Chapter 2
Traditions and Changes in the Teaching and Learning of Mathematics in Germany

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Abstract This chapter investigates different effects on mathematics teaching in the Western and Eastern part of Germany after the fall of the Berlin Wall in 1989 and the reunification of the two German states. Striking developments and discussions on mathematics teaching are analyzed in their historical context, which is mainly affected by the different design of mathematics education and mathematics teacher training in the two German states [Federal Republic of Germany (FRG) and German Democratic Republic (GDR)]. Recurrent pendulum swing patterns can be observed in both the thematic emphases of mathematics teaching and the preferred approaches to teaching mathematical content. The aim of this chapter is to reproduce, in as fact-based a manner as possible, the recent history of German mathematics education, which is regarded as part of German social developments as a whole.

Keywords Mathematics education · German history · Cultural changes

1 Introduction

General mathematics teaching in Germany has an eventful history in terms of content and methodological orientation. These aspects were influenced by far-reaching historical events and social changes. After the Second World War (1945) Germany was divided into different occupation zones, see Fig. 2.1. In 1949 there was an administrative separation between the western part of Germany, named the Federal Republic of Germany (FRG) that contained the American, French, and British zones and the eastern part composed of the Soviet occupation zone, named the German Democratic Republic (GDR). The Western Allies (USA, Great Britain, and France) supported the construction of a Western democracy in the FRG. In the GDR, a

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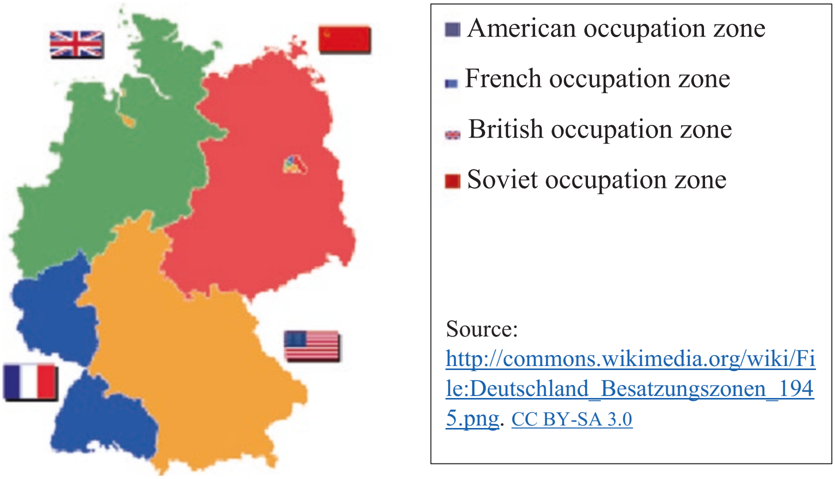

Fig. 2.1 Occupation zones of Germany 1945

communist system similar to Poland, Hungary, and Czechoslovakia was established under Soviet influence.

In the FRG the individual federal states had educational sovereignty and were able to decide on the school types, timetables, and curricula themselves. Coordination was carried out by the Standing Conference of the Ministers of Education and Cultural Affairs of the States in the Federal Republic of Germany, known as the Kultusministerkonferenz (KMK). The KMK was a voluntary association and dealt with issues of cultural policy of superregional importance with the aim of forming a common opinion and will and the representation of common concerns (https://www.kmk.org/kmk/information-in-english.html).

In the GDR, a centralized state, a uniform educational system was established. In this system all children from the first grade up to the end of their eighth school year were taught together (Polytechnische Oberschule). In the Soviet occupation zone, professorships and courses in the “Methodology of Mathematics Teaching” were awarded and established as early as 1946 (Borneleit 2006). For more details on the different developments in education and mathematics education in divided Germany between 1945 and 1990, see works by Borneleit (2003), Bruder et al. (2013), Einsiedler (2015), Henning and Bender (2003), Jahnke et al. (2017), Neigenfind (1970), Schubring (2014), Wuschke (2018), and Zabel (2009) and the brief overview in Sect. 3.

The separation between the East and West of Germany manifested in 1961 with the construction of a massive border wall, which also ran through the capital, Berlin. This wall fell in 1989 as the result of a peaceful revolution in the GDR and led in 1990 to the reunification of Germany with the accession of the GDR by the FRG, see Fig. 2.2.

As a result of the rapid adaptation of the structures of the new federal states (the former GDR) to the educational system of the old federal states in 1990, there were

Fig. 2.2 The German federal states since reunification in 1991 and the neighboring countries of Germany. (Source: Panther Media GmbH/Alamy Stock Photo, JB50N7)

major changes to the framework for mathematics education and to the entire environment of education and teacher training in the new federal states (see also Sect. 4). However, educational leaders of the old federal states saw no need to make changes to their own mathematics education and teacher training.

The public perception and discussion about mathematics education in the entire country changed only a few years after reunification due to the weak German scores in some international studies, Trends in International Mathematics and Science Study (TIMSS) and Program for International Student Assessment (PISA).

> The participation in PISA was decided by the Conference of Education Ministers (Kultusministerkonferenz) in 1997 and then commissioned. …And so, with PISA 2000, a continuous performance evaluation of the German school system began through an international comparison, and the performance comparison of the federal states began. (Tillmann 2004, p. 478)[1]

This so-called "empirical turn" had a strong influence on the general conditions and the orientation of mathematics teaching and didactic research in united Germany. Not only were mathematical comparison tests launched for the diagnosis of basic competencies, but also large-scale projects for empirical educational research, such as the National Education Panel and technology-based testing (Buchhaas-Birkholz 2010). Output and competence orientation formed the main thread of these projects (for more details see Sect. 4).

This chapter will focus on the changes in German mathematics teaching after the fall of the Berlin Wall in 1989 and the reunification of the two German states.

Over the last 30 years there have been very few studies or publications dealing with the effects of the reunification of Germany on the teaching and learning of mathematics in the old and the new federal states (former GDR). The nature of the difficulty of conducting such studies is similar to what the Eastern German sociologist Hansgünter Meyer described 1 week after the GDR's accession to the FRG in his discipline:

> Whoever deals with the problem… is put into the role of a chronicler who speaks about a fact without historical-systematic processing, i.e. is very dependent on personal points of view. … The sociology that originated in the GDR still exists, the GDR no longer does. (Meyer 1991, p. 69)

These issues can be better understood and classified if differences in the social framework conditions and factors influencing mathematics teaching in both parts of Germany after 1945 are taken into account. These differences and factors are discussed in Sect. 3.

The influencing factors apply both to mathematics didactic research in the former GDR and to mathematics teaching itself, which was shaped by the specific socialization of teachers, pupils, and their parents in the new federal states. Some aspects of the current interior and exterior views of German mathematics teaching since 1990 are described in Sect. 4.

[1] All translations from German are by the author.

Mathematics didactic research results from the GDR period are briefly dealt with in Sect. 5. At the beginning of the 1990s there were many representatives from the old federal states, who were interested in a scientific exchange and the development of mathematics teaching and practice-oriented didactic research in the former GDR, according to the reports of the joint meetings in 1990 and 1996 in Henning and Bender (2003).

Sometimes, however, decisions on educational policy are also in reaction to developments in scientific discourse, even if they are conducted publicly. The eventful history of paradigms in educational science has always been heavily influenced by developments in the field of mathematics (let us recall New Math) and social changes.

Here, for example, interesting recurring patterns can be seen in the thematic emphases in mathematics teaching and the preferred approaches to learning mathematical content. In the next section, these patterns are described as pendulum movements. This section is positioned ahead of following sections, to provide a framework.

2 Pendulum Movements in the Main Focuses of Mathematics Teaching Since the Nineteenth Century—Using the Example of the German-Speaking World

The specialization of mathematics in the nineteenth century and the lack of practical relevance in pure mathematics prompted the establishment of technical colleges and the conception of "engineering mathematics." The development of mathematics education led, in turn, to the creation of "school mathematics." This area of teaching methodology was then named "Stoffdidaktik"—literally "subject matter didactics," a concept that combines mathematical theory, its application in real-world situations and the student's pedagogical and psychological state of development. Stoffdidaktik concentrated on the mathematical content of the subject matter to be taught (Sträßer 2019). It attempted to be as close as possible to the discipline of mathematics. A major aim was to make mathematics accessible and understandable to the learner:

> In the development of the didactics of mathematics as a professional field in Germany, subject-related approaches played an important role. Felix Klein created a model that has been referred to for a long time. A general goal was to develop approaches for representing mathematical concepts and knowledge in a way that corresponded to the cognitive abilities and personal experiences of the students while simultaneously simplifying the material without disturbing the mathematical substance. A fundamental claim was that such simplifications should be "intellectually honest" and "upwardly compatible" (Kirsch 1977). (as cited in Jahnke et al. 2017, p. 307)

Until the 1980s, content orientations in mathematics education were closely linked to developments in mathematics itself.

Picker (1991) discusses the effects of New Mathematics on mathematics lessons in elementary school and shows an interesting phenomenon in the choice of learning content, which dated back to the sixteenth century. He describes the following dispute about the right way of teaching lessons on calculating:

> Visual methodology or counting methodology …?
> Quantity handling … or series formation …?
> Cardinal number … or ordinal number …?
>
> This dualism is as old as Plato … and Aristotle …
> They form no contrast, but are complementary mutual as are material and formal education. (Picker 1991, p. 335)

While Adam Riese advocated for material education around 1522, Pestalozzi advocated for formal education in 1801. In 1814, Harnisch called for a combination of material and formal education in schools.

In 1838, Diesterweg favored the viewing and recognition of point patterns without counting in primary school arithmetic lessons. While in 1888, Hartmann explicitly referred to counting as an intuitive access to quantity concepts. Kühnel (1916) recommended in his proposals for the "new building of the arithmetic lesson" to allow both: looking and counting.

While Wittmann, in 1929, proposed the concept of cardinal numbers and a set theoretical basis for initial instruction, Breidenbach, in 1947, advocated starting with ordinal numbers on the basis of children's first experience of counting off numbers. Here, too, 20 years later, a solomonic solution was found: Fricke recommended that both number aspects be taken into account from the outset: cardinal number and ordinal number.

The challenge behind these examples lies in a balanced use of the findings and proposals from personalities and various interest groups in educational policy. This applies to the selection of learning content and the associated objectives (e.g., formal and material education in the past and later around 1970 in the opposition between learning to think and learning to calculate). A balanced approach is also key to understanding different professional approaches and to partially contrary methodical ways of learning selected learning contents. Schneider writes:

> Between the poles of divinity and usefulness has always fluctuated the occupation with mathematics and thus also the teaching of arithmetic, space theory and mathematics. In the process, there have been and continue to be clubs of opinions up to the present day. (Schneider 1989, p. 7)

The history of alternative approaches to mathematical topics of (German) math lessons are characterized by pendulum swings.

The search for a better mathematical approach to subjects in schools played a central role in the discussion about mathematics education until the 1990s and repeatedly led to partially contradictory curricular changes. One could ask why these oscillations were not recognized early and why educators did not learn from them, but that is just a rhetorical question. A dialectical approach using multiple perspectives and taking into consideration the advantages and disadvantages of exposed technical approaches in regard to a particular goal or a particular method could (if done in conjunction with empirical evidence) perhaps have avoided extreme pendulum swings. Schneider, however, also gave some perspective:

> Are we questioning history to find answers for us today or are we just looking to history for confirmation of our current views? We can always find something there. The aim is probably only to track down progressive tendencies and to justify why they brought progress, and also, in the case of regressive views, to prove why they had an inhibitory effect, whatever their advocates subjectively wanted. (Schneider 1989, p. 4)

At the same time, teaching methodology in mathematics is still a very young academic discipline that is rooted in the field of mathematics. The new generation of professors who filled the few available places for teacher education in colleges and universities were mainly mathematicians—not all of whom had teaching experience in schools. The science of "teaching methodology"—known as *Fachdidaktik* in the western part of Germany and as *Fachmethodik* in the GDR—has only existed as an independent academic area since the 1960s. Teaching methodology in mathematics in the German-speaking world has its origins in the development of mathematics schooling. And for *Stoffdidaktik*, new challenges arose again. Jahnke et al. (2017) wrote about the resulting challenges to *Stoffdidaktik* and its current development:

> Concepts and explanations should be taught to students with sufficient mathematical rigor in a manner that connects with and expands their knowledge of the subject. For this reason, subject-matter didactics placed value on constructing viable and robust mental representations (Grundvorstellungen) to capture mathematical concepts and procedures as they are represented in the mental realm. In the 80s, views of the nature of learning as well as objects and methods of research in mathematics education changed and the perspective was widened and opened towards new directions and gave more attention to the learners' perspective. This shift of view issued new challenges to subject-related considerations that have been enhanced by the recent discussion about professional mathematical knowledge for teaching. (Jahnke et al. 2017, p. 307)

But the pattern of pendulum movements was repeated. Around the 1970s and 1980s, more attention was paid to the students' needs, own goals, and individual difficulties. That was a significant development, especially in the FRG. In the GDR, the centrally set goals for mathematics lessons were not up for debate, so teaching focused more on individual support, both to overcome deficits and to support endowments.

The growing interest in researching individual learning processes in the FRG had to do with various social developments. In addition to questions and new answers about the content of mathematics teaching (e.g., New Math and problem-solving), more questions were asked about the ways that classes were organized. The Sputnik shock and the 1968 movement spurred many of these questions (Schubring 2014, 2016) and led to the propagation of antiauthoritarian education. The institutionalized education of mathematics teachers at colleges and universities was influenced by developments in mathematics as well as education in both German states.

Parallel to the social developments that gave the individual and his or her needs more space, constructivist notions of learning received great attention and replaced previous behaviorist concepts. However, there were differences between developments in the FRG and in the GDR until 1990. In the GDR, the 1968 movement had little influence on schools, which were controlled by the state; as a result, the pendulum swing toward antiauthoritarian education was much less pronounced and behavioral ideas about learning were not thrown overboard so quickly. For example, in the new federal states the repetition of basic knowledge and basic skills has always been a common practice. So-called "daily exercises" or as we say now "mixed mental exercises" are very well suited to keep the basics alive. Such methods were hardly known in the FRG.

Some of the characteristic features (not only as a goal, but visible in the reality) of math classes in the eastern part of Germany were:

- application of mathematical knowledge
- close connection of mathematics with other subjects
- discovering theorems and proving them.

Of course, there was still great potential for development, and these aspects were subjects of teacher training. In the GDR, colleges of education and universities were solely responsible for teacher training, including practical phases. School-practical studies accompanied academic study.

In the old federal states, a two-phase teacher education was favored with a traineeship run by the state school administration after students finished university studies. However, there was also a pilot project for single-phase teacher training in Oldenburg and Osnabrück, which was comparable to the model in the GDR (Daxner et al. 1979). This model was not pursued for political reasons. The two-phase teacher education prevailed. There was no time for a discussion of alternatives. At the time of the accession of the GDR by the FRG, all structures of the old federal states were introduced in the east without examination.

The concept of single-phase teacher education was introduced at the same time, and a nationwide dispute over the introduction of comprehensive schooling began. The question was whether the schools should be differentiated—by age or by school type. The debate was also an example of the abovementioned pendulum swings. Now in united Germany there is a certain variety of schools, but, for the most part, only the names in the different federal states differ, not the concepts. The question of the "best" type of school and school system continues to smolder in the background and repeatedly comes up in political discussions.

At several universities and colleges for mathematics teacher training in the GDR, the idea of learning was shaped by the theory of social-constructivist-oriented activity as described by Vygotsky (1978), Luria, Davydow and Galperin—and later Kossakowski and Lompscher (Lompscher 1985; Giest and Lompscher 2006). After that, learners were understood as subjects rather than objects of instruction. However, the idea of the subject position of the learners did not lead to a methodological pluralism in the teaching of mathematics. In the teacher education a moderate constructivist approach to learning was associated with the idea that underdefined conditions there would be an optimal way to design a learning unit for all students so that everyone could benefit from it. Forms of internal differentiation were part of such concepts so as to overcome and avoid failures.

In the old federal states, constructivist learning concepts had to have many facets. Support was given to a constructivist view of learning through many qualitative analyses of the mathematical learning processes (Bruder et al. 2013; Schreiber et al. 2015). While mathematics teachers in the old federal states could learn to be aware of individual student difficulties with mathematics, the teachers in the GDR received more practical methodical solutions for dealing with the entire learning group. In the FRG curricula learning goals and content were given, but not any methods or concepts for mediation. The teachers had more freedom to design their lessons and less supervision by the school inspectorate than in the former GDR. How the teachers used these free spaces is another question. Of course, any description will be incomplete. It can only be a question of indicating general trends and not describing the situation of individual teachers or students.

Currently, constructivism is establishing individualized learning environments and discovery learning is regarded as the best instruction design. This current trend is visible in the expectations and requirements for the teaching of mathematics in the second phase of teacher education (traineeship) and in the questionnaires used in school inspection. Most of the federal states have school inspection now, but not as often, as strictly or as ideologically based as in the GDR. Inspections focus more on the exclusion of allegedly obsolete theoretical ideas of learning. Hamburg, for example, assessed the following for teaching, which clearly describes the expectations:

> The lesson observations should be about recording the quality of the lesson, not the assessment of the teaching teacher. When recording the quality, the superficial quality is meant. "Professionalism cannot play a role in recording the lesson with the lesson observation sheet." (Leist et al. 2016)
>
> Item 15 of the lesson observation sheet:

The students are encouraged to actively participate in the lessons, or they actively participate in the lessons The students are given responsibility to actively participate in the lessons. The methodology is specified by the teacher, the content, however, determines the students	Core indicators • Students are responsible for the learning process of their classmates and the learning content, whereas the methodology is given in the form of cooperative learning forms by the teacher • The students change from a learning role to a learning facilitator role and practice their own teaching functions, such as communicating, supporting, and securing results
	Additional indicators • The teacher acts as a role model and conducts an activity related to the class during the student-centered lesson time. The teacher gives, for example, "tutoring" for students, which are weaker, corrects tests, gives private lessons for students who have missed something, etc. • The teacher refers to inquiries and requests for help of students first to classmates before they intervene • There is neither teacher-centered teaching nor individual work

Such concerns led to a strong pendulum swing of school practice in the direction of expected instructional design in the classroom, ignoring empirical knowledge about the conditions for successful discovery learning or inquiry-based learning (Hattie 2009; Bruder and Prescott 2013). It is well known from the research on teaching that enacting different learning goals effectively requires different methodological implementation (Weinert 1999; Bruder and Roth 2017).

The structural changes in the educational system combined with new instructional design had a lot of consequences, for example, for the kind and the understanding of kindergarten care as well as for the education of kindergarten teachers, and for the level and kind of learning in primary schools (visible in the new curricula) and also for the special schools with an advanced course of study in the area of mathematics and science. If there were no role models in the old federal states, it was very difficult to import specific GDR structures. In particular, this applies to schools for the gifted, especially in mathematics. A reconciliation between the sometimes opposite practices of the FRG and GDR had no chance in 1990.

Another consequence, founded in constructivist notions of learning, is noticeable today in textbooks, which often only contain tasks and dispense with coherent explanations. Other versions of textbook design are described in Sect. 4.

From these historical developments one can learn to separate but not to exclude different aspects of mathematical content, goals, and methods. A goal for further discussion regarding educational policy could be the linking of different aspects in dialectical consideration. We will come back to this phenomenon when we describe striking discourses and developments in mathematics education since 1990.

3 Different Framework Conditions and Developments in Teaching and Learning Mathematics in Eastern Germany from 1945 to 1990

This section presents aspects of the development of the school system of the former GDR, which are important for the understanding of the changes after 1990. The school system in the FRG after 1945 and its various influencing factors are already well documented in English (Schubring 2014).

The following remarks are a translation by the author of a piece from Porges (2017, p. 217):

> The development of the general school system of the Soviet occupation zone/GDR in retrospect reveals three phases (Köhler 2008). Social change after 1945 necessitated a structural change in the school system, specifically comprehensive denazification and new appointments. …The need for a school reform that would provide educational opportunities for all was expressed by representatives of various parties as early as 1945. Consequently, a new beginning of the school system developed and was called the antifascist democratic school reform. This change led to a law for the democratization of German Schools, which came into force on September 1, 1946. It regulated the objectives of the school reform and the

> tasks of the school administration, and it defined the structure of the democratic unitary school. The law was based on the guiding principles of the uniform structure of the education and training system, the right to education at all educational institutions and the transfer of school affairs to the state. (Lang 1946)

In this context, private schools were facing dissolution, integration, or transformation to prevent "any offside education and segregation in youth education" (Lang 1946, p. 11). Students would attend an 8-year primary school designed as a democratic standard school for all, after which they could proceed to a 3-year vocational school or a 4-year secondary school. This scheme replaced the previous tripartism of the school system (parallel Hauptschule, Realschule and Gymnasium with selection after primary school, which was also the standard structure in the FRG). The aim of primary school teaching was "to overcome traditional popular education and to lead all children into the realm of educational opportunities which, from a proletarian point of view […] were regarded as a privilege of social elites" (Geissler et al. 1996, p. 19). The secondary school consisted of a course system that was divided into new language, old language, and mathematical-scientific branches (Köhler 2008). In preparation for high school, grades seven and eight differed in core and course instruction. The introduction of 10-year schools in 1951 expanded the educational landscape. In this type of school, the instructors taught according to the curricula of the mathematics and science branch of the secondary schools (Köhler 2008). The aim was to shorten secondary education to 2 years and to provide direct access to technical schools. Only 2 years later this experiment was stopped. In 1955, a new form of the 10-year school system was introduced under the name "Mittelschule" (secondary school). The law regarding the socialist development of the school system in the GDR introduced the third structural change in December 1959 and required students to attend school for 10 years instead of eight in order to graduate (Köhler 2008).

A 1965 law on the uniform socialist education required structural changes which resulted in a 10-year general polytechnic secondary school (POS) dividing grades in the lower, one through four, and higher, five through ten. A 4-year extended secondary school (EOS), which led to the Abitur was developed (Rockstuhl 2011). These schools remained in existence until 1981.

Some further differences between GDR mathematics lessons and those in the FRG were:

- Centrally organized examinations: German, Mathematics, Russian, Science
- Systems of subject commissions created in every district and of circles (clubs) in each major subject in every school
- Supervisors for all subjects, regular visits and inspections of math lessons every 2 years at all schools
- Mandatory retraining for all teachers every 5 years in several fields: pedagogy, subject to be taught (say, mathematics), psychology, and philosophy.

For comparison here is an overview of the FRG school system before reunification (see also Fig. 2.3).

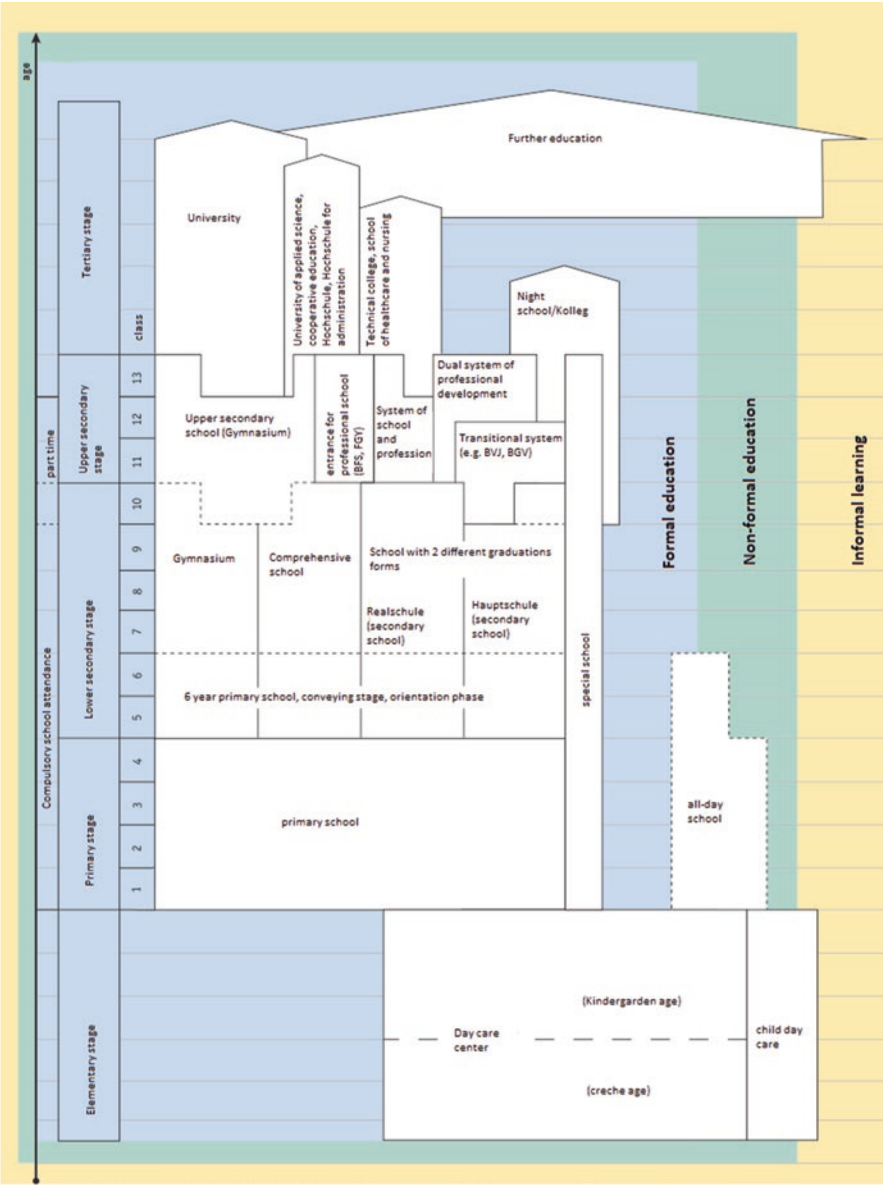

Fig. 2.3 Educational venues and learning worlds in Germany. (Source: Authors' group educational reporting—http://www.bildungsbericht.de/daten2012/vorspann_web2012.pdf)

The main organizational and structural differences between the school systems in East and West Germany before the reunification were as follows:

- Thirteen school years required for graduation in the FRG compared to 12 in the GDR

- Nationwide afternoon offerings and care in the Polytechnische Oberschule (GDR)
- Higher number of lessons in science, technology, engineering, and mathematics (STEM) subjects in the GDR
- Unitary school in the GDR, which enabled more than 8 years of joint learning
- Systematic talent promotion in the GDR with some compulsory participation in the first stage of the Mathematical Olympiad.

A collection of documents on regulations for teacher training in the GDR can be found in Richter (1972). Brislinger et al. (1998) published social science data from the GDR and the new federal states 1968–1996.

Porges (2017, p. 223) writes:

> In line with the curricula, teaching aids for teachers and textbooks for pupils were provided. Both were considered planning aids. Founded in 1945, the *Volk und Wissen* publishing house published all school textbooks without competition. In 1945 there was already a central picture and teaching material office for teaching materials. From 1948 onwards, the State Office for Teaching Materials and School Furniture acted as a sales organisation. In addition, the German Television Broadcasting Service (DFF) began broadcasting school television in 1964. In the year 1971/72 the total demand plan for teaching aids were published with the aim of creating unity between demand plan, curriculum, textbook and teaching aids.

Wuschke (2018) describes the development of the content of mathematics lessons in the Soviet occupation zone and early GDR from 1945 to 1959. Although there are no systematic overview studies for the later years up to 1990, reports on the school system and mathematics teaching in the GDR (Birnbaum 2003), on the conditions for school and science policy (Weber 2003) and on contributions to individual areas of mathematics teaching are available from the perspectives of those involved. These include a contribution by Sill (2018) to the didactics of geometry teaching and a contribution by Borneleit (2003) to curriculum and textbook development.

In fact, there have been some significant differences in the orientation of research in mathematics education. By the end of the 1980s in the FRG, interpretative teaching research had already become accepted as the prevailing method for study in the field. The preferred method was the case study and the explorative “small scale study,” and interpretative methods were predominant. In a fundamental contribution to the development of the didactics of mathematics, Griesel (1975) describes the “development of practicable courses” as its most important task.

Some examples of performance assessments from 1990 until 2004 are comparative studies in German and English on mathematics teaching. Performance tests were also conducted in some cases.

Performance assessments of a single class or school or something larger always played a major role in the history of mathematics teaching in the GDR. The social system of the GDR defined itself as a performance-oriented society. School was to lay the foundations for a high level of performance and commitment from every citizen.

To inspect the performance of a school, a “comprehensive, discriminating system of centralised performance controls, analyses and control mechanisms, from giving grades, tests, centralised and local in-school extracurricular performance comparisons was created" (Döbert and Geißler 2000).

The teachers' performance was seen as the main factor in students' performance. Consequently, measures to improve the students' performance were targeted at the work of teachers. The conception of humanity was dominated by the ideal of a socialistic personality "which was attainable by all." Because of this perception, the cause of shortcomings in the knowledge and ability of pupils was seen as the result of the work of the teachers.

The empirical studies done in schools in the GDR were not isolated independent actions merely for the purposes of collecting data. They were invariably conducted with the aim of deducing necessary concrete changes to the school or to evaluate the effectiveness of measures already initiated. So, as a rule, these studies were linked to the introduction of new curricula and textbooks. Accordingly, the extensive empirical studies of the 1980s were referred to as stress tests for the new materials.

Overall, an almost diametrically opposed development of empirical research in mathematics teaching in the GDR and the FRG is evident. While large-scale empirical studies and field experiments were carried out in the GDR, focusing on the quantitative assessment of students' performance, empirical studies in the FRG concentrated increasingly on isolated, high-quality case studies.

Some of the personnel-related and social reasons for the development of empirical research in the FRG until the mid-1990s were:

- The roots of most didacticians lay in the fields of mathematics.
- The distinction in the 1960s and 1970s between practice and theory was based on separation of the academic and practical phases of teacher training.
- There was long-lasting trauma from the failed reform of mathematics teaching in the 1960s and 1970s.
- The educational administration had not, until that time, required the compiling of teaching results.
- Consequently, there were no regular, centralized performance reviews in most states.

The ignorance in both parts of Germany of the developments of the other side is regrettable. In scientific publications in the GDR dealing with mathematics learning, developments in the FRG were not discussed. And, even today, empirical studies which were done in the GDR are hardly known in the unified Germany.

Significant time had to elapse between historical developments in German education in order for authors to be able to present and describe the developments in Eastern and Western Germany without biases and prejudices. Thus, reflective contributions to mathematics teaching and research on teaching and learning mathematics in the two Germanies have only been available since about 2003. These sorts of publications began with the assessment on the proceedings of two meetings of East and West German didacticians held in 1996 (Henning and Bender 2003). This point in time seems connected with the weak results of the PISA study in 2000, which was particularly surprising to Germany's education leaders and triggered the so-called PISA shock. With PISA 2000 began a continuous performance evaluation of the German school system as well as international comparisons. It also began the performance comparison of the federal states (Tillmann 2004, p. 478). This completely new situation for Germany was met with great public interest which was strongly echoed in the media.

4 Aspects of the Current Interior and Exterior Views of German Mathematics Teaching since 1990

The period of change in 1990 was marked in the GDR by the dissolution and decentralization of existing structures. There were also profound changes in educational offerings both in content and in forms of mediation (Schneider 2003). Schneider reports that proposals for the reform of the East German educational structure were developed and discussed as early as the autumn of 1989. In it, the merits of the unitary school for the majority of children were to be combined with a stronger differentiation that would set in earlier. However, these ideas were neither discussed nor applied after 1990. All changes were politically motivated by the respective majorities; they were not organically grown, nor were they adequately prepared.

In the 1990s, mathematics instruction in the old federal states was still struggling with the aftereffects of “New Math,” and discussions were held about comprehensive schools. The previous education system in the new federal states was adapted to the administrative structures and legal frameworks of the old federal states relatively silently. The decision-making positions within the new education administration were often filled by people from the old federal states. Schneider (2003) and others report on this phase of great uncertainty for both mathematics teachers at schools and representatives of didactics at universities and colleges.

The new federal states used their newly gained freedoms and leeway to set different priorities. While some introduced the 13th school year at Gymnasiums and, as in the state of Brandenburg, adapted curricular content and structure in a short amount of time due to their geographical proximity to Berlin, others remained with the 12-year school system (e.g., Saxony and Thuringia), but opened up to the variety of teaching and learning materials now available. The new federal states each had one partner from the old federal states whose influence was already evident in the structure of the new curricula (Schneider 2003, p. 260). After reunification, mathematics teachers in the new federal states were confronted with completely new textbook choices. With the publisher *Volk und Wissen* there was only one textbook in the GDR, the development and evaluation of which involved at least in some way all areas of methodology in the GDR. After 1990, all textbook publishers in the old federal states were producing special editions. *Volk und Wissen* also published a slightly modified and then revised edition of its textbook series. At a publishing house founded by former employees of the dissolved Academy of Pedagogical Sciences, a new textbook series was developed for secondary level I (e.g., Sill 2002). The aim was to develop new concepts as a result of analysis of textbooks from the old federal states and taking into account the experiences of the GDR. The structure of the book was not based on Lietzmann's suggestion for a methodical book but, rather, on a guideline suitable for pupils and a collection of tasks. The description of Lietzmann's book, *Das Wesen der Mathematik* (1949), provides essential background for understanding the new approach:

> The older mathematics textbooks began to answer the question of the nature of mathematics and its individual branches, and indeed to provide definitions of these terms. We have gotten away from it today. Rightly! For such a question does not belong to the beginning, but to the conclusion of the study of mathematics. Only when one already has learned something about mathematics, does it seem appropriate to get clear about the mathematical method, the structure of the teaching material and its basis. Numerous previous curricula for higher schools have moved a "repetitive structure of the number concept" into the upper classes. The Merano proposals and, according to them, other modern plans for materials have been taken as the conclusion of mathematical instruction: "Retrospect based on historical and philosophical considerations." The Prussian guidelines of 1925 emphasized philosophically-deepened retrospectives in both the methodological remarks and the curricula: "Logic and knowledge theory find a place in mathematics. Even the psychological foundations of mathematical thinking should touch the lesson. Individual questions such as numerical and spatial representations should, if possible, be deepened philosophically"—so it is said in them. The Marienauer proposals (1945), to name at least one of the new plans, demand: "Structure and basis of mathematics: development of the concept of number and function, axiomatic method of foundation by the example of geometry, prospects of logic and epistemology."

Another challenge for teachers were short-term changes to all curricula in some of the new Länder, which were in several ways due to changes in school structures. In order to adapt to the plans of the old federal states, content was deleted, such as explicit demands for evidence and derivations or for descriptive geometry, which was taught previously in the GDR. Also, new content was included, such as increased description of statistics and of probability theory, which were not included in the GDR curriculum. Since the existing system of mandatory teacher training in the GDR collapsed during this time, teachers were largely on their own when it came to implementing the new requirements.

The effects of these changes on the real practice of mathematics teaching in the new states have not yet been researched. This is partly due to the low level of textbook research in Germany. A search in the MathEduc database using the search terms "U20" (Textbooks. Analysis of textbooks, development and evaluation of textbooks. Textbook use in the classroom) and "German" returns only 82 contributions, of which only 3 deal with the comparative analyses of textbooks. Curricula are also only a marginal subject of didactic research (Sill 2018).

The international PISA tests and the German supplementary test in 2003 showed large differences between the federal states and the old and new states in particular. An analysis of the causes of these differences is difficult because the socioeconomic framework conditions varied in many ways. Nevertheless, it is clear that those new federal states that made only minor structural changes achieved significantly better results on the mathematics test than those federal states that had already been exposed to multiple structural changes—from new school types to educational transitions (say, changing the beginning of lower secondary school from grade 5 to grade 7) to changed timetables and the abolition of advanced courses in upper schools.

The different results of the old and new federal states offer an opportunity to deal with mathematics teaching in the GDR. Sill found that in the *Handbuch der Mathematikdidaktik* (Bruder et al. 2015), for example, only 18 of the 1700 bibliographical references are to teaching works from the GDR.

In 2012, almost 44,600 pupils from the ninth grades of all school types and from all federal states took part in a large learning status survey in mathematics and natural sciences on behalf of the KMK. The comparison between the federal states delivered such headlines in the media as here at SPIEGEL online 11/10/2013:

Maths and Natural Sciences

- The performance gap between students in the East and West is serious:

The East has model students: Saxony and Thuringia lead the nationwide school comparison in mathematics and science. Laggards are the city states and North Rhein-Westphalia. There, students are up to 2 years behind (Fig. 2.4).

This large-scale investigation shows that the performance of students in mathematics and natural sciences is very strongly dependent on the respective federal state in all four subjects examined.

The clear winners in the new study are the federal states of Saxony, Thuringia, Saxony-Anhalt, and with a few slight exceptions, such as Brandenburg, they are significantly above the German average in all four subjects.

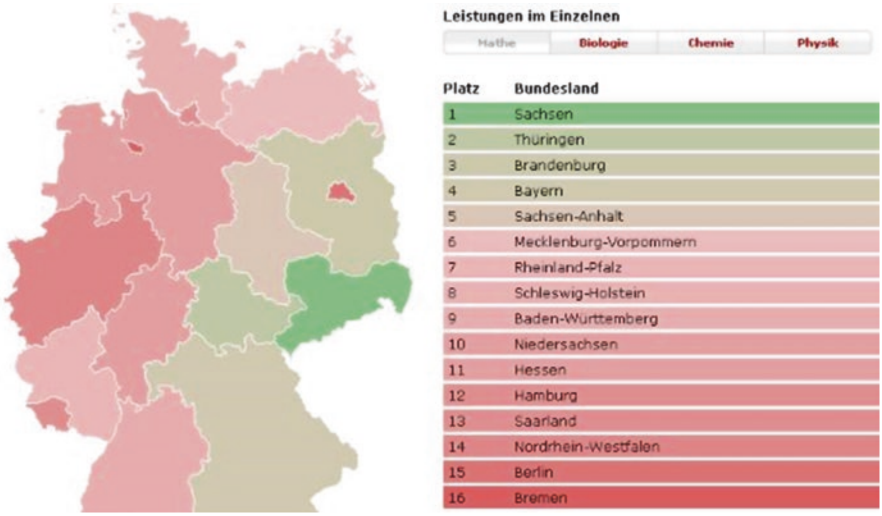

Fig. 2.4 School comparison of the federal states of Germany

Origin Makes a Difference in Performance of 3 Years

- Researchers are bothered about the extent to which educational success in Germany depends on the parents' home. …Students from better-off families score an average of 82 points more in mathematics than children from weaker families, a difference of almost 3 school years. …. Children with two parents born abroad, regardless of their social background, had significantly worse results than classmates with only one or no immigrant parent.

The first explanation given by education experts for the good performance of the East German states is the GDR's tradition of mathematics and science. Polytechnic secondary schools there focused on these subjects.

Educational disadvantages based on immigration status, poverty and educational distance are the biggest problems that the authors of the study point out (Titz 2013)

Even after these media reports emerged, there were hardly any efforts to examine the phenomenon of the GDR school traditions more closely. Rather, the lower proportion of migration among pupils in the new states was identified as a reason for the differences in performance. This, however, overlooks some important experiences in the GDR school tradition, which could be helpful to solving current problems, for example, individual support for all-day schooling.

> Looking at the educational policy process according to PISA from the perspective of "output" (the decisions made and implemented), an ambivalent assessment is appropriate: on the one hand, meaningful and long overdue measures (e.g. more all-day schools) have finally gotten off the ground; on the other hand, all activities to reduce selectivity in secondary schools are excluded. And the measures that are most consistently implemented in primary and secondary education in all the states – setting standards and evaluation – are particularly controversial among educators. (Tillmann 2004, p. 483)

Another topic that has received a great deal of media coverage and is linked to the conclusions drawn from the PISA results is the transition between school and university with regard to mathematical fundamentals and the ability of school leavers to study:

> First of all, the point of departure at the interface between school and university is dominated by political framework conditions and changes to the education system. In our view (shared by numerous colleagues), the shortening of schooling from 13 to 12 years and the shortening of the number of hours in mathematics…. led to a decrease in the level of mathematics at school.
>
> The politically motivated increase in the number of university students additionally reduces the average mathematical competencies of new university students. As a result of all this, poor mathematical skills of students are diagnosed by those lecturing at universities, and administrations criticize high dropout rates in mathematics-intensive subjects. This applies, in particular, to engineering courses but also to mathematics courses for a teaching or a specialized bachelor's degree. This development is even more worrying now, as ministries and universities have started to move to financing concepts dependent on graduation rates.
>
> With a view to schools, the university side laments the fact that today's students mainly have shortcomings in the subjects taught primarily in the lower secondary level. This fact is evidenced by tests in the study entry phase, e.g. at bridge courses (Greefrath et al. 2018). In an open letter in 2017, the unsatisfactory situation concerning the gap between mathematics results in schools and the expectations at colleges was taken as an opportunity to criticize the introduction of educational standards and the orientation of teaching to competencies

> instead of prescriptive knowledge (Baumann 2017). This public discussion on possible causes received a lot of media coverage. (DMV 2017)
>
> Based on the experiences of universities, a great number of school leavers show deficits in knowledge of fractional arithmetic and other calculation techniques; also, logical speech comprehension is often inadequately developed. This would indicate deficits in the sustainability of the transfer of knowledge at school. In addition, the university side has also noticed deficits in the general competencies of new university students, such as self-organization, self-assessment and a willingness to work hard. (Kramer 2010)

An interesting example of a constructive approach to educational policy with regard to student test results and empirical data on mathematics teaching is shown by the city-state of Hamburg (1.8 million inhabitants). Hamburg was not satisfied with the results of the comparative tests in mathematics and therefore used an Expert Commission of mathematics didacticians and human scientists to analyze the state of mathematics teaching. On the basis of extensive empirical data, the Expert Commission developed proposals for six kinds of action, ranging from the establishment of preschool programs to the further education and training of mathematics teachers working in Hamburg. The central message that runs through all fields of action is the need for subject-related and subject-didactic professionalization measures on the part of pedagogical staff (in day-care centers) and teachers (in schools) in order to increase mathematical competence on the part of children and young people. The report of the Expert Commission was published at the end of 2018 (Expert Commission Hamburg 2018).

The recommendations for action in Hamburg address both structural and organizational questions (all-day schools with support programs that are tailored to development of gifted students) as well as content and teaching-related methodological aspects. These include well-balanced goals for the curriculum that take both application and argumentation within mathematics seriously. But there must also be regular repetition and awakening of basic knowledge and basic skills.

If you look at these recommendations for action, there are some parallels to the differences between the FRG/GDR school systems (see Sect. 3) and to the idea of an individually supported mathematics lesson, which also gives sufficient space to specialized content and the general learning potential of students of mathematics.

From today's point of view, with an interval of about 30 years, some of the rapid changes and adaptations to structural conditions in the new federal states appear under a different light:

- In the kindergartens, the opportunities for all children included not only state-supported aspects, to build up the cult of personality related to the political leadership in the GDR, but also carefully thought-out learning and preparation for mathematical thinking and working in school. Today, the potential of early childhood education in the field of mathematics is (again) being considered in the old federal states as well (see also the recommendations of the Expert Commission on Mathematics Education in Hamburg 2018).

- In the afternoon care (after-school care) at the general education schools in the GDR, which was intensively used due to large number of working parents, there were a variety of supports offered both to pupils with learning difficulties and to high-performing pupils, as well as offers for leisure activities. After-school working groups in mathematics to promote gifted children were also a part of this but were not continued after joining the FRG due to the lack of a structural fit and ideological reservations.
- In some places there were private initiatives established with the support of associations [e.g., *Brandenburgischer Landesverein zur Förderung mathematisch-naturwissenschaftlich-technisch interessierter Schüler e.V* (BLIS) in the state of Brandenburg] that were able to continue to train students for the Mathematics Olympiad. The Mathematics Olympiads were continued mainly due to the activities of the mathematicians of the University of Rostock (and other colleagues, see Kugel 2019) and then extended to all federal states (Engel 1990). Since 1994 exist the Union of Mathematics Olympics e.V. (https://www.mathematik-olympiaden.de/). Only about 20 years later did the changed social framework conditions of the Federal Republic of Germany require more afternoon care at schools to answer the growing demand for a better work-life balance. New structures are now being set up, for which there were already possible role models with many years of experience from the GDR. However, these were not well enough known.

Since the 1990s, the promotion of gifted children was still little accepted in the old federal states from the point of view of educational policy. The special schools for mathematics and natural sciences working at a high technical level in the GDR had a hard time surviving with the FRG's concepts of holistic personal development of highly gifted children. In the meantime, the promotion of gifted students has received the status of an official educational goal of the United Nations. The PISA 2015 test results for mathematics in Germany showed that the proportion of students with very good results decreased and had been falling behind the rest of Europe for many years. These results were one of the reasons why the 2016 Standing Conference of the Ministers of Education and Cultural Affairs of the States and the Federal Government decided to launch a "Joint Initiative of the Federal Government and the States for the Promotion of Highly Efficient and Potentially Very Highly Efficient Pupils" focusing on mathematics, natural sciences, and languages. In this initiative, school development toward the direction of talent advancement was to be strongly supported.

After the results of the Third International Mathematics and Science Study (TIMSS) became known in Germany in 1997, a broad discussion about the quality of mathematics and science teaching began. Based on the disappointing TIMSS results, the *Bund-Länder-Kommission model programme* for increasing the efficiency of mathematics and science teaching (SINUS) was implemented at schools between 1998 and 2013 with a wide range of teacher training courses and material developments (see IPN 2003, among others). At the same time, binding educational standards for all federal states were adopted by the Conference of Ministers of

Education and Cultural Affairs in 2004 and 2005 for primary and lower secondary schools and in 2012 for upper secondary schools (KMK 2004a, b, 2005, 2012). Educational standards were described as performance standards, and performance was defined by competencies.

According to Weinert (2001, p. 27), competencies are "the cognitive abilities and skills available to or to be learned by individuals in order to solve certain problems." The focus should be on what pupils actually know and are able to do and not just what they are supposed to learn. Thus, educational standards constitute a pragmatic response to problems in traditional debates on education and curricula.

The first educational standards for mathematics were developed in a relatively short time by experts from the education ministries and were based on existing models, such as the American NCTM standards (2000), the Danish KOM project (Niss 2003), or on the international framework for PISA 2003 (OECD 2004).

The PISA studies and the development of standards in Germany were accompanied by clearly audible criticism within the didactic community (German Society for Didactics of Mathematics—GDM). Compare, for example, the critical analysis of educational standards in Sill (2008) and the German Mathematicians Association (DMV). On the one hand, there was agreement that mathematics teaching should be improved, but the path of controlling output represented by education policy through the standards and tests of PISA studies was ambivalently adopted by the scientific community. The criticism of the introduction of educational standards for mathematics teaching, combined with the new orientation toward competencies, reached a high point in March 2017 with an open letter that received a great deal of media coverage. This letter identified alarming symptoms of a crisis in mathematics education in schools. A central point of the criticism was formulated around the subject matter that mathematics in education had been thinned out to such an extent that the mathematical knowledge of many first-year university students was no longer sufficient for economic, mathematical or scientific-technical studies (see Baumann 2017). In the reactions to this letter the problems and issues were acknowledged but their explanation was questioned.

Dissatisfaction with the fact that experts from mathematics and mathematics teaching were not heard much regarding the implementation of educational standards at the state level led three associations, the DMV, GDM, and MNU (Association of the teachers for the mathematically scientific lessons/Verband der Lehrkräfte für den mathematisch-naturwissenschaftlichen Unterricht), to establish a joint Commission on Current Issues in Mathematical Education in the Transition from School to University in 2011. In particular, this Commission generates and communicates recommendations for the design of the school-university transition (Greefrath et al. 2018) and serves as a contact for the education administration. For activities and statements see http://www.mathematik-schule-hochschule.de/.

At the beginning of 2019, this Mathematics Commission presented the public with 19 recommendations for action to facilitate the transition from school to university. These recommendations for action are also a reaction to the urgent discussion in 2017, which called for, among other things, 4 h of mathematics lessons per week at each grade level, high-quality further training for teachers, a concretization of educational standards and a central examination section for the graduation examination (named *Abitur* in Germany or *Matura* in Austria and Switzerland) that is free of aids (Mathematik Kommission Schule-Hochschule 2019).

In the last 2 years, the quality debate in education has shown new developments, especially in mathematics. The so-called "empirical turnaround" in educational research initiated by the TIMSS and PISA studies increasingly led to the question of whether there was empirical evidence for proposed answers to the open questions—for example on the performance of different school types. Even if expectations cannot all be fulfilled at present, empirically working mathematics didacticians have been able to make themselves heard more. They are more in demand today when decisions on educational policy are to be made.

There are recurring discussions on the goals and content of mathematics as a teaching subject in general schools that are linked to social developments (Neubrand 2015). According to Heymann (1996, p. 50ff), the societal demands on general mathematics instruction are reflected in the "Seven Tasks of General Education Schools" at very different levels of quality: life preparation; foundation of cultural coherence; world orientation; guidance for the critical use of reason; development of willingness to take responsibility; practice in understanding and cooperation; strengthening the student's self.

However, such analytical categories must first be consciously associated with concrete subjects and situations in mathematics lessons. To this end, Winter (1995, p. 37f) formulated three "basic experiences" that share a consensus in Germany today:

> Mathematics teaching should aim to enable the following three basic experiences, which are interlinked in many ways:
>
> - to perceive and understand in a specific way phenomenon of the world around us that concern or should concern us all, from nature, society, and culture
> - to know and understand mathematical objects and facts (represented in language, symbols, images, and formulas) as spiritual creations, as a deductively ordered world of one's own kind
> - to acquire problem-solving skills beyond mathematics (heuristic skills) in dealing with tasks.

The finding of a harmonious balance between experience-based situational learning and systematic, cumulative knowledge acquisition proves to be a central problem of school-based learning, especially in mathematics:

> If one aims at cumulative knowledge acquisition within a specific field of knowledge, for example in mathematics or a scientific subject, the empirical findings prove the effectiveness of systematic, cognitively abstract learning: a well-organized knowledge base is the best prerequisite for subsequent learning within a domain.
>
> […] If one rather aims at lateral transfer, at the transfer of what has been learned to parallel but distinct application situations, then situated learning proves its strength. In school, both perspectives of learning are important. The structural strength of the school undoubtedly lies in the organization of systematic, long-term knowledge acquisition processes…. The regulatory idea of school teaching is the long-term cumulative acquisition of knowledge using varying, and if possible also authentic, application situations, with a constantly new balance to be found between casuistry and systematics." (Bund Länder Kommision–BLK 1997, pp. 19–20)

This statement comes from the expert opinion of a Bund Länder Kommision (BLK 1997) project group in response to the results of the TIMSS study (Klieme and Baumert 1998), which founded the teacher training program SINUS.

At the end of the 1990s, the curriculum revisions of individual federal states focused more on student activities in connection with a more process-oriented and less product-oriented view of mathematics (Klieme et al. 2003, p. 45). There was a stronger orientation toward interdisciplinary action competence, including professional competence, methodological competence, personal competence, and social competence.

As an educational policy response to the public debate on the disappointing achievements in the international comparative studies, especially PISA 2000 (Baumert 2001), transnational educational standards were introduced via the *Kultusministerkonferenz* (KMK). Among other things, the "expertise for development of national educational standards" (Klieme et al. 2003) recommended the description of minimum standards and the development of core curricula. But the KMK did not follow all aspects of this recommendation—now we have so-called *Regelstandards* (rule standards) in Germany.

> However, schools will need additional guidance, support and counseling to be able to deal productively with educational standards.
>
> It follows from this that teacher training, curriculum work, school supervision, and other instances of educational administration must take up the impulses of educational standards and assume new functions. (Klieme et al. 2003, p. 90)

5 Aspects of Dealing with Mathematics Didactic Research and Development Results from the GDR Period

Already in 1990 there was a meeting of didactics representatives from East and West Germany in Ohrbeck for an exchange about didactic research in both parts of Germany, organized by Hans-Georg Steiner (IDM, Bielefeld) (Bruder and Winkelmann 1991). In 1996 there were two follow-up meetings in Osnabrück and Magdeburg, for which proceedings were not published until 2003 (Henning and Bender 2003).

It is a fact that until today East German research results have found only little entry into the all-German research on the development of mathematics teaching. There are still noticeable prejudices due to the undeniable proximity of research and development in mathematics teaching in the GDR to the failed East German political system. However, there are other reasons for the poor reception of the results to date. As Sill (2018) describes, after 1990 there were very few East German didactics professors who were still active and could continue or communicate previous research traditions. The greatest difference between the teaching and researching of subject didacticians from the old and new federal states are visible in their practical relevance. A (theory-based) practical relevance in the GDR was much stronger. This discrepancy led to a different degree of appreciation of the subject of didactics in the

1990s in the East and in the West of Germany. (This assessment is based on personal experience of the author during many years of teaching and in-service trainings in both the East and West.)

The lack of access to East German dissertations made the reception of East German research and developments in mathematics teaching more difficult; they were not published due to scarce resources and were essentially only available at specific research locations through individual specimen copies. Until 1990 there was also no domain-specific research journal in the GDR like the *Zentralblatt for Didactics of Mathematics* (ZDM) and the *Journal of Mathematics Didactics* (JMD) in the FRG. There were, however, special editions and interdisciplinary scientific university journals in small print runs at some universities and colleges. Unfortunately, these articles and materials are still not digitally available today. Sill writes further on the possible causes for the widespread disregard of the results of mathematical methodology in the GDR:

> The most interesting results for today are not to be found mainly in the journal ‘Mathematik in der Schule’ or the teaching aids, but in the dissertations, the scientific journals of the institutions and the so-called gray literature. The qualification work is available in only a few copies in the libraries of the institutions or in the German National Library. (Sill 2018, p. 5)

Two international conferences in recent years on teaching and learning mathematics, which were organized in Germany, opened up the possibility of presenting developments in mathematics education and its research from both the old and the new federal states. This opportunity was used at the Psychology in Mathematics Education conference in Kiel (Bruder et al. 2013) and at the ICME 2016 in Hamburg (Bruder and Schmitt 2016).

Jahnke et al. (2017) described and analyzed developments that have taken place in German mathematics education research during the last 40 years. The 16 authors are experts and identify eight themes, which “were characteristic for the discussion on how Germany was influenced by and how it interacted with the international community” (Jahnke et al. 2017, p. 305). The authors show:

> the profound changes that have taken place in German-speaking mathematics education research during the last 40 years. The development comes near to a sort of revolution—not very typical for Germany. The only themes that could have appeared in the program of the Karlsruhe Congress in 1976 are subject-matter didactics and, with qualifications, design science and *Allgemeinbildung*. All other topics, especially modelling, theory traditions, classroom studies, and empirical research represent for Germany completely new fields of activity. Today, they define the stage on which German mathematics educators have to act. (Jahnke et al. 2017, p. 317)

But theoretical traditions (e.g., the development of subject didactical theories) concerning typical teaching and learning situations (Steinhöfel et al. 1978) as well as for structuring math lessons, classroom studies (e.g., about using hand held computers, see Fanghänel 1985) and empirical research about initial differentiation or problem-solving (Bruder et al. 2013) were very well established in the GDR, but not in the old states. And that hardly has changed. Since most of the professorships for specialized didactics in mathematics now come from the old federal states and naturally bring along their own training culture, there is a danger that theoretical foundations and research results of the former GDR will become even less visible in the future.

Hopefully, future subject-didactic German-language research on teaching and learning mathematics will:

- become aware of historical oscillations and provide for a timely necessary balance to extreme positions by keeping a memory of research results from more than 30 years ago
- be, above all, responsible for its service function in the further development of real mathematics education and
- engage in open discourse with reference disciplines and without ideological reservations, valuing and taking part in all forms of insight gained at various research locations.

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