Chapter 5
All That Glitters Is Not Gold

All that glisters is not gold;
Often have you heard that told:
Many a man his life hath sold
But my outside to behold:
Gilded tombs do worms enfold.

William Shakespeare (1564–1616)
The Merchant of Venice, Act 2, scene 7

Despite the temptations of gold alluded to in Shakespeare's verse above from *The Merchant of Venice*, the pursuit of *mathematical gold* leads, not to gilded tombs, but to the paradise of the Elysian fields of ancient Greece. Our journey in this chapter takes us back to the days of Phidias (480–430 BC), a Greek sculptor and mathematician who is said to have helped with the design of the Parthenon. The approach in this chapter uses a simple artifice—the ratio of two line segments.

5.1 The Golden Ratio

Consider a line consisting of two line segments. The first segment has a length of 1 unit and the other has length $1-\epsilon$ where $0<\epsilon<1$. By construction, the second length is the smaller of the two. We are going to select the value of ϵ that equalizes two ratios. The first ratio, r_1, is the length of the total line segment to that of the larger section, thus

$$r_1 = \frac{1+1-\epsilon}{1} = 2-\epsilon \tag{5.1}$$

R. Nelson, *A Brief Journey in Discrete Mathematics*,
https://doi.org/10.1007/978-3-030-37861-5_5

The second ratio, r_2, is the length of the larger segment to that of the smaller segment

$$r_2 = \frac{1}{1-\epsilon} \tag{5.2}$$

For some value $\epsilon^\star$ the two ratios are equal. It is not hard to derive an equation for $\epsilon^\star$:

$$2 - \epsilon^\star = \frac{1}{1-\epsilon^\star}$$

or that

$$(\epsilon^\star)^2 - 3\epsilon^\star + 1 = 0$$

Using the quadratic formula yields the solution to this quadratic polynomial

$$\epsilon^\star = \frac{3 \pm \sqrt{5}}{2}$$

The solution corresponding to the positive square root is larger than 1 which lies outside the bound on the length of the second segment so[1]

$$\epsilon^\star = \frac{3-\sqrt{5}}{2} = 0.381966011250105\ldots$$

The value of $\epsilon^\star$ can be calculated using (5.1) and shows that

$$r_1 = 2-\epsilon^\star = 2-\frac{3-\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2} = \phi = 1.61803398874989\ldots \tag{5.3}$$

This calculated value, typically denoted by ϕ, is the *golden ratio* that was admired by Phidias who used it in the design of the shape of the Parthenon.[2]

Equation (5.2) expresses ϕ differently

[1] The appearance of $\sqrt{5}$ implies that $\epsilon^\star$ is irrational, see the proof on page 171.

[2] The Internet is replete with interesting historical facts dealing with this ratio.

$$r_2 = \frac{1}{1-\epsilon^\star} = \frac{1}{1-\frac{3-\sqrt{5}}{2}} = \frac{1}{\frac{-1+\sqrt{5}}{2}} = \frac{2}{\sqrt{5}-1} = \phi \tag{5.4}$$

These two ratios provide two equations for the golden ratio. Straightforward algebra shows that

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \Longrightarrow \qquad \sqrt{5} = 2\phi - 1 \tag{5.5}$$

Using this in (5.4) yields

$$\phi = \frac{2}{\sqrt{5}-1} = \frac{2}{2\phi-2} = \frac{1}{\phi-1}$$

and thus

$$\phi^2 = \phi + 1$$

This shows that the golden ratio is *one of the solutions* to the quadratic equation $x^2 = x + 1$, an equation which, in this chapter, is termed *the defining equation*. A key observation to make about this equation is that the left-hand side, x^2, corresponds to a multiplication whereas the right-hand side, $x+1$, is an addition. In essence, the equation converts multiplication to addition. What can such an observation yield?

5.1.1 Fibonacci Numbers

Since ϕ is one solution to the defining equation this means all occurrences of ϕ^2 can be replaced with $\phi + 1$ without changing the value of an expression. This can be used to calculate an expression for ϕ^3:

$$\phi^3 = \phi\phi^2 = \phi(\phi+1) = \phi^2 + \phi = \phi + 1 + \phi = 2\phi + 1$$

Using this to calculate the next power shows that

$$\phi^4 = \phi\phi^3 = \phi(2\phi+1) = 2\phi^2 + \phi = 2(\phi+1) + \phi = 3\phi + 2$$

Continuing with this progression yields

$$\phi^5 = \phi\phi^4 = \phi(3\phi + 2) = 3\phi^2 + 2\phi = 3(\phi + 1) + 2\phi = 5\phi + 3$$

$$\phi^6 = \phi\phi^5 = \phi(5\phi + 3) = 5\phi^2 + 3\phi = 5(\phi + 1) + 3\phi = 8\phi + 5$$

$$\phi^7 = \phi\phi^6 = \phi(8\phi + 5) = 8\phi^2 + 5\phi = 8(\phi + 1) + 5\phi = 13\phi + 8$$

$$\phi^8 = \phi\phi^7 = \phi(13\phi + 8) = 13\phi^2 + 8\phi = 13(\phi + 1) + 8\phi = 21\phi + 13$$

The numbers have an intriguing progression and to see the pattern more clearly consider the following table:

Power	Multiple of ϕ	Constant
1	1	0
2	1	1
3	2	1
4	3	2
5	5	3
6	8	5
7	13	8
8	21	13

The first thing to note is that the multiplier equals the constant for the following power. In other words the values are just shifted versions of each other. There is another observation that comes from looking at the sequence of numbers

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21, \ldots$$

Observe that after starting with two, 1's, the next number is the sum of the previous 2. This sequence of numbers is the famous *Fibonacci sequence* named for the Italian mathematician, Leonardo of Pisa (1170–1240?) around the year 1200. The Internet is replete with the history and myriad applications of this sequence including the mating characteristics of rabbits.

Let $f_i,\ i = 0, \ldots$ denote the Fibonacci numbers so that $f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 5$, and $f_n = f_{n-1} + f_{n-2},\quad n = 4, \ldots$. It is customary to start the sequence with $f_0 = 0$. The above table suggests that

$$\phi^n = f_n\phi + f_{n-1},\quad n = 2, \ldots \tag{5.6}$$

To prove this, use the technique of substituting $\phi+1$ for all occurrences of ϕ^2:

$$\begin{aligned}\phi^{n+1} &= \phi\phi^n = \phi(f_n\phi + f_{n-1}) \\ &= f_n\phi^2 + \phi f_{n-1} = f_n(\phi + 1) + \phi f_{n-1} = (f_n + f_{n-1})\phi + f_n\end{aligned}$$

Since $f_{n+1} = f_n + f_{n-1}$ the last expression can be rewritten as

$$\phi^{n+1} = f_{n+1}\phi + f_n \tag{5.7}$$

which shows that the pattern continues to the $n + 1$'st case. There is an analogy in the recurrence relationship of the Fibonacci numbers with the golden ratio seen by writing patterns side by side:

$$f_n = f_{n-1} + f_{n-2} \qquad\qquad \phi^n = \phi^{n-1} + \phi^{n-2}$$

To derive the second expression, write

$$\phi^n = \phi^{n-2}\phi^2 = \phi^{n-2}(\phi + 1) = \phi^{n-1} + \phi^{n-2},\quad n = 3, \ldots$$

5.1.2 A Closed Form Solution

To derive a closed form equation for f_n, return to the other solution besides ϕ to the defining quadratic equation $x^2 = x + 1$. The second solution from the quadratic formula is

$$\psi = \frac{1 - \sqrt{5}}{2} = -0.61803398874989\ldots \tag{5.8}$$

This solution necessarily shares properties similar to ϕ since it also satisfies $\psi^2 = \psi + 1$. This implies that the pattern derived above for powers of ϕ also holds for powers of ψ

$$\psi^n = f_n\psi + f_{n-1}, \quad n = 2, \ldots \tag{5.9}$$

This equation gives the key in finding an expression for f_n. Subtract ψ^n of equation (5.9) from ϕ^n of (5.6) to get:

$$\phi^n - \psi^n = f_n\phi + f_{n-1} - (f_n\psi + f_{n-1}) = f_n(\phi - \psi)$$

and, like picking a rabbit out of hat, this shows that

$$f_n = \frac{\phi^n - \psi^n}{\phi - \psi}$$

This almost seems too easy a way to get such a difficult result. This expression can be simplified since

$$\phi - \psi = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \sqrt{5}$$

which yields the compact formula

$$f_n = \frac{\phi^n - \psi^n}{\sqrt{5}} \tag{5.10}$$

This equation was first derived along different lines by Jacques Binet (1786–1856).

Before ending this section, observe that substitution (5.5) into equation (5.8) shows that

$$\psi = \frac{1-\sqrt{5}}{2} = \frac{1-(2\phi-1)}{2} = \frac{2-2\phi}{2} = 1-\phi$$

which yields

$$\phi + \psi = 1$$

This is useful in deriving an expression for the sum of ϕ^n and ψ^n. Using equations (5.7) and (5.9) yields

$$\begin{aligned}\phi^n + \psi^n &= f_n\phi + f_{n-1} + (f_n\psi + f_{n-1})\\ &= f_n(\phi+\psi) + 2f_{n-1} = f_n + 2f_{n-1}\end{aligned}$$

This can be rewritten in an easier form

$$\phi^n + \psi^n = f_n + 2f_{n-1} = f_n + f_{n-1} + f_{n-1} = f_{n+1} + f_{n-1} \tag{5.11}$$

which leads to the sequence of numbers[3]:

$$2,\ 1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29, \ldots \tag{5.12}$$

The Fibonacci recurrence relationship invites one to derive identities between the values. For example, write the equation for f_{2n}

$$f_{2n} = \frac{\phi^{2n} - \psi^{2n}}{\sqrt{5}} = \frac{(\phi^n)^2 - (\psi^n)^2}{\sqrt{5}}$$

and use the identity $x^2 - y^2 = (x-y)(x+y)$ and the equation for $\phi^n + \psi^n$ above in (5.11) to obtain

$$f_{2n} = \frac{\phi^n - \psi^n}{\sqrt{5}} (\phi^n + \psi^n) = f_n(f_{n+1} + f_{n-1}) \tag{5.13}$$

To derive the same value along a different line write

$$f_{2n} = \frac{\phi^{2n} - \psi^{2n}}{\sqrt{5}} = \frac{\left(\phi^2\right)^n - \left(\psi^2\right)^n}{\sqrt{5}}$$

Use the fact that $\phi^2 = 1+\phi$ and $\psi^2 = 1+\psi$, and the binomial theorem to express f_{2n} as

$$f_{2n} = \frac{(1+\phi)^n - (1+\psi)^n}{\sqrt{5}} = \sum_{k=0}^{n} \binom{n}{k} \frac{\phi^k - \psi^k}{\sqrt{5}} = \sum_{k=0}^{n} \binom{n}{k} f_k \tag{5.14}$$

Equating the two identities yields the relationships

$$f_{2n} = f_n(f_{n+1} + f_{n-1}) = \sum_{k=0}^{n} \binom{n}{k} f_k$$

[3]This sequence is termed the Lucas sequence and will be revisited later in the chapter, see equation (5.23).

Equation (5.14) corresponds to a binomial expansion of the values f_k. Using binomial inversion thus leads to another identity involving Fibonacci numbers:

$$f_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f_{2k} \tag{5.15}$$

To consider another identity, let $\alpha_n = f_{n+1}f_{n-1} - f_n^2$ and observe that $\alpha_1 = -1$ and $\alpha_2 = 1$. This suggests that $\alpha_n = (-1)^2$. The following set of manipulations shows that

$$\begin{aligned}
\alpha_n &= f_{n+1}f_{n-1} - f_n^2 \\
&= (f_n + f_{n-1})f_{n-1} - f_n^2 \\
&= f_{n-1}^2 - f_n(f_{n-1} + f_{n-2} - f_{n-1}) \\
&= f_{n-1}^2 - f_n f_{n-1} \\
&= -\alpha_{n-1}
\end{aligned}$$

which thus yields the identity

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n \tag{5.16}$$

One more identity can easily be obtained. Consider the product of two successive Fibonacci numbers

$$f_i f_{i-1} = (f_{i-1} + f_{i-2})f_{i-1} = f_{i-1}^2 + f_{i-1}f_{i-2}$$

This recursive equation is easily solved, leading to the identity

$$f_i f_{i-1} = \sum_{j=1}^{i-1} f_j^2, \quad i = 1, \ldots \tag{5.17}$$

The process of creating identities for Fibonacci numbers could go on almost endlessly since there are literally tens of thousands of such relationships.

The equation $\phi^2 = \phi + 1$ leads to some beautiful equations. Take the square root of this to yield

$$\phi = \sqrt{\phi^2} = \sqrt{\phi + 1}$$

Substituting the square root expression for every occurrence of ϕ shows that:

$$\phi = \sqrt{1+\phi} = \sqrt{1+\sqrt{1+\phi}} = \sqrt{1+\sqrt{1+\sqrt{1+\phi}}}$$

Clearly this continues without end and thus ϕ arises from the infinite cascade of square roots:

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}} \tag{5.18}$$

If the defining equation is rewritten as

$$\phi = 1 + \frac{1}{\phi}$$

then continual substitution for ϕ leads to an infinite continued fraction expansion of ϕ[4]:

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}} \tag{5.19}$$

The golden ratio $\phi \approx 1.6180339$ is one of a handful of fundamental constants like $\pi \approx 3.1415926$, the base of the natural logarithm $e \approx 2.7182818$, or Euler's constant $\gamma \approx 0.5772156$.

[4] An alternative derivation is found in (10.14).

5.2 An Alternate Derivation

Viewing Fibonacci numbers through the eyes of a different model reveals a strikingly different closed form expression. Consider the number of possible sequences of the integers 1 or 2 that, when summed, equals n. Let $\rho(n)$ be the number of different ways to do this. For example, $\rho(2) = 2$ since the sequences 11 and 2 are the only possibilities. Other cases include $\rho(3) = 3$, (111, 12 and 21) and $\rho(4) = 5$, (1111, 112, 121, 211, and 22).

Comparing these values to Fibonacci numbers suggests that $\rho(n) = f_{n+1}$. To show that this is the case it suffices to establish the recurrence

$$\rho(n) = \rho(n-1) + \rho(n-2) \tag{5.20}$$

The small examples above show that this is satisfied for values of $n \leq 4$. The equation is established by induction. For $n > 4$, consider a sequence that adds up to n that starts with a 1. In this case, the remaining integers of the sequence necessarily add up to $n-1$ which we know amounts to $\rho(n-1)$ possible sequences. Similarly for those sequences that start with 2 the remaining numbers must add to $n-2$ which equals a total of $\rho(n-2)$ sequences. These two cases account for all possibilities and thus establishes (5.20).

This perspective opens up a different form of a closed form expression for $\rho(n)$ and thus also for f_{n+1}. Partition all sequences of 1s and 2s by their length. There is only one sequence of length n which consists of all 1s. Consider a sequence having i, 2s, in it. Such a sequence has $n - 2i \geq 0$, 1s, since the sum equals n. This implies the sequence length is $n - 2i + i = n - i$. There are $\binom{n-i}{i}$ ways of choosing the places for the i, 2s, in such sequences. Since $n - 2i > 0$, it must be the case that $i \leq \lfloor n/2 \rfloor$ where $\lfloor x \rfloor$ is the integer portion of x. Summing all possibilities shows that

$$\rho(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} \qquad \left(f_{n+1} = \frac{\phi^{n+1} - \psi^{n+1}}{\sqrt{5}} \right) \tag{5.21}$$

In equation (5.21) the previous closed form expression (5.10) is shown to contrast the striking difference between the two derivations. Such contrasts often happen in mathematics in the form of equations that arise for the same mathematical quantity when viewed from different perspectives. Adjusting the indices of (5.21) to be more natural provides the following identity:

$$f_{2n+1} = \sum_{i=0}^{n} \binom{n+i}{2i} = \sum_{i=0}^{n} \binom{n+i}{n-i} \tag{5.22}$$

5.3 Generalized Fibonacci Numbers

Clearly the starting values for the Fibonacci series are arbitrary and have little influence on any basic properties derived from the essential recurrence $f_n = f_{n-1} + f_{n-2}$. For example, if the sequence starts with a, b (they both cannot be 0) then these *generalized Fibonacci numbers* satisfy

$$g_n = af_{n-1} + bf_n$$

The widely studied *Lucas* sequence (named after Édouard Lucas (1842–1891)), that arises with the selection $a = 2$ and $b = 1$, yields[5]

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \ldots \tag{5.23}$$

A closed form solution for g_n is a modification of equation (5.10)

$$\begin{aligned} g_n &= a\left(\frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}}\right) + b\left(\frac{\phi^n - \psi^n}{\sqrt{5}}\right) \\ &= \frac{\phi^n(b + a\phi^{-1}) - \psi^n(b + a\psi^{-1})}{\sqrt{5}} \\ &= \frac{\phi^n(b - a\psi) - \psi^n(b - a\phi)}{\sqrt{5}} \end{aligned}$$

The last simplification uses the equation $\phi\psi = -1$.

[5]Equation (5.11) also generated this sequence.

5.4 *k*-Bonacci Numbers

Recall that Fibonacci numbers arise from the defining equation $x^2 = x + 1$. Consider the next highest defining equation given by $x^3 = x^2 + x + 1$. Mimicking the previous derivation of Fibonacci numbers in Section 5.1.1 by deriving powers of x yields the following pattern:

Power	x^2 term	x term	1 term
x^3	1	1	1
x^4	2	2	1
x^5	4	3	2
x^6	7	6	4
x^7	13	11	7

Notice that the numbers in each row are the sum of the previous *three* numbers in the preceding rows. Such numbers are said to be *Tribonacci numbers* and the differences in sequences shown in the columns arise, like the difference between Fibonacci and Lucas numbers, from their different initial values. In particular, the first and third column correspond to the sequence starting with $0, 0, 1$,

$$0, 0, 1, 1, 2, 4, 7, 13, \cdots$$

whereas the second column corresponds to the sequence starting with $0, 1, 0$,

$$0, 1, 0, 1, 2, 3, 6, 11, \cdots$$

In all cases, the general term of the recurrence satisfies $t_n = t_{n-1} + t_{n-2} + t_{n-3}$. This generalization of Fibonacci numbers also leads to closed form expressions that have an algebraic and a combinatoric representation although they are substantially more complicated. The combinatoric representation for t_n for the starting value $0, 0, 1$, for example, is given by

$$t_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n - i - 2k}{i + k} \binom{i + k}{k}$$

Such sequences easily generalize to the *k-bonacci numbers* defined by

$$b_n = b_{n-1} + \cdots + b_{n-k} \tag{5.24}$$

with starting values given by $k-1$ zeros followed a 1. A table of these numbers up to $k = 5$ follows:

k	Name	Initial Values	Rest of Sequence										
2	Fibonacci	$0, 1$	1	1	2	3	5	8	13	21	34	55	...
3	Tribonacci	$0, 0, 1$	1	1	2	4	7	13	24	44	81	149	...
4	Tetranacci	$0, 0, 0, 1$	1	1	2	4	8	15	29	56	108	208	...
5	Pentanacci	$0, 0, 0, 0, 1$	1	1	2	4	8	16	31	61	120	236	...

The defining equation for k-bonacci numbers is $x^k = x^{k-1} + \cdots + x + 1$.

It can be shown that every integer has at least one way to write it as the sum of k-bonacci integers. To illustrate this, consider the Fibonacci case where 100 can be written in terms of Fibonacci numbers as $3+8+89$, $1+2+8+89$, $3+8+34+55$ and $3+8+89$. Edouard Zeckendorf (1901–1983) showed for the $k = 2$ case that there is only one way to write such a representation that does not use adjacent Fibonacci numbers ($100 = 3+8+89$ above). It is trivial to construct a *Zeckendorf representation* for small values of n by enumeration. Assume that such a representation is possible for all integers up to n. If $n+1$ is a Fibonacci integer, then it already has such a representation (a Fibonacci number is its own representation). Therefore consider the case where $n+1$ not Fibonacci. This implies that there is a value j that satisfies

$$f_j < n + 1 < f_{j+1}$$

By assumption, the value m defined by $m = n+1-f_j$ has a Zeckendorf representation since it is less than n. Using this representation shows that $m + f_j$ is a representation for the integer $n + 1$ that uses only Fibonacci numbers. But this representation might consist of adjacent Fibonacci numbers and thus not be a Zeckendorf representation. To show that this cannot be the case, use the general recurrence for Fibonacci integers to write

$$m + f_j < f_{j+1} = f_j + f_{j-1} \quad \Longrightarrow \quad m < f_{j-1}$$

5.5 Generalization of the Fibonacci Recurrence

Consider a generalization along different lines. Since the Fibonacci sequence arises from the number of sequences of the numbers 1 and 2 that sum to n, it is only natural to ask what type of numbers arise from patterns only using the numbers 1 and k. Denote the number of such sequences by $\rho_k(n)$. Solving the general case, using a similar argument to the derivation of equation (5.21), yields

$$\rho_k(n) = \sum_{i=0}^{\lfloor n/k \rfloor} \binom{n-(k-1)i}{i}$$

Interestingly, the recurrence equation corresponding to these values is given by

$$g_n = g_{n-1} + g_{n-k}, \qquad n = k+1, \ldots$$

where the initial portion of the sequence consists of k, 1's, followed by a 2. A table of these numbers is given by

k	Initial Values	Rest of Sequence												
2	$1,1,2$	3	5	8	13	21	34	55	89	144	233	377	610	...
3	$1,1,1,2$	3	4	6	9	13	**19**	**28**	**41**	60	**88**	129	189	...
4	$1,1,1,1,2$	3	4	5	7	10	14	19	26	36	50	69	95	...
5	$1,1,1,1,1,2$	3	4	5	6	8	11	15	20	26	34	45	60	...

Another generalization arises if the defining equation is changed so that $x^2 = ax + 1$ for some positive integer a. An analysis similar to that found in the beginning of the chapter shows that this creates a *Fibonacci type sequence* which satisfies $h_i = ah_{i-1} + h_{i-2}$. With initial conditions $h_0 = 0$, $h_1 = 1$, and $h_2 = a$ we find that the n'th type *Hibonacci number* (coining the term) is given by

$$h_{n,a} = \frac{(h_a^+)^n - (h_a^-)^n}{\sqrt{a^2+4}} \tag{5.25}$$

where

$$h_a^+ = \frac{a+\sqrt{a^2+4}}{2} \quad \text{and} \quad h_a^- = \frac{a-\sqrt{a^2+4}}{2} \tag{5.26}$$

It is straightforward to derive equation (5.25) by mimicking the steps leading to equation (5.10). Some illustrative values of such sequences are given in the following table:

a	Hibonacci Sequence								
1	1	1	2	3	5	8	13	21	34
2	1	2	5	12	29	70	169	408	985
3	1	3	10	33	109	360	1,189	3,927	12,970
4	1	4	17	72	305	1,292	5,473	23,184	98,209
5	1	5	26	135	701	3,640	18,901	98,145	509,626
6	1	6	37	228	1,405	8,658	53,353	328,776	2,026,009
7	1	7	50	357	2,549	18,200	129,949	927,843	6,624,850

The continued fraction expansion of h_a^+ is a generalization of ϕ found in equation (5.19) and is given by[6]

$$h_a^+ = a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{\ddots}}}}} \tag{5.27}$$

Also observe that the generalization of equation (5.18) for Hibonacci numbers is given by

$$h_a^+ = \sqrt{1 + a\sqrt{1 + a\sqrt{1 + a\sqrt{1 + \cdots}}}}$$

Clearly there are endless generalizations and results to be found along these, and other, lines.

[6]Derivation of equation (5.27) is found in equation (10.13) and the values in the table found on page 137.