Chapter 8
Low Dimensional Semiconductors

Quasi two-dimensional semiconductor layers offer a lot of spectacular properties of which the most famous is the Quantum Hall effect. We limit ourselves, however, only to the presentation of the simplest, well understood, but nevertheless surprising peculiarities of the classical and quantum mechanical two-dimensional motion of Coulomb interacting electrons in the presence of a strong transverse magnetic field.

In the last decades progresses in semiconductor technology produced ultra-thin semiconductor layer systems in which by a suitable choice of the layers, the electrons and holes are restricted to a two-dimensional motion. An ultra-thin layer of a semiconductor is inserted between two thick layers of another semiconductor with a larger band gap and therefore, both the electrons and holes in the ultra-thin layer are in a quantum well and their lowest states are discrete as it is shown in Fig. 8.1.

At sufficiently low temperatures the electrons and holes sit on their lowest levels (shown here in red and green). It means, the transverse motion at low temperatures is "frozen" in its lowest lying state $\phi_0(z)$ in the potential well created by the adjoining layers and the wave function of an electron, in a good approximation is given by

$$\Psi(x, y, z) = \psi(x, y)\phi_0(z),$$

with the coordinates x, y lying in the plane and z being the transverse coordinate. This is the experimental realization of a two dimensional (2D) electron–hole system, which has a lot of very interesting properties.

Already the free motion in 2D has a peculiarity, namely the one-electron state density is constant above $\epsilon = 0$

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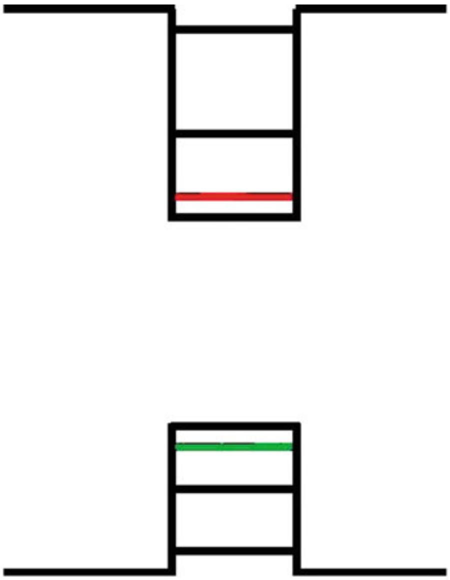

Fig. 8.1 A semiconductor quantum well

$$z(\epsilon) = \frac{1}{(2\pi)^2} \int d\mathbf{k} \delta(\epsilon - \frac{\hbar^2 k^2}{2m}) = \frac{m}{2\pi\hbar^2} \theta(\epsilon).$$

Such systems show above all spectacular properties in the presence of a strong magnetic field transverse to the plane. The most famous one is the Quantum Hall effect. We will not try to describe here the different, sometimes contradicting theories of this effect. Nevertheless, we want to bring the attention to the kind of strange physics we encounter in two dimensions (2D).

8.1 Exciton in 2D

Let us consider an electron–hole pair with Coulomb attraction in a 2D semiconductor. In the effective mass approximation, the relative motion in cylindrical coordinates is described by the Hamiltonian

$$H = -\frac{\hbar^2}{2m} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) - \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{e^2}{\epsilon \rho},$$

where m is the reduced mass of the pair and ϵ is the dielectric constant (supposed here as being the same in the 2D layer as in the surrounding semiconductor). The eigenstates of this Hamiltonian are characterized by two quantum numbers $n =$

$0, 1, 2 \ldots$ and $\mu = \pm 0, \pm 1, \pm 2, \ldots$. The lowest eigenstate ($n = 0,\ \mu = 0$) is

$$\psi_{0,0} = \frac{4}{\sqrt{2\pi} a} e^{-\frac{2\rho}{a_B}},$$

while the ground state energy is

$$E_{0,0} = -4E_R.$$

where we used as parameter the Bohr radius a_B and the Rydberg energy E_R of the 3D exciton. A comparison with the ground state exciton wave function in 3D (see Sect. 3.1.2) shows that the exciton radius in 2D is twice smaller than in 3D and its binding energy is four times bigger. Contrary to any expectations, the transversely compressed exciton shrinks also in the still allowed two dimensions.

8.2 Motion of a 2D Electron in a Strong Magnetic Field

According to the discussed 3D motion of an electron in a homogeneous magnetic field (see Sect. 2.2.2) the stationary states in the Landau gauge are given by a plane wave in the field direction (along the z axis) and in the transverse plane by the Landau states of discrete energies. If one restricts the motion to the plane x,y, then the energy of such a Landau state is just

$$\epsilon_{n,X,} = \hbar\omega_c(n + \frac{1}{2}) \qquad (n = 0, 1, 2, \ldots).$$

Since these energies do not depend on the quantum number X (the x-coordinate of the center of the cyclotron motion), they are (in the absence of boundary conditions!) infinitely degenerate.
In what follows we consider very strong magnetic fields at very low temperatures and therefore we may consider that all the spins are aligned along the magnetic field. We shall consider the motion of such a 2D electron in a strong magnetic field perpendicular to the plane of motion in the presence of an external potential $U(\mathbf{r})$.

The simplest approach to consider is to ignore the higher lying Landau levels and, if the potential is weak, to consider its projection on the lowest lying Landau state ($n = 0$) within first order perturbation for the energy

$$E(X) = \frac{1}{2}\hbar\omega_0 + \langle 0, X|U(x, y)|0, X\rangle.$$

Kubo's more profound approach considers the limit of ultra-strong magnetic fields for arbitrary potentials and an arbitrary kinetic energy T. The 2D Hamilton operator is

$$H = T(\boldsymbol{\pi}) + U(\mathbf{r}),$$

where the generalized momenta $\boldsymbol{\pi}$ in the presence of the vector potential $\mathbf{A}$ are

$$\boldsymbol{\pi} \equiv \mathbf{p} + \frac{e}{c}\mathbf{A}(\mathbf{r})) \ ; \qquad \mathbf{p} \equiv \imath\hbar\nabla.$$

Within the set of operators $(\boldsymbol{\pi}, \mathbf{r})$ the only non-vanishing commutators are

$$[\pi_x, \pi_y] = \frac{\hbar e}{\imath c}B; \qquad [\pi_x, x] = [\pi_y, y] = \frac{\hbar}{\imath}.$$

Defining new operators

$$\xi = \frac{c}{eB}\pi_y, \quad \eta = \frac{c}{eB}\pi_x$$

and

$$X = x - \xi, \quad Y = y - \eta$$

it results that

$$[\xi, \eta] = \frac{\hbar c}{\imath eB} \equiv \frac{l_B^2}{\imath} ,$$

$$[X, Y] = -\frac{\hbar c}{\imath eB} \equiv -\frac{l_B^2}{\imath} ,$$

while the other commutators of the new operators vanish. The kinetic energy T depends only on the new operators ξ and η. Also, in the absence of the potential U the operators X and Y do not change in time. They are assimilated with the coordinates of the cyclotron motion, while the operators ξ and η are the relative coordinates.

In the presence of the external potential, either in the frame of quantum mechanics or within classical mechanics, the center of the cyclotron motion moves according to

$$\dot{X} = \frac{c}{eB}\frac{\partial U}{\partial y}, \qquad \dot{Y} = -\frac{c}{eB}\frac{\partial U}{\partial x}.$$

According to the above canonical commutation rules one has the uncertainty relations

$$\triangle X \triangle Y = 2\pi l_B^2$$

and since ultra-strong magnetic fields imply $l_B \to 0$, a classical description with classical coordinates X, Y is appropriate. On the other hand, if the potential is slowly varying on the scale of the magnetic length, one may ignore the relative motion and to a very good approximation one could use the classical equations of motion

$$\dot{X} = \frac{c}{eB}\frac{\partial U(X,Y)}{\partial Y}, \qquad \dot{Y} = -\frac{c}{eB}\frac{\partial U(X,Y)}{\partial X}.$$

It follows that

$$\frac{\partial Y}{\partial X} = \frac{\dot{Y}}{\dot{X}} = -\frac{\frac{\partial U(X,Y)}{\partial X}}{\frac{\partial U(X,Y)}{\partial Y}}.$$

However, this corresponds to the motion on a curve defined by $U(x, y) = U_0$. Indeed, by using the definition of the implicit derivative one gets

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial U(x,y)}{\partial x}}{\frac{\partial U(x,y)}{\partial y}}.$$

To conclude, in the limit of ultra-strong magnetic fields the motion of the cyclotron center of electrons in 2D with arbitrary kinetic energy in the presence of an external potential is just a classical one along the equipotential curves of the potential.

8.3 Coulomb Interaction in 2D in a Strong Magnetic Field

8.3.1 Classical Motion

A stranger aspect of the 2D motion in a strong magnetic field is how Coulomb forces act. Let us consider first the classical problem of the motion of two particles of opposite charges (electron and hole). We are interested only in the relative motion, therefore one of the particles we may keep fixed in the origin. A numerical solution of the corresponding Newton equations shows the trajectory in Fig. 8.2. This is a somewhat complicated picture, but the particles, as expected seem to attract each other and stick together.

To our surprise, even two identically charged particles in 2D, having repulsive Coulomb forces, stick together, showing an effective attraction, as it is illustrated in Fig. 8.3. Of course, the Coulomb force tries to accelerate the electrons in the repulsive manner, but the accelerated electron is returned by the bending in the magnetic field. The only escape would have been in the now forbidden transverse direction.

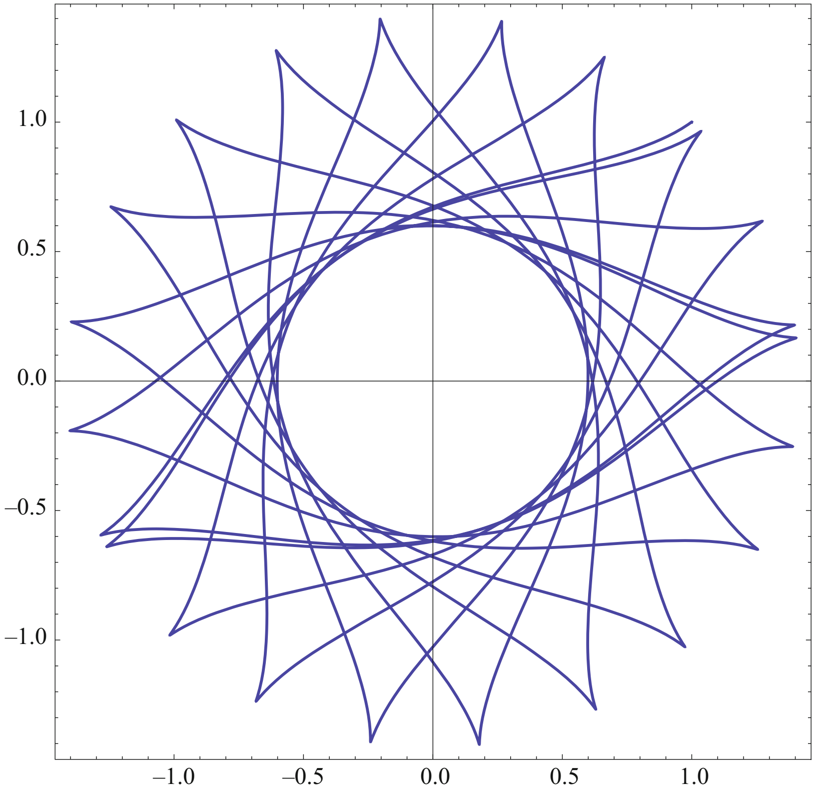

Fig. 8.2 Classical relative motion of opposite charged particles in 2D in the presence of a transverse magnetic field

If one considers a whole cluster of electrons these are sticking together in a cluster, as it is shown in Fig. 8.4.

8.3.2 Quantum Mechanical States

The quantum mechanical analysis of the motion of two Coulomb repulsive particles confirms also the existence of bound electron–electron states in 2D.
The quantum mechanical Hamiltonian for the 3D relative motion of two electrons in the presence of a magnetic field **B** is

$$H^{3D} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e\hbar}{mc}\imath\left(\mathbf{A}(\mathbf{r}) \,.\, \nabla + \frac{1}{2}\nabla\mathbf{A}(\mathbf{r})\right) + \frac{e^2}{2mc^2}\mathbf{A}(\mathbf{r})^2 + \frac{e^2}{r}.$$

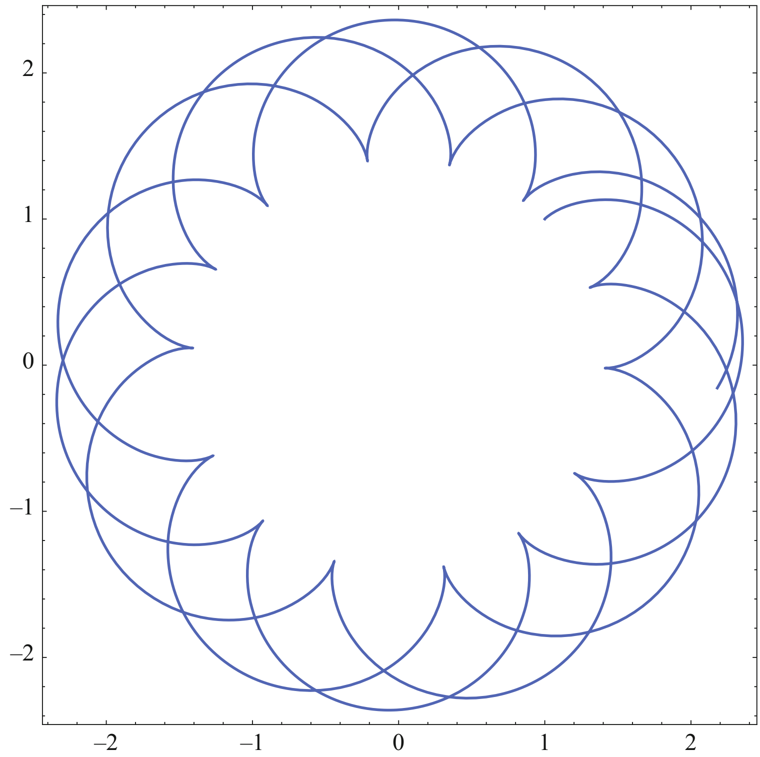

Fig. 8.3 Classical relative motion of identically charged particles in 2D electron in the presence of a transverse magnetic field

If one chooses the divergenceless vector potential (with the magnetic field in the z-direction)

$$\mathbf{A}(\mathbf{r}) \equiv \left(-\frac{1}{2}By,\ \frac{1}{2}Bx,\ 0\right),$$

one may write the 2D Hamiltonian describing the in-plane motion in cylinder coordinates $\rho,\ \phi$ as

$$H^{2D} = -\frac{\hbar^2}{2m}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} - \imath\frac{e\hbar B}{2mc}\frac{\partial}{\partial\phi} + \frac{e^2B^2}{8mc^2}\rho^2 + \frac{e^2}{\rho}.$$

One looks, as usual, for the eigenfunctions as

$$\psi(\rho,\phi) = u(\rho)e^{\imath\mu\phi} \qquad (\mu = 0, 1, 2, \ldots).$$

Then the eigenvalue problem for the radial part is

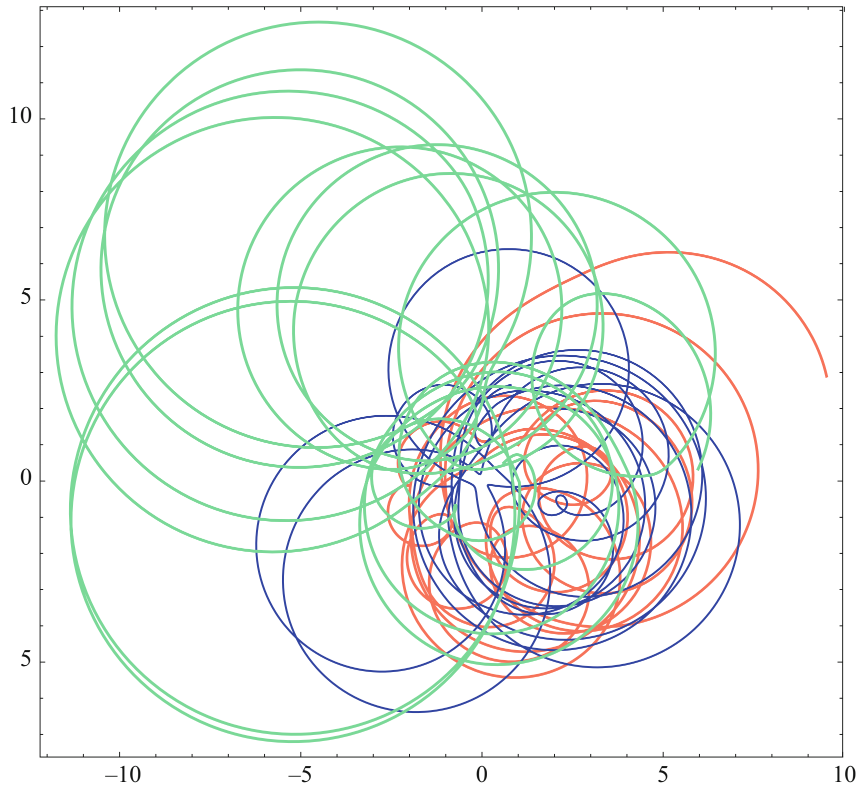

Fig. 8.4 Classical motion of 4 electrons (one is kept fixed in the origin) in 2D in the presence of the magnetic field

$$-\frac{\hbar^2}{2m}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\left(\frac{\hbar^2}{2m}\frac{\mu^2}{\rho^2}+\frac{e\hbar B}{2mc}\mu+\frac{e^2B^2}{8mc^2}\rho^2+\frac{e^2}{\rho}\right)u=E_\mu u.$$

This Schrödinger equation differs (up to a shift in the energy with $\frac{e\hbar B}{2mc}\mu$) from that of a radial 2D oscillator just due to the Coulomb term $\frac{e^2}{\rho}$. Since in a very strong magnetic field the spins are supposed to be aligned along the magnetic field and the wave functions must be anti-symmetrical for fermions; only odd angular momenta μ are of interest. In terms of the dimensionless parameter

$$\xi=\frac{eB}{2c\hbar}\left(\frac{\hbar^2}{2me^2}\right)^2$$

an ultra-strong magnetic field corresponds to $\xi \gg 1$. The radial wave function of the lowest $\mu = 1$ state of the 2D oscillator is

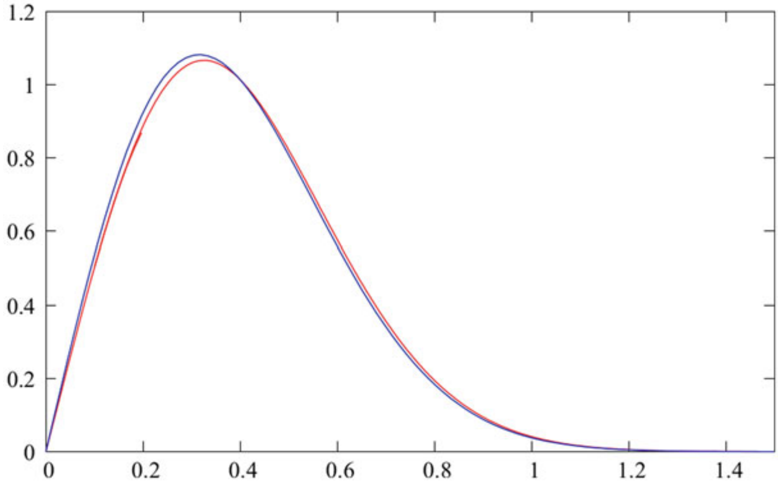

Fig. 8.5 The lowest $\mu = 1$ eigenfunction with and without Coulomb repulsion

$$R_{1,0}(\rho) = \frac{\xi}{\sqrt{\pi}} \rho e^{-\frac{1}{2}\xi\rho^2}; \qquad 2\pi \int_0^\infty \rho |R(\rho)|^2 d\rho = 1,$$

where for convenience the radius ρ is measured in units of the length $\lambda = \frac{\hbar^2}{2me^2}$.

A numerical solution of the lowest eigenfunction including Coulomb repulsion for $\mu = 1$ and $\xi = 10$ is shown red in Fig. 8.5, while the corresponding oscillator function is shown in blue.

The two wave functions are surprisingly close to each other. The contribution of the Coulomb potential to the energy may be approximated by first order perturbation theory as its average over the oscillator function. The result again lies surprisingly close to the exact value (in our example, up to four digits).

On the other hand, without the Coulomb potential the wave function would have been centered at any arbitrary position in the plane. The existence of the repulsive center, however, fixes its position! This suggested the construction of the so-called Laughlin-wave functions for many electrons out of oscillator wave functions.

It is worth to remark also that the lowest s-wave ($\mu = 0$) state shows in concordance with the formerly discussed theory of Kubo a motion concentrated on a circle (equipotential line for the Coulomb potential).