Chapter 2
Let Me Count the Ways

How do I love thee? Let me count the ways.
I love thee to the depth and breadth and height
My soul can reach, when feeling out of sight
For the ends of being and ideal grace.

Elizabeth Barrett Browning (1806–1861)

Elizabeth Browning probably didn't realize that she was really talking about mathematics when she penned her 43rd sonnet, *How Do I Love Thee?* This chapter provides a more comprehensive answer to this question than Browning was able to present in the remaining stanzas where she enumerates the ways she loves the veiled object of her sonnet. With the power of mathematics, equations are derived that provide a thorough enumeration, leaving no stone untouched. This is done through the simple expedient of selecting a set of items from a set. It is surprising, as when one falls in love, how fast innocent simplicity explodes into a tangled web of complexity. Perhaps this is what makes love stories, and mathematics, so enduringly interesting.

Assume there are n distinguishable items in a set from which k items are selected.[1] The object of this chapter is to count the number of possible ways to make such a selection. There are four different counting paradigms that depend upon the order items are selected and whether selected items are removed or returned to the set. *Selection with replacement* occurs when selected items are returned, otherwise the selection is termed *selection without replacement*. A *permutation* occurs when the order of selected items is maintained, otherwise the selection corresponds to a *combination*.

[1] For example, different colored balls are *distinguishable* whereas electrons which have no discernible differences are *indistinguishable*.

R. Nelson, *A Brief Journey in Discrete Mathematics*,
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To illustrate the four different selection paradigms, consider the case where 2 items are selected from the set $\{1, 2, 3\}$. A selection can be represented by an integer so that 31 corresponds to first selecting item 3 followed by item 1. The first line of the table below shows that there are 6 possible permutations when items are not returned to the set (possibilities represented by $\{12, 21, 13, 31, 23, 32\}$). When order is not maintained (so that the selection 12 is counted as being the same as the selection 21) then the number of possibilities reduces to 3 (possibilities represented by $\{12, 13, 23\}$). When items are returned after selection, the number of possibilities in each of the above cases increases by 3 corresponding to the addition of possibilities given by $\{11, 22, 33\}$.

	#	Permutation	#	Combination
Without Replacement	6	$\{12, 21, 13, 31, 23, 32\}$	3	$\{12, 13, 23\}$
With Replacement	9	$\{12, 21, 13, 31, 23, 32, 11, 22, 33\}$	6	$\{12, 13, 23, 11, 22, 33\}$

The rest of this chapter derives equations for the number of possible selections for each of the four counting paradigms, establishes relationships between them, and derives identities that arise from the resultant equations.[2] Throughout this chapter, let the number of different items in a set be denoted by n from which k items are selected.

2.1 Permutations: With and Without Replacement

We first consider permutations when items are not returned to the set, a quantity that is denoted by $p^r_{k,n}$.[3] This value satisfies the recurrence

$$p^r_{k,n} = np^r_{k-1,n}, \quad n \geq 1, \; k \geq 1 \tag{2.1}$$

Initial values of the recurrence are $p^r_{0,n} = 1$ and $p^r_{k,n} = 0$ for $k < 0$ or $n < 0$. To explain (2.1) note that the first of the k selections can be done in n different ways, leaving $k - 1$ items left to be selected. Since

[2]See the Appendix for a review of using recurrence to solve problems.

[3]The "r" superscript means *with replacement* rather than being a numeric index value.

the selected item is returned to the step, this leaves $p^r_{k-1,n}$ remaining possibilities. Recurrence (2.1) can be solved to yield

$$p^r_{k,n} = n^k \tag{2.2}$$

Values of $p^r_{k,n}$ for small parameter values are given in the following table:

	Values of $p^r_{k,n} = n^k$							
$k\backslash n$	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9
2		9	16	25	36	49	64	81
3			64	125	216	343	512	729
4				625	1,296	2,401	4,096	6,561
5					7,776	16,807	32,768	59,049
6						117,649	262,144	531,441
7							2,097,152	4,782,969
8								43,046,721

The number of different possibilities for selecting permutations without replacement, denoted by $p_{k,n}$, satisfies the recursion

$$p_{k,n} = np_{k-1,n-1}, \quad n \geq 1, \; k \geq 1 \tag{2.3}$$

with initial values of $p_{0,n} = 1$ and $p_{k,n} = 0$ for $k < 0$ of $n < 0$.[4] To explain this, note again that the first selection can be done in n different ways. Since the selected item now is removed from the set, this leaves the remaining $k-1$ items to be selected from a set of $n-1$ items, thus the quantity $p_{k-1,n-1}$. This recurrence can be solved to yield

$$p_{k,n} = n^{\underline{k}} \tag{2.4}$$

where the *lower factorial* is defined by

$$n^{\underline{k}} = n(n-1)\cdots(n-k+1) \tag{2.5}$$

[4]The restrictions to have non-negative arguments for $p^r_{k,n}$ and $p_{k,n}$ can be relaxed but will not be considered in this book.

As an aside, note that two algebraic identities follow directly definition (2.5):

$$n^{\underline{k}} = n^{\underline{\ell}}\ (n-\ell)^{\underline{k-\ell}}, \quad \ell \le k \tag{2.6}$$

and

$$n\ n^{\underline{k}} = n^{\underline{k+1}} + kn^{\underline{k}} \tag{2.7}$$

Values of $p_{k,n}$ for small parameter values are given in the following table (notice the size differences between this and the table above):

	Values of $p_{k,n} = n^{\underline{k}}$							
$k \backslash n$	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9
2		6	12	20	30	42	56	72
3			24	60	120	210	336	504
4				120	360	840	1,680	3,024
5					720	2,520	6,720	15,120
6						5,040	20,160	60,480
7							40,320	181,440
8								362,880

So far the analysis is straightforward; however, a counterintuitive result follows from these the two defining equations (2.3) and (2.4). The *birthday problem* is typically stated by calculating the probability that, from a set of k people, at least two people have the same birthday. To calculate this, assume that birthdays occur uniformly throughout the year.[5] If all birthdays are unique, then a selection from 365 days of k items is a permutation without replacement. Setting $n = 365$ in equation (2.4) shows that there are $365^{\underline{k}}$ such permutations. The number of total possible selections from 365 days that allows duplicate days is equivalent to selecting k days with replacement which is given

[5] A smoothed plot of the frequency of birthdays ascends from a low around January until it reaches a peak in September. There is thus a greater chance that two or more people have a common birthday than what is calculated from equation (2.8).

by equation (2.2), 365^k. Thus, the fraction of random k selections where there are no duplicate birthdays equals[6]

$$\frac{p_{k,n}}{p^r_{k,n}} = \prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right) \tag{2.8}$$

To determine the probability that there are multiple birthdays, subtract (2.8) from 1. The numerical results are surprising as shown in the following table:

k	10	15	20	25	30	35	40	45	50	55
$1 - p_{k,n}/p^r_{k,n}$	.117	.253	.411	.569	.706	.814	.891	.941	.97	.986

The break-even point occurs when $k = 23$, showing that there is a 50.72% probability that two people have the same birthday! The table shows how fast this percentage increases with k so that the 99% level is breached at $k = 57$. A fun way to see mathematics in action is to grab the microphone at a typical wedding and ask people having a birthday today to raise their hands. It will almost never fail that at least a couple of hands shoot up!

2.1.1 Dearrangements

To explain another problem that can be solved only using equations (2.3) and (2.4), consider a list of the integers $(1, 2, \ldots, n)$ that are permuted to a new ordering $(q_1, q_2, \ldots, q_n)$ so that no integer is in its original position, $q_i \neq i,\ i = 1, \ldots, n$. Such a rearrangement is termed a *dearrangement*. Denote the number of such possibilities, by g_n which has boundary values: $g_0 = 1$ and $g_1 = 0$. Consider integer j and assume that after permutation it is in position k, and hence $g_k = j$. There are $n-1$ possibilities to select k if $k \neq j$. The number of remaining dearrangements that are possible depend on where integer k is permuted. If k exchanges position with j, so that $g_j = k$, then this leaves $n - 1$ remaining items in the dearrangement, a value given by

[6] The value of $p_{k,n}$ and $p^r_{k,n}$ soon swamps a computer's floating point range for large argument values. This is the reason why the value is computed as the product of simple ratios.

g_{n-1}. If this is not the case and integer k is found in position $\ell \neq j$ so that $g_\ell = k$, then two slots of the permutation are taken leaving $n-2$ items left in the dearrangement, a value given by g_{n-2}. Summing these disjoint possibilities shows that

$$g_n = (n-1)(g_{n-1} + g_{n-2}) \tag{2.9}$$

The number of dearrangements grows quickly with n as shown in the following table:

n	2	4	6	8	10	12	14
g_n	1	9	265	14,833	1,334,961	176,214,841	32,071,101,049

A billion dearrangements are surpassed with $n = 13$ (2,290,792,932 ways to be exact). Thus, if you take a fresh pack of cards, separate out one suit in its numeric order and shuffle these 13 cards thoroughly, then about 36.787% of 13! possibilities will correspond to dearrangements. In fact, it is not too difficult to show that as n gets large, the fraction of random permutations of n items that are dearrangements converges to $1/e$ where $e \approx 2.718281828$ is called *Euler's number* after the Swiss mathematician Leonard Euler (1707–1783) who, among his vast achievements, studied properties of the exponential function.[7]

The difficulty with solving recurrence (2.9) lies in the multiplicative factor $(n-1)$ found in the equation. To counteract this, consider a scaled version where $f_n = g_n/n!$. This produces the recurrence

$$\begin{aligned} f_n = \frac{g_n}{n!} &= \frac{n-1}{n!}\,(g_{n-1} + g_{n-2}) \\ &= \frac{n-1}{n!}\,((n-1)!f_{n-1} + (n-2)!f_{n-2}) \\ &= \left(1 - \frac{1}{n}\right) f_{n-1} + \frac{1}{n} f_{n-2} \end{aligned}$$

Rewriting this reveals a difference that can be formed between successive index values

$$f_n - f_{n-1} = -\frac{1}{n}\,(f_{n-1} - f_{n-2})$$

[7] Convergence is quick. The value of $|g_{13}/13! - 1/e|$ is about 10^{-11}.

This can be iterated to yield

$$f_n - f_{n-1} = (-1)^n \frac{1}{n!}$$

This telescopes to the boundary $f_0 = 1$ leading to the following summation:

$$f_n = \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

Converting back to the original recursion produces a lovely result: the number of dearrangements equals an alternating sum of permutations:

$$g_n = n! f_n = \sum_{i=0}^{n} (-1)^i p_{i,n} \tag{2.10}$$

2.2 Combinations: Without Replacement

Consider a particular combination obtained from selecting k items from a set of size n. The ordering of the items in this combination can be permuted in $k!$ ways without adding to the number of combinations. Thus an equation for the number of possible combinations, denoted by $c_{k,n}$, is given by

$$c_{k,n} = \frac{n^{\underline{k}}}{k!} = \prod_{j=0}^{k-1} \frac{n-j}{k-j} \tag{2.11}$$

It is customary to write this using a *binomial coefficient*[8]

$$c_{k,n} = \binom{n}{k} = \frac{n!}{(n-k)!\ k!} \tag{2.12}$$

[8] The product expansion (2.11) is used when computing the value of a binomial coefficient.

Values of $c_{k,n}$ for small parameter values are given in the following table:

	Values of $c_{k,n} = \binom{n}{k}$								
$k\backslash n$	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9
2		1	3	6	10	15	21	28	36
3			1	4	10	20	35	56	84
4				1	5	15	35	70	126
5					1	6	21	56	126
6						1	7	28	84
7							1	8	36
8								1	9
9									1

A recursive derivation of (2.11) is instructive and proves to be useful in future derivations. Initial conditions are easily calculated: $c_{k,n} = 0$ for $k > n$ or $k < 0$, $c_{1,n} = n$ and $c_{n,n} = 1$. To derive a general recurrence for $c_{k,n}$, partition selections into two disjoint sections. If item j is selected, then the remaining $k-1$ items must be selected from the $n-1$ remaining items, a quantity given by $c_{k-1,n-1}$. If item j is not selected, then it is equivalent to not being in the set, a quantity given by $c_{k,n-1}$. The total number of combinations without replacement is the sum of these two disjoint possibilities and thus equals

$$c_{k,n} = c_{k-1,n-1} + c_{k,n-1} \tag{2.13}$$

Simple algebra establishes that equation (2.11) (or equation (2.12)) satisfies recursion (2.13). The value of $c_{k,n}$ equals the number of ways to pick k items from a set of n or, equivalently, equals the number of possible subsets of size k that can be formed from a set of n items.

2.2.1 Binomial Identities

Straightforward algebra establishes the following identities between binomial coefficients:

$$\binom{n}{k} = \binom{n}{n-k} \tag{2.14}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{2.15}$$

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} = \frac{n-k+1}{k}\binom{n}{k-1} \tag{2.16}$$

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j} \tag{2.17}$$

Identity (2.14) shows the symmetry of binomials coefficients and identity (2.15) is a direct restatement of recurrence (2.13). Identity (2.16) follows from the product expansion of (2.11) and from setting $j = 1$ in (2.17). A typical application of (2.17) is to separate variables j and k in a summation. As an example, consider the following derivation which uses (2.17) in the first step:

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} \frac{a^k}{1+k} &= \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k+1} a^k \\ &= \frac{1}{n+1} \sum_{j=1}^{n+1} \binom{n+1}{j} a^{j-1} \\ &= \frac{(1+a)^{n+1}-1}{a(1+n)} \end{aligned} \tag{2.18}$$

Two special cases of this identity arise when $a = 1$ or $a = -1$:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{1+k} = \frac{2^{n+1}-1}{n+1} \tag{2.19}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{1+k} = \frac{1}{n+1} \tag{2.20}$$

Summations of binomial coefficients are typically derived using induction from an easily calculated base class. There are two ways to interpret the recursion of equation (2.13). The *backward view* starts from $c_{k,n}$ and recurses to a lower value of n with the values $c_{k-1,n-1}$ and $c_{k,n-1}$. This generates an identity which starts from the base case, $c_{0,2} + c_{1,2} + c_{2,2} = 4$, suggesting that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{2.21}$$

Assume this is true for all values less than or equal to n. Using identity (2.15) implies that

$$\begin{aligned}\sum_{k=0}^{n+1}\binom{n+1}{k} &= \sum_{k=0}^{n+1}\binom{n}{k-1}+\binom{n}{k}\\ &= \sum_{k=0}^{n+1}\binom{n}{k-1}+2^n\\ &= \sum_{k=0}^{n}\binom{n}{k}+\binom{n}{-1}+2^n\\ &= 2^n+0+2^n\\ &= 2^{n+1}\end{aligned}$$

thus proving the claim.

A variation of identity (2.21) involves summing over even or odd indexed values of k. Notice that small examples include: $c_{0,3}+c_{2,3}=4$ and $c_{0,4}+c_{2,4}+c_{4,4}=8$ which suggest that

$$\sum_{k \text{ even}}\binom{n}{k}=2^{n-1} \tag{2.22}$$

Also note that $c_{1,3}+c_{3,3}=4$ and $c_{1,4}+c_{3,4}=8$ which suggests that

$$\sum_{k \text{ odd}}\binom{n}{k}=2^{n-1} \tag{2.23}$$

Assume that both of these assumptions holds for all values up to n and calculate

$$\begin{aligned}\sum_{k \text{ even}}\binom{n+1}{k} &= \sum_{k \text{ even}}\binom{n}{k-1}+\binom{n}{k}\\ &= 2^{n-1}+\sum_{k \text{ even}}\binom{n}{k-1}\\ &= 2^{n-1}+\sum_{k \text{ odd}}\binom{n}{k}\\ &= 2^{n-1}+2^{n-1}=2^n\end{aligned}$$

A similar argument holds if the initial summation takes place over odd indices. This establishes the double induction and proves identities (2.22) and (2.23).

Equations (2.21), (2.22), and (2.23) are all special cases of a much deeper result—*the binomial theorem*. This states that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{2.24}$$

This is easily established for small n values. Assume it holds for all values up to n. Then identity (2.15) shows that the pattern continues:

$$\begin{aligned}
(x+y)^{n+1} &= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k} \\
&= \sum_{k=0}^{n+1} \left(\binom{n}{k} + \binom{n}{k-1} \right) x^k y^{n+1-k} \\
&= y \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} + x \sum_{k=1}^{n+1} \binom{n}{k-1} x^{k-1} y^{n+1-k} \\
&= y(x+y)^n + x \sum_{\ell=0}^{n} \binom{n}{\ell} x^k y^{n-k} \\
&= y(x+y)^n + x(x+y)^n = (x+y)^{n+1}
\end{aligned}$$

There are a countless number of identities that arise by varying parameters of the binomial theorem besides those just mentioned. Listing just a few:

$$\left(1 - \frac{1}{\ell}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{-1}{\ell}\right)^k \tag{2.25}$$

$$(x+a)^n - (x-a)^n = 2 \sum_{k \text{ odd}} \binom{n}{k} x^{n-k} a^k \tag{2.26}$$

$$\left(\frac{1}{x} + \frac{1}{y}\right)^n = \frac{1}{y^n} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{y}{x}\right)^k \tag{2.27}$$

$$\sum_{k=0}^{n-i}(-1)^k \binom{n-i}{k} = \begin{cases} 0, \, i = 0, \ldots, n-1 \\ 1, \, i = n \end{cases} \tag{2.28}$$

A further identity can be obtained by forming a telescoping summation combined with the binomial theorem. To derive this, observe that

$$\begin{aligned} n &= \sum_{k=1}^{n} \binom{n}{k} k - \binom{n}{k-1} (k-1) \\ &= \sum_{k=1}^{n} \binom{n}{k-1} (n-2k+2) \qquad \text{From Identity (2.16)} \\ &= n + \sum_{k=1}^{n+1} \binom{n}{k-1} (n-2k+2) \\ &= n + \sum_{k=0}^{n} \binom{n}{k} (n-2k) \end{aligned}$$

Thus

$$\sum_{k=0}^{n} \binom{n}{k} (n-2k) = 0$$

which, using the binomial theorem, establishes the identity

$$\sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1} \tag{2.29}$$

A similar telescoping summation can be used to calculate

$$\sum_{k=0}^{n} \binom{n}{k} k(k-1) = n(n-1)2^{n-2} \tag{2.30}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} k^2 = n(n+1)2^{n-2} \tag{2.31}$$

The next identity we derive is the *forward view* of recurrence (2.13) that proceeds from $n-1$ to n and corresponds to a summation of the numerator of the combinatorial coefficient. This summation follows directly from the listing recursions from (2.13):

$$\begin{aligned} c_{k+1,n+1} &= c_{k,n} + c_{k+1,n} \\ c_{k+1,n} &= c_{k,n-1} + c_{k+1,n-1} \\ c_{k+1,n-1} &= c_{k,n-2} + c_{k+1,n-2} \\ \vdots \quad &\quad \vdots \qquad\quad \vdots \\ c_{k+1,k+2} &= c_{k,k+1} + c_{k+1,k+1} \end{aligned}$$

Collecting these yields

$$c_{k+1,n+1} = c_{k,n} + c_{k,n-1} + \cdots + c_{k,k+1} + 1$$

implies that

$$\binom{n+1}{k+1} = \sum_{\ell=k}^{n} \binom{\ell}{k} \tag{2.32}$$

With the index substitution, $j = \ell - k$, the right-hand side of identity (2.32) can be rewritten as

$$\sum_{\ell=k}^{n} \binom{\ell}{k} = \sum_{j=0}^{n-k} \binom{k+j}{k} \tag{2.33}$$

Making the substitution $m - 1 = n - k$, with equations (2.32) and (2.33), shows that[9]

$$\sum_{r=0}^{m-1} \binom{k+r}{k} = \binom{m+k}{k+1} \tag{2.34}$$

Combinatoric arguments can often provide insight into identities without the need for algebraic manipulation. Consider selecting k items

[9] This equation is termed Pascal's equation after the French mathematician Blaise Pascal (1623–1662).

from a list of n. Divide the list n into two sublists of size m and $n-m$ where $0 \leq m \leq n$ and suppose that j items are selected from the first list and $k-j$ items from the second list. There are $c_{j,m}c_{k-j,n-m}$ ways in which such a selection can be made. Accounting for all such possibilities leads to the identity

$$\binom{n}{k} = \sum_{j=0}^{n} \binom{m}{j}\binom{n-m}{k-j}, \quad 0 \leq m \leq n \tag{2.35}$$

The conditions we have placed on binomial terms so that $c_{k,n} = 0$ if $k < 0$, $n < 0$ or $k > n$ allow us to write the right-hand side of equation (2.35) without needing to put restrictions on the binomial coefficients contained in the summation. This identity is termed *Vandermonde convolution* named after Alexandre-Théophile Vandermonde (1735–1796), a French mathematician who supported the French revolution of 1789 and is most known for his mathematical work in determinants.

2.3 Combinations with Replacement

This leaves the last problem—to calculate the number of combinations obtained when one uses a replacement strategy. To derive an equation, let $c^r_{k,n}$ denote the number of combinations obtained when selecting k items from a set of n items using replacement.[10] We set $c^r_{0,1} = 1$ and it is clear that $c^r_{1,n} = n$ and $c^r_{k,1} = 1$. To derive a general equation for $c^r_{k,n}$, partition selections into two disjoint sections. If item j is selected, then the number of combinations with replacement is the same as if it had been selected on the first selection, a quantity given by $c^r_{k-1,n}$. On the other hand, if item j is not selected, then it is equivalent to not being in the set, a quantity given by $c^r_{k,n-1}$. The total number of combinations is the sum of these two disjoint possibilities and thus equals

$$c^r_{k,n} = c^r_{k-1,n} + c^r_{k,n-1} \tag{2.36}$$

[10] Again, the r superscript means *replacement* rather than being an integer index.

It is interesting to compare equations (2.13) and (2.36). They differ only in the indices of the first term. The n index of $c^r_{k-1,n}$ as compared to the $n-1$ index of $c_{k-1,n-1}$ is a result of putting a selected item back in the list when the replacement policy is utilized. This results in a substantial increase in the number of possibilities that a replacement policy has in comparison to a non-replacement policy.

To proceed with the derivation, consider equation (2.36) where $k = 2$:

$$\begin{aligned} c^r_{2,n} &= c^r_{1,n} + c^r_{2,n-1} \\ &= c^r_{1,n} + c^r_{1,n-1} + c^r_{2,n-2} \end{aligned}$$

Observe that $c^r_{2,n-2}$ is of the same form as $c^r_{2,n}$ except it moves 2 down on the n value which suggests iterating

$$\begin{aligned} c^r_{2,n} &= c^r_{1,n} + c^r_{1,n-1} + \cdots + c^r_{1,2} + c^r_{1,1} \\ &= n + (n-1) + \cdots + 2 + 1 \\ &= \binom{n+1}{2} \end{aligned}$$

This case suggests the following guess for a general solution:

$$c^r_{k,n} = \binom{n+k-1}{k}$$

Assume this equation holds for all values less than or equal to some value k. Then, using the recurrence (2.36) permits

$$\begin{aligned} c^r_{k+1,n} &= c^r_{k,n} + c^r_{k+1,n-1} \\ &= c^r_{k,n} + c^r_{k,n-1} + c^r_{k+1,n-2} \\ &= c^r_{k,n} + c^r_{k,n-1} \cdots + c^r_{k,2} + c^r_{k+1,1} \\ &= \binom{k+n-1}{k} + \cdots + \binom{k+2}{k} + \binom{k+1}{k} + 1 \\ &= \sum_{r=0}^{n-1} \binom{k+r}{k} = \binom{n+(k+1)-1}{k+1} \end{aligned}$$

where the last equation uses the identity (2.34). This shows that the induction is satisfied and thus that

$$c^r_{k,n} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \tag{2.37}$$

where *binomial-R coefficients* are defined by

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \binom{n+k-1}{k} \tag{2.38}$$

Values of $c^r_{k,n}$ for small parameter values are given in the following table:

	Values of $c^r_{k,n} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$								
$k \backslash n$	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9
2	1	3	6	10	15	21	28	36	45
3	1	4	10	20	35	56	84	120	165
4	1	5	15	35	70	126	210	330	495
5	1	6	21	56	126	252	462	792	1,287
6	1	7	28	84	210	462	924	1,716	3,003
7	1	8	36	120	330	792	1,716	3,432	6,435
8	1	9	45	165	495	1,287	3,003	6,435	12,870
9	1	10	55	220	715	2,002	5,005	11,440	24,310

2.3.1 Binomial-R Identities

Some identities that are easy to verify include

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \left\langle \begin{matrix} k+1 \\ n-1 \end{matrix} \right\rangle \tag{2.39}$$

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle + \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle$$

$$\left\langle {n \atop k} \right\rangle = \frac{n}{k} \left\langle {n+1 \atop k-1} \right\rangle$$

$$\binom{n}{k} = \left\langle {n+1-k \atop k} \right\rangle$$

Adding one more term to (2.34) produces a similar identity expressed in terms of binomial-R coefficients:

$$\begin{aligned} \sum_{r=0}^{n} \binom{k+r}{k} &= \sum_{r=0}^{n-1} \binom{k+r}{k} + \binom{n+k}{k} \\ &= \binom{n+k}{k+1} + \binom{n+k}{k} \\ &= \binom{n+k+1}{k+1} \\ &= \left\langle {n+1 \atop k+1} \right\rangle \end{aligned} \tag{2.40}$$

The flow of this derivation uses equations from (2.16) and (2.34). One can also derive an identity involving the sum binomial-R coefficients given by

$$\sum_{k=0}^{m} \left\langle {n \atop k} \right\rangle = \left\langle {n+1 \atop m} \right\rangle \tag{2.41}$$

The size difference between combinations without and with replacement can be quantified, similar to that of equation (2.8), by forming their ratio:

$$\frac{c_{k,n}}{c^r_{k,n}} = \prod_{j=1}^{k-1} \left(1 - \frac{k}{n+k-j}\right) \tag{2.42}$$

With the same values as in the birthday problem, $k = 23$ and $n = 365$, the ratio of combinations given in equation (2.42) yields an answer of around 25% in comparison to 50% found with permutations.

2.3.2 Polynomial Solutions to Combinatorial Problems

To view the previous analysis within a general framework, suppose there are a set of n integers, m_i, $i = 1, \ldots, n$, each between 0 and k having a total sum that equals k:

$$m_1 + m_2 + \cdots + m_n = k, \qquad 0 \leq m_i \leq k$$

How many ways can such numbers be selected to satisfy these constraints? This problem can be thought as the number of n partitions of the integer k. For example, if $n = 3$ and $k = 2$, then there are six possible partitions:

$$\{2,0,0\},\ \{1,1,0\},\ \{1,0,1\},\ \{0,2,0\},\ \{0,1,1\},\ \{0,0,2\} \tag{2.43}$$

To answer the general question, suppose that k balls are thrown randomly into n buckets. The value of m_i counts the total number of balls that land in bucket i. Some thought shows that this is equivalent to the number of combinations for selecting k items from a set of n where a replacement strategy is used. To see this, associate a bucket with each item in the set of n items. With this association, selecting the i'th item in the set is equivalent to throwing a ball into the i'th bucket. The constraints on the buckets mean that only up to k balls can land in any particular bucket. Thus the solution to the posed question is that the number of possible partitions equals $c^r_{k,n}$. This simple solution leads to an extremely useful concept which will be derived in the following paragraphs.

The great thing about being a mathematician is that your work, which is really like play, can be done almost anywhere so there is never a danger of becoming bored. Perhaps, for instance, you are stuck in the middle of a theater during a particularly uninteresting play. Then, as long as you have a pen, you can play with equations on the back or margins of the program. You might, during one of these occasions, jot down a simple infinite polynomial like

$$f(x) = 1 + x + x^2 + x^3 + \cdots$$

If x is between 0 and 1, then such a sequence convergences and its sum is given by

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$

The linkage between this and combinatorics arises when f is raised to a power

$$f^n(x) = \left(\frac{1}{1-x}\right)^n$$

It is clear that this expression is another infinite polynomial

$$f^n(x) = \sum_{i=0}^{\infty} a_{i,n} x^i$$

with integer coefficients $a_{i,n}$. What are these coefficients?

As a concrete example, suppose that $n = 3$ and $k = 2$. Consider the value of $a_{2,3}$ which corresponds to the coefficient of x^2 in $f^3(x)$. To facilitate the argument, write

$$f^3(x) = \underbrace{(x^0 + x^1 + x^2 + \cdots)}_{\text{first group}} \times \underbrace{(x^0 + x^1 + x^2 + \cdots)}_{\text{second group}} \times \underbrace{(x^0 + x^1 + x^2 + \cdots)}_{\text{third group}}$$

Let the exponent of x selected in the first group be denoted by m_1 and similarly define m_2 and m_3. When the sum of these x exponents, $m_1 + m_2 + m_3$, equals 2 then those factors contribute to the value of $a_{2,3}$. Since the coefficients of $f(x)$ are all 1, the value of $a_{2,3}$ is the number of ways three non-negative integers sum to 2. In our example, this corresponds to the sets given in (2.43). The solution to the problem posed a couple of paragraphs back showed that this quantity equals $c^r_{2,3}$. Clearly this argument generalizes, and thus $a_{i,n} = c^r_{i,n}$ which establishes the equation:

$$f^n(x) = \sum_{i=0}^{\infty} \left\langle {n \atop i} \right\rangle x^i \tag{2.44}$$

To link this analysis back to counting, recall that $c^r_{k,n}$ is the number of ways that $m_1 + m_2 + \cdots + m_n = k$ where m_i was bounded below by 0 and above by k. Suppose that the value of m_i has a different set of constraints. For example, consider a simple case where $n = 3$ and $k = 2$ that has the following constraints:

$$m_1 \in \{0, 1\}, \quad m_2 \in \{1, 2\}, \quad m_3 \in \{0, 2\}$$

The number of combinations where $m_1 + m_2 + m_3 = 2$ can be easily listed

$$\{1, 1, 0\}, \quad \{0, 2, 0\} \tag{2.45}$$

But is there a way to calculate that there are 2 possibilities without actually listing them all?

The answer lies in the previous analysis. Associate the polynomial $x^0 + x^1$ with the first value of m_1 which mathematically incorporates the constraint that $m_1 \in \{0, 1\}$. Similarly associate the polynomial $x^1 + x^2$ with m_2 and $x^0 + x^2$ with m_3. The polynomial $g(x)$ defined by the product of all these sub polynomials is given by

$$g(x) = (x^0 + x^1)(x^1 + x^2)(x^0 + x^2) = x + 2x^2 + 2x^3 + 2x^4 + x^5 \tag{2.46}$$

By construction, the coefficient of x^ℓ corresponds to the number of possible combinations that result when $m_1 + \ldots + m_k = \ell$. Hence, the expansion in equation (2.46) shows that the number of such possibilities equals 2 when $\ell = 2$, thus confirming the enumeration given in (2.45). Additionally, the expansion shows that there are no ways to sum to 0, one way to sum to 1 or 5, and two ways to sum to 2, 3, or 4.

To state the general case, suppose that m_i can only have values

$$m_i \in \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\} \tag{2.47}$$

and define

$$g_i(x) = x^{v_{i,1}} + x^{v_{i,2}} + \cdots + x^{v_{i,n_i}}$$

from which the following product is formed:

$$g(x) = \prod_{i=1}^{n} g_i(x)$$

Then ,the coefficient of x^ℓ in the polynomial $g(x)$ equals the number of ways that $m_1 + m_2 + \cdots + m_n = \ell$ where each m_i satisfies its particular set of constraints (2.47). In essence, this technique solves an entire class of difficult problems—a strikingly deep result considering that it arises from a simple combinatoric derivation.

As a final example, consider the number of different values that can be obtained by summing combinations of the first n primes, $p_i, i = 1, \ldots, n$. Each prime can either be selected in the sum or not and thus $g_i(x) = 1 + x^{p_i},\ i = 1, \ldots, n$, and

$$g(x) = \prod_{i=1}^{n} 1 + x^{p_i}$$

For the special case of $n = 4$ this expansion yields

$$g(x) = 1 + x^2 + x^3 + 2x^5 + 2x^7 + x^8 + x^9 + 2x^{10} + 2x^{12} + x^{14} + x^{15} + x^{17}$$

There are multiple results contained in this expression: there are two ways to sum the first four primes (2,3,5,7) leading to integers in set $\{5, 7, 10, 12\}$ and one way for integers in set $\{0, 2, 3, 8, 9, 14, 15, 17\}$. The set of integers $\{4, 6, 11, 13, 16, 18, 19, \ldots\}$ has been left out of the set of possibilities and there are 12 different sums that are possible since this equals the number of terms contained in $g(x)$. The number of prime numbers contained in the set of possibilities equals 5 (the set $\{2, 3, 5, 7, 17\}$) and the largest consecutive sequence of numbers has four members (the series 7, 8, 9, 10). Clearly, these observations open new questions concerning how they scale as n increases. Is the longest consecutive series, for instance, bounded if n increases indefinitely?

2.4 Transforms and Identities

Let $\boldsymbol{x} = (x_0, \ldots, x_n)$ be a vector of length $n + 1$ and define functions $f_\ell(\boldsymbol{x})$ and $g_\ell(\boldsymbol{x})$ as follows:

$$f_\ell(\boldsymbol{x}) = \sum_{k=0}^{\ell} \binom{\ell}{k} x_k \tag{2.48}$$

$$g_\ell(\boldsymbol{x}) = \sum_{k=0}^{\ell} (-1)^{\ell-k} \binom{\ell}{k} x_k, \qquad \ell = 0, \ldots, n \tag{2.49}$$

Define $\boldsymbol{a} = (a_0, \ldots, a_n)$ and $\boldsymbol{b} = (b_0, \ldots, b_n)$ and set $b_\ell = f(\boldsymbol{a}),\ \ell = 0, \ldots, n$. Expanding (2.49) yields

$$
\begin{aligned}
g_\ell(\boldsymbol{b}) &= \sum_{k=0}^{\ell} (-1)^{\ell-k} \binom{\ell}{k} b_k \\
&= \sum_{k=0}^{\ell} (-1)^{\ell-k} \binom{\ell}{k} \sum_{i=0}^{k} \binom{k}{i} a_i \\
&= \sum_{i=0}^{\ell} a_i \binom{\ell}{i} \sum_{k=i}^{\ell} \binom{\ell-i}{k-i} (-1)^{\ell-k}, \qquad \text{From identity (2.17)} \\
&= \sum_{i=0}^{\ell} a_i \binom{\ell}{i} \sum_{j=0}^{\ell-i} \binom{\ell-i}{j} (-1)^{\ell-j-i} \\
&= \sum_{i=0}^{\ell} a_i (-1)^{\ell-i} \binom{\ell}{i} \sum_{j=0}^{\ell-i} \binom{\ell-i}{j} (-1)^{j} \\
&= a_\ell \qquad \text{From identity (2.28)}
\end{aligned}
$$

The paired equations $b_\ell = f_\ell(\boldsymbol{a})$ and $a_\ell = g_\ell(\boldsymbol{b})$ show that functions f_ℓ and g_ℓ are inverse functions of each other. This observation creates a useful transformation termed *binomial transformation*. To distinguish these functions, f_ℓ is typically termed a *binomial transform* and g_ℓ a *binomial inverse*.

Binomial transformation can be used to create a paired set of identities whenever a sequence satisfies either (2.48) or (2.49). For example, equation (2.31) corresponds to the binomial transform of $a_\ell = \ell^2$. This implies that $b_\ell = \ell(\ell+1)2^{\ell-2},\ \ell = 0, \ldots, n$ thus creating the paired identity

$$n^2 = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k(k+1) 2^{k-2} \tag{2.50}$$

The inverse transforms of (2.21), (2.25), (2.29), and (2.30) are given by

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^k = 1 \tag{2.51}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left(1 - \frac{1}{\ell}\right)^k = \left(\frac{-1}{\ell}\right)^n \tag{2.52}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k 2^{k-1} = n \tag{2.53}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k(k-1) 2^{k-2} = n(n-1) \tag{2.54}$$

The inverse transform of (2.18) provides the paired identity

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{(1+a)^{k+1}-1}{k+1} = \frac{a^{n+1}}{n+1} \tag{2.55}$$

A special case of (2.18) and (2.55) for $a = 2$ creates the following two identities:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{2^{k+1}}{k+1} = \frac{3^{n+1}-1}{n+1} \tag{2.56}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{3^{k+1}-1}{k+1} = \frac{2^{n+1}}{n+1} \tag{2.57}$$

Another form of a binomial transform is given by

$$h_\ell(\boldsymbol{x}) = \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} x_k \tag{2.58}$$

which defines an *involution*, a function that is its own inverse. To establish this, let $b_\ell = h_\ell(\boldsymbol{a})$ and proceed as follows:

$$\begin{aligned} h_\ell(\boldsymbol{b}) &= \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} b_k \\ &= \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} a_i \\ &= \sum_{i=0}^{\ell} a_i \binom{\ell}{i} \sum_{k=i}^{\ell} \binom{\ell-i}{k-i} (-1)^{k+i} \end{aligned}$$

$$= \sum_{i=0}^{\ell} a_i \binom{\ell}{i} \sum_{j=0}^{\ell-i} \binom{\ell-i}{j} (-1)^j$$
$$= a_\ell$$

To illustrate a use of this form of binomial transformation, substitute $a = -b$ in (2.18) which then implies the two identities

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{b^k}{k+1} = -\frac{(1-b)^{n+1} - 1}{b(n+1)} \tag{2.59}$$

and

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(1-b)^{k+1} - 1}{b(k+1)} = -\frac{b^n}{(n+1)} \tag{2.60}$$

As a special case of these identities, set $b = 2$ which then creates identities

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^k}{k+1} = \begin{cases} 0, & n \text{ odd}, \\ \frac{1}{n+1}, & n \text{ even} \end{cases} \tag{2.61}$$

$$\sum_{k=0,\ k \text{ even}}^{n} (-1)^k \binom{n}{k} \frac{1}{(k+1)} = \frac{2^n}{n+1} \tag{2.62}$$