Chapter 7
Sums of the Powers of Successive Integers

Not only could nobody but Gauss have produced it, but it would never have occurred to any but Gauss that such a formula was possible

Albert Einstein (1879–1955)

What happens when you sum successive powers of integers? To investigate this, define

$$S_{k,n} = 1 + 2^k + 3^k + \cdots + n^k = \sum_{i=1}^{n} i^k, \qquad k = 0, 1, \ldots \tag{7.1}$$

An easy program generates the following table of numeric values for small k and n:

Table of Values of $S_{k,n}$

k/n	1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	**6**	7	8	9	10
1	1	3	6	10	15	**21**	28	36	45	55
2	**1**	**5**	**14**	**30**	55	**91**	140	204	285	385
3	1	9	36	100	225	**441**	784	1,296	2,025	3,025
4	1	17	98	354	979	2,275	4,676	8,772	15,333	25,333
5	1	33	276	1,300	4,425	12,201	29,008	61,776	120,825	220,825
6	1	65	794	4,890	20,515	67,171	184,820	446,964	978,405	1,978,405
7	1	129	2,316	18,700	96,825	376,761	1,200,304	3,297,456	8,080,425	18,080,425

The $k = 0$ case, shown above, is immediate since

$$S_{0,n} = 1^0 + 2^0 + \cdots + n^0 = n$$

R. Nelson, *A Brief Journey in Discrete Mathematics*,
https://doi.org/10.1007/978-3-030-37861-5_7

Supposedly, the case for $k = 1$ was assigned as a teacher's punishment for the child prodigy, Carl Friedrich Gauss (1777–1855). Gauss was told to sum the numbers from 1 to 100 and, instead of laboring for an hour or two, he quickly responded 5,050 to the consternation of his teacher. How did he do it so quickly?

The young Gauss, who later grew up to be a famous mathematician, probably noticed that a *backwards* version of $S_{1,n}$ given by

$$S_{1,n} = n + (n-1) + \cdots + 1 = \sum_{i=1}^{n} (n+1-i)$$

could be added to the *forward* version to yield

$$2S_{1,n} = \sum_{i=1}^{n} (n+1-i) + \sum_{i=1}^{n} i = \sum_{i=1}^{n} (n+1) = n(n+1)$$

quickly giving

$$S_{1,n} = \frac{n(n+1)}{2} \tag{7.2}$$

The precocious Gauss saw this pattern, did the numerical calculation, and thus bypassed his teacher's punishment.

7.1 A General Equation

A key observation to make on the above approach is that by canceling the i in the forward and backward versions of the $k = 1$ case, the solution only required the equation for the previous, $k = 0$, case. Following on this logic, consider another way to solve for $S_{1,n}$ which arises by forming an equation for the next higher dimension, $k = 2$, and having the i^2 term conveniently cancel out. To illustrate this, consider the shifted sequence given by $(i+1)^2$. Summing this from 1 to n creates an addition term of $(n+1)^2$ but lacks the first term when compared to $S_{2,n}$. Thus

$$\sum_{i=1}^{n} (i+1)^2 = S_{2,n} + (n+1)^2 - 1$$

Expanding $(i+1)^2$ and subtracting i^2 of the original sequence yields $2i+1$ which suggests that subtracting the original sequence from its shifted version

$$S_{2,n} + (n+1)^2 - 1 - S_{2,n} = \sum_{i=1}^{n} (i+1)^2 - \sum_{i=1}^{n} i^2$$
$$= \sum_{i=1}^{n} 2i + 1 = 2S_{1,n} + n$$

Thus,

$$(n+1)^2 - 1 = 2S_{1,n} + n$$

which yields

$$S_{1,n} = \frac{(n+1)^2 - 1 - n}{2} = \frac{n(n+1)}{2}$$

as before.

A pattern emerges which is made clear with one more example. Writing

$$(i+1)^3 = i^3 + 3i^2 + 3i + 1$$

and using the fact that

$$\sum_{i=1}^{n} (i+1)^3 = S_{3,n} + (n+1)^3 - 1$$

implies that

$$S_{3,n} + (n+1)^3 - 1 - S_{3,n} = \sum_{i=1}^{n} 3i^2 + 3i + 1 = 3S_{2,n} + 3S_{1,n} + n$$

This shows that

$$(n+1)^3 - 1 = 3S_{2,n} + 3S_{1,n} + n$$

which, when substituting the above expression for $S_{1,n}$, solves to

$$
\begin{aligned}
S_{2,n} &= \frac{(n+1)^3 - (1+n) - 3n(n+1)/2}{3} \\
&= \frac{(n+1)}{6}\left(2(n+1)^2 - 2 - 3n\right) \\
&= \frac{(n+1)}{6}\left(2n^2 + n\right) \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned} \tag{7.3}
$$

The pattern suggests that the solution of the k case emerges when considering the case for one dimension higher, $k+1$. An equation for $S_{k,n}$ can then be determined by arranging for the cancellation of the term, i^{k+1}. The binomial theorem implies that

$$
(i+1)^{k+1} - i^{k+1} = \sum_{\ell=0}^{k+1} \binom{k+1}{\ell} i^{\ell} - i^{k+1} = \sum_{\ell=0}^{k} \binom{k+1}{\ell} i^{\ell} \tag{7.4}
$$

Summation of (7.4) produces a telescoping sum on the left-hand side of the equation yielding the general equation

$$
\begin{aligned}
(n+1)^{k+1} - 1 &= \sum_{i=1}^{n} \sum_{\ell=0}^{k} \binom{k+1}{\ell} i^{\ell} \\
&= \sum_{\ell=0}^{k} \binom{k+1}{\ell} \sum_{i=1}^{n} i^{\ell} \\
&= \sum_{\ell=0}^{k} \binom{k+1}{\ell} S_{\ell,n}
\end{aligned} \tag{7.5}
$$

Also observe that the binomial theorem shows that

$$
(n+1)^{k+1} - 1 = \sum_{\ell=1}^{k+1} \binom{k+1}{\ell} n^{\ell}
$$

which, after substitution into (7.5) yields

$$\sum_{\ell=0}^{k} \binom{k+1}{\ell} S_{\ell,n} = \sum_{\ell=1}^{k+1} \binom{k+1}{\ell} n^{\ell}$$

This can be rewritten as

$$n + \sum_{\ell=1}^{k} \binom{k+1}{\ell} S_{\ell,n} = \sum_{\ell=1}^{k} \binom{k+1}{\ell} n^{\ell} + n^{k+1} \tag{7.6}$$

The right-hand side of (7.6) is a polynomial of order $k+1$ which implies that the left-hand side is also a polynomial of this order. This implies that the function $S_{\ell,n}$ can be expressed as a polynomial (previous calculations show is of order $\ell+1$) so that

$$S_{\ell,n} = a_{1,\ell}\, n + a_{2,\ell}\, n^2 + \cdots + a_{\ell+1,\ell}\, n^{\ell+1}$$

The coefficients of this polynomial can be determined by isolating powers of n. The constant coefficient above is missing since $S_{\ell,0} = 0$. Note that

$$\begin{aligned}\sum_{\ell=1}^{k} \binom{k+1}{\ell} S_{\ell,n} &= \sum_{\ell=1}^{k} \binom{k+1}{\ell} \sum_{\jmath=1}^{\ell+1} a_{j,\ell} n^j \\ &= n \sum_{\ell=1}^{k} \binom{k+1}{\ell} a_{1,\ell} + \sum_{j=2}^{k+1} n^j \sum_{\ell=j-1}^{k} \binom{k+1}{\ell} a_{j,\ell}\end{aligned}$$

The defining equation (7.6) can now be expressed to expose the powers of n on each side of the equation:

$$\begin{aligned}&n + n \sum_{\ell=1}^{k} \binom{k+1}{\ell} a_{1,\ell} + \sum_{j=2}^{k+1} n^j \sum_{\ell=j-1}^{k} \binom{k+1}{\ell} a_{j,\ell} \\ &\quad = \sum_{\ell=1}^{k} \binom{k+1}{\ell} n^{\ell} + n^{k+1}\end{aligned}$$

The $k+1$ equations that arise from matching powers of n^j on each side of the equation can be delineated as

Power	Equation
j=1	$1+\sum_{\ell=1}^{k}\binom{k+1}{\ell}a_{1,\ell}=k+1$
$j=2,\ldots,k$	$\sum_{\ell=j-1}^{k}\binom{k+1}{\ell}a_{j,\ell}=\binom{k+1}{j}$
$j=k+1$	$(k+1)a_{k+1,k}=1$

The last entry shows that

$$a_{k+1,k}=1/(k+1) \tag{7.7}$$

The remaining coefficients can be found iteratively. Two cases illustrate how this is done: For $j=1$, we can simplify the equation found in the table

$$\sum_{\ell=1}^{k}\binom{k+1}{\ell}a_{1,\ell}=k$$

and proceed sequentially:

$$\begin{aligned}
k=1:&\binom{2}{1}a_{1,1}=1\\
&\Longrightarrow a_{1,1}=1/2\\
k=2:&\binom{3}{1}1/2+\binom{3}{2}a_{1,2}=2\\
&\Longrightarrow a_{1,2}=1/6\\
k=3:&\binom{4}{1}1/2+\binom{4}{2}1/6+\binom{4}{3}a_{1,3}=3\\
&\Longrightarrow a_{1,3}=0\\
k=4:&\binom{5}{1}1/2+\binom{5}{2}1/6+\binom{5}{3}0+\binom{5}{4}a_{1,4}=4\\
&\Longrightarrow a_{1,4}=-1/30
\end{aligned}$$

A similar procedure can be used for $j = 2$:

$$k = 1: \binom{2}{1} a_{2,1} = 1$$

$$\implies a_{2,1} = 1/2$$

$$k = 2: \binom{3}{1} 1/2 + \binom{3}{2} a_{2,2} = \binom{3}{2}$$

$$\implies a_{2,2} = 1/2$$

$$k = 3: \binom{4}{1} 1/2 + \binom{4}{2} 1/2 + \binom{4}{3} a_{2,3} = \binom{4}{2}$$

$$\implies a_{2,3} = 1/4$$

$$k = 4: \binom{5}{1} 1/2 + \binom{5}{2} 1/2 + \binom{5}{3} 1/4 + \binom{5}{4} a_{2,4} = \binom{5}{2}$$

$$\implies a_{2,4} = 0$$

The sequence of operations outlined above applies to all $j \leq k$ and is easily programmed. The results, for the first seven cases, are summarized in the following table (the column headed by d is a denominator):

Table of Values of $a_{j,k}$

$k\backslash j$	1	2	3	4	5	6	7	8	d
0	1	0	0	0	0	0	0	0	1
1	1	1	0	0	0	0	0	0	2
2	1	3	2	0	0	0	0	0	6
3	0	1	2	1	0	0	0	0	4
4	-1	0	10	15	6	0	0	0	30
5	0	**-1**	0	**5**	**6**	**2**	0	0	12
6	1	0	-7	0	21	21	6	0	42
7	0	2	0	-7	0	14	12	3	24

We have highlighted the non-zero entries in bold that are required to write the equation for $S_{5,n}$ given by

$$S_{5,n} = \frac{-n^2 + 5n^4 + 6n^5 + 2n^6}{12}$$

Using the above coefficients we can easily calculate the polynomial equation for $S_{k,n}$ but this sheds no light on the relationship between the numbers found in the table of values given at the beginning of the chapter. What is this relationship? Can one start with a simple case and generate the remainder of the numbers algorithmically?

7.1.1 Iterative Approach

If there exists such an algorithm, then it must be the case that the value of $S_{k,n}$ can be written in terms of elements that occur previous to it in the table. Typically this can be done either by a calculation of terms in the previous row or the previous column. In this way, values can be generated starting from easily calculated small values of k and n. It makes sense then to consider two different summations: one along the n axis and one along the k axis.

To follow this approach, first consider the following summation:

$$\sum_{\ell=1}^{n} S_{k,\ell} = 1 + (1 + 2^k) + (1 + 2^k + 3^k) + \cdots + (1 + 2^k + \cdots + n^k)$$

This shows that 1 appears in all n of the sums, 2^k appears in $n-1$ of them, and ℓ^k appears in $n - \ell + 1$ of the summations. Hence,

$$S_{k,n} = nS_{k-1,n} - \sum_{i=1}^{n-1} S_{k-1,i} \tag{7.8}$$

This shows that $S_{k,n}$ can be calculated using the previous row in the table. To illustrate this, consider the case $k = 3$ and $n = 4$. The equation above yields the value of 100 that is underlined in the table:

$$100 = 4 \cdot 30 - (1 + 5 + 14)$$

Now consider the column-wise summation along the k axis. A straightforward summation like that above yields

$$\begin{aligned}\sum_{\ell=0}^{k} S_{\ell,n} &= \sum_{\ell=0}^{k}\sum_{i=1}^{n} i^{\ell} \\ &= \sum_{i=1}^{n} 1 + i^1 + i^2 + \cdots + i^k \\ &= S_{0,n} + S_{1,n} + \cdots + S_{k,n}\end{aligned}$$

which gets nowhere. A key insight that often proves to be useful arises by inserting a combinatorial term in the above sum, thus allowing the binomial theorem to be used to obtain a closed form equation. With this thought in mind consider the binomial transform of $S_{\ell,n}$

$$\begin{aligned}\sum_{\ell=0}^{k}\binom{k}{\ell} S_{\ell,n} &= \sum_{\ell=0}^{k}\binom{k}{\ell}\sum_{i=1}^{n} i^{\ell} \\ &= \sum_{i=1}^{n}\sum_{\ell=0}^{k}\binom{k}{\ell} i^{\ell} \\ &= \sum_{i=1}^{n}(i+1)^k \\ &= S_{k,n+1} - 1\end{aligned} \tag{7.9}$$

This works like a charm and rewriting the above equation more directly leads to the identity

$$S_{k,n} = 1 + \sum_{r=0}^{k}\binom{k}{r} S_{r,n-1} \tag{7.10}$$

This equation provides a method that uses the columns to determine the value of $S_{k,n}$. Illustrating this with $k = 3$ and $n = 7$ results in the underlined value of 784 found in the table

$$784 = 1 + 6 + 3 \cdot 21 + 3 \cdot 91 + 441$$

These two methods can be used to generate all the values of $S_{k,n}$ starting from initial beginning values.

This section ends with two easily calculated identities. Forming the binomial inverse of (7.10) leads to the identity

$$S_{k,n} = \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell} (S_{\ell,n+1} - 1) = \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell} S_{\ell,n+1} \tag{7.11}$$

The sum of integer powers leads to

$$S_{k,n}^2 = \left(\sum_{i=1}^{n} i^k\right)^2 = S_{2k,n} + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (ij)^k$$

Thus, a closed form equation for the cross product terms contained in the summation:

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (ij)^k = \frac{S_{k,n}^2 - S_{2k,n}}{2} \tag{7.12}$$

7.2 Triangular Numbers

The values of $S_{1,n}$ given by $1, 3, 6, 10, 15, 21, 28, \ldots$ correspond to the number of items needed to fill out a triangle like a rack of billiard balls. Because of this analogy, they are called *triangular numbers*. Simplify notation and define

$$T_n = \frac{n(n+1)}{2} = \binom{n+1}{2}, \quad n = 0, 1, \ldots \tag{7.13}$$

Like Fibonacci numbers, triangular numbers have a multitude of interesting properties. One property was discovered by Gauss who might have had an affinity for these numbers since they allowed his school boy escape. In one of his notebooks, Gauss wrote *Eureka!* (actually his note book said, *EYPHKA*:num=$\Delta + \Delta + \Delta$) after he discovered that all numbers could be written as the sum of three triangular numbers. Hence, for any integer k, Gauss showed that it is possible to find integers $m_i,\ i = 1, 2, 3$, such

$$k = T_{m_1} + T_{m_2} + T_{m_3}$$

An example

$$30 = 1 + 1 + 28 = 3 + 6 + 21 = 0 + 15 + 15 = 10 + 10 + 10$$

shows that there might be multiple ways to write this sum. We say a number, n, is *represented* by a general expression if an equality for n can be found by varying the terms of that expression. Thus, Gauss's Eureka moment came when he saw that three triangular numbers represented all non-negative integers.

Endless diversion can be found in the relationships between triangular numbers. For example, straightforward algebra establishes the following identities that result in a polynomial in n:

$$\begin{aligned} T_n + T_{n+1} &= (n+1)^2 \\ T_{n+1} - T_n &= n+1 \\ T_{n+1}^2 - T_n^2 &= (n+1)^3 \\ 8T_n + 1 &= (2n+1)^2 \\ T_{2n+1} - T_{2n} &= 2n+1 \\ T_{2n-1} - 2T_{n-1} &= n^2 \end{aligned} \tag{7.14}$$

An identity that arises from viewing triangular numbers combinatorially is given by

$$T_{n+k} = T_n + T_k + nk \tag{7.15}$$

To prove this, consider two sets: one consisting of n elements and the other containing k elements. We are interested in calculating the number of ways to select a pair of items from these two sets, T_{n+k}. There are three mutually different ways this can be done: both elements can be selected from the set of n elements in T_n different ways, from the set of k elements in T_k different ways, and one from each set in nk different ways. Summing these yields the equation above.

By varying values of k, a variety of relations can be derived from this equation including

$$\begin{aligned} T_{n(n+1)} &= T_n + T_{n^2} + n^3 && (k = n^2) \\ T_{n+2} &= T_n + T_2 + 2n && (k = 2) \\ T_{2n} &= 2T_n + n^2 && (k = n) \\ T_{T_n} &= T_n + T_{T_{n-1}} + nT_{n-1} && (k = T_{n-1}) \end{aligned} \tag{7.16}$$

By selecting $k = T_n$ in the defining equation $T_{k-1} + k = T_k$, we obtain the identity

$$T_{T_n} = T_{T_n - 1} + T_n \tag{7.17}$$

Equating the last two identities yields the tongue twisting result

$$T_{T_n - 1} = T_{T_{n-1}} + nT_{n-1} \tag{7.18}$$

There is a similarity between the two identities previously stated that contain an n^2 term. Stated again, these identities reveal the curious relationship

$$n^2 = T_{2n-1} - 2T_{n-1} = T_{2n} - 2T_n$$

If we assume that $T_i = 0$ for $i \leq 0$, then both of these equations are special cases of a family of identities given by

$$n^2 = T_{2n-k} + T_{k-1} - 2T_{n-k}, \quad k = 0, \ldots, n \tag{7.19}$$

A product formula, analogous to the summation identity stated above, is given by

$$T_{nk} = T_{n-1}T_{k-1} + T_nT_k \tag{7.20}$$

Again, varying k leads to a variety of results:

$$\begin{aligned} T_{n^2} &= T_{n-1}^2 + T_n^2, & (k = n) \\ T_{n^3} &= T_{n-1}T_{n^2-1} + T_nT_{n^2}, & (k = n^2) \\ T_{2T_n} &= T_n\left(T_{n-1} + T_{n+1}\right), & (k = n+1) \\ T_{2n} &= T_1T_{n-1} + T_2T_n, & (k = 2) \end{aligned} \tag{7.21}$$

A triangle is a 2-dimensional object which, when raised to three dimensions, becomes a pyramid. In this case, instead of racking billiard balls, one stacks cannon balls. The sequence of values then corresponds to $1, 4, 10, 20, 35, \ldots$ which are given by values of T_{2n}. The numeric sequence can be written as $4i - 1$ and thus another identity for T_{2n} emerges into view

$$T_{2n} = \sum_{i=1}^{n} 4i - 1 = 4T_n - n \tag{7.22}$$

The algebra to calculate the sum of the squares of triangular numbers is a bit tricky. Linking back to the equation derived for the sums of power of integers reveals the curious relationship $S_{3,n} = T_n^2$. Using this, and the row-wise summation previously derived, shows that

$$\sum_{i=1}^{n} T_i^2 = \sum_{i=1}^{n} S_{3,i} = (n+1)S_{3,n} - S_{4,n} \tag{7.23}$$

Writing this in a different way leads to

$$\sum_{i=1}^{n} T_i^2 = \frac{1}{4}\sum_{i=1}^{n} i^2(i+1)^2 = \frac{1}{4}\sum_{i=1}^{n} i^4 + 2i^3 + i^2 = \frac{1}{4}\left(S_{4,n} + 2S_{3,n} + S_{2,n}\right) \tag{7.24}$$

Equating these last two equations leads to the identity

$$2(2n+1)S_{3,n} = S_{2,n} + 5S_{4,n}$$

The general case for the summing the k'th power of triangular numbers can be written as

$$\begin{aligned} \sum_{i=1}^{n} T_i^k &= \sum_{i=1}^{n} \frac{i^k(i+1)^k}{2^k} = \sum_{i=1}^{n} \frac{i^k}{2^k} \sum_{\ell=0}^{k} \binom{k}{\ell} i^\ell \\ &= \frac{1}{2^k} \sum_{\ell=0}^{k} \binom{k}{\ell} \sum_{i=1}^{n} i^{k+\ell} = \frac{1}{2^k} \sum_{\ell=0}^{k} \binom{k}{\ell} S_{k+\ell,n} \end{aligned} \tag{7.25}$$

This identity can also be recast as[1]

$$\sum_{i=1}^{n} i^k(i+1)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} S_{k+\ell,n} \tag{7.26}$$

We can analyze the fine structure of T_i^k by using a telescoping sum to write

[1] This corresponds to the sum of powers of variety-2 integers, see equation (3.3).

$$
\begin{aligned}
(7.27) \qquad T_i^k &= \sum_{j=1}^{i} T_j^k - T_{j-1}^k \\
&= \sum_{j=1}^{i} \left(\frac{j(j+1)}{2}\right)^k - \left(\frac{(j-1)j}{2}\right)^k \\
&= \frac{1}{2^k} \sum_{j=1}^{i} j^k \left((j+1)^k - (j-1)^k\right) \\
&= \frac{1}{2^k} \sum_{j=1}^{i} j^k \sum_{\ell=0}^{k} \binom{k}{\ell} j^\ell \left(1 - (-1)^{k-\ell}\right) \\
&= \frac{1}{2^k} \sum_{\ell=0}^{k} \binom{k}{\ell} \left(1 - (-1)^{k-\ell}\right) \sum_{j=1}^{i} j^{k+\ell} \\
&= \frac{1}{2^k} \sum_{\ell=0}^{k} \binom{k}{\ell} \left(1 - (-1)^{k-\ell}\right) S_{k+\ell,i}
\end{aligned}
$$

The term $1 - (-1)^{k-\ell}$ above equals 0 if $k - \ell$ is even and equals 2 if $k - \ell$ is odd. In light of this parity, let $\mathcal{I}_k$ be the set of odd (even, respectively) integers less than k if k is an even (respectively, odd) integer. Then, summing the above equation yields

$$
\begin{aligned}
(7.28) \qquad \sum_{i=1}^{n} T_i^k &= \frac{1}{2^{k-1}} \sum_{i=1}^{n} \sum_{\ell \in \mathcal{I}_k} \binom{k}{\ell} S_{k+\ell,i} \\
&= \frac{1}{2^{k-1}} \sum_{\ell \in \mathcal{I}_k} \binom{k}{\ell} \sum_{i=1}^{n} S_{k+\ell,i} \\
&= \frac{1}{2^{k-1}} \sum_{\ell \in \mathcal{I}_k} \binom{k}{\ell} \left((n+1)S_{k+\ell,n} - S_{k+\ell+1,n}\right)
\end{aligned}
$$

The last identity of this section displays a curious symmetry

$$
(7.29) \qquad n^2 T_k + n T_{k-1} = k^2 T_n + k T_{n-1}
$$

7.3 Cauchy's Theorem

As stated before, Gauss showed that all numbers could be written as the sum of three triangular numbers. This generalizes to squares with a theorem by Joseph-Louis Lagrange (1736–1813) which shows that four squares summed together are sufficient to represent all non-negative integers (like triangular numbers we consider 0 to be a member of this set). A generalization of these results is best stated in terms of polygonal numbers.

The n'th number from the set of k polygonal numbers is given by

$$P_{k,n} = \frac{(k-2)n^2 - (k-4)n}{2}$$

Triangular numbers correspond to $k = 3$ in the above equation and square numbers to $k = 4$. Like racking billiard balls, these numbers arise when you rack k-gons. Polygonal numbers can be written multiple ways in terms of triangular numbers after some algebraic manipulation

$$P_{k,n} = (k-2)T_{n-1} + n = (k-3)T_{n-1} + T_n = (k-4)T_{n-1} + n^2 \tag{7.30}$$

Some conjectures are so pure and beautiful that one feels that something would be wrong with the world if they were not true. We are facing one example here because the natural generalization of the two cases above is: *all integers can be written as the sum of k polygonal numbers of order k*. It turns out that this is not just a lofty conjecture that implies the perfection of mathematics, it is actually a theorem that was first proved by Augustin-Louis Cauchy (1789–1857).

It might seem that there are good reasons to not expect Cauchy's result since the distance between successive polygonal numbers increases with k and n

$$P_{k,n} - P_{k,n-1} = n(k-2) + 3 - k$$

For example, the $k = 8$ sequence corresponds to an *octagon* whose first few values are

$$0,\ 1,\ 8,\ 21,\ 40,\ 96,\ 133,\ 176,\ 225,\ 280$$

whereas for squares we have

$$0,\ 1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81$$

The increased spacing of the octagonal case over the square case is offset by the need to have eight, rather than four, values added together so that all integers can be represented. This is a deep and beautiful result.

It is hard to resist answering one further question that arises: *are some* k polygonal *numbers also triangular numbers?* Recall the identity $n^2 = T_{2n-1} - 2T_{n-1}$ of (7.14). Using this equation and the identity $P_{k,n} = (k-4)T_{n-1} + n^2$ shows that these two equations share the common terms of n^2 and T_{n-1}. Equating them

$$n^2 = T_{2n-1} - 2T_{n-1} = P_{k,n} - (k-4)T_{n-1}$$

thus uncovers that $k = 6$ satisfies the equation and thus hexagonal k-gons are also triangular,

$$P_{6,n} = T_{2n-1}$$

What other forms of equations can be used to represent sets of numbers? How about the number of cubes needed to represent all numbers or, for that matter, the number of k powers that are required. Edward Waring (1736–1798) posed this question by defining a function $g(k)$ to be the minimum number of k powers that were required to represent all of the integers. Lagrange's four square theorem showed that $g(2) = 4$. Currently we only know the values $g(3) = 9,\ g(4) = 19$, $g(5) = 37$, and $g(6) = 73$. Bounds and properties of g have been extensively studied but, as of yet, the functional form of g remains unknown.