac{F_{n+1}}{F_n} = \phi^2 \approx 2.618033987, \tag{6}$$

and we conclude that C_∞^N can be constructed using the circumferences $\varGamma_1$ and $\varGamma_{\phi^2}$.

Another approach to directly study the "limit octagon" C_∞ instead of the "limit normalized octagon" C_∞^N, could come by using the computational system introduced for example in [18–20] and applied as well to limit curves, limit polytopes, fractals and similar geometric shapes in [1,3–5,20] (or even to Fibonacci numbers in [9]).

Aknowledgments. This work is partially supported by the research projects *"IoT&B, Internet of Things and Blockchain"*, CUP J48C17000230006, POR Calabria FESR-FSE 2014–2020.

[3] The reader certainly remembers the well know property $\phi^2 = 1 + \phi$ of the golden ratio that causes the coincidence of the fractional parts of (5) and (6).

References

1. Antoniotti, L, Caldarola, F., Maiolo, M.: Infinite numerical computing applied to Hilbert's, Peano's, and Moore's curves. Mediterr. J. Math. (in press)
2. Ball, D.: Equiangular polygons. Math. Gaz. **86**(507), 396–407 (2002)
3. Caldarola, F.: The Sierpiński curve viewed by numerical computations with infinities and infinitesimals. Appl. Math. Comput. **318**, 321–328 (2018). https://doi.org/10.1016/j.amc.2017.06.024
4. Caldarola, F.: The exact measures of the Sierpiński d-dimensional tetrahedron in connection with a Diophantine nonlinear system. Commun. Nonlin. Sci. Numer. Simul. **63**, 228–238 (2018). https://doi.org/10.1016/j.cnsns.2018.02.026
5. Caldarola, F., Maiolo, M., Solferino, V.: A new approach to the Z-transform through infinite computation. Commun. Nonlin. Sci. Numer. Simul. **82**, 105019 (2020). https://doi.org/10.1016/j.cnsns.2019.105019
6. Caldarola, F., Cortese, D., d'Atri, G., Maiolo, M.: Paradoxes of the infinite and ontological dilemmas between ancient philosophy and modern mathematical solutions. In: Sergeyev, Y., Kvasov, D. (eds.) NUMTA 2019. LNCS, vol. 11973, pp. 358–372. Springer, New York (2020)
7. De Villiers, M.: Equiangular cyclic and equilateral circumscribed polygons. Math. Gaz. **95**, 102–107 (2011)
8. Koshy, T.: Fibonacci and Lucas Numbers with Applications. Wiley, New York (2001)
9. Margenstern, M.: Fibonacci words, hyperbolic tilings and grossone. Commun. Nonlin. Sci. Numer. Simul. **21**(1–3), 3–11 (2015)
10. Pirillo, G.: A characterization of Fibonacci numbers. Chebyshevskii Sbornik **19**(2), 259–271 (2018)
11. Pirillo, G.: Figure geometriche su un portale del Duomo di Prato. Prato Storia e Arte **121**, 7–16 (2017). (in Italian)
12. Pirillo, G.: La scuola pitagorica ed i numeri di Fibonacci. Archimede **2**, 66–71 (2017). (in Italian)
13. Pirillo, G.: L'origine pitagorica dei numeri di Fibonacci. Periodico di Matematiche **9**(2), 99–103 (2017). (in Italian)
14. Pirillo, G.: Some recent results of Fibonacci numbers, Fibonacci words and Sturmian words. Southeast Asian Bull. Math. **43**(2), 273–286 (2019)
15. Pirillo, G.: Fibonacci numbers and words. Discret. Math. **173**(1–3), 197–207 (1997)
16. Pirillo, G.: Inequalities characterizing standard Sturmian and episturmian words. Theoret. Comput. Sci. **341**(1–3), 276–292 (2005)
17. Pirillo, G.: Numeri irrazionali e segmenti incommensurabili. Nuova Secondaria **7**, 87–91 (2005). (in Italian)
18. Sergeyev, Y.D.: Arithmetic of Infinity. Edizioni Orizzonti Meridionali, Cosenza (2003)
19. Sergeyev, Y.D.: Lagrange lecture: methodology of numerical computations with infinities and infinitesimals. Rend. Semin. Matematico Univ. Polit. Torino **68**(2), 95–113 (2010)
20. Sergeyev, Y.D.: Numerical infinities and infinitesimals: methodology, applications, and repercussions on two Hilbert problems. EMS Surv. Math. Sci. **4**(2), 219–320 (2017)