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Birational Geometry and Moduli Spaces

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Preface

This volume collects contributions from speakers at the INdAM Workshop "Birational Geometry and Moduli Spaces", which was held in Rome on 11–15 June 2018.

The workshop was devoted to the interplay between birational geometry and moduli spaces, two central topics in Algebraic Geometry that have always attracted the interest of national and international researchers.

A longstanding problem in geometry is the classification of geometric objects. The starting point is a geometric object, such as a curve, a variety or a sheaf, defined by some equations or given by some conditions. Then, one can deform it to get a new geometric object related to the old one. This expresses the concept of a family of an object and the main purpose is the classification of these families, in such a way that the classifying space is a reasonable geometric space. This space is the so-called moduli space and its geometric points parametrise the objects that we are considering. One is interested in classifying geometric objects up to isomorphism, so that the moduli space has a variety structure, or geometric objects with their automorphisms, thus meaning that the moduli space has a stack structure. In order to understand the local and global structure of moduli spaces, it is important to handle well techniques from deformation theory. Indeed, deformation theory can be regarded as a tool to understand the local geometry of moduli spaces.

There is also interest in classifying geometric objects up to birational morphism. Many results are known for the birational study of algebraic surfaces, while for higher dimension there are many open problems that can be tackled via the minimal model program. Great progress has been made in recent years, supporting the development of new techniques and innovative ideas.

The workshop focused on these research aspects and their interactions and offered the possibility to disseminate the knowledge of advanced results and key techniques used to solve open problems in the area.

This volume covers many topics in the wide research area of birational geometry and moduli spaces, and includes both surveys and original research papers. In particular, it contains works on irreducible holomorphic symplectic manifolds, Severi varieties, degeneration of Calabi–Yau varieties, uniruled threefolds, toric Fano threefolds, mirror symmetry, canonical bundle formula, the Lefschetz principle, birational transformation, and deformation of diagrams of algebras.

We are indebted to Francesco Bastianelli and Antonio Rapagnetta for their help in organising the workshop and to Emilia Mezzetti for her support as a member of the Scientific Committee.

On behalf of the Scientific and Organising Committees, we would like to thank the Foundation Compositio Mathematica for their support.

We are all very grateful to the Istituto Nazionale di Alta Matematica "Francesco Severi" (INdAM) for the generous support and help that made all this possible.

Finally, we would like to offer special thanks to all the participants and speakers of the INdAM Workshop, and especially to the authors who accepted the invitation to publish here.

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Pisa, Italy — Rita Pardini

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About the Editors

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Negative Rational Curves and Their Deformations on Hyperkähler Manifolds

Ekaterina Amerik

Abstract We survey some results about rational curves on hyperkähler manifolds, explaining how to prove a certain deformation-invariance statement for loci covered by rational curves with negative Beauville–Bogomolov square.

Keywords Hyperkähler manifold · Rational curve · MBM classes

This text is an expanded version of a talk which I gave at the INDAM workshop in Rome in June 2018. It largely follows the paper [2] and our other joint works.

1 Hyperkähler Manifolds

A compact Kähler manifold X is **irreducible holomorphic symplectic (IHSM)**, or **hyperkähler** (see next section for a brief explanation of the term), if it is simply-connected and $H^{2,0}(X) = \mathbb{C}\sigma$ for some nowhere degenerate ("symplectic") form σ. Such a manifold is even-dimensional (we set $2n = dim(X)$) and has trivial canonical bundle. Such manifolds can be seen as higher-dimensional analogues of K3 surfaces. The most-studied IHSM are actually closely related to K3 surfaces, namely those are the deformations of their punctual Hilbert schemes; we call them

The results presented at the INDAM workshop and in this survey have been obtained jointly with M. Verbitsky.
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IHSM of K3 type. By Bogomolov's decomposition theorem, together with the abelian and the Calabi–Yau manifolds the hyperkähler manifolds are building blocks for the varieties with zero first Chern class.

The main feature that the hyperkähler manifolds share with surfaces is the presence of a natural integral quadratic form q on their second cohomology: **the Beauville–Bogomolov form**. It can be defined by integrating the differential forms, but we omit the explicit definition and only mention the following **Fujiki's formula**, which stresses the topological nature of q.

$$(q(\alpha))^n = C\alpha^{2n},$$

where C is a positive constant which only depends on the deformation type of X. It is well-known that this form has the same signature as the intersection form on a surface would have, that is, $(3, b_2 - 3)$ on $H^2(X, \mathbb{R})$ and $(1, b_2 - 3)$ on $H^{1,1}(X, \mathbb{R})$. We refer to [5] or [6] for details (another classical reference [13] is suggested by the referee). The cone of classes of strictly positive square in $H^{1,1}(X, \mathbb{R})$ has two connected components. We call the **positive cone** $\mathsf{Pos}(X)$ the one containing the Kähler classes.

2 Rational Curves

On a K3 surface S, for every integral $(1, 1)$-class z of square -2, either z or $-z$ is effective by Riemann–Roch formula. If z is represented by an irreducible curve, we call it a **−2-curve**. By adjunction, such a curve is smooth and rational. These curves are important for the understanding of the geometry of S: for instance, by the work of Pyatetski-Shapiro and Shafarevich [14] in the algebraic setting, or Looijenga and Peters [9] in the Kähler one, the ample cone of S is bounded, within the positive cone, by the orthogonals to -2-curves.

2.1 Lower Bound for the Dimension of a Family

A sufficiently general (in particular non-projective) hyperkähler manifold has no curves at all, indeed the deformation theory implies that it has no integral classes of type $(1, 1)$. When it does, one would like to understand the deformations of such a rational curve $C \subset X$ inside and outside of X. A lower bound for the dimension of the deformation space of maps $f : \mathbb{P}^1 \to X$ is provided by the Euler characteristics of the pullback of the tangent space, computed by the Riemann–Roch formula

$$\chi(f^*T_X) = -K_X C + dim(X) = dim(X)$$

as soon as K_X is trivial, so that C deforms in a family of dimension $2n-3$ (where 3 accounts for the automorphisms of $\mathbb{P}^1$).

Z. Ran noticed in [15] that on a hyperkähler X, there is a better bound: a rational curve deforms in a family of dimension at least $2n-2$. We have learned the following beautiful proof of this fact from Eyal Markman. Recall first that a hyperkähler manifold comes together with a natural deformation of complex structure, called **the twistor deformation**. It appears as follows: by Yau's theorem, on an irreducible holomorphic symplectic manifold there is a **hyperkähler metric**, that is a Riemannian metric g with three complex structures I, J, K, which are Kähler with respect to g (i.e. the corresponding forms are Kähler) and multiply as quaternions. For any (a, b, c) with $a^2+b^2+c^2=1$, $aI+bJ+cK$ is also a Kähler structure, so that X is a member of a family $\mathcal{X}_t,\ t \in S^2$. $\mathcal{X}$ is in fact a complex manifold and its projection π to $S^2=\mathbb{CP}^1$ is holomorphic.

Now Markman's idea is as follows: let $C \subset X$ be a rational curve, that is the image of some (say generically injective) map $f : \mathbb{P}^1 \to X$. By Riemann–Roch again, it must move within $\mathcal{X}$ in a family of dimension at least $-K_{\mathcal{X}}C + dim(\mathcal{X}) - 3 = 2n-2$. But it is well-known that the neighbouring fibers X_t of X in $\mathcal{X}$ do not contain any curves at all, whereas a deformation of a curve in a fiber of π obviously must be contained in a fiber of π (the intersection number of a curve and a fiber is zero when the curve is contained in a fiber and strictly positive when not). So in fact all deformations of C in $\mathcal{X}$ remain in X.

2.2 Minimality

If there is only a finite number of rational curves through a general point of the locus they cover, then this locus is of dimension $2n-1$, that is, a divisor (remark that X itself is not uniruled). However in general this does not have to be true. Already on manifolds of K3 type examples show that the uniruled loci can have any dimension between n and $2n-1$. As for the dimension of the family of deformations of a rational curve C, it can be arbitrarily large if one does not impose some minimality assumption, for example, that C is of minimal degree among the rational curves through the general point of the corresponding locus. Behind such a minimality assumption is of course the famous bend-and-break lemma.

Lemma 2.2.1 (Mori) *In any one-parameter family of rational curves through two distinct points x and y, there is a reducible member.*

So the minimality of the degree is equivalent to the following.

Definition A rational curve C is called minimal if there is only a finite number of deformations of C through two general points of the locus they cover.

By bend-and-break lemma, any irreducible uniruled subvariety Z of X is generically covered by a family of minimal rational curves, which are deformations of some minimal curve $C \subset Z$. We say that Z is a *locus* of C.

If C is a minimal rational curve, let Z be a locus of C and consider the **rational quotient** $r : Z \dashrightarrow Q$. The restriction of the form σ to Z at its general point has kernel along the fiber of r, since there are no holomorphic k-forms on a rationally connected manifold for $k > 0$. One derives the following

Proposition 2.2.2 *The codimension k of Z is equal to the relative dimension of r, and any minimal rational curve deforms in a family of dimension exactly $2n-2$ in X.*

Proof Indeed since σ is non-degenerate on X, its kernel in restriction to Z is at most of rank k, on the other hand we must have at least a $k-1$-parameter family of rational curves through a general point of Z, so the fibers of rational quotient are at least k-dimensional. This also shows that Z is **coisotropic**.

Applying Markman's trick to the local universal family instead of the twistor deformation, one gets the following

Corollary 2.2.3 *A minimal rational curve in X deforms to a neighbouring IHSM X_t if and only if its cohomology class remains of type $(1, 1)$ on X_t.*

It is interesting for several reasons to understand the loci of minimal rational curves and their behaviour under global deformations. One encounters two problems here. First of all, under a global deformation a minimal curve can cease to be minimal. Secondly, the family of deformations of C may have several components, C being minimal in some of them but not in the others. So the geometry of uniruled loci is still not well-understood even on much studied IHSM. For instance one conjectures but does not know that a projective deformation of the Hilbert square of a K3 contains a rational surface.

The point of the present talk is to indicate that all such problems have a satisfactory solution in the case when the Beauville–Bogomolov square of C is negative. Here **the Beauville–Bogomolov square** of a curve is a rational number, obtained by considering $H_2(X, \mathbb{Z})$ as an overlattice of $H^2(X, \mathbb{Z})$ by duality.

The loci covered by minimal rational curves of negative square are especially interesting as these are loci of birational contractions, at least in the projective case.

3 Deformation Spaces

3.1 Teichmüller and Torelli

Deformations of IHSM are unobstructed by the work of Todorov [17]. Thus the local deformation space $Def(X)$ of X is identified with a neighbourhood of zero in $H^1(X, T_X)$, which is in turn identified with $H^1(X, \Omega^1_X)$ by means of the symplectic form. The locus of deformations where a class λ remains of type $(1, 1)$ is the hyperplane orthogonal to λ with respect to the Beauville–Bogomolov form. This local picture globalizes as follows.

Let M be the underlying differentiable manifold and $\mathsf{Comp}(M)$ the space of complex structures of Kähler type (by Kodaira–Spencer stability theorem, this is an open subset in the space of all complex structures). We consider **the Teichmüller space**

$$\mathsf{Teich} = \mathsf{Teich}(M) = \mathsf{Comp}(M)/\mathsf{Diff}^0(M),$$

where $\mathsf{Diff}^0(M)$ is the group of isotopies: the action of the whole diffeomorphism group $\mathsf{Diff}(M)$ on $\mathsf{Comp}(M)$ is "too bad" for taking any reasonable quotient. Instead we consider only the action of the mapping class group $\Gamma = \mathsf{Diff}(M)/\mathsf{Diff}^0(M)$ on $\mathsf{Teich}(M)$. Huybrechts in [7] proved that $\mathsf{Teich}(M)$ has only finitely many connected components when M carries a hyperkähler structure. By abuse of notation, we shall use the notation $\mathsf{Teich}(M)$ for the component containing our chosen complex structure.

The space $\mathsf{Teich}(M)$ is only generically Hausdorff. Verbitsky in [18] proved the **Torelli theorem for hyperkähler manifolds**, namely that the Hausdorff quotient $\mathsf{Teich}_b(M)$ is isomorphic to the **period domain**. The latter is defined as follows:

$$\mathsf{Per}(M) = \{l \in \mathbb{P}H^2(M, \mathbb{C}) | q(l, l) = 0, q(l, \bar{l}) > 0\}$$

and the **period map** *per* from $\mathsf{Teich}(M)$ to $\mathsf{Per}(M)$, realizing the up-to-nonhausdorffness isomorphism, is defined by sending the parameter point for a complex manifold X to the complex line generated by the class of σ.

It is often convenient (e.g. in Verbitsky's proof) to see $\mathsf{Per}(M)$ as the grassmannian of positive (with respect to q) 2-planes in the real cohomology $Gr_{++}(2, H^2(M, \mathbb{R}))$ (the transition from the open subset of the complex quadric as above is simply by taking the real and the imaginary part).

Define the subspace $\mathsf{Teich}_z = \mathsf{Teich}_z(M) \subset \mathsf{Teich}(M)$ as the locus where an integral class z remains of type $(1, 1)$. It is easy to see that this is the inverse image by *per* of the hyperplane $z^{\perp}$ (the q-orthogonal to z). It is clear from the description in the following section that Teich_z is not Hausdorff even generically and splits naturally in two "halves".

3.2 Kähler Chambers and MBM Classes

The following theorem summarizes the main results of [1] (similar results to some of those have been independently obtained by Mongardi in [11]).

Theorem 3.2.1 *The Kähler cone $K(X)$ is a connected component of the complement to a union of hyperplanes $z^{\perp}$, $z \in R \cap H^{1,1}(X)$, in $\mathsf{Pos}(X)$, where R is the set of negative integral classes in $H^2(X, \mathbb{Z})$, called **MBM classes**, which can be characterized by the following equivalent conditions: z is MBM iff Teich_z containes no twistor curve iff a rational multiple of z is represented by a rational curve on* *a "generic"*[1] *deformation of X within* Teich_z *(that is, such deformation that only the multiples of z are integral classes of type* $(1, 1)$*). The orthogonal hyperplane to an MBM class of type* $(1, 1)$ *supports the wall of the Kähler cone of X, or of a birational model X', or the monodromy image of such a wall.*

The orthogonals to MBM classes of type $(1, 1)$ thus divide the positive cone into chambers, called **Kähler-Weil chambers** by Markman. Each chamber is the Kähler cone of a birational model X' or its monodromy translate.

Theorem 3.2.2 (Markman) *The points in $per^{-1}(per(X))$, that is, the inseparable points of* Teich *one of which corresponds to X, are in one-to-one correspondence with the Kähler-Weil chambers of* $\mathsf{Pos}(X)$.

In particular, Teich_z is non-separated even at its general point: at such a point z is the only MBM class of type $(1, 1)$ and $z^\perp$ divides the positive cone in two chambers. Choosing an orientation, we naturally separate Teich_z in two halves $\mathsf{Teich}_z^\pm$ (the complex structures which have the Kähler cone "to the left" and "to the right" of $z^\perp$).

The next step is to take the part of, say, Teich_z^+ corresponding to chambers adjacent to $z^\perp$ (that is, such that $z^\perp$ contains a wall, rather than a lower-dimensional face, of the chamber in question). We denote this part by Teich_z^{min}. Its separable quotient coincides with Teich_z^+ itself, but fibers over some points of Per are slightly smaller than those in Teich_z^+: indeed we only keep the chambers adjacent to $z^\perp$, so that on the varieties parameterized by Teich_z^{min}, a multiple of the class z is represented by a minimal rational curve (as $z^\perp$ gives a part of the boundary of the Kähler cone with non-empty interior), whereas on the varieties in the complement $\mathsf{Teich}_z^+ - \mathsf{Teich}_z^{min}$ this shall be true only for some monodromy transform of z and only on some birational model, whereas z itself may be e.g. the class of a reducible rational curve.

We are going to be concerned with deformations of rational curves in the class z over Teich_z^{min}.

4 Main Result

If X is in Teich_z^{min} we consider the locus Z covered by curves whose class is proportional to z. When X is projective, it follows from Kawamata's base point free theorem that Z is the contraction locus of a birational morphism. Indeed take a sufficiently general integral class in that part of $z^\perp$ which is a wall of the Kähler cone of X. By Lefschetz theorem it is the class of a line bundle L which is nef since it is on the boundary of the Kähler cone, big since by definition its square is strictly positive, and therefore eventually base-point-free by Kawamata'a theorem.

[1]On an arbitrary deformation, an MBM class would be, up to a multiple, represented by a possibly reducible rational curve, but the converse is in general false.

The morphism given by the sections of the line bundle $L^{\otimes m}$ for m large enough contracts exactly the curves in the classes proportional to z. In particular Z has only finitely many irreducible components, and also Z is uniruled since all contraction loci are uniruled (see e.g. [8]).

The finiteness (and, post factum, the uniruledness) carries over to the non-projective case, since the loci $Z_t \subset X_t$ of curves in a class proportional to z on X_t are deformations of $Z \subset X$. We note that a recent work of Bakker and Lehn [4] seems to imply a much less obvious fact that Z is contractible even if X is non-projective.

Our main results can be summarized as follows.

Theorem 4.1 *Let $\mathcal{X}$ be the universal family of IHSM with second Betti number greater than 5 over Teich_z^{min} and $\mathcal{Z}$ the subvariety covered by curves of class proportional to z. All fibers Z_t of $\mathcal{Z}$ are homeomorphic and stratified diffeomorphic, with the possible exception of t corresponding to the complex structures with the maximal Picard number. Moreover the homeomorphisms (and stratified diffeomorphisms) can be chosen in such a way that they respect rational curves, so that they induce maps between Zariski-open subsets of fibers of rational quotients on Z_t and Z_t', $t, t' \in \mathsf{Teich}_z^{min}$. These maps are bimeromorphisms.*

Remark 4.2 The complex manifolds X_t with maximal Picard number are isolated points of Teich_z^{min}. Indeed the image of such a complex structure by the period map is in the intersection of $h^{\perp}$ for all $h \in Pic(X_t)$. The results of Bakker and Lehn seem to imply a strengthening of ours, eliminating the nonmaximality assumption and eventually giving a real analytic isomorphism of the loci Z_t inducing a biholomorphism of fibers of the rational quotient. This is a work in progress which we hope to describe in a forthcoming paper.

5 Ergodicity

Our main tool is Verbitsky's ergodicity theorem for the action of the mapping class group on the Teichmüller space ([19, 20]). An action on a space with a measure is **ergodic** if any invariant measurable subset or its complement has measure zero. Most of the orbits of such an action on a manifold are dense.

Theorem 5.1 (Verbitsky) *Let X be an IHSM with $b_2(X) \geq 5$. There are three types of orbits of the action of the mapping class group Γ on Teich:*

(1) closed orbits of complex structures with rational period plane, i.e. such that Picard number is maximal;
(2) dense orbits of complex structures such that the period plane contains no rational vector;

(3) *"intermediate orbits": orbits of complex structures whose period planes contain a single rational vector v. The orbit closure then consists of all period planes containing v and is* ***totally real****, that is, neither contained in a complex subvariety nor contains a positive-dimensional one, even locally.*

Sketch of proof It is done in two steps. The mapping class group acts on the period domain and one first proves these statements for period planes (the orbit of a period plane is dense whenever the plane contains no rational vector, etc). This is obtained by Ratner theory which describes actions of groups generated by unipotents on homogeneous spaces $\Gamma\backslash G$, where Γ is a lattice in a simple Lie group G. Indeed another result by Verbitsky, a corollary of a general theorem by Sullivan, affirms that the image of the mapping class group Γ in $G = SO^+(3, b_2 - 3)$ (the automorphism group of the second cohomology of X) is a lattice. Let $G = SO^+(3, b_2 - 3)$ and $H = SO^+(1, b_2 - 3)$, so that G/H is fibered in $SO(2)$ over Per and H is generated by unipotents.

Ratner theory (see [16], but also [12] for a very accessible exposition) states that $\overline{xH}$ is again an orbit under a closed intermediate subgroup S, also generated by unipotents and in which $\Gamma \cap S$ is a lattice (this particular result is known as Ratner's orbit closure theorem and applies to actions of subgroups generated by unipotents on a quotient of a Lie group by a lattice). From the study of Lie group structure on G we derive that the subgroup must be either H itself (the orbit is closed), or the whole of G (the orbit is dense), or the stabilizer of an extra vector $\cong SO^+(2, b_2 - 3)$ (the third case). One concludes passing in an obvious way (via the double quotient) from an H-action on $\Gamma\backslash G$ to a Γ-action on G/H.

The other step is to prove that the period map commutes with taking orbit closures, in the following way. A point of Teich can be seen as a pair of a complex structure and a Kähler chamber. Introduce the space Teich_K which consists of pairs (I, ω) where $I \in \mathsf{Teich}$ and ω is a Kähler form of square 1. The period map is injective on Teich_K. In other words, Teich_K is naturally embedded in Per_K, the homogeneous manifold of all pairs consisting of a period point $I \in \mathsf{Per}$ and an element ω of square one in its positive cone (which indeed depends only on the period point, not on the complex structure itself). The latter is a homogeneous space, so we can try to apply Ratner theory to prove the following result, which clearly implies what we need: for any I, the closure of the Γ-orbit of $(I, K(I)) \subset \mathsf{Teich}_K \subset \mathsf{Per}_K$ contains the orbit of $(\mathsf{Per}I, \mathsf{Pos}(I))$ (here by an orbit of the subset we mean the union of its translates and $K(I)$ denotes the Kähler cone of the complex structure I). Now one can construct orbits of one-parameter subgroups which are entirely contained in $(I, K(I))$ and such that the closure of their projection to Per_K/Γ contains the projection of the positive cone. Indeed, one deduces from the non-maximality of the Picard number that $K(I)$ has a "round part". This is used to find many horocycles in $K(I)$ tangent to the round part of the boundary. The horocycle is an orbit of a one-parameter unipotent subgroup. Applying Ratner theory to a sufficiently general horocycle of this type, one sees that the closure of its image in Per_K/Γ contains an entire $SO(H^{1,1}(I))$-orbit ([20], Proposition 3.5), which is the positive cone $\mathsf{Pos}(I)$. □

For our purposes, we are concerned with the space Teich_z^{min} which maps onto Per_z, that is, the hyperplane $z^\perp$ in Per. On these spaces there is an action of Γ_z, the subgroup of the mapping class group whose action on the second cohomology fixes z. The first step of the argument, concerning the orbits of period planes, carries over verbatim, we only have to suppose $b_2 > 5$ rather than $b_2 \geq 5$ as we are now working in a hyperplane. The nature of the considerations in the second step is such that they naturally carry over to Teich_z^{min} rather than Teich_z or Teich_z^+.

Theorem 5.2 *Assume $b_2(M) > 5$ and let $z \in H^2(M, \mathbb{Z})$ be an MBM class and Γ_z the subgroup of the mapping class group consisting of all elements whose action on the second cohomology fixes z. Then Γ_z acts on $\mathsf{Teich}_z^{\min}$ ergodically, and there are the same three types of orbits of this action as above.*

Proof It proceeds along the same lines. We introduce the spaces $\mathsf{Per}_{K,z}$ consisting of pairs

$$\{(\mathsf{Per}(I), \omega \in \mathsf{Pos}(I) \cap z^\perp), q(\omega, \omega) = 1\}$$

and $\mathsf{Teich}_{K,z}$ consisting of pairs $(I \in \mathsf{Teich}_z^{\min}, \omega) \in \mathsf{Per}_{K,z}$ where ω belongs to the wall of the Kähler cone given by $z^\perp$. We denote such a wall by $K(I)_z$, though of course its elements are not Kähler forms in the complex structure I, but rather semipositive limits of those. Since the complex structures in $\mathsf{Teich}_z^{\min}$ which have the same period point are in one-to-one correspondence with the walls of the Kähler chambers in which the other MBM classes partition $z^\perp$, $\mathsf{Teich}_{K,z}$ again embeds naturally in $\mathsf{Per}_{K,z}$. We fix a complex structure I with non-maximal Picard number. We need to prove that the closure of the Γ_z-orbit of the subset $(I, K(I)_z)$ contains the orbit of $(\mathsf{Per}(I), \mathsf{Pos}(I) \cap z^\perp)$. This is done exactly in the same way as in Verbitsky's theorem. We take a general three-dimensional subspace W in $z^\perp$, the intersection of W with our wall $K(I)_z$ contains horocycles, and we deduce from Ratner orbit closure theorem and Proposition 3.5 of [20] that the closure of the projection of such a horocycle to $\mathsf{Per}_{K,z}/\Gamma_z$ is large, containing an $SO(H^{1,1}(I) \cap z^\perp)$-orbit, which is the projection of $\mathsf{Pos}(I) \cap z^\perp$.

6 Proof of the Main Result

We are concerned with the family of smooth manifolds $\mathcal{X}$ over $\mathsf{Teich}_z^{\min}$ and its analytic subset $\mathcal{Z}$, and we wish to show that $\mathcal{Z}$ has no degenerations except possibly over points of maximal Picard number. It is well-known that an analytic subset of a complex manifold admits a "nice" stratification (Whitney stratification or Thom-Mather stratification; see [10]). Recall also the following **first isotopy lemma** by Thom (we refer to [10] for precise definitions and proofs).

Lemma ([10], Proposition 11.1) *Let $f : Y \to B$ be a smooth mapping of smooth manifolds and W a closed subset of Y admitting Whitney stratification, such that $f : W \to B$ is proper. If the restriction of f to each stratum of W is a submersion then W is locally trivial over B.*

In our setting we deduce that the family $\mathcal{Z}$ is locally trivial on the complement $B \subset \mathsf{Teich}_z^{min}$ to a union (possibly countable, but finite in a neighbourhood of any point in the base) $P \subset \mathsf{Teich}_z^{min}$ of proper analytic subsets. If $x \in \mathsf{Teich}_z^{min}$ is a point with dense Γ-orbit, then its orbit hits B and therefore Z_x is naturally stratified diffeomorphic to Z_b, $b \in B$. If the orbit of $x \in \mathsf{Teich}_z^{min}$ does not hit B, itself and its closure must be contained in P. But we know about the orbit closures that even locally they are not contained in a proper complex subvariety, except possibly when x corresponds to a manifold X with maximal Picard number. Thus the degenerations in the family $\mathcal{Z}$ can only happen over such points.

The same is true for other families over Teich_z^{min} coming from the geometry of IHSM over it. We obtain stratified diffeomorphisms preserving rational curves by applying the Γ-orbit argument to Barlet spaces parameterizing rational curves with cohomology class proportinal to z, and then to the incidence varieties. Note that these spaces are not naturally embedded into smooth manifolds, so it is not quite the first isotopy lemma which applies, but there are versions of it suitable for our situation (e.g. [21], corollaire 5.1).

Finally, at a general point of a fiber of the rational quotient, the holomorphic tangent space is generated by the tangents to rational curves. A diffeomorphism preserving the holomorphic tangent space is a biholomorphism.

Note Added in Proof During the INDAM workshop, A. Rapagnetta attracted our attention to a recent work by Bakker and Lehn [4], which in the meantime allowed us to considerably strengthen the above result. The main point is that it indeed follows from Bakker and Lehn's work that minimal rational curves can be contracted even in the non-projective case. An interested reader may consult our recent preprint [3].

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Moduli Spaces of Cubic Threefolds and of Irreducible Holomorphic Symplectic Manifolds

Chiara Camere

Abstract In this survey, based on joint work of the author and S. Boissière and A. Sarti, we will describe an isomorphism between the moduli space of smooth cubic threefolds, as described by Allcock, Carlson and Toledo, and the moduli space of fourfolds of $K3^{[2]}$-type with a special non-symplectic automorphism of order three; then, I will show some consequences of this isomorphism concerning degenerations of non-symplectic automorphisms. Finally we will explore possible generalizations of the problem to higher dimensions and other moduli spaces of cubic threefolds.

Keywords Irreducible holomorphic symplectic manifold · Hyperkähler manifold · Non-symplectic automorphism · Ball quotient · Cubic threefold · Moduli space of cubic threefold · Occult period map · Pfaffian cubic threefold · Degeneration of irreducible holomorphic symplectic manifold · Degeneration of automorphism · Fano variety of a cubic fourfold

1 Introduction

The aim of this survey is to illustrate the results of a joint work by the author and S. Boissière and A. Sarti [17]; as a consequence, almost all the material here is neither original nor new, with the exception of a part of Sect. 4.

Irreducible holomorphic symplectic (denoted *IHS*) manifolds are smooth compact Kähler manifolds X which are simply connected and which carry a holomorphic symplectic two-form ω_X such that $H^{2,0}(X) = \mathbb{C}\omega_X$. These manifolds, which also carry a hyperkähler structure, are one of the building blocks of Beauville–Bogomolov decomposition theorem [8], and in particular they have $c_1(X)_{\mathbb{R}} = 0$. One of the main properties of these manifolds is the fact that the group $H^2(X, \mathbb{Z})$

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can be endowed with the structure of an integral non-degenerate indefinite lattice of signature $(3, b_2(X) - 3)$ by means of the so-called Beauville–Bogomolov quadratic form q_X. The global Torelli theorem of Huybrechts [25], Markman [33], and Verbitsky [40] asserts that this lattice controls in many senses the geometry of the manifold.

One deformation family of IHS manifolds is that of *fourfolds of* $K3^{[2]}$*-type*: these are deformations of the Hilbert scheme of two points of a smooth $K3$ surface. In particular, all these fourfolds X have $b_2(X) = 23$, the maximal one for an IHS in dimension four, and the Beauville–Bogomolov lattice is isometric to $L := L_{K3} \oplus \langle -2 \rangle$, where L_{K3} is the $K3$ lattice $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Among the projective elements of this deformation family, Beauville and Donagi show in [11] that the Fano variety $F(Y)$ of lines contained in a smooth cubic fourfold $Y \subset \mathbb{P}^5$ is a fourfold of $K3^{[2]}$-type which always carries a polarization of degree six.

An automorphism $\sigma \in \mathrm{Aut}(X)$ of an IHS manifold X induces an isometry $\sigma^* \in O(H^2(X, \mathbb{Z}))$, more precisely one induced by a monodromy operator on it (see [33, Theorem 1.3]), and an isomorphism of Hodge structures on $H^2(X, \mathbb{C})$. As a consequence, the line $H^{2,0}(X)$ is invariant under σ^*: if $\sigma^*\omega_X = \omega_X$, we say that σ is *symplectic*; else, if $\sigma^*\omega_X = \xi\omega_X$ with ξ a non-trivial root of unity, we say that σ is *non-symplectic*. From work of Beauville [7, Proposition 6] we know that the existence of such a non-symplectic automorphism forces the manifold X to be projective (there exists an invariant ample class); moreover, the order n of σ, if it is finite, is bounded by the inequality $\varphi(n) \leq b_2(X) - \rho(X)$, where φ denotes Euler's totient function and $\rho(X)$ is the Picard number of X.

Starting with the paper [14], we have been interested in classifying non-symplectic automorphisms of prime order acting on fourfolds of $K3^{[2]}$-type. Because of the properties just mentioned above, when X is a projective fourfold of $K3^{[2]}$-type and $\sigma \in \mathrm{Aut}(X)$ is non-symplectic of prime order p, we have $2 \leq p \leq 23$. The key idea of [14] is to classify the isometries of L induced by such automorphisms, exactly as it is done in the case of $K3$ surfaces by Nikulin [36] and by Artebani–Sarti–Taki [5]. As in the case of $K3$ surfaces, in the case of fourfolds X of $K3^{[2]}$-type the map $\mathrm{Aut}(X) \to O(H^2(X, \mathbb{Z}))$ is injective, so that σ^* is non-trivial if σ is. Building up on fundamental work contained in [13], we show that these actions are classified by a pair of primitive sublattices: the *invariant lattice* $T := H^2(X, \mathbb{Z})^{\sigma^*}$ and its orthogonal complement S inside L. This allows the partial classification contained in [14]: we could obtain all the cases with $p \neq 5, 23$; $p = 23$ was then solved in [15] and $p = 5$ in [39].

Nowadays, the state of the art of the classification is the following:

- Most cases can be constructed starting from a non-symplectic automorphism τ of the same order p on a smooth $K3$ surface Σ, either by considering the *natural* automorphism $\sigma = \tau^{[2]}$ on $\Sigma^{[2]}$ (see for example [12] for further details), or by considering the *(twisted) induced* automorphism on a moduli space of stable (twisted) sheaves on Σ (see [35, 38] and [19] for further details).
- For $p = 2$, there is the first (historically) remarkable exception to the above cases when $T = \langle 2 \rangle$: this is Beauville's famous non-natural non-symplectic involution

defined on the Hilbert scheme of two points of a smooth quartic surface not containing a line [7]. This example is in fact a special instance of the non-symplectic cover involution on O'Grady's double EPW sextics (see [22, 37] and references therein). There are a few more cases which cannot come from $K3$ surfaces and are degenerations of this one.

- For $p = 3$, two exceptions are given by $T = \langle 6\rangle$ and $T = \langle 6\rangle \oplus E_6^*(3)$: in both cases the only examples that we can explicitly construct are obtained on Fano varieties of lines of cubic fourfolds endowed with additional symmetries. The first case is the main object of [17], whereas the second one has still to be further understood.
- For $p = 23$, the only possibility is $T = \langle 46\rangle$: in [15] we show that there exists only one fourfold of $K3^{[2]}$-type carrying such a non-symplectic automorphism of order 23 inside the 20-dimensional family of $\langle 46\rangle$-polarized fourfolds of $K3^{[2]}$-type once we fix the action on L. Unfortunately, the proof is not constructive but uses in a crucial way the global Torelli theorem for IHS manifolds, and the explicit construction of such a fourfold remains an open question.

As already mentioned, the main object of [17], and of this survey as well, is the case $p = 3$, $T = \langle 6\rangle$: we will see that in this case we obtain a relation between the moduli space of marked fourfolds of $K3^{[2]}$-type with such a non-symplectic automorphism of order three and the classical GIT moduli space of smooth cubic threefolds. The existence of such a relation is not so surprising once one sees the first example of such a non-symplectic automorphism, perhaps more surprising is the fact that there is an isomorphism between these two moduli spaces; indeed, both are uniformized by a ten-dimensional complex ball quotient.

In Sect. 2 we will explain how to obtain the isomorphism and some of the immediate consequences, after illustrating how the two moduli spaces appearing are constructed, respectively in [16] and in [2]. More precisely, we will see that both moduli spaces are isomorphic to the complement of a hyperplane arrangement $\mathcal{H}$ inside a ten-dimensional complex ball quotient.

Section 3 then deals with understanding what happens on hyperplanes inside $\mathcal{H}$: this is completely understood in the case of cubic threefolds, since we obtain semistable points in the GIT quotient, corresponding either to nodal cubics or to the chordal cubic, as shown in [2]. What we investigate in Sect. 3 is what happens to the non-symplectic automorphism of order three and how it degenerates. The final outcome of this analysis will be that the two subarrangements are in fact birational to moduli spaces of $K3^{[2]}$-fourfolds carrying a non-symplectic automorphism of order three of different type in the classification.

The picture that we obtained is thus the analogue in higher dimension of the isomorphism of the moduli space of cubic surfaces with the moduli space of $K3$ surfaces with a special non-symplectic automorphism of order three, illustrated by Dolgachev, van Geemen and Kondō in [21] building on the construction of Allcock, Carlson and Toledo in [1] (see also [9] for a nice survey on this subject).

Finally, in Sect. 4, we explore related problems. As we observed above, non-natural automorphisms on Hilbert schemes of $K3$ surfaces are extremely difficult to construct; in [17], we also proved that there exist Pfaffian cyclic cubic fourfolds $Y \subset \mathbb{P}^5$ such that $\psi : F(Y) \simeq \Sigma^{[2]}$ for a certain $K3$ surface Σ of degree 14 not containing a line, by means of Beauville–Donagi's isomorphism [11]. On $\Sigma^{[2]}$ we then obtain a non-natural automorphism by conjugation with ψ of the automorphism of order three, which we construct in Sect. 2, on the Fano variety of any cyclic cubic fourfold. Another interesting question is what happens to higher dimensional manifolds of $K3^{[n]}$-type related to cyclic cubic fourfolds, for example Lehn–Lehn–Sorger–van Straten's eightfolds.

2 An Isomorphism of Moduli Spaces

In this section we will review the main constructions of [2] and [16] of the moduli spaces, and then prove the isomorphism among the two moduli spaces. Before describing the two moduli spaces involved, we first of all recall the main example of a non-symplectic automorphism of order three which we will consider here.

2.1 Fano Varieties of Cyclic Cubic Fourfolds

Recall that, for any smooth cubic fourfold $Y \subset \mathbb{P}^5$, Beauville and Donagi show in [11] that the Fano variety $F(Y)$ of lines contained in Y is of $K3^{[2]}$-type; moreover, they prove that the Plücker polarization H_F has degree six, so that $\mathrm{Pic}(F(Y)) \supset \langle 6\rangle$, and that the Abel–Jacobi map $A : H^4(Y,\mathbb{Z}) \to H^2(F(Y),\mathbb{Z})$ maps H_Y^2, where H_Y is the ample class on Y, to H_F and induces an anti-isometry on primitive cohomology (the orthogonal of the polarization), i.e. an isometry $A_0 : H_0^4(Y,\mathbb{Z})(-1) \to H_0^2(F(Y),\mathbb{Z})$. As a consequence, $H_0^2(F(Y),\mathbb{Z}) \cong S := U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$ as lattices, where U denotes the standard hyperbolic plane and $E_8(-1)$, $A_2(-1)$ are the negative definite lattices associated to the Dynkin diagrams E_8 and A_2.

Let $Y \subset \mathbb{P}^5$ be the triple cover of $\mathbb{P}^4$ branched along a smooth cubic threefold $C \subset \mathbb{P}^4$; such a fourfold is what is usually called a *cyclic cubic fourfold*. If $G(X_0 : \ldots : X_4) = 0$ is an equation defining C, then Y is the zero locus defined by $X_5^3 + G = 0$, and it is smooth if and only if C is smooth. We choose $\sigma \in \mathrm{Aut}(Y)$ to be the cover automorphism obtained as restriction to Y of the map $X_5 \mapsto \xi X_5$, for ξ a fixed (for the rest of the paper) primitive third root of unity. Let σ be the induced automorphism on $F(Y)$: in [14, Example 4.6] we show that it is a non-symplectic automorphism of order three, that the fixed locus is the Fano surface $F(C)$ of lines contained in the cubic threefold C and that the invariant lattice of σ is $T := \langle 6\rangle$, embedded in L in such a way that its orthogonal complement is isometric to S (this is the so-called embedding *of non-split type* of an ample class of square six, see [23] for further details). It is clear that such an automorphism cannot be natural nor induced on a moduli space of stable sheaves on a K3 surface by a non-symplectic automorphism of the underlying K3 surface, since these can only exist on moduli spaces X of stable sheaves on K3 surfaces with $\rho(X) \geq 2$.

2.2 An "Occult Period Map" for Cubic Threefolds

We now start to recall Allcock–Carlson–Toledo's construction of a period map for the moduli space of smooth cubic threefolds, which has been later on defined "occult" by Kudla and Rapoport in [27].

Let $\mathcal{C}_3^s := |\mathcal{O}_{\mathbb{P}^4}(3)| \mathbin{/\mkern-6mu/} PGL_5(\mathbb{C})$ be the GIT moduli space of stable cubic threefolds; it contains as an open subset the moduli space $\mathcal{C}_3^{sm}$ of smooth cubic threefolds. Classical work of Clemens and Griffiths gives a map $\mathcal{C}_3^{sm} \to \mathcal{A}_5$ by associating to C its Intermediate Jacobian $IJ(C) := H^{2,1}(C, \mathbb{C})^*/H^3(C, \mathbb{Z})$, which is a ppav of dimension 5; the problem is that these two moduli spaces are respectively of dimension 10 and 15, so there is no hope to get an isomorphism. This is the reason for which Allcock, Carlson and Toledo came up with a different idea to construct a period map for smooth cubic threefolds by using the period map of smooth cubic fourfolds, and in particular by showing that it gives an isomorphism with a 10-dimensional complex ball quotient. We will now review the main ideas of their construction.

Let $C \in \mathcal{C}_3^{sm}$ and let $Y \subset \mathbb{P}^5$ be the associated cyclic cubic fourfold. Recall that a *marking* of any cubic fourfold Y is an isometry $\eta_0 : H_0^4(Y, \mathbb{Z}) \to S(-1)$ and that the *period* of the marked pair (Y, η_0) is $[\eta_0(H^{3,1}(Y))] \in \mathbb{P}(S(-1) \otimes \mathbb{C})$. The key idea is now to use the restriction of the period map to cyclic cubic fourfolds to define a period map for cubic threefolds: when Y is cyclic, its period is an eigenvector for the action of σ^*, so that $[\eta_0(H^{3,1}(Y))] \in \mathbb{P}(S(-1)_\xi)$, where $S(-1)_\xi$ is the eigenspace of $\eta_0 \circ \sigma^* \circ \eta_0^{-1}$ relative to ξ inside $S(-1) \otimes \mathbb{C}$.

Fix as initial data a smooth cubic threefold $\bar{C} \in \mathcal{C}_3^{sm}$, and let $\bar{Y}, \bar{\sigma}$ and $\bar{\eta_0}$ be respectively the cyclic cubic fourfold associated to $\bar{C}$, the cover automorphism as defined above and a chosen marking of $\bar{Y}$. We define $\rho_0 := \bar{\eta_0} \circ \bar{\sigma} \circ \bar{\eta_0}^{-1} \in O(S(-1))$ to be the abstract isometry induced by $\bar{\sigma}$, and we use it to move the action around the locally trivial family of cubic fourfolds.

A *framing* of $C \in \mathcal{C}_3^{sm}$ is the equivalence class of a marking η_0 of Y such that $\eta_0 \circ \rho_0 = \rho_0 \circ \eta_0$, up to the action of $\mu_6 := \{\pm \mathrm{id}_S, \pm\rho_0, \pm\rho_0^2\}$: indeed, $S(-1)$ has a structure of $\mathbb{Z}[\xi]$-module induced by ρ_0, and μ_6 is exactly the group of units in $\mathbb{Z}[\xi]$. We define the moduli space of framed pairs up to isomorphism:

$$\mathcal{F}_3^{sm} := \{(C, \eta_0) | C \in \mathcal{C}_3^{sm},\ \eta_0 \text{ is a frame}\}/\simeq .$$

The group $\Gamma_0 := \{\gamma \in O(S(-1)) | \gamma \circ \rho_0 = \rho_0 \circ \gamma\}$ acts on a framed pair by $(C, \eta_0) \mapsto (C, \gamma \circ \eta_0)$, but because of our definition of framings as equivalence classes, the action is not free. We thus introduce $\mathbb{P}\Gamma := \Gamma_0/\mu_6$, and we recover the moduli space of smooth cubic threefolds as the quotient $\mathcal{C}_3^{sm} \simeq \mathcal{F}_3^{sm}/\mathbb{P}\Gamma$.

We can now study the map induced by the period map of cubic fourfolds on $\mathcal{F}_3^{sm}$: as we already observed, the period $[\eta_0(H^{3,1}(Y))] \in \mathbb{P}(S(-1)_\xi)$ and it is well-defined even for the framing, since composing with any element in μ_6 corresponds to multiplying the period for a non-zero scalar. A standard argument shows that

$$[\eta_0(H^{3,1}(Y))] \in \mathbb{B}_{10} \simeq \{x \in \mathbb{P}(S(-1)_\xi) | (x, \bar{x}) < 0\};$$

equivalently, the period domain is a ten-dimensional complex ball.

Theorem 1 *The period map sending $(C, \eta_0) \in \mathcal{F}_3^{sm}$ to $[\eta_0(H^{3,1}(Y))] \in \mathbb{B}_{10}$ defines an isomorphism onto the image and is equivariant with respect to the action of $\mathbb{P}\Gamma$, so that we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{F}_3^{sm} & \xrightarrow{\simeq} & \mathbb{B}_{10} \setminus \mathcal{H} \\ \downarrow & & \downarrow \\ \mathcal{C}_3^{sm} & \xrightarrow{\mathcal{P}_3} & (\mathbb{B}_{10} \setminus \mathcal{H})/\mathbb{P}\Gamma \end{array}$$

where $\mathcal{H}$ is the union of hyperplanes in $\mathbb{B}_{10}$ orthogonal to classes $\delta \in S(-1)$ with $\delta^2 = 2$.

2.3 Moduli Spaces of (ρ, T)-Polarized IHS Fourfolds

In this section, we recall the constructions of [16] and [17] of moduli spaces of IHS manifolds of $K3^{[n]}$-type carrying a non-symplectic automorphism of prime order associated to a given pair of sublattices (T, S), but instead of working in full generality we illustrate everything in the case of interest here. We fix a primitive embedding $j : S \hookrightarrow L$ and we call $\theta \in L$ a generator of $j(S)^\perp$, which is of degree six as we have already discussed.

A *marking* of a fourfold of $K3^{[2]}$-type X is an isometry $\eta : H^2(X, \mathbb{Z}) \to L$. Let $\mathcal{M}_L$ be the moduli space of marked pairs (X, η) of $K3^{[2]}$-type modulo isomorphism. The choice of the initial data made in Sect. 2.2 produces also initial data for this construction: indeed, we consider $F(\bar{Y})$ the Fano variety of the cyclic cubic fourfold $\bar{Y}$ and we define a marking by extending $\bar{\eta_0} \circ A^{-1}$ to a map $\bar{\eta} : H^2(F(\bar{Y}), \mathbb{Z}) \to L$ which sends $H_{F(\bar{Y})}$ to θ. In this way we obtain a point of $\mathcal{M}_L$; let $\mathcal{M}_L^+$ be the connected component containing $(F(\bar{Y}), \bar{\eta})$. We also get a non-symplectic automorphism $\bar{\sigma} \in \mathrm{Aut}(F(\bar{Y}))$ of order three and a corresponding isometry $\rho := \bar{\eta} \circ \bar{\sigma}^* \circ \bar{\eta}^{-1} \in O(L)$ of order three which satisfies the following additional properties:

1. $\rho_{|S} = \rho_0$;
2. $\rho \in \mathrm{Mon}^2(L) := \bar{\eta} \circ \mathrm{Mon}^2(F(\bar{Y}) \circ \bar{\eta}^{-1}$ is a monodromy operator on L, so by [23, §3] it has real spinor norm $\mathrm{sn}_L^{\mathbb{R}}(\rho) = 1$ (see *loc. cit.* for the definition of $\mathrm{sn}_L^{\mathbb{R}}$).

Definition 1 A $(\rho, \langle 6 \rangle)$*-polarized pair* (X, η) is the data of a marked pair $(X, \eta) \in \mathcal{M}_L^+$ and $\sigma \in \mathrm{Aut}(X)$ respectively, for which there exists $\iota : \langle 6 \rangle \hookrightarrow \mathrm{Pic}(X)$ a primitive embedding of lattices such that $\eta \circ \iota = j$ and such that $\eta \circ \sigma \circ \eta^{-1} = \rho$.

This definition turns out to be the counterpart of the notion of a framing for a cubic threefold. We define $\mathcal{M}_{\rho,\langle 6 \rangle}$ to be the subspace of isomorphism classes of $(\rho, \langle 6 \rangle)$-polarized pairs inside $\mathcal{M}_L^+$. The stabilizer of $\mathcal{M}_{\rho,\langle 6 \rangle}$ is

$$\mathrm{Mon}^2(L)_\rho := \{g \in \mathrm{Mon}^2(L) | g \circ \rho = \rho \circ g\};$$

we define the quotient space $\mathcal{N}_{\rho,\langle 6 \rangle} := \mathcal{M}_{\rho,\langle 6 \rangle} / \mathrm{Mon}^2(L)_\rho$.

Let $\Gamma_\rho := r_S(\mathrm{Mon}^2(L)_\rho)$ be the image of $\mathrm{Mon}^2(L)_\rho$ via the restriction map $r_S : O(L) \to O(S)$, which acts on the periods in $\mathbb{P}(S \otimes \mathbb{C})$. One of the key ingredients of our main result is the following

Lemma 1 *There is an isomorphism of groups* $\Gamma_\rho \simeq \Gamma_0 / \{\pm \mathrm{id}_S\}$.

The period map of marked pairs of $K3^{[2]}$-type restricts to a holomorphic map $\mathcal{M}_{\rho,\langle 6 \rangle} \to \mathbb{B}_{10}$, since again $[\eta(H^{2,0}(X))] \in \mathbb{P}(S_\xi)$ (this is nothing but the equivariance of the Abel–Jacobi map of a cyclic cubic fourfold with respect to the action of σ). In order to describe the image we need crucially the notion of *MBM class* $\delta \in L$ introduced by Amerik and Verbitsky in [3], which we do not recall in detail here since, as we will see in Sect. 2.4, it is possible to characterize these classes lattice-theoretically.

Theorem 2 *The period map restricts to an isomorphism of the subspace of isomorphism classes of* $(\rho, \langle 6 \rangle)$*-polarized pairs* $\mathcal{M}_{\rho,\langle 6 \rangle}$ *onto its image inside* $\mathbb{B}_{10}$*, and it is equivariant with respect to the action of* $\mathrm{Mon}^2(L)_\rho$*, so that it descends to the quotients. We have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{M}_{\rho,\langle 6 \rangle} & \xrightarrow{\simeq} & \mathbb{B}_{10} \setminus \mathcal{H}' \\ \downarrow & & \downarrow \\ \mathcal{N}_{\rho,\langle 6 \rangle} & \xrightarrow{\mathcal{P}_6} & (\mathbb{B}_{10} \setminus \mathcal{H}') / \Gamma_\rho \end{array}$$

where $\mathcal{H}'$ *is the union of hyperplanes in* $\mathbb{B}_{10}$ *orthogonal to MBM classes* $\delta \in S$.

2.4 The Isomorphism

We now conclude the proof of the main result: we are left with comparing the two hyperplane arrangements, since it immediately follows from Lemma 1 that the two arithmetic subgroups $\mathbb{P}\Gamma$ and Γ_ρ have the same orbits on $\mathbb{B}_{10}$.

It is known from work of Bayer–Hassett–Tschinkel [6] and Mongardi [34] that MBM classes for fourfolds of $K3^{[2]}$-type are exactly classes $\delta \in L$ either with $\delta^2 = -2$ or with $\delta^2 = -10$ and $\mathrm{div}_L(\delta) = 2$, where the *divisibility* of x in a lattice L is defined as the generator of the ideal (x, L).

Lemma 2 *All MBM classes $\delta \in S$ satisfy $\delta^2 = -2$, and $\mathcal{H} = \mathcal{H}'$.*

This follows easily from the fact that $\mathrm{div}_S(\delta)$ is a common divisor of $\det(S) = 3$ and of $\det(L) = 2$, so it cannot be two.

Theorem 3 *There is an isomorphism between the moduli spaces of smooth cubic threefolds $\mathcal{C}_3^{sm}$ and the arithmetic quotient $\mathcal{N}_{\rho,\langle 6\rangle}$ of the subspace of isomorphism classes of $(\rho, \langle 6\rangle)$-polarized pairs inside $\mathcal{M}_L^+$, such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{N}_{\rho,\langle 6\rangle} & \xrightarrow{\ \simeq\ } & \mathcal{C}_3^{sm} \\ & {\scriptstyle \mathcal{P}_6}\searrow \quad \swarrow{\scriptstyle \mathcal{P}_3} & \\ & (\mathbb{B}_{10} \setminus \mathcal{H}')/\,\Gamma_\rho & \end{array}$$

Whilst a priori we did not know whether any element of $\mathcal{N}_{\rho,\langle 6\rangle}$ was exactly the Fano variety of a cyclic cubic fourfold, this follows immediately from Theorem 3.

Corollary 1 *Any $(X, \eta) \in \mathcal{M}_{\rho,\langle 6\rangle}$ is of the form $(F(Y), \eta)$ for some Y cyclic cubic fourfold.*

A second interesting consequence is that we obtain a more classically-flavoured period map also in the case of fourfolds of $K3^{[2]}$-type.

Corollary 2 *There is an open embedding of $\mathcal{N}_{\rho,\langle 6\rangle}$ inside $\mathcal{A}_5$ given by sending (X, η) into $\mathrm{Alb}(X^\sigma)$.*

Indeed, this follows immediately from the fact that $\mathrm{Alb}(F(C)) \cong IJ(C)$.

3 Degenerations

Degenerations of smooth cubic threefolds are well-studied; in particular it is proven in [2] that the hyperplane arrangement $\mathcal{H}$ is the union of two sub-arrangements $\mathcal{H}_n$ and $\mathcal{H}_c$ respectively corresponding to stable cubics with at least one nodal singularity and to the semistable chordal cubic. From a lattice theoretical viewpoint, the two cases correspond respectively to *nodal* and *chordal* roots $\delta \in S(-1)$ of square 2.

In the case of cubic threefolds, it is nowadays very well-understood what happens when the period hits these hyperplanes; here, we want to answer the same question in the case of IHS manifolds. In particular the aim of this section is to explain what happens to the non-symplectic automorphism of order three when the period of (X, η) is in $\mathcal{H}$. This has been already explained by Dolgachev and Kondō in [20] for $K3$ surfaces. We know that in the fiber of the period map of fourfolds of $K3^{[2]}$-type over a point in the hyperplane arrangement $\mathcal{H}$ we can still find smooth fourfolds; in order to understand what happens to the (ρ, T)-polarization we apply [33, Theorem 1.6] and obtain the following statement:

Proposition 1 *Let $\omega \in \delta^{\perp} \subset \mathcal{H}$ and let (X, η) be any element in $\mathcal{P}_6^{-1}(\omega)$; then there exists $\beta \in \mathrm{Bir}(X)$ and $w \in W_{\mathrm{exc}}(X)$ such that $\eta^{-1} \circ \rho \circ \eta = w \circ \beta^*$, where $W_{\mathrm{exc}}(X) \subset O(H^2(X, \mathbb{Z}))$ is the subgroup generated by reflections in prime exceptional divisors, (i.e. reduced and irreducible effective divisors of negative degree).*

As a consequence of Proposition 1, one might be tempted to expect that the isometry ρ no longer induces an automorphism but only a birational map of order three with a non-empty indeterminacy locus. In fact this is not the case, and the automorphism changes in the sense that it has a different action on the lattice. In order to explain this phenomenon we need to recall some definitions from [20].

Definition 2 Given $\omega \in \mathcal{H}$, the degeneracy sublattice of ω is the primitive sublattice of S spanned by MBM classes which are orthogonal to ω.

The key remark is that for any MBM class in S, we have $\delta^{\perp} \cap \mathbb{P}(S_{\xi}) = \rho(\delta)^{\perp} \cap \mathbb{P}(S_{\xi})$. From now on we suppose $\omega \in H \subset \mathcal{H}$ for only one hyperplane $H = \delta^{\perp}$; in this case the degeneracy lattice is $R_{\delta} := \langle \delta, \rho(\delta) \rangle \cong A_2(-1)$; this admits, up to isometry, two primitive embeddings into S, according to the two cases $R_{\delta}^{\perp} = U \oplus U(a) \oplus E_8(-1)^{\oplus 2}$ for $a = 1, 3$; we speak respectively about chordal and nodal roots δ in the two cases $a = 1, 3$, since the following holds:

Lemma 3 *Let $\delta \in S$ have square -2 and let $\omega \in \delta^{\perp} \subset \mathcal{H}$ be very general; then, $R_{\delta}^{\perp}$ is unimodular if and only if $\omega \in \mathcal{H}_c$, if and only if $a = 1$.*

We now want to look at how the isometry ρ degenerates along the hyperplanes. We call T_{δ} the saturation of $\langle 6 \rangle \oplus R_{\delta}$ in L, so that $(T_{\delta})_L^{\perp} \subset S_{\delta} := (R_{\delta})_S^{\perp}$.

Lemma 4 *The lattice T_{δ} is respectively isometric to $T_n := U(3) \oplus \langle -2 \rangle \cong \langle 6 \rangle \oplus A_2(-1)$ for $\omega \in \mathcal{H}_n$ (case $a = 3$), and to $T_c := U \oplus \langle -2 \rangle$ for $\omega \in \mathcal{H}_c$ (case $a = 1$).*

We define ρ_{δ} as the isometry of L obtained by gluing $\mathrm{id}_{T_{\delta}}$ and $\rho_{|S_{\delta}}$: it is the degeneration of the isometry ρ to $\delta^{\perp}$, and by construction we have $L^{\rho_{\delta}} = T_{\delta}$.

Proposition 2 *The isometry $\rho_{\delta} \in \mathrm{Mon}^2(L)$ and we have $\rho_{\delta} = r_{\rho(\delta)} \circ r_{\delta} \circ \rho$, where $r_v \in O(L)$ denotes the reflection in v, given by $x \mapsto x - 2\frac{(x,v)}{(v,v)}v$.*

In order to distinguish the two cases, we denote by ρ_n and ρ_c the isometries ρ_δ respectively when δ is a nodal and a chordal root.

3.1 The Nodal Locus

A semistable nodal cubic threefold, whose period maps to the very general $\omega \in \delta^\perp$ with δ a nodal root, is of the form $C : X_0 Q + M_3 = 0$ with an A_1-singularity in $p = [1 : 0 : \ldots : 0]$; hence, the associated cubic fourfold has equation $Y : X_0 Q + M_3 + X_3^5 = 0$ with an A_2-singularity in $p = [1 : 0 : \ldots : 0]$. Allcock, Carlson and Toledo extend $\mathcal{P}_3$ to semistable points by using the period of $\Sigma : Q = M_3 + X_5^3 = 0$: when the singularity of C is A_1, this is a smooth $K3$ surface, endowed with a non-symplectic automorphism τ of order three, with invariant lattice $U(3) \subset H^2(\Sigma, \mathbb{Z})$ (see [4]).

Indeed, it is a classical fact that $F(Y)$ is singular in this situation and that the singular locus is contained in $F(Y, p)$, defined as the surface of lines contained in Y which pass through one of the isolated singularities p (compare with [30, 37] and [41]). The surface Σ is the minimal resolution of $F(Y, p)$ in this case, and Lehn shows in [30] that $\Sigma^{[2]} \to F(Y)$ is a resolution of singularities.

Moreover, the natural automorphism $\tau^{[2]}$ on $\Sigma^{[2]}$ is induced by the original automorphism of $\mathbb{P}^5$ which we had chosen at the beginning, and its invariant sublattice is exactly T_n.

Proposition 3 *There exists a marking $\eta : H^2(\Sigma^{[2]}, \mathbb{Z}) \to L$ such that $(\Sigma^{[2]}, \eta)$ is (ρ_n, T_n)-polarized. In particular, $\eta^{-1} \circ \rho_n \circ \eta = (\tau^{[2]})^*$.*

We now choose a very general point $(\Sigma^{[2]}, \eta)$ as above and such that $\mathrm{Pic}(\Sigma^{[2]}) \cong T_n$, and we define $K(T_n) := \eta(\mathcal{K}_{\Sigma^{[2]}})$. Following the construction of [16] and the techniques first developed in [26], we say that $(X, \eta) \in \mathcal{M}^{\xi}_{\rho_n, T_n}$ is $K(T_n)$*-general* if $\eta(\mathcal{K}_X) \cap (T_n)_{\mathbb{R}} = K(T_n)$; the moduli space of $K(T_n)$-general (ρ_{T_n}, T_n)-polarized fourfolds of $K3^{[2]}$-type is $\mathcal{M}^{\xi}_{K(T_n),\rho_n}$ and its quotient for the action of $\mathrm{Mon}^2(L, \rho_n)$ is $\mathcal{N}^{\xi}_{K(T_n),\rho_n}$.

Making the above reasoning global, in [17] we obtain the following

Theorem 4 *The stable locus $\Delta_3^s = C_3^s \setminus C_3^{sm}$ is birational to $\mathcal{N}^{\xi}_{K(T_n),\rho_n}$.*

3.2 The Chordal Locus

In the case of the stable chordal locus Ξ_3^s the picture is more complicated: associated to one semistable chordal cubic we find a hyperplane arrangement. The fact is, as Allcock, Carlson and Toledo show, that the variation of Hodge structure along a 1-parameter family hitting $\mathcal{H}_c$ depends on the path chosen, so that from one semistable point in the moduli space one gets in fact an entire hyperplane arrangement. Again, they show that the variety realizing the degeneration of the Hodge structure is a $K3$ surface Σ carrying a non-symplectic automorphism τ of order three having fixed lattice isometric to U (see [4]). The final result, though through a less geometrical proof, is the following

Theorem 5 *The stable chordal locus Ξ_3^s is birational to $\mathcal{N}^{\xi}_{K(T_c),\rho_c}$.*

It remains an open question to understand the geometry of the (singular) Fano variety of lines of the cyclic fourfold corresponding to the chordal cubic threefold, and which relation, if any, it has with the Hilbert schemes of $K3$ surfaces in $\mathcal{N}^{\xi}_{K(T_c),\rho_c}$.

Theorems 4 and 5 in some sense shed also new light on why smooth cubic threefolds are degenerating to $K3$ surfaces: in dimension four the picture becomes much more uniform and the degeneration is explained as degeneration of non-symplectic automorphisms of order three on fourfolds of $K3^{[2]}$-type.

4 Further Developments and Open Questions

4.1 Pfaffian Cyclic Cubic Fourfolds

Other natural loci to study inside the arithmetic quotient $\mathcal{N}_{\rho,\langle 6\rangle} = \mathcal{M}_{\rho,\langle 6\rangle} / \mathrm{Mon}^2(L)_\rho$ are Heegner divisors, and in particular those coming from the intersection among the family of cyclic cubic fourfolds and the so-called Hassett's divisors $\mathcal{H}_d \in C_4^{sm}$ (see [24] for the exact definition). In [17] we have studied the closely-related question about the existence of a cyclic Pfaffian cubic fourfold not containing a plane (indeed, the closure of the Pfaffian locus is $\mathcal{H}_{14}$, and cubics containing a plane are exactly the elements of $\mathcal{H}_8$).

Recall the following construction from classical Pfaffian geometry, as described in [11]: let V be a six-dimensional vector space and let $\mathcal{P}f \subset \mathbb{P}(\Lambda^2 V^*)$ be the hypersurface of degenerate skew-symmetric 2-forms. Any Pfaffian cubic fourfold Y is then of the form $\mathcal{P}f \cap \mathbb{P}(W)$ for $W \subset \Lambda^2 V^*$ six-dimensional. The *dual K3 surface* of Y is $\Sigma := G(2, V) \cap \mathbb{P}(W^\perp)$, of degree 14.

Proposition 4 *There exists a smooth cyclic Pfaffian cubic fourfold $Y \subset \mathbb{P}^5$ defined over $\mathbb{Q}$ which does not contain a plane, and such that the dual $K3$ surface Σ does not contain a line.*

The interest of Proposition 4 is that Beauville and Donagi proved in [11] that under these special assumptions, the Fano variety of a Pfaffian cubic fourfold is isomorphic to the Hilbert scheme of two points on Σ. As a consequence, the automorphism is defined on $\Sigma^{[2]}$ and we have

Corollary 3 *There exists a non-natural non-symplectic automorphism of order three on $\Sigma^{[2]}$.*

See [17] for a geometrical description. We remark that more in general, one obtains a non-natural automorphism on a Hilbert square whenever there exists a $K3$ surface Σ such that $\Sigma^{[2]} \cong F(Y)$ for a cyclic cubic fourfold Y, but the question of studying the intersection, which happens in positive codimension, among Hassett's divisors and the locus of cyclic cubic fourfolds remains open.

4.2 Higher Dimensional IHS Manifolds

Fourfolds of $K3^{[2]}$-type are not the only IHS manifolds that can be associated to cubic fourfolds: a recent example are the *Lehn–Lehn–Sorger–van Straten (LLSvS)* eightfolds of $K3^{[4]}$-type, defined as contraction Z of the Hilbert scheme of generalized twisted cubic curves contained in a smooth cubic fourfold Y which does not contain a plane [31].

In [18, §6] we show that there is a non-symplectic automorphism ζ of order three on the LLSvS eightfold Z associated to a smooth cyclic cubic fourfold not containing a plane; we can thus define the isometry $\rho_Z \in \mathrm{Mon}^2(L_{K3} \oplus \langle -6 \rangle)$ associated to his action (starting from a fixed initial example as above) the invariant lattice of ζ is $\langle 2 \rangle$ and its orthogonal complement is again S inside $L_{K3} \oplus \langle -6 \rangle$. Exactly the same theory illustrated in 2.3 following [16] leads to the construction of the moduli space $\mathcal{M}_{\rho_Z, \langle 2 \rangle}$ of $(\rho_Z, \langle 2 \rangle)$-polarized eightfolds of $K3^{[4]}$-type; in particular, [16, Theorem 1.2] implies the following

Corollary 4 *Let $\mathcal{H}''$ be the hyperplane arrangement in $\mathbb{P}(S \otimes \mathbb{C})$ consisting of hyperplanes orthogonal to MBM classes in S. The period map*

$$\mathcal{P}_2 : \mathcal{M}_{\rho_Z, \langle 2 \rangle} \to \mathbb{B}_{10} \setminus \mathcal{H}''$$

is an isomorphism, equivariant with respect to the action of $\mathrm{Mon}^2(L_{K3} \oplus \langle -6 \rangle)_{\rho_Z}$.

It follows from [34] that in dimension eight the MBM classes that can be contained in S are classes $\delta \in S$ either such that $\delta^2 = -2$ or such that $\delta^2 = -6$ or $\delta^2 = -24$ and div $= 3$. As a consequence, $\mathcal{H}''$ is strictly bigger than $\mathcal{H}$, though this was to be expected because the construction of the LLSvS eighfold is only possible in the open subset of cubic fourfolds not containing a plane. Another difference which arises with the increase in dimension is that the group of isometries induced by monodromy operators is

$$\mathrm{Mon}^2(L_{K3} \oplus \langle -6 \rangle) = \{g \in O(L_{K3} \oplus \langle -6 \rangle) \mid \mathrm{sn}^{\mathbb{R}}(g) = 1,\ \bar{g} = \mathrm{id} \in O(q_{A_{L_{K3} \oplus \langle -6 \rangle}}\},$$

so a priori it could be of index two inside $\mathbb{P}\Gamma$.

In any case, ideally one obtains a finite rational map

$$\mathcal{M}_{\rho_Z, \langle 2 \rangle} / \mathrm{Mon}^2(L_{K3} \oplus \langle -6 \rangle)_{\rho_Z} \dashrightarrow C_3^{sm}.$$

Moreover, as it was explained in another talk of this workshop by Lahoz, forthcoming work of Bayer, Lahoz, Macrì, Neuer, Perry and Stellari includes the construction of many other families of higher dimensional polarized manifolds of $K3^{[n]}$-type associated to cubic fourfolds (see [32, Theorem 5.11]), so it is expected that there is a "tower" of rational maps from moduli spaces of manifolds of $K3^{[n]}$-type with a non-symplectic automorphism of order three having invariant lattice of rank one to the moduli space of smooth cubic threefolds.

4.3 Another Case

In the recent work [29], the authors use the same techniques of Allcock, Carlson and Toledo in order to study the moduli space $\mathcal{W}$ of pairs (C, H) consisting of a smooth cubic threefold $C \in C_3^{sm}$ and of a hyperplane H. In particular, if $G = 0$ is an equation of C and $L = 0$ is an equation of H, they consider the cubic fourfold Y defined by the equation $G + X_5^2 L = 0$, and they consider the open subset $\mathcal{W}^0 \subset \mathcal{W}$ given by pairs for which Y is smooth, so that they can study the restriction of the period map $\mathcal{P}_4$ to it. Again, this construction allows to relate the GIT compactification $\bar{\mathcal{W}}$ of $\mathcal{W}^0$ and the Baily–Borel compactification of the period domain.

Again, if one considers the Fano variety of lines $F(Y)$, this has a non-symplectic involution ι induced by $X_5 \mapsto -X_5$, as observed in [10]. The computations contained in [29] on the transcendental part of $H^4(Y, \mathbb{Z})$, which is isomorphic to $U^{\oplus 2} \oplus D_4^{\oplus 3}$, shed new light on this involution: since $F(Y)$ is general, by dimension count, in the family carrying this kind of involution, we have that the orthogonal complement of the invariant lattice M of ι is $N := U^{\oplus 2} \oplus D_4(-1)^{\oplus 3}$ and that M is a 2-elementary overlattice of $\langle 6 \rangle \oplus E_6(-2)$. A standard computation shows that $M \cong \langle 2 \rangle \oplus \langle -2 \rangle^{\oplus 6}$: we remark that in [14, Figure 2] this case is erroneously omitted, though it cannot be realized by a natural automorphism, because N has no primitive embedding in L_{K3} (see [28, Theorem 3.4]).

Fix a primitive embedding $j_M : M \subset L$ such that $j_M(M)^\perp \cong N$; the fourfolds of $K3^{[2]}$-type with such a non-symplectic involution fixing M are exactly all the lattice polarized fourfolds $(X, \eta) \in \mathcal{M}_L$ such that there exists $i : M \subset H^2(X, \mathbb{Z})$ primitive with $\eta \circ i = j_M$ and such that $i(M) \cap \mathcal{K}_X \neq \emptyset$, as explained in [26]. We denote by $\mathcal{M}_M \subset \mathcal{M}_L$ the subspace of these lattice-polarized pairs and by $\mathcal{M}_{M,K}$ the connected component containing a pair $(F(Y), \eta)$ with Y smooth as above such that $\mathrm{Pic}(F(Y)) \cong M$, where $K := \eta(\mathcal{K}_{F(Y)} \cap \mathrm{Pic}(F(Y)))$.

Forthcoming work of the author will try to answer the question whether $\mathcal{W}$ and $\mathcal{M}_{M,K}$ are isomorphic, and, if the answer is positive, to check whether the isomorphism extends to the compactifications.

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A Note on Severi Varieties of Nodal Curves on Enriques Surfaces

Ciro Ciliberto, Thomas Dedieu, Concettina Galati, and Andreas Leopold Knutsen

Abstract Let $|L|$ be a linear system on a smooth complex Enriques surface S whose general member is a smooth and irreducible curve of genus p, with $L^2 > 0$, and let $V_{|L|,\delta}(S)$ be the Severi variety of irreducible δ-nodal curves in $|L|$. We denote by $\pi : X \to S$ the universal covering of S. In this note we compute the dimensions of the irreducible components V of $V_{|L|,\delta}(S)$. In particular we prove that, if C is the curve corresponding to a general element $[C]$ of V, then the codimension of V in $|L|$ is δ if $\pi^{-1}(C)$ is irreducible in X and it is $\delta - 1$ if $\pi^{-1}(C)$ consists of two irreducible components.

Keywords Severi varieties · Moduli · Nodal curves · Enriques surfaces

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1 Introduction

Let S be a smooth complex projective surface and L a line bundle on S such that the complete linear system $|L|$ contains smooth, irreducible curves (such a line bundle, or linear system, is often called a *Bertini system*). Let

$$p := p_a(L) = \frac{1}{2}L \cdot (L + K_S) + 1,$$

be the arithmetic genus of any curve in $|L|$.

For any integer $0 \leq \delta \leq p$, consider the locally closed, functorially defined subscheme of $|L|$

$$V_{|L|,\delta}(S) \quad \text{or simply} \quad V_{|L|,\delta}$$

parameterizing irreducible curves in $|L|$ having only δ nodes as singularities; this is called the *Severi variety* of δ-nodal curves in $|L|$. We will let $g := p - \delta$, the geometric genus of the curves in $V_{|L|,\delta}$.

It is well-known that, if $V_{|L|,\delta}$ is non-empty, then all of its irreducible components V have dimension $\dim(V) \geq \dim |L| - \delta$. More precisely, the Zariski tangent space to $V_{|L|,\delta}$ at the point corresponding to C is

$$T_{[C]}V_{|L|,\delta} \simeq H^0(L \otimes \mathcal{J}_N)/ < C >, \tag{1}$$

where $\mathcal{J}_N = \mathcal{J}_{N|S}$ is the ideal sheaf of subscheme N of S consisting of the δ nodes of C (see, e.g., [4, §1]). Thus, $V_{|L|,\delta}$ is *smooth of dimension* $\dim |L| - \delta$ at $[C]$ if and only if the set of nodes N imposes independent conditions on $|L|$. In this case, $V_{|L|,\delta}$ is said to be *regular* at $[C]$. An irreducible component V of $V_{|L|,\delta}$ will be said to be *regular* if the condition of regularity is satisfied at any of its points, equivalently, if it is smooth of dimension $\dim |L| - \delta$.

The *existence and regularity problems of* $V_{|L|,\delta}(S)$ have been studied in many cases and are the most basic problems one may ask on Severi varieties. We only mention some of known results. In the case $S \simeq \mathbb{P}^2$, Severi proved the existence and regularity of $V_{|L|,\delta}(S)$ in [13]. The description of the tangent space is due to Severi and later to Zariski [15]. The existence and regularity of $V_{|L|,\delta}(S)$ when S is of general type has been studied in [4] and [3]. Further regularity results are provided in [10]. More recently Severi varieties on K3 surfaces have received a lot of attention for many reasons. In this case Severi varieties are known to be regular (cf. [14]) and are nonempty on general K3 surfaces by Mumford and Chen (cf. [2, 12]).

As far as we know, Severi varieties on Enriques surfaces have not been studied yet, apart from [8, Thm. 4.12] which limits the singularities of a general member of the Severi variety $V^g_{|L|}$ of irreducible genus g curves in $|L|$, and gives a sufficient condition for the density of the latter in the Severi variety $V_{|L|,p-g}$ of $(p-g)$-nodal curves. In particular, the existence problem is mainly open and we intend to treat it in a forthcoming article. The result of this paper is Proposition 1, which answers the regularity question for Severi varieties of nodal curves on Enriques surfaces.

2 Regularity of Severi Varieties on Enriques Surfaces

Let S be a smooth Enriques surface, i.e. a smooth complex surface with nontrivial canonical bundle $\omega_S \not\simeq \mathcal{O}_S$, such that $\omega_S^{\otimes 2} \simeq \mathcal{O}_S$ and $H^1(\mathcal{O}_S) = 0$. We denote linear (resp. numerical) equivalence by $\sim$ (resp. $\equiv$).

Let L be a line bundle on S such that $L^2 > 0$. It is well-known that $|L|$ contains smooth, irreducible curves if and only if it contains irreducible curves (see [5, Thm. 4.1 and Prop. 8.2]); in other words, *on Enriques surfaces the Bertini linear systems are the linear systems that contain irreducible curves*. Moreover, by [6, Prop. 2.4], this is equivalent to L being nef and not of the form $L \sim P + R$, with $|P|$ an elliptic pencil and R a smooth rational curve such that $P \cdot R = 2$ (in which case $p = 2$). If $|L|$ is a Bertini linear system, the adjunction formula, the Riemann–Roch theorem, and Mumford vanishing yield that

$$L^2 = 2(p-1) \quad \text{and} \quad \dim|L| = p - 1$$

(see, e.g., [5, 7]).

Let K_S be the canonical divisor. It defines an étale double cover

$$\pi : X \longrightarrow S \tag{2}$$

where X is a smooth, projective $K3$ surface (that is, $\omega_X \simeq \mathcal{O}_X$ and $H^1(\mathcal{O}_X) = 0$), endowed with a fixed-point-free involution ι, which is the universal covering of S. Conversely, the quotient of any $K3$ surface by a fixed-point-free involution is an Enriques surface.

Let $C \subset S$ be a reduced and irreducible curve of genus $g \geq 2$. We will henceforth denote by $\nu_C : \widetilde{C} \to C$ the normalization of C and define $\eta_C := \mathcal{O}_C(K_S) = \mathcal{O}_C(-K_S)$, a nontrivial 2-torsion element in $\mathrm{Pic}^0 C$, and $\eta_{\widetilde{C}} := \nu_C^* \eta_C$. The fact that η_C is nontrivial follows from the cohomology of the restriction sequence

$$0 \longrightarrow \mathcal{O}_S(K_S - C) \longrightarrow \mathcal{O}_S(K_S) \longrightarrow \eta_C \longrightarrow 0,$$

which yields $h^0(\eta_C) = h^1(K_S - C) = h^1(C) = 0$, the latter vanishing as C is big and nef. One has the fiber product

$$\begin{array}{ccc} (\pi^{-1}C) \times_C \widetilde{C} & \longrightarrow & \widetilde{C} \\ \downarrow & & \downarrow \nu_C \\ (\pi^{-1}C) & \xrightarrow[\pi|_{\pi^{-1}(C)}]{} & C, \end{array}$$

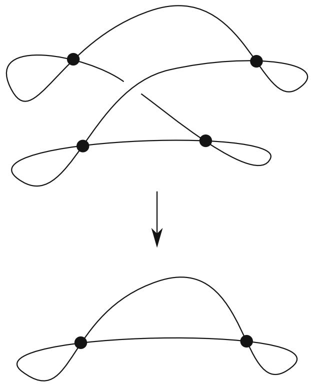

Fig. 1 $\eta_{\tilde{C}} = \nu_C^*(\eta_C) \neq 0$

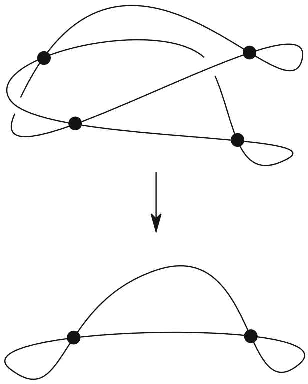

Fig. 2 $\eta_{\tilde{C}} = \nu_C^*(\eta_C) = 0$

where $\pi|_{|_{\pi^{-1}(C)}}$ and the upper horizontal map are the double coverings induced respectively by η_C and $\eta_{\tilde{C}}$. By standard results on coverings of complex manifolds (cf. [1, Sect. I.17]), two cases may happen:

- $\eta_{\tilde{C}} \not\cong \mathcal{O}_{\tilde{C}}$ and $\pi^{-1}C$ is irreducible, as in Fig. 1;
- $\eta_{\tilde{C}} \simeq \mathcal{O}_{\tilde{C}}$ and $\pi^{-1}C$ consists of two irreducible components conjugated by the involution ι. These two components are *not* isomorphic to C, as η_C is nontrivial, as in Fig. 2 (each component of $\tilde{C}$ is a partial normalization of C).

As mentioned in the Introduction, it is well-known that *any* irreducible component of a Severi variety on a $K3$ surface is regular when nonempty (see, e.g., [4, Ex. 1.3]; see also [8, §4.2]). The corresponding result on Enriques surfaces is the following.

First note that, in the above notation, the dimension of the Severi variety of genus $g = p_g(C)$ curves in $|L| = |C|$ at the point $[C]$ satisfies the inequalities

$$g - 1 \leq \dim_{[C]}\bigl(V^g_{|L|}\bigr) \leq h^0(\omega_{\tilde{C}} \otimes \eta_{\tilde{C}}) = \begin{cases} g - 1 & \text{if } \eta_{\tilde{C}} \not\simeq \mathcal{O}_{\tilde{C}} \\ g & \text{if } \eta_{\tilde{C}} \simeq \mathcal{O}_{\tilde{C}} \end{cases} \tag{3}$$

(see [8, ineq. (2.6) and Lem. 2.3]). Our result implies that the second inequality in (3) is in fact an equality when C is nodal, and gives a concrete geometric description of the situation in both cases.

Proposition 1 *Let L be a Bertini linear system, with $L^2 > 0$, on a smooth Enriques surface S. Then the Severi variety $V_{|L|,\delta}(S)$ is smooth and every irreducible component $V \subseteq V_{|L|,\delta}(S)$ has either dimension $g-1$ or g; in the former case the component is regular. Furthermore, with the notation introduced above,*

1. *for any curve C in a $(g-1)$-dimensional irreducible component V, $\pi^{-1}C$ is irreducible (whence an element in $V_{|\pi^*L|,2\delta}(X)$);*
2. *for any g-dimensional component V, there is a line bundle L' on X with $(L')^2 = 2(p-d)-2$ and $L' \cdot \iota^*L' = 2d$ for some integer d satisfying*

$$\frac{p-1}{2} \leq d \leq \delta,$$

such that $\pi^*L \simeq L' \otimes \iota^*L'$, and the curves parametrized by $V \subseteq V_{|L|,\delta}(S)$ are the birational images by π of the curves in $V_{|L'|,\delta-d}(X)$ intersecting their conjugates by ι transversely (in $2d$ points). In other words, for any $[C] \in V$, we have $\pi^{-1}C = Y + \iota(Y)$, with $[Y] \in V_{|L'|,\delta-d}(X)$ and $[\iota(Y)] \in V_{|\iota^*L'|,\delta-d}(X)$ intersecting transversely.

Furthermore, if $L' \simeq \iota^*L'$, which is the case if S is general in moduli, then $d = \frac{p-1}{2}$ and $L \sim 2M$, for some $M \in \operatorname{Pic} S$ such that $M^2 = d$.

We will henceforth refer to components of dimension $g-1$ as *regular* and the ones of dimension g as *nonregular*. Note however that from a parametric perspective the Severi variety has the expected dimension and is smooth in both cases, as the fact that (3) is an equality indicates; we do not dwell on this here, and refer to [8] for a discussion of the differences between the parametric and Cartesian points of view (the latter is the one we adopted in this text).

Note that Proposition 1 does not assert that the Severi variety $V_{|L|,\delta}$ is necessarily non-empty: in such a situation, $V_{|L|,\delta}$ does not have any irreducible component and the statement is empty.

Proof Pick any curve C in an irreducible component V of $V_{|L|,\delta}(S)$. Let $f : \widetilde{S} \to S$ be the blow-up of S at N, the scheme of the δ nodes of C, denote by $\mathfrak{e}$ the (total) exceptional divisor and by $\widetilde{C}$ the strict transform of C. Thus $f_{|\widetilde{C}} = \nu_C$ and we have

$$K_{\widetilde{S}} \sim f^*K_S + \mathfrak{e} \ \text{ and } \ \widetilde{C} \sim f^*C - 2\mathfrak{e}.$$

From the restriction sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{S}}(\mathfrak{e}) \longrightarrow \mathcal{O}_{\widetilde{S}}(\widetilde{C} + \mathfrak{e}) \longrightarrow \omega_{\widetilde{C}}(\eta_{\widetilde{C}}) \longrightarrow 0$$

we find

$$\begin{aligned}\dim T_{[C]} V_{|L|,\delta}(S) = \dim |L \otimes \mathcal{J}_N| &= h^0(L \otimes \mathcal{J}_N) - 1 = h^0(f^*L - \mathfrak{e}) - 1 \\ &= h^0(\mathcal{O}_{\widetilde{S}}(\widetilde{C} + \mathfrak{e})) - 1 = h^0(\omega_{\widetilde{C}}(\eta_{\widetilde{C}})) \\ &= \begin{cases} g - 1, & \text{if } \eta_{\widetilde{C}} \not\simeq \mathcal{O}_{\widetilde{C}}, \\ g, & \text{if } \eta_{\widetilde{C}} \simeq \mathcal{O}_{\widetilde{C}}. \end{cases}\end{aligned} \tag{4}$$

In the upper case, by (1), we have that $V_{|L|,\delta}$ is smooth at $[C]$ of dimension $g - 1 = p - \delta - 1 = \dim |L \otimes \mathcal{J}_N|$.

Assume next that we are in the lower case. Then, by the discussion prior to the proposition, we have $\pi^{-1}C = Y + \iota(Y)$ for an irreducible curve Y on X, such that π maps both Y and $\iota(Y)$ birationally, but not isomorphically, to C. In particular, Y and $\iota(Y)$ have geometric genus $p_g(Y) = p_g(\iota(Y)) = p_g(C) = p - \delta = g$. Set $L' := \mathcal{O}_X(Y)$ and $2d := Y \cdot \iota(Y)$. Note that d is an integer because, if $y = \iota(x) \in Y \cap \iota(Y)$, then $\iota(y) = x \in Y \cap \iota(Y)$. Since $Y \simeq \iota(Y)$ and π is étale, both Y and $\iota(Y)$ are nodal with $\delta - d$ nodes and they intersect transversely at $2d$ points, which are pairwise conjugate by ι, and therefore map to d nodes of C. Hence $d \leq \delta$. We have

$$p_a(Y) = p_a(\iota(Y)) = g + \delta - d = p - \delta + \delta - d = p - d, \tag{5}$$

whence

$$(L')^2 = 2(p - 1 - d).$$

By the Hodge index theorem, we have

$$4(p - 1 - d)^2 = \left((L')^2\right)^2 = (L')^2(\iota^*L')^2 \leq \left(L' \cdot \iota^*L'\right)^2 = 4d^2,$$

whence $p - 1 \leq 2d$.

By the regularity of Severi varieties on $K3$ surfaces, any irreducible component of $V_{|L'|,\delta-d}(X)$ has dimension $\dim |L'| - (\delta - d) = p_g(Y) = g$. Hence, V is g-dimensional; more precisely, the curves parameterized by V are the (birational) images by π of the curves in an irreducible component of $V_{|L'|,\delta-d}(X)$ intersecting their conjugates by ι transversely (in $2d$ points). By (4), it also follows that $\dim V = \dim T_{[C]} V_{|L|,\delta}(S)$, so that $[C]$ is a smooth point of $V_{|L|,\delta}(S)$.

To prove the final assertion of the proposition, observe that, by the regularity of Severi varieties on $K3$ surfaces, we may deform Y and $\iota(Y)$ on X to irreducible curves Y' and $\iota(Y')$ with any number of nodes $\leq \delta - d$ and intersecting transversally in $2d$ points; in particular, we may deform Y and $\iota(Y)$ to *smooth* curves Y' and $\iota(Y')$. Thus, $C' := \pi(Y')$ is a member of $V_{|L|,d}$, whence of geometric genus $p - d$.

Since $\dim|Y'| = p_a(Y') = p_g(C') = p_a(C') - d = p - d$, the component of $V_{|L|,d}$ containing $[C']$ has dimension $\dim|L| - d + 1 = p - d$. We thus have $\dim|L \otimes \mathcal{J}_{N'}| = \dim|L| - d + 1$, where N' is the set of d nodes of C', hence N' does not impose independent conditions on $|L|$.

Assume now that $L' \simeq \iota^* L'$, which—as is well-known (see, e.g., [9, §11])—is the case occurring for generic S, as then Pic X is precisely the invariant part under ι of $H_2(X, \mathbb{Z})$. Then $2d = L' \cdot \iota^* L' = (L')^2 = 2(p - 1 - d)$, so that $p - 1 = 2d$. Since $L^2 = 2(p - 1) = 4d$ and N' does not impose independent conditions on $|L|$, by [11, Prop. 3.7] there is an effective divisor $D \subset S$ containing N' satisfying $L - 2D \geq 0$ and

$$L \cdot D - d \leq D^2 \overset{(i)}{\leq} \frac{1}{2} L \cdot D \overset{(ii)}{\leq} d, \qquad (6)$$

with equality in (i) or (ii) only if $L \equiv 2D$; moreover, since $L - 2D \geq 0$, the numerical equivalence $L \equiv 2D$ implies the linear equivalence $L \sim 2D$. Now since $N' \subset D$, we must have $L \cdot D = C' \cdot D \geq 2d$, hence the inequalities in (6) are all equalities, and thus $D^2 = d$ and $L \sim 2D$.

The following corollary is a straightforward consequence of Prop. 1 and the fact that the nodes on curves in a regular component in a Severi variety (on any surface and in particular on a K3 surface) can be independently smoothened.

Corollary 1 *If a Severi variety $V_{|L|,\delta}$ on an Enriques surface has a regular (resp., nonregular) component, then for any $0 \leq \delta' \leq \delta$ (resp., $d \leq \delta' \leq \delta$, with d as in Prop. 1), also $V_{|L|,\delta'}$ contains a regular (resp., nonregular) component.*

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A Travel Guide to the Canonical Bundle Formula

Enrica Floris and Vladimir Lazić

Abstract We survey known results on the canonical bundle formula and its applications in algebraic geometry.

Keywords Canonical bundle formula · Moduli divisor · Minimal Model Program

1 Introduction

The Minimal Model Program (MMP) predicts that every projective pair with mild singularities is birationally built out of three classes of pairs: those whose log canonical classes are ample, numerically trivial or anti-ample.

More precisely, let (X, Δ) be a log canonical pair. Then there should exist a birational contraction $\varphi\colon (X, \Delta) \dashrightarrow (X_{\min}, \Delta_{\min})$ together with a fibration $f\colon (X_{\min}, \Delta_{\min}) \to X_{\mathrm{can}}$ so that $K_{X_{\min}} + \Delta_{\min} \sim_{\mathbb{Q}} f^*A$, for a suitable ample $\mathbb{Q}$-divisor A on X_{can}. Note that the Iitaka dimension of $K_X + \Delta$ restricted to a general fibre of the composed map $f \circ \varphi$ is zero.

It is a natural and important question to determine whether singularities of the MMP are preserved in this process. The singularities of $(X_{\min}, \Delta_{\min})$ are the same

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as those of (X, Δ). On the other hand, it remains an open problem whether there exists a boundary divisor Δ_{can} on X_{can} such that $(X_{\mathrm{can}}, \Delta_{\mathrm{can}})$ is log canonical and $K_{X_{\min}} + \Delta_{\min} \sim_{\mathbb{Q}} f^*(X_{\mathrm{can}} + \Delta_{\mathrm{can}})$; in other words, whether singularities of (X, Δ) *descend* to the canonical model X_{can}.

When the singularities of (X, Δ) are klt, it is known by a work of Ambro and Kawamata that such a divisor exists; this result has already had numerous consequences in birational geometry. However, the proof is not constructive: more precisely, one loses control of the coefficients of Δ at the last step. It is desirable that the singularities of $(X_{\mathrm{can}}, \Delta_{\mathrm{can}})$ reflect in a canonical way the singularities of (X, Δ).

In general, with notation as above, it is known that

$$A \sim_{\mathbb{Q}} K_{X_{\mathrm{can}}} + B_{X_{\mathrm{can}}} + M_{X_{\mathrm{can}}},$$

where $B_{X_{\mathrm{can}}}$—the *discriminant*—is closely related to the singularities of f, and the divisor $M_{X_{\mathrm{can}}}$—the *moduli divisor*—conjecturally carries information on the birational variation of the fibres of f. A formula of this form is called the *canonical bundle formula*.

This paper is an attempt to give an account of all the known results on the canonical bundle formula, and to serve as a guide to those wishing to study this important subject.

2 Lc-Trivial Fibrations

We work over $\mathbb{C}$. We denote by $\equiv$, $\sim$ and $\sim_{\mathbb{Q}}$ the numerical, linear and $\mathbb{Q}$-linear equivalence of divisors respectively.

For a Weil $\mathbb{Q}$-divisor $D = \sum d_i D_i$, for a real number r we denote $D^{\leq r} := \sum_{d_i \leq r} d_i D_i$. If $f: X \to Y$ is a proper surjective morphism between normal varieties and D is a Weil $\mathbb{R}$-divisor on X, then D_v and D_h denote the vertical and the horizontal part of D with respect to f. In this setup, we say that D is *f-exceptional* if $\operatorname{codim}_Y \operatorname{Supp} f(D) \geq 2$.

In this section we introduce the main topic of this survey—*lc-trivial fibrations*. In this section we define them and give some examples which will accompany us through the paper.

We first need to introduce singularities of pairs. This is by now a standard topic in higher dimensional birational geometry, and good references are [27] and [26].

A *pair* (X, Δ) consists of a normal variety X and a Weil $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. A pair (X, Δ) is *log smooth* if X is smooth and the support of Δ is a simple normal crossings divisor.

A *log resolution* of a pair (X, Δ) is a birational morphism $f: Y \to X$ such that the exceptional locus $\mathrm{Exc}(f)$ is a divisor and the pair $\big(Y, \operatorname{Supp}(f_*^{-1}\Delta + \mathrm{Exc}(f))\big)$ is log smooth.

If (X, Δ) be a pair and if $\pi : Y \to X$ is a birational morphism with Y normal, we can write

$$K_Y \sim_{\mathbb{Q}} \pi^*(K_X + \Delta) + \sum a(E_i, X, \Delta) \cdot E_i,$$

where $E_i \subseteq Y$ are distinct prime divisors and the numbers $a(E_i, X, \Delta) \in \mathbb{Q}$ are called *discrepancies*. The order of vanishing at the generic point of each E_i defines a *geometric valuation* on $\mathbb{C}(X)$. The pair (X, Δ) is *klt*, respectively *log canonical*, if $a(E, X, \Delta) > -1$, respectively $a(E, X, \Delta) \geq -1$, for every geometric valuation E over X.

Much of what we say in this paper can be generalised to pairs (X, Δ), where Δ is allowed to have real coefficients. We stick to rational divisors mostly for reasons of clarity and simplicity.

2.1 Definition and First Examples

The objects for which we can write a canonical bundle formula are called *lc-trivial fibrations*.

Definition 2.1 Let (X, Δ) be a pair. A morphism $f : (X, \Delta) \to Y$ to a normal projective variety Y is a *klt-trivial*, respectively *lc-trivial*, fibration if:

(a) f is a surjective morphism with connected fibres,
(b) (X, Δ) has klt, respectively log canonical, singularities over the generic point of Y,
(c) there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that

$$K_X + \Delta \sim_{\mathbb{Q}} f^*D,$$

(d) there exists a log resolution $\pi' : X' \to X$ of (X, Δ) such that, if $\mathcal{E}$ is the set of all geometric valuations over X which are defined by a prime divisor E on X' such that $a(E, X, \Delta) > -1$, and if we denote $\Xi' = \sum_{E \in \mathcal{E}} a(E, X, \Delta) \cdot E$, then

$$\operatorname{rk}(f \circ \pi')_* \mathcal{O}_{X'}(\lceil \Xi' \rceil) = 1.$$

Terminology 2.2 In [2], klt-trivial fibrations as in Definition 2.1 are called lc-trivial fibrations.

Remark 2.3 We make a few comments on the condition (d) in Definition 2.1. For simplicity, assume that the pair (X, Δ) is klt. Note that then

$$\Xi' \sim_{\mathbb{Q}} K_{X'} - \pi'^*(K_X + \Delta).$$

The divisor $\lceil \Xi' \rceil$ is effective on the generic fibre of the morphism $f \circ \pi'$ by (b), hence $\mathrm{rk}(f \circ \pi')_* \mathcal{O}_{X'}(\lceil \Xi' \rceil) \geq 1$. Therefore, the point of (d) is the opposite inequality.

The most important case to keep in mind is that when the divisor Δ is effective on the generic fibre: indeed, in that case the divisor $\lceil \Xi' \rceil$ is an effective exceptional divisor on the generic fibre of $f \circ \pi'$, and the condition (d) is immediate. However, in order to be able to study lc-trivial fibrations by applying basic operations of birational geometry in Sect. 2.2, it is crucial to allow divisors Δ with negative coefficients.

One more thing to notice is that if (d) holds for a log resolution π', then it holds on any log resolution $\pi'' : X'' \to X$ which factors through π'. Indeed, define the divisor Ξ'' on X'' analogously as in Definition 2.1, and let $\theta : X'' \to X'$ be the induced morphism. Since X' and X'' are smooth, there exists an integral effective divisor E such that $K_{X''} \sim \theta^* K_{X'} + E$. Thus,

$$\begin{aligned} f_* \pi''_* \mathcal{O}_{X''}(\lceil \Xi'' \rceil) &= f_* \pi''_* \mathcal{O}_{X''}(\lceil \theta^* \Xi' + E \rceil) = f_* \pi''_* \mathcal{O}_{X''}(\lceil \theta^* \Xi' \rceil + E) \\ &\subseteq f_* \pi''_* \mathcal{O}_{X''}(\theta^* \lceil \Xi' \rceil + E) = f_* \pi'_* \mathcal{O}_{X'}(\lceil \Xi' \rceil), \end{aligned}$$

where we used that $\lceil \theta^* \Xi' \rceil \leq \theta^* \lceil \Xi' \rceil$. Therefore, $\mathrm{rk}(f \circ \pi'')_* \mathcal{O}_{X''}(\lceil \Xi'' \rceil) = 1$.

In general, one can show that (d) is independent of the choice of the resolution by using [14, Lemma 2.7].

Now we can formulate the canonical bundle formula associated to an lc-trivial fibration.

Definition 2.4 Let $f : (X, \Delta) \to Y$ be an lc-trivial fibration such that $K_X + \Delta \sim_{\mathbb{Q}} f^* D$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y. If $P \subseteq Y$ is a prime divisor, the *log canonical threshold* of $f^* P$ with respect to (X, Δ) is

$$\gamma_P = \sup\{t \in \mathbb{R} \mid (X, \Delta + t f^* P) \text{ is log canonical over the generic point of } P\}.$$

The condition that $(X, \Delta + t f^* P)$ is *log canonical over the generic point of* P means that for every geometric valuation E over X which surjects onto P, we have $a(E, X, \Delta + t f^* P) \geq -1$. The *discriminant* of f is

$$B_Y = \textstyle\sum_P (1 - \gamma_P) P.$$

Fix $\varphi \in \mathbb{C}(X)$ and the smallest positive integer r such that $K_X + \Delta + \frac{1}{r} \operatorname{div} \varphi = f^* D$. Then there exists a unique Weil $\mathbb{Q}$-divisor M_Y, the *moduli part* of f, such that

$$K_X + \Delta + \tfrac{1}{r} \operatorname{div} \varphi = f^*(K_Y + B_Y + M_Y).$$

This formula is the *canonical bundle formula* associated to f.

Remark 2.5 The definition of the discriminant as above first appeared in [23, Theorem 2]. The discriminant is a Weil $\mathbb{Q}$-divisor on Y, and it is effective if Δ

is effective. We notice that the discriminant is defined as an actual divisor, while the moduli part is defined only up to $\mathbb{Q}$-linear equivalence: it depends on the choice of D. Further, if G is a $\mathbb{Q}$-divisor on Y, then $f: (X, \Delta + f^*G) \to Y$ is an lc-trivial fibration with discriminant $B_Y + G$ and moduli divisor M_Y.

Note that if we are interested in proving properties of the moduli divisor of an lc-trivial fibrations as above, we may always assume that the pair (X, Δ) is log canonical by [12, Remark 3.6], although we may not assume that Δ is effective.

Example 2.6 Assume that X is smooth, that $\Delta = 0$, that Y is a curve, that $f^{-1}(P)$ is smooth and that $f^*P = mf^{-1}(P)$ is a multiple fibre. Then $\gamma_P = \frac{1}{m}$.

Example 2.7 This example is historically the first example of a canonical bundle formula. Let $f: X \to C$ be an elliptic fibration, that is, X is a smooth surface, C is a smooth curve and a general fibre of f is a smooth elliptic curve. We assume furthermore that f is relatively minimal: there are no (-1)-curves contained in the fibres of f. Kodaira in [24, Theorem 6.2] classified the singular fibres of f. Kodaira's canonical bundle formula reads as

$$K_X \sim f^*(K_C + B_C + M_C),$$

where B_C is defined in terms of the classification of the singular fibres and by [24, 32] we have $12M_C = j^*\mathcal{O}_{\mathbb{P}^1}(1)$, with $j: C \to \mathbb{P}^1$ being the j-invariant. For a detailed account on Kodaira's canonical bundle formula see [4, Chapter V, §7–§13].

Example 2.8 Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let D be a reduced divisor of bidegree (d, k) with $d \geq 2$. Let $\Delta = \frac{2}{d}D$ and let $f: (X, \Delta) \to \mathbb{P}^1$ be the projection onto the second factor. Then f is an lc-trivial fibration. Indeed, $K_X + \Delta$ has bidegree $(-2, -2) + \frac{2}{d}(d, k) = (0, -2 + \frac{2k}{d})$ and therefore is the pullback of a divisor from $\mathbb{P}^1$.

2.2 Base Change Property

In this subsection we investigate how canonical bundle formulas behave under base change. This will help improve the properties of the moduli part of a canonical bundle formula, at least on a sufficiently high birational model.

If $f: (X, \Delta) \to Y$ is a klt-trivial fibration (respectively lc-trivial), and if we consider a base change diagram

$$\begin{CD} (X', \Delta') @>{\tau}>> (X, \Delta) \\ @V{f'}VV @VV{f}V \\ Y' @>>{\rho}> Y, \end{CD} \tag{1}$$

where ρ is a proper generically finite morphism, X' is the normalisation of the fibre product and Δ' is defined so that we have

$$K_{X'} + \Delta' = \tau^*(K_X + \Delta),$$

then $f' \colon (X', \Delta') \to Y'$ is also a klt-trivial (respectively lc-trivial) fibration. In the rest of the paper, we implicitly refer to this klt-trivial fibration when writing $M_{Y'}$ and $B_{Y'}$ for the moduli part and discriminant.

If ρ is birational, then $\rho_* M_{Y'} = M_Y$ and $\rho_* B_{Y'} = B_Y$; in other words, these collections of divisors form *b-divisors*, see for instance [2, Section 1.2].

The following is the *base change property* of canonical bundle formulas.

Theorem 2.9 *Let $f \colon (X, \Delta) \to Y$ be an lc-trivial fibration. Then there exists a proper birational morphism $Y' \to Y$ such that for every proper birational morphism $\pi \colon Y'' \to Y'$ we have:*

(i) $K_{Y'} + B_{Y'}$ is a $\mathbb{Q}$-Cartier divisor and $K_{Y''} + B_{Y''} = \pi^*(K_{Y'} + B_{Y'})$,
(ii) $M_{Y'}$ is a nef $\mathbb{Q}$-Cartier divisor and $M_{Y''} = \pi^* M_{Y'}$.

The first version of Theorem 2.9 is [23, Theorem 2], which essentially shows the nefness of the moduli part; this is also the point of the proof where the condition (d) in Definition 2.1 is used. Theorem 2.9 has been proved in this form for klt-trivial fibrations by Ambro [2, Theorem 0.2]. For lc-trivial fibrations, it was proved by Kollár [25, Theorem 8.3.7] and [18, Theorem 3.6], with an alternative proof in [11].

In the context of the previous theorem, we say that $M_{Y'}$ *descends* to Y', and we call any such Y', where additionally $B_{Y'}$ has simple normal crossings support, an *Ambro model* for f.

We give a brief sketch of the proof of the nefness of the moduli divisor in the previous theorem; very good references are [2, Lemma 5.2] and especially [25, Theorem 8.5.1 and §8.10], where many more details are given. By the proof of [25, Theorem 8.5.1], it suffices to show the claim after making a suitable generically finite base change and taking the cyclic cover of X associated to $\sqrt[r]{\varphi}$. The base change as in (1) that we are aiming for is a composition of a log resolution with a Kawamata cover such that, on an open subset of $U' \subseteq Y'$ whose complement has codimension at least 2 in Y', the local systems $R^i f'_* \mathbb{C}_{X'}|_{U'}$ have unipotent monodromies; this is the content of [25, 8.10.7–8.10.10]. We may also assume that $M_{Y'}$ is a Cartier divisor. If f is a klt-trivial fibration, then by [25, Theorem 8.5.1] the line bundle $\mathcal{O}_{Y'}(M_{Y'})$ is a quotient of the locally free sheaf $f'_* \omega_{X'/Y'}$. Then one applies [22, Theorem 5], which asserts that $f'_* \omega_{X'/Y'}$ is the canonical extension of the bottom piece of the Hodge filtration of $R^{\dim X - \dim Y} f'_* \mathbb{C}_{X'}|_{U'}$, and hence its quotients are nef by the same result. A similar statement holds also for lc-trivial fibrations.

Recently, a stronger statement was shown in [17, Theorem 1.1]. The result shows that [22, Theorem 5] can be improved to show that not only any quotient of $f'_* \omega_{X'/Y'}$ is nef, but moreover, it carries a singular metric whose Lelong numbers are all zero.

This is much stronger than being nef, as it implies, in particular, that the multiplier ideal associated to this metric is trivial.

We summarise this in the following result.

Theorem 2.10 *Let $f\colon (X,\Delta)\to Y$ be a klt-trivial fibration. Then there exists a proper birational morphism $Y'\to Y$ such that for every proper birational morphism $\pi: Y''\to Y'$ we have:*

(i) $K_{Y'}+B_{Y'}$ is a $\mathbb{Q}$-Cartier divisor and $K_{Y''}+B_{Y''}=\pi^*(K_{Y'}+B_{Y'})$,
(ii) $M_{Y'}$ is a $\mathbb{Q}$-Cartier divisor carrying a singular metric whose all Lelong numbers are zero, and $M_{Y''}=\pi^*M_{Y'}$.

The proof of part (ii) of the theorem is the same as the sketch of the proof of Theorem 2.9(ii) above, since a $\mathbb{Q}$-divisor carries a singular metric whose all Lelong numbers are zero if and only if its pullback by a proper surjective map carries a singular metric whose all Lelong numbers are zero by [7, Corollary 4].

2.3 Inversion of Adjunction

In order to appreciate the following result and to see why base change property is important, let us go back to the construction of the canonical bundle formula. Recall that the discriminant divisor was constructed in terms of *local* log canonical thresholds, that is, log canonical thresholds over the generic point of a prime divisor; in particular, with notation from Definition 2.4, for some prime divisor P on Y, the pair $(X,\Delta+\gamma_P f^*P)$ does not have to be *globally* log canonical. However, the following *inversion of adjunction* [2, Theorem 3.1] states that this is precisely what happens on an Ambro model.

Theorem 2.11 *Let $f\colon (X,\Delta)\to Y$ be an lc-trivial fibration, and assume that Y is an Ambro model for f. Then (Y,B_Y) has klt, respectively log canonical, singularities in a neighbourhood of a point $y\in Y$ if and only if (X,Δ) has klt, respectively log canonical, singularities in a neighbourhood of $f^{-1}(y)$.*

Note that Theorem 2.11 is stated for klt-trivial fibrations in [2], but the proof extends verbatim to the lc-trivial case by using Theorem 2.9.

We finish this subsection with the following nice result [3, Proposition 3.1] which we will apply in the proof of Theorem 2.16 below.

Theorem 2.12 *Let $f\colon (X,\Delta)\to Y$ be a klt-trivial fibration, and assume that Y is an Ambro model for f. Then for every base change by a proper generically finite morphism $w\colon W\to Y$ we have $K_W+B_W\sim_{\mathbb{Q}} w^*(K_Y+B_Y)$ and $M_W\sim_{\mathbb{Q}} w^*M_Y$.*

2.4 Coefficients of the Moduli Divisor

Often in applications one needs to bound the denominators of M_Y. For instance, such bounds were used in [10, Theorem 1.2].

Theorem 2.13 *For each nonnegative integer b there exists an integer N depending on b such that the following holds.*

Let $f\colon (X,\Delta) \to Y$ be a klt-trivial fibration with a general fibre F, and let r be the smallest positive integer such that $r(K_F+\Delta|_F) \sim 0$. Let $E \to F$ be the associated r-th cyclic cover and let $\overline{E}$ be a resolution of singularities of E. If $\dim H^{\dim \overline{E}}(\overline{E},\mathbb{C}) = b$, then the divisor NM_Y is integral.

The result was proved in [19, Theorem 3.1] when $\Delta = 0$, but the same proof works for klt-trivial fibrations [10, Theorem 5.1].

A more refined result holds when a general fibre is a rational curve, [9, Theorem 1.6(2)].

Theorem 2.14 *Fix a positive integer r. For a prime number q set $s(q) = \max\{s \mid q^s \leq 2r\}$ and define*

$$N = \prod_{q\ prime} q^{s(q)}.$$

(a) Let $f\colon (X,\Delta) \to Y$ be an lc-trivial fibration whose general fibre F is a rational curve, and assume that r is the smallest positive integer such that $r(K_F+\Delta|_F) \sim 0$. Then the divisor NM_Y is integral.

(b) Assume r is odd. Then there exists an lc-trivial fibration $f\colon (X,\Delta) \to Y$ such that if v is the smallest integer for which the divisor vM_Y is integral, then $v = N/r$.

2.5 Goodness of Moduli Divisors

Now we come to the central topic of this survey, already announced in the introduction: the descent of singularities. Since we already know the nefness of the moduli divisor by Theorem 2.9, if it were additionally big, then this would allow to conclude in many cases. Bigness is too much to ask; however, the following result of Ambro [3, Theorem 3.3 and Proposition 4.4] turns out to be almost as good.

Theorem 2.15 *Let $f\colon (X,\Delta) \to Y$ be a klt-trivial fibration between normal projective varieties such that Δ is effective over the generic point of Y. Then there exists a diagram*

$$\begin{array}{ccccc} (X,\Delta) & & & & (X^+,\Delta^+) \\ \downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle f^+} \\ Y & \xleftarrow[\tau]{} & W & \xrightarrow[\rho]{} & Y^+ \end{array}$$

such that:

(i) $f^+\colon (X^+,\Delta^+) \to Y^+$ *is a klt-trivial fibration,*
(ii) τ *is generically finite and surjective, and* ρ *is surjective,*
(iii) *if* M_Y *and* M_{Y^+} *are the moduli divisors of* f *and* f^+ *respectively, then* M_{Y^+} *is big and, after possibly a birational base change, we have* $\tau^* M_Y = \rho^* M_{Y^+}$*,*
(iv) *there exists a non-empty open set* $U \subseteq W$ *and an isomorphism*

$$\begin{array}{ccc} (X,\Delta)\times_Y U & \xrightarrow{\simeq} & (X^+,\Delta^+)\times_{Y^+} U \\ & \searrow \quad \swarrow & \\ & U, & \end{array}$$

(v) *if there exists an isomorphism*

$$\Phi\colon (X,\Delta)\times_Y U \to (F,\Delta_F)\times U$$

over a non-empty open subset $U \subseteq Y$*, then* Φ *extends to an isomorphism over*

$$Y^0 = Y \setminus \big(\operatorname{Supp} B_Y \cup \operatorname{Sing}(Y) \cup f(\operatorname{Supp} \Delta_v^{\leq 0})\big).$$

Note that in (v) one does not need that Δ is effective on the generic fibre of f.

For us, the most important part of this result is (iii). Its immediate consequence is the descent of klt singularities [3, Theorem 0.2].

Theorem 2.16 *Let (X,Δ) be a projective klt pair with Δ effective, and let $f\colon X \to Y$ be a surjective morphism to a normal projective variety such that $K_X + \Delta \sim_{\mathbb{Q}} f^* D$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y. Then there exists an effective $\mathbb{Q}$-divisor Δ_Y on Y such that the pair (Y,Δ_Y) is klt and*

$$K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Y + \Delta_Y).$$

Proof We use the notation from Theorem 2.15, which clearly applies in our situation. We have the base change diagram

$$\begin{array}{ccc} (W, \Delta_W) & \xrightarrow{w} & (X, \Delta) \\ {\scriptstyle f_W}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ W & \xrightarrow[\tau]{} & Y. \end{array}$$

We may additionally assume that τ factors through an Ambro model $\pi\colon Y' \to Y$ of f, and denote by $\sigma\colon W \to Y'$ the induced morphism. By replacing W by a suitable log resolution, by an easy argument with the Stein factorisation of σ together with Theorem 2.12 we may assume that W is an Ambro model of f_W.

Now, $\tau^*D \sim_{\mathbb{Q}} K_W + B_W + M_W$, and by the inversion of adjunction, Theorem 2.11, the pair (W, B_W) is klt. Since the divisor M_{Y^+} is nef and big, by using a version of Kodaira's trick [27, Proposition 2.61] together with Bertini's theorem, we may find an effective $\mathbb{Q}$-divisor E_W on W such that $M_W \sim_{\mathbb{Q}} E_W$ and such that the pair $(W, B_W + E_W)$ is klt.

By Theorem 2.12 we have $K_W + B_W \sim_{\mathbb{Q}} \sigma^*(K_{Y'} + B_{Y'})$ and $M_W \sim_{\mathbb{Q}} \sigma^* M_{Y'}$. Hence, if we denote $E_{Y'} = \frac{1}{\deg \sigma}\sigma_* E$, we have $M_{Y'} \sim_{\mathbb{Q}} E_{Y'}$ and

$$K_W + B_W + E \sim_{\mathbb{Q}} \sigma^*(K_{Y'} + B_{Y'} + E_{Y'}).$$

Then the pair $(Y', B_{Y'} + E_{Y'})$ is klt by [27, Proposition 5.20]. Setting

$$\Delta_Y = \pi_*(B_{Y'} + E_{Y'}) = B_Y + \pi_* E,$$

we have $K_Y + \Delta_Y \sim_{\mathbb{Q}} D$, and (Y, Δ_Y) is a klt pair. Since Δ is effective, the divisor B_Y is effective, thus Δ_Y is effective. $\square$

Finally, we mention that Theorem 2.15(i)–(iii) was generalised to lc-trivial fibrations where $\Delta \geq 0$ and (X, Δ) is log canonical in [18, Lemma 1.1]. The proof involves running a careful MMP in order to reduce to a situation where one has a klt-trivial fibration.

Example 2.17 Theorem 2.15(iv) does not hold for lc-trivial fibrations. For instance, let $X = \mathbb{P}^1 \times \mathbb{P}^1$, let f be the second projection, let δ be the diagonal and set $\Delta = \delta + \frac{1}{2}\{0\} \times \mathbb{P}^1 + \frac{1}{2}\{\infty\} \times \mathbb{P}^1$. Then $f\colon (X, \Delta) \to \mathbb{P}^1$ is an lc-trivial fibration with discriminant supported on $\{0\} \cup \{\infty\}$. By considering log resolutions, one calculates that the discriminant is equal to $\frac{1}{2}0 + \frac{1}{2}\infty$, hence the moduli divisor is torsion. Indeed, we have

$$K_X + \Delta \sim_{\mathbb{Q}} f^*\big(K_{\mathbb{P}^1} + \tfrac{1}{2}0 + \tfrac{1}{2}\infty + M_{\mathbb{P}^1}\big),$$

but the divisor $K_X + \Delta$ has bi-degree $(0, -1)$.

However, f is not birational to a product. Indeed, it induces a family of rational curves with 4 marked points parametrised by $\mathbb{P}^1$: for $t \in \mathbb{P}^1$, set $p_1(t) = 0$, $p_2(t) = 1$, $p_3(t) = p_4(t) = t$. This family of marked curves is not trivial, therefore the fibration cannot be locally a product.

3 B-Semiampleness Conjectures

We saw above in Theorem 2.16 that klt singularities descend along a klt-trivial fibration. However, even if one had the full analogue of Theorem 2.15 in the log canonical setting one could not follow the same strategy to show that log canonical singularities descend. Moreover, one sees that the use of Kodaira's trick in the proof of Theorem 2.16 obliterated the connection of the coefficients of the divisor Δ_Y to that of the divisor Δ. In order to remedy the situation, something more is needed.

The main conjecture on the canonical bundle formula predicts something much stronger: that the moduli part is *semiample* on an Ambro model of the fibration. We discuss it in this section.

There are two versions of the conjecture; the stronger one was proposed in [31, Conjecture 7.13.3]. We start with the stronger, *effective* version.

Effective B-Semiampleness Conjecture *Fix positive integers d and r. Then there exists an integer m depending only on d and r, such that for any lc-trivial fibration $f: (X, B) \to Y$ with the generic fibre F, if* $\dim F = d$ *and r is the smallest positive integer such that $r(K_F + B|_F) \sim 0$, there exists an Ambro model Y' of f such that $mM_{Y'}$ is base point free.*

More generally, any conjecture as above, in which m depends on some invariants of the generic fibre of the fibration, goes under the name of *effective b-semiampleness*.

In the original statement [31, Conjecture 7.13.3], the constant m depended on $\dim X$ and r. The main result of [10] is that it suffices to show the Effective B-Semiampleness Conjecture in the case where Y is a curve. This led to the formulation of the Effective B-Semiampleness Conjecture above.

The conjecture is widely open. We list below the cases where it is known. They all make use of some notion of moduli space for the fibres.

Theorem 3.1 *The Effective B-Semiampleness Conjecture holds in the following cases:*

(1) if general fibres are elliptic curves by [24, 32]*; we have $m = 12$;*
(2) if general fibres are rational curves [31, Theorem 8.1]*;*
(3) if general fibres are K3 surfaces or abelian varieties of dimension d by [13, Theorem 1.2]*; then we have $m = 19k$, respectively $m = k(d + 1)$, where k is a weight associated to the Baily-Borel-Satake compactification of the period domain.*

In the weaker version of the conjecture we lose control on the constant m.

B-Semiampleness Conjecture *Let $f\colon (X, B) \to Y$ be an lc-trivial fibration. Then there exists an Ambro model Y' of f such that $M_{Y'}$ is semiample.*

More is known about this conjecture than about its effective version above, although the progress has been limited to the cases where either the bases Y or the fibres of f are low dimensional. We summarise the situation for low dimensional fibres in the following result.

Theorem 3.2 *Apart from the cases listed in Theorem 3.1, the B-Semiampleness Conjecture holds in the following cases:*

(1) if the fibres are surfaces of Kodaira dimension 0 by [13, Lemma 4.1 and Corollary 6.4] *and* [29, Part I, (5.15.9)(ii)]*;*
(2) if f is a klt-trivial fibration and the generic fibre is a uniruled surface not isomorphic to $\mathbb{P}^2$ by [8, Theorem 1.7].

The B-Semiampleness Conjecture holds for another important family of fibrations:

Theorem 3.3 *Let $f\colon (X, \Delta) \to Y$ be an lc-trivial fibration, and assume that the moduli part M_Y descends to Y. If $M_Y \equiv 0$, then $M_Y \sim_{\mathbb{Q}} 0$.*
In particular, if $\dim Y = 1$, then M_Y is semiample.

Theorem 3.3 is [3, Theorem 3.5] for klt-trivial fibrations and [10, Theorem 1.3] for lc-trivial fibrations. Theorem 1.2 in [10] states that Effective B-Semiampleness Conjecture is true for klt-trivial fibrations with numerically trivial moduli part.

Another partial result is [5, Theorem 3.2]. They prove that a small perturbation of the moduli part in a specific direction is semiample, under some effectivity hypotheses for $K_Y + B_Y$.

3.1 Restrictions to Divisors

As we saw in the previous subsection, the progress on the B-Semiampleness Conjecture has been concentrated on the cases of either the low dimension of the base Y or the low dimension of the generic fibre of the fibration f.

In [12, Theorem A] we obtained the following result towards the conjecture valid in every dimension. Note that the phrase *the B-semiampleness Conjecture in dimension n* means that we consider the conjecture in the case when the dimension of the base Y is n.

Theorem 3.4 *Assume the B-Semiampleness Conjecture in dimension $n-1$.*
Let (X, Δ) be a log canonical pair and let $f\colon (X, \Delta) \to Y$ be an lc-trivial fibration to an n-dimensional variety Y, where the divisor Δ is effective over the generic point of Y. Assume that Y is an Ambro model for f.

Then for every birational model $\pi: Y' \to Y$ and for every prime divisor T on Y' with the normalisation T^ν and the induced morphism $\nu: T^\nu \to Y'$, the divisor $\nu^*\pi^*M_Y$ is semiample on T^ν.

As a corollary, combining with Theorem 3.3, we obtain that the restriction of the moduli part to every prime divisor on every sufficiently high birational model of Y is semiample if Y is a surface.

We comment on the proof of Theorem 3.4, as it will be useful in the following subsection. We first apply a base change to Y and modify (X, Δ) by blowing up suitably, but we try to remember (X, Δ) along the proof. We then run a suitable relative MMP over Y, which contracts many "bad" components of Δ (in particular, those with negative coefficients); as a result, we obtain a new lc-trivial fibration $g: (W, \Delta_W) \to Y$ with $\Delta_W \geq 0$ and with the same moduli divisor M_Y. Choosing a minimal log canonical centre S of (W, Δ_W) which surjects onto T, we obtain an induced *klt*-trivial fibration $g|_S: (S, \Delta_S) \to T'$, where T' is obtained from the Stein factorisation of the morphism $S \to T$. Then we first show that $M_Y|_{T'}$ is *almost* $M_{T'}$. Even though at this step we may not deduce equality between these two divisors, we can control their difference in a very precise manner. After a suitable further blowup of Y, we can force this difference to disappear and we conclude by induction on the dimension.

3.2 Reduction Result

As we mentioned above, in the setup of lc-trivial fibrations $f: (X, \Delta) \to Y$ one does not assume that Δ is effective. Furthermore, often it is much more difficult to work with lc-trivial fibrations than with klt-trivial fibrations.

In [12] the B-Semiampleness Conjecture is reduced to the following conjecture with much weaker hypotheses.

Conjecture 3.5 *Let (X, Δ) be a log canonical pair and let $f: (X, \Delta) \to Y$ be a klt-trivial fibration to an n-dimensional variety Y. If Y is an Ambro model of f and if the moduli divisor M_Y is big, then M_Y is semiample.*

We show in [12, Theorem E]:

Theorem 3.6 *Assume Conjecture 3.5 in dimensions at most n. Then the B-Semiampleness Conjecture holds in dimension n.*

The proof is similar to the proof of Theorem 3.4 sketched above.

3.3 Generalisation

In the papers [16, 20] the authors consider slc-trivial fibrations. Those are completely analogous to lc-trivial fibrations, the difference being that the ambient space X is not irreducible, but the pair (X, Δ) is *slc* on the generic fibre of the fibration; such a setup appears occasionally in inductive proofs. The precise statement is [16, Definition 4.1].

Then one can define, as in the case of lc-trivial fibrations, the moduli divisor and the discriminant. Then [16, Theorem 1.2] proves the analogue of Theorem 2.9 for slc-trivial fibrations, and [20, Theorem 1.3] shows the analogue of Theorem 3.3 in this context.

4 Parabolic Fibrations

Finally, in this section we discuss a more general situation than that of lc-trivial fibrations.

Let $g\colon (X, \Delta) \to Z$ be a surjective morphism, where (X, Δ) is a klt projective pair and Z is a projective variety. Assume that $g_*\mathcal{O}_X\big(m(K_X + \Delta)\big) \neq 0$ for some positive integer m, and consider the relative Iitaka fibration $f\colon X \dashrightarrow Y$ associated to $K_X + \Delta$. Possibly by blowing up further, one may assume that (X, Δ) is log smooth and that f is a morphism. Then if F is a general fibre of f, we have $\kappa\big(F, (K_X + \Delta)|_F\big) = 0$, however $K_X + \Delta$ is not necessarily a pullback from Y. One still wonders if there is a canonical bundle formula for the map f.

The resulting formula is the canonical bundle formula of Fujino and Mori [19]. We first need a definition, which is justified from the setup above.

Definition 4.1 A *parabolic fibration* is a fibration $f\colon (X, \Delta) \to Y$, where (X, Δ) is a projective klt pair, Y is a smooth projective variety and if F is the generic fibre of f, then $\kappa\big(F, (K_X + \Delta)|_F\big) = 0$.

The following is [19, Section 4], the *canonical bundle formula* of Fujino and Mori.

Theorem 4.2 *Let $f\colon (X, \Delta) \to Y$ be a parabolic fibration, where Δ is effective. Then there is a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\tau'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\tau]{} & Y, \end{array}$$

where τ and τ' are birational, X' and Y' are smooth, and f' has connected fibres, such that the following holds.

There exist effective $\mathbb{Q}$-divisors B^+ and B^- on X' without common components, a $\mathbb{Q}$-divisor $\Delta' \geq 0$ on X' and $\mathbb{Q}$-divisors $B_{Y'}$ and $M_{Y'}$ on Y such that

$$K_{X'} + \Delta' + B^- \sim_{\mathbb{Q}} f'^*(K_{Y'} + B_{Y'} + M_{Y'}) + B^+,$$

with the following properties:

(i) the pair (X', Δ') is klt and log smooth, and there exists an effective exceptional divisor E on X' such that

$$K_{X'} + \Delta' \sim_{\mathbb{Q}} \tau'^*(K_X + \Delta) + E,$$

(ii) $f'_*\mathcal{O}_{X'}(\lfloor nB^+ \rfloor) \simeq \mathcal{O}_{Y'}$ for all $n \in \mathbb{N}$,
(iii) B^- is f'-exceptional and τ'-exceptional,
(iv) the induced map $f' \colon (X', \Delta' + B^- - B^+) \to Y'$ is a klt-trivial fibration, and $B_{Y'}$ and $M_{Y'}$ are the corresponding discriminant and moduli divisors,
(v) the pair $(Y', B_{Y'})$ is klt, $B_{Y'} \geq 0$ and $M_{Y'}$ is nef,
(vi) for every $n \in \mathbb{N}$ sufficiently divisible we have

$$H^0\big(X, n(K_X + \Delta)\big) \simeq H^0\big(X', n(K_{X'} + \Delta')\big) \simeq H^0\big(Y', n(K_{Y'} + B_{Y'} + M_{Y'})\big).$$

There are several non-trivial parts of this formula which do not follow from considerations in the previous sections: the existence of divisors B^+ and B^- with the properties (ii) and (iii) above, as well as the fact that $B_{Y'}$ is effective.

Part (vi) follows immediately from (i), (ii) and (iii). We sketch how (iv) follows from (i), (ii) and (iii), following [1, Lemma 4.2]. Let F' be a general fibre of f', and we define the divisor Ξ' with respect to $f' \colon (X', \Delta' + B^- - B^+) \to Y'$ as in Definition 2.1(d). We may assume that $\Xi' = -\Delta' - B^- + B^+$, and $B^-|_{F'} = 0$ by (iii).

We have $(K_{X'} + \Delta' + B^- - B^+)|_{F'} \sim_{\mathbb{Q}} 0$ by construction and $\kappa\big(F', (K_{X'} + \Delta')|_{F'}\big) = 0$ by (i), hence

$$\kappa(F', B^+|_{F'}) = \kappa\big(F', (B^+ - B^-)|_{F'}\big) = 0.$$

Since there exists a positive integer b such that $\lceil B^+ \rceil|_{F'} \leq bB^+|_{F'}$, this implies $\kappa\big(F', \lceil B^+ \rceil|_{F'}\big) = 0$. Therefore,

$$\begin{aligned}\kappa\big(F', \lceil \Xi' \rceil|_{F'}\big) &= \kappa\big(F', \lceil -\Delta' - B^- + B^+ \rceil|_{F'}\big)\\ &\leq \kappa\big(F', \lceil -\Delta' \rceil|_{F'} + \lceil -B^- \rceil|_{F'} + \lceil B^+ \rceil|_{F'}\big)\\ &\leq \kappa\big(F', \lceil B^+ \rceil|_{F'}\big) = 0.\end{aligned}$$

This shows part (d) of Definition 2.1, and the rest is easy.

In order to apply the previous result, we recall that for a log canonical pair (X, Δ), the ring

$$R(X, K_X + \Delta) = \bigoplus_{n\in\mathbb{N}} H^0\big(X, \lfloor n(K_X + \Delta)\rfloor\big)$$

is the *canonical ring* of (X, Δ). We also recall that for a graded ring $R = \bigoplus_{n\in\mathbb{N}} R_n$, the *d-th Veronese subring* of R is defined as $R^{(d)} := \bigoplus_{n\in\mathbb{N}} R_{dn}$.

An immediate consequence of Theorem 4.2 is the following result [19, Theorem 5.2], which has widespread use in the Minimal Model Program. It often allows to pass from a pair (X, Δ) with $\kappa(X, K_X + \Delta) \geq 0$ to a pair (X', Δ') on which $K_{X'} + \Delta'$ is big.

Theorem 4.3 *Let (X, Δ) be a projective klt pair with $\kappa(K_X + \Delta) = \ell \geq 0$. Then there exist an ℓ-dimensional klt pair (X', Δ') with $\kappa(X', K_{X'} + \Delta') = \ell$ and positive integers d and d' such that*

$$R(X, K_X + \Delta)^{(d)} \simeq R(X', K_{X'} + \Delta')^{(d')}.$$

The proof follows immediately from Theorem 4.2(vi), by combining it with the proof of Theorem 2.16; see also the proof of Theorem 4.4 below.

Assume now that (X, Δ) has simple normal crossings and let Δ^+ and Δ^- be effective divisors without common components such that $\Delta = \Delta^+ - \Delta^-$. Then if $\kappa\big(F, (K_X + \Delta^+)|_F\big) = 0$ for a general fibre F of f, and if there exists a good model of $(F, \Delta^+|_F)$, then it was shown in [15, Theorem 3.13] that the moduli b-divisor is b-nef and good in the sense of [3, Definition 3.2]; this is an application of MMP techniques from [18] and [3, Theorem 3.3].

We finish the paper with the following result, which can sometimes be used in order to avoid running a Minimal Model Program; for instance, compare the proofs of [21, Lemma 4.4] and [28, Theorem 5.3]. Note that $\nu(X, L)$ denotes the *numerical dimension* of a divisor L on a projective variety X, see for instance [28, §2.2] for basic properties and related references.

Theorem 4.4 *Let (X, Δ) be a projective klt pair and let $f\colon (X, \Delta) \to Y$ be a parabolic fibration such that $\nu\big(F, (K_X + \Delta)|_F\big) = 0$ for a general fibre F of f. Then there exists a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\ \pi'\ } & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ \pi\ ]{} & Y, \end{array}$$

where X' and Y' are smooth, f' has connected fibres, π and π' are birational, and such that, if we write

$$K_{X'} + \Delta' \sim_{\mathbb{Q}} \pi'^*(K_X + \Delta) + E',$$

where Δ' and E' have no common components, then:

(i) we have

$$K_{X'} + \Delta' + B^- \sim_{\mathbb{Q}} f'^*(K_{Y'} + \Delta_{Y'}) + B^+,$$

where the pair $(Y', \Delta_{Y'})$ is klt, and the divisors B^+ and B^- are effective and have no common components,

(ii) B^- is π'-exceptional and f'-exceptional,

(iii) we have $f'_*\mathcal{O}_{X'}(\lfloor \ell B^+ \rfloor) \simeq \mathcal{O}_{Y'}$ for all positive integers ℓ.

Moreover, if $K_X + \Delta$ is pseudoeffective, then $K_{Y'} + \Delta_{Y'}$ is pseudoeffective.

Proof By [6, Corollaire 3.4] and [30, Corollary V.4.9] we have

$$\kappa\big(F, (K_X + \Delta)|_F\big) = \nu\big(F, (K_X + \Delta)|_F\big) = 0. \quad (2)$$

By Theorem 4.2 there exists a diagram as in the theorem such that (ii) and (iii) hold, as well as

$$K_{X'} + \Delta' + B^- \sim_{\mathbb{Q}} f'^*(K_{Y'} + B_{Y'} + M_{Y'}) + B^+,$$

where $(Y', B_{Y'})$ is klt and $M_{Y'}$ is the moduli part of the associated klt-trivial fibration $f' \colon (X', \Delta' + B^- - B^+) \to Y'$. Then analogously as in the proof of Theorem 2.16 one shows that there exists an effective $\mathbb{Q}$-divisor $\Delta_{Y'} \sim_{\mathbb{Q}} B_{Y'} + M_{Y'}$ such that the pair $(Y', \Delta_{Y'})$ is klt, which gives (i).

Finally, if F' is a general fibre of f', we have $\nu\big(F', (K_{X'} + \Delta')|_{F'}\big) = 0$ by (2) and by [28, Lemma 2.3]. Therefore, there exists a good model of $(F', \Delta'|_{F'})$ by [6, Corollaire 3.4] and [30, Corollary V.4.9], hence $(K_{X'} + \Delta')|_{F'}$ is geometrically abundant in the sense of [30, Definition V.2.23]. Then by [30, Lemma V.2.27] for an ample divisor A on Y' and for any positive rational number ε, the divisor $K_{X'} + \Delta' + \varepsilon f'^*A$ is geometrically abundant. In particular, $\kappa(X', K_{X'} + \Delta' + \varepsilon f'^*A) \geq 0$, and hence by (i) and by (iii) we have

$$\begin{aligned}\kappa\big(Y', K_{Y'} + \Delta_{Y'} + \varepsilon A) &= \kappa\big(X', f'^*(K_{Y'} + \Delta_{Y'} + \varepsilon A) + B^+\big)\\ &= \kappa(X', K_{X'} + \Delta' + B^- + \varepsilon f'^*A) \geq 0.\end{aligned}$$

Since this holds for any positive rational number ε, we conclude that $K_{Y'} + \Delta_{Y'}$ is pseudoeffective, as desired. □

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Some Examples of Calabi–Yau Pairs with Maximal Intersection and No Toric Model

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Abstract It is known that a maximal intersection log canonical Calabi–Yau surface pair is crepant birational to a toric pair. This does not hold in higher dimension: this article presents some examples of maximal intersection Calabi–Yau pairs that admit no toric model.

Keywords Calabi–Yau · Fano varieties · Minimal model program

1 Introduction and Motivation

A Calabi–Yau (CY) pair (X, D_X) consists of a normal projective variety X and a reduced sum of integral Weil divisors D_X such that $K_X + D_X \sim_{\mathbb{Z}} 0$.

The class of CY pairs arises naturally in a number of problems and comprises examples with very different birational geometry. Indeed, on the one hand, a Gorenstein Calabi–Yau variety X can be identified with the CY pair $(X, 0)$. On the other hand, if X is a Fano variety, and if D_X is an effective reduced anticanonical divisor, then (X, D_X) is also a CY pair.

Definition 1

(a) A pair (X, D_X) is (t, dlt) (resp. (t, lc)) if X is $\mathbb{Q}$-factorial, terminal and (X, D_X) divisorially log terminal (resp. log canonical).

(b) A birational map $(X, D_X) \overset{\varphi}{\dashrightarrow} (Y, D_Y)$ is volume preserving or crepant birational if $a_E(K_X + D_X) = a_E(K_Y + D_Y)$ for every geometric valuation E with centre on X and on Y.

The dual complex of a dlt pair $(Z, D_Z = \sum D_i)$ is the regular cell complex obtained by attaching an $(|I|-1)$-dimensional cell for every irreducible component of a non-empty intersection $\bigcap_{i\in I} D_i$.

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The dual complex encodes the combinatorics of the lc centres of a dlt pair and [5] shows that its PL homeomorphism class is a volume preserving birational invariant.

By [3, Theorem 1.9], a (t, lc) CY pair (X, D_X) has a volume preserving (t, dlt) modification $(\widetilde{X}, D_{\tilde{X}}) \to (X, D_X)$, and the birational map between two such modifications is volume preserving.

Abusing notation, I call dual complex the following volume preserving birational invariant of a (t, lc) CY pair (X, D_X).

Definition 2 $\mathcal{D}(X, D_X)$ is the PL homeomorphism class of the dual complex of a volume preserving (t, dlt) modification of (X, D_X).

As the underlying varieties of CY pairs range from CY to Fano varieties, they can have very different birational properties. However, X being Fano is not a volume preserving birational invariant of the pair (X, D_X). Following [14], I consider the following volume preserving birational invariant notion:

Definition 3 A (t, lc) CY pair (X, D_X) has maximal intersection if $\dim \mathcal{D}(X, D_X) = \dim X - 1$.

In other words, (X, D_X) has maximal intersection if there is a volume preserving (t, dlt) modification of (X, D_X) with a 0-dimensional log canonical centre. Maximal intersection CY pairs have some Fano-type properties; Kollár and Xu show the following:

Theorem 1 *Let (X, D_X) be a dlt maximal intersection CY pair, then:*

1. *[14, Proposition 19] X is rationally connected,*
2. *[14, Theorem 21] there is a volume preserving map $(X, D_X) \overset{\varphi}{\dashrightarrow} (Z, D_Z)$ such that D_Z fully supports a big and semiample divisor.*

Remark 1 The expression "Fano-type" should be understood with a pinch of salt. Having maximal intersection is a degenerate condition: a general (t, lc) CY pair (X, D_X) with X Fano and D_X a reduced anticanonical section need not have maximal intersection.

Definition 4 A toric pair (X, D_X) is a (t, lc) CY pair formed by a toric variety and the reduced sum of toric invariant divisors.

A toric model is a volume preserving birational map to a toric pair.

Remark 2

(a) If X is a normal toric variety and D_X the reduced sum of toric invariant divisors, (X, D_X) is log canonical [4, Corollary 11.4.25], so that (X, D_X) is a toric pair precisely when X is terminal and $\mathbb{Q}$-factorial.
(b) If a CY pair (X, D_X) has a toric model, X is rational.

Example 1 A CY pair with a toric model has maximal intersection.

Remark 3 In dimension 2, the converse holds: maximal intersection CY surface pairs are precisely those with a toric model [7, Proposition 1.3].

The characterisation of CY pairs with a toric model is an open and difficult problem. A characterisation of toric pairs was conjectured by Shokurov and is proved in [1], but it is not clear how to refine it to get information on the existence of a toric model. A motivation to better understand the birational geometry of CY pairs and their relation to toric pairs comes from mirror symmetry.

The mirror conjecture extends from a duality between Calabi–Yau varieties to a correspondence between Fano varieties and Landau-Ginzburg models, i.e. non-compact Kähler manifolds endowed with a superpotential. Most known constructions of mirror partners rely on toric features such as the existence of a toric model or of a toric degeneration. In an exciting development, Gross, Hacking and Keel conjecture the following construction for mirrors of maximal intersection CY pairs.

Conjecture 1 ([7, Section 0.4] [8, Conjecture 1.9]) Let (Y, D_Y) be a simple normal crossings maximal intersection CY pair and denote by U the complement $Y \setminus D_Y$. Assume that D_Y supports an ample divisor, let R be the ring $k[\mathrm{Pic}(Y)^\times]$, Ω the canonical volume form on U and

$$U^{\mathrm{trop}}(\mathbb{Z}) = \Big\{\text{divisorial valuations } v\colon k(U) \setminus \{0\} \to \mathbb{Z} \text{ with } v(\Omega) < 0\Big\} \cup \{0\}.$$

Then, the free R-module V with basis $U^{\mathrm{trop}}(\mathbb{Z})$ has a natural finitely generated R-algebra structure whose structure constants are non-negative integers determined by counts of rational curves on U.

Denote by K the torus $\mathrm{Ker}\{\mathrm{PicY} \to \mathrm{Pic}(U)\}$. The fibration

$$p\colon \mathrm{Spec}(V) \to \mathrm{Spec}(R) = T_{\mathrm{Pic}(Y)}$$

is a T_K-equivariant flat family of affine maximal intersection log CY varieties. The quotient

$$\mathrm{Spec}(V)/T_K \to T_{\mathrm{Pic}(U)}$$

only depends on U and is the mirror family of U.

Versions of Conjecture 1 are proved for cluster varieties in [9], but relatively few examples are known.

This note presents examples of maximal intersection CY pairs that do not admit a toric model and for which one can hope to construct the mirror partner proposed in Conjecture 1. Table 1 summarises the properties of the Examples of (t, lc) CY pairs constructed in Sects. 3 and 4.

Table 1 Examples of (t, lc) CY pairs that do not admit a toric model

Boundary	Reference	X	$\mathrm{Sing}(D_X)$	$\dim \mathcal{D}(X, D_X)$
Normal	Example 2	Smooth $X_4 \subset \mathbb{P}^4$	Cusp $T_{4,4,4}$	2
	Example 3	Nodal $X_4 \subset \mathbb{P}^4$	Cusp $T_{3,3,4}$ and	2
		$\#\mathrm{Sing}(X) = 3$	3 odps	
	Example 4	Smooth $X_4 \subset \mathbb{P}^4$	Cusp $T_{3,4,4}$	2
	Example 8	Smooth $X_4 \subset \mathbb{P}^4$	Two cusps $T_{2,3,6}$	1
Non-normal	Example 5	Smooth $X_3 \subset \mathbb{P}^4$	Nodal plane cubic curve	2
			Double pinch point at node	
	Example 6	Nodal $X_4 \subset \mathbb{P}^4$	$L \cup L'$	2
		$\#\mathrm{Sing}(X) = 6$	Double pinch point at $L \cap L'$	
	Example 7	Nodal $X_4 \subset \mathbb{P}^4$	$L \cup L'$	1
		$\#\mathrm{Sing}(X) = 6$	$L \cap L' = \emptyset$	

2 Auxiliary Results on Threefold CY Pairs

The examples in Sect. 3 are threefold maximal intersection CY pairs whose underlying varieties are birationally rigid. In particular, such pairs admit no toric model; this shows that the results in [7] on maximal intersection surface CY pairs do not extend to higher dimensions. In this section, I first recall some results on birational rigidity of Fano threefolds. Then, I introduce the (t, dlt) modifications suited to the construction outlined in Conjecture 1 and discuss the singularities of the boundary D_X.

2.1 Birational Rigidity

Let X be a terminal $\mathbb{Q}$-factorial Fano threefold. When X has Picard rank 1, X is a Mori fibre space, i.e. an end product of the classical MMP.

Definition 5 A birational map $Y/S \overset{\varphi}{\dashrightarrow} Y'/S'$ between Mori fibre spaces Y/S and Y'/S' is square if it fits into a commutative square

$$\begin{array}{ccc} Y & \overset{\varphi}{\dashrightarrow} & Y' \\ \downarrow & & \downarrow \\ S & \overset{g}{\dashrightarrow} & S' \end{array}$$

where g is birational and the restriction $Y_\eta \overset{\varphi_\eta}{\dashrightarrow} Y'_\eta$ is biregular, where η is the function field of the base $k(S)$.

A Mori fibre space Y/S is (birationally) rigid if for every birational map $Y/S \overset{\varphi}{\dashrightarrow} Y'/S'$ to another Mori fibre space, there is a birational self map $Y/S \overset{\alpha}{\dashrightarrow} X/S$ such that $\varphi \circ \alpha$ is square.

In particular, if X is a rigid Mori fibre space, then X is non-rational and no (t, lc) CY pair (X, D_X) admits a toric model.

Non-singular quartic hypersurfaces $X_4 \subset \mathbb{P}^4$ are probably the most famous examples of birationally rigid threefolds [11]. Some mildly singular quartic hypersurfaces are also known to be birationally rigid, in particular, we have:

Proposition 1 ([2, 18]) *Let $X_4 \subset \mathbb{P}^4$ be a quartic hypersurface with no worse than ordinary double points. If $|\mathrm{Sing}(X)| \leq 8$, then X is $\mathbb{Q}$-factorial (in particular, X is a Mori fibre space) and is birationally rigid.*

2.2 Singularities of the Boundary

I now state some results on the singularities of the boundary of a threefold (t, lc) CY pair. Let (X, D_X) be a threefold (t, lc) CY pair and $(\widetilde{X}, D_{\tilde{X}})$ a (t, dlt) modification. A stratum of $(\widetilde{X}, D_{\tilde{X}})$ is an irreducible component of a non-empty intersection of components of $D_{\tilde{X}}$. Given a stratum W, there is a divisor $\mathrm{Diff}_W D_{\tilde{X}}$ on W such that $(W, \mathrm{Diff}_W D_{\tilde{X}})$ is a lc CY pair and

$$K_W + \mathrm{Diff}_W D_{\tilde{X}} \sim_{\mathbb{Q}} \left(K_{\tilde{X}} + D_{\tilde{X}}\right)_{|W}.$$

When $K_{\tilde{X}} + D_{\tilde{X}}$ is Cartier and $D_{\tilde{X}}$ reduced, $\mathrm{Diff}_W D_{\tilde{X}}$ is the sum of the restrictions of the components of $D_{\tilde{X}}$ that do not contain W.

In particular, for any irreducible component S of $D_{\tilde{X}}$, the link of $[S]$ in $\mathcal{D}(X, D_X)$ is the dual complex $\mathcal{D}(S, \mathrm{Diff}_S D_{\tilde{X}})$. Therefore, if (X, D_X) has maximal intersection, so does $(S, \mathrm{Diff}_S D_{\tilde{X}})$. By the results of [7], $(S, \mathrm{Diff}_S D_X)$ then has a toric model.

As X has terminal singularities, X is normal and Cohen–Macaulay. Any Cartier component S of the boundary D_X is Cohen–Macaulay and satisfies Serre's condition S_2. By [15, Proposition 16.9], $(S, \mathrm{Diff}_S D_X)$ is semi log canonical (slc). As a special case, if X is Gorenstein and D_X is irreducible, this shows that D_X has slc singularities.

I am particularly interested in producing examples of (t, lc) CY pairs for which the mirror partners proposed in Conjecture 1 can be constructed. The discussion in [10] motivates the following definition:

Definition 6 A (t, dlt) modification $(\widetilde{X}, D_{\tilde{X}}) \to (X, D_X)$ is good if $(\widetilde{X}, D_{\tilde{X}})$ is a log smooth variety (in the sense of log geometry), i.e. if $(\widetilde{X}, D_{\tilde{X}})$ is a toroidal pair.

Remark 4 The properties of (t, dlt) pairs and those of toroidal pairs are somehow different. On the one hand, $(\widetilde{X}, D_{\tilde{X}})$ being dlt implies that the pair has simple normal crossings at the generic point of each stratum, which is not in general satisfied by toroidal pairs. On the other hand, as $\widetilde{X}$ has terminal singularities and is analytically isomorphic to a toric variety, by [4, Theorem 11.4.8], $\widetilde{X}$ only has finite quotient singularities, i.e. $\widetilde{X}$ is an orbifold (in particular, $\widetilde{X}$ has no Gorenstein singularities).

Normal Singularities Let $p \in \mathrm{Sing}(D_i)$ be an isolated singularity lying on a single component of the boundary. Since $K_X + D_X \sim_{\mathbb{Z}} 0$, the indices of K_X and of D_X at p are the same. After taking a canonical cover, assume that X is Gorenstein at p, and without loss of generality (for the purpose of this discussion) assume that $D_X = D_i$ is irreducible and that $\mathrm{Sing}(D_X) = \{p\}$. Then, since D_X is Cartier, by [15, Proposition 16.4], we have $K_{D_X} \sim_{\mathbb{Q}} (K_X + D_X)_{|D_X}$, so that D_X is Gorenstein.

If $f\colon (\widetilde{X}, D_{\tilde{X}}) \to (X, D_X)$ is a good modification, $\widetilde{X}$ is smooth (because it is Gorenstein) and up to composing with the minimal resolution of $D_{\tilde{X}}$, we may assume that $f_{|D_{\tilde{X}}}\colon D_{\tilde{X}} \to D_X$ is a resolution. Write:

$$K_{D_{\tilde{X}}} = (K_{\tilde{X}} + D_{\tilde{X}})_{|D_{\tilde{X}}} = (f_{|D_{\tilde{X}}})^*(K_{D_X}) - E_{|D_{\tilde{X}}}$$

where E is defined by $K_{\tilde{X}} + f_*^{-1}D_X + E = f^*(K_X + D_X)$. We see that p is canonical if $E \cap D_{\tilde{X}} = \emptyset$, and elliptic otherwise. Indeed, let

$$D_{\tilde{X}} \overset{q}{\to} \overline{D}_X \overset{\mu}{\to} D_X$$

be the factorisation of $f_{|D_{\tilde{X}}}$ through the minimal resolution of $(p \in D_X)$. Then, q is either an isomorphism or an isomorphism at the generic point of each component of $E_{|D_{\tilde{X}}}$ by [3, Lemma 2.5] because f is volume preserving. We have: $K_{\overline{D}_X} = \mu^* K_{D_X} - Z$, where the effective cycle $Z = q_*(E_{D_{\tilde{X}}})$ is either empty (and p is canonical) or a reduced sum of μ-exceptional curves (and p is elliptic). In the second case, $Z \sim -K_{D_{\tilde{X}}}$ is the fundamental cycle of $(p \in D_X)$. If Z is irreducible, it is reduced and has genus 1; if not, every irreducible component of Z is a smooth rational curve of self-intersection -2.

When p is elliptic, Z is reduced and p is a Kodaira singularity [12, Theorem 2.9], i.e. a resolution is obtained by blowing up points of the singular fibre in a degeneration of elliptic curves; further, in Arnold's terminology, the singularity p is uni or bimodal.

Further, $p \in D_i$ is a hypersurface singularity (resp. a codimension 2 complete intersection, resp. not a complete intersection) when $-3 \leq Z^2 \leq -1$ (resp. $Z^2 = -4$, resp. $Z^2 \leq -5$) [16]. When $-4 \leq Z^2 \leq -1$, normal forms are known for $p \in D_i$: Table 2 lists normal forms of slc hypersurface singularities, while normal forms of codimension 2 complete intersections elliptic singularities are given in [20].

Table 2 Dimension 2 slc hypersurface singularities [17]

Type	Name	Symbol	Equation $f \in \mathbb{C}[x, y, z]$		$\text{mult}_0 f$
Terminal	Smooth	A_0	x		1
Canonical	du Val	A_n	$x^2 + y^2 + z^{n+1}$	$n \geq 1$	2
		D_n	$x^2 + z(y^2 + z^{n-2})$	$n \geq 4$	2
		E_6	$x^2 + y^3 + z^4$		2
		E_7	$x^2 + y^3 + yz^3$		2
		E_8	$x^2 + y^3 + z^5$		2
lc	Simple elliptic	$X_{1,0}$	$x^2 + y^4 + z^4 + \lambda xyz$	$\lambda^4 \neq 64$	2
		$J_{2,0}$	$x^2 + y^3 + z^6 + \lambda xyz$	$\lambda^6 \neq 432$	2
		$T_{3,3,3}$	$x^3 + y^3 + z^3 + \lambda xyz$	$\lambda^3 \neq -27$	3
	Cusp	$T_{p,q,r}$	$x^p + y^q + z^r + xyz$	$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$	2 or 3
slc	Normal crossing	A_∞	$x^2 + y^2$		2
	Pinch point	D_∞	$x^2 + y^2 z$		2
	Degenerate cusp	$T_{2,\infty,\infty}$	$x^2 + y^2 + z^2$		2
		$T_{2,q,\infty}$	$x^2 + y^2(z^2 + y^{q-2})$	$q \geq 3$	2
		$T_{\infty,\infty,\infty}$	xyz		3
		$T_{p,\infty,\infty}$	$xyz + x^p$	$p \geq 3$	3
		$T_{p,q,\infty}$	$xyz + x^p + y^q$	$q \geq p \geq 3$	3

3 Examples of Rigid Maximal Intersection Threefold CY Pairs

All the examples below are (t, lc) CY pairs (X, D_X) which admit no toric model. Except for Example 5, all underlying varieties X are birationally rigid quartic hypersurfaces by Proposition 1; the underlying variety in Example 5 is a smooth cubic threefold, and therefore non-rational.

3.1 Examples with Normal Boundary

Example 2 Consider the CY pair (X, D_X) where X is the nonsingular quartic hypersurface

$$X = \{x_1^4 + x_2^4 + x_3^4 + x_0x_1x_2x_3 + x_4(x_0^3 + x_4^3) = 0\}$$

and D_X is its hyperplane section $X \cap \{x_4 = 0\}$.

(continued)

Example 2 (continued)
The quartic surface D_X has a unique singular point $p = (1{:}0{:}0{:}0{:}0)$, and using the notation of Table 1, p is locally analytically equivalent to a $T_{4,4,4}$ cusp

$$0 \in \{x^4 + y^4 + z^4 + xyz = 0\}.$$

D_X is easily seen to be rational: the projection from the triple point p is

$$D_X \dashrightarrow \mathbb{P}^2_{x_1,x_2,x_3};$$

this map is the blowup of the 12 points $\{x_1^4 + x_2^4 + x_3^4 = x_1x_2x_3 = 0\}$, of which 4 lie on each coordinate line $L_i = \{x_i = 0\}$, for $i = 1, 2, 3$.

I treat this example in detail and construct explicitly a good (t, dlt) modification of the pair (X, D_X).

Let $f\colon X_p \to X$ be the blowup of p, then X_p is non-singular, the exceptional divisor E satisfies $(E, \mathcal{O}_E(E)) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$, and if D denotes the proper transform of D_X, we have:

$$K_{X_p} + D + E = f^*(K_X + D).$$

Explicitly, the blowup $\mathcal{F} \to \mathbb{P}^4$ of $\mathbb{P}^4$ at p is the rank 2 toric variety $\mathrm{TV}(I, A)$, where $I = (u, x_0) \cap (x_1, \dots, x_4)$ is the irrelevant ideal of $\mathbb{C}[u, x_0, \dots, x_4]$ and A is the action of $\mathbb{C}^* \times \mathbb{C}^*$ with weights:

$$\begin{pmatrix} u & x_0 & s_1 & s_2 & s_3 & s_4 \\ 1 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \qquad (1)$$

The equation of X_p is

$$X_p = \{u^2\big(u(s_1^4 + s_2^4 + s_3^4) + x_0s_1s_2s_3\big) + s_4(x_0^3 + u^3s_4^3) = 0\},$$

while $E = \{u = 0\}$ and $D = \{u(s_1^4 + s_2^4 + s_3^4) + x_0s_1s_2s_3 = 0\}$. By construction, E is the projective plane with coordinates s_1, s_2, s_3. Note that $(X_p, D + E)$ is not dlt because $D \cap E = \{x_0s_1s_2s_3 = 0\}$ consists of 3 concurrent lines C_1, C_2, C_3.

Consider $g_1\colon X_1 \to X_p$ the blowup of the nonsingular curve

$$C_1 = \{u = s_1 = s_4 = 0\} \subset X_p.$$

(continued)

Example 2 (continued)
The exceptional divisor of g_1 is a surface $E_1 \simeq \mathbb{P}(\mathcal{N}_{C_1/X_p})$, and since $C_1 \simeq \mathbb{P}^1$, the restriction sequence of normal bundles gives

$$\mathcal{N}_{C_1/X_p} \simeq \mathcal{N}_{C_1/E} \oplus (\mathcal{N}_{E/X_p})_{|C_1} \simeq \mathcal{O}_{C_1}(1) \oplus \mathcal{O}_E(-1)_{|C_1},$$

so that $E_1 = \mathbb{F}_2$. Further,

$$K_{X_1} + D + E + E_1 = g_1^*(K_{X_p} + D + E)$$

where, abusing notation, I denote by D and E the proper transforms of the divisors D and E. The "restricted pair" on E_1 is a surface CY pair $(E_1, (D + E)_{|E_1})$ by adjunction. By construction, $E \cap E_1$ is the negative section σ. The curve $\Gamma = D \cap E_1$ is irreducible, and since $(D + E)_{|E_1}$ is anticanonical, we have

$$\Gamma \sim \sigma + 4f \text{ where } f \text{ is a fibre of } \mathbb{F}_2 \to \mathbb{P}^1, \text{ and } \Gamma^2 = 6, \Gamma \cdot E_{|E_1} = 2.$$

The divisors D, E, E_1 meet in two points, the dual complex $\mathcal{D}(X_1, D + E + E_1)$ is not simplicial because it is a sphere S^2 whose triangulation is given by 3 vertices on an equator. While not strictly necessary, we consider a further blowup to obtain a (t, dlt) pair with simplicial dual complex.

Denote by C_2 the proper transform of the curve

$$\{u = s_2 = s_4 = 0\}.$$

Then $C_2 \subset E \cap D$ is rational, and as above

$$\mathcal{N}_{C_2/X_1} \simeq \mathcal{N}_{C_2/E} \oplus (\mathcal{N}_{E/X_2})_{|C_2} = \mathcal{O}_{C_2}(1) \oplus \mathcal{O}_{C_2}(-2).$$

Let $g_2 \colon X_2 \to X_1$ be the blowup of C_2, then the exceptional divisor of g_2 is a Hirzebruch surface

$$E_2 \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{N}_{C_2/X_1}) \simeq \mathbb{F}_3.$$

Still denoting by D, E, E_1 the strict transforms of D, E, E_1, we have:

$$K_{X_2} + D + E + E_1 + E_2 = g_2^*(K_{X_1} + D + E + E_1).$$

The pair $(X_2, D + E + E_1 + E_2)$ is dlt; the composition

$$g_2 \circ g_1 \circ f \colon (\widetilde{X}, D_{\tilde{X}}) = (X_2, D + E + E_1 + E_2) \to (X, D_X)$$

(continued)

Example 2 (continued)
is a good (t, dlt) modification.

The "restrictions" of $(\widetilde{X}, D_{\tilde{X}})$ to the component of the boundary are the following surface anticanonical pairs:

- On D: $(E+E_1+E_2)_{|D}$ is a cycle of (-3)-curves, the morphism $D \to D_X$ is the familiar resolution of the $T_{4,4,4}$ cusp singularity;
- On E: $(D + E_1 + E_2)_{|E}$ is the triangle of coordinate lines with self-intersections $(1, 1, 1)$;
- On E_1: $(D + E + E_2)_{|E_1}$ is an anticanonical cycle with self-intersections $(5, -3, -1)$;
- On E_2: $(D + E + E_1)_{|E_2}$ is an anticanonical cycle with self-intersections $(5, -3, 0)$ (as above, $E_{|E_2} \sim \sigma$ is a negative section, $E_{1|E_2} \sim f$ a fibre of $\mathbb{F}_3 \to \mathbb{P}^1$, and $D_{|E_2} \sim 4f + \sigma$).

It follows that the dual complex $\mathcal{D}(X, D_X)$ is PL homeomorphic to a tetrahedron and (X, D_X) has maximal intersection. Note that $(0 \in D_X)$ is a maximal intersection lc point, and since D_X is a rational surface, it has a toric model.

Example 3 Let X be the hypersurface

$$X = \{x_3(x_0^3 + x_1^3) + x_2^4 + x_0x_1x_2x_3 + x_4(x_3^3 + x_4^3) = 0\},$$

and D_X its hyperplane section $X \cap \{x_4 = 0\}$.

The quartic X has 3 ordinary double points at the intersection points

$$L \cap \{x_0^3 + x_1^3 = 0\},$$

where L is the line $\{x_2 = x_3 = x_4 = 0\}$. The singular locus of D_X is $\mathrm{Sing}(X) \cup \{p\}$, where $p = (0{:}0{:}0{:}1{:}0)$ is a $T_{3,3,4}$ cusp, i.e. locally analytically equivalent to

$$0 \in \{x^3 + y^3 + z^4 + xyz = 0\}.$$

The quartic surface D_X is rational; the projection of D_X from p is

$$D_X \dashrightarrow \mathbb{P}^2_{x_0,x_1,x_2};$$

this map is defined outside of the 12 points (counted with multiplicity) defined by $\{x_2^4 = x_0^3 + x_1^3 + x_0x_1x_2 = 0\}$.

(continued)

Example 3 (continued)

If $\tilde{X} \xrightarrow{f} X$ is the composition of the blowups at the ordinary double points and at p, $\widetilde{X}$ is smooth and $D_{\tilde{X}}$ is non-singular, so that f is a good (t, dlt) modification.

The minimal resolution of $p \in D_X$ is a rational curve with self intersection $C^2 = -3$. Explicitly, taking the blowup of X at p, the proper transform is a rational surface D. The exceptional curve is the preimage of a nodal cubic in $\mathbb{P}^2$ blown up at 12 points counted with multiplicities. Note that $(\widetilde{X}, D + E)$ is not dlt, but in order to obtain a (t, dlt) modification, we just need to blowup the node of $D \cap E$ which is a nonsingular point of $\widetilde{X}$, D and E. The (t, dlt) modification of (X, D_X) in a neighbourhood of p is good and the associated dual complex is 2-dimensional.

The pair (X, D_X) has maximal intersection; but as in the previous examples, X is rigid, so that (X, D_X) can have no toric model.

Example 4 Let X be the nonsingular quartic hypersurface

$$X = \{x_0^3x_3 + x_1^4 + x_2^4 + x_0x_1x_2x_3 + x_4(x_3^3 + x_4^3) = 0\} \subset \mathbb{P}^4$$

and D_X its hyperplane section $X \cap \{x_4 = 0\}$.

The surface D_X has a unique singular point $p = (0{:}0{:}0{:}1{:}0)$, which is a cusp $T_{3,4,4}$, i.e. is locally analytically equivalent to

$$0 \in \{x^3 + y^4 + z^4 + xyz = 0\}.$$

As in Example 2, X is non-singular, and finding a good (t, dlt) modification of (X, D_X) will amount to taking a minimal resolution of the singular point of D_X. Let $X_p \to X$ be the blowup of X at p; X_p is non-singular and if D denotes the proper transform of D_X, and E the exceptional divisor, $D \cap E$ consists of 2 rational curves of self intersection -3 and -4. These curves are the proper transforms of $\{x_0 = 0\}$ and of $\{x_0^2 + x_1x_2 = 0\}$ under the blow up of $\mathbb{P}^2_{x_0,x_1,x_2}$ at the points

$$\{x_1^4 + x_2^4 = x_0(x_1x_2 + x_0^2) = 0\}.$$

The dual complex consists of 3 vertices that are joined by edges and span 2 distinct faces: $\mathcal{D}(X, D_X)$ is PL homeomorphic to a sphere S^2 whose triangulation is given by 3 vertices on an equator. The CY pair (X, D_X) has maximal intersection but no toric model.

3.2 Examples with Non-normal Boundary

Example 5 This example is due to R. Svaldi. Consider the cubic threefold

$$X = \{x_0x_1x_2 + x_1^3 + x_2^3 + x_3q + x_4q' = 0\} \subset \mathbb{P}^4$$

where q, q' are homogeneous polynomials of degree 2 in $x_0, \cdots, x_4$. If the quadrics q and q' are general and if

$$(q(1, 0, 0, 0, 0), q'(1, 0, 0, 0, 0)) \neq (0, 0),$$

then X and $S = \{x_3 = 0\} \cap X$ and $T = \{x_4 = 0\} \cap X$ are nonsingular.

Let D_X be the anticanonical divisor $S + T$. The curve $C = S \cap T = \Pi \cap X$ for $\Pi = \{x_3 = x_4 = 0\}$ is a nodal cubic. It follows that both (S, C) and (T, C) are log canonical, and therefore so is (X, D_X).

Since S and T are smooth, $\mathrm{Sing}(D_X) = S \cap T = C$, and if p is the node of C, we have:

$$\begin{aligned}(p \in D_X) &\sim 0 \in \left\{(xy + x^3 + y^3 + z)(xy + x^3 + y^3 + t) = 0\right\}\\ &\sim 0 \in \left\{(xy + z)(xy + t) = 0\right\} \sim 0 \in \left\{(xy + z)(xy - z) = 0\right\},\end{aligned}$$

where $\sim$ denotes analytic equivalence. Thus, $p \in D_X$ is a double pinch point, i.e. p is locally analytically equivalent to $0 \in \{x^2y^2 - z^2 = 0\}$.

We now construct a good (t, dlt) modification of (X, D_X). Let $f : X_C \to X$ be the blowup of X along C; $\mathrm{Sing}(X_C)$ is an ordinary double point.

Indeed, let $\Pi = \{x_3 = x_4 = 0\}$, then f is the restriction to X of the blowup $\mathcal{F} \to \mathbb{P}^4$ of $\mathbb{P}^4$ along Π. Note that $\mathcal{F}$ is the rank 2 toric variety $\mathrm{TV}(I, A)$, where $I = (u, x_0, x_1, x_2) \cap (x_3, x_4)$ is the irrelevant ideal of $\mathbb{C}[u, x_0, \ldots, x_4]$ and A is the action of $\mathbb{C}^* \times \mathbb{C}^*$ with weights:

$$\begin{pmatrix} u & x_0 & x_1 & x_2 & x_3 & x_4 \\ 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The equation of X_C is

$$\{x_0x_1x_2 + x_1^3 + x_2^3 + u(x_3q + x_4q') = 0\},$$

(continued)

Example 5 (continued)
so that X_C has a unique singular point at

$$x_0 - 1 = u = x_1 = x_2 = x_3 q(1,0,0,0,0) + x_4 q'(1,0,0,0,0) = 0,$$

and this is a threefold ordinary double point. In addition, denoting by $E_f = \{u = 0\} \cap X_C$ the exceptional divisor, we have

$$K_{X_C} + \tilde{S} + \tilde{T} + E_f = K_X + S + T,$$

so that the pair $(X_C, \tilde{S} + \tilde{T} + E_f)$ is a (t, lc) CY pair.

The pair $(X_C, \tilde{S} + \tilde{T} + E_f)$ is not dlt as the boundary has multiplicity 3 along the fibre F over the node of $S \cap T$. The blowup of F is not $\mathbb{Q}$-factorial, therefore in order to obtain a good (t, dlt) modification, we consider the divisorial contraction $g\colon \tilde{X} \to X_C$ centred along F. This is obtained by (a) blowing up the node, (b) then blowing up the proper transform of F, (c) flopping a pair of lines with normal bundle $(-1,-1)$ and (d) contracting the proper transform of the $\mathbb{P}^1 \times \mathbb{P}^1$ above the node to a point $\frac{1}{2}(1,1,1)$. The exceptional divisor of g is denoted by E_g.

The pair $(\tilde{X}, \tilde{S} + \tilde{T} + \tilde{E}_f + E_g)$ is the desired (t, dlt) modification of (X, D_X), and it has maximal intersection. The dual complex is PL homeomorphic to a tetrahedron.

Example 6 Let X be the quartic hypersurface

$$X = \{x_1^2 x_2^2 + x_1 x_2 x_3 l + x_3^2 q + x_4 f_3 = 0\} \subset \mathbb{P}^4,$$

where l (resp. q) is a general linear (resp. quadratic) form in $x_0, \cdots, x_3$, and f_3 a general homogeneous form of degree 3 in $x_0, \cdots, x_4$. Let D_X be the hyperplane section $X \cap \{x_4 = 0\}$.

As l, q and f_3 are general, X has 6 ordinary double points. Indeed, denote by $L = \{x_1 = x_3 = x_4 = 0\}$ and $L' = \{x_2 = x_3 = x_4 = 0\}$, then

$$\operatorname{Sing}(X) = \big\{L \cap \{f_3 = 0\}\big\} \cup \big\{L' \cap \{f_3 = 0\}\big\} = \{q_1, q_2, q_3\} \cup \{q_1', q_2', q_3'\}$$

(continued)

Example 6 (continued)
which consists of 3 points on each of the lines. In the neighbourhood of each point q_i (resp. q_i') for $i = 1, 2, 3$, the equation of X is of the form

$$0 \in \{xy + zt = 0\}$$

(and $D_X = \{t = 0\}$) so that all singular points of X are ordinary double points. The quartic hypersurface X is birationally rigid by Proposition 1.

The surface D_X is non-normal as it has multiplicity 2 along L and L'. The point $p = L \cap L'$ is locally analytically equivalent to

$$0 \in \{x^2y^2 + z^2 = 0\},$$

so that $p \in D_X$ is a double pinch point. We conclude that the surface D_X has slc singularities, and hence (X, D_X) is a (t, lc) CY pair.

We construct a good (t, dlt) modification as follows.

First, since $\mathrm{Sing}(X) \cap L$ (resp. $\mathrm{Sing}(X) \cap L'$) is non-empty, the blowup of X along L (resp. along L') is not $\mathbb{Q}$-factorial. In order to remain in the (t, dlt) category, we consider the divisorial extraction $f\colon X_L \to X$ centered on L (resp. L').This is obtained by (a) blowing up the 3 nodes lying on L, (b) blowing up the proper transform of L, (c) flopping 3 pairs of lines with normal bundle $(-1, -1)$ and (d) contracting the proper transforms of the three exceptional divisors $\mathbb{P}^1 \times \mathbb{P}^1$ lying above the nodes to points $\frac{1}{2}(1, 1, 1)$. The exceptional divisor of f is denoted by E. Let $p\colon \widetilde{X} \to X$ denote the morphism obtained by composing the divisorial extraction centered on L with that centered on L' (in any order), and let E, E' denote the exceptional divisors of the divisorial extractions. Then

$$K_{\widetilde{X}} + \widetilde{D} + E + E' = p^*(K_X + D)$$

is a (t, dlt) modification of (X, D_X) and it has maximal intersection. The dual complex $\mathcal{D}(X, D_X)$ is PL homeomorphic to a sphere S^2 whose triangulation is given by 3 vertices on an equator.

4 Further Results on Quartic Threefold CY Pairs: Beyond Maximal Intersection

This section concentrates on (t, lc) CY pairs (X, D_X), where X is a factorial quartic hypersurface in $\mathbb{P}^4$ and D is an irreducible hyperplane section of X. I give some more detail on the possible dual complexes of such pairs.

As explained in Sect. 2.2, D_X is slc because (X, D_X) is lc. In order to completely study the dual complexes of such (t, lc) CY pairs, one needs a good understanding of the normal forms of slc singularities that can lie on D_X. In the case of a general Fano X, this step would require additional work, but here, D_X is a quartic surface in $\mathbb{P}^3$ and the study of singularities of such surfaces has a rich history. I recall some results directly relevant to the construction of degenerate CY pairs (X, D_X). The classification of singular quartic surfaces in $\mathbb{P}^3$ can be broken in three independent cases.

(a) Quartic surfaces with no worse than rational double points: the minimal resolution is a $K3$ surface. Possible configurations of canonical singularities were studied by several authors using the moduli theory of $K3$ surfaces; there are several thousands possible configurations. The pair (X, D_X) is (t, dlt) and the dual complex of (X, D_X) is reduced to a point.
(b) Non-normal quartic surfaces were classified by Urabe [19]; there are a handful of cases recalled in Theorem 2.
(c) Non-canonical quartic surfaces with isolated singularities. These are studied by Wall [21] and Degtyarev [6] among others; their results are recalled in Theorem 3.

Theorem 2 ([19]) *A non-normal quartic surface $D \subset \mathbb{P}^3$ is one of:*

1. *the cone over an irreducible plane quartic curve with a singular point of type A_1 or A_2.*
2. *a ruled surface over a smooth elliptic curve G, $D = \varphi_{\mathcal{L}}(Z)$, where:*

   (a) M is a line bundle of degree 2 over G, N is a non-trivial line bundle of degree 0 over G, $\pi : Z = \mathbb{P}(\mathcal{O}_G \oplus N) \to G$ is the $\mathbb{P}^1$-bundle associated to $\mathcal{O}_G \oplus N$, C_1 and C_2 are the sections of π associated to $\mathcal{O}_G \oplus N \to \mathcal{O}_G$ and to $\mathcal{O}_G \oplus N \to N$, and $\mathcal{L} = \mathcal{O}_Z(C_1) \otimes \pi^*M$, $\varphi_{\mathcal{L}}$ is the map associated to the linear system $|\mathcal{L}|$. Denoting by L_i the image by $\varphi_{\mathcal{L}}$ of C_i, we have $\mathrm{Sing}(D) = L_1 \cup L_2$.

   (b) M is a line bundle of rank 2 over G, E is a rank 2 vector bundle over G which fits in a non-splitting exact sequence

$$0 \to \mathcal{O}_G \to E \to \mathcal{O}_G \to 0,$$

   $\pi : Z = \mathbb{P}(E) \to G$ is the $\mathbb{P}^1$-bundle associated to E, C is the section of π associated to $E \to \mathcal{O}_G$, $\mathcal{L} = \mathcal{O}_Z(C) \otimes \pi^*M$, $\varphi_{\mathcal{L}}$ is the map associated to the linear system $|\mathcal{L}|$. Denoting by L the image by $\varphi_{\mathcal{L}}$ of C, we have $\mathrm{Sing}(D) = L$.

3. *a rational surface $D \subset \mathbb{P}^3$ which is*

   (a) the image of a smooth $S \subset \mathbb{P}^5$ under the projection from a line disjoint from S; D has no isolated singular point and

   – *$S = v_2(\mathbb{P}^2)$, where v_2 is the Veronese embedding; D is the Steiner Roman surface;*

- *$S = \varphi(\mathbb{P}^1 \times \mathbb{P}^1)$, where φ is the embedding defined by $|l_1 + 2l_2|$ for $l_{1,2}$ the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$;*
- *$S = \varphi(\mathbb{F}_2)$, where φ is the embedding defined by $|\sigma + 3f|$ for σ the negative section and f the fibre of $\mathbb{F}_2$.*

(b) the image of a surface $\hat{D} \subset \mathbb{P}^4$ with canonical singularities under the projection from a point not lying on it; $\hat{D}$ is a degenerate dP4 surface which is the blowup of $\mathbb{P}^2$ in 5 points in almost general position.

(c) a rational surface embedded by a complete linear system on its normalisation $\hat{D}$; the non-normal locus of D is a line L and D may have isolated singularities outside L. The minimal resolution of the normalisation of D is a blowup of $\mathbb{P}^2$ in 9 points. The normalisation of D has at most two rational triple points lying on the inverse image of the non-normal locus; their images on D are also triple points.

Remark 5 D is not slc in case 1.

Corollary 1 *Let (X, D_X) be a (t, lc) quartic CY pair with non-normal boundary. Then, (X, D_X) has maximal intersection except in the cases described in 2.(a) and (b) of Theorem 2.*

Example 7 Consider the pair (X, D_X) where:

$$X = \{x_0^2 x_3^2 + x_1^2 x_2 x_3 + x_2^2 q(x_0, x_1) + x_4 f_3 = 0\},\ D_X = X \cap \{x_4 = 0\},$$

where q is a general quadratic form in (x_1, x_2) and f_3 a general cubic in $x_0, \cdots, x_4$.

Since q and f_3 are general, the quartic hypersurface X has 6 ordinary double points. Indeed, denote by $L = \{x_0 = x_1 = x_4 = 0\}$, and by $L' = \{x_2 = x_3 = x_4 = 0\}$, then $\mathrm{Sing}(X)$ consists of points of intersection of $\{f_3 = 0\}$ with $L \cup L'$; there are 3 such points $\{q_1, q_2, q_3\}$ on L and 3 points $\{q_1', q_2', q_3'\}$ on L' because f_3 is general. In the neighbourhood of each point q_i (resp. q_i') for $i = 1, 2, 3$, the equation of X is of the form

$$0 \in \{xy + zt = 0\}$$

(and $D_X = \{t = 0\}$) so that all singular points of X are ordinary double points. The nodal quartic X is terminal and $\mathbb{Q}$-factorial because it has less than 9 ordinary double points; X is birationally rigid by [2, 18].

Taking the divisorial extraction of the lines L and L' is enough to produce a dlt modification $(\widetilde{X}, D_{\widetilde{X}} + E + E')$ of (X, D_X). As the lines L, L' are skew, E and E' are disjoint and (X, D_X) does not have maximal intersection. The dual complex has two 1-strata, The quartic surface D_X is an example of case 2.(a) in Theorem 2.

Theorem 3 ([21]) *A normal quartic surface $D \subset \mathbb{P}^3$ with at least one non-canonical singular point is one of:*

1. *D has a single elliptic singularity and D is rational, or*
2. *D is a cone, or*
3. *D is elliptically ruled and*
   (a) *D has a double point p with tangent cone z^2, the projection away from p is the double cover of $\mathbb{P}^2$ branched over a sextic curve Γ. The curve Γ is the union of 3 conics in a pencil that also contains a double line. When this line is a common chord, D has two $T_{2,3,6}$ singularities; when this line is a common tangent, D has one singularity of type $E_{4,0}$. In the first case, D may have an additional A_1 singular point.*
   (b) *D is $\{(x_0x_3 + q(x_1, x_2))^2 + f_4(x_1, x_2, x_3) = 0\}$ and $\{f_4 = 0\}$ is four concurrent lines. Depending on whether $L = \{x_3 = 0\}$ is one of these lines or not and on whether the point of concurrence lies on L, D has either two $T_{2,4,4}$ singular points or one trimodal elliptic singularity. The surface may have additional canonical points A_n for $n = 1, 2, 3$ or $2A_1$.*

Example 8 Let X be the nonsingular quartic hypersurface

$$X = \{x_0^2x_3^2 + x_0x_1^3 + x_3x_2^3 + x_0x_1x_2x_3 + x_4(x_0^3 + x_3^3 + x_4^3) = 0\}$$

and D_X its hyperplane section $X \cap \{x_4 = 0\}$. The surface D_X is normal,

$$\mathrm{Sing}(D_X) = \{p, p'\} = \{(1{:}0{:}0{:}0{:}0), (0{:}0{:}0{:}1{:}0)\},$$

and each singular point is simple elliptic $J_{2,0} = T_{2,3,6}$, i.e. is locally analytically equivalent to $0 \in \{x^2 + y^3 + z^6 + xyz = 0\}$.

Here X is nonsingular and D_X is irreducible and normal, and as I explain below, finding a good (t, dlt) modification amounts to constructing a minimal resolution of D_X. Let $\widetilde{X} \to X$ be the composition of the weighted blowups at $p = (1{:}0{:}0{:}0{:}0)$ with weights $(0, 2, 1, 3, 1)$ and at $p' = (0{:}0{:}0{:}1{:}0)$ with weights $(3, 1, 2, 0, 1)$, and denote by E and E' the corresponding exceptional divisors. Note that $\widetilde{X}$ is terminal and $\mathbb{Q}$-factorial by [13, Theorem 3.5] and has no worse than cyclic quotient singularities. The morphism

$$(\widetilde{X}, D + E + E') \xrightarrow{f} (X, D)$$

is volume preserving and the intersection of D with each exceptional divisor is a smooth elliptic curve $C_6 \subset \mathbb{P}(1, 1, 2, 3)$ not passing through the singular points of E and E'; f is a good (t, dlt) modification.

(continued)

Example 8 (continued)
The dual complex $\mathcal{D}(X, D_X)$ is 1-dimensional, it has 3 vertices and 2 edges; (X, D_X) does not have maximal intersection. The quartic surface D_X is an example of case 3(a) in Theorem 3.

Corollary 2 *Let (X, D_X) be a (t, lc) quartic CY pair. Assume that D_X is normal, has non-canonical singularities but is not a cone. Then (X, D_X) has maximal intersection except in cases 3.(a) and (b) of Theorem 3.*

Remark 6 When $\dim \mathcal{D}(X, D_X) = 1$, D_X either has two $T_{2,3,6}$ or two $T_{2,4,4}$ singularities. Indeed, as is explained in Sect. 2.2, singular points $p \in D$ are Kodaira singularities, and in particular are at worst bimodal. The description of cases 3.(a) and (b) of Theorem 3 immediately implies the result, because a surface singularity of type $E_{4,0}$ is trimodal.

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On Deformations of Diagrams of Commutative Algebras

Emma Lepri and Marco Manetti

Abstract In this paper we study classical deformations of diagrams of commutative algebras over a field of characteristic 0. In particular we determine several homotopy classes of DG-Lie algebras, each one of them controlling this above deformation problem: the first homotopy type is described in terms of the projective model structure on the category of diagrams of differential graded algebras, the others in terms of the Reedy model structure on truncated Bousfield-Kan approximations.

The first half of the paper contains an elementary introduction to the projective model structure on the category of commutative differential graded algebras, while the second half is devoted to the main results.

Keywords Model categories · Deformation theory · Differential graded algebras

1 Introduction

Let $\mathbb{K}$ be a fixed field, S a Noetherian commutative $\mathbb{K}$-algebra and $X = \operatorname{Spec}(S)$ the associated affine scheme. It is well known that every deformation of X, in the category of schemes over $\mathbb{K}$, is affine, hence the deformation theory of X is the same of the deformation theory of S inside the category $\mathbf{Alg}_{\mathbb{K}}$ of unitary commutative $\mathbb{K}$-algebras. Similarly, the deformation theory of a separated Noetherian scheme X over $\mathbb{K}$ is the same as the deformation theory of a diagram in $\mathbf{Alg}_{\mathbb{K}}$. More precisely, if $\mathcal{N}$ is the nerve of an affine open cover $\{U_i\}$ of X, it is not difficult to prove that the deformations of X (up to isomorphism) are the same as the deformations (up to isomorphism) of the diagram

$$S_{\bullet}\colon \mathcal{N} \to \mathbf{Alg}_{\mathbb{K}}, \qquad S_{i_0,\ldots,i_n} = \Gamma(U_{i_0} \cap \cdots \cap U_{i_n}, \mathcal{O}_X)\,,$$

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where $\mathcal{N}$ is considered as a poset and as a small category in the obvious way, inside the category of diagrams $\mathrm{Fun}(\mathcal{N}, \mathbf{Alg}_{\mathbb{K}}) = \{\mathcal{N} \to \mathbf{Alg}_{\mathbb{K}}\}$.

It is well known (see e.g. [5, 13, 15]) that, if $\mathbb{K}$ has characteristic 0, then every (commutative) deformation of an algebra $S \in \mathbf{Alg}_{\mathbb{K}}$ is isomorphic to $H^0(R')$, where R' is obtained by perturbing the differential of a fixed Tate resolution $R \to S$ [20]. This easily implies that the deformations of S are controlled, in the sense of [12], by the differential graded Lie algebra of derivations of R. A short introduction to differential graded algebras is given here in Sect. 2.

It is possible to prove that the above strategy generalises to arbitrary Noetherian separated schemes, where Tate resolution is replaced by the algebraic analogue of Palamodov's resolvent [16, 17]. This is possible because the nerve $\mathcal{N}$ of a covering is a direct Reedy category, i.e., there exists a degree function $\deg\colon \mathcal{N} \to \mathbb{N}$ such that every non identity arrow increases degree.

The aim of this paper is to study deformations of diagrams $\mathcal{D} \to \mathbf{Alg}_{\mathbb{K}}$ for a general small category $\mathcal{D}$. The first result is to extend the above strategy by detecting what is the correct notion of Tate resolution of a diagram (= the correct notion of Palamodov's resolvent for a diagram). In doing this it is extremely convenient to work in the framework of model structures, briefly recalled in Sect. 3.

The category $\mathbf{Alg}_{\mathbb{K}}$ can be considered in an obvious way as a full subcategory of $\mathbf{CDGA}_{\mathbb{K}}$ (resp.: $\mathbf{CDGA}_{\mathbb{K}}^{\le 0}$), the category of commutative differential graded algebras (resp.: in non-positive degrees).

By a classical result of Bousfield and Gugenheim [1] the category $\mathbf{CDGA}_{\mathbb{K}}$ admits a model structure where weak equivalences are the quasi-isomorphisms and fibrations are the surjective maps, cf. [4].

The category $\mathbf{CDGA}_{\mathbb{K}}^{\le 0}$ carries a similar model structure, where weak equivalences are the quasi-isomorphisms and fibrations are the surjective maps in negative degrees. Due to the lack of appropriate references, in Sect. 4 we provide an elementary proof of this fact, based on the properties of free and semifree extensions.

Section 5 is devoted to some technical lemmas that are probably well known to experts. In Sect. 6 we prove the main result of this paper (Theorem 4), namely that the deformation theory of a diagram $S_\bullet\colon \mathcal{D} \to \mathbf{Alg}_{\mathbb{K}}$ is controlled by the differential graded Lie algebra of derivations of a cofibrant replacement of $S_\bullet$ in the model category of diagrams $\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}_{\mathbb{K}}^{\le 0})$, equipped with the projective model structure.

Unfortunately, for general index categories $\mathcal{D}$, cofibrant replacements in the projective model structure are difficult to describe from the constructive point of view. For this reason, in the last sections we propose a different approach by describing a countable family of functors between small categories (Definition 15)

$$\epsilon_k\colon N(\mathcal{D})_{\le k} \to \mathcal{D}, \qquad k = 2, 3, \ldots, \infty,$$

such that for every k in the above range:

1. every diagram $S_\bullet \colon \mathcal{D} \to \mathbf{Alg}_{\mathbb{K}}$ has the same isomorphism classes of deformations as $S_\bullet \circ \epsilon_k$;
2. $N(\mathcal{D})_{\leq k}$ is a Reedy category (see Sect. 8) and the projective model structure on the category $\mathrm{Fun}(N(\mathcal{D})_{\leq k}, \mathbf{CDGA}_{\mathbb{K}}^{\leq 0})$ is the same as the Reedy model structure, hence with cofibrations described constructively in terms of latching objects and cofibrations in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$.

In our construction the functor ϵ_∞ is the forgetful functor from the simplex category of $\mathcal{D}$ (see Sect. 7), and ϵ_k is its restriction to the full subcategory of p-simplexes, with $p \leq k$. The composition map

$$\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}) \xrightarrow{-\circ\epsilon_\infty} \mathrm{Fun}(N(\mathcal{D})_{\leq\infty}, \mathbf{CDGA}_{\mathbb{K}}^{\leq 0})$$

is called Bousfield-Kan approximation and plays an important role in the homotopy theory of diagrams [3].

Putting together all the above facts, the main result of this paper is:

Theorem 1 (=Theorem 4 + Corollary 1) *Let $\mathcal{D}$ be a small category, $S_\bullet \colon \mathcal{D} \to \mathbf{Alg}_{\mathbb{K}}$ a diagram of unitary commutative algebras.*

1. *Let $R_\bullet \to S_\bullet$ be a cofibrant replacement in* $\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}_{\mathbb{K}}^{\leq 0})$ *with respect to the projective model structure. Then the DG-Lie algebra $L = \mathrm{Der}^*_{\mathbb{K}}(R_\bullet, R_\bullet)$ controls the deformations of $S_\bullet$.*
2. *For every $k = 2, \dots, \infty$, let $\epsilon_k \colon N(\mathcal{D})_{\leq k} \to \mathcal{D}$ be the functor defined in Definition 15 and let $R_{\bullet k} \to S_\bullet \circ \epsilon_k$ be a Reedy cofibrant replacement in* $\mathrm{Fun}(N(\mathcal{D})_{\leq k}, \mathbf{CDGA}_{\mathbb{K}}^{\leq 0})$. *Then the DG-Lie algebra $L_k = \mathrm{Der}^*_{\mathbb{K}}(R_{\bullet k}, R_{\bullet k})$ controls the deformations of $S_\bullet$.*

1.1 Notation and Setup

Throughout this paper we will work over a fixed field $\mathbb{K}$ of characteristic 0. Unless otherwise specified, every (graded) vector space is assumed over $\mathbb{K}$ and the symbol $\otimes$ denotes the tensor product over $\mathbb{K}$. If $V = \oplus_{n\in\mathbb{Z}} V^n$ is a graded vector space, we denote by $\overline{a}$ the degree of a non-zero homogeneous element a: in other words $\overline{a} = n$ whenever $a \neq 0$ and $a \in V^n$. It is implicitly assumed that if a mathematical formula contains the degree symbols $\overline{a}, \overline{b}, \dots$ then all the elements $a, b, \dots$ involved are homogeneous and different from 0. As usual, for every complex of vector spaces V, we shall denote by $Z^n(V)$, $B^n(V)$ and $H^n(V)$ the space of n-cocycles, the space of n-coboundaries and the nth cohomology group, respectively. We denote by **Set** the category of sets, by **Grp** the category of groups, by $\mathbf{Alg}_{\mathbb{K}}$ the category of unitary commutative $\mathbb{K}$-algebras and by $\mathbf{Art}_{\mathbb{K}} \subset \mathbf{Alg}_{\mathbb{K}}$ the full subcategory of local Artin algebras with residue field $\mathbb{K}$. Finally, in order to avoid an excessive length we assume that the reader has a basic knowledge of differential graded Lie algebras and of the associated deformation functors: for instance, the papers [12, 14] contain everything needed for the comprehension of this paper.

2 Commutative Differential Graded Algebras

In the first four sections of this paper we shall give a short survey, addressed to a wide mathematical audience, of some homotopical algebra that we use in the second part of the paper. We begin by recalling the definition and the first properties of unitary commutative differential graded algebras (DG-algebras for short) over $\mathbb{K}$.

Definition 1 A unitary commutative graded algebra is a graded vector space $A = \bigoplus_{n\in\mathbb{Z}} A^n$ with a product $A^i \times A^j \longrightarrow A^{i+j}$ which is $\mathbb{K}$-linear, associative and graded commutative, i.e., such that $ab = (-1)^{\overline{a}\,\overline{b}}ba$ for every $a, b \in A$. Moreover there exists a unit $1 \in A^0$ such that $1a = a1 = a$ for every $a \in A$.

A morphism of unitary commutative graded algebras is a morphism of graded vector spaces that commutes with products and preserves the units. We denote by $\mathbf{CGA}_{\mathbb{K}}$ the category of unitary commutative graded algebras. In the above definition it is allowed that $1 = 0$, and this happens if and only if $A = 0$.

The usual construction of polynomials extends without difficulties to the graded case. Given a unitary commutative graded algebra A and a set $\{x_i\}$, $i \in I$, of indeterminates, each one equipped with a degree $\overline{x_i} \in \mathbb{Z}$, the polynomial algebra $A[\{x_i\}]$ is defined as the graded vector space generated by the monomials in x_i with coefficients in A, subject to the relations $x_i x_j = (-1)^{\overline{x_i}\,\overline{x_j}} x_j x_i$ and $a x_i = (-1)^{\overline{x_i}\,\overline{a}} x_i a$, $a \in A$. For instance, if $\overline{x} = 0$ and $\overline{y} = 1$, then $xy = yx$, $y^2 = 0$ and therefore $\mathbb{K}[x, y] = \mathbb{K}[x] \oplus \mathbb{K}[x]y$.

Given $A \in \mathbf{CGA}_{\mathbb{K}}$, a derivation of degree k of A is a linear map $\alpha\colon A \to A$ such that $\alpha(A^n) \subset A^{n+k}$ for every n, satisfying the *(graded) Leibniz identity*:

$$\alpha(ab) = \alpha(a)b + (-1)^{k\overline{a}} a\alpha(b)\,.$$

The vector space of derivations of degree k is denoted $\operatorname{Der}^k_{\mathbb{K}}(A, A)$.

If $A = \mathbb{K}[\{x_i\}]$, by the Leibniz identity every derivation $\alpha \in \operatorname{Der}^k_{\mathbb{K}}(A, A)$ is uniquely defined by the values $\alpha(x_i) \in A^{\overline{x_i}+k}$.

Definition 2 A commutative differential graded algebra (DG-algebra for short) is a graded commutative algebra A equipped with a derivation $d \in \operatorname{Der}^1_{\mathbb{K}}(A, A)$, called *differential*, such that $d^2 = 0$. In other words:

1. $d(A^n) \subseteq A^{n+1}$,
2. $d^2 = 0$,
3. (Graded Leibniz identity) $d(ab) = d(a)b + (-1)^{\overline{a}} a d(b)$.

A morphism of commutative differential graded algebras is a morphism of commutative graded algebras that commutes with differentials.

We denote by $\mathbf{CDGA}_{\mathbb{K}}$ the category of commutative differential graded algebras. Notice that $d(1) = 0$, and that $\mathbb{K}$ and 0 are respectively the initial and the final object in the category $\mathbf{CDGA}_{\mathbb{K}}$. It is easy to see that this category is complete and cocomplete.

Definition 3 (Free Extensions) Let $A \in \mathbf{CDGA}_{\mathbb{K}}$ and x_i, $i \in I$, a set of indeterminates of degree $\overline{x_i} \in \mathbb{Z}$. Consider a parallel set of indeterminates dx_i, with $\overline{dx_i} = \overline{x_i} + 1$ and the polynomial extension $A \to A[\{x_i, dx_i\}]$. The differential d on A can be extended to a differential on $A[\{x_i, dx_i\}]$ by setting $d(x_i) = dx_i$ and $d(dx_i) = 0$.

The name free extension is motivated by the following property: for every morphism $f \colon A \to B$ in $\mathbf{CDGA}_{\mathbb{K}}$ and every subset $\{b_i\} \subset B$ with $b_i \in B^{\overline{x_i}}$ for every i, there exists a unique morphism of DG-algebras $g \colon A[\{x_i, dx_i\}] \to B$ extending f and such that $g(x_i) = b_i$ for every i. Clearly $g(dx_i) = d(b_i)$.

Lemma 1 *Every free extension of DG-algebras is a quasi-isomorphism, i.e., the inclusion* $A \to A[\{x_i, dx_i\}]$ *induces an isomorphism in cohomology.*

Proof Since every element of $A[\{x_i, dx_i\}]$ is a polynomial in a finite number of indeterminates, we can assume the set of variables finite, say $x_1, \ldots, x_n$, and proceed by induction on n. Therefore it is sufficient to show that the inclusion $A \to A[x, dx]$ is a quasi-isomorphism.

If x has even degree, then $(dx)^2 = 0$ and every homogeneous element of the quotient $A[x, dx]/A$ is of type

$$v = \sum_{i=0}^{n} x^{i+1} a_i + x^i dx\, b_i, \qquad a_i, b_i \in A\,.$$

If $dv = 0$ then $d(b_i) = (i+1)a_i$ and therefore (notice the assumption $\mathrm{char}(\mathbb{K}) = 0$)

$$v = d\left(\sum_{i=0}^{n} \frac{x^{i+1}}{i+1}\, b_i\right).$$

If x has odd degree, then $x^2 = 0$ and every homogeneous element of the quotient $A[x, dx]/A$ is of type

$$u = \sum_{i=0}^{n} (dx)^{i+1}\, a_i + x(dx)^i b_i, \qquad a_i, b_i \in A\,.$$

If $du = 0$ then $da_i + b_i = 0$ for every i, and we can write

$$u = d\left(\sum_{i=0}^{m} x(dx)^i a_i\right).$$

Thus we have proved that $A[x, dx]/A$ is an acyclic complex of vector spaces. □

Definition 4 Let $f : A \to B$ be a morphism in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, with $A = A^0 \in \mathbf{Alg}_{\mathbb{K}}$. We shall say that f is *flat*, or that B is a flat A-algebra if B is a complex of flat A-modules.

Clearly the above definition extends the usual notion of flat morphism of algebras. It is worth pointing out that there also exists a good notion of flatness for every morphism in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ [15].

For every $A \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ we shall denote by $\mathbf{CDGA}_{A}^{\leq 0}$ the undercategory of maps $A \to B$: the morphisms in $\mathbf{CDGA}_{A}^{\leq 0}$ are the commutative triangles. The following lemma is completely standard, see e.g. [19, Lemma A.4 and Theorem A.10].

Lemma 2 *Let $A \in \mathbf{Art}_{\mathbb{K}}$ and let $f : B \to C$ be a morphism in $\mathbf{CDGA}_{A}^{\leq 0}$:*

1. *if C is flat over A and the induced map $B \otimes_A \mathbb{K} \to C \otimes_A \mathbb{K}$ is an isomorphism, then f is also an isomorphism;*
2. *if B, C are flat A-algebras and the induced map $B \otimes_A \mathbb{K} \to C \otimes_A \mathbb{K}$ is a quasi-isomorphism, then f is also a quasi-isomorphism;*
3. *if B is flat over A and $H^i(B \otimes_A \mathbb{K}) = 0$ for every $i < 0$, then $H^0(B)$ is a flat A-algebra and the natural map $H^0(B) \to H^0(B \otimes_A \mathbb{K})$ induces an isomorphism $H^0(B) \otimes_A \mathbb{K} = H^0(B \otimes_A \mathbb{K})$.*

3 A Very Short Introduction to Model Structures

We briefly recall the definition of model category and some few basic results about them; the reader may consult [6, 7] for a deeper and more complete exposition of the subject. Throughout this section $\mathbf{M}$ will denote a fixed category.

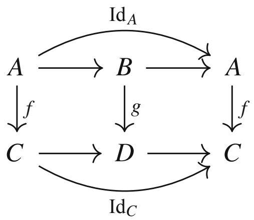
Definition 5 (Lifting Properties) Consider two morphisms $i : A \to B$, $f : C \to D$ in $\mathbf{M}$. If for every solid commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ B & \longrightarrow & D \end{array}$$

there exists the dotted arrow that makes both triangles commute, we shall say that the map i has the left lifting property with respect to f, and the map f has the right lifting property with respect to i.

For instance, in the category of sets, every injective map i has the left lifting property with respect to any surjective map f. The same holds in the category of vector spaces.

Definition 6 (Retracts) A morphism f in **M** is called a retract of a morphism g in **M** if there exists a commutative diagram:



Definition 7 A model structure on **M** is the data of three classes of maps: weak equivalences, fibrations and cofibrations, which satisfy the following axioms:

(M1) (2-out-of-3) If f and g are morphisms in **M** such that the composition gf is defined, and two out of the three f, g and gf are weak equivalences, so is the third.

(M2) (Retracts) If f and g are maps in **M** such that f is a retract of g, and g is a weak equivalence, a cofibration or a fibration, then so is f.

(M3) (Lifting) A trivial fibration is map which is both a fibration and a weak equivalence; a trivial cofibration is map which is both a cofibration and a weak equivalence.

  (a) Trivial fibrations have the right lifting property with respect to cofibrations.
  (b) Trivial cofibrations have the left lifting property with respect to fibrations.

(M4) (Factorisation) Every morphism g in **M** admits two factorisations:

  (CW, F): $g = qj$, where j is a trivial cofibration and q is a fibration,
  (C, FW): $g = pi$, where i is a cofibration and p is a trivial fibration.

Definition 8 A model category is a complete and cocomplete category equipped with a model structure.

In particular every model category has an initial object $\coprod_\emptyset$ and a final object $\prod_\emptyset$; an object X is called cofibrant in the morphism $\coprod_\emptyset \to X$ is a cofibration; it is called fibrant if the morphism $X \to \prod_\emptyset$ is a fibration. A *cofibrant replacement* of an object Y is a trivial fibration $X \to Y$ with X cofibrant. The factorisation axiom guarantees that cofibrant replacements always exist.

For notational simplicity we shall denote by $\mathcal{W}, \mathcal{F}$ and $\mathcal{C}$ the classes of weak equivalences, fibrations and cofibrations, respectively. We shall denote by $\mathcal{FW} = \mathcal{F} \cap \mathcal{W}$ the class of trivial fibrations and by $\mathcal{CW} = \mathcal{C} \cap \mathcal{W}$ the class of trivial cofibrations.

Lemma 3 *If j has the left (right) lifting property with respect to f, and i is a retract of j, then i has the left (right) lifting property with respect to f.*

For a proof, see [6, 7.2.8].

Proposition 1 (Retract Argument) *Let g be a map which can be factored as $g = pi$*

1. *If g has the left lifting property with respect to p then g is a retract of i.*
2. *If g has the right lifting property with respect to i then g is a retract of p.*

For a proof, see [7, 1.1.9].

Lemma 4 *Let* **M** *be a model category:*

1. *A map in* **M** *that has the left lifting property with respect to all trivial fibrations is a cofibration.*
2. *A map in* **M** *that has the right lifting property with respect to all trivial cofibrations is a fibration.*

For a proof, see [7, 1.1.10].

Remark 1 It follows from the previous lemma that isomorphisms belong to all three classes of maps. Furthermore, two of the three classes $\mathcal{W}, \mathcal{C}, \mathcal{F}$ determine the third. Pay attention to the fact that, for example, if $\mathcal{W}$ and $\mathcal{F}$ are two classes satisfying (M1) and (M2), in general they do not extend to a model structure.

The following lemma is clear.

Lemma 5 *If $h = gf$ and both f, g have the left (right) lifting property with respect to p, then h has the left (right) lifting property with respect to p.*

Remark 2 The previous lemmas show that the three classes of fibrations, cofibrations and weak equivalences are closed by composition.

Model categories were introduced by Quillen [18] under the name of (complete and cocomplete) closed model categories. Nowadays many authors (e.g. [6, 7]) assume that the (C,FW) and (CW,F) factorisations are functorial. Since in algebraic geometry it is often important to resolve minimally algebraic structures, we prefer here to adopt the original Quillen's assumption, and require only the existence of factorisations.

3.1 Pre-model Structures

In several concrete cases, a convenient way to describe model structures is in terms of *pre-model structures*. Since two out of the three classes $\mathcal{W}, \mathcal{C}, \mathcal{F}$ determine the third, it is typical to try to construct a model structure by establishing two out of the three classes and seeing whether they extend to a model structure. Often it happens that two classes have an easy description and the third is more complicated, but it has a nice subclass sufficiently large to ensure axiom (M4). The notion of left pre-model structure applies when one has fixed the weak equivalences, fibrations and two more classes, as in the next definition.

Definition 9 A *left pre-model structure* on **M** is the data of four classes of maps: $\mathcal{W}, \mathcal{F}, \mathcal{C}', \mathcal{CW}'$ such that:

1. (2-out-of-3) The maps in $\mathcal{W}$ satisfy the 2-out-of-3 property;
2. (Retracts) The classes $\mathcal{W}$ and $\mathcal{F}$ are closed under retracts;
3. $\mathcal{CW}' \subseteq \mathcal{W}$;
4. (Lifting) The maps in $\mathcal{C}'$ have the left lifting property with respect to the maps in $\mathcal{F} \cap \mathcal{W}$; the maps in $\mathcal{CW}'$ have the left lifting property with respect to the maps in $\mathcal{F}$;
5. (Factorisation) Every map g in **M** has two factorisations:
   (a) $g = qj$, where j is in $\mathcal{C}'$ and q is in $\mathcal{F} \cap \mathcal{W}$,
   (b) $g = pi$, where i is in $\mathcal{CW}'$ and p is in $\mathcal{F}$.

Theorem 2 *Given a left pre-model structure $\mathcal{W}, \mathcal{F}, \mathcal{C}', \mathcal{CW}'$ there exists a unique model structure where the weak equivalences are the maps in $\mathcal{W}$ and the fibrations are the maps in $\mathcal{F}$. Notably, the cofibrations are the retracts of $\mathcal{C}'$, and the trivial cofibrations are the retracts of $\mathcal{CW}'$.*

Proof We set $\mathcal{C}$ the retracts of $\mathcal{C}'$ and check that $\mathcal{C}, \mathcal{F}, \mathcal{W}$ satisfy the model category axioms (Definition 7). Axioms (M1) and (M2) follow immediately from the definition of pre-model structure and of $\mathcal{C}$. As above, for notational simplicity we shall denote by $\mathcal{CW} = \mathcal{C} \cap \mathcal{W}$ and $\mathcal{FW} = \mathcal{F} \cap \mathcal{W}$.

We first show that $\mathcal{CW}' \subset \mathcal{C}$. Let $g\colon A \to B$ be a morphism in $\mathcal{CW}'$ and consider a factorisation $g\colon A \xrightarrow{\mathcal{C}'} X \xrightarrow{\mathcal{FW}} B$. Since g has the left lifting property with respect to maps in $\mathcal{FW}$ it follows by the retract argument that g is a retract of an element of $\mathcal{C}'$. Since $\mathcal{CW}' \subset \mathcal{W}$ by assumption, we have $\mathcal{CW}' \subset \mathcal{CW}$.

We now show that every map in $\mathcal{CW}$ is a retract of a map in $\mathcal{CW}'$. Let $f\colon A \longrightarrow B$ be a map in $\mathcal{CW}$, using the factorisation axiom $f\colon A \xrightarrow{\mathcal{CW}'} X \xrightarrow{\mathcal{F}} B$, and by the 2-out-of-3 axiom $X \longrightarrow B$ is in $\mathcal{FW}$. Therefore by the lifting axiom and the retract argument (1), $A \longrightarrow B$ is a retract of $A \longrightarrow X$, so it is the retract of a map in $\mathcal{CW}'$.

By definition of pre-model structure, maps in $\mathcal{C}'$ have the left lifting property with respect to maps in $\mathcal{FW}$; then by Lemma 3 maps in $\mathcal{C}$ have the left lifting property with respect to maps in $\mathcal{FW}$. Similarly, maps in $\mathcal{CW}'$ have the left lifting property with respect to maps in $\mathcal{F}$, so we have that maps in $\mathcal{CW}$ have the left lifting property with respect to maps in $\mathcal{F}$.

The factorisation axiom (M4) is clear since we have already proved that $\mathcal{C} \subset \mathcal{C}'$ and $\mathcal{CW}' \subseteq \mathcal{CW}$. □

It is plain that one can also give the analogous notion of right pre-model structure, simply working in the opposite category and exchanging the role of $\mathcal{C}$ and $\mathcal{F}$. Finally, the reader should be aware that some authors use the name of pre-model structure for a completely different concept.

4 Model Structure on DG-Algebras

It is well known that the category $\mathbf{CDGA}_{\mathbb{K}}$ admits a model structure where weak equivalences are the quasi-isomorphisms and fibrations are the surjective maps [1, 4]. Consequently, by Lemma 4 the cofibrations are the morphisms that have the left lifting property with respect to the class of surjective quasi-isomorphisms.

Similarly, the category $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ of DG-algebras concentrated in non-positive degree admits a model structure where weak equivalences are the quasi-isomorphisms and fibrations are the surjective maps in negative degree. It is worth noticing that the existence of the model structure on $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ is an immediate consequence of [11, Proposition 4.5.4.6] applied to the standard (cofibrantly generated) model structure on the category of non-positively graded DG-vector spaces. Moreover, by Lurie's result also follows that the model structure on $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ is combinatorial and cofibrantly generated.

In this section, following the ideas of [8], we give an elementary proof of the above mentioned model structure on $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, which relies on the notion of semifree extension.

Definition 10 (Semifree Extension) Let $A \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, I be a set, and let $x_i, i \in I$ be indeterminates of non-positive degree $\overline{x_i} \in \mathbb{Z}_{\leq 0}$. Any inclusion of DG-algebras of type

$$A \longrightarrow A[\{x_i\}],$$

regardless of the differential on $A[\{x_i\}]$, is called a *semifree extension*.

Recall that a differential on $A[\{x_i\}]$ is determined by the differential on A and by the values $d(x_i)$. Every free extension is also semifree.

Theorem 3 *There exists a model structure on the category* $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$*, where weak equivalences are the quasi-isomorphisms and fibrations are the maps surjective in negative degree. Moreover:*

1. *cofibrations are the retracts of semifree extensions,*
2. *trivial cofibrations are the retracts of free extensions,*
3. *trivial fibrations are the surjective quasi-isomorphisms.*

The proof that every trivial fibration is surjective is a simple argument in basic homological algebra. In fact, if $f : A \longrightarrow B$ is a quasi-isomorphism which is surjective in negative degree, for every $x \in B^0$, since $dx = 0$ and $f : H^0(A) \to H^0(B)$ is bijective, there exist $y \in A^0$ and $z \in B^{-1}$ such that $f(y) = x + dz$. Since $f : A^{-1} \longrightarrow B^{-1}$ is surjective there exists $u \in A^{-1}$ such that $f(u) = z$ and therefore $x = f(y - du)$.

Now the proof of Theorem 3 follows, according to Theorem 2, from the fact that the four classes:

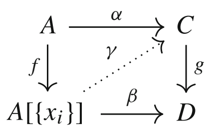
1. $\mathcal{W}$ quasi-isomorphisms,
2. $\mathcal{F}$ maps surjective in negative degree,

3. $\mathcal{C}'$ semifree extensions,
4. $\mathcal{CW}'$ free extensions.

form a left pre-model structure. We have already proved in Lemma 1 that $\mathcal{CW}' \subset \mathcal{W}$. The 2-out-of-3 axiom for $\mathcal{W}$ is clear, and the retract axiom for $\mathcal{W}, \mathcal{F}$ is also obvious: the retract of a injective (surjective) map is also injective (surjective).

Proposition 2 *The maps in $\mathcal{C}'$, i.e., the semifree extensions, have the left lifting property with respect to all trivial fibrations.*

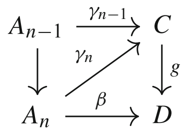
Proof Consider the following solid commutative diagram



where g is a trivial fibration and f a semifree extension. For every integer $n \geq -1$ consider the DG-subalgebra of $A[\{x_i\}]$

$$A_n = A[\{x_i \mid \overline{x_i} \geq -n\}]\,.$$

We have $A_{-1} = A$, $\cup_n A_n = A[\{x_i\}]$ and therefore, setting $\gamma_{-1} = \alpha$ it is sufficient to prove by induction that for every $n \geq 0$ we have a commutative diagram



where the left vertical arrow is the inclusion $A_{n-1} \subset A_n$. If $n = 0$, then for every x_i with $\overline{x_i} = 0$ we have $dx_i = 0$. Since g is surjective there exists $c_i \in C$ such that $g(c_i) = \beta(x_i)$. We define γ_0 by setting $\gamma_0(x_i) = c_i$.

Assume now $n > 0$ and γ_{n-1} already defined. If $\overline{x_i} = -n$, then $\overline{d(x_i)} = -n + 1$ and therefore $dx_i \in A_{n-1}$. We have that

$$d\gamma_{n-1}(dx_i) = \gamma_{n-1}(d^2 x_i) = 0, \qquad g\gamma_{n-1}(dx_i) = \beta(dx_i) = d\beta(x_i)$$

and since g is injective in cohomology we have $\gamma_{n-1}(dx_i) = dy_i$ with $y_i \in C$. Setting $z_i = \beta(x_i) - g(y_i)$ we have

$$dz_i = \beta(dx_i) - g(dy_i) = \beta(dx_i) - g\gamma_{n-1}(dx_i) = 0\,.$$

Since g is a surjective quasi-isomorphism there exists $c_i \in C$ such that $dc_i = 0$ and $g(c_i) = z_i$. We can now define $\gamma_n(x_i) = y_i + c_i$. □

Proposition 3 *Maps in $\mathcal{CW}'$ have the left lifting property with respect to all fibrations.*

Proof Consider the following solid commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \alpha\ } & C \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle g} \\ A[\{x_i, dx_i\}] & \xrightarrow{\ \beta\ } & D \end{array}$$

with g surjective in negative degrees. If $A = 0$ there is nothing to prove; otherwise, since $A[\{x_i, dx_i\}] \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, every x_i has negative degree and there exists $c_i \in C$ such that $g(c_i) = \beta(x_i)$. We set $h(x_i) = c_i$, $h(dx_i) = dc_i$, and $h|_A = \alpha$. □

Proposition 4 *Every map in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ can be factored as a free extension followed by a fibration.*

Proof Let $f : A \to B$ be a map in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. For every homogeneous element $b \in B$ of strictly negative degree we add two indeterminates x_b and dx_b to A, with x_b of degree $\overline{b}$ and dx_b of degree $\overline{b} + 1$, obtaining the free extension

$$A \xrightarrow{i} A[\{x_b, dx_b\}]$$

We define $\pi : A[\{x_b, dx_b\}] \to B$ in the following way:

1. π is equal to f on A,
2. $\pi(x_b) = b$,
3. $\pi(dx_b) = db$.

The map π is obviously a fibration and the composition

$$A \xrightarrow{i} A[\{x_b, dx_b\}] \xrightarrow{\pi} B$$

is equal to f. □

Proposition 5 *Every map in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ can be factored as a semifree extension followed by a trivial fibration.*

Proof Since semifree extensions are closed by composition, according to Proposition 4 it is sufficient to prove that every morphism $f \colon A \to B$ factors as a composition of a semifree extension and a quasi-isomorphism.

We use the differential graded analog of the classical argument about the existence of Tate-Tyurina resolutions: we construct recursively a countable sequence of semifree extensions

$$A = A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots$$

together with morphisms of DG-algebras $f_n\colon A_n \to B$ such that $f_0 = f$ and:

1. $A_{n+1} = A_n[\{x_i\}]$, with $\overline{x_i} = -n$;
2. f_{n+1} extends f_n;
3. $f_n\colon Z^i(A_n) \to Z^i(B)$ is surjective for every $i > -n$;
4. $f_n\colon H^i(A_n) \to H^i(B)$ is bijective for every $i > -n+1$.

Let $\{b_i\}$, $i \in I$, be a set of generators of B^0 as a A^0-algebra; then we may define

$$A_1 = A[\{x_i\}], \qquad \overline{x_i} = 0, \quad dx_i = 0, \quad f_1(x_i) = b_i\,.$$

Assume now $n > 0$ and $f_n\colon A_n \to B$ defined. By choosing a suitable set of generators of $Z^{-n}(B)$ as A_n^0-module we can first consider a factorisation $f_n\colon A_n \subset C \xrightarrow{g} B$ such that

$$C = A_n[\{x_i\}], \qquad \overline{x_i} = -n, \quad dx_i = 0, \quad f_n(x_i) \in Z^{-n}(B),$$

and such that $g\colon Z^{-n}(C) \to Z^{-n}(B)$ is surjective. If $g\colon H^{-n+1}(C) \to H^{-n+1}(B)$ is bijective we can define $A_{n+1} = C$ and $f_{n+1} = g$. Otherwise let $\{c_i\}$ be a set of elements in $Z^{-n+1}(C)$ whose cohomology classes generate the kernel of $g\colon H^{-n+1}(C) \to H^{-n+1}(B)$, choose elements $b_i \in B^{-n}$ such that $db_i = g(c_i)$ and consider the factorisation

$$A_{n+1} = C[\{x_i\}], \qquad \overline{x_i} = -n, \quad dx_i = c_i, \quad f_{n+1}(x_i) = b_i.$$

It is easy to verify that the map $f_{n+1}\colon A_{n+1} \to B$ has the required properties. Finally, since $H^i(A_n) = H^i(A_{n+1})$ for every $i > -n+1$, the colimit of the sequence $f_{n+1}\colon A_{n+1} \to B$ gives the required factorisation. □

Remark 3 Let f be a morphism in $\mathbf{CDGA}_{\mathbb{K}}^{\le 0}$. We have already proved that f is a trivial fibration in $\mathbf{CDGA}_{\mathbb{K}}^{\le 0}$ if and only if it is a trivial fibration in $\mathbf{CDGA}_{\mathbb{K}}$. Since the truncation functor

$$\tau\colon \mathbf{CDGA}_{\mathbb{K}} \to \mathbf{CDGA}_{\mathbb{K}}^{\le 0}, \qquad (\tau A)^n = \begin{cases} A^n & n < 0,\\ Z^0(A) & n = 0,\\ 0 & n > 0,\end{cases}$$

is right adjoint to the faithful natural inclusion $\mathbf{CDGA}_{\mathbb{K}}^{\le 0} \subset \mathbf{CDGA}_{\mathbb{K}}$ and preserves trivial fibrations, by Lemma 4 it follows that f is cofibration in $\mathbf{CDGA}_{\mathbb{K}}^{\le 0}$ if and only if it is a cofibration in $\mathbf{CDGA}_{\mathbb{K}}$.

The notions of semifree extension and left pre-model structure apply to many other contexts, for instance cochain complexes over a commutative ring, DG-algebras, DG-Lie algebras etc.: full details will appear in the forthcoming thesis of the first author.

5 Modules and Derivations

Let (A, d_A) be in $\mathbf{CDGA}_{\mathbb{K}}$, an A-module is a differential graded vector space (M, d_M) together with an associative and distributive $\mathbb{K}$-linear left multiplication map $A \times M \to M$, with the properties:

1. $A^i M^j \subset M^{i+j}$,
2. $d_M(am) = d_A(a)\, m + (-1)^{\overline{a}} a\, d_M(m)$ for every $a \in A, m \in M$.

A morphism of A-modules is a morphism of differential graded vector spaces commuting with multiplications. Since A is graded commutative, we can also define an associative right multiplication map $M \times A \to M$ by setting $ma = (-1)^{\overline{a}\,\overline{m}} am$, $a \in A, m \in M$. Notice that $a(mb) = (am)b$ for every $a, b \in A, m \in M$.

The *trivial extension* of a DG-algebra A by the A-module M is the direct sum of complexes $A \oplus M$ equipped with the product:

$$(a, m)(b, n) = (ab, mb + an).$$

It is immediate to see that $A \oplus M \in \mathbf{CDGA}_{\mathbb{K}}$, the projection $A \oplus M \to A$ is a morphism of DG-algebras and M is a square-zero ideal of $A \oplus M$.

For a given graded vector space M and an integer n we shall denote by $M[-n]$ the same space with the degrees shifted by $-n$, namely $M[-n]^i = M^{i-n}$, and by $s^n \colon M \to M[-n]$ the tautological (bijective) map of degree n. In other words, s^n is the essentially the identity and its only effect is changing the degree:

$$s^n \colon M^i \to M[-n]^{i+n}, \qquad x \mapsto s^n x\,.$$

If M is an A-module, then $M[-n]$ is also an A-module, where the differential and the product are defined accordingly to the Koszul sign rule:

$$d(s^n x) = (-1)^n s^n d(x), \qquad a(s^n x) = (-1)^{n\overline{a}} s^n(ax), \qquad (s^n x)a = s^n(xa)\,.$$

Definition 11 Let M be an A-module. A $\mathbb{K}$-linear map $\alpha \colon A \to M$ is a derivation of degree $j \in \mathbb{Z}$ if $\alpha(A^n) \subset M^{n+j}$ and it satisfies Leibniz's law:

$$\alpha(ab) = \alpha(a)b + (-1)^{\overline{a}j} a\alpha(b)$$

The vector space of derivations of degree j from A to M is denoted $\mathrm{Der}^j_{\mathbb{K}}(A, M)$. The graded vector space $\mathrm{Der}^*_{\mathbb{K}}(A, M) = \bigoplus_{j\in\mathbb{Z}} \mathrm{Der}^j_{\mathbb{K}}(A, M)$ has a natural structure of A-module, with multiplication $(a\alpha)(x) = a(\alpha(x))$ and differential $(d\alpha)(x) = d(\alpha(x)) - (-1)^{\overline{\alpha}}\alpha(dx)$. Observe that for every integer n there is a natural isomorphism of A-modules

$$\mathrm{Der}^*_{\mathbb{K}}(A, M[-n]) \to \mathrm{Der}^*_{\mathbb{K}}(A, M)[-n]\,.$$

Every morphism of DG-algebras $f\colon A \to B$ induces in the natural way an A-module structure on B. In this case the module of derivations will be denoted $\mathrm{Der}^*_{\mathbb{K}}(A, B; f)$: a $\mathbb{K}$-linear map $\alpha\colon A \to B$ is an f-derivation of degree k if $\alpha(A^n) \subset B^{n+k}$ and $\alpha(ab) = \alpha(a)f(b) + (-1)^{k\overline{a}} f(a)\alpha(b)$.

Remark 4 Let $f: A \to B$ be a morphism of DG-algebras, $I \subset B$ a square-zero ideal and $\pi: B \to B/I$ the quotient map. Then I is a B/I-module and then also an A-module via the morphism πf. It is immediate to check that if $g\colon A \to B$ is a morphism of graded algebras such that $\pi g = \pi f$ then $g - f\colon A \to I$ is a derivation of degree 0. Conversely, if $\alpha \in \mathrm{Der}^0_{\mathbb{K}}(A, I)$, then $f + \alpha$ is a morphism of graded algebras, and it is a morphism of DG-algebras if and only if $\alpha \in Z^0(\mathrm{Der}^*_{\mathbb{K}}(A, I))$.

Lemma 6 *Let $A \in$ $\mathbf{CDGA}_{\mathbb{K}}$ be a cofibrant algebra and $f: M \to N$ a surjective quasi-isomorphism of A-modules. Then the map*

$$f_*\colon \mathrm{Der}^*_{\mathbb{K}}(A, M) \to \mathrm{Der}^*_{\mathbb{K}}(A, N), \qquad \alpha \mapsto f\alpha\,,$$

is a surjective quasi-isomorphism.

Proof Since $f: M[-n] \to N[-n]$ is a surjective quasi-isomorphism for every integer n it is sufficient to prove that:

1. $f_*\colon \mathrm{Der}^0_{\mathbb{K}}(A, M) \to \mathrm{Der}^0_{\mathbb{K}}(A, N)$ is surjective;
2. $f_*\colon H^0(\mathrm{Der}^*_{\mathbb{K}}(A, M)) \to H^0(\mathrm{Der}^*_{\mathbb{K}}(A, N))$ is bijective.

Let's denote by $C(P) = P \oplus P[1]$ the mapping cone of the identity of an A-module P, with the differential defined by the formula $d(x + s^{-1}y) = dx + y - s^{-1}dy$; notice that $C(P)$ is acyclic and the natural projection $C(P) \to P[1]$ is a morphism of A-modules.

For every linear map $\alpha\colon A \to P$ of degree 0 we shall denote

$$\widetilde{\alpha}\colon A \to A \oplus C(P), \qquad \widetilde{\alpha}(a) = a + \alpha(a) + s^{-1}(\alpha(da) - d\alpha(a))\,.$$

It is straightforward to check that $\widetilde{\alpha}$ is a morphism of complexes and that every morphism of complexes $A \to A \oplus C(P)$ lifting the identity on A is obtained this way. Moreover, α is a derivation if and only if $\widetilde{\alpha}$ is a morphism in $\mathbf{CDGA}_{\mathbb{K}}$.

Since $A \oplus C(M) \to A \oplus C(N)$, $a + x + s^{-1}y \mapsto a + f(x) + s^{-1}f(y)$, is a trivial fibration, the lifting of a derivation $\alpha \in \mathrm{Der}^0_{\mathbb{K}}(A, N)$ is obtained by taking the lifting of the morphism of DG-algebras $\widetilde{\alpha}\colon A \to A \oplus C(N)$. This proves the first item.

If K is the kernel of f, then we have an exact sequence of complexes

$$0 \to \mathrm{Der}^*_{\mathbb{K}}(A, K) \to \mathrm{Der}^*_{\mathbb{K}}(A, M) \to \mathrm{Der}^*_{\mathbb{K}}(A, N) \to 0$$

and in order to prove the second item it is sufficient to show that $\mathrm{Der}^*_{\mathbb{K}}(A, K)$ is acyclic. By the shifting degree argument it is sufficient to prove that

$H^1(\mathrm{Der}^*_{\mathbb{K}}(A, K)) = 0$. Given $\beta \in Z^1(\mathrm{Der}^*_{\mathbb{K}}(A, K))$, the map

$$\widehat{\beta}\colon A \to A \oplus K[1], \qquad \widehat{\beta}(a) = a + s^{-1}\beta(a),$$

is a morphism of DG-algebras and the proof that $\beta \in B^1(\mathrm{Der}^*_{\mathbb{K}}(A, K))$ follows immediately by considering a lifting of $\widehat{\beta}$ along the trivial fibration $A \oplus C(K) \to A \oplus K[1]$. □

6 Deformations of Diagrams via Projective Cofibrant Resolutions

Throughout this section we shall denote by $\mathcal{D}$ a fixed small category. For every category $\mathbf{M}$ we shall denote by $\mathrm{Fun}(\mathcal{D}, \mathbf{M})$ the category of diagrams $\mathcal{D} \to \mathbf{M}$. For every local Artin $\mathbb{K}$-algebra A with residue field $\mathbb{K}$ we shall denote by $\mathbf{Alg}_A$ the category of unitary commutative A-algebras. For simplicity of notation, if

$$P_\bullet \in \mathrm{Fun}(\mathcal{D}, \mathbf{Alg}_A), \qquad \mathcal{D} \ni a \mapsto P_a\,,$$

is a diagram of A-algebras and $A \to B$ is a morphism of algebras, we shall denote $P_\bullet \otimes_A B$ the diagram $(P_\bullet \otimes_A B)_a = P_a \otimes_A B$, $a \in \mathcal{D}$.

Here we are interested in studying the deformation theory of a diagram $S_\bullet\colon \mathcal{D} \to \mathbf{Alg}_{\mathbb{K}}$ of unitary commutative algebras.

Definition 12 A *deformation* over $A \in \mathbf{Art}_{\mathbb{K}}$ of a diagram $S_\bullet\colon \mathcal{D} \to \mathbf{Alg}_{\mathbb{K}}$ is the data of a diagram $S_{\bullet A}\colon \mathcal{D} \to \mathbf{Alg}_A$ of *flat* A-algebras and a morphism of diagrams of algebras $\phi\colon S_{\bullet A} \to S_\bullet$ inducing an isomorphism $S_{\bullet A} \otimes_A \mathbb{K} \simeq S_\bullet$.

Two deformations $\phi\colon S_{\bullet A} \to S_\bullet$ and $\psi\colon S'_{\bullet A} \to S_\bullet$ are isomorphic if there exists an isomorphism of diagrams of A-algebras $\eta\colon S_{\bullet A} \to S'_{\bullet A}$ such that $\phi = \psi\eta$.

It is possible to prove, see e.g. [10, A.2], that for every small category $\mathcal{D}$ there exist model structures on $\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}^{\le 0}_{\mathbb{K}})$ and $\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}_{\mathbb{K}})$, called *projective model structures*, such that a morphism of diagrams $F \to G$ is a weak equivalence (resp.: fibration) if and only if $F_a \to G_a$ is a weak equivalence (resp.: fibration) for every $a \in \mathcal{D}$. The same argument used in Remark 3 shows that a morphism f in $\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}^{\le 0}_{\mathbb{K}})$ is a weak equivalence, cofibration, trivial fibration in $\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}^{\le 0}_{\mathbb{K}})$ if and only if it is a weak equivalence, cofibration, trivial fibration in $\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}_{\mathbb{K}})$, respectively.

The notions of module and derivation extend naturally to the context of diagrams. For every diagram $R_\bullet\colon \mathcal{D} \to \mathbf{CDGA}^{\le 0}_{\mathbb{K}}$ the DG-Lie algebra of derivations is

$$\mathrm{Der}^*_{\mathbb{K}}(R_\bullet, R_\bullet) = \left\{ \{\alpha_a\} \in \prod_{a \in \mathcal{D}} \mathrm{Der}^*_{\mathbb{K}}(R_a, R_a) \mid \alpha_b R_f = R_f \alpha_a,\ \forall\, a \xrightarrow{f} b \right\}.$$

It is plain that $\mathrm{Der}^*_{\mathbb{K}}(R_\bullet, R_\bullet)$ is a DG-Lie subalgebra of $\prod_{a \in \mathcal{D}} \mathrm{Der}^*_{\mathbb{K}}(R_a, R_a)$.

An $R_\bullet$-module $M_\bullet$ is a diagram of differential graded vector spaces over $\mathcal{D}$ such that M_a is an R_a-module for every $a \in \mathcal{D}$ and, for every arrow $a \xrightarrow{f} b$ in $\mathcal{D}$, the map $M_f : M_a \to M_b$ is a morphism of R_a-modules, where M_b is considered as a R_a-module via the morphism of DG-algebras $R_f : R_a \to R_b$. A morphism $g: M_\bullet \to N_\bullet$ of $R_\bullet$-modules is a morphism of diagrams of DG-vector spaces such that $g_a : M_a \to N_a$ is a morphism of R_a-modules for every $a \in \mathcal{D}$.

The differential graded vector space of derivations is

$$\operatorname{Der}^*_{\mathbb{K}}(R_\bullet, M_\bullet) = \left\{ \{\alpha_a\} \in \prod_{a \in \mathcal{D}} \operatorname{Der}^*_{\mathbb{K}}(R_a, M_a) \mid \alpha_b R_f = M_f \alpha_a,\ \forall\, a \xrightarrow{f} b \right\} .$$

The same argument used in the proof of Lemma 6 works, mutatis mutandis, also for diagrams and gives the following result.

Lemma 7 *Let $R_\bullet \in \operatorname{Fun}(\mathcal{D}, \mathbf{CDGA}_{\mathbb{K}})$ be a projective cofibrant diagram and $f: M_\bullet \to N_\bullet$ a morphism of $R_\bullet$-modules such that $f_a : M_a \to N_a$ is a surjective quasi-isomorphism for every $a \in \mathcal{D}$. Then the map*

$$f_* : \operatorname{Der}^*_{\mathbb{K}}(R_\bullet, M_\bullet) \to \operatorname{Der}^*_{\mathbb{K}}(R_\bullet, N_\bullet), \qquad \alpha \mapsto f\alpha ,$$

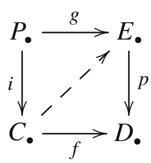
is a surjective quasi-isomorphism.

The main goal of this section is to prove the following theorem.

Theorem 4 *Let $\mathcal{D}$ be a small category and $S_\bullet : \mathcal{D} \to \mathbf{Alg}_{\mathbb{K}}$ a diagram of unitary commutative algebras. Let $R_\bullet \to S_\bullet$ be a cofibrant replacement in* $\operatorname{Fun}(\mathcal{D}, \mathbf{CDGA}^{\le 0}_{\mathbb{K}})$ *with respect to the projective model structure. Then the DG-Lie algebra* $\operatorname{Der}^*_{\mathbb{K}}(R_\bullet, R_\bullet)$ *controls the deformations of $S_\bullet$.*

In other words, the functor of isomorphism classes of deformations of $S_\bullet$ is isomorphic to the functor of Maurer-Cartan solutions in $\operatorname{Der}^*_{\mathbb{K}}(R_\bullet, R_\bullet)$ modulus gauge equivalence. We shall prove Theorem 4 after a certain number of preliminary results. Unless otherwise specified we always equip the categories $\operatorname{Fun}(\mathcal{D}, \mathbf{CDGA}^{\le 0}_{\mathbb{K}})$ and $\operatorname{Fun}(\mathcal{D}, \mathbf{CDGA}_{\mathbb{K}})$ with the projective model structure. Therefore $R_\bullet \to S_\bullet$ is a cofibrant resolution also in the model category $\operatorname{Fun}(\mathcal{D}, \mathbf{CDGA}_{\mathbb{K}})$ and we can apply Lemma 7 to the diagram $R_\bullet$.

Lemma 8 *Consider a commutative square of solid arrows*



in Fun($\mathcal{D}$, $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$). *If i is a cofibration and $p_a\colon E_a \to D_a$ is surjective for every $a \in \mathcal{D}$, then there exists a lifting $\gamma\colon C_\bullet \to E_\bullet$ in the category of diagrams of graded algebras.*

Proof Consider the contractible polynomial algebra $\mathbb{K}[d^{-1}] \in \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, where $\overline{d^{-1}} = -1$ and $d(d^{-1}) = 1$, and notice that the natural inclusion $\alpha\colon \mathbb{K} \to \mathbb{K}[d^{-1}]$ is a morphism of DG-algebras, while the natural projection $\beta\colon \mathbb{K}[d^{-1}] \to \mathbb{K}$ is a morphism of graded algebras; moreover $\beta\alpha$ is the identity on $\mathbb{K}$. Now, the morphism

$$E_\bullet \otimes_{\mathbb{K}} \mathbb{K}[d^{-1}] \xrightarrow{p\otimes \mathrm{Id}} D_\bullet \otimes_{\mathbb{K}} \mathbb{K}[d^{-1}]$$

is a trivial fibration, and so there exists a commutative square

$$\begin{array}{ccc} P_\bullet & \xrightarrow{(\mathrm{Id}\otimes\alpha)g} & E_\bullet \otimes_{\mathbb{K}} \mathbb{K}[d^{-1}] \\ {\scriptstyle i}\downarrow & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle p\otimes\mathrm{Id}} \\ C_\bullet & \xrightarrow[(\mathrm{Id}\otimes\alpha)f]{} & D_\bullet \otimes_{\mathbb{K}} \mathbb{K}[d^{-1}] \end{array}$$

in Fun($\mathcal{D}$, $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$). It is now sufficient to take $\gamma = (\mathrm{Id}\otimes\beta)\varphi$. □

Lemma 9 *Let $A \in \mathbf{Art}_{\mathbb{K}}$ and let $N_{\bullet A}\colon \mathcal{D} \to \mathbf{CDGA}_{A}^{\leq 0}$ be a diagram of flat A-algebras. Then every cofibrant replacement $f\colon P_\bullet \to N_{\bullet A} \otimes_A \mathbb{K}$ in* Fun($\mathcal{D}$, $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$) *lifts to an A-linear differential on $P_\bullet \otimes A$ and to a trivial fibration $P_\bullet \otimes A \to N_{\bullet A}$ in the category* Fun($\mathcal{D}$, $\mathbf{CDGA}_{A}^{\leq 0}$). *The above lifting is unique up to A-linear algebra isomorphisms of $P_\bullet \otimes A$ lifting the identity on $P_\bullet$.*

Proof *(Existence)* We proceed by induction on the length of the Artin ring. Since for $A = \mathbb{K}$ there is nothing to prove, we may assume $A \in \mathbf{Art}_{\mathbb{K}}$ of length $l(A) > 1$ and then there exists a non-trivial element $t \in A$ annihilated by the maximal ideal $\mathfrak{m}_A$, giving a small extension

$$0 \to \mathbb{K} \xrightarrow{\cdot t} A \to B \to 0, \qquad l(B) = l(A) - 1\,.$$

By induction there exist a B-linear differential on $P_\bullet \otimes_{\mathbb{K}} B$ and a commutative square in Fun($\mathcal{D}$, $\mathbf{CDGA}_{B}^{\leq 0}$):

$$\begin{array}{ccc} P_\bullet \otimes_{\mathbb{K}} B & \xrightarrow{q} & N_{\bullet A} \otimes_A B \\ \downarrow & & \downarrow \\ P_\bullet & \xrightarrow{f} & N_{\bullet A} \otimes_A \mathbb{K} \end{array}$$

In view of the embedding $\mathbb{K} \subset B$, the B-linear morphism of diagrams q is uniquely determined by its restriction $q_{|P_\bullet} : P_\bullet \to N_{\bullet A} \otimes_A B$, which is a morphism in $\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}_{\mathbb{K}}^{\leq 0})$. By Lemma 8 we can lift $q_{|P_\bullet}$ to a morphism of diagrams of graded algebras $P_\bullet \to N_{\bullet A}$ and then we get a commutative diagram with (pointwise) exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_\bullet & \xrightarrow{t} & P_\bullet \otimes_{\mathbb{K}} A & \xrightarrow{\pi} & P_\bullet \otimes_{\mathbb{K}} B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \\ 0 & \longrightarrow & N_{\bullet A} \otimes_A \mathbb{K} & \xrightarrow{t} & N_{\bullet A} & \longrightarrow & N_{\bullet A} \otimes_A B & \longrightarrow & 0 \end{array}$$

Since f and q are trivial fibrations, to conclude the proof it is sufficient to show that there exists a lifting of the differential of $P_\bullet \otimes_{\mathbb{K}} B$ to an A-linear differential of $P_\bullet \otimes_{\mathbb{K}} A$ making p a morphism of diagrams of DG-algebras. Let $d = \{d_a : P_a \to P_a \mid a \in \mathcal{D}\}$ be the differential of $P_\bullet$, since the A-linear derivations of $P_\bullet \otimes_{\mathbb{K}} A$ of degree 1 lifting d are of type $d + \eta$ with $\eta \in \mathrm{Der}^1_{\mathbb{K}}(P_\bullet, P_\bullet) \otimes \mathfrak{m}_A$, we can lift the differential of $P_\bullet \otimes_{\mathbb{K}} B$ to an A-linear derivation $\delta : P_\bullet \otimes_{\mathbb{K}} A \to P_\bullet \otimes_{\mathbb{K}} A$ of degree 1.

Now it is sufficient to prove that there exists a derivation in $\xi \in \mathrm{Der}^1_{\mathbb{K}}(P_\bullet, P_\bullet)$ such that:

1. $(\delta + t\xi)^2 = 0$,
2. $p(\delta + t\xi) = d_{N_\bullet} p$,

and consider $\delta + t\xi$ as the differential of $P_\bullet \otimes_{\mathbb{K}} A$. Since t is annihilated by the maximal ideal $\mathfrak{m}_A$, the condition $(\delta + t\xi)^2 = 0$ is equivalent to $d\xi + \xi d = 0$, i.e., the above condition (1) holds if and only if $\xi \in Z^1(\mathrm{Der}^*_{\mathbb{K}}(P_\bullet, P_\bullet))$. The map $\psi = d_{N_\bullet} p - p\delta$ is A-linear and its image is contained in $t(N_{\bullet A} \otimes_A \mathbb{K})$, hence it factors to a derivation $\phi \in Z^1(\mathrm{Der}^*_{\mathbb{K}}(P_\bullet, N_{\bullet A} \otimes_A \mathbb{K}; f))$ and the above condition (2) is equivalent to $\phi = f\xi$. It is now sufficient to observe that since f is a trivial fibration, by Lemma 7 the morphism

$$f : \mathrm{Der}^*_{\mathbb{K}}(P_\bullet, P_\bullet) \to \mathrm{Der}^*_{\mathbb{K}}(P_\bullet, N_{\bullet A} \otimes_A \mathbb{K}; f)$$

is a surjective quasi-isomorphism and therefore

$$f : Z^1(\mathrm{Der}^*_{\mathbb{K}}(P_\bullet, P_\bullet)) \to Z^1(\mathrm{Der}^*_{\mathbb{K}}(P_\bullet, N_{\bullet A} \otimes_A \mathbb{K}; f))$$

is a surjective map.

Unicity Let δ, δ' be two A linear differentials on $P_\bullet \otimes_{\mathbb{K}} A$ lifting the differential d on $P_\bullet$ and let

$$p : (P_\bullet \otimes_{\mathbb{K}} A, \delta) \to N, \qquad q : (P_\bullet \otimes_{\mathbb{K}} A, \delta') \to N,$$

be two morphisms in $\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}_A^{\le 0})$ lifting the trivial fibration $f\colon P_\bullet \to N_{\bullet A} \otimes_A \mathbb{K}$. We need to prove that there exists an isomorphism of diagrams of differential graded A-algebras $\phi\colon (P_\bullet \otimes_{\mathbb{K}} A, \delta) \to (P_\bullet \otimes_{\mathbb{K}} A, \delta')$ such that $p = q\phi$.

By induction on the length we can assume that there exists an isomorphism of diagrams of differential graded B-algebras $\phi'\colon (P_\bullet \otimes_{\mathbb{K}} B, \delta) \to (P_\bullet \otimes_{\mathbb{K}} B, \delta')$ such that $p = q\phi'$. By Lemma 8 we can lift ϕ' to an isomorphism of diagrams of graded A-algebras $\psi'\colon (P_\bullet \otimes_{\mathbb{K}} A, \delta) \to (P_\bullet \otimes_{\mathbb{K}} A, \delta')$; therefore, replacing δ' with $(\psi')^{-1}\delta'(\psi)$ and q with $q\psi'$ if necessary, it is not restrictive to assume ϕ' equal to the identity. The derivation $p - q\colon P_\bullet \to t(N_{\bullet A} \otimes_A \mathbb{K})$ can be lifted to a derivation $\alpha \in \mathrm{Der}^0_{\mathbb{K}}(P_\bullet, P_\bullet)$ and then, replacing q with $q(\mathrm{Id} + t\alpha)$ and δ' with $(\mathrm{Id} - t\alpha)\delta'(\mathrm{Id} + t\alpha)$ if necessary, it is not restrictive to assume $p = q$. This implies in particular that $\delta' = \delta + t\xi$, for some $\xi \in Z^1(\mathrm{Der}^*_{\mathbb{K}}(P_\bullet, P_\bullet))$. Since the kernel of $f\colon \mathrm{Der}^*_{\mathbb{K}}(P_\bullet, P_\bullet) \to \mathrm{Der}^*_{\mathbb{K}}(P_\bullet, N_{\bullet A} \otimes_A \mathbb{K}; f))$ is acyclic and $p\delta = p\delta'$, we have $f\xi = 0$ and therefore $\xi = [d, \alpha]$ for some $\alpha \in \mathrm{Der}^0_{\mathbb{K}}(P_\bullet, P_\bullet)$ such that $f\alpha = 0$. Now $e^{t\alpha}$ is the required isomorphism. □

Proof of Theorem 4 We assume that the reader has a certain familiarity with the theory of deformation functors associated to DG-Lie algebras; the basic facts exposed in [12, 14] are sufficient for our needs.

Let $\mathcal{D}$ be a small category and $S_\bullet\colon \mathcal{D} \to \mathbf{Alg}_{\mathbb{K}}$ a diagram of unitary commutative algebras. Let $R_\bullet \to S_\bullet$ be a cofibrant replacement in $\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}_{\mathbb{K}}^{\le 0})$ with respect to the projective model structure and consider the DG-Lie algebra $L = \mathrm{Der}^*_{\mathbb{K}}(R_\bullet, R_\bullet)$.

Denoting by $\mathrm{Def}_{S_\bullet}\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ the functor of isomorphism classes of deformations we want to describe an isomorphism

$$\phi\colon \mathrm{Def}_L \to \mathrm{Def}_{S_\bullet}\ .$$

Denoting by $d \in \mathrm{Der}^1_{\mathbb{K}}(R_\bullet, R_\bullet)$ the differential of $R_\bullet$, for every $A \in \mathbf{Art}_{\mathbb{K}}$ with maximal ideal $\mathfrak{m}_A$, a Maurer-Cartan element

$$\xi \in \mathrm{MC}_L(A) = \left\{ x \in L^1 \otimes \mathfrak{m}_A \;\middle|\; d_L x + \frac{1}{2}[x, x] = 0 \right\}$$

is exactly a derivation $\xi \in \mathrm{Der}^1_{\mathbb{K}}(R_\bullet, R_\bullet \otimes \mathfrak{m}_A)$ such that $(R_\bullet \otimes A, d + \xi)$ is a flat diagram in $\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}_A^{\le 0})$. Moreover $\xi, \eta \in \mathrm{MC}_L(A)$ are gauge equivalent if and only if there exists an isomorphism of diagrams of DG-algebras $(R_\bullet \otimes A, d + \xi) \simeq (R_\bullet \otimes A, d + \eta)$, lifting the identity over $R_\bullet$. According to Lemma 2 the map

$$\mathrm{MC}_L(A) \to \mathrm{Def}_{S_\bullet}(A), \qquad \xi \mapsto H^0(R_\bullet \otimes A, d + \xi),$$

is properly defined and factors to a natural transformation $\phi\colon \mathrm{Def}_L \to \mathrm{Def}_{S_\bullet}$. Finally Lemma 9 implies immediately that ϕ is an isomorphism. □

In the situation of Theorem 4, according to Lemma 7, the natural map $L = \mathrm{Der}^*_{\mathbb{K}}(R_\bullet, R_\bullet) \to \mathrm{Der}^*_{\mathbb{K}}(R_\bullet, S_\bullet)$ is a quasi-isomorphism of complexes, hence $H^i(L) = 0$ for every $i < 0$. The $S_\bullet$-module $\mathrm{Der}^*_{\mathbb{K}}(R_\bullet, S_\bullet)$, defined up to quasi-isomorphism, is called the *tangent complex* of $S_\bullet$. Its cohomology groups are denoted by $T^i(S_\bullet)$. According to [12, 14], an immediate consequence of Theorem 4 is that the space of first order deformations of the diagram $S_\bullet$ is $H^1(\mathrm{Der}^*_{\mathbb{K}}(R_\bullet, R_\bullet)) = T^1(S_\bullet)$, and obstructions to deformations are contained in the space $H^2(\mathrm{Der}^*_{\mathbb{K}}(R_\bullet, R_\bullet)) = T^2(S_\bullet)$.

Although in principle Theorem 4 gives a complete answer to our initial problem, for diagrams over a general small category $\mathcal{D}$ it may be very difficult to concretely describe a cofibrant replacement, since projective cofibrations are described either as maps satisfying the left lifting property with respect to trivial fibrations, or as transfinite compositions of certain elementary cofibrations.

A possible strategy to overcome this difficulty is to give an explicit functor of small categories $\epsilon\colon \mathcal{N} \to \mathcal{D}$ such that:

1. for every diagram $S_\bullet \in \mathrm{Fun}(\mathcal{D}, \mathbf{Alg}_{\mathbb{K}})$, the deformation theory of $S_\bullet$ is the same as the deformation theory of $S_\bullet \circ \epsilon \in \mathrm{Fun}(\mathcal{N}, \mathbf{Alg}_{\mathbb{K}})$;
2. cofibrations in $\mathrm{Fun}(\mathcal{N}, \mathbf{CDGA}^{\le 0}_{\mathbb{K}})$ admit a constructive description.

In the next sections we follow this strategy by setting as ϵ a simplified version of the Bousfield-Kan approximation [3]. In our construction the category $\mathcal{N}$ will be in particular a Reedy category (see Sect. 8) and the projective model structure in $\mathrm{Fun}(\mathcal{N}, \mathbf{CDGA}^{\le 0}_{\mathbb{K}})$ will be the same as the Reedy model structure, hence with a simpler description of cofibrations.

7 Simplex Categories

Let $\mathbf{\Delta}$ be the category with objects the finite ordinals $[n] = \{0, 1, \cdots, n\}$ and morphisms non-decreasing maps, also known as the *simplex category*. We denote by $\delta_k\colon [n-1] \to [n]$, and by $\sigma_k\colon [n+1] \to [n]$, $k = 0, \ldots, n$, the usual face and degeneracy maps:

$$\delta_k\colon [n-1] \to [n], \qquad \delta_k(p) = \begin{cases} p & \text{if } p < k \\ p+1 & \text{if } p \ge k \end{cases}, \qquad k = 0, \ldots, n,$$

$$\sigma_k\colon [n+1] \to [n], \qquad \sigma_k(p) = \begin{cases} p & \text{if } p \le k \\ p-1 & \text{if } p > k \end{cases}, \qquad k = 0, \ldots, n,$$

They satisfy the cosimplicial identities:

$$\sigma_i\sigma_j = \sigma_j\sigma_{i+1} \quad \text{for } i \geq j$$

$$\delta_i\delta_j = \delta_{j+1}\delta_i \quad \text{for } i \leq j$$

$$\sigma_i\delta_j = \begin{cases} \delta_{j-1}\sigma_i, & \text{if } j > i+1 \\ \mathrm{Id}, & \text{if } j = i, i+1 \\ \delta_j\sigma_{i-1}, & \text{if } j < i. \end{cases}$$

We recall that a cosimplicial group is a functor $G\colon \mathbf{\Delta} \to \mathbf{Grp}$, $[n] \mapsto G_n$; in the sequel we shall need the following proposition, which is an easy generalisation of a well known result about cosimplicial groups, cf. [2, Prop. X.4.9].

Proposition 6 *Let G be a cosimplicial group, let $n \geq 1$, and $I \subseteq [n]$. Assume there are given elements $x_i \in G_n$, $i \in I$, such that $\sigma_{i-1}x_j = \sigma_j x_i$ for all $i > j$ and $i, j \in I$. Then there exists $x \in G_{n+1}$ such that $\sigma_i x = x_i$ for all $i \in I$.*

Proof Writing $I = \{i_0 < i_1 < \cdots < i_k\} \subseteq [n]$, consider the sequence $z_{i_0}, \ldots, z_{i_k}$ defined recursively by the formula:

$$z_{i_k} = \delta_{i_k}x_{i_k}, \qquad z_{i_p} = z_{i_{p+1}} \cdot (\delta_{i_p}\sigma_{i_p}z_{i_{p+1}})^{-1} \cdot (\delta_{i_p}x_{i_p}), \quad p < k\,.$$

For later use we point out that in the construction of this sequence we have only used the group homomorphisms

$$\sigma_i\colon G_{n+1} \to G_n, \qquad \delta_i\colon G_n \to G_{n+1}, \qquad i \in I\,.$$

We claim that $x = z_{i_0}$ is the required element: we show by induction on $k - p$ that that $\sigma_{i_m}z_{i_p} = x_{i_m}$ for all $m \geq p$. For $m > p$,

$$\begin{aligned}\sigma_{i_m}z_{i_p} &= (\sigma_{i_m}z_{i_{p+1}}) \cdot (\sigma_{i_m}\delta_{i_p}\sigma_{i_p}z_{i_{p+1}})^{-1} \cdot (\sigma_{i_m}\delta_{i_p}x_{i_p}) \\ &= x_{i_m} \cdot (\delta_{i_p}\sigma_{i_m-1}\sigma_{i_p}z_{i_{p+1}})^{-1} \cdot (\delta_{i_p}\sigma_{i_m-1}x_{i_p}) \\ &= x_{i_m} \cdot (\delta_{i_p}\sigma_{i_p}\sigma_{i_m}z_{i_{p+1}})^{-1} \cdot (\delta_{i_p}\sigma_{i_p}x_{i_m}) \\ &= x_{i_m} \cdot (\delta_{i_p}\sigma_{i_p}x_{i_m})^{-1} \cdot (\delta_{i_p}\sigma_{i_p}x_{i_m}) = x_{i_m}\,,\end{aligned}$$

and for $m = p$

$$\begin{aligned}\sigma_{i_p}z_{i_p} &= (\sigma_{i_p}z_{i_{p+1}}) \cdot (\sigma_{i_p}\delta_{i_p}\sigma_{i_p}z_{i_{p+1}})^{-1} \cdot (\sigma_{i_p}\delta_{i_p}x_{i_p}) \\ &= (\sigma_{i_p}z_{i_{p+1}}) \cdot (\sigma_{i_p}z_{i_{p+1}})^{-1} \cdot (x_{i_p}) = x_{i_p}\,.\end{aligned}$$

□

The simplex category $\mathbf{\Delta}$ admits the following useful generalisation. Let $\mathcal{B}$ be a small category and consider, for every $n \geq 0$, the set $N(\mathcal{B})_n$ of n-simplexes of the nerve of $\mathcal{B}$: every element of $N(\mathcal{B})_n$ is a string

$$x = [x_0 \xrightarrow{\alpha_1} x_1 \cdots x_{n-1} \xrightarrow{\alpha_n} x_n]$$

of n morphisms of $\mathcal{B}$. The *simplex category* $N(\mathcal{B})$ of $\mathcal{B}$ is defined in the following way: the set of objects is the disjoint union of $N(\mathcal{B})_n$, $n \geq 0$. Given two objects

$$x = [x_0 \xrightarrow{\alpha_1} x_1 \cdots x_{n-1} \xrightarrow{\alpha_n} x_n], \qquad y = [y_0 \xrightarrow{\beta_1} y_1 \cdots y_{m-1} \xrightarrow{\beta_m} y_m],$$

a morphism $f\colon x \to y$ is a monotone map $f\colon [n] \to [m]$ such that $y_{f(i)} = x_i$ for every $i \in [n]$, and for every $0 \leq i \leq n$ the morphism α_i is the composition of β_j, for $f(i-1) < j \leq f(i)$

$$\alpha_i\colon\; x_{i-1} = y_{f(i-1)} \xrightarrow{\beta_{f(i-1)+1}} \cdots \longrightarrow y_{f(i)-1} \xrightarrow{\beta_{f(i)}} y_{f(i)} = x_i\,.$$

Notice that the equality $f(i-1) = f(i)$ implies $x_i = x_{i-1}$ and $\alpha_i = \mathrm{Id}$. For example, if $[x \xrightarrow{\alpha} y \xrightarrow{\beta} z] \in N(\mathcal{B})_2$, then we have in the category $N(\mathcal{B})$ the following morphisms:

$$\begin{array}{ccccc}
[z] & \xrightarrow{\delta_0} & [x \xrightarrow{\beta\alpha} z] & & \\
\Big\downarrow{\scriptstyle \delta_0} & & \Big\downarrow{\scriptstyle \delta_1} & & \\
[y \xrightarrow{\beta} z] & \xrightarrow{\delta_0} & [x \xrightarrow{\alpha} y \xrightarrow{\beta} z] & \xleftarrow{\delta_2} & [x \xrightarrow{\alpha} y] \\
{\scriptstyle \sigma_0}\Big\uparrow & & & & {\scriptstyle \sigma_1}\Big\uparrow \\
[y \xrightarrow{\mathrm{Id}} y \xrightarrow{\beta} z] & & & & [x \xrightarrow{\alpha} y \xrightarrow{\mathrm{Id}} y]\,.
\end{array}$$

Notice that the simplex category of the singleton $\mathcal{B} = \{*\}$ is exactly $\mathbf{\Delta}$.

Definition 13 A morphism $f\colon x \to y$ in $N(\mathcal{B})$:

$$x \in N(\mathcal{B})_n, \quad y \in N(\mathcal{B})_m, \quad f\colon [n] \to [m],$$

is called an *anchor* if $f(n) = m$.

Definition 14 For every $k \in \mathbb{N} \cup \{+\infty\}$ we shall denote by $N(\mathcal{B})_{\leq k}$ the full subcategory of $N(\mathcal{B})$ with objects the (disjoint) union of $N(\mathcal{B})_i$ for $i \leq k$, and by

$$\mathrm{Fun}^{\bigstar}(N(\mathcal{B})_{\leq k}, \mathbf{M}) \subseteq \mathrm{Fun}(N(\mathcal{B})_{\leq k}, \mathbf{M})$$

the full subcategory of diagrams $F\colon N(\mathcal{B})_{\le k} \to \mathbf{M}$ such that $F(f)$ is an isomorphism for every anchor map f.

Definition 15 The forgetful functor $\epsilon\colon N(\mathcal{B}) \to \mathcal{B}$ is defined by setting

$$\epsilon([x_0 \xrightarrow{\alpha_1} x_1 \ \cdots \ x_{n-1} \xrightarrow{\alpha_n} x_n]) = x_n$$

on the objects. For any morphism

$$f\colon [x_0 \xrightarrow{\alpha_1} x_1 \ \cdots \ x_{n-1} \xrightarrow{\alpha_n} x_n] \to [y_0 \xrightarrow{\beta_1} y_1 \ \cdots \ y_{m-1} \xrightarrow{\beta_m} y_m],$$

we have

$$\epsilon(f) = \beta_m \circ \cdots \circ \beta_{f(n)+1}\colon \ x_n = y_{f(n)} \to y_m \,.$$

In particular, $\epsilon(f) = \mathrm{Id}$ for any anchor f. It is clear that the composition with the functor ϵ gives, for every k, a natural transformation:

$$\epsilon^*\colon \ \mathrm{Fun}(\mathcal{B}, \mathbf{M}) \to \mathrm{Fun}^{\bigstar}(N(\mathcal{B})_{\le k}, \mathbf{M}), \qquad \epsilon^*(F) = F \circ \epsilon_{|N(\mathcal{B})_{\le k}} \,. \tag{1}$$

If $k \ge 2$ we also have a natural transformation

$$\tau\colon \ \mathrm{Fun}^{\bigstar}(N(\mathcal{B})_{\le k}, \mathbf{M}) \to \mathrm{Fun}(\mathcal{B}, \mathbf{M}) \tag{2}$$

defined in the following way: given $G \in \mathrm{Fun}^{\bigstar}(N(\mathcal{B})_{\le k}, \mathbf{M})$ and an object $x \in \mathcal{B}$ we set

$$\tau(G)(x) = G([x])\,.$$

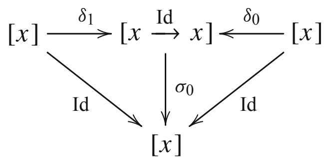
Given a morphism $x \xrightarrow{\alpha} y$ in $\mathcal{B}$ we have

$$G([x]) \xrightarrow{G(\delta_1)} G([x \xrightarrow{\alpha} y]) \xleftarrow{G(\delta_0)} G([y])$$

and, since $G(\delta_0)$ is an isomorphism we can define

$$\tau(G)(\alpha) = G(\delta_0)^{-1} G(\delta_1)\colon G([x]) \to G([y])\,.$$

We need to prove that $\tau(G)$ is a functor: applying G to the commutative diagram



we prove that $\tau(G)$ preserves the identities. Given $[x \xrightarrow{\alpha} y \xrightarrow{\beta} z] \in N(\mathcal{B})_2$, applying G to the commutative diagram

$$\begin{array}{ccccc} [z] & \xrightarrow{\delta_0} & [x \xrightarrow{\beta\alpha} z] & \xleftarrow{\delta_1} & [x] \\ \downarrow{\scriptstyle \delta_0} & & \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_1} \\ [y \xrightarrow{\beta} z] & \xrightarrow{\delta_0} & [x \xrightarrow{\alpha} y \xrightarrow{\beta} z] & \xleftarrow{\delta_2} & [x \xrightarrow{\alpha} y] \\ & \nwarrow{\scriptstyle \delta_1} & & \nearrow{\scriptstyle \delta_0} & \\ & & [y] & & \end{array}$$

we obtain $\tau(G)(\beta\alpha) = \tau(G)(\beta) \circ \tau(G)(\alpha)$. Therefore τ is properly defined and its functoriality is clear.

Proposition 7 *The above functors ϵ^* and τ are equivalences of categories.*

Proof It is immediate from the definition that $\tau \circ \epsilon^*$ is the identity. On the other hand every anchor map of type

$$[x_n] \to [x_0 \to \cdots \to x_n]$$

induces, for every $G \in \mathrm{Fun}^{\bigstar}(N(\mathcal{B})_{\leq k}, \mathbf{M})$, a canonical isomorphism

$$\epsilon^*\tau(G)([x_0 \to \cdots \to x_n]) = \tau(G)(x_n) = G([x_n]) \xrightarrow{\simeq} G([x_0 \to \cdots \to x_n])\,.$$

□

Proposition 8 *Let $S_\bullet \in \mathrm{Fun}(\mathcal{B}, \mathbf{Alg}_{\mathbb{K}})$ be a diagram of commutative algebras and $k \geq 2$. Then the isomorphism classes of deformations of $S_\bullet$ are the same as the isomorphism classes of deformations of $\epsilon^* S_\bullet \in \mathrm{Fun}(N(\mathcal{B})_{\leq k}, \mathbf{Alg}_{\mathbb{K}})$.*

Proof It is immediate from the definition that for every $A \in \mathbf{Art}_{\mathbb{K}}$ the equivalences of categories

$$\mathrm{Fun}(\mathcal{B}, \mathbf{Alg}_A) \underset{\tau}{\overset{\epsilon^*}{\rightleftarrows}} \mathrm{Fun}^{\bigstar}(N(\mathcal{B})_{\leq k}, \mathbf{Alg}_A)$$

preserve flatness. Therefore, for every deformation $\phi\colon S_{A,\bullet} \to S_\bullet$ of $S_\bullet$ the map $\epsilon^*\phi\colon \epsilon^* S_{A,\bullet} \to \epsilon^* S_\bullet$ is a deformation of the diagram $\epsilon^* S_\bullet$.

In order to conclude the proof we only need to show that, if $R_\bullet\colon N(\mathcal{B})_{\leq k} \to \mathbf{Alg}_A$ is a deformation of $\epsilon^* S_\bullet$, and $f : \alpha \to \beta$ is an anchor in $N(\mathcal{B})_{\leq k}$, then R_f is an isomorphism. This implies that $R_\bullet \in \mathrm{Fun}^{\bigstar}(N(\mathcal{B})_{\leq k}, \mathbf{Alg}_A)$. We have by definition that $\phi\colon R_\bullet \to \epsilon^* S_\bullet$ induces an isomorphism $R_\bullet \otimes_A \mathbb{K} \to \epsilon^* S_\bullet$ and, since $\epsilon^* S_\alpha \to \epsilon^* S_\beta$ is the identity, by the commutativity of the diagram

$$\begin{array}{ccc} R_\alpha \otimes_A \mathbb{K} & \longrightarrow & R_\beta \otimes_A \mathbb{K} \\ \cong\downarrow\phi & & \cong\downarrow\phi \\ \epsilon^* S_\alpha & \xrightarrow[\text{Id}]{} & \epsilon^* S_\beta \end{array}$$

we obtain that $R_\alpha \otimes_A \mathbb{K} \to R_\beta \otimes_A \mathbb{K}$ is also an isomorphism. By Lemma 2, $R_\alpha \to R_\beta$ is an isomorphism too. □

8 Reedy Model Structures

We briefly recall the notion of Reedy category, for more details see [6]. We do so in view of Theorem 5, which yields a model structure on the category $\mathrm{Fun}(\mathcal{D}, \mathbf{M})$ of diagrams on a model category $\mathbf{M}$ indexed by a Reedy category $\mathcal{D}$.

Definition 16 A Reedy category is a small category $\mathcal{D}$ together with two subcategories $\overrightarrow{\mathcal{D}}$, $\overleftarrow{\mathcal{D}}$, such that:

1. $\mathrm{Ob}(\mathcal{D}) = \mathrm{Ob}(\overrightarrow{\mathcal{D}}) = \mathrm{Ob}(\overleftarrow{\mathcal{D}})$
2. Every morphism f in $\mathcal{D}$ has a unique factorisation $f = gh$, where g is in $\overrightarrow{\mathcal{D}}$ and h is in $\overleftarrow{\mathcal{D}}$.
3. There exists a function $\deg\colon \mathrm{Ob}(\mathcal{D}) \to \mathbb{N}$ such that every non-identity morphism in $\overrightarrow{\mathcal{D}}$ raises degree and every non-identity morphism in $\overleftarrow{\mathcal{D}}$ lowers degree.

It is easy to see that in a Reedy category every isomorphism is an identity: if f is an isomorphism and $f = g_1h_1$, $f^{-1} = g_2h_2$, $h_1g_2 = g_3h_3$ are factorisations as in (2), then g_1g_3 must be the identity. A Reedy category is called direct if $\mathcal{D} = \overrightarrow{\mathcal{D}}$, or equivalently if $\overleftarrow{\mathcal{D}}$ contains only the identities.

For instance, we have the following examples of Reedy categories:

1. A category whose only morphisms are the identities is called discrete. Every discrete category is trivially a Reedy category.
2. Let I be a finite poset such that there exists a function $\deg\colon I \to \mathbb{N}$ such that $\deg(x) > \deg(y)$ for every $x > y$. Then I is a direct Reedy category.
3. The simplex category $\mathbf{\Delta}$ is a Reedy category, with $\deg([n]) = n$, $\overrightarrow{\mathbf{\Delta}}$ the injective maps and $\overleftarrow{\mathbf{\Delta}}$ the surjective maps.
4. If $\mathcal{C}$ and $\mathcal{D}$ are Reedy categories then so is the product $\mathcal{C} \times \mathcal{D}$, with $\overrightarrow{\mathcal{C} \times \mathcal{D}} = \overrightarrow{\mathcal{C}} \times \overrightarrow{\mathcal{D}}$, $\overleftarrow{\mathcal{C} \times \mathcal{D}} = \overleftarrow{\mathcal{C}} \times \overleftarrow{\mathcal{D}}$ and $\deg(c, d) = \deg(c) + \deg(d)$.

If a is an object of a category $\mathcal{D}$ we denote by $a \downarrow \mathcal{D}$ the undercategory of maps $a \to b$ in $\mathcal{D}$ and by $\mathcal{D} \downarrow a$ the overcategory of maps $b \to a$ in $\mathcal{D}$.

Definition 17 Let $\mathcal{D}$ be a Reedy category and a an object in $\mathcal{D}$.

1. The matching category $M_a\mathcal{D}$ of $\mathcal{D}$ at a is the full subcategory of $a \downarrow \overleftarrow{\mathcal{D}}$ containing all objects except the identity map of a.
2. The latching category $L_a\mathcal{D}$ of $\mathcal{D}$ at a is the full subcategory of $\overrightarrow{\mathcal{D}} \downarrow a$ containing all objects except the identity map of a.

Definition 18 Let $\mathcal{D}$ be a Reedy category, let $\mathbf{M}$ be a complete and cocomplete category, let X be a $\mathcal{D}$-diagram in $\mathbf{M}$, and a be an object in $\mathcal{D}$. For notational simplicity X also denotes the induced $M_a\mathcal{D}$-diagram, with $X_{a\to b} = X_b$, and the induced $L_a\mathcal{D}$-diagram, with $X_{b\to a} = X_b$.

1. The matching object of X at a is $M_aX = \lim_{M_a\mathcal{D}} X$.
2. The latching object of X at a is $L_aX = \operatorname{colim}_{L_a\mathcal{D}} X$.

There are natural morphisms $L_aX \longrightarrow X$ and $X \longrightarrow M_aX$.

The main use of Reedy categories originates from the following theorem: the category of diagrams in a model category indexed by a Reedy category has a model category structure.

Theorem 5 (Reedy-Kan) *Let $\mathcal{D} = (\overrightarrow{\mathcal{D}}, \overleftarrow{\mathcal{D}})$ be a Reedy category, and $\mathbf{M}$ a model category. There is a model structure on* $\mathrm{Fun}(\mathcal{D}, \mathbf{M})$ *where a map $f\colon X \longrightarrow Y$ is:*

1. *a weak equivalence iff $X_i \longrightarrow Y_i$ is a weak equivalence for all $i \in \mathcal{D}$;*
2. *a fibration iff $X_i \longrightarrow M_iX \times_{M_iY} Y_i$ is a fibration for all $i \in \mathcal{D}$;*
3. *a cofibration iff $X_i \coprod_{L_iX} L_iY \longrightarrow Y_i$ is a cofibration for all $i \in \mathcal{D}$.*

For a proof, see [6, 15.3].

We call Reedy weak equivalences, Reedy fibrations and Reedy cofibrations the weak equivalences, fibrations and cofibrations of this model structure, to avoid confusion with other model structures on the same category. For example, the commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

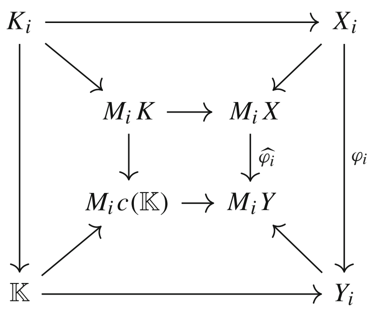
may be considered as a diagram over the Reedy poset of subsets of $\{0, 1\}$. Then it is a Reedy fibrant diagram if and only if A, B, C, D are fibrant objects; it is a Reedy cofibrant diagram if and only if A is a cofibrant object and the three maps f, g and $B \amalg_A C \to D$ are cofibrations.

We say that a map $f\colon X \to Y$ in $\mathrm{Fun}(\mathcal{D}, \mathbf{M})$ is a pointwise weak equivalence (cofibration, fibration) if $f_i : X_i \to Y_i$ is a weak equivalence (cofibration, fibration) for all $i \in \mathcal{D}$.

Lemma 10 *Let $\mathcal{D}$ be a Reedy category. Then the Reedy model structure on* $\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}_{\mathbb{K}}^{\leq 0})$ *coincides with the projective model structure if and only if every object is Reedy fibrant.*

Proof By definition the two model structures have the same weak equivalences. Let $\varphi\colon X \to Y$ be a morphism in $\mathrm{Fun}(\mathcal{D}, \mathbf{CDGA}_{\mathbb{K}}^{\leq 0})$. If φ is a Reedy fibration, then it is not difficult to prove it is a pointwise fibration ([6], 15.3.11); therefore if every pointwise fibration is a Reedy fibration then the two model structures coincide.

Assume that every object is Reedy fibrant and that $\varphi_i\colon X_i \longrightarrow Y_i$ is a fibration in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ for all $i \in \mathcal{D}$; we want to prove that φ is a Reedy fibration. We denote by $c(\mathbb{K})\colon \mathcal{D} \to \mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$ the constant diagram $i \mapsto \mathbb{K}$, and by $K = c(\mathbb{K}) \times_Y X$ the fibre product of φ and the initial morphism $c(\mathbb{K}) \to Y$. Note that $M_i c(\mathbb{K})$ is concentrated in degree 0. Since the fibre product and the matching objects are both limits, they commute by Fubini's theorem [9, Prop. 6.2.8], and we have $M_i(K) = M_i(c(\mathbb{K}) \times_Y X) \cong M_i c(\mathbb{K}) \times_{M_i Y} M_i X$. Thus we have a morphism of cartesian squares:



The map $\varphi_i\colon X_i \to Y_i$ induces $\widehat{\varphi_i}\colon M_i X \to M_i Y$; let $Y_i \times_{M_i Y} M_i X$ be the fibre product of $\widehat{\varphi_i}$ and the natural map $Y_i \to M_i Y$. We have to show that the map $X_i \to Y_i \times_{M_i Y} M_i X$ is surjective in strictly negative degree. Let $(\alpha, \beta) \in Y_i \times_{M_i Y} M_i X$, with $\overline{\alpha} = \overline{\beta} < 0$. Without loss of generality, because of the surjectivity of φ_i, we can assume $\alpha = 0$, so $\widehat{\varphi_i}(\beta) = 0$, which means β lifts to $(0, \beta) \in M_i K$, and then to K_i, since the map $K_i \to M_i K$ is by hypothesis a fibration. By the commutativity of the above diagram, we have the thesis.

Conversely, if the Reedy and projective model structures coincide, an object is Reedy fibrant if and only if it is point-wisely fibrant, and that is clearly true in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$. □

The following result is clear.

Lemma 11 *The simplex category $N(\mathcal{B})$ is a Reedy category, where the direct subcategory $\overrightarrow{N(\mathcal{B})}$ is defined by injective maps and the inverse subcategory $\overleftarrow{N(\mathcal{B})}$ by surjective maps. The same applies to its subcategories $N(\mathcal{B})_{\leq k}$.*

In particular, when $\mathcal{B} = \{*\}$ we recover the usual Reedy structure on $\mathbf{\Delta}$. By Theorem 5 we have the Reedy model structure on the category $\mathrm{Fun}(N(\mathcal{B}), \mathbf{CDGA}_{\mathbb{K}}^{\leq 0})$; we show that the Reedy and projective model structures on this category coincide, using Lemma 10.

Theorem 6 *Every object in* $\mathrm{Fun}(N(\mathcal{B})_{\leq k}, \mathbf{CDGA}_{\mathbb{K}}^{\leq 0})$ *is Reedy fibrant, and so the Reedy model structure coincides with the projective model structure.*

Proof In view of the definition of fibrations in $\mathbf{CDGA}_{\mathbb{K}}^{\leq 0}$, it is sufficient to show that for every $X \in \mathrm{Fun}(N(\mathcal{B})_{\leq k}, \mathbf{Grp})$ the map $X_\alpha \to M_\alpha X$ is surjective for all $\alpha \in N(\mathcal{B})_{\leq k}$. Fix $n \leq k$ and let $\alpha^{(m)} \in N(\mathcal{B})_n$,

$$\alpha^{(m)} = [x_0 \xrightarrow{h_1} x_1 \xrightarrow{h_2} \cdots \xrightarrow{h_n} x_n],$$

where m is the number of morphisms h_i equal to the identity, $0 \leq m \leq n$. If $m = 0$ there is nothing to prove, because the matching category of $\alpha^{(0)}$ is empty, so every $X_{\alpha^{(0)}}$ is automatically fibrant.

In case $n = m = 1$, we have

$$[a \to a] \;\underset{\delta_0,\ \delta_1}{\overset{\sigma_0}{\rightleftarrows}}\; [a]$$

with $\sigma_0\delta_0 = \sigma_0\delta_1 = \mathrm{Id}$ by the cosimplicial identities, so $X(\sigma_0)\colon X_{\alpha^{(1)}} \to M_{\alpha^{(1)}}X$ has a section and hence is surjective.

In general, assume $n \geq 2$ and $0 < m \leq n$; let $I = \{i \in \{0, \cdots n-1\} \mid h_{i+1} = \mathrm{Id}\}$, $|I| = m$. For every $i \in I$ we have a degeneracy map $\sigma_i : \alpha^{(m)} \to \alpha_i^{(m-1)}$, where the $\alpha_i^{(m-1)}$ are suitable objects in $N(\mathcal{B})_{n-1}$. From each $\alpha_i^{(m-1)}$ there are $m - 1$ degeneracy maps to other objects $\alpha_{i,j}^{(m-2)} \in N(\mathcal{B})_{n-2}$, and so on. For example, for $n = 3,\ m = 2,\ I = \{0, 2\}$:

$$\begin{array}{ccccc} & & [a \to a \to b \to b] & & \\ & \swarrow^{\sigma_0} & & \searrow^{\sigma_2} & \\ [a \to b \to b] & & & & [a \to a \to b] \\ & \searrow_{\sigma_1} & & \swarrow_{\sigma_0} & \\ & & [a \to b] & & \end{array}$$

For every map $\sigma_i : \alpha^{(m)} \to \alpha_i^{(m-1)}$ we also have two sections $\delta_i, \delta_{i+1} : \alpha_i^{(m-1)} \to \alpha^{(m)}$, so using an identical computation to Lemma 6, we have that $X_{\alpha^{(m)}}$ maps surjectively into

$$V := \{x_l \in \alpha_l^{(m-1)},\ l \in I \mid \sigma_{i-1}x_j = \sigma_j x_i \ \text{ for every } i, j \in I,\ i > j\}.$$

It is clear that the map $X_{\alpha^{(m)}} \to V$ factors through the inclusion $M_{\alpha^{(m)}}X \to V$, so $X_{\alpha^{(m)}}$ also maps surjectively into $M_{\alpha^{(m)}}X$, and we have the thesis. □

Finally, the following corollary is an immediate consequence of the above results.

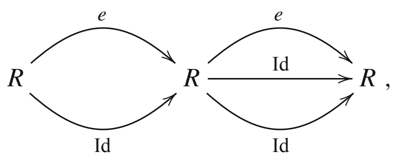
Corollary 1 *Let $\mathcal{B}$ be a small category and $S_\bullet \colon \mathcal{B} \to \mathbf{Alg}_{\mathbb{K}}$ a diagram of unitary commutative algebras. Let $\epsilon \colon N(\mathcal{B})_{\le k} \to \mathcal{B}$ be the functor defined in Definition 15 for some $k \ge 2$ and let $R_\bullet \to S_\bullet \circ \epsilon$ be a Reedy cofibrant replacement in* $\mathrm{Fun}(N(\mathcal{B})_{\le k}, \mathbf{CDGA}_{\mathbb{K}}^{\le 0})$. *Then the DG-Lie algebra* $\mathrm{Der}^*_{\mathbb{K}}(R_\bullet, R_\bullet)$ *controls the deformations of $S_\bullet$.*

An example of deformation problem which is naturally encoded by a diagram over a non-Reedy category is the case of deformations of pairs (algebra, idempotent), cf. [16, Example 5.1].

Let $e \colon R \to R$ be an idempotent morphism of an algebra $R \in \mathbf{Alg}_{\mathbb{K}}$, then the deformations of the pair (R, e) can be interpreted as the deformations of the diagram

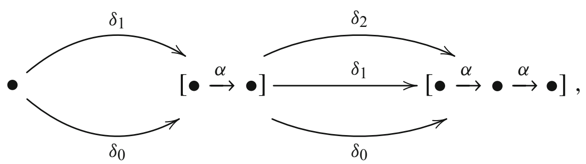
$R_\bullet:$ 

over the (non-Reedy) category $\mathcal{B}$ that has one object $\bullet$ and two morphisms $\{\mathrm{Id}, \alpha\}$, with $\alpha^2 = \alpha$. By Proposition 8, the diagrams $R_\bullet$ and $\epsilon^* R_\bullet \in \mathrm{Fun}(N(\mathcal{B})_{\le 2}, \mathbf{Alg}_{\mathbb{K}})$ have the same deformation theory. Moreover, since $\epsilon^* R_\bullet \in \mathrm{Fun}^\bigstar(N(\mathcal{B})_{\le 2}, \mathbf{Alg}_{\mathbb{K}})$, it is easy to see that the diagram $\epsilon^* R_\bullet$ has the same deformation theory of the diagram



$$R \in \mathbf{M}, \quad e^2 = e\,.$$

of algebras over the following (direct Reedy) subcategory of $N(\mathcal{B})_{\le 2}$:



$$\delta_0^2 = \delta_1\delta_0\,, \quad \delta_0\delta_1 = \delta_2\delta_0\,, \quad \delta_1^2 = \delta_2\delta_1\,.$$

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The Lefschetz Principle in Birational Geometry: Birational Twin Varieties

César Lozano Huerta and Alex Massarenti

Abstract Inspired by the Weak Lefschetz Principle, we study when a smooth projective variety fully determines the birational geometry of some of its subvarieties. In particular, we consider the natural embedding of the space of complete quadrics into the space of complete collineations and we observe that their birational geometry, from the point of view of Mori theory, fully determines each other. When two varieties are related in this way, we call them birational twins. We explore this notion and its various flavors for other embeddings between Mori dream spaces.

Keywords Mori dream spaces · Cox rings · Spherical varieties · Complete collineations and quadrics

2010 Mathematics Subject Classification Primary 14E30; Secondary 14J45, 14N05, 14E07, 14M27

1 Introduction

The celebrated Lefschetz hyperplane theorem asserts that if there is a smooth complex projective variety Y with the inclusion of a smooth ample hypersurface $i : X \to Y$, then the ambient variety Y often fully determines many of the topological invariants of X. Such invariants include homology, cohomology (compatible with Hodge structures) and homotopy groups among others. This phenomenon leads to the question: what are some algebro-geometric invariants to which a result of this

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type applies? In other words, we are asking if the Weak Lefschetz Principle holds beyond topological invariants.

The purpose of this paper is to investigate the Weak Lefschetz Principle in the context of birational geometry. In other words, if we consider two varieties X, Y as well as an embedding between them $i : X \to Y$, then we ask about the birational invariants X which are determined by those of Y. Such invariants may include effective and ample cones, or finer information as we now explain.

Let us recall that a normal $\mathbb{Q}$-factorial projective variety X with finitely generated Picard group is a Mori dream space if and only if its Cox ring is finitely generated [11, Proposition 2.9]. The birational geometry of these varieties is encoded in this ring and we may ask whether the Weak Lefschetz Principle applies to it. This question has recently been studied in the case of hypersurfaces [10, 12, 20, 22], and for Mori dream spaces of Picard rank two [13]. In addition to considering the Cox ring, let us recall that the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$ of a projective variety X with irregularity zero, such as a Mori dream space, can be decomposed into chambers depending on the stable base locus of the corresponding linear series. This decomposition is called *stable base locus decomposition* and in general it is coarser than the so-called *Mori chamber decomposition*; which encodes the isomorphism type of the birational models of X. Mori dream spaces and these decompositions have been widely studied by many authors recently, for instance see [1, 7, 17, 18] and the references therein.

We may also ask whether the previous two decompositions satisfy the Weak Lefschetz Principle. We then summarize the previous discussion in the following definition. Let us denote by $\mathrm{SBLD}(X)$ the stable base locus decomposition of $\overline{\mathrm{Eff}}(X)$.

Definition 1.1 Let X, Y be $\mathbb{Q}$-factorial projective varieties and $i : X \to Y$ be an embedding. These varieties are said *Lefschetz divisorially equivalent* if the pull-back $i^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ induces an isomorphism such that

$$i^* \mathrm{Eff}(Y) = \mathrm{Eff}(X), \quad i^* \mathrm{Mov}(Y) = \mathrm{Mov}(X), \quad i^* \mathrm{Nef}(Y) = \mathrm{Nef}(X)$$

Furthermore, X and Y are said *strongly Lefschetz divisorially equivalent* if they are Lefschetz divisorially equivalent and in addition $i^* \mathrm{SBLD}(Y) = \mathrm{SBLD}(X)$, with $h^1(X, \mathcal{O}_X) = h^1(Y, \mathcal{O}_Y) = 0$.

It is an immediate consequence of the Lefschetz hyperplane theorem that a smooth variety Y, of Picard rank one and dimension at least four, is Lefschetz divisorially equivalent to any of its smooth ample divisors. If the assumption that the Picard rank is one is removed in this case, B. Hassett et al. [10] exhibited an example of a variety and an ample divisor $D \subset Y$ whose Picard groups are isomorphic by the Lefschetz hyperplane theorem, but such that the nef cone is not preserved; hence D and Y are not Lefschetz divisorially equivalent.

The previous definition is not restricted to subvarieties of codimension one. In fact, the results in this paper involve classical spaces and subvarieties of high codimension, such that the birational geometry of both objects can now be linked using Definition 1.1. Let us recall those spaces so we can state our results precisely.

Let V, W be two K-vector spaces of dimensions $n+1$ and $m+1$, respectively, with $n \leqslant m$. The field K is algebraically closed of characteristic zero. We will denote by $\mathcal{X}(n,m)$ the space of complete collineations $V \to W$ and by $\mathcal{Q}(n)$ the space of complete $(n-1)$-dimensional quadrics of V. In [23, 24], I. Vainsencher showed that these spaces can be understood as sequences of blow-ups along the subvariety parametrizing rank one matrices and the strict transforms of its secant varieties. Such blowups are central characters of this paper, so let us be more precise about them. Given an irreducible and reduced non-degenerate variety $X \subset \mathbb{P}^N$, and a positive integer $h \leqslant N$, the *h-secant variety* $\mathbb{S}ec_h(X)$ of X is the subvariety of $\mathbb{P}^N$ obtained as the closure of the union of all $(h-1)$-planes spanned by h general points of X. Spaces of matrices and symmetric matrices admit a natural stratification dictated by the rank and observe that a general point of the h-secant variety of a Segre, or a Veronese, corresponds to a matrix of rank h. More precisely, let $\mathbb{P}^N$ be the projective space parametrizing $(n+1)\times(n+1)$ matrices modulo scalars, $\mathbb{P}^{N_+}$ the subspace of symmetric matrices, $\mathcal{S} \subset \mathbb{P}^N$ the Segre variety, and $\mathcal{V} \subset \mathbb{P}^{N_+}$ the Veronese variety. Set $\mathcal{X}(n) := \mathcal{X}(n,n)$ and denote by $\mathcal{X}(n)_i$, $\mathcal{Q}(n)_i$ respectively the varieties obtained at the step i in Vainsencher's constructions 3.4; which roughly says that $\mathcal{X}(n)_i$, $\mathcal{Q}(n)_i$ are the blow-ups respectively of $\mathbb{P}^N$ and $\mathbb{P}^{N_+}$ along all the j-secant varieties of $\mathcal{S}$ and $\mathcal{V}$ for $j \leq i$. Since $\mathbb{S}ec_h(\mathcal{V}) = \mathbb{S}ec_h(\mathcal{S}) \cap \mathbb{P}^{N_+}$, the natural inclusion $\mathbb{P}^{N_+} \hookrightarrow \mathbb{P}^N$ lifts to an embedding $\mathcal{Q}(n)_i \hookrightarrow \mathcal{X}(n)_i$ for any i.

These families of spaces $\mathcal{X}(n)_i$ and $\mathcal{Q}(n)_i$ are very different in may aspects, for instance their dimensions are different. However, when it comes to certain birational invariants they are indistinguishable. Indeed, by Proposition 3.12 and Lemma 3.18, we have our first result.

Theorem A *The varieties $\mathcal{X}(n)_i$ and $\mathcal{Q}(n)_i$ are Lefschetz divisorially equivalent via the usual embedding for any i. In particular, the spaces of complete collineations $\mathcal{X}(n)$ and quadrics $\mathcal{Q}(n)$ are Lefschetz divisorially equivalent via the usual embedding.*

A more thorough version of the Weak Lefschetz Principle in our context would be the following: a small modification of the ambient variety Y restricts to a small modification, possibly an isomorphism, of the subvariety X, and all the small modifications of X can be obtained in this way. This is stronger than determining invariants and it is the content of the following definition. We will test this definition on some Mori dream spaces. The Mori chamber decomposition of $\mathrm{Eff}(X)$, of a Mori dream space X, will be denoted by $\mathrm{MCD}(X)$.

Definition 1.2 Let X, Y be as in the previous definition, we say that they are *birational twins* if they are Lefschetz divisorially equivalent Mori dream spaces and in addition $i^*\,\mathrm{MCD}(Y) = \mathrm{MCD}(X)$. Also, they are said to be *strong birational twins* if they are birational twins and in addition $i^*\,\mathrm{SBLD}(Y) = \mathrm{SBLD}(X)$.

Our next result shows that the relationship between the birational geometry of complete collineations and complete quadrics, in some cases, goes deeper than Theorem A. The following result follows from Theorem 3.20 and Corollary 3.22.

Theorem B *The varieties* $\mathcal{X}(n)_3$ *and* $\mathcal{Q}(n)_3$ *are birational twins for any* $n \geqslant 1$*. Furthermore,* $\mathcal{X}(3)$ *and* $\mathcal{Q}(3)$ *are strong birational twins.*

In general, the spaces $\mathcal{X}(n)_i$, $\mathcal{Q}(n)_i$ are Mori dream spaces for any i; we can actually count the number of chambers in the Mori chamber decomposition in various cases. Furthermore, the ample, movable and effective cones are preserved by Theorem A. However, we do not know if the structure of the Mori chambers is also preserved. In other words, if Theorem B holds in general.

Question 1.3 Are $\mathcal{X}(n)_i$ and $\mathcal{Q}(n)_i$ birational twins for $n \geqslant 4$ and any $i \geq 4$?

We organize this paper as follows. In Sect. 2, we discuss Definitions 1.1 and 1.2 and the relations among the various flavors of these definitions. In particular, Theorem 2.6 provides examples of Mori dream spaces that are strongly Lefschetz divisorially equivalent but not birational twins, and of Mori dream spaces that are birational twins but not strong birational twins. In Sect. 3, we introduce the spaces of complete collineations and quadrics, and we study their birational geometry from a Mori theoretic viewpoint. Finally, in Sect. 4 we recall the modular description of the unique flip of $\mathcal{Q}(3)$ given in [14, Section 5.2], and based on it, we give a conjectural description of the unique flip of $\mathcal{X}(3)$.

2 Lefschetz Divisorially Equivalent Varieties and Birational Twins

This section contains preliminaries and explores various aspects of Definitions 1.1 and 1.2. In particular, Theorem 2.6 and Proposition 2.9 are the main results of this section.

Let X be a normal projective variety over an algebraically closed field of characteristic zero. We denote by $N^1(X)$ the real vector space of $\mathbb{R}$-Cartier divisors modulo numerical equivalence. The *nef cone* of X is the closed convex cone $\mathrm{Nef}(X) \subset N^1(X)$ generated by classes of nef divisors. The *movable cone* of X is the convex cone $\mathrm{Mov}(X) \subset N^1(X)$ generated by classes of *movable divisors*; these are Cartier divisors whose stable base locus has codimension at least two in X. The *effective cone* of X is the convex cone $\mathrm{Eff}(X) \subset N^1(X)$ generated by classes of *effective divisors*. We have inclusions $\mathrm{Nef}(X) \subset \overline{\mathrm{Mov}(X)} \subset \overline{\mathrm{Eff}(X)}$.

We will denote by $N_1(X)$ the real vector space of numerical equivalence classes of 1-cycles on X. The closure of the cone in $N_1(X)$ generated by the classes of irreducible curves in X is called the *Mori cone* of X, we will denote it by $\mathrm{NE}(X)$.

A class $[C] \in N_1(X)$ is called *moving* if the curves in X of class $[C]$ cover a dense open subset of X. The closure of the cone in $N_1(X)$ generated by classes of moving curves in X is called the *moving cone* of X and we will denote it by $\mathrm{mov}(X)$. We refer to [9, Chapter 1] for a comprehensive treatment of these topics.

We say that a birational map $f : X \dashrightarrow X'$ to a normal projective variety X' is a *birational contraction* if its inverse does not contract any divisor. We say that it is a *small $\mathbb{Q}$-factorial modification* if X' is $\mathbb{Q}$-factorial and f is an isomorphism in codimension one. If $f : X \dashrightarrow X'$ is a small $\mathbb{Q}$-factorial modification, then the natural pull-back map $f^* : N^1(X') \to N^1(X)$ sends $\mathrm{Mov}(X')$ and $\mathrm{Eff}(X')$ isomorphically onto $\mathrm{Mov}(X)$ and $\mathrm{Eff}(X)$, respectively. In particular, we have $f^*(\mathrm{Nef}(X')) \subset \overline{\mathrm{Mov}(X)}$.

Definition 2.1 A normal projective $\mathbb{Q}$-factorial variety X is called a *Mori dream space* if the following conditions hold:

- $\mathrm{Pic}\,(X)$ is finitely generated, or equivalently $h^1(X, \mathcal{O}_X) = 0$,
- $\mathrm{Nef}\,(X)$ is generated by the classes of finitely many semi-ample divisors,
- there is a finite collection of small $\mathbb{Q}$-factorial modifications $f_i : X \dashrightarrow X_i$, such that each X_i satisfies the second condition above, and $\mathrm{Mov}\,(X) = \bigcup_i f_i^*(\mathrm{Nef}\,(X_i))$.

By [3, Corollary 1.3.2] smooth Fano varieties are Mori dream spaces. In fact, there is a larger class of varieties called log Fano varieties which are Mori dream spaces as well. By work of M. Brion [5] we have that $\mathbb{Q}$-factorial spherical varieties are Mori dream spaces. An alternative proof of this result can be found in [21, Section 4].

The collection of all faces of all cones $f_i^*(\mathrm{Nef}\,(X_i))$ in Definition 2.1 forms a fan which is supported on $\mathrm{Mov}(X)$. If two maximal cones of this fan, say $f_i^*(\mathrm{Nef}\,(X_i))$ and $f_j^*(\mathrm{Nef}\,(X_j))$, meet along a facet, then there exist a normal projective variety Y, a small modification $\varphi : X_i \dashrightarrow X_j$, and $h_i : X_i \to Y$ and $h_j : X_j \to Y$ small birational morphism of relative Picard number one such that $h_j \circ \varphi = h_i$. The fan structure on $\mathrm{Mov}(X)$ can be extended to a fan supported on $\mathrm{Eff}(X)$ as follows.

Definition 2.2 Let X be a Mori dream space. We describe a fan structure on the effective cone $\mathrm{Eff}(X)$, called the *Mori chamber decomposition*. We refer to [11, Proposition 1.11] and [19, Section 2.2] for details. There are finitely many birational contractions from X to Mori dream spaces, denoted by $g_i : X \dashrightarrow Y_i$. The set $\mathrm{Exc}(g_i)$ of exceptional prime divisors of g_i has cardinality $\rho(X/Y_i) = \rho(X) - \rho(Y_i)$. The maximal cones $\mathcal{C}$ of the Mori chamber decomposition of $\mathrm{Eff}(X)$ are of the form: $\mathcal{C}_i = \left\langle g_i^*\big(\mathrm{Nef}(Y_i)\big), \mathrm{Exc}(g_i)\right\rangle$. We call $\mathcal{C}_i$ or its interior $\mathcal{C}_i^\circ$ a *maximal chamber* of $\mathrm{Eff}(X)$.

Definition 2.3 Let X be a normal projective variety with finitely generated divisor class group $\mathrm{Cl}(X) := \mathrm{WDiv}(X)/\mathrm{PDiv}(X)$, in particular $h^1(X, \mathcal{O}_X) = 0$. The *Cox sheaf* and *Cox ring* of X are defined as

$$\mathcal{R} := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{O}_X(D) \qquad\qquad \mathrm{Cox}(X) := \Gamma(X, \mathcal{R})$$

Recall that $\mathcal{R}$ is a sheaf of $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebras, whose multiplication maps are discussed in [2, Section 1.4]. In case the divisor class group is torsion-free, one can just take the direct sum over a subgroup of $\mathrm{WDiv}(X)$, isomorphic to $\mathrm{Cl}(X)$ via the quotient map, getting immediately a sheaf of $\mathcal{O}_X$-algebras. Denote by $\widehat{X}$ the relative spectrum of $\mathcal{R}$ and by $\overline{X}$ the spectrum of $\mathrm{Cox}(X)$. The $\mathrm{Cl}(X)$-grading induces an action of the quasitorus $H_X := \mathrm{Spec}\,\mathbb{C}[\mathrm{Cl}(X)]$ on both spaces. The inclusion $\mathcal{O}_X \to \mathcal{R}$ induces a good quotient $p_X \colon \widehat{X} \to X$ with respect to this action. Summarizing, we have the following diagram

$$\begin{array}{ccc} \widehat{X} & \subseteq & \overline{X} \\ {\scriptstyle p_X}\downarrow & & \\ X & & \end{array}$$

and we will refer to it as the *Cox construction* of X. In case $\mathrm{Cox}(X)$ is a finitely generated algebra, the complement of $\widehat{X}$ in the affine variety $\overline{X}$ has codimension $\geqslant 2$. This subvariety is the *irrelevant locus* and its defining ideal is the *irrelevant ideal* $\mathcal{J}_{\mathrm{irr}}(X) \subseteq \mathrm{Cox}(X)$.

Let X be a normal $\mathbb{Q}$-factorial projective variety, and let D be an effective $\mathbb{Q}$-divisor on X. The stable base locus $\mathbf{B}(D)$ of D is the set-theoretic intersection of the base loci of the complete linear systems $|sD|$ for all positive integers s such that sD is integral. In other words,

$$\mathbf{B}(D) = \bigcap_{s>0} B(sD)$$

Since stable base loci do not behave well with respect to numerical equivalence, we will assume that $h^1(X, \mathcal{O}_X) = 0$. This implies that linear and numerical equivalence of $\mathbb{Q}$-divisors coincide.

Since numerically equivalent $\mathbb{Q}$-divisors on X have the same stable base locus, then $\overline{\mathrm{Eff}}(X)$ the pseudo-effective cone of X can be decomposed into chambers depending on the stable base locus of the corresponding linear series. This decomposition is called *stable base locus decomposition*.

If X is a Mori dream space, satisfying then the condition $h^1(X, \mathcal{O}_X) = 0$, determining the stable base locus decomposition of $\mathrm{Eff}(X)$ is a first step in order to compute its Mori chamber decomposition.

Remark 2.4 Recall that two divisors D_1, D_2 are said to be *Mori equivalent* if $\mathbf{B}(D_1) = \mathbf{B}(D_2)$ and the following diagram of rational maps is commutative

$$\begin{array}{ccc} & X & \\ {\scriptstyle \phi_{D_1}}\swarrow & & \searrow{\scriptstyle \phi_{D_2}} \\ X(D_1) & \xrightarrow[\sim]{} & X(D_2) \end{array}$$

where the horizontal arrow is an isomorphism. Therefore, the Mori chamber decomposition is a refinement of the stable base locus decomposition.

Let X be a Mori dream space with Cox ring $\mathrm{Cox}(X)$ and grading matrix Q. The matrix Q defines a surjection

$$Q\colon E \to \mathrm{Cl}(X)$$

from a free, finitely generated, abelian group E to the divisor class group of X. Denote by γ the positive quadrant of $E_{\mathbb{Q}} := E \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $e_1, \dots, e_r$ be the canonical basis of $E_{\mathbb{Q}}$. Given a face $\gamma_0 \preceq \gamma$, we say that $i \in \{1, \dots, r\}$ is a cone index of γ_0 if $e_i \in \gamma_0$. The face γ_0 is an $\mathfrak{F}$*-face* if there exists a point of $\overline{X} = \mathrm{Spec}(\mathrm{Cox}(X))$ whose i-th coordinate is non-zero exactly when i is a cone index of γ_0 [2, Construction 3.3.1.1]. The set of these points is denoted by $\overline{X}(\gamma_0)$. Given the Cox construction of X we denote by $X(\gamma_0) \subseteq X$ the image of $\overline{X}(\gamma_0)$, and given an $\mathfrak{F}$-face γ_0 its image $Q(\gamma_0) \subseteq \mathrm{Cl}(X)_{\mathbb{Q}}$ is an *orbit cone* of X. The set of all orbit cones of X is denoted by Ω. Accordingly to [2, Definition 3.1.2.6] an effective class $w \in \mathrm{Cl}(X)$ defines the *GIT chamber*

$$\lambda(w) := \bigcap_{\{\omega \in \Omega \,:\, w \in \omega\}} \omega \tag{2.5}$$

If w is an ample class of X, then the corresponding GIT chamber is the semi-ample cone of X. The variety X can be reconstructed from the pair $(\mathrm{Cox}(X), \Phi)$ formed by the Cox ring together with a *bunch of cones*, consisting of certain subsets of the orbit cones [2, Definition 3.1.3.2].

We now explore different aspects of Definitions 1.1 and 1.2. We exhibit two varieties which are birational twins but fail to be strong birational twins. We do this inspired by [13, Theorem 1.1.]. We also exhibit two Mori dream spaces which are strongly Lefschetz divisorially equivalent but fail to be birational twins.

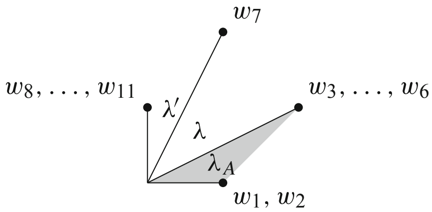
Theorem 2.6 *Let Z be the toric variety with Cox ring $K[T_1, \dots, T_{11}]$ whose grading matrix and irrelevant ideal are the following*

$$Q = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad \mathcal{J}_{\mathrm{irr}}(Z) = \langle T_1, T_2 \rangle \cap \langle T_3, \dots, T_{11} \rangle$$

and let $F = T_3T_8 + T_4T_9 + T_1T_7$, $G = T_5T_9 + T_6T_{11} + T_2T_7$. Then the ring $\frac{K[T_1,\dots,T_{11}]}{(F,G)}$ is the Cox ring of a projective normal $\mathbb{Q}$-factorial Mori dream space $X \subset Z$ of Picard rank two. The varieties X and Z are birational twins but fail to be strong birational twins.

Furthermore, the ring $\frac{K[T_1,\dots,T_{11}]}{(F,G,T_7+T_1T_{11}^2+T_2T_8T_{11}+T_2T_{10}^2+T_2T_8T_9)}$ is the Cox ring of a projective normal $\mathbb{Q}$-factorial Mori dream space $Y \subset X$ of Picard rank two. The varieties Y and X are strongly Lefschetz divisorially equivalent but fail to be birational twins.

Proof Let us denote by $w_i \in \mathbb{Z}^2$ the degree of T_i. We will denote by $V(T_{i_1}, \dots, T_{i_r}) := \{T_{i_1} = \dots = T_{i_r} = 0\}$. The following picture represents the degrees of the generators of the Cox ring



where $\mathrm{Eff}(Z) = \mathrm{Mov}(Z) = \lambda_A \cup \lambda \cup \lambda'$ and $\mathrm{Nef}(Z) = \lambda_A$. Let $\overline{Z} = K^{11}$ be the spectrum of the Cox ring of Z and let $\overline{X}$ be the affine subvariety defined by $\overline{X} = \{F = G = 0\} \subset \overline{Z}$.

A standard computation shows that $\overline{X}$ is irreducible and $\mathrm{codim}_{\overline{X}}(\mathrm{Sing}(\overline{X})) \geqslant 2$, and then Serre's criterion on normality yields that $\overline{X}$ is normal. Let $p_Z : \widehat{Z} \to Z$ be the characteristic space morphism of Z, and let $\widehat{X} := \overline{X} \cap \widehat{Z}$. Note that the irrelevant locus with respect to the chamber λ_A has two components given by $\Gamma_1 = \{T_1 = T_2 = 0\}$ and $\Gamma_2 = \{T_3 = \dots = T_{11} = 0\}$. Hence $\widehat{Z} = K^{11} \setminus \{\Gamma_1 \cup \Gamma_2\}$.

The image of $\widehat{X}$ via p_Z is a subvariety X of Z, let $i : X \to Z$ be the inclusion. Since $\overline{X}$ is irreducible and normal, and X is a GIT quotient of $\overline{X}$ by a reductive group [6, Theorem 1.24 (vi)] yields that X is irreducible and normal as well.

Let Z' be the smooth locus of Z. Note that Z' contains the open subset Z'' of Z obtained by removing the union of all the toric subvarieties of the form $p_Z(V(T_i, T_j))$. Since the Zariski closure of $p_Z^{-1}(X \cap Z'')$ in $\widehat{Z}$ is $\widehat{X}$ we conclude that $\widehat{X}$ is the Zariski closure of $p_Z^{-1}(X \cap Z')$ in $\widehat{Z}$.

Note that, according to the grading matrix Q, the $K^* \times K^*$ action on $\overline{X}$ is given by

$$\begin{array}{ccc} (K^* \times K^*) \times \overline{X} & \longrightarrow & \overline{X} \\ ((\lambda, \mu), (T_1, \dots, T_{11})) & \longmapsto & (\lambda T_1, \lambda T_2, \lambda^2 \mu T_3, \dots, \lambda \mu^2 T_7, \mu T_8, \dots, \mu T_{11}) \end{array}$$

Therefore, if $T_2 \neq 0, T_8 \neq 0$ we may set $\lambda = \mu = 1$, and write $T_3 = -\frac{T_4 T_9 + T_1 T_7}{T_8}$, $T_7 = -\frac{T_5 T_9 + T_6 T_{11}}{T_2}$. Therefore, if we remove the images of $V(T_2) \cup V(T_8)$ from X, the resulting variety is isomorphic to an affine space. Therefore, $\mathrm{Cl}(X)$ is generated by the classes of the images of the two irreducible divisors $V(T_i) \cap \overline{X}$, with $i \in \{2, 8\}$, and $\rho(X) \leqslant 2$. Moreover, crossing the wall corresponding to w_1, w_2 we get a morphism $f : Z \to \mathbb{P}^1$. Furthermore, $X \subset Z$ is not contained in any fiber of f, and hence f restricts to a surjective morphism $f_{|X} : X \to \mathbb{P}^1$. This forces $\rho(X) \geqslant 2$. Finally, we conclude that the images of $V(T_2), V(T_8)$ form a basis of $\mathrm{Cl}(X)$. So $\rho(X) = 2$ and the pull-back $i^* : \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$, induced by the inclusion, is an isomorphism. Now, note that $\Gamma_2 \subset \overline{X}$ has codimension greater that two in $\overline{X}$, and $\Gamma_1 \cap \overline{X} = \{T_1 = T_2 = T_3 T_8 + T_4 T_9 =$

$T_5T_9 + T_6T_{11} = 0\}$ has codimension two in $\overline{X}$. Hence $\operatorname{codim}_{\overline{X}}(\overline{X} \setminus \widehat{X}) \geqslant 2$ and [2, Corollary 4.1.1.5] yields that the Cox ring of X is

$$\operatorname{Cox}(X) = \frac{K[T_1, \dots, T_{11}]}{I(\overline{X})}$$

In general, if X is a Mori dream space such that $\operatorname{Cl}(X)$ has rank two we can fix a total order on the classes in the effective cone $w \leqslant w'$ if w is on the left of w'. Given two convex cones λ, λ' contained in the effective cone we will write $\lambda \leqslant \lambda'$ if $w \leqslant w'$ for any $w \in \lambda$ and $w' \in \lambda$. Note that

$$V(f_i : w_i \geqslant \lambda_A) = \{T_1 = T_2 = \widetilde{F} = \widetilde{G} = 0\}$$

where $\widetilde{F} = T_3T_8 + T_4T_9$, $\widetilde{G} = T_5T_9 + T_6T_{11}$. Furthermore

$$V(f_i : w_i \leqslant \lambda) = \{T_7 = T_8 = T_9 = T_{10} = T_{11} = 0\};$$
$$V(f_i : w_i \leqslant \lambda') = \{T_1 = T_2 = T_8 = T_9 = T_{10} = T_{11} = 0\} \cup \{T_7 = T_8 = T_9 = T_{10} = T_{11} = 0\}.$$

So $V(f_i : w_i \geqslant \lambda_A) \supset \{T_1 = T_2 = T_8 = T_9 = T_{10} = T_{11} = 0\}$ contains a component of the set $V(f_i : w_i \leqslant \lambda')$. Finally, [13, Lemmas 4.3, 4.4, 4.5] yields that $\operatorname{MCD}(X) = i^*\operatorname{MCD}(Z)$, and that $\operatorname{SBLD}(X)$ has just two chambers while $\operatorname{SBLD}(Z) = \operatorname{MCD}(Z)$ has three chambers.

Let $\overline{Y} \subset \overline{Z} = K^{11}$ be the subvariety cut out by

$$\overline{Y} = \{F = G = T_7 + T_1T_{11}^2 + T_2T_8T_{11} + T_2T_{10}^2 + T_2T_8T_9 = 0\}$$

We have that $\overline{Y}$ is irreducible of dimension eight and $\operatorname{codim}_{\overline{Y}}(\operatorname{Sing}(\overline{Y})) \geqslant 2$, so Serre's criterion on normality yields that $\overline{Y}$ is normal, and [6, Theorem 1.24 (vi)] yields that the GIT quotient Y is irreducible and normal as well.

If $T_2 \neq 0, T_8 \neq 0$ we may write $T_3 = -\frac{T_4T_9+T_1T_7}{T_8}$, $T_7 = -\frac{T_5T_9+T_6T_{11}}{T_2}$ and $T_9 = -\frac{T_2T_{10}^2+T_2T_8T_{11}+T_1T_{11}^2+T_7}{T_2T_8}$. Therefore, $\overline{Y} \setminus (V(T_2) \cup V(T_8))$ is isomorphic to an affine space and $\operatorname{Cl}(Y)$ is generated by the classes of the images of the two irreducible divisors $V(T_i) \cap \overline{Y}$, with $i \in \{2, 8\}$, and $\rho(Y) \leqslant 2$. Again crossing the wall corresponding to w_1, w_2 we get a morphism $f : Z \to \mathbb{P}^1$ and since $Y \subset Z$ is not contained in any fiber of f this morphism restricts to a surjective morphism $f_{|Y} : Y \to \mathbb{P}^1$. This forces $\rho(Y) \geqslant 2$. We conclude that the images of $V(T_2)$, $V(T_8)$ form a basis of $\operatorname{Cl}(Y)$. So $\rho(Y) = 2$ and the pull-back $j^* : \operatorname{Cl}(X) \to \operatorname{Cl}(Y)$, induced by the inclusion $j : Y \to X$, is an isomorphism. As before $\Gamma_2 \subset \overline{Y}$ has codimension greater that two in $\overline{Y}$, and $\Gamma_1 \cap \overline{Y}$ has codimension two in $\overline{Y}$. Hence [2, Corollary 4.1.1.5] yields that the Cox ring of Y is

$$\operatorname{Cox}(Y) = \frac{K[T_1, \dots, T_{11}]}{I(\overline{Y})}$$

Now, for the Mori dream space Y we have

$$V(f_i : w_i \leqslant \lambda) = V(f_i : w_i \leqslant \lambda') = \{T_7 = T_8 = T_9 = T_{10} = T_{11} = 0\};$$
$$V(f_i : w_i \geqslant \lambda) = V(f_i : w_i \geqslant \lambda') = \{T_1 = T_2 = T_3 = T_4 = T_5 = T_6 = T_7 = 0\}.$$

Hence [13, Lemma 4.3] yields that $\mathrm{MCD}(Y)$ has two chambers while $\mathrm{MCD}(X) = i^* \mathrm{MCD}(Z)$ has three chambers. □

Remark 2.7 Note that any pair of varieties among the Mori dream spaces $Y \subset X \subset Z$ in Theorem 2.6 gives a pair of Lefschetz divisorially equivalent varieties. On the other hand, $i^* \mathrm{SBLD}(Z) \neq \mathrm{SBLD}(X)$ and $(i \circ j)^* \mathrm{MCD}(Z) \neq \mathrm{MCD}(Y)$. Therefore, Lefschetz divisorially equivalent does not imply strongly Lefschetz divisorially equivalent nor birational twins.

However, when a Mori dream space inherits the stable base locus decomposition from an ambient toric variety something more can be said.

Remark 2.8 Given a Mori dream space X, then there is an embedding $i : X \to \mathcal{T}_X$ into a simplicial projective toric variety $\mathcal{T}_X$ such that $i^* : \mathrm{Pic}(\mathcal{T}_X) \to \mathrm{Pic}(X)$ is an isomorphism which induces an isomorphism $\mathrm{Eff}(\mathcal{T}_X) \to \mathrm{Eff}(X)$, [11, Proposition 2.11]. Furthermore, the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{T}_X)$ is a refinement of the Mori chamber decomposition of $\mathrm{Eff}(X)$. Indeed, if $\mathrm{Cox}(X) \cong \frac{K[T_1,\dots,T_s]}{I}$ where the T_i are homogeneous generators with non-trivial effective $\mathrm{Pic}(X)$-degrees then $\mathrm{Cox}(\mathcal{T}_X) \cong K[T_1, \dots, T_s]$.

Proposition 2.9 *Let $X \to \mathcal{T}_X$ be a Mori dream space embedded in its canonical toric embedding. Assume that both X and $\mathcal{T}_X$ have Picard rank two. If X and $\mathcal{T}_X$ are strongly Lefschetz divisorially equivalent then they are strong birational twins.*

Proof By Remark 2.8 $\mathrm{MCD}(\mathcal{T}_X)$ refines $\mathrm{MCD}(X)$, and moreover $\mathrm{MCD}(X)$ refines $\mathrm{SBLD}(X)$. By our hypothesis $\mathrm{SBLD}(X) = i^* \mathrm{SBLD}(\mathcal{T}_X)$, where $i : X \to \mathcal{T}_X$ is the inclusion. Now, since $\mathcal{T}_X$ is a toric projective variety of Picard rank two [13, Theorem 1.2] yields $\mathrm{SBLD}(\mathcal{T}_X) = \mathrm{MCD}(\mathcal{T}_X)$. Summing up, $\mathrm{MCD}(\mathcal{T}_X) = \mathrm{SBLD}(\mathcal{T}_X)$ refines $\mathrm{MCD}(X)$ which in turn refines $\mathrm{SBLD}(X) = \mathrm{SBLD}(\mathcal{T}_X) = \mathrm{MCD}(\mathcal{T}_X)$, and this forces $\mathrm{MCD}(X) = i^* \mathrm{MCD}(\mathcal{T}_X)$. □

3 Birational Twin Varieties: Complete Collineations and Quadrics

This section is the core of the paper. It contains the definitions and crucial properties of the spaces of complete collineations and complete quadrics. It also contains the details of the proofs of Theorems A and B listed in the introduction.

Let V and W be K-vector spaces of dimension $n + 1$ and $m + 1$, respectively with $n \leqslant m$. Let $\mathbb{P}^N$ denote the projective space whose points parametrize non-zero linear maps $W \to V$ up to a scalar multiple. We recall that a point in this spaces is called a collineation from W to V.

Observe that a full-rank collineation $p \in \mathrm{Hom}(W, V)$ induces another collineation $\wedge^k W \to \wedge^k V$, for any $k \le n+1$, by taking the wedge product. Consider the rational map

$$\begin{array}{cccc} \phi : \mathbb{P}(\mathrm{Hom}(W, V)) & \dashrightarrow & \mathbb{P}(\mathrm{Hom}(\bigwedge^2 W, \bigwedge^2 V)) \times \cdots \times \mathbb{P}(\mathrm{Hom}(\bigwedge^{n+1} W, \bigwedge^{n+1} V)) \\ p & \longmapsto & (\wedge^2 p, \dots, \wedge^{n+1} p) \end{array} \tag{3.1}$$

Definition 3.2 The *space of complete collineations* $\mathcal{X}(n, m)$ from W to V is defined as the closure of the graph of the rational map (3.1).

In order to understand the closure of the image of ϕ, we now recall an alternative description of the space $\mathcal{X}(n, m)$.

Given an irreducible, reduced and non-degenerate variety $X \subset \mathbb{P}^N$, for any $h < N$, we can consider the *h-secant variety* of X, which is defined as the closure of the union of all $(h-1)$-planes spanned by h general points in X. We denote this variety by $\mathbb{S}ec_h(X)$. Observe that any point $p \in \mathbb{P}^N = \mathbb{P}(\mathrm{Hom}(W, V))$ can be represented by a $(n+1)\times(m+1)$ matrix Z, and that the locus of matrices of rank one is the Segre variety $\mathcal{S} \subset \mathbb{P}^N$. More generally, any $p \in \mathbb{S}ec_h(S)$ can be represented by a matrix Z which is a linear combination of h matrices of rank one, and conversely. In other words, a point p is contained in $\mathbb{S}ec_h(S)$ if and only if the rank of its representing matrix is at most h.

Setting up notation, we will think of the point $p = [z_{0,0} : \cdots : z_{n,m}] \in \mathbb{P}^N$ as the matrix

$$Z = \begin{pmatrix} z_{0,0} & \cdots & z_{0,m} \\ \vdots & \ddots & \vdots \\ z_{n,0} & \cdots & z_{n,m} \end{pmatrix} \tag{3.3}$$

We can now describe the space $\mathcal{X}(n, m)$ as a sequence of blow-ups as follows.

Construction 3.4 Let us consider the following sequence of blow-ups:

- $\mathcal{X}(n, m)_1$ is the blow-up of $\mathcal{X}(n, m)_0 := \mathbb{P}^N$ along the Segre variety $\mathcal{S}$;
- $\mathcal{X}(n, m)_2$ is the blow-up of $\mathcal{X}(n, m)_1$ along the strict transform of $\mathbb{S}ec_2(\mathcal{S})$;

  $\vdots$
- $\mathcal{X}(n, m)_i$ is the blow-up of $\mathcal{X}(n, m)_{i-1}$ along the strict transform of $\mathbb{S}ec_i(\mathcal{S})$;

  $\vdots$
- $\mathcal{X}(n, m)_n$ is the blow-up of $\mathcal{X}(n, m)_{n-1}$ along the strict transform of $\mathbb{S}ec_n(\mathcal{S})$.

It follows from [24] that the variety $\mathcal{X}(n, m)_n$ is isomorphic to $\mathcal{X}(n, m)$, the space of complete collineations from W to V. Furthermore, we have that for any $i = 1, \dots, n$ the variety $\mathcal{X}(n, m)_i$ is smooth, the strict transform of $\mathbb{S}ec_i(\mathcal{S})$ in $\mathcal{X}(n, m)_{i-1}$ is smooth, and the divisor $E_1 \cup E_2 \cup \cdots \cup E_{i-1}$ in $\mathcal{X}(n, m)_{i-1}$ is simple normal crossing. When $n = m$ we will write $\mathcal{X}(n)$ for $\mathcal{X}(n, n)$.

Setting up notation, let $f_i : \mathcal{X}(n,m)_i \to \mathcal{X}(n,m)_{i-1}$ be the blow-up morphism and $E_i = \mathrm{Exc}(f_i)$ its exceptional divisor. By abusing notation, we will also denote by E_i the strict transform in the subsequent blow-ups. Let H denote the pull-back to $\mathcal{X}(n,m)_i$ of the hyperplane section of $\mathbb{P}^N$. We will denote by $f : \mathcal{X}(n,m) \to \mathbb{P}^N$ the composition of the f_i's.

If $n = m$, then the last blow-up in Construction 3.4 is the blow-up of a Cartier divisor and hence $\mathcal{X}(n)_n \cong \mathcal{X}(n)_{n-1}$. Furthermore, we have a distinguished linear subspace in $\mathbb{P}^N = \mathbb{P}(\mathrm{Hom}(V,V))$. Indeed, the space of symmetric matrices is defined by $\mathbb{P}^{N_+} := \mathbb{P}(\mathrm{Sym}^2 V) = \{z_{i,j} = z_{j,i} \,|\, i \neq j\}$, where $N_+ = \binom{n+2}{2} - 1$. Furthermore, the space $\mathbb{P}^{N_+}$ cuts out scheme-theoretically on $\mathcal{S}$ the Veronese variety $\mathcal{V} \subseteq \mathbb{P}^{N_+}$; which parametrizes $(n+1)\times(n+1)$ symmetric matrices of rank one. More generally, the space $\mathbb{P}^{N_+}$ cuts out scheme-theoretically on $\mathbb{S}ec_h(\mathcal{S})$ the h-secant variety $\mathbb{S}ec_h(\mathcal{V})$.

Observe that a full-rank symmetric matrix $q \in \mathrm{Sym}^2(V)$ induces another symmetric matrix $\wedge^k q$, for any $k \leq n+1$, by taking the wedge product. There is a rational map

$$\begin{array}{cccc} \rho : \mathbb{P}(\mathrm{Sym}^2 V) & \dashrightarrow & \mathbb{P}(\mathrm{Sym}^2 \bigwedge^2 V) \times \cdots \times \mathbb{P}(\mathrm{Sym}^2 \bigwedge^n V) \\ q & \longmapsto & (\wedge^2 q, \ldots, \wedge^n q) \end{array} \tag{3.5}$$

Definition 3.6 The *space of complete quadrics* in $\mathbb{P}^n$ is defined as the closure of the graph of the map (3.5). We denote this space by $\mathcal{Q}(n)$.

One can apply Construction 3.4 in the symmetric setting to the spaces $\mathbb{P}^{N_+}$. In doing so, we obtain the following blow-up construction for the space of complete quadrics [23, Theorem 6.3].

Construction 3.7 Let us consider the following sequence of blow-ups:

- $\mathcal{Q}(n)_1$ is the blow-up of $\mathcal{Q}(n)_0 := \mathbb{P}^{N_+}$ along the Veronese variety $\mathcal{V}$;
- $\mathcal{Q}(n)_2$ is the blow-up of $\mathcal{Q}(n)_1$ along the strict transform of $\mathbb{S}ec_2(\mathcal{V})$;

  $\vdots$
- $\mathcal{Q}(n)_i$ is the blow-up of $\mathcal{Q}(n)_{i-1}$ along the strict transform of $\mathbb{S}ec_i(\mathcal{V})$;

  $\vdots$
- $\mathcal{Q}(n)_n$ is the blow-up of $\mathcal{Q}(n)_{n-1}$ along the strict transform of $\mathbb{S}ec_n(\mathcal{V})$.

It follows from [23] that the variety $\mathcal{Q}(n)_{n-1} \cong \mathcal{Q}(n)_n$ is isomorphic to the space of complete $(n-1)$-dimensional quadrics $\mathcal{Q}(n)$. Furthermore, the variety $\mathcal{Q}(n)_i$ is smooth, the strict transform of $\mathbb{S}ec_i(\mathcal{V})$ in $\mathcal{Q}(n)_{i-1}$ is smooth, and the divisor $E_1^+ \cup E_2^+ \cup \cdots \cup E_{i-1}^+$ in $\mathcal{Q}(n)_{i-1}$ is simple normal crossing for any $i = 1, \ldots, n$.

Setting up notation, let $f_i^+ : \mathcal{Q}(n)_i \to \mathcal{Q}(n)_{i-1}$ be the blow-up morphism and $E_i^+ = \mathrm{Exc}(f_i^+)$ its exceptional divisor. We will also denote by E_i^+ the strict transforms of this divisor in the subsequent blow-ups, if confusion does not arise. Let H^+ stand for the pull-back to $\mathcal{Q}(n)_i$ of the hyperplane section of $\mathbb{P}^{N_+}$. We will denote by $f^+ : \mathcal{Q}(n) \to \mathbb{P}^{N_+}$ the composition of the f_i^+'s.

3.1 Birational Geometry of the Intermediate Spaces

In this section we will investigate the birational geometry of the intermediate blow-ups $\mathcal{X}(n)_i$, $\mathcal{Q}(n)_i$ appearing in Constructions 3.4 and 3.7. We begin by recalling the notion of spherical variety.

Definition 3.8 A *spherical variety* is a normal variety X together with an action of a connected reductive affine algebraic group $\mathscr{G}$, a Borel subgroup $\mathscr{B} \subset \mathscr{G}$, and a base point $x_0 \in X$ such that the $\mathscr{B}$-orbit of x_0 in X is a dense open subset of X.

Let $(X, \mathscr{G}, \mathscr{B}, x_0)$ be a spherical variety. We distinguish two types of $\mathscr{B}$-invariant prime divisors: a *boundary divisor* of X is a $\mathscr{G}$-invariant prime divisor on X, a *color* of X is a $\mathscr{B}$-invariant prime divisor that is not $\mathscr{G}$-invariant. We will denote by $\mathcal{B}(X)$ and $\mathcal{C}(X)$ respectively the set of boundary divisors and colors of X.

Notation 3.9 For any $i = 1, \ldots, n+1$ let us denote by D_i the strict transform of the divisor in $\mathbb{P}^N$ given by

$$\det \begin{pmatrix} z_{n-i+1,m-i+1} & \cdots & z_{n-i+1,m} \\ \vdots & \ddots & \vdots \\ z_{n,m-i+1} & \cdots & z_{n,m} \end{pmatrix} = 0 \tag{3.10}$$

in the intermediate spaces of complete collineations. Similarly, we will denote by D_i^+ the analogous divisors in the intermediate spaces of complete quadrics.

Lemma 3.11 *Let $\mathcal{X}(n,m)_i$ be the intermediate space appearing at the step $1 \leqslant i \leqslant n-1$ of Construction 3.4. Then $\mathcal{X}(n,m)_i$ is spherical. Furthermore,* $\mathrm{Pic}(\mathcal{X}(n,m)_i) = \mathbb{Z}[H, E_1, \ldots, E_i]$ *and*

$$\mathcal{B}(\mathcal{X}(n,m)_i) = \begin{cases} \{E_1, \ldots, E_i\} & \text{if } n < m \\ \{E_1, \ldots, E_i, D_{n+1}\} & \text{if } n = m \end{cases} \qquad \mathcal{C}(\mathcal{X}(n,m)_i) = \begin{cases} \{D_1, \ldots, D_{n+1}\} & \text{if } n < m \\ \{D_1, \ldots, D_n\} & \text{if } n = m \end{cases}$$

Finally, for the intermediate spaces $\mathcal{Q}(n)_i$ in Construction 3.7 we have

$$\mathcal{B}(\mathcal{Q}(n)_i) = \{E_1^+, \ldots, E_i^+, D_{n+1}^+\}, \quad \mathcal{C}(\mathcal{Q}(n)_i) = \{D_1^+, \ldots, D_n^+\}.$$

Proof Note that $\mathcal{X}(n,m)_i$ is spherical, even though it is not wonderful, with respect to the action of $\mathscr{G} = SL(n+1) \times SL(m+1)$, with the Borel subgroup $\mathscr{B}$ given by pair of upper triangular matrices. Indeed, by [24, Theorem 1] $\mathcal{X}(n,m)$ is a wonderful variety and $\mathscr{B}$ acts on it with a dense open orbit, and since the actions of $\mathscr{B}$ on $\mathcal{X}(n,m)$ and $\mathcal{X}(n,m)_i$ are compatible with the blow-up maps in Construction 3.4 we conclude that $\mathscr{B}$ acts on $\mathcal{X}(n,m)_i$ with a dense open orbit as well. For the definition and the basic properties of wonderful varieties we refer to [2, Section 4.5.5].

Since the actions of $\mathscr{G}$ on $\mathcal{X}(n,m)$ and $\mathcal{X}(n,m)_i$ are compatible with the blow-up morphism $\mathcal{X}(n,m) \to \mathcal{X}(n,m)_i$, the intermediate space $\mathcal{X}(n,m)_i$ has at most the same number of boundary divisors and colors as $\mathcal{X}(n,m)$. Now, to compute $\mathcal{B}(\mathcal{X}(n,m)_i)$ and $\mathcal{C}(\mathcal{X}(n,m)_i)$ it is enough to note that the ones listed in the statement are clearly boundary divisors and colors, and to apply [15, Proposition 3.6]. □

Proposition 3.12 *Let $n < m$. Then $D_1 \sim H$ and*

$$D_k \sim kH - \sum_{h=1}^{k-1}(k-h)E_h \tag{3.13}$$

for $k = 2, \dots, n+1$. Furthermore, $\mathrm{Eff}(\mathcal{X}(n,m)_i) = \langle E_1, \dots, E_i, D_{n+1}\rangle$ *and* $\mathrm{Nef}(\mathcal{X}(n,m)_i) = \langle D_1, \dots, D_{i+1}\rangle$.

If $n = m$ we have $D_1 \sim H$, $D_1^+ \sim H^+$ and

$$D_k \sim kH - \sum_{h=1}^{k-1}(k-h)E_h, \quad D_k^+ \sim kH^+ - \sum_{h=1}^{k-1}(k-h)E_h^+ \tag{3.14}$$

for $k = 2, \dots, n$, and for the boundary divisors E_n and E_n^+ we have

$$E_n \sim (n+1)H - \sum_{h=1}^{n-1}(n-h+1)E_h, \quad E_n^+ \sim (n+1)H^+ - \sum_{h=1}^{n-1}(n-h+1)E_h^+. \tag{3.15}$$

Furthermore, $\mathrm{Eff}(\mathcal{X}(n)_i) = \langle E_1, \dots, E_i, D_{n+1}\rangle$, $\mathrm{Nef}(\mathcal{X}(n)_i) = \langle D_1, \dots, D_{i+1}\rangle$ *and similarly* $\mathrm{Eff}(\mathcal{Q}(n)_i) = \left\langle E_1^+, \dots, E_i^+, D_{n+1}^+\right\rangle$, $\mathrm{Nef}(\mathcal{Q}(n)_i) = \left\langle D_1^+, \dots, D_{i+1}^+\right\rangle$ *for $i = 1, \dots, n-1$.*

Proof Consider the case $n < m$, a similar argument works for the case $n = m$. Let $f : \mathcal{X}(n,m) \to \mathbb{P}^N$ be the blow-up morphism in Construction 3.4. We have

$$D_k = \deg(f_*D_k)H - \sum_{h=1}^{i}\mathrm{mult}_{\mathbb{S}ec_h(\mathcal{S})}(f_*D_k)E_h$$

Now, since f_*D_k is the zero locus of a $k \times k$ minor of the matrix Z then $\deg(f_*D_k) = k$. Furthermore, [15, Lemma 3.10] yields $\mathrm{mult}_{\mathbb{S}ec_h(\mathcal{S})}(f_*D_k) = k - h$ if $h = 1, \dots k-1$ and $\mathrm{mult}_{\mathbb{S}ec_h(\mathcal{S})}(f_*D_k) = 0$ if $h \geqslant k$, and hence we get (3.13).

By [2, Proposition 4.5.4.4] and [4, Section 2.6] the generators of the effective and the nef cones are respectively generated by the boundary divisors and the colors. Lemma 3.11 then concludes the proof. □

3.1.1 Generators of $\mathrm{Cox}(\mathcal{X}(n,m)_i)$

Let $I = \{i_0, \dots, i_k\}$, $J = \{j_0, \dots, j_k\}$ be two ordered sets of indexes with $0 \leqslant i_0 \leqslant \cdots \leqslant i_k \leqslant n$ and $0 \leqslant j_0 \leqslant \cdots \leqslant j_k \leqslant m$, and denote by $Z_{I,J}$ the $(k+1) \times (k+1)$ minor of Z built with the rows indexed by I and the columns indexed by J. Similarly, let $Z^+_{I,J}$ be the $(k+1) \times (k+1)$ minor of the symmetrization of Z built with the rows indexed by I and the columns indexed by J.

Proposition 3.16 *Let $T_{I,J}$ be the canonical section associated to the strict transform of the hypersurface $\{\det(Z_{I,J}) = 0\} \subset \mathbb{P}^N$, and let S_j be the canonical section associated to the exceptional divisor E_j in Construction 3.4. Then $\mathrm{Cox}(\mathcal{X}(n,m))_i$ is generated by the $T_{I,J}$ for $1 \leqslant |I|, |J| \leqslant n+1$ and the S_j for $j = 1, \dots, i$.*

Now, consider $\mathcal{Q}(n)_i$. Let $T^+_{I,J}$ be the canonical section associated to the strict transform of the hypersurface $\{\det(Z^+_{I,J}) = 0\} \subset \mathbb{P}^{N_+}$, and let S^+_j be the canonical section associated to the exceptional divisor E^+_j in Construction 3.7. Then $\mathrm{Cox}(\mathcal{Q}(n)_i)$ is generated by the $T^+_{I,J}$ for $1 \leqslant |I|, |J| \leqslant n+1$ and the S^+_j for $j = 1, \dots, i$.

Proof By [2, Theorem 4.5.4.6] if $\mathscr{G}$ is a semi-simple and simply connected algebraic group and $(X, \mathscr{G}, \mathscr{B}, x_0)$ is a spherical variety with boundary divisors $E_1, \dots, E_r$ and colors $D_1, \dots, D_s$, then $\mathrm{Cox}(X)$ is generated as a K-algebra by the canonical sections of the E_i's and the finite dimensional vector subspaces $\mathrm{lin}_K(\mathscr{G} \cdot D_i) \subseteq \mathrm{Cox}(X)$ for $1 \leqslant i \leqslant s$.

Consider the case $n < m$. For $\mathcal{X}(n)_i$ and $\mathcal{Q}(n)_i$ an analogous argument works. By Lemma 3.11 we have that $\mathcal{B}(\mathcal{X}(n,m)_i) = \{E_1, \dots, E_i\}$ and $\mathcal{C}(\mathcal{X}(n,m)_i) = \{D_1, \dots, D_{n+1}\}$. Recall that for any $k = 0, \dots, n$ the divisor D_{k+1} is the strict transform of the hypersurface in $\mathbb{P}^N$ defined by the determinant of the $(k+1) \times (k+1)$ most right down minor of the matrix Z. Let us denote by $Z^{rd}_{I,J}$ such minor.

Note that $\det(Z^{rd}_{I,J}) \in I(\mathbb{S}ec_k(\mathcal{S}))$. Therefore, considering the action of $\mathscr{G} = SL(n+1) \times SL(m+1)$ we have that $g \cdot \det(Z^{rd}_{I,J}) \in I(\mathbb{S}ec_k(\mathcal{S}))$ for any $g \in \mathscr{G}$, and hence $\mathrm{lin}_K(\mathscr{G} \cdot \det(Z^{rd}_{I,J})) \subseteq I(\mathbb{S}ec_k(\mathcal{S}))$. To conclude it is enough to recall that $I(\mathbb{S}ec_k(\mathcal{S}))$ is generated by the $(k+1) \times (k+1)$ minors of Z. □

3.2 The Mori Chamber Decomposition of the Intermediate Space of Picard Rank Three

In this section, using the techniques introduced in [15, 16], we will study the Mori chamber and stable base locus decomposition for the spaces $\mathcal{X}(n)_3$ in Construction 3.4.

Notation 3.17 We will denote by $\langle v_1, \dots, v_s\rangle$ the cone in $\mathbb{R}^n$ generated by the vectors $v_1, \dots, v_s \in \mathbb{R}^n$. Given two vectors v_i, v_j we set $(v_i, v_j] := \langle v_i, v_j\rangle \setminus \mathbb{R}_{>0}v_i$ and $(v_i, v_j) := \langle v_i, v_j\rangle \setminus (\mathbb{R}_{>0}v_i \cup \mathbb{R}_{>0}v_j)$.

Lemma 3.18 *Consider the natural inclusion* $i : \mathcal{Q}(n)_i \to \mathcal{X}(n)_i$. *Then* $i^*\operatorname{Mov}(\mathcal{X}(n)_i) = \operatorname{Mov}(\mathcal{Q}(n)_i)$.

Proof By Proposition 3.16 $\operatorname{Cox}(\mathcal{Q}(n)_i)$ is generated by the pull-backs of the generators of $\operatorname{Cox}(\mathcal{X}(n)_i)$. Hence the statement follows from [2, Proposition 3.3.2.3]. □

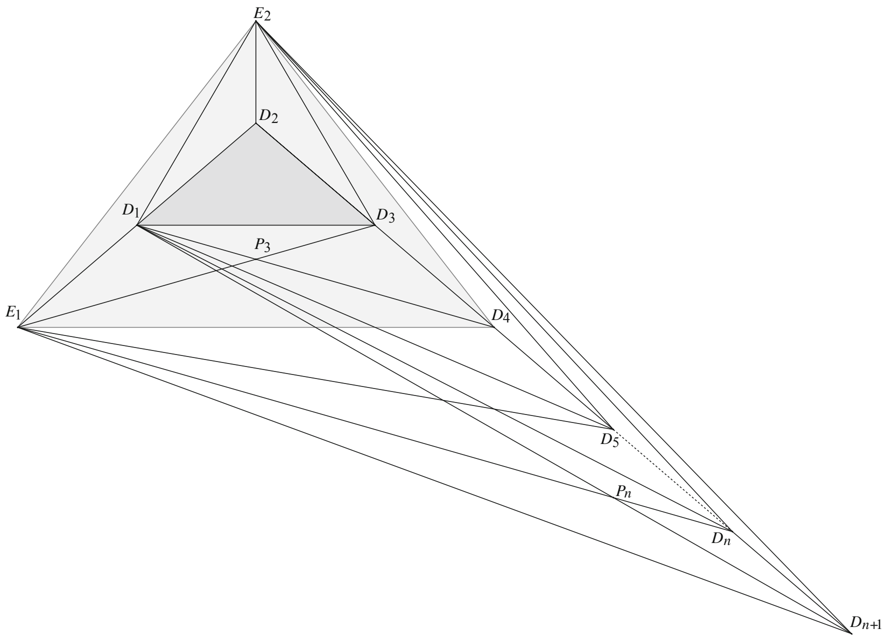
Proposition 3.19 *The movable cone of* $\mathcal{X}(n)_3$ *is generated by* $D_1 \sim H, D_2 \sim 2H - E_1, D_n \sim nH + (1-n)E_1 + (2-n)E_2, P_n \sim n(n-1)H + n(2-n)E_1 + (n-1)(2-n)E_2$.

Similarly, $\operatorname{Mov}(\mathcal{Q}(n)_3)$ *is generated by* $D_1^+ \sim H^+, D_2^+ \sim 2H^+ - E_1^+, D_n^+ \sim nH^+ + (1-n)E_1^+ + (2-n)E_2^+, P_n^+ \sim n(n-1)H^+ + n(2-n)E_1^+ + (n-1)(2-n)E_2^+$.

Proof We use [2, Proposition 3.3.2.3] which gives a way of computing the movable cone starting from the generators of the Cox ring. More specifically, in order to compute the movable cone it is enough to intersect all the cones generated by the sets of vectors obtained by removing just one element from the set of generators of the Cox ring.

In our specific case, using the generators of $\operatorname{Cox}(\mathcal{X}(n)_3)$ in Proposition 3.16, it is immediate to see that this procedure give us the vectors $(1, 0, 0)$, $(2, -1, 0)$, $(n, 1-n, 2-n)$ and the vector $(n(n-1), n(2-n), (n-1)(2-n))$ that is the intersection of the plane generated by $(1, 0, 0)$, $(n+1, -n, -n+1)$ and the plane generated by $(0, 1, 0)$, $(n, 1-n, 2-n)$. Finally, the statement on $\operatorname{Mov}(\mathcal{Q}(n)_3)$ follows from Lemma 3.18. □

Theorem 3.20 *Let $f(n)$ be the number of chambers in the Mori chamber decomposition of $\mathcal{X}(n)_3$. Then $f(n+1) = f(n) + n + 1$. Furthermore, the decomposition can be described by the following 2-dimensional cross-section of* $\operatorname{Eff}(\mathcal{X}(n)_3)$



Furthermore, the same statements hold, by replacing the relevant divisors with their pull-backs via the embedding $i : \mathcal{Q}(n)_3 \to \mathcal{X}(n)_3$, *for the intermediate space of quadrics* $\mathcal{Q}(n)_3$.

Proof By [15, Theorem 6.11] the statement holds for $n = 3$. Consider the case $n = 4$. Then the Cox ring of $\mathcal{X}(4)_3$ has additional generators belonging to the class of D_5.

Now, let $\mathcal{T}_{\mathcal{X}(4)_3}$ be a projective toric variety as in Remark 2.8. Then there is an embedding $i : \mathcal{X}(4)_3 \to \mathcal{T}_{\mathcal{X}(4)_3}$ such that $i^* : \mathrm{Pic}(\mathcal{T}_{\mathcal{X}(4)_3}) \to \mathrm{Pic}(\mathcal{X}(4)_3)$ is an isomorphism inducing an isomorphism $\mathrm{Eff}(\mathcal{T}_{\mathcal{X}(4)_3}) \to \mathrm{Eff}(\mathcal{X}(4)_3)$.

Since $\mathcal{T}_{\mathcal{X}(4)_3}$ is toric, the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{T}_{\mathcal{X}(4)_3})$ can be computed by means of the Gelfand-Kapranov-Zelevinsky decomposition [2, Section 2.2.2]. Roughly speaking such decomposition is given by all the convex cones we can construct using all the generators of $\mathrm{Cox}(\mathcal{T}_{\mathcal{X}(4)_3})$. Starting from the decomposition for $\mathcal{T}_{\mathcal{X}(4)_3}$ we are allowed to add just two more segments, namely $[E_2, D_5], [E_1, D_5]$ which introduce a new chamber each, and $[D_1, D_5]$ which gives rise two three new chambers by dividing $\langle D_1, D_3, E_1\rangle, \langle E_1, D_3, D_4\rangle, \langle E_1, D_4, D_5\rangle$. Note that we get exactly $f(4) = f(3) + 4 + 1 = 9 + 4 + 1 = 14$ chambers. On the other hand, a priori this is just a refinement of the Mori chamber decomposition of $\mathrm{Eff}(\mathcal{X}(4)_3)$. We will prove that it is indeed the Mori chamber decomposition.

Note that by considering a general $\mathbb{P}^3 \times \mathbb{P}^3 \subset \mathbb{P}^4 \times \mathbb{P}^4$ we get a natural embedding $j : \mathcal{X}(3)_3 \to \mathcal{X}(4)_3$. Furthermore, this embedding preserves the generators of the Picard groups in Lemma 3.11 and the generators of the Cox rings in Proposition 3.16. Therefore, all the chambers of $\mathrm{MCD}(\mathcal{X}(3)_3)$ appear in $\mathrm{MCD}(\mathcal{X}(4)_3)$. Furthermore, the segments $[E_2, D_5], [E_1, D_5]$ must appear in $\mathrm{MCD}(\mathcal{X}(4)_3)$ simply because by Proposition 3.12 they define two external walls of $\mathrm{Eff}(\mathcal{X}(4)_3)$, and the segment $[D_1, D_5]$ must appear as well since it is needed to intercept the vector P_4 which by Proposition 3.19 is an extremal ray of $\mathrm{Mov}(\mathcal{X}(4)_3)$.

Now, assuming that the statement holds for $\mathcal{X}(n-1)_3$ we can prove it for $\mathcal{X}(n)_3$ arguing exactly as we did in order to pass from $\mathcal{X}(3)_3$ to $\mathcal{X}(4)_3$. Indeed, arguing exactly as in the previous part of the proof we see that all the chambers of $\mathrm{MCD}(\mathcal{X}(n-1)_3)$ must appear in $\mathrm{MCD}(\mathcal{X}(n)_3)$. Moreover, we have to add the segments $[E_1, D_{n+1}], [E_2, D_{n+1}]$ which by Proposition 3.12 give external walls of $\mathrm{Eff}(\mathcal{X}(n)_3)$, and $[D_1, D_{n+1}]$ needed to intercept the vectors P_n which by Proposition 3.19 is an extremal ray of $\mathrm{Mov}(\mathcal{X}(n)_3)$.

Finally, note that $n-2$ of the chambers in $\mathcal{X}(n-1)_3$ are subdivided in two chambers each by $[D_1, D_{n+1}]$, the new cone $\langle E_1, D_n, D_{n+1}\rangle$ also is subdivided in two chambers each by $[D_1, D_{n+1}]$, and we must add also the chamber $\langle E_2, D_n, D_{n+1}\rangle$. Hence the number of chambers of $\mathrm{Eff}(\mathcal{X}(n)_3)$ is given by $f(n) = f(n-1) + n - 2 + 2 + 1 = f(n-1) + n + 1$. □

Remark 3.21 By [15, Theorem 6.11] $\mathcal{X}(3)_3$ has nine Mori chambers but just eight stable base locus chambers. Indeed, within the non-convex chamber determined by the divisors D_1, D_2, D_3, E_2 the stable base locus is E_2. Note that this last fact holds true more generally for the space $\mathcal{X}(n)_3$ for $n \geqslant 3$. Similarly, this holds also for the spaces of quadrics $\mathcal{Q}(n)_3$.

Corollary 3.22 *The varieties $\mathcal{X}(n)_i$ and $\mathcal{Q}(n)_i$ are Lefschetz divisorially equivalent, and moreover $\mathcal{X}(n)_3$ and $\mathcal{Q}(n)_3$ are birational twins for any $n \geqslant 1$. Furthermore, $\mathcal{X}(3)_3$ and $\mathcal{Q}(3)_3$ are strong birational twins.*

Proof The first statement follows from Proposition 3.12 and Lemma 3.18. The second statement follows then from Theorem 3.20. Finally, the third statement is a consequence of Remark 3.21. □

4 The Flip of the Spaces of Complete Quadric Surfaces

Theorem 3.20 and Corollary 3.22 imply that each of the spaces $\mathcal{X}(3)$, $\mathcal{Q}(3)$ of complete collineations of $\mathbb{P}^3$ and complete quadric surfaces, admits a single flip. Moreover, such flips are strongly related to each other. Indeed, by computing the Mori chamber decomposition in both cases, we observe that $i^*\mathrm{MCD}(\mathcal{X}(3)) = \mathrm{MCD}(\mathcal{Q}(3))$, thus these two spaces are birational twins and the unique flip on $\mathcal{X}(3)$ induces that of $\mathcal{Q}(3)$.

In this section we exhibit the flip of $\mathcal{Q}(3)$, denoted by $\mathcal{Q}(3)^+$, the space of complete quadric surfaces following [14]. Towards the end of the section, we conjecture the geometry of the flip of $\mathcal{X}(3)$. We construct $\mathcal{Q}(3)^+$ by analyzing a $\mathbb{Z}/2$-action on the following Hilbert scheme.

Proposition 4.1 *Let* **Hilb** $=$ $\mathbf{Hilb}^{2x+1}(\mathbb{G}(1,3))$ *denote the Hilbert scheme parametrizing subschemes of* $\mathbb{G}(1,3) \subset \mathbb{P}^5$ *whose Hilbert polynomial is* $P(x) = 2x+1$. *This space is isomorphic to the following blow-up*

$$\mathbf{Hilb} \cong Bl_{\mathbb{OG}}\mathbb{G}(2,5)$$

where $\mathbb{OG} \subset \mathbb{G}(2,5)$ *denotes the orthogonal Grassmannian inside the Grassmannian of* 2*-planes in* $\mathbb{P}^5$.

Proof Observe that a generic smooth curve with Hilbert polynomial $P(x) = 2x+1$ in $\mathbb{P}^5$ is a plane conic C. Thus, its ideal $I_C \subset k[p_0, \ldots, p_5]$ is generated by a quadric F and three independent linear forms L_1, L_2, L_3. Since $C \subset \mathbb{G} = \mathbb{G}(1,3)$, the equation F is the quadric corresponding to the Grassmannian $\mathbb{G} \subset \mathbb{P}^5$ under the Plücker embedding. This description gives rise to a rational map

$$f : \mathbb{G}(2,5) \dashrightarrow \mathbf{Hilb}$$

by assigning the 2-plane P defined by the independent linear forms (L_1, L_2, L_3) to the ideal $\langle L_1, L_2, L_3\rangle + \langle F\rangle \subset k[p_0, \ldots, p_5]$. Observe that the exceptional locus of f consists of planes in $\mathbb{P}^5$ such that there is a containment of ideals $\langle F\rangle \subset \langle L_1, L_2, L_3\rangle$, that is planes P which are contained in the quadric $\mathbb{G} \subset \mathbb{P}^5$. The locus parametrizing such planes is exactly the orthogonal Grassmannian $\mathbb{OG}$. Now, we resolve the rational map f,

$$\begin{array}{ccc} Bl_{\mathbb{OG}}\mathbb{G}(2,5) & & \\ \downarrow \pi & \overset{\tilde{f}}{\searrow} & \\ \mathbb{G}(2,5) & \overset{f}{\dashrightarrow} & \mathbf{Hilb} \end{array}$$

The morphism $\tilde{f}$ is an isomorphism. Indeed, the rational map f is birational as it has an inverse morphism $g : \mathbf{Hilb} \to \mathbb{G}(2,5)$ defined as follows: let $[C] \in \mathbf{Hilb}$ be a point, then the ideal $I(C) = (F) + (\text{plane}) \overset{g}{\mapsto} (\text{plane}) \in \mathbb{G}(2,5)$. It is clear that $f \circ g = Id$, hence f, and consequently $\tilde{f}$, is birational. Furthermore, $\tilde{f}$ is a bijection. Indeed, since the exceptional divisor $E \subset Bl_{\mathbb{OG}}\mathbb{G}(2,5)$ is a $\mathbb{P}^5$-bundle over $\mathbb{OG}$, whose points can be thought of as pairs (P, C), where $P \subset \mathbb{P}^5$ is a 2-plane and $C \subset P$ is a plane conic. Thus, Zariski's Main Theorem implies that $\tilde{f}$ is an isomorphism. □

Corollary 4.2 *Let* **Hilb** *be as above, then* Pic(**Hilb**) $\cong \langle H^+, E_2^+, E_{1,1}^+\rangle$ *where* H^+ *is the pullback of the unique generator of the group* $A^1(\mathbb{G}(2,5))$ *and the* E^+*'s are the exceptional divisors of the blow-up.*

Proof The orthogonal Grassmannian $\mathbb{OG}$ has two components, hence the result follows. □

If the base field K is algebraically closed, then for a given smooth quadric $Q \subset \mathbb{P}^3$, the Fano variety of lines $F_1(Q) \subset \mathbb{G}(1,3)$ consists of two smooth conics. We get a $\mathbb{Z}/2$-action on $\mathbf{Hilb}^{2x+1}(\mathbb{G}(1,3))$ by exchanging such conics.

Lemma 4.3 *There is a nontrivial globally defined $\mathbb{Z}/2$-action on* $\mathbf{Hilb}^{2x+1}(\mathbb{G}(1,3))$.

Proof Let $Q \subset \mathbb{P}^3$ be a smooth quadric hypersurface. The Fano variety of lines $F_1(Q)$ is the zero locus of a section of the following bundle,

$$\begin{array}{c} \mathrm{Sym}^2(S^*) \\ {\scriptstyle Q_{|L}} \uparrow \downarrow {\scriptstyle \pi} \\ \mathbb{G}(1,3) \end{array}$$

where S^* is the dual of the tautological bundle S over $\mathbb{G}(1,3)$. A smooth conic in $\mathbb{P}^5$ determines uniquely a 2-plane, thus in the Plücker embedding $\mathbb{G}(1,3) \subset \mathbb{P}^5$, we have that

- $F_1(Q)$ determines two 2-planes if $\mathrm{rank}(Q)$ is either 2 or 4,
- $F_1(Q)$ determines a single 2-plane if $\mathrm{rank}(Q)$ is either 1 or 3.

Exchanging such planes gives rise to a $\mathbb{Z}/2$-action on $\mathbb{G}(2,5)$, the Grassmannian of 2-planes in $\mathbb{P}^5$. The second condition above, says that such a $\mathbb{Z}/2$-action on $\mathbb{G}(2,5)$ preserves the orthogonal Grassmannian $\mathbb{OG}$, hence inducing a $\mathbb{Z}/2$-action on the blow-up $\mathbf{Hilb}^{2x+1}(\mathbb{G}(1,3))$. □

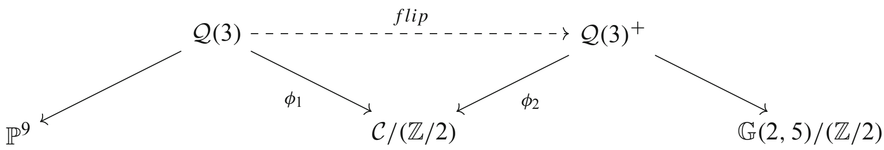
Observe that there is an $SL_4(\mathbb{C})$-action on **Hilb** induced by the action of SL_4 on $\mathbb{P}^3$. This action stratifies **Hilb** into SL_4-orbits compatible with the exceptional divisors E_2^+, $E_{1,1}^+$. Notice that $\mathbb{Z}/2$ acts trivially on SL_4-orbits of codimension 2. In codimension 1, we have that $\mathbb{Z}/2$ acts as the identity on the exceptional divisors E_2^+ and $E_{1,1}^+$. Consider the quotient

$$\mathcal{Q}(3)^+ := \mathbf{Hilb}/(\mathbb{Z}/2)$$

where the $\mathbb{Z}/2$-action is defined in the previous Lemma 4.3.The main result of this section is the following.

Theorem 4.4 *There is a flip $f : \mathcal{Q}(3) \dashrightarrow \mathcal{Q}(3)^+$ over the Chow variety* $\mathbf{Chow}^{2x+1}(\mathbb{G}(1,3))$*, and the flipping locus in $\mathcal{Q}(3)$ is the intersection the divisors $E_1 \cap E_3$.*

The previous result can summarized in the following diagram



where ϕ_1 and ϕ_2 are small contractions and the other maps, except for the flip, are all divisorial contractions described in previous sections.

In order to show that $\mathcal{Q}(3)$ and $\mathcal{Q}(3)^+$ are related through a flip, we lift the maps so we avoid the action of $\mathbb{Z}/2$. In this scenario, we need to describe the flip for the Hilbert scheme $\mathbf{Hilb}^{2x+1}(\mathbb{G}(1,3))$. In doing so, we are lead to consider the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{G},2)$ of degree two stable maps into the Grassmannian $\mathbb{G} = \mathbb{G}(1,3)$.

Lemma 4.5 *There is a nontrivial globally defined $\mathbb{Z}/2$-action on the Kontsevich space $\overline{\mathcal{M}}_{0,0}(\mathbb{G}(1,3),2)$.*

Proof We have a generic 2 to 1 morphism from the Kontsevich moduli space $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{0,0}(\mathbb{G}(1,3),2) = \{(C, C^*)\}$ to the space $\mathcal{Q}(3)$ of complete quadric surfaces defined as follows

$$(C, C^*) \mapsto \left(\bigcup_{L\in C} L, C^*\right)$$

where the notation (S, C^*) means a surface S, and a curve C^* as its marking. This map is 2 to 1 over the open subset parametrizing smooth quadric surfaces as well as over the divisor of complete quadrics of rank two. Indeed, if $\bigcup_{L\in C} L$ sweeps out a smooth quadric S, then L is a ruling of S. The other ruling induces another conic C' which gets mapped to S. The situation is similar over the locus of complete quadrics of rank two. Note that this map is 1 to 1 outside two such regions. We now define the $\mathbb{Z}/2$-action on $\overline{\mathcal{M}}$ by identifying the two curves C and C'. □

Corollary 4.6 *The quotient of $\overline{\mathcal{M}}_{0,0}(\mathbb{G},2)$ by the $\mathbb{Z}/2$-action is isomorphic to $\mathcal{Q}(3)$. In particular, the quotient is smooth.*

Proof Let Z denote the quotient of $\overline{\mathcal{M}}$ by the $\mathbb{Z}/2$-action defined above. Observe that X_3 and Z are birational and there is a bijection between them. Zariski's Main Theorem implies now the corollary. □

It follows from [8] that $\overline{\mathcal{M}}_{0,0}(\mathbb{G},2)$ and $\mathbf{Hilb}^{2x+1}(\mathbb{G}(1,3))$ are related through a flip over the Chow variety. Furthermore, the maps in such a flip diagram between these two spaces are $\mathbb{Z}/2$-equivariant [14, Section 5.2]. The result in Theorem 4.4 now follows from the previous lemmas.

Following the construction of the space $\mathcal{Q}(3)^+$ above, we conjecture a modular interpretation for the flip of the space of complete collineations $\mathcal{X}(3)$. Let $A \in \mathcal{X}(3)$

be a generic complete collineation of $\mathbb{P}^3$. Then, we associate to it a hypersurface of $X = \mathbb{P}^3 \times \mathbb{P}^3$ of bidegree $(1, 1)$, denoted by Y_A; this is an element of the complete linear system $|\mathcal{O}_X(1, 1)| \cong \mathbb{P}^{15}$. The singular locus of Y_A, when non-empty, is a product of projective spaces. Hence, the space $\mathcal{X}(3)$ parametrizes hypersurfaces Y_A with a marking: complete collineation of the type Y_Q, where $Q \in \mathcal{X}(i)$, for some $i < 3$. The hypersurface Y_A is smooth for generic $A \in \mathcal{X}(3)$.

On another hand, let $F(Y_A) \subset \mathbb{G}(1, 3) \times \mathbb{G}(1, 3)$ be the Fano scheme of ruled surfaces, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, embedded in Y_A. This is the zero locus of a global section induced by Y_A,

$$\begin{array}{c} S^* \times S^* \\ {\scriptstyle Y_A}\big\uparrow\big\downarrow{\scriptstyle \pi} \\ \mathbb{G}(1, 3) \times \mathbb{G}(1, 3) \end{array}$$

where S^* is the dual of the tautological bundle over $\mathbb{G}(1, 3)$.

We claim that the subscheme $F(Y_A)$ fully determines the hypersurface Y_A. Hence, the Hilbert scheme $\mathbf{Hilb}_{1,1}$ parametrizing subschemes $F(Y_A)$ inside the product $\mathbb{G}(1, 3) \times \mathbb{G}(1, 3)$ is birational to $\mathcal{X}(3)$. In other words, $\mathbf{Hilb}_{1,1} \cong \mathcal{X}(3)$. We conjecture that this rational map is a flip over the Chow variety that parametrizes the cycle classes of $F(Y_A)$.

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What is the Monodromy Property for Degenerations of Calabi-Yau Varieties?

Luigi Lunardon

Abstract In this survey, we discuss the state of art about the monodromy property for Calabi-Yau varieties. We explain what is the monodromy property for Calabi-Yau varieties and then discuss some examples of Calabi-Yau varieties that satisfy this property. After this recap, we discuss a possible approach to future research in this area.

Keywords Degenerations of Calabi-Yau varieties · Motivic integration · Motivic monodromy conjecture · Motivic zeta function · Triple-points-free models

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1 Introduction

Our aim is the study of the monodromy of degenerations of Calabi-Yau varieties, and we are interested, in particular, in the so-called motivic monodromy conjecture. While degenerations and monodromy are intuitive concepts in complex geometry, translating these ideas in the setting of algebraic geometry is not trivial. In this section we try to give an intuitive description of our topic; hopefully, this picture will guide the reader through the understanding of this paper.

Denote by $D \subset \mathbb{C}$ the set $\{x \in \mathbb{C} \colon |x| < 1\}$. For the moment, a model for a degeneration of a Calabi-Yau is the datum of: a Calabi-Yau variety X over $\mathbb{C}$, a

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smooth complex variety $\mathcal{X}$, endowed with a proper holomorphic map $\pi : \mathcal{X} \to D$ which is smooth on $D^* = D \setminus \{0\}$ and satisfying the following additional properties:

(1) there exists $z_0 \in D,\ z_0 \neq 0$ such that $\pi^{-1}(z_0) = \mathcal{X}_{z_0} \cong X$;
(2) X_0 is a strict normal crossing divisor on $\mathcal{X}$—by this we mean that the irreducible components of $(X_0)_{red}$ are smooth and with transverse intersection.

The divisor $X_0 = \sum_{i \in I} N_i E_i$ contains some of the geometrical data of the degeneration, but it is not enough to determine the motivic zeta function. As we see in Sect. 3 some additional information is required.

Up to homotopy, we have a natural action of $\mathbb{Z} = \pi_1(D^*)$ on the underlying topological space of X. This action induces an action on the cohomology groups $H^i(X, \mathbb{Q})$. We can think of this as the cohomological datum of the degeneration.

A natural question is how the geometrical and the cohomological data of a degeneration are related. The motivic monodromy conjecture suggests a possible answer to this question; this conjecture states that poles of the motivic zeta function of a degeneration of Calabi-Yau varieties are linked to monodromy eigenvalues. The motivic zeta function is a power series which depends on $\mathcal{X}^*$—the restriction of the model $\mathcal{X}$ over D^*—and on a relative volume form ω on $\mathcal{X}^*$. The motivic zeta function encodes all these data and some more information about the geometry of the special fiber. A degeneration of X satisfies the monodromy property if any pole of the motivic zeta function determines an eigenvalue in monodromy. The motivic monodromy conjecture states that Calabi-Yau varieties satisfy the monodromy property.

The motivic monodromy conjecture is the more recent of a series of similar conjectures in different settings. The first conjecture of this series is known as the p-adic monodromy conjecture and it was suggested by Igusa. Given a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$, this conjecture suggests a connection between the associated p-adic zeta function and the local monodromy eigenvalues of $f : \mathbb{C}^n \to \mathbb{C}$. After Kontsevich's work on motivic integration, Denef and Loeser proposed an upgrade of this conjecture in [6]. Given a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, they introduced the associated motivic zeta function and formulated the motivic monodromy conjecture for hypersurface singularities, which statement is analogue to the one of the p-adic monodromy conjecture.

The main aim of this survey is to present this topic in a rigorous way. In Sect. 2 we give the definition of a degeneration of a Calabi-Yau varieties and we explain what a model is. In Sect. 3 we recall some basic definitions in motivic integration and state the monodromy property for Calabi-Yau varieties. The main technical issue is that the motivic zeta function is a power series on a certain localization of an equivariant Grothendieck ring of varieties, so we have to be careful when we talk about poles of the function. For the definition of the monodromy zeta function and the A'Campo formula we refer to [16]. The equivariant version of the Grothendieck ring of varieties was introduced in [10]. The definition of the motivic zeta function and the Denef and Loeser's formula are the ones given in [9]. The definitions of poles of a power series with coefficients in the Grothendieck ring of varieties were given in [17]. In Sect. 4 we talk about abelian varieties, Calabi-Yau varieties which admit an equivariant Kulikov model and Kummer surfaces (we refer to [9] for the first two cases and to [18] for the last one). In Sect. 5 we discuss triple-points-free models of K3 surfaces (we refer to [11, 12] and [15]). Finally, in Sect. 6 we briefly discuss some directions of the research in this topic.

2 Models for Calabi-Yau Varieties over *K*

In the introduction we talked about degenerations of Calabi-Yau varieties, the picture we described was intuitive and geometric, however, it was not rigorous. In this section we define rigorously what we mean by degenerations of Calabi-Yau varieties and what a model is.

We fix the following notations $k = \mathbb{C}$, $K = \mathbb{C}((t))$ and $R = \mathbb{C}[[t]]$. For every positive integer d, we have $K(d) = \mathbb{C}((\sqrt[d]{t}))$, $R(d) = \mathbb{C}[[\sqrt[d]{t}]]$. We set $\overline{K} = \mathbb{C}\{\{t\}\}$, the field of Puiseux series:

$$\mathbb{C}\{\{t\}\} = \bigcup_{d=1}^{\infty} K(d);$$

this is an algebraic closure of K. The group $\mathrm{Gal}(K(d)/K)$ is canonically isomorphic to μ_d, the group of d-th roots of unity in k. The group $\widehat{\mu}$ is the profinite group of roots of unity in k, it is obtained as $\varprojlim \mu_d$. We have that $\widehat{\mu} \cong \mathrm{Gal}(\overline{K}/K)$.

Definition 2.1 (Calabi-Yau Varieties) A Calabi-Yau variety X over K is a smooth, proper, geometrically connected variety with trivial canonical sheaf.

Definition 2.2 (Abelian Variety) An abelian variety A over K is a smooth, proper, geometrically connected commutative group-scheme over K.

Remark 2.3 All abelian varieties are Calabi-Yau varieties. In fact, since they are group varieties, their tangent bundle is trivial. Thus it follows that the top exterior power of the cotangent bundle is the trivial line bundle.

Definition 2.4 (K3 Surfaces) A K3 surface over K is a 2-dimensional Calabi-Yau variety X over K with $H^1(X, \mathcal{O}_X) = 0$.

Remark 2.5 If we restrict out attention to surfaces over $\overline{K}$, it follows by the Enriques-Kodaira classification, that a 2 dimensional Calabi-Yau variety is either an abelian or a K3 surface.

Definition 2.6 (Model) Let X be a proper and smooth K-scheme; a model for X over R is a flat R-algebraic space $\mathcal{X}$ endowed with an isomorphism of K-schemes: $\mathcal{X}_K = \mathcal{X} \times_R K \to X$. We say that $\mathcal{X}$ is a strict normal crossing model (snc model) for X if it is regular and proper over R, and $\mathcal{X}_k = \mathcal{X} \times_R k$ is a strict normal crossing divisor on $\mathcal{X}$. The variety $\mathcal{X}_k$ is called special fiber, while $\mathcal{X}_K$ is called generic fiber.

The special fiber, under this definition, need not to be reduced. If it is reduced, then the model is called semistable.

Remark 2.7 In this remark we compare this definition of a model to the one that we gave in the introduction. The space Spec$R$ corresponds to the disk D; the generic fiber corresponds to the degeneration over the punctured disk, while the special fiber corresponds to the fiber over 0. The generic fiber over $\overline{K}$ corresponds to the universal fiber of the degeneration, i.e. the base change to a universal covering space of D^*.

Remark 2.8 Snc models always exist by Hironaka's resolution of singularities, while semistable models do not, in general. However, by the semi-stable reduction Theorem [13], given any proper model $\mathcal{X}$ of X there exist a positive integer d and a semistable model $\mathcal{X}_d$ of $X \times_K K(d)$ that dominates $\mathcal{X} \times_R R(d)$. If, moreover, X is projective, then also $\mathcal{X}_d$ may be taken projective. It is important to notice that $\mathcal{X}_d$ is not a model for X.

Notation 2.9 Choose an snc model $\mathcal{X}$ for the Calabi-Yau variety X. Express the special fiber as the divisor $\mathcal{X}_k = \sum_{i \in I} N_i E_i$. For the rest of the paper we fix the following notation: for any $J \subset I$, we define $E_J = \bigcap_{j \in J} E_j$ and $E_J^\circ = E_J \setminus \left(\bigcup_{i \notin J} E_i \right)$.

Example 2.10 Let X be the K3 surface in $\mathbb{P}^3_K$ given by the equation:

$$x^2w^2 + y^2w^2 + z^2w^2 + x^4 + y^4 + z^4 + tw^4 = 0.$$

Let $\mathcal{X}$ be the closed subscheme of $\mathbb{P}^3_R$ given by the same equation. The scheme $\mathcal{X}$ is regular, but the special fiber $\mathcal{X}_k$ is a singular surface. The only singular point is $P = [0, 0, 0, 1]$; the singularity at that point is a canonical singularity of type A_1. If we blow up $\mathcal{X}$ at P, we obtain an snc model for X, call it $\mathcal{X}'$.

The special fiber of this model is non-reduced, so in particular it is not a semistable model. As a divisor the special fiber of this model is $\mathcal{X}'_k = D + 2E$, where D is the proper transform of $\mathcal{X}_k$, which is a smooth K3 surface, and $E \cong \mathbb{P}^2_k$. The intersection of D and E is transverse and it is a smooth conic in E.

3 Monodromy Property for Calabi-Yau Varieties

In this section we explain what the motivic monodromy property for Calabi-Yau varieties is. First we explain what is the monodromy action; then we introduce the motivic zeta function and explain how they are, conjecturally, related.

3.1 The Monodromy Action

Recall that we fixed $\overline{K} = \mathbb{C}\{\{t\}\}$, the field of Puiseux series; let σ be the canonical topological generator of the Galois group $\mathrm{Gal}(\overline{K}/K)$. The generator σ can be described as $\sigma = \left(\exp\left(\frac{2\pi i}{d}\right)\right)_{d>0}$ and is called the monodromy operator.

Definition 3.1 (Monodromy Eigenvalue) If X is a smooth, proper variety over K, then, for all i, the monodromy operator σ acts on the l-adic étale cohomology group $H^i_{ét}(X \times_K \overline{K}, \mathbb{Q}_l)$. We say that λ is a monodromy eigenvalue if there exists an index i such that λ is an eigenvalue for the action of σ on $H^i_{ét}(X \times_K \overline{K}, \mathbb{Q}_l)$.

Definition 3.2 (Monodromy Zeta Function) The monodromy zeta function of X is defined as

$$\zeta_X(T) = \prod_{i>0} \left(\det\left(T \cdot \mathrm{Id} - \sigma|_{H^i_{ét}(X\times_K \overline{K}, \mathbb{Q}_l)}\right)\right)^{(-1)^{i+1}} \in \mathbb{Q}_l(T).$$

Remark 3.3 The monodromy zeta function does not encode all the information about the monodromy eigenvalues, in fact, some cancelations may occur. Moreover, there is an even more natural function that encodes all the monodromy eigenvalue—namely, the product of all the characteristic polynomials. One of the main reason why the monodromy zeta function is such a useful tool to study the monodromy eigenvalues is that Theorem 3.5 gives an alternative, and often easier, way to compute this function, while computing all the characteristic polynomials of the monodromy action is usually more complicated.

Remark 3.4 If X is a K3 surface over K, then $H^i_{ét}(X \times_K \overline{K}\, \mathbb{Q}_l)$ is trivial in odd degrees; as a consequence, the monodromy zeta function of a K3 surface has all the monodromy eigenvalues as poles.

Theorem 3.5 (A'Campo's Formula, [16] Theorem 2.6.2) *Let X be a smooth and proper K-variety. Fix an snc model $\mathcal{X}$ for X with special fiber $\mathcal{X}_k = \sum_{i\in I} N_i E_i$. Then the monodromy zeta function is given by*

$$\zeta_X(T) = \prod_{i\in I} \left(T^{N_i} - 1\right)^{-\chi_{top}(E_i^\circ)},$$

where χ_{top} is the topological Euler characteristic and $E_i^\circ = E_i \setminus \bigcup_{j\neq i} E_j$.

Remark 3.6 If follows from A'Campo formula that $\zeta_X(T) \in \mathbb{Q}(T) \subset \mathbb{Q}_l(T)$.

3.2 The Motivic Zeta Function

Next we discuss the second main ingredient to understand the monodromy property: the motivic zeta function. First we need some background in motivic integration. Since we want to keep track of the Galois action of $\mathrm{Gal}(\overline{K}/K)$, instead of working with the usual Grothendieck ring of varieties, we use an equivariant version of it.

Definition 3.7 (Category of G-schemes Over k) Given any group G, we define the category of G-schemes over k, denote by $\mathbf{Sch}_{k,G}$, as the category that has:

(1) as objects separated k-schemes of finite type with good G-action;
(2) as morphisms G-equivariant morphisms of k-schemes.

We say that the action of the group G on the scheme X is good if X has a finite partition in G-stable affine subschemes.

Remark 3.8 The definition of good action of the group G on the scheme X that we use is the one given in [9, (2.2.1)], and it is weaker than the one that is commonly used, i.e. that X can be covered by G-stable affine open subschemes. The reason we preferred our definition is that it can be generalized to algebraic spaces, and it gives rise to the same Grothendieck ring as the usual one.

Definition 3.9 (Equivariant Grothendieck Ring, [10] Definition 4.1) Fix a finite group G. The equivariant Grothendieck ring of G-varieties over k is the ring generated as an abelian group by the isomorphism classes of objects $X \in \mathbf{Sch}_{k,G}$, with the ring structure given by the fiber product over k, with the additional relations:

(1) (Scissor relation) Given a G-scheme X and a closed G-subscheme Y, then

$$[X] = [Y] + [X \setminus Y].$$

(2) Given A_1 and A_2 two G-equivariant affine bundles of rank d over a scheme $S \in \mathbf{Sch}_{k,G}$ we have $[A_1] = [A_2]$.

We denote this ring by $K_{0,k}^G$.

Remark 3.10 An algebraic space X with good G-action defines a class in the equivariant Grothendieck ring. In fact, since X admits a partition in G-stable affine schemes, it is possible to use the scissor relation to construct the class in the equivariant Grothendieck ring.

For our purposes, the ring $K_{0,k}^G$ is not enough, we have to invert an element, namely the class $\mathbb{L} = [\mathbb{A}^1]$, with trivial action of the group G. Thus we define the ring $\mathcal{M}_k^G$ to be the ring

$$\mathcal{M}_k^G = K_{0,k}^G[\mathbb{L}^{-1}].$$

If we have a profinite group $\widehat{G} = \varprojlim G_i$, with all the groups G_i finite, then we define:

$$\mathcal{M}_k^{\widehat{G}} = \varinjlim \mathcal{M}_k^{G_i}$$

Definition 3.11 (Equivariant Weak Néron Model) Let X be a smooth proper K-scheme. For every $d > 0$, set $X(d) = X \times_K K(d)$. There is an action of μ_d on $X(d)$. An equivariant weak Néron model for $X(d)$ is a separated and smooth $R(d)$-scheme $\mathcal{X}$, with a good action of μ_d. Moreover, we require that there exists an isomorphism of $K(d)$-schemes $\mathcal{X}_{K(d)} \to X(d)$ which is μ_d-equivariant and such that the natural map $\mathcal{X}(R(d)) \to X(K(d))$ is a bijection.

Remark 3.12 An equivariant weak Néron model always exists, as explained in [9, 2.2.3]. If we have $\mathcal{X}$ an snc model for a Calabi-Yau variety X, then it is possible to construct an equivariant weak Néron model for $X(d)$. We have just to normalize the pullback of $\mathcal{X}$ on the new base, apply a μ_d-equivariant resolution of singularities and then restrict to the $R(d)$-smooth locus.

Now we are ready to talk about motivic integration. We fix our variety X over K with trivial canonical bundle, and choose a volume form ω on X. Denote by ω_d the pullback of ω to $X(d)$. Choose a weak equivariant Néron model $\mathcal{X}$ for $X(d)$. For every connected component C of the special fiber $\mathcal{X}_k$, the order of ω_d along C is the unique integer n such that $t^{-n/d}\omega_d$ is a generator for the sheaf $\omega_{\mathcal{X}/R(d)}$ locally at the generic point of C. For every integer i, let $C(i)$ the union of the connected components of $\mathcal{X}_k$ of order i. It is important to remark that $C(i)$ is stable under the action of μ_d.

Definition 3.13 (Motivic Integral) With the above notation the motivic integral of ω_d on $X(d)$ is defined as:

$$\int_{X(d)} |\omega_d| = \sum_{i \in \mathbb{Z}} [C(i)]\mathbb{L}^{-i} \in \mathcal{M}_k^{\widehat{\mu}}.$$

Proposition 3.14 ([9], Proposition 2.2.5) *The motivic integral of ω_d on $X(d)$ is independent of the choice of the weak Néron model $\mathcal{X}$.*

Definition 3.15 (Motivic Zeta Function) Let X be a Calabi-Yau variety over K, choose ω a volume form on X. The motivic zeta function of X with respect to the volume form ω is:

$$Z_{X,\omega}(T) = \sum_{d>0} \left(\int_{X(d)} |\omega_d| \right) T^d \in \mathcal{M}_k^{\widehat{\mu}}[[T]].$$

As in the case of the monodromy zeta function, once we fix an snc model, there exists an alternative way to compute the motivic zeta function. Assume that $\mathcal{X}$ is an snc model for X, and let $\mathcal{X}_k = \sum_{i\in I} N_i E_i$ be the special fiber, the reduced special fiber is $\mathcal{X}_{k,red} = \sum_{i\in I} E_i$. The volume form ω on X defines a rational section of the line bundle $\omega_{\mathcal{X}/R}(\mathcal{X}_{k,red} - \mathcal{X}_k)$ on $\mathcal{X}$. To this section we can associate a divisor supported on the special fiber. This divisor is of the form $\sum_{i\in I} \nu_i E_i$. By the numerical data of E_i we mean the couple (N_i, ν_i). For any subset $J \subset I$ consider the varieties E_J and E_J°, defined as in Notation 2.9, and set $N_J = \gcd\{N_j | j \in J\}$. Let $\mathcal{X}(N_J)$ be the normalization of $\mathcal{X} \times_R R(N_J)$. Then $\widetilde{E_J^\circ} = \mathcal{X}(N_J) \times_{\mathcal{X}} E_J^\circ$ is a Galois cover of E_J° (an explicit description of this cover can be found in Section 2.3 of [16]). Now we are ready to give an alternative description of the motivic zeta function.

Theorem 3.16 (Denef and Loeser's Formula, [2], Corollary 4.3.2) *In the above setting we have:*

$$Z_{X,\omega}(T) = \sum_{\emptyset \neq J \subset I} [\widetilde{E_J^\circ}](\mathbb{L}-1)^{|J|-1} \prod_{j\in J} \frac{\mathbb{L}^{-\nu_j} T^{N_j}}{1 - \mathbb{L}^{-\nu_j} T^{N_j}} \in \mathcal{M}_k^{\widehat{\mu}}[[T]]$$

It is clear from Definition 3.15 that the motivic zeta function is a power series with coefficients in the ring $\mathcal{M}_k^{\widehat{\mu}}$. However, the definition of poles of this function is not immediate.

Definition 3.17 (Poles, [17]) Let $Z(T) \in \mathcal{M}_k^{\widehat{\mu}}\left[T, \frac{1}{1-\mathbb{L}^a T^b}\right]_{(a,b)\in S}$ be a rational function over $\mathcal{M}_k^{\widehat{\mu}}$, and choose $q \in \mathbb{Q}$. We say that the rational number q is a pole of order at most $m \geq 1$ for the function $Z(T)$ if there exists a set $\mathcal{P}$ whose elements are multisets contained in $\mathbb{Z} \times \mathbb{Z}_{>0}$ such that:

(1) each multiset $P \in \mathcal{P}$ contains at most m elements (a, b) such that $\frac{a}{b} = q$ and
(2) $Z(T)$ is an element in the $\mathcal{M}_k^{\widehat{\mu}}[T]$-submodule of $\mathcal{M}_k^{\widehat{\mu}}[[T]]$ generated by the elements of the form

$$\prod_{(a,b)\in P} \frac{1}{(1 - \mathbb{L}^a T^b)},$$

for all multisets $P \in \mathcal{P}$.

The order of a pole at q is the minimal m such that $Z(T)$ has a pole at q of order at most m.

Remark 3.18 The reason why the definition of a pole is so involved is that the ring $\mathcal{M}_k^{\widehat{\mu}}$ is not a domain. Actually in [19] it is proven that, if the base field has characteristic zero, not even the usual Grothendieck ring of varieties $K_{0,k}$ is a domain. Moreover, if $k = \mathbb{C}$ it was proven in [1] that $\mathbb{L}$ is a zero-divisor in $K_{0,k}$. It was proven in [7] that $\mathcal{M}_k = K_{0,k}[\mathbb{L}^{-1}]$ is not a domain, the background to this result is presented in Appendix A of [3]; since $\mathcal{M}_k$ injects into $\mathcal{M}_k^{\widehat{\mu}}$ (as varieties with trivial $\widehat{\mu}$ action) it follows that the latter is not a domain either.

At this point we have all the background needed to explain what the monodromy property for Calabi-Yau varieties is.

Definition 3.19 ([9], Definition 2.3.5) Given a Calabi-Yau variety X over K and a volume form ω on X we say that the couple (X, ω) satisfies the monodromy property if there exists a finite set $S \subset \mathbb{Z} \times \mathbb{Z}_{>0}$ such that $Z_{X,\omega}(T)$ is an element of the sub-ring

$$\mathcal{M}_k^{\widehat{\mu}}\left[T, \frac{1}{1-\mathbb{L}^a T^b}\right]_{(a,b)\in S} \subset \mathcal{M}_k^{\widehat{\mu}}[[T]],$$

and, for any couple $(a, b) \in S$, we have that $\exp\left(2\pi\sqrt{-1}\frac{a}{b}\right)$ is a monodromy eigenvalue as in definition 3.1; which means that it is an eigenvalue of the action of $\mathrm{Gal}(\overline{K}/K)$ on $H^i_{ét}(X \times_K \overline{K}, \mathbb{Q}_l)$ for some $i \geq 0$ and every embedding of $\mathbb{Q}_l$ in $\mathbb{C}$.

4 Some Well-Understood Cases

We are far from a complete understanding of whether or not Calabi-Yau varieties satisfy the monodromy property. It was proven for certain families, under some more restrictive hypothesis on the type of Calabi-Yau variety or on the type of degeneration. The conjecture was first proven for abelian surfaces in [8], this proof uses the theory of Néron models. A generalization of this result was obtained in [9]; in this paper it was proven that degenerations of Calabi-Yau varieties which admit equivariant Kulikov models satisfy the monodromy property and that abelian varieties admit such models. Using the aforementioned result of [9], it was also possible to prove the monodromy property for Kummer surfaces: it was showed in [18] that they admit equivariant Kulikov models. All the examples we mentioned have in common that the motivic zeta function has a unique pole. This is not by chance, in fact, we have the following, more general, result.

Theorem 4.1 ([9, Theorem 3.3.3]) *Let X be a Calabi-Yau variety of dimension n over K, with a volume form ω. Choose an snc model for X, let $\mathcal{X}_k = \sum_i N_i E_i$ be the special fiber of this model and denote with ν_i the vanishing order of ω on E_i. Let $\min(\omega) = \min_i \frac{\nu_i}{N_i}$. Then $\exp(-2\pi i \min(\omega))$ is an eigenvalue of the action of σ on $H^n_{ét}(X \times_K \overline{K}, \mathbb{C})$.*

For any Calabi-Yau varieties X and any volume form ω on X, we have that $1 - \min(\omega)$ is always a pole of $Z_{X,\omega}$. Indeed, in all the cases we mentioned at the beginning of this section, it was proven that $1 - \min(\omega)$ is the unique pole, and since $\exp(-2\pi i \min(\omega))$ is a monodromy eigenvalue, then the monodromy property holds. In Sect. 5 we present a family of degenerations of K3 surfaces that satisfies the monodromy property, but whose motivic zeta functions may have more than one pole, most of these results are the work of [11] and [12].

Theorem 4.2 ([9, Theorem 4.2.2]) *Fix an abelian variety A with a volume form ω. Choose an snc model $\mathcal{A}$, and let $\mathcal{A}_k = \sum_i N_i E_i$ be the special fiber. We denote with ν_i the vanishing order of ω on E_i. The motivic zeta function $Z_{A,\omega}(T)$ has a unique pole at $q = 1 - \min(\omega)$. More precisely:*

$$Z_{A,\omega}(T) \in \mathcal{M}_k^{\widehat{\mu}}\left[T, \frac{1}{1-\mathbb{L}^a T^b}\right]_{(a,b)\in S;\ a/b=q} \subset \mathcal{M}_k^{\widehat{\mu}}[[T]].$$

It follows that the monodromy property holds for abelian varieties from Theorems 4.2 and 4.1.

Now we are ready to introduce the definition of equivariant Kulikov models for Calabi-Yau varieties.

Definition 4.3 (Equivariant Kulikov Model) Let X be a Calabi-Yau variety over K, and fix a positive integer d. A Kulikov model for X over $R(d)$ is a regular, proper and flat algebraic space $\mathcal{X}$ over $R(d)$ such that:

(1) there is an isomorphism of $K(d)$-schemes:

$$\begin{array}{ccccc} \mathcal{X}_{K(d)} & \xrightarrow{\sim} & X \times_K K(d) & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & \operatorname{Spec}(K(d)) & \longrightarrow & \operatorname{Spec} K \end{array},$$

(2) the special fiber $\mathcal{X}_k$ is a divisor with normal crossing
(3) the logarithmic relative canonical bundle $\omega_{\mathcal{X}/R(d)}(\mathcal{X}_{k,red} - \mathcal{X}_k)$ is trivial.

We say that the Kulikov model $\mathcal{X}$ is equivariant if the Galois action of μ_d on $X \times_K K(d)$ extends to $\mathcal{X}$.

Remark 4.4 Not all snc models are Kulikov models; for instance, the model we constructed in Example 2.10 is not a Kulikov model over K. It is easy to check that the K3 surface of Example 2.10 admits a Kulikov model over $K(2)$, however, we will show in Example 5.8 that this model is not equivariant. Indeed, in Example 5.8 we show something stronger, i.e. that the K3 surface of Example 2.10 does not admit an equivariant Kulikov model for any d.

Theorem 4.5 ([9, Theorem 5.3.2]) *Let X be a Calabi-Yau variety, with a volume form ω. Assume that X admits an equivariant Kulikov model over $R(d)$ for some positive d. Then the motivic zeta function of (X,ω) has a unique pole at $q = 1 - \min_i \frac{\nu_i}{N_i}$. More precisely:*

$$Z_{X,\omega}(T) \in \mathcal{M}_k^{\widehat{\mu}}\left[T, \frac{1}{1-\mathbb{L}^a T^b}\right]_{(a,b)\in S, a/b=q} \subset \mathcal{M}_k^{\widehat{\mu}}[[T]].$$

Remark 4.6 As in the previous case, the monodromy property for Calabi-Yau varieties admitting an equivariant Kulikov model follows from Theorems 4.5 and 4.1. This result generalize Theorem 4.2, in fact it was proven that abelian varieties admit an equivariant Kulikov model in [9, Theorem 5.1.6].

Another consequence of Theorem 4.5 is that the monodromy property holds for Kummer surfaces, this was proven in [18].

Definition 4.7 (Kummer Surface) Let A be an abelian surface over K, consider an involution ι, and call A_ι the fixed point scheme of ι. Let $\widetilde{A}$ be the blow of A at A_ι; the involution ι acts regularly on $\widetilde{A}$. Call X the quotient of $\widetilde{A}$ by the action of ι; X is a smooth K3 surface over K. Any K3 surface that can be obtained in this way is called a Kummer surface.

Theorem 4.8 ([18, Theorem 6.2]) *Let X be a Kummer surface, then there exists a minimal $d_0 > 0$ such that $X(d_0)$ has an equivariant Kulikov model, moreover, if $d > 0$ is such that $X(d)$ admits an equivariant Kulikov model, then $d_0|d$. In particular, Kummer surfaces satisfy the monodromy property.*

Theorem 4.1 was also used in [9] to prove the monodromy property for some Calabi-Yau varieties which were not in any of these families. As far as we know, the cases we listed in this section are the only ones where Theorem 4.1 was used to prove the monodromy property.

5 Triple-Points-Free Models of K3 and Why they are Interesting

In this section we discuss triple-points-free models of K3 surfaces. If a K3 surface admits a triple-points-free model, it satisfies the monodromy property, however, the motivic zeta function of this K3 surface may have more then a single pole. In particular K3 surfaces admitting a triple-points-free model may not admit an equivariant Kulikov model.

In [4] Crauder and Morrison described special the fiber of relatively minimal, triple-points-free snc models of surfaces with trivial pluricanonical bundles. Many additional results to describe the combinatorics of the special fiber were given by Jaspers in [12]. The motivic zeta function and the monodromy property for these surfaces were studied in [11] and [12]. In Corollary 4.2.4 of [12] it was shown that poles of the motivic zeta function may be recovered from the combinatorial data of the model, in particular it was shown that besides $1 - \min(\omega)$, there are additional poles as soon as the triple-points-free model has a so-called conic flower.

Regarding the monodromy property, it was proven that under some additional conditions on the special fiber, K3 surfaces admitting a triple-points-free model satisfy the monodromy property. In Appendix B of [12] there is a classification of the possible special fibers of triple-points-free models that do not satisfy the monodromy property; such surfaces are known as combinatorial countercandidates. In [15] we proved that these combinatorial countercandidates do not exist. As a consequence, the monodromy property holds for K3 admitting triple-points-free models.

Definition 5.1 (Triple-Points-Free Model) Given a K3 surface X, a triple-points-free model $\mathcal{X}$ of X is a relatively minimal snc model such that given any three distinct irreducible components E_i, E_j, E_k of the special fiber $\mathcal{X}_k$,

$$E_i \cap E_j \cap E_k = \emptyset.$$

Since there are not triple intersections, the dual complex of the special fiber of this model is a graph, we call it Γ; to each vertex of Γ we associate the weight

$$\rho_i = \frac{\nu_i}{N_i} + 1;$$

we denote by $\Gamma_{\min}$ the subgraph of Γ of components of minimal weight.

There is a very explicit description of possible special fibers of triple-points-free models for K3 surfaces.

Theorem 5.2 (Crauder-Morrison Classification for K3 Surfaces, [4] and [12]) *Let X be a smooth, proper $K3$ surface over K, let $\mathcal{X}$ be a relatively minimal triple-points-free model of X, then $\mathcal{X}$ has the following properties:*

(1) $\Gamma_{\min}$ *is a connected subgraph of* Γ. *It is either a vertex or a chain.*
(2) *Each connected component of* $\Gamma \setminus \Gamma_{\min}$ *is a chain (called flower)* $F_0 - F_1 - \cdots - F_l$ *where only* F_l *meets* $\Gamma_{\min}$. *The weights strictly decrease along these flowers,* F_0 *being the one with maximal weight. The surface* F_0 *is either minimal ruled or isomorphic to* $\mathbb{P}^2$. *If it is isomorphic to* $\mathbb{P}^2$ *then* $F_0 \cap F_1$ *is either a line or a conic. The other components are minimal ruled surfaces, and* $F_i \cap F_{i+1}$ *and* $F_i \cap F_{i-1}$ *are both sections of the ruling.*
(3) *If* $\Gamma_{\min}$ *is a single vertex, there are three possible cases for the corresponding surface. It is either isomorphic to* $\mathbb{P}^2$ *or it is a ruled surface or the canonical divisor is numerically trivial. If* $\Gamma_{\min}$ *is a point, we call the model a flowerpot degeneration.*
(4) *If* $\Gamma_{\min}$ *is a chain* $V_0 - V_1 - \cdots - V_{k+1}$, *then we can describe the components of the chain. If* $i \neq 0, k+1$, V_i *is an elliptic ruled surface, and* $V_{i-1} \cap V_i$ *and* $V_i \cap V_{i+1}$ *are both sections of the ruling; if* $i = 0, k+1$, *then* V_i *is isomorphic to* $\mathbb{P}^2$, *or it is a, rational or elliptic, ruled surface. If* $\Gamma_{\min}$ *is a chain we call the model a chain degeneration.*

Remark 5.3 In [4] there is an even stronger result. Indeed, they classified the special fiber of triple-points-free models with generic fiber $\mathcal{X}_K$ with trivial pluricanonical bundle. In [4] there is also a complete classification of the possible flowers, divided in 21 combinatorial classes. If $\mathcal{X}_K$ is a K3 surface, this classification was refined in Chapter 3 of [12].

One of the main results of [12] is the description of the motivic zeta functions of these models. This description is extremely explicit and using it, it was possible to study the poles. It turned out that poles of the motivic zeta function are closely related to the presence of conic flowers (i.e. F_0 is isomorphic to $\mathbb{P}^2$ and $F_0 \cap F_1$ is a conic).

Theorem 5.4 ([12, Theorem 4.3.8]) *Let X be a K3 surface over K with a volume form ω. Choose a triple-points-free model $\mathcal{X}$, whose special fiber is $\mathcal{X}_k = \sum_i N_i E_i$, with numerical data (N_i, ν_i). Then $q \in \mathbb{Q}$ is a pole of $Z_{X,\omega}(T)$ if and only if there exists an i such that the numerical data of E_i satisfy $q = -\nu_i/N_i$ and such that:*

(1) *either ρ_i is minimal*
(2) *or E_i is the top of a conic flower.*

Moreover, while in the second case the pole is always of order 1, in the first one it is of order 1 if $\mathcal{X}$ is a flowerpot degeneration and of order 2 if it is a chain degeneration.

Remark 5.5 In [12, Appendix A] there is a Python code that describes the contribution of each flower to the motivic zeta function.

Remark 5.6 In [12] it was also discussed whether or not a K3 surface X admitting a triple-points-free model $\mathcal{X}$ satisfies the monodromy property. In [12, Theorem 5.2.1] it was proven that flowerpot degenerations satisfy the monodromy property. In [12, Theorem 5.3.1] it was proven that chain degenerations with some extra assumptions satisfy the monodromy property. However, it was not clear if those assumptions were enough to prove the monodromy property for any chain degeneration.

The strategy adopted in [12] was to try to prove the conjecture by contradiction. The assumption was that there exists a chain degeneration that does not satisfy the monodromy property, this mean that some poles of the motivic zeta function of this degeneration do not correspond to monodromy eigenvalues. The first step was to use A'Campo's formula and Denef and Loeser's formula to deduce some information on the geometry of the special fiber of this triple-points-free model. The last step would have been to prove that these surfaces—called combinatorial countercandidates—do not appear as the special fiber of any triple-points-free model. This would have proven the monodromy property for all K3 surfaces admitting a triple-points-free model.

In [12, Section 6] there is a description of some topological properties of these combinatorial countercandidates, and this description become even more explicit in [12, Appendix B], where all the possible countercandidates are listed. The kind of information we have about the numerical countercandidates is:

(1) the central fiber of the degeneration is a chain degeneration, the chain is

$$V_0 - V_1 - \cdots - V_{k+1}.$$

(2) The surface V_0 satisfies the following properties:

   (a) it contains a smooth elliptic curve D such that $D = -K_{V_0}$;
   (b) it is a smooth rational ruled non-minimal surface obtained from an Hirzebruch surface by l_0 blow-ups whose centers lies in the image of D;
   (c) it contains at least h smooth rational curves of self intersection -2, we denote them by C_i. The curves C_i are all disjoint, moreover from the adjunction formula it follows that they do not intersect D;

(3) the other internal components of the chain are elliptic or rational ruled surfaces, and they are obtained from minimal surface by some blow-up along the intersection of two components. Moreover, we know that there is a certain number of -2 rational curves, disjoint from the sections $V_i \cap V_{i+1}$.

Whether or not any combinatorial countercandidate exists was left as an open problem in [12]. It was proven in [15] that they do not exist, all the information required to prove this result are the one regarding the surface V_0.

Theorem 5.7 ([12, 15]) *Let X be a K3 surface that admits a triple-points-free model $\mathcal{X}$. Then the monodromy property holds for X.*

To conclude this section we come back to the quartic surface of Example 2.10.

Example 5.8 ([9, 11, 12]) Let X be the K3 surfaces in $\mathbb{P}^3_K$, given by the equation:

$$x^2w^2 + y^2w^2 + z^2w^2 + x^4 + y^4 + z^4 + tw^4 = 0.$$

From Example 2.10 we already know that there exists a model $\mathcal{X}'$ such that the special fiber is $\mathcal{X}'_{\mathbb{C}} = D + 2E$, where D is a K3 surface and $E \cong \mathbb{P}^2_{\mathbb{C}}$. The curve $C = D \cap E$ is a smooth conic in E; thus E is a conic flower of type 2B. We can now choose a volume form on X (for instance the natural volume form induced by the embedding in $\mathbb{P}^3$). This form extends to a relative volume form on $\mathcal{X}'$. We can now compute the numerical data of this relative volume form and we obtain $\nu_D = 0$ and $\nu_E = 1$. The motivic zeta function of this K3 is:

$$Z_{X,\omega}(T) = [\widetilde{D}^\circ]\frac{T}{1-T} + [\widetilde{E}^\circ]\frac{\mathbb{L}^{-1}T^2}{1-\mathbb{L}^{-1}T^2} + [\widetilde{C}](\mathbb{L}-1)\frac{\mathbb{L}^{-1}T^3}{(1-T)(1-\mathbb{L}^{-1}T^2)}.$$

The motivic zeta function of this K3 has two poles, namely 0 and $\frac{1}{2}$.

Remark 5.9 The quartic surface of Example 5.8 does not admit any equivariant Kulikov model. This follows from Theorem 4.5, since its motivic zeta function has two poles.

6 What's Next

In this section we describe a possible approach to future research in this area. Our idea is to construct further examples of K3 surfaces whose motivic zeta function has more than one pole. Once we have done this, the natural question is whether or not these K3 surfaces satisfy the monodromy property.

From the results of Theorem 5.4 it is clear that, in the case of triple-points-free models, additional poles come only from conic flowers. However, it is not immediately clear what distinguishes these flowers geometrically. To obtain some understanding of this, we tried to study what happens to the model if we contract some flowers. The idea of contracting flowers already appears in [5], in this paper there is also a list of the singularities caused by the contraction of some family of flowers.

From the computations of Example 5.8, we can see that the contraction of the flower of type 2B gives a regular model with a special fiber which is irreducible but with a singular point. The situation with non-conic flowers is different, L. Halle showed me that, up to a finite base change, it is possible to contract them smoothly—by this we mean that the resulting model is an snc model. This suggested to us that there might be a relation between poles of the motivic zeta function and singularities of the models.

Assume now that X is a K3 surface over K, ω a volume form on X and let $\mathcal{X}$ be a model for X. To X we can associate a motivic zeta function $Z_{X,\omega}$; however, it may not be possible to use Denef and Loeser's formula with the model $\mathcal{X}$. In general it is not regular, and even if it were, the irreducible components of the special fiber $\mathcal{X}_k$ might be singular. Of course, by Hironaka's resolution of singularities it is possible to construct an snc model $\mathcal{Y}$ which dominates $\mathcal{X}$, but what we would like to understand is whether or not it is possible to deduce the presence of additional poles of $Z_{X,\omega}$ from the singularities of $\mathcal{X}$. This problem, however, is very generic, and thus pretty tough to approach. So we had to restrict the class of singularities we would consider.

Definition 6.1 (Rational Double Points) Given X a normal surface over $\mathbb{C}$, we say that a point $x \in X$ is a rational double point (or ADE or Du Val singularity) if it is a canonical singularity. Given a rational double point singularity, we have that étale locally around the point the surface is isomorphic to the closed subset of $\mathbb{A}^3$ given by one of the following equations:

(1) $x^2 + y^2 + z^{n+1} = 0$; these are A_n singularities;
(2) $x^2 + y^2 z + z^{n-1} = 0$, with $n > 3$; these are D_n singularities;
(3) $x^2 + y^3 + z^4 = 0$; this is the E_6 singularity;
(4) $x^2 + y^3 + z^3 y = 0$; this is the E_7 singularity;
(5) $x^2 + y^3 + z^5 = 0$; this is the E_8 singularity.

Fix a K3 surface X which admits a regular model $\mathcal{X}$ whose special fiber $\mathcal{X}_k$ has some rational double points (for instance, the K3 surface of Example 5.8). Assume, furthermore, that none of the rational double points lie the intersection of some irreducible components of the special fiber and call the set of rational double points S. Then we can construct an snc model $\mathcal{Y}$ for X, with a morphism $\pi : \mathcal{Y} \to \mathcal{X}$ blowing-up $\mathcal{X}$ at some smooth points and at some smooth curves contained in $\mathcal{X}_k$. Once we have constructed such an snc model, it is possible to compute the motivic zeta function using Denef and Loeser's formula. We can split the motivic zeta function $Z_{X,\omega}$ in two parts, the one which depends on the strata contained in $\pi^{-1}(\mathcal{X}_k \setminus S)$ and the one which depends on the strata contained in $\pi^{-1}(S)$. We denote the second part by $Z_{X,\omega,sing}$, and we refer to it as the contribution of the singularities to $Z_{X,\omega}$.

Example 6.2 For instance, in Example 5.8, the contribution of the A_1 singularity is:

$$Z_{X,\omega,sing} = [\widetilde{E}^{\circ}]\frac{\mathbb{L}^{-1}T^2}{1-\mathbb{L}^{-1}T^2} + [\widetilde{C}](\mathbb{L}-1)\frac{\mathbb{L}^{-1}T^3}{(1-T)(1-\mathbb{L}^{-1}T^2)};$$

since the motivic zeta function in this example has more than one pole, then X can't have good reduction over K; in particular we have that the action of $\mathrm{Gal}(\overline{K}/K)$ is not trivial.

We are studying the contributions of ADE singularities to the motivic zeta function. In the last part of this paper we briefly discuss some of the questions we would like to answer. Questions 6.3, 6.4, and 6.5 may be considered short term goals, while Question 6.7 is much wilder, and should be considered a long term goal. Question 6.9 was for us a further motivation to study this topic, and a natural follow-up question to Question 6.7.

Question 6.3 Assume that X is a K3 surface that admits a regular model whose special fiber has rational double points, is it true that the motivic zeta function of X has at least two poles? If not, for which singularities is this true?

A positive answer to this question would provide more examples of Calabi-Yau varieties whose motivic zeta function has multiple poles.

Question 6.4 In the above setting, can we describe explicitly the contributions of the various singularities?

Having an explicit description of the motivic zeta function would be not only interesting by itself, but it is a necessary step in a better understanding of the monodromy property for K3 surfaces. This brings us to the next question.

Question 6.5 In the above setting, does X satisfy the monodromy property? If not, which obstructions are encountered?

Requiring that the model $\mathcal{X}$ is regular is a strong assumption; we would like to relax this hypothesis. The natural class of threefold singularities that extends Du Val singularities is known as compound Du Val singularities.

Definition 6.6 (Compound Du Val Singularities) Given X a normal threefold over $\mathbb{C}$, we say that a point $x \in X$ is a compound Du Val singularity if for some general hyperplane section H through x we have that $x \in H$ is a Du Val singularity.

Question 6.7 Assume that X is a K3 surface that admits a model $\mathcal{X}$ whose singularities are compound Du Val singularities. What can we say of the analogs of Questions 6.3, 6.4, 6.5 under these less restrictive hypothesis?

There exists a criterion to show that a K3 surface X has good reduction.

Theorem 6.8 ([14], Theorem 6.1) *Let X be a K3 surface over K, then X has good reduction if and only if the action of* $Gal(\overline{K}/K)$ *on* $H^2_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_l)$ *is trivial.*

An interesting problem is to find sufficient conditions to prove that a K3 surface does not have good reduction.

Question 6.9 Assume that X is a K3 surface that admits a model $\mathcal{X}$ whose singularities are compound Du Val singularities or a regular model whose special fiber has rational double points. Is that enough to prove that X does not have good reduction? If X satisfies the monodromy property, can this obstruction be detected from the motivic zeta function?

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Examples of Irreducible Symplectic Varieties

Arvid Perego

Abstract Irreducible symplectic manifolds are one of the three building blocks of compact Kähler manifolds with numerically trivial canonical bundle by the Beauville-Bogomolov decomposition theorem. There are several singular analogues of irreducible symplectic manifolds, in particular in the context of compact Kähler orbifolds, and in the context of normal projective varieties with canonical singularities. In this paper we will collect their definitions, analyze their mutual relations and provide a list of known examples.

Keywords Irreducible symplectic varieties · Moduli spaces of sheaves · K3 surfaces

1 Introduction

A central problem in complex geometry is the classification of Ricci-flat compact Kähler manifolds. By Yau's theorem [51], these are exactly the compact Kähler manifolds whose first Chern class is zero in $H^2(X, \mathbb{R})$ or, equivalently, whose canonical bundle is numerically trivial. This implies that the Kodaira dimension is zero.

In dimension 1, compact Kähler manifolds of Kodaira dimension zero are exactly elliptic curves. The birational classification of compact complex surfaces shows that compact Kähler surfaces of Kodaira dimension zero are K3 surfaces, 2-dimensional complex tori, Enriques surfaces and bielliptic surfaces.

Among compact Kähler manifolds of Kodaira dimension zero, a very special role is played by those manifolds whose canonical bundle is trivial (which are sometimes called *Calabi-Yau manifolds*).

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The first family of examples is given by *complex tori*, i.e. quotients of a complex vector space V of dimension n by a rank $2n$ lattice Γ in V. A projective complex torus is called *Abelian variety*: in dimension 1 all complex tori are projective, and they are exactly the elliptic curves. In higher dimension there are complex tori which are not projective. In any case, if X is a complex torus of dimension n, then $\pi_1(X) \simeq \mathbb{Z}^{2n}$.

A second family of examples is given by *special unitary manifolds*, i.e. compact Kähler manifolds with trivial canonical bundle, whose every finite étale covering has no non-trivial holomorphic p-form for $0 < p < n$ (where n is the complex dimension). They are projective for $n \geq 3$, and have finite fundamental group, which is trivial if n is even (see Proposition 2 of [2]). Simply connected special unitary manifolds are called *irreducible Calabi-Yau manifolds*.

A third family of examples is given by irreducible symplectic manifolds, that will be our main interest. We recall that if X is a complex manifold, a *holomorphic symplectic form* on X is a closed, holomorphic 2-form σ on X which is everywhere non-degenerate. A *holomorphic symplectic manifold* is a complex manifold admitting a holomorphic symplectic form. Among holomorphic symplectic manifolds we find all even-dimensional complex tori.

If X is a holomorphic symplectic manifold, then its complex dimension is even. If $2n$ is the complex dimension of X, and σ is a holomorphic symplectic form on X, then σ^n is a nowhere vanishing section of K_X: it follows that a holomorphic symplectic manifold has trivial canonical bundle. Moreover, as σ is closed, then it defines a nontrivial cohomology class in $H^0(X, \Omega^2_X)$.

Definition 1 An *irreducible symplectic manifold* is a compact, Kähler, holomorphic symplectic manifold X which is simply connected and such that $h^{2,0}(X) = 1$.

Remark 1 As shown in Propositions 3 and 4 of [2], for an irreducible symplectic manifold X of dimension n and for every $0 \leq p \leq n$ we have

$$h^{p,0}(X) = \begin{cases} 0, \text{ if } p \text{ is odd} \\ 1, \text{ otherwise} \end{cases} \tag{1}$$

Conversely, by Proposition A.1 of [21] every compact, Kähler, holomorphic symplectic manifold X of complex dimension n such that $h^{p,0}(X)$ is as in (1) is simply connected, and hence an irreducible symplectic manifold.

Remark 2 As shown in Proposition 4 of [2], a compact Kähler manifold of complex dimension $2n$ is a holomorphic symplectic manifold if and only if its holonomy group is contained in the symplectic group $Sp(r)$. The holonomy group is precisely $Sp(r)$ if and only if X is an irreducible symplectic manifold.

As already recalled, Ricci-flat compact Kähler surfaces are K3 surfaces, complex tori of dimension 2, Enriques surfaces and bielliptic surfaces. Among them, K3 surfaces are both irreducible Calabi-Yau and irreducible symplectic.

An Enriques surface is a finite quotient of a K3 surface by a fixed point free involution, while a bielliptic surface is quotient of an Abelian surface (product of two elliptic curves) by the free action of a finite Abelian group. It follows that a Ricci-flat compact Kähler surface has a finite étale covering which is either an irreducible symplectic (Calabi-Yau) surface or a complex torus of dimension 2.

The same phenomenon occurs for higher dimensional Ricci-flat compact Kähler manifolds, giving to complex tori, irreducible Calabi-Yau manifolds and irreducible symplectic manifolds a special role in the classification. This is the content of the following, which goes under the name of *Beauville-Bogomolov Decomposition Theorem*, and whose proof (based on several results of differential geometry) is contained in [6] and [2].

Theorem 1 *Let X be a Ricci-flat compact Kähler manifold. Then X has a finite étale covering $f : Y \longrightarrow X$, where Y is a product of complex tori, irreducible Calabi-Yau manifolds and irreducible symplectic manifolds.*

While it is not difficult to provide examples of complex tori and irreducible Calabi-Yau manifolds, it is a hard problem to give examples of irreducible symplectic manifolds. The list of all the known deformation classes is very short:

1. Irreducible symplectic surfaces are exactly K3 surfaces.
2. For $n \geq 2$, the Hilbert scheme $Hilb^n(S)$ of n points on a K3 surface S is an irreducible symplectic manifold of dimension $2n$ (see Théorème 3 of [2]).
3. For $n \geq 2$, let T be a complex torus of dimension 2 and $s : Hilb^{n+1}(T) \longrightarrow T$ the sum morphism. Then $Kum^n(T) := s^{-1}(0_S)$, called *generalized Kummer variety*, is an irreducible symplectic manifold of dimension $2n$ (see Théorème 4 of [2]).
4. Two more deformation classes, OG_6 of dimension 6 and OG_{10} of dimension 10, were constructed by O'Grady (in [44] and [43] respectively) as a symplectic resolution of some singular moduli spaces of semistable sheaves on a projective K3 or on an Abelian surface.

The previous examples form different deformation classes because even if they can have the same dimension, they have different second Betti number (which is, following the previous ordering, 22, 23, 7, 8 and 24, see [2, 43, 44] and [48]).

Remark 3 Different constructions of examples of irreducible symplectic manifolds were presented in [4, 9, 22, 45] (deformation equivalent to $Hilb^2(K3)$) and [28] (deformation equivalent to $Hilb^4(K3)$). Moduli spaces of stable sheaves or of Bridgeland stable complexes on projective K3 surfaces or on Abelian surfaces give rise to examples of irreducible symplectic manifolds which are deformation equivalent to Hilbert schemes of points on K3 surfaces or to generalized Kummer varieties on Abelian surfaces.

Theorem 1 is stated only for compact Kähler manifolds. Anyway, if X is a complex projective manifold with Kodaira dimension zero, the minimal model Y of X (whose conjectural existence is predicted by the Minimal Model Program) is a projective variety which is birational to X, and has terminal singularities and nef canonical divisor. Assuming the abundance conjecture, it follows that a multiple of the canonical divisor K_Y of Y is trivial.

For the classification of projective varieties whose Kodaira dimension is 0 it is then central to extend Theorem 1 to normal projective varieties having terminal singularities and torsion (i.e. numerically trivial by Theorem 8.2 of [26]) canonical divisor. This implies the need for a definition of singular analogues of irreducible Calabi-Yau and symplectic manifolds.

Various definitions have been proposed and studied over the years, and the main purpose of this survey is to present a list of definitions of irreducible Calabi-Yau and symplectic varieties which can be found in the literature, together with a list of known example.

2 Irreducible Symplectic Varieties

There are two main settings we will consider: orbifolds and varieties with canonical singularities. The second one is more natural for the purposes of the Minimal Model Program, while the first one has the advantage to be more similar to the smooth case (and it was the first one to be considered).

2.1 Orbifolds

A first generalization of the decomposition theorem was obtained by Campana in [8], in the setting of orbifolds. We refer the reader to [10] or to [15] (where orbifolds are called *V-manifolds*) for precise definitions and results on these analytic spaces.

An *orbifold* is a connected, para-compact analytic space X such that for every $x \in X$ there is an open neighborhood U of x in X, an open subset V of $\mathbb{C}^n$ (with respect to the Euclidean topology) and a finite group G of automorphisms of V such that there is an isomorphism $\phi : V/G \longrightarrow U$. The composition $\pi : V \longrightarrow U$ of the projection $V \longrightarrow V/G$ with ϕ is called *uniformization map*, and the triple (V, G, π) a *uniformizing system* for U.

It follows from Proposition 1.3 of [5] and by Proposition 5.15 of [27] that an orbifold is normal, $\mathbb{Q}$-factorial, Cohen-Macaulay and has rational singularities.

On an orbifold X of dimension n one can then always find a uniformizing open cover $\{U_i\}_{i\in I}$, i.e. for each $i \in I$ there is a uniformizing system (V_i, G_i, π_i) for U_i. The sheaf of differential p-forms on X is denoted $\mathcal{A}^p_X$, and by definition it is the sheaf which restricted to U_i is $\pi_{i*}(\mathcal{A}^p_{V_i})^{G_i}$, i.e. the push-forward under π_i of the G_i-invariant part of the sheaf of differential p-forms on V_i. Similarly one defines the sheaf of holomorphic p-forms on X, denoted Ω^p_X.

A differential 2-form $\omega \in \mathcal{A}^2_X(X)$ is a *Kähler form* for X if for every $x \in X$ and every uniformizing system (V, G, π) of an open neighborhood U of x, we have that $\omega_{|U} \in (\mathcal{A}^2_V)^G(V)$ is a Kähler form on V.

Moreover, one can still define the notion of first Chern class for orbifolds: the canonical sheaf K_X (which, restricted to every uniformized open subset U with uniformizing system (V, G, π), is just $\pi_* K_V$) is not in general locally free, but K_X^m is if m is a multiple of $|G|$. It follows that if X is compact, then there is $m \gg 0$ such that K_X^m is locally free, and one defines $c_1(X) := -\frac{1}{m}c_1(K_X^m)$.

As shown by Theorem 1.1 of [8], if X is a compact, connected Kähler orbifold such that $c_1(X) = 0$ in $H^2(X, \mathbb{R})$, every Kähler class is represented by a unique Ricci-flat Kähler metric on X.

2.1.1 The Decomposition Theorem for Orbifolds

As in the smooth case, one has two important kinds of Ricci-flat compact Kähler orbifolds which arise. The first one generalizes irreducible Calabi-Yau manifolds.

Definition 2 An *irreducible Calabi-Yau orbifold* is a compact, connected Kähler orbifold X having simply connected smooth locus, $K_X \simeq \mathcal{O}_X$ and $H^0(X, \Omega_X^p) = 0$ for every $0 < p < \dim(X)$.

By Proposition 6.6 of [8] an irreducible Calabi-Yau orbifold of dimension n is equivalently a compact, connected Kähler orbifold whose smooth locus is simply connected, and which has a Ricci-flat Kähler metric whose holonomy group is $SU(n)$.

The second kind of Ricci-flat compact Kähler orbifolds is a generalization of irreducible symplectic manifolds. A closed holomorphic 2-form on X, i.e. a global section of Ω_X^2, which is everywhere non-degenerate is called *symplectic form*. An orbifold admitting a holomorphic symplectic form is called *holomorphic symplectic orbifold*. As for holomorphic symplectic manifolds, a holomorphic symplectic orbifold has even complex dimension and trivial canonical sheaf.

Definition 3 An *irreducible symplectic orbifold* is a compact, connected Kähler orbifold which is holomorphic symplectic, has simply connected smooth locus and $h^0(X, \Omega_X^p) = 1$.

By Proposition 6.6 of [8] an irreducible symplectic orbifold of dimension $2n$ is equivalently a compact, connected Kähler orbifold whose smooth locus is simply connected, and which has a Ricci-flat Kähler metric whose holonomy group is $Sp(n)$.

The decomposition theorem for Ricci-flat compact Kähler orbifolds is the following (see Théorème 6.4 of [8]):

Theorem 2 *Let X be a Ricci-flat compact Kähler orbifold. Then X has a finite quasi-étale covering $f : Y \longrightarrow X$, where Y is a product of complex tori, irreducible Calabi-Yau orbifolds and irreducible symplectic orbifolds.*

A finite quasi-étale morphism is a finite morphism which is étale in codimension 1. The statement of Théorème 6.4 of [8] is more precise about the finite quasi-étale covering: it is indeed a finite orbifold covering (see Définition 5.1 of [8]).

The proof of Theorem 2 relies first on a generalization to orbifolds of the de Rham decomposition theorem (Proposition 5.4 of [8]), which provides a decomposition of the universal orbifold covering $\widetilde{X}$ of X as a product $M_0 \times M_1 \times \cdots \times M_k$, where M_0 is a Euclidean space, and $M_1, \ldots, M_k$ have all irreducible holonomy representation. The orbifold version of the Cheeger-Gromoll theorem is provided by Borzellino and Zhu (see [7]), and implies that $M_1, \ldots, M_k$ are all compact and their Ricci curvature is zero. The remaining part of the proof is similar to the one for compact manifolds.

2.1.2 Related Notions and Examples

There are other definitions which are related to irreducible symplectic orbifolds.

Definition 4 A *primitively symplectic V-manifold* is a holomorphic symplectic orbifold X such that $h^{2,0}(X) = 1$. An *irreducible symplectic V-manifold* is a compact, connected Kähler holomorphic symplectic orbifold X which is simply connected and such that $h^{2,0}(X) = 1$.

Primitively symplectic V-manifolds are introduced in (section 2.1, [15]), while irreducible symplectic V-manifolds are introduced in (Definition 1.3.(iv), [30]). Clearly an irreducible symplectic V-manifold is a compact, connected Kähler primitively symplectic V-manifold which is simply connected. Moreover, an irreducible symplectic orbifold is an irreducible symplectic V-manifold.

We will discuss in the following several examples of irreducible symplectic V-manifolds/orbifolds of dimension 4 and 6 which appear in the literature. In particular, we will see that irreducible symplectic V-manifolds are not always irreducible symplectic orbifolds.

Symmetric Products of K3 Surfaces

If S is a K3 surface and $m \geq 2$, the symmetric product $X := Sym^m(S)$ is an irreducible symplectic V-manifold. Indeed, it is a compact, connected Kähler orbifold having $Y := Hilb^m(S)$ as a resolution of the singularities. As Y is an irreducible symplectic manifold, it is simply connected and has $h^{2,0}(Y) = 1$: it follows that X is simply connected and $h^{2,0}(X) = 1$ (see as instance Proposition 2.13 of [15]).

We notice that the smooth locus X^s of X is not simply connected, since $\pi_1(X^s) \simeq \Sigma_m$. It follows that X is not an irreducible symplectic orbifold.

Fujiki's Examples

Tables I and II in section 13 of [15] present a list of 18 examples of irreducible symplectic V-manifolds of dimension 4 constructed as follows: let S be a K3 surface or a 2-dimensional complex torus, H a finite group acting symplectically on S (i.e.

each element $h \in H$ preserves the holomorphic symplectic form of S) and τ an automorphism of H of order 2. Then H acts on $S \times S$ via the action mapping $(h, (s, t))$ to $(h(s), \tau(h)(t))$. Let $G(H)$ be the subgroup of $Aut(S \times S)$ generated by H and by the involution ι mapping (s, t) to (t, s), and consider $Y := S \times S/G(H)$.

The fixed locus of the action of $G(H)$ may have 2-dimensional components, and apart from that it has only isolated points: the singular locus of Y is then given by a (possibly empty) 2-dimensional locus Σ and by a finite number of points. Blowing up Σ, one gets a V-manifold X whose singular locus is given by a finite number of points. Theorem 13.1 of [15] shows that X is an irreducible symplectic V-manifold.

We reproduce here Tables I and II for the convenience of the reader: here b_2 is the second Betti number of the variety, and a_k is the number of singular points of type $\widehat{A}_k = (\mathbb{C}^4/g_k, 0)$, where $g_k = (\zeta_k, \zeta_k, \zeta_k^{-1}, \zeta_k^{-1})$ and $\zeta_k := e^{\frac{2\pi i}{k}}$ (the sum $a_2 + a_3 + a_4 + a_6$ is the number of singular points of the variety). The varieties X_p are obtained starting from a K3 surface S, while the varieties Y_n are obtained starting from a 2-dimensional complex torus.

Table 1 Fujiki's examples

Symbol	H	b_2	a_2	a_3	a_4	a_6
X_1	id_S	23	0	0	0	0
X_2	$\mathbb{Z}/2\mathbb{Z}$	16	28	0	0	0
X_3	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	14	36	0	0	0
X_4	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	16	28	0	0	0
X_5	$\mathbb{Z}/3\mathbb{Z}$	11	0	15	0	0
X_6	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	7	0	12	0	0
X_7	$\mathbb{Z}/4\mathbb{Z}$	10	10	0	6	0
X_8	$(\mathbb{Z}/4\mathbb{Z})^{\oplus 2}$	8	12	0	0	0
X_9	$\mathbb{Z}/6\mathbb{Z}$	8	7	6	0	1
X_{10}	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$	10	16	0	4	0
X_{11}	$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/6\mathbb{Z})$	8	6	6	0	0
Y_1	$\mathbb{Z}/3\mathbb{Z}$	7	0	36	0	0
Y_2	$(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$	7	0	27	0	0
Y_3	$(\mathbb{Z}/3\mathbb{Z})^{\oplus 3}$	7	0	0	0	0
Y_4	$\mathbb{Z}/4\mathbb{Z}$	8	54	0	6	0
Y_5	$(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$	10	52	0	4	0
Y_6	$(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	14	48	0	0	0
Y_7	$\mathbb{Z}/6\mathbb{Z}$	8	35	16	0	0

In all the previous examples, the singular locus of $S \times S/G(H)$ has always at least one irreducible component of dimension 2: we have 1 component for X_1, X_5, X_6, Y_1, Y_2 and Y_3; 2 components for X_2, X_7, X_9, Y_4 and Y_7; 4 components for X_3, X_8, X_{10}, X_{11} and Y_5; 8 components for X_4 and Y_6.

The varieties X_1 and Y_3 are the only smooth examples we get, and they are both irreducible symplectic manifolds: X_1 is the Hilbert scheme of two points on S, while

Y_3 is deformation equivalent to $Kum^2(T)$ (see Remark 13.2.3 and Proposition 14.3 of [15]).

If $p \neq 1$ and $n \neq 3$, then X_p and Y_n are singular symplectic varieties whose singular locus has codimension 4: by Corollary 1 of [39] they all have terminal singularities, and by the Main Theorem of [41] their deformations are locally trivial. It follows that if $p \neq q$ and $(p, q) \neq (2, 4)$ then X_p and X_q are not deformation equivalent; if $n \neq m$ then Y_n is not deformation equivalent to Y_m, and for every p, n we have that X_p is not deformation equivalent to Y_n.

It is not known if X_2 and X_4 give different deformation classes. Anyway, the Fujiki examples provide at least 17 different deformation classes of irreducible symplectic V-manifolds in dimension 4, 15 of which are singular. The values of b_2 of these examples are 7, 8, 10, 11, 14, 16 and 23. It is still an open problem to determine if X_p and Y_n are irreducible symplectic orbifolds if $p \neq 1$ and $n \neq 3$.

Quotients of $Hilb^n(K3)$

We now consider quotients of $Hilb^n(S)$ where S is a projective K3 surface. We start with the case $n = 2$, so that the fixed locus $Fix(G)$ of G is a union of a finite number of points (depending on the order of G) and a (possibly empty) holomorphic symplectic surface, and the same holds for the singular locus of $Hilb^2(S)/G$. By blowing up the singular surface one then gets an irreducible symplectic V-manifold M_G (see [15]).

A particular case is when $G = \langle \phi \rangle$, where ϕ is a symplectic automorphism of prime order. By Corollary 2.13 of [36], the order p of ϕ can only be 2, 3, 5, 7 or 11. The case of $p = 2$, i.e. ϕ is a symplectic involution, is studied in [30] and [32]. The case $p = 3$ is studied in [32]. The case $p = 11$ is studied in [33]. The cases $p = 5, 7$ are treated in [35].

Example 1 (Quotient by a Symplectic Involution) If $G = \langle \phi \rangle$ where ϕ is a symplectic involution, then $M_2 := M_G$ is an irreducible symplectic V-manifold with 28 isolated singular points, and by Theorem 2.5 of [32] we have $b_2(M_2) = 16$. By Corollary 3.23 of [34] and Proposition 5.1 of [30] we have that the topological Euler number of M_2 is 268.

It follows that M_2 is not deformation equivalent to any of the Fujiki examples above: the only examples in Table 1 having 28 singular points are X_2 and X_4 (for both of which $b_2 = 16$), whose topological Euler number is 226 (see Remark 13.2.4 of [15]).

Example 2 (Quotient by a Natural Symplectic Automorphism of Order 3) If $G = \langle\phi\rangle$ where ϕ is a natural symplectic automorphism of order 3, then $Fix(G)$ is given by 27 isolated points. We get that $M_3 := M_G$ is an irreducible symplectic V-manifold with 27 isolated singular points.

This example is then not deformation equivalent to M_2, and neither to any of the Fujiki examples in Table 1: the only one of that list having 27 singular points is Y_2, whose $b_2 = 7$. But by Theorem 1.3 of [33] we have $b_2(M_3) = 11$, so M_3 is a new deformation class.

Example 3 (Quotient by a Symplectic Automorphism of Order 11) Examples 4.5.1 and 4.5.2 of [36] provide two K3 surfaces S_1 and S_2 such that $Hilb^2(S_i)$ has an automorphism σ_i of order 11. Let $G_i := \langle\sigma_i\rangle$: in both cases $Fix(G_i)$ is given by 5 isolated points. Then $M_{11}^i := M_{G_i}$ is an irreducible symplectic V-manifold with 5 singular points.

By [33] we have $b_2(M_{11}^i) = 3$ and that M_{11}^1 and M_{11}^2 are not deformation equivalent, so they provide two more deformation classes.

If $G = \phi$ where ϕ has order 5, then $M_5 := M_G$ is an irreducible symplectic V-manifold with 14 singular points and $b_2(M_5) = 7$. If $G = \phi$ where ϕ has order 7, then $M_7 := M_G$ is an irreducible symplectic V-manifold with 9 singular points and $b_2(M_7) = 5$ (see Theorems 1.2 and 1.3 of [35] and Examples 4.3.1 and 4.4.1 of [36]). We then get 6 more deformation classes of singular irreducible symplectic V-manifolds of dimension 4 to add to the 15 classes presented by Fujiki. If we now consider a K3 surface S and $Hilb^n(S)$ for $n \geq 3$, let ϕ be a symplectic involution on $Hilb^n(S)$. The quotient H_n of $Hilb^n(S)$ by the action of ϕ is again an irreducible symplectic V-manifold (see [15], or Lemma 1 below).

We are in the position to describe which of these examples are irreducible symplectic orbifolds, as the following shows:

Proposition 1 *Let S be a projective K3 surface and G a finite group of automorphisms of $Hilb^n(S)$ acting symplectically.*

1. *If the codimension of $Fix(G)$ is at least 4, then $M_G := Hilb^2(S)/G$ is not an irreducible symplectic orbifold.*
2. *If $n = 2$ and G is generated by a symplectic involution, then the singular locus of $Hilb^2(S)/G$ is given by 28 isolated points and a K3 surface Σ. Then the partial resolution M_G of $Hilb^2(S)/G$ obtained by blowing-up Σ is an irreducible symplectic orbifold.*

In particular M_2 is an irreducible symplectic orbifold, while M_3, M_5, M_7, M_{11}^1, M_{11}^2 and H_n are irreducible symplectic V-manifolds which are not irreducible symplectic orbifolds.

Proof Suppose first that $Fix(G)$ has codimension at least 4 in $Hilb^n(S)$, so the singular locus of M_G has codimension at least 4. We know that M_G is an irreducible symplectic V-manifold: we let U be its smooth locus and $f : Hilb^n(S) \longrightarrow M_G$ the quotient morphism. Hence $f : f^{-1}(U) \longrightarrow U$ is a finite étale covering whose degree is the order of G: hence U is not simply connected, and M_G is not an irreducible symplectic orbifold.

We notice that for M_3, M_5, M_7, M^1_{11} and M^2_{11} we have that $Fix(G)$ is given by isolated points (see Examples 2 and 3 above, and Examples 4.3.1 and 4.4.1 of [36]), so they are not irreducible symplectic orbifolds. The fixed locus $Fix(G)$ in the case of H_n has codimension at least 4 by Theorem 1.1 of [24], so that H_n is not an irreducible symplectic orbifold.

The case of $n = 2$ and G generated by a symplectic involution is due to Menet (see Remark 3.22 of [34]). We need to show that the smooth locus U of M_G is simply connected. Let D be the exceptional divisor of M_G coming from the blow-up $b : M_G \longrightarrow X_G := Hilb^2(S)/G$.

Now, let Σ' be the 2-dimensional component of the singular locus of X_G, and U' the smooth locus of X_G. We notice that the blow-up morphism b gives an isomorphism between $U \setminus D$ and U'. Moreover the quotient morphism $f : Hilb^2(S) \longrightarrow X_G$ gives a double étale covering $Hilb^2(S) \setminus Fix(G) \longrightarrow U'$. As $Fix(G)$ has codimension 2 in $Hilb^2(S)$, and $Hilb^2(S)$ is smooth, it follows that $\pi_1(Hilb^2(S)) \simeq \pi_1(Hilb^2(S) \setminus Fix(G))$, so that $Hilb^2(S) \setminus Fix(G)$ is simply connected.

As a consequence, we see that $\pi_1(U \setminus D) \simeq \pi_1(U') \simeq \mathbb{Z}/2\mathbb{Z}$. Hence, there is an étale covering $\pi' : Y' \longrightarrow U \setminus D$ of degree 2, which extends to a finite covering $\pi : Y \longrightarrow U$ branched along D. Notice that Y' is simply connected, hence it follows that Y is simply connected as well.

Let now $x_0 \in D$ and consider a loop γ in U pointed at x_0. Let $\widetilde{\gamma}$ be a lift of γ to Y: notice that as x_0 is a branching point, the fiber of π over x_0 is given by a unique point y_0, so $\widetilde{\gamma}$ is a loop in Y pointed at y_0. Since Y is simply connected, it follows that the homotopy class of $\widetilde{\gamma}$ is zero in $\pi_1(Y, y_0)$, so the homotopy class of γ is zero as well in $\pi_1(U, x_0)$, and we are done.

Quotients of $Kum^2(S)$

Similar considerations are done in [25] for quotients of $Kum^2(S)$ for an Abelian surface S. If σ is a symplectic involution on $Kum^2(S)$, by Theorem 7.5 of [25] the fixed locus of σ is given by 36 points together with a K3 surface Σ.

The quotient $Kum^2(S)/\sigma$ has then a singular locus given by 36 points and a K3 surface. Blowing up the image of Σ in $Kum^2(S)/\sigma$ one gets a V-manifold K which has 36 singular points. Again K is an irreducible symplectic V-manifold, and $b_2(K) = 8$ (see [25]), hence K is not deformation equivalent to M_2, M_3, M^1_{11} and M^2_{11}.

Moreover, the only examples in Table 1 having 36 singular points are X_3 and Y_1, whose b_2 is 14 and 7 respectively, so K is not deformation equivalent to any of the Fujiki examples: we then have another deformation class of singular irreducible symplectic V-manifolds to add the previously mentioned 21 classes. The same proof of point 2 of Proposition 1 gives the following:

Proposition 2 *The variety K is an irreducible symplectic orbifold.* □

Examples of Markushevich-Tikhomirov

Markushevich and Tikhomirov present in [30] a different construction of irreducible symplectic V-manifolds of dimension 4. Let X be a del Pezzo surface which is the double cover of $\mathbb{P}^2$ branched along a smooth quartic B_0 with 28 bitangent lines, and let S be the double cover of X branched along the curve Δ_0 such that $\Delta_0 + i(\Delta_0)$ is the inverse image of a smooth quartic curve of $\mathbb{P}^2$ which is totally tangent to B_0 at eight distinct points (here i is the involution on X induced by the double cover $X \longrightarrow \mathbb{P}^2$).

The surface S is a K3 surface, and let M_k be the moduli space of torsion sheaves on S with first Chern class H (the pull-back of $-K_X$) and Euler character $k - 2$ for $k \in 2\mathbb{Z}$, i.e. of Mukai vector $v = (0, H, k-2)$. We notice that M_k is a projective variety of dimension 6: as k is even, M_k has exactly 28 singular points (see Proposition 1.12.(iii) of [30]). Moreover, the moduli space M_k has an involution σ mapping $L \in M_k$ to $\mathcal{E}xt^1_{\mathcal{O}_S}(L, \mathcal{O}_S(-H))$.

If τ is the involution on S induced by the double cover $S \longrightarrow X$, we let $\kappa := \tau^* \circ \sigma$: the fixed locus of κ has a 4-dimensional irreducible component, denoted $\mathcal{P}^k$. The morphism mapping $L \in \mathcal{P}^k$ to $L(H) \in \mathcal{P}^{k+2}$ is an isomorphism, hence we get at most two non-isomorphic varieties $\mathcal{P}^0$ and $\mathcal{P}^2$.

By Markushevich and Tikhomirov [30] we know that $\mathcal{P}^0$ and $\mathcal{P}^2$ are both irreducible symplectic V-manifolds having 28 singular points. By Lemma 5.2 and Corollary 5.7 of [30] we have that $\mathcal{P}^0$ is birational to M_2 via a Mukai flop, and by Corollary 3.23 of [34]: it follows that $\mathcal{P}^0$ is an irreducible symplectic orbifold.

It is not known if $\mathcal{P}^0$ and $\mathcal{P}^2$ are birational, isomorphic nor deformation equivalent, and it is still an open question if $\mathcal{P}^2$ is an irreducible symplectic orbifold.

Example of Matteini

A similar construction to that of Markushevich and Tikhomirov is presented in [31]. Take a K3 surface S which is a double cover of a generic cubic surface Y with involution τ, and take M to be the moduli space of semistable torsion sheaves with first Chern class H (the pull-back of $-K_Y$) and Euler character -3: this is a singular projective variety of dimension 8.

One still has an involution σ on M obtained as before, and we let $\kappa := \tau^* \circ \sigma$. The fixed locus of κ has a 6-dimensional irreducible component $\mathcal{P}$, whose singular locus is the union of 27 singular K3 surfaces. In [31] it is shown that $\mathcal{P}$ is an irreducible symplectic V-manifold of dimension 6. It is not known if this example is an irreducible symplectic orbifold.

2.2 Varieties with Canonical Singularities

A further generalization of the Beauville-Bogomolov decomposition theorem was obtained for projective varieties with canonical singularities by the works of Druel, Greb, Guenancia, Höring, Kebekus and Peternell, in particular [11, 12, 16, 18] and [19].

2.2.1 The Decomposition for Singular Projective Varieties

We introduce the following notation: if X is a normal variety and X_{reg} is the smooth locus of X whose open embedding in X is $j : X_{reg} \longrightarrow X$, for every $p \in \mathbb{N}$ such that $0 \leq p \leq \dim(X)$ we let

$$\Omega_X^{[p]} := j_* \Omega_{X_{reg}}^p = \left(\wedge^p \Omega_X \right)^{**},$$

whose global sections are called *reflexive p-forms* on X.

Notice that by definition of $\Omega_X^{[p]}$ we have $H^0(X, \Omega_X^{[p]}) = H^0(X_{reg}, \Omega_{X_{reg}}^p)$. Theorem 1.4 of [17] shows that if X is a quasi-projective variety with klt singularities and $\pi : \widetilde{X} \longrightarrow X$ is a log-resolution, then for every $p \in \mathbb{N}$ such that $0 \leq p \leq \dim(X)$ the sheaf $\pi_* \Omega_{\widetilde{X}}^p$ is reflexive. This implies in particular that $H^0(X, \Omega_X^{[p]}) \simeq H^0(\widetilde{X}, \Omega_{\widetilde{X}}^p)$ (see Observation 1.3 therein).

As shown in [17], if $f : Y \longrightarrow X$ is a finite, dominant morphism between two irreducible normal varieties, then there is a morphism of reflexive sheaves $f^* \Omega_X^{[p]} \longrightarrow \Omega_Y^{[p]}$, induced by the usual pull-back morphism of forms on the smooth loci, giving a morphism $f^{[*]} : H^0(X, \Omega_X^{[p]}) \longrightarrow H^0(Y, \Omega_Y^{[p]})$, called *reflexive pull-back morphism.*

We recall the definitions of symplectic form and symplectic variety (see [3]).

Definition 5 Let X be a normal variety.

1. A *symplectic form* on X is a closed reflexive 2-form σ on X which is non-degenerate at each point of X_{reg}.
2. If σ is a symplectic form on X, the pair (X, σ) is a *symplectic variety* if for every resolution $f : \widetilde{X} \longrightarrow X$ of the singularities of X, the holomorphic symplectic form $\sigma_{reg} := \sigma_{|X_{reg}}$ extends to a holomorphic 2-form on $\widetilde{X}$.
3. If (X, σ) is a symplectic variety and $f : \widetilde{X} \longrightarrow X$ is a resolution of the singularities over which σ_{reg} extends to a holomorphic symplectic form, we say that f is a *symplectic resolution.* □

A normal variety having a symplectic form and whose singular locus has codimension at least 4 is a symplectic variety (see [14]), and a symplectic variety has terminal singularities if and only if its singular locus has codimension at least 4 (Corollary 1 of [39]).

We now define irreducible Calabi-Yau and irreducible symplectic varieties following [18]. If X and Y are two irreducible normal projective varieties, a *finite quasi-étale morphism* $f : Y \longrightarrow X$ is a finite morphism which is étale in codimension one.

Definition 6 Let X be an irreducible normal projective variety with trivial canonical divisor and canonical singularities, of dimension $d \geq 2$.

1. The variety X is *irreducible Calabi-Yau* if for every $0 < p < d$ and for every finite quasi-étale morphism $Y \longrightarrow X$, we have $H^0(Y, \Omega_Y^{[p]}) = 0$.
2. The variety X is *irreducible symplectic* if it has a symplectic form $\sigma \in H^0(X, \Omega_X^{[2]})$, and for every finite quasi-étale morphism $f : Y \longrightarrow X$ the exterior algebra of reflexive forms on Y is spanned by $f^{[*]}\sigma$. □

A description of irreducible Calabi-Yau and irreducible symplectic varieties in terms of holonomy is available by the work of Greb et al. [16]. More precisely, if H is an ample divisor on X, by Eyssidieux et al. [13] there is a singular Ricci-flat Kähler metric ω_H in $c_1(H)$, inducing a Riemannian metric g_H on X_{reg}.

We let $Hol(X_{reg}, g_H)$ be the holonomy group of this metric. Proposition F of [18] shows that a normal projective variety of dimension n with klt singularities and trivial canonical bundle is an irreducible Calabi-Yau variety if and only if $Hol(X_{reg}, g_H)$ is isomorphic to $SU(n)$, and it is an irreducible symplectic variety if and only if $Hol(X_{reg}, g_H)$ is isomorphic to $Sp(n/2)$.

The decomposition theorem for singular projective varieties is the following:

Theorem 3 *Let X be a normal projective varieties with klt singularities and numerically trivial canonical bundle. Then X has a finite quasi-étale covering $f : Y \longrightarrow X$, where Y is a normal projective variety with canonical singularities which is a product of complex tori, irreducible Calabi-Yau varieties and irreducible symplectic varieties.* □

This is Theorem 1.15 of [19], and the proof can be found therein. It consists of three major parts: one is the holonomy decomposition obtained by Greb, Guenancia and Kebekus in [16] (namely Theorem B and Proposition D therein); a second one is an algebraic integrability theorem of Druel, which is Theorem 1.4 of [11]; the final ingredient is Theorem 1.1 of [19]. Less general versions of the Bogomolov decomposition theorem in the projective singular setting were previously obtained in [11, 18] and [12]. The complete proof can be found in section 4 of [19] (the proof of Theorem 1.5 therein).

Remark 4 We notice that even if the statement of Theorem 1.15 of [19] gives the existence of a quasi-étale covering $f : Y \longrightarrow X$, this is consistent with the statement of Theorem 3 since they define quasi-étale morphisms as finite morphisms whose ramification divisor is empty. □

2.2.2 Relation Between the Previous Notions

The first result we state is about the relation between irreducible symplectic manifolds, orbifolds and varieties.

Proposition 3 *The following properties hold.*

1. *Irreducible symplectic manifolds are irreducible symplectic orbifolds.*
2. *Projective irreducible symplectic orbifolds are irreducible symplectic varieties.*
3. *Smooth irreducible symplectic varieties are irreducible symplectic manifolds.*

□

Proof The first point is trivial, and the last is a consequence of Proposition A.1 of [21].

Suppose that X is a projective irreducible symplectic orbifold. Then X is an normal projective variety with rational Cohen-Macaulay singularities and trivial canonical bundle. It follows that X has rational Gorenstein singularities. Moreover, it has a holomorphic symplectic form on its singular locus, so by Theorem 6 of [40] it follows that X is a projective symplectic variety. In particular it has canonical singularities (see [3]).

By Theorem 3 there is then a finite quasi-étale covering $f : Y \longrightarrow X$, where Y is a product of Abelian varieties, irreducible Calabi-Yau varieties and irreducible symplectic varieties. As X_{reg} is simply connected by definition of irreducible symplectic orbifold, it follows that f is an isomorphism. As a consequence Y is simply connected, so it has no factor which is an Abelian variety, and it is a symplectic variety, hence it has no factor which is an irreducible Calabi-Yau variety.

Hence Y is a product of irreducible symplectic varieties. But as $H^0(X, \Omega_X^{[2]})$ is one dimensional by definition of irreducible symplectic orbifold, the same holds for Y. If Y is a product of m irreducible symplectic varieties, we have that $H^0(Y, \Omega_Y^{[2]})$ has dimension m: it follows that $m = 1$, so Y is an irreducible symplectic variety. As X is isomorphic to Y, we are done.

Irreducible symplectic V-manifolds are not necessarily irreducible symplectic varieties. A first example of this is given by symmetric products of K3 surfaces: if S is a K3 surface and $m \in \mathbb{N}$, $m \geq 2$, then $X := Sym^m(S)$ is an irreducible symplectic V-manifold. Anyway it has a finite quasi-étale covering $S^m \longrightarrow X$, and $h^0(S^m, \Omega^2_{S^m}) = m$, so X is not an irreducible symplectic variety.

By Theorem I of [18] we know that all irreducible symplectic varieties are simply connected, so if X is a primitively symplectic V-manifold which is an irreducible symplectic variety, then X is an irreducible symplectic V-manifold. Anyway, as symplectic singularities are not, in general, quotient singularities, we cannot expect that an irreducible symplectic variety is an irreducible symplectic V-manifold: an example will be given in the last section.

The following is a criterion to guarantee that some quotients of an irreducible symplectic manifold are irreducible symplectic varieties (and irreducible symplectic V-manifolds as well).

Lemma 1 *Let X be an irreducible symplectic manifold, $G \subseteq Aut(X)$ a finite subgroup acting symplectically on X and $Y := X/G$.*

1. *The quotient Y is an irreducible symplectic V-manifold.*
2. *If Y has terminal singularities and X is projective, then Y is an irreducible symplectic variety.* □

Proof The proof of the first part is basically contained in [15], but we present it here for the reader's sake. The fact that X is a compact, connected Kähler orbifold is Lemma 1.4 of [15], and by Lemma 2.4 of [15] we have that Y is symplectic. As X is simply connected, by Lemma 1.2 of [15] it follows that Y is simply connected as well. Finally, we have $H^{2,0}(Y) \simeq H^{2,0}(X)^G$, where $H^{2,0}(X)^G$ is the space of G-invariant sections. But as G acts symplectically and X is an irreducible symplectic manifold, it follows that $H^{2,0}(Y)$ is 1-dimensional, so that Y is an irreducible symplectic V-manifold.

For the second part, as we know that Y is a normal projective symplectic variety with terminal singularities, in order to show that it is an irreducible symplectic variety we just need to look at its finite quasi-étale coverings. So, let $f : Y' \longrightarrow Y$ be a finite quasi-étale covering of Y. As the singularities of Y are terminal and as X is simply connected (being an irreducible symplectic manifold), it follows that X is the universal covering of Y.

In particular, it follows that we have a finite quasi-étale covering $\pi' : X \longrightarrow Y'$ given by the quotient by a subgroup G' of G, and if we let $\pi : X \longrightarrow Y$, then $\pi = f \circ \pi'$. As X is an irreducible symplectic manifold, its exterior algebra of holomorphic forms is spanned by the symplectic form on X, which is the reflexive pull-back of the one on Y.

Moreover, we notice that $H^0(Y', \Omega_{Y'}^{[p]}) \simeq H^{p,0}(X)^{G'}$, and as X is an irreducible symplectic manifold and G (and hence G') acts symplectically we see that if p is even then $H^0(Y', \Omega_{Y'}^{[p]})$ is spanned by $f^{[*]}\sigma^{p/2}$ (where σ is the symplectic form on Y) and if p is odd then $H^0(Y', \Omega_{Y'}^{[p]}) = 0$. But this shows that Y is an irreducible symplectic variety.

As an application of this, we see that the examples M_3, M_5, M_7, M_{11}^1, M_{11}^2 and H_n presented in Sect. 2.1.2 are all examples of irreducible symplectic V-manifolds which are irreducible symplectic varieties (but not irreducible symplectic orbifolds).

The partial resolution M_2 of the quotient of $Hilb^2(S)$ by a symplectic involution (where S is a K3 surface), the partial resolution K_2 of the quotient of $Kum^2(T)$ by a symplectic involution (where T is a 2-dimensional complex torus), and the example $\mathcal{P}^0$ of [30] (which is deformation equivalent to M_2) are all irreducible symplectic varieties as they are irreducible symplectic orbifolds.

For all other examples in Sect. 2.1.2 (those of Fujiki, the example $\mathcal{P}^2$ of [30] and the example of Matteini), it is not known if they are irreducible symplectic varieties.

2.2.3 Related Notions

Irreducible symplectic varieties appear in several papers under different definitions. A first one appears in [1], and it is defined as follows (the name given to these varieties in [1] is *irreducible symplectic varieties*).

Definition 7 A *resolvable symplectic variety* is a normal, compact Kähler space X whose smooth locus has a holomorphic symplectic form σ, and which has a symplectic resolution of the singularities which is an irreducible symplectic manifold. □

A projective resolvable symplectic variety is not always an irreducible symplectic variety: if S is a projective K3 surface and $m \geq 2$, then $Sym^m(S)$ is a projective resolvable symplectic variety (since it is a normal projective symplectic variety having $Hilb^m(S)$ as a symplectic resolution), but it is not an irreducible symplectic variety.

Similarly, singular Kummer surfaces (i.e. a surface S obtained as quotient of an Abelian surface A by the involution mapping $p \in A$ to $-p \in A$) are resolvable symplectic surfaces (they have a symplectic resolution which is a K3 surface), irreducible symplectic V-manifolds but not irreducible symplectic varieties (as the quotient map $A \longrightarrow S$ is a finite quasi-étale covering and $h^{1,0}(A) \neq 0$).

We will see in Sect. 2.2.4 examples of irreducible symplectic varieties which are not resolvable symplectic varieties, and of resolvable symplectic varieties which are not irreducible symplectic V-manifolds. Anyway we have the following:

Proposition 4 *If X is an irreducible symplectic variety (resp. an irreducible symplectic V-manifold) having a symplectic resolution Y, then Y is an irreducible symplectic manifold. In particular, X is a resolvable symplectic variety.* □

Proof If X is an irreducible symplectic variety, this is Remark 1.16 of [47]. If X is an irreducible symplectic V-manifold, then X has canonical singularities. By Takayama [50] we get $\pi_1(X) \simeq \pi_1(Y)$, so Y is simply connected. Finally, by Theorem 1.4 of [17] we have $h^0(Y, \Omega^2_Y) = h^0(X, \Omega^{[2]}_X)$, which is 1 by definition, and we are done.

A further definition of irreducible symplectic variety appears in [49], where it is defined as a projective symplectic variety X such that $h^1(X, \mathcal{O}_X) = 0$ and $h^0(X, \Omega^{[2]}_X) = 1$. It is called Namikawa symplectic variety if it is moreover $\mathbb{Q}$-factorial and its singular locus has codimension at least 4 (see Definition 1 therein).

Definition 8 We will call *Namikawa symplectic variety* a normal, compact Kähler complex space X such that $h^1(X, \mathcal{O}_X) = 0$ and $h^0(X, \Omega^{[2]}_X) = 1$. □

Namikawa symplectic varieties are the most general kind of varieties we will deal with. Namely:

Proposition 5 *Irreducible symplectic varieties, resolvable symplectic varieties and irreducible symplectic V-manifolds are all Namikawa symplectic varieties.* □

Proof The proof that irreducible symplectic and resolvable symplectic varieties are Namikawa symplectic is given in Propositions 1.9 and 1.10 of [47]. If X is an irreducible symplectic V-manifold, then X has rational Gorenstein singularities and has a symplectic form on its smooth locus, hence by Theorem 6 of [40] it is a symplectic variety.

Moreover, if $f : \widetilde{X} \longrightarrow X$ is a resolution of the singularities, as X has rational singularities we have an isomorphism between $H^1(X, \mathcal{O}_X)$ and $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. As X has klt singularities we have $\pi_1(X) \simeq \pi_1(\widetilde{X})$ (by Takayama [50]), and since X is simply connected it follows that $\widetilde{X}$ is simply connected. As a consequence of this we see that $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$, and hence $H^1(X, \mathcal{O}_X) = 0$. As by definition of irreducible symplectic V-manifold we have that $H^0(X, \Omega_X^{[2]})$ is one dimensional, we are done.

No example of Namikawa symplectic variety which is not an irreducible symplectic variety, nor an irreducible symplectic V-manifold, nor a resolvable symplectic variety is known.

2.2.4 Examples

We now introduce two families of examples of irreducible symplectic varieties. In what follows S will denote a projective K3 surface or an Abelian surface, and we let $\epsilon(S) := 1$ if S is K3, and 0 if S is Abelian. We let $\rho(S)$ be the rank of the Néron-Severi group $NS(S)$ of S.

An element $v \in \widetilde{H}(S, \mathbb{Z}) := H^{2*}(S, \mathbb{Z})$ will be written $v = (v_0, v_1, v_2)$, where $v_i \in H^{2i}(S, \mathbb{Z})$, and $v_0, v_2 \in \mathbb{Z}$. It will be called *Mukai vector* if $v_0 \geq 0$, $v_1 \in NS(S)$ and if $v_0 = 0$, then either v_1 is the first Chern class of an effective divisor, or $v_1 = 0$ and $v_2 > 0$.

The $\mathbb{Z}$-module $\widetilde{H}(S, \mathbb{Z})$ has a pure weight-two Hodge structure and a lattice structure with respect to the Mukai pairing $(.,.)$ (see [20], Definitions 6.1.5 and 6.1.11). We let $v^2 := (v, v)$ for every Mukai vector v, and we refer to $\widetilde{H}(S, \mathbb{Z})$ as the *Mukai lattice* of S. We will always write $v = mw$, where $m \in \mathbb{N}$ and w is a primitive Mukai vector on S.

To any coherent sheaf $\mathcal{F}$ on S we associate a Mukai vector

$$v(\mathcal{F}) := ch(\mathcal{F})\sqrt{td(S)} \in \widetilde{H}(S, \mathbb{Z}).$$

Taking v a Mukai vector on S and supposing that H is a v-generic polarization (see as instance section 2.1 of [47] for the precise definition), we consider the moduli space $M_v(S, H)$ (resp. $M_v^s(S, H)$) of Gieseker H-semistable (resp. H-stable) sheaves on S with Mukai vector v. Then M_v is a projective variety and $M_v^s \subseteq M_v$ is open (see [20]).

The following properties hold.

1. If S is a K3 surface and $v = mw$ for a primitive Mukai vector w, then $M_v \neq \emptyset$ if and only if $w^2 \geq -2$ (see [38] and [52]). If S is an Abelian surface, then $M_v \neq \emptyset$ if and only $w^2 \geq 0$ (see [53]).
2. If S is a K3 surface and $v = mw$ for a primtive Mukai vector w such that $w^2 = -2$, then M_v is a point (see [38]).
3. If S is a K3 surface, $v = mw$ and $w^2 = 0$, then $M_v \simeq Sym^m(S')$ for a projective K3 surface S' (see [38] and [23]). If $m = 1$ we then get a projective K3 surface, while if $m \geq 2$ we then get an irreducible symplectic V-manifold, which is a resolvable symplectic variety but which is neither an irreducible symplectic variety nor an irreducible symplectic orbifold.
4. If S is an Abelian surface, $v = mw$ and $w^2 = 0$, then $M_v \simeq Sym^m(T)$ for an Abelian surface T (see [38] and [23]). If $m = 1$ we then get an Abelian surface, and hence a K3 surface via the Kummer construction. If $m \geq 2$ we consider the natural sum morphism $s : M_v \longrightarrow T$, and let $K := s^{-1}(0)$: then K is an irreducible symplectic V-manifold, which is a resolvable symplectic variety but which is neither an irreducible symplectic variety nor an irreducible symplectic orbifold (see Example 1.13 of [47]).
5. If S is a K3 surface or an Abelian surface and $v = mw$ for a primitive Mukai vector w such that $w^2 > 0$, then M_v is a normal, irreducible projective variety of dimension $v^2 + 2$ whose smooth locus is M_v^s (see [52]). By [37] M_v has a symplectic form.
6. If S is an Abelian surface and $v = mw$ for a primitive Mukai vector w such that $w^2 > 0$, by section 4.1 of [53] we have a dominant isotrivial fibration $a_v : M_v(S, H) \longrightarrow S \times \widehat{S}$, where $\widehat{S}$ is the dual of S. We let $K_v := a_v^{-1}(0_S, \mathcal{O}_S)$, and $K_v^s := K_v \cap M_v^s$. The restriction of the symplectic form of M_v to K_v is a symplectic form (see [53]).

The moduli spaces M_v and K_v described above give us examples of irreducible symplectic varieties if $v^2 > 0$. This is the content of the following result, which is Theorem 1.19 of [47]:

Proposition 6 *Let S be a projective K3 surface of an Abelian surface, v a Mukai vector on S such that $v = mw$ for a primitive Mukai vector w with $w^2 > 0$, and H a v-generic polarization.*

1. *If S is K3, then $M_v(S, H)$ is an irreducible symplectic variety.*
   a. *If $m = 1$, it is an irreducible symplectic manifold.*
   b. *If $m = 2$ and $w^2 = 2$, it has a symplectic resolution which is an irreducible symplectic manifold.*
   c. *In all other cases it has terminal singularities.*
2. *Suppose that S is Abelian. If $m = 1$ and $w^2 = 2$ then $K_v(S, H)$ is a point. In all other cases $K_v(S, H)$ is an irreducible symplectic variety.*
   a. *If $m = 1$ and $w^2 > 2$, it is an irreducible symplectic manifold.*

b. *If $m = 2$ and $w^2 = 2$, it has a symplectic resolution which is an irreducible symplectic manifold.*
c. *In all other cases it has terminal singularities.* □

The case $m = 1$ was proved in its final form by Yoshioka in [52] and [53] (but with important steps towards the complete proof given in [2, 37, 42]). The case $m = 2$, $w^2 = 2$ was studied first by O'Grady in [43] and [44] for $v = 2(1, 0, -1)$, where it was shown that M_v (resp. K_v) has a symplectic resolution which is an irreducible symplectic manifold.

In [29] it is shown that the symplectic resolution exists for all $v = 2w$, where w is primitive and $w^2 = 2$. In [46] it is shown that such a symplectic resolution is an irreducible symplectic manifold, deformation equivalent to OG_{10} (resp. OG_6). The proof of the statement for all other cases is contained in [47].

The cases $m = 1$ and $m = 2$, $w^2 = 2$ then recover all the known deformation classes of irreducible symplectic manifolds. Moreover, the case $m = 2$, $w^2 = 2$ gives examples of irreducible symplectic varieties which are resolvable symplectic varieties. We notice in particular that if S is Abelian, then the smooth locus of K_v in this case is not simply connected (see Theorem 3.7 of [47]), giving then an example of irreducible symplectic variety which is not an irreducible symplectic orbifold.

The remaining cases give examples of irreducible symplectic varieties having no symplectic resolution, hence they are not resolvable symplectic varieties. Moreover, we have the following result:

Proposition 7 *Let S_1 and S_2 be two projective K3 surfaces (resp. Abelian surfaces), $v_i = m_i w_i$ a Mukai vector on S_i for $m_i > 0$ and w_i primitive Mukai vector, and H_i a v_i-generic polarization on S_i. Then $M_{v_1}(S_1, H_1)$ (resp. $K_{v_1}(S_1, H_1)$) is deformation equivalent to $M_{v_2}(S_2, H_2)$ (resp. $K_{v_2}(S_2, H_2)$) if and only if $m_1 = m_2$ and $w_1^2 = w_2^2$.* □

The sufficient condition is essentially proved by Yoshioka in [54] and [55] (see even [47]). The converse is proved in a forthcoming paper of the author and Rapagnetta. As a consequence we see that in dimension $2n$ we find a deformation class of irreducible symplectic varieties of dimension $2n$ for each pair (m, k) such that $m^2k + 1 = n$ or $m^2k - 1 = n$ (just take the moduli space of semistable sheaves of Mukai vector $(m, 0, -mk)$).

Remark 5 The deformation classes in dimension 4 and 6 we obtain in this way are different from those of the examples of singular irreducible symplectic V-manifolds we presented in Sect. 2.1.2. Indeed, if S is K3 we have $\dim(M_v) = 4, 6$ only if $v^2 = 2, 4$, so v must be primitive and M_v is then smooth. If S is Abelian we have $\dim(K_v) = 4, 6$ only if $v^2 = 6, 8$. If $v^2 = 6$ then v is primitive and K_v smooth; if $v^2 = 8$ then either v is primitive and K_v is smooth, or $v = 2w$ for w primitive with w^2, and K_v has a symplectic resolution. □

Finally, we remark that the singular moduli spaces in the statement of Proposition 6 give examples of irreducible symplectic varieties which are not irreducible symplectic V-manifolds, as their singularities are not quotient singularities.

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An Example of Mirror Symmetry for Fano Threefolds

Andrea Petracci

Abstract In this note we illustrate the Fanosearch programme of Coates, Corti, Galkin, Golyshev, and Kasprzyk in the example of the anticanonical cone over the smooth del Pezzo surface of degree 6.

Keywords Mirror symmetry · Fano varieties · Toric varieties · Deformation theory

2010 Mathematics Subject Classification 14J45, 14M25, 14J33, 14D15

1 Introduction

1.1 Aim

The Fanosearchprogramme of Coates et al. [8] studies Fano varieties via Mirror Symmetry. In this context it is crucial to study toric degenerations of smooth Fano varieties, or conversely smoothings of toric Fano varieties. Toric Fano varieties are associated to certain lattice polytopes, called Fano polytopes; some combinatorial input on a Fano polytope conjecturally allows to construct a deformation of the corresponding toric Fano. This is also reflected by Mirror Symmetry, where the combinatorial input is encoded by certain special Laurent polynomials. The goal of this note is to illustrate this programme in a specific example where two different combinatorial inputs on the same polytope produce two different smoothings of the same toric Fano threefold.

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1.2 The Example

The example we consider is the projective cone over the anticanonical embedding of the smooth del Pezzo surface of degree 6. This threefold, denoted by X, is a toric Fano and has an isolated Gorenstein canonical non-terminal singularity at the vertex of the cone. The deformations of this singularity have been studied by Altmann [4]; we will recall Altmann's results in Sect. 2.3. In Sect. 2.4 we will see that the base of the miniversal deformation (or equivalently the Kuranishi family) of the projective threefold X has two irreducible components, which deform X to two different smooth Fano threefolds, namely:

- a general element X_2 of the linear system $|\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 1)|$,
- $X_3 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

(The reason for the subscripts 2 and 3 will be evident later.) These two smooth Fanos are connected via deformation through X, but cannot be connected via a deformation with smooth fibres, as their Betti numbers are different.

As X is toric, by means of toric geometry, we can associate to X a 3-dimensional lattice polytope $P \subseteq \mathbb{R}^3$ which is a hexagonal pyramid (see the precise definition in (2) and Fig. 1). The hexagonal facet of P is denoted by F (see (1) and the left part of Fig. 2). In Proposition 2.2 we will see that the two smoothings of X are associated to some combinatorial additional data on the polytope P. More precisely, they correspond to the two maximal Minkowski decompositions of the hexagon F (see (3) and (4), and Fig. 2). We will introduce the notion of Minkowski sum and Minkowski decomposition in Sect. 2.3.

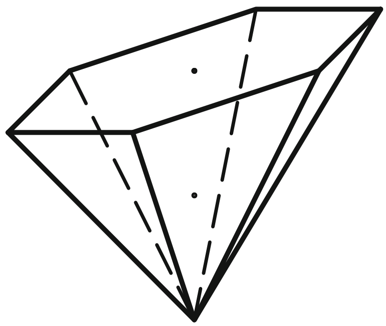


Fig. 1 The 3-dimensional lattice polytope P associated to X

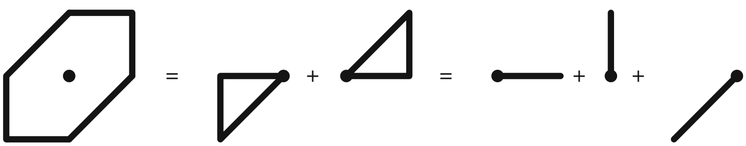


Fig. 2 The two maximal Minkowski decompositions of the hexagon F

Now we consider the Laurent polynomials in 3 variables which are supported on P, i.e. Laurent polynomials $f \in \mathbb{C}[x^{\pm}, y^{\pm}, z^{\pm}]$ such that if the monomial $x^i y^j z^k$ appears in f then the point $(i, j, k) \in \mathbb{Z}^3$ lies in P. Among these Laurent polynomials, we consider those which have coefficient 1 on the vertices of P and have coefficient 0 on the origin of $\mathbb{R}^3$; this gives rise to the following 1-dimensional family:

$$f_a = z\left(a + x + xy + y + \frac{1}{x} + \frac{1}{xy} + \frac{1}{y}\right) + \frac{1}{z}$$

with parameter $a \in \mathbb{C}$.

One can show that f_2 is mirror to X_2 and f_3 is mirror to X_3. In other words, a certain generating function for some Gromov–Witten invariants of X_2, called *quantum period* (see Sect. 3.1), is equal to a certain power series, called *classical period* (see Sect. 3.3), associated to f_2, and the same holds for X_3 and f_3. Here we are using the formulation of the Mirror Symmetry correspondence between Fanos and Landau–Ginzburg models that is given in [8, 31, 32] and summarised in Sect. 3.4.

We will see that the Laurent polynomial f_2 is closely related to the combinatorial input given by the Minkowski decomposition of the hexagon F which is associated to the smoothing of X to X_2. Analogously, f_3 is closely related to the Minkowski decomposition of the hexagon F which is associated to the smoothing of X to X_3.

1.3 The General Picture

What we have described in the case of the projective cone over the del Pezzo surface of degree 6 is an instance of the following conjecture, which is still slightly vague.

Conjecture 1.1 ([8]) Let Q be a Fano polytope of dimension 3 and let X_Q be the corresponding toric Fano threefold. Assume that X_Q has Gorenstein singularities. From some "combinatorial input" on Q one constructs

(i) a smoothing V of X_Q and
(ii) a Laurent polynomial f supported on Q

such that f is mirror to V.

The definition of "mirror" that we are using comes from [8, 31, 32] and is given in Definition 3.4.

If the toric variety X_Q is smooth (there are 18 cases), then the polytope Q has only triangular facets which are standard simplices and X_Q is rigid. Thus $V = X_Q$ and f is uniquely determined by insisting that it has coefficient 1 on vertices of Q and coefficient 0 on the origin. This case was already known by Givental [17, 18] who proved that f is mirror to X_Q.

In the example considered in this note, the combinatorial input on P is the choice of a maximal Minkowski decomposition of the facet F of P. There are two such choices which lead to two different smoothings of X and to two different Laurent polynomials.

An interesting case, which is not too restrictive, is the following: the combinatorial input is the choice of a Minkowski decomposition of each facet of Q into A-triangles. Here an A-triangle is either a unitary segment or a lattice triangle which is $\mathbb{Z}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$-equivalent to the convex hull of the points $(0, 0)$, $(0, 1)$, $(\ell, 0)$, for some integer $\ell \geq 1$. For example, both maximal Minkowski decompositions of the hexagon F are decompositions into A-triangles. In these circumstances one can easily construct a Laurent polynomial f which is supported on Q and depends on the choice of the Minkowski decompositions of the facets of Q (see [1], where such Laurent polynomial f is called a *Minkowski polynomial*). In joint work with Corti and Hacking [11], we construct a smoothing V of X_Q, under a slight additional assumption which is necessary by Petracci [29]. It is conjectured that f is mirror to V. However, even in this situation we completely lack a conceptual way to prove that f is mirror to V.

Unfortunately, there exist polytopes Q which have facets without Minkowski decompositions into A-triangles. So, at the moment, it is not clear what sort of combinatorial input we should consider on Q in the general case.

Another approach to construct smoothings of the toric Fano variety X_Q is pursued by Coates et al. [10, 30]; they embed X_Q into a bigger toric variety Z and try to deform it inside Z. This works very well in many explicit examples, but a general framework has yet to be discovered.

Finally, it is worth mentioning that Conjecture 1.1 can be stated in all dimensions. Therefore, this might be a way to classify smooth Fano varieties that admit a toric degeneration.

1.4 Notation and Conventions

In a real vector space of finite dimension, a polyhedron is the intersection of finitely many closed half-spaces and a polytope is a compact polyhedron; equivalently, a polytope is the convex hull of a finite set. We denote by $\mathrm{conv}\{\cdot\}$ the convex hull of a set.

All varieties and schemes are defined over $\mathbb{C}$. We always use the following notation.

dP_6	the smooth del Pezzo surface of degree 6
X	the projective cone over the anticanonical embedding of dP_6
U	the affine cone over the anticanonical embedding of dP_6
X_2	a general effective divisor of type $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$
X_3	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
F	the lattice polygon associated to dP_6 (see (1) and the left part of Fig. 2)

P	the lattice polytope associated to X (see (2) and Fig. 1)
f_a	$z\left(a + x + xy + y + x^{-1} + x^{-1}y^{-1} + y^{-1}\right) + z^{-1}$, for each $a \in \mathbb{C}$
Q	an arbitrary Fano polytope (see Definition 2.1)
X_Q	the Fano toric variety associated to the Fano polytope Q

2 The Geometry of X

2.1 Toric Geometry

We now recall the basics of Fano toric varieties. We refer the reader to [12, §8.3], [15, p. 25], and [22].

Definition 2.1 Let N be a lattice of rank n. A *Fano polytope* in N is an n-dimensional polytope $Q \subseteq N_{\mathbb{R}}$ such that the origin $0 \in N$ lies in the interior of Q and every vertex of Q is a primitive lattice point of N.

The *spanning fan* of a Fano polytope Q in N is the complete fan whose cones are the cones over the proper faces of Q. We denote by X_Q the toric variety associated to the spanning fan of a Fano polytope Q.

For brevity, we say that X_Q is associated to Q, and conversely. If Q is a Fano polytope of dimension n, then X_Q is an n-dimensional complete toric variety which is Fano, i.e. its anticanonical divisor is $\mathbb{Q}$-Cartier and ample. Every Fano toric variety arises in this way from a Fano polytope.

For example, consider the hexagon

$$F = \operatorname{conv}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right\} \subseteq \mathbb{R}^2, \tag{1}$$

which is depicted on the left of Fig. 2. It is clear that F is a Fano polytope in $\mathbb{Z}^2$. The toric variety associated to its spanning fan is the smooth del Pezzo surface of degree 6, denoted by dP_6. The anticanonical map of dP_6 is a closed embedding into $\mathbb{P}^6$.

Now imagine to put the hexagon F into the plane $\mathbb{R}^2 \times \{1\}$ in $\mathbb{R}^3$ and create the pyramid over it with apex at the point $(0, 0, -1)$: this is the polytope

$$P = \operatorname{conv}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\} \subseteq \mathbb{R}^3 \tag{2}$$

and is depicted in Fig. 1. It is clear that P is a Fano polytope in $\mathbb{Z}^3$. Let X be the toric variety associated to the spanning fan of P. Let U be the affine toric open subscheme of X associated to the hexagonal face of P, i.e. U is the affine toric variety associated to the cone $\mathbb{R}_{\geq 0}(F \times \{1\})$. Hence X (resp. U) is the projective (resp. affine) cone over the anticanonical embedding of dP_6. We have that X is a Fano threefold with an isolated non-terminal canonical Gorenstein singularity at the vertex of the cone.

2.2 Equations

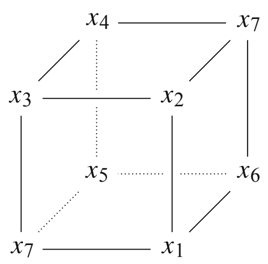
The equations of the three closed embeddings $\mathrm{dP}_6 \subseteq \mathbb{P}^6$, $U \subseteq \mathbb{A}^7$ and $X \subseteq \mathbb{P}^7$ are the same and can be conveniently described in two ways. Here, $x_1, \dots, x_7$ denote the homogeneous coordinates of $\mathbb{P}^6$, the affine coordinates of $\mathbb{A}^7$ and the last homogeneous coordinates of $\mathbb{P}^7$, as x_0 is the remaining homogeneous coordinate of $\mathbb{P}^7$.

The first way is:

$$\operatorname{rank}\begin{pmatrix} x_7 & x_1 & x_2 \\ x_4 & x_7 & x_3 \\ x_5 & x_6 & x_7 \end{pmatrix} \leq 1.$$

Note the repetition of x_7 on the diagonal. If two of the x_7's had been two extra variables, these would have been the equations of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$. This shows that X is the intersection of the projective cone over the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ with two hyperplanes of $\mathbb{P}^9$ passing through the vertex.

Now consider the cube



where at the vertices there are the variables $x_1, \dots, x_7$. Note the repetition of x_7. The second way to describe the equations is to consider the determinants of all rectangles which can be formed with edges of the cube or with diagonals of faces of the cube. If one of the x_7's had been an extra variable, these would have been the equations of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^7$. This shows that X is the intersection of the projective cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with a hyperplane of $\mathbb{P}^8$ passing through the vertex.

The equations above also appear in [20, Example 3.3]. Moreover, these two ways of describing the equations of X in $\mathbb{P}^7$ are called Tom and Jerry, respectively, in [5].

2.3 Minkowski Sums and Deformations of U

We first define the notion of Minkowski sum of polyhedra (for instance see [34, §1.1]). If $\Pi_1, \dots, \Pi_r$ are polyhedra in a real vector space, we define their *Minkowski sum* to be the polyhedron

$$\Pi_1 + \cdots + \Pi_r := \{p_1 + \cdots + p_r \mid p_1 \in \Pi_1, \dots, p_r \in \Pi_r\}.$$

When we have $\Pi = \Pi_1 + \cdots + \Pi_r$, we say that we have a *Minkowski decomposition* of the polyhedron Π. We consider Minkowski decompositions up to translation: for instance, we consider the Minkowski decomposition $(p + \Pi_1) + (-p + \Pi_2)$ to be equivalent to $\Pi_1 + \Pi_2$ for every vector p. Moreover, in what follows we require that the summands Π_j are lattice polyhedra, i.e. their vertices belong to a fixed lattice.

The hexagon F has two maximal Minkowski decompositions (see Fig. 2): one into 3 unitary segments

$$F = \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\} + \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} + \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\} \tag{3}$$

and one into 2 triangles

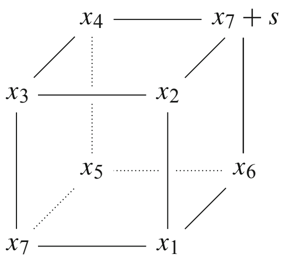
$$F = \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\} + \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}. \tag{4}$$

Altmann [3] has noticed that Minkowski decompositions of polytopes induce deformations of affine toric varieties (see also [25] and [28]). More precisely, from a Minkowski decomposition of a polytope Π it is possible to construct an unobstructed deformation of the affine toric variety associated to the cone $\mathbb{R}_{\geq 0}(\Pi \times \{1\})$.

In the case at hand, the Minkowski decomposition (3) induces the deformation of U over $\operatorname{Spec} \mathbb{C}[u, v]$ given by the equations

$$\operatorname{rank}\begin{pmatrix} x_7 & x_1 & x_2 \\ x_4 & x_7 + u & x_3 \\ x_5 & x_6 & x_7 + v \end{pmatrix} \leq 1.$$

The Minkowski decomposition (4) induces the deformation of U over $\operatorname{Spec} \mathbb{C}[s]$ given by the equations obtained by taking minors of rectangles on edges and diagonals of faces of the following cube.



Moreover, Altmann [4] shows that the miniversal deformation of U is (the completion of) the union of these two deformations and its base is $\mathbb{C}[\![s, u, v]\!]/(su, sv)$.

2.4 The Two Smoothings of X

Now we want to study deformations of X.

Proposition 2.2 *The base of the miniversal deformation of X is* $\mathbb{C}[\![s, u, v]\!]/(su, sv)$ *and has two irreducible components. The 2-dimensional component* $(s = 0)$ *is associated to the Minkowski decomposition* (3) *and deforms X to a general divisor* $X_2 \in |\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1, 1)|$. *The 1-dimensional component* $(u = v = 0)$ *is associated to the Minkowski decomposition* (4) *and deforms X to* $X_3 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Proof Consider the local-to-global spectral sequence for $\mathrm{Ext}^\bullet_X(\Omega_X, \mathcal{O}_X)$: the second page is $E_2^{p,q} = \mathrm{H}^q(X, \mathcal{E}xt^p_X(\Omega_X, \mathcal{O}_X))$. As X has an isolated singularity, for all $p \geq 1$, the sheaf $\mathcal{E}xt^p_X(\Omega_X, \mathcal{O}_X)$ is supported on the singular point of X; therefore, for all $p \geq 1$ and $q \geq 1$, $E_2^{p,q} = 0$.

Let $j : W \hookrightarrow X$ be the smooth locus. The sheaves $\mathcal{H}om_X(\Omega_X, \mathcal{O}_X)$ and $j_*\Omega^2_W \otimes \mathcal{O}_X(-K_X)$ are the same, because they are both reflexive and coincide on W. As X is toric and $-K_X$ is ample, by Bott–Steenbrink–Danilov vanishing [12, Theorem 9.3.1] (see also [6, 14, 26]) one has $E_2^{0,q} = \mathrm{H}^q(X, \mathcal{H}om_X(\Omega_X, \mathcal{O}_X)) = 0$ for all $q \geq 1$. This argument comes from the proof of [33, Theorem 5.1].

Therefore E_2 is zero outside the line $q = 0$. This implies that, for all $p \geq 0$, the natural map

$$\mathrm{Ext}^p_X(\Omega_X, \mathcal{O}_X) \to \mathrm{H}^0(X, \mathcal{E}xt^p_X(\Omega_X, \mathcal{O}_X))$$

is an isomorphism. Since the unique singular point of X is contained in U and U is affine, we have $\mathrm{H}^0(X, \mathcal{E}xt^p_X(\Omega_X, \mathcal{O}_X)) = \mathrm{Ext}^p_U(\Omega_U, \mathcal{O}_U)$ for all $p \geq 1$. This implies that, for all $p \geq 1$, the natural map

$$\phi_p : \mathrm{Ext}^p_X(\Omega_X, \mathcal{O}_X) \to \mathrm{Ext}^p_U(\Omega_U, \mathcal{O}_U),$$

is an isomorphism.

We now consider the functors of infinitesimal deformations of X and U: Def_X and Def_U, which are covariant functors from the category of local finite $\mathbb{C}$-algebras to the category of sets (see [24, §3]). There is an obvious map $\phi\colon Def_X \to Def_U$, which restricts a deformation of X to U. Since X is normal, $\mathrm{Ext}^1_X(\Omega_X, \mathcal{O}_X)$ is the tangent space of Def_X and $\mathrm{Ext}^2_X(\Omega_X, \mathcal{O}_X)$ is an obstruction space for Def_X, and a similar statement holds for U. Since ϕ_1 is bijective and ϕ_2 is injective, we have that ϕ induces an isomorphism on tangent spaces and an injection on obstruction spaces. Therefore, by [24, Remark 4.12], ϕ is smooth and induces an isomorphism on tangent spaces. In particular, the two functors Def_X and Def_U have the same hull, i.e. the bases of the miniversal deformations of X and U are the same.

The equations of the two deformations of U, given in Sect. 2.3, can be projectivised to construct deformations of X: it is enough to replace s, u and v by sx_0, ux_0 and vx_0. These are the two components of the miniversal deformation of X. The fact that they are associated to the two Minkowski decompositions (3) and (4) of the hexagon F follows from the discussion in Sect. 2.3.

From Sect. 2.2 we know that X is the intersection of the projective cone over the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ with two hyperplanes of $\mathbb{P}^9$ passing through the vertex of the cone. On the component $(s = 0)$, in the deformation we are moving these two hyperplanes away from the vertex. Therefore, the general fibre over this component is X_2, a general $(1, 1)$-divisor in $\mathbb{P}^2 \times \mathbb{P}^2$.

Recall that X is the intersection of the projective cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with a hyperplane of $\mathbb{P}^8$ passing through the vertex. On the component $(u = v = 0)$, in the deformation we are moving this hyperplane of $\mathbb{P}^8$ away from the vertex. Therefore, the general fibre on this component is $X_3 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. □

3 Mirror Symmetry

3.1 Gromov–Witten Invariants and Quantum Periods

The quantum period of a smooth Fano variety is a generating function for some genus zero Gromov–Witten invariants. The regularised quantum period is a slightly different version, which is convenient for our description of Mirror Symmetry.

Definition 3.1 ([8, 9, 31]) The *quantum period* and the *regularised quantum period* of a smooth Fano variety V are the following power series

$$G_V(t) = 1 + \sum_{\beta \in \mathrm{H}_2(V,\mathbb{Z})} \langle [\mathrm{pt}]\psi^{-K_V\cdot\beta-2}\rangle^V_{0,1,\beta}\, t^{-K_V\cdot\beta} \in \mathbb{Q}[\![t]\!]$$

$$\widehat{G}_V(t) = 1 + \sum_{\beta \in \mathrm{H}_2(V,\mathbb{Z})} (-K_V\cdot\beta)!\langle [\mathrm{pt}]\psi^{-K_V\cdot\beta-2}\rangle^V_{0,1,\beta}\, t^{-K_V\cdot\beta} \in \mathbb{Q}[\![t]\!]$$

where $\langle[\mathrm{pt}]\psi^{-K_V\cdot\beta-2}\rangle^V_{0,1,\beta}$ denotes the 1-marked genus zero Gromov–Witten invariant of curve class β associated to the cohomology class of a point in V and gravitational descendant of order $-K_V\cdot\beta-2$.

Roughly speaking, $\langle[\mathrm{pt}]\psi^{-K_V\cdot\beta-2}\rangle^V_{0,1,\beta}$ is the number of rational curves in V of class β passing through a fixed general point of V and satisfying a certain condition on their complex structure. Therefore, the quantum period G_V gives information about rational curves in V. The series G_V is a symplectic invariant of V, so it does not change if V is deformed to another smooth Fano variety through a deformation with smooth fibres.

If the anticanonical line bundle $\mathcal{O}_V(-K_V)$ is divisible by a positive integer m inside the Picard group of the smooth Fano variety V, then only powers of t^m appear in the (regularised) quantum period of V.

It is also possible to define quantum periods for Fano varieties with quotient singularities [27, §3.3].

It is known how to compute the quantum period of smooth Fano varieties which are either toric or complete intersections in smooth Fano toric varieties [7, 17]. The quantum periods of all smooth Fano varieties of dimension ≤ 3 have been computed by Coates et al. [9]. In particular, we have the following formulae for X_2 and X_3.

Proposition 3.2 ([9]) *The quantum periods and the regularised quantum periods of X_2 and X_3 are the following.*

$$G_{X_2}(t)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(l+m)!}{(l!)^3(m!)^3}t^{2l+2m}$$

$$G_{X_3}(t)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(l!)^2(m!)^2(n!)^2}t^{2l+2m+2n}$$

$$\widehat{G}_{X_2}(t)=1+4t^2+60t^4+1120t^6+24220t^8+567504t^{10}+\cdots$$

$$\widehat{G}_{X_3}(t)=1+6t^2+90t^4+1860t^6+44730t^8+1172556t^{10}+\cdots$$

3.2 Laurent Polynomials

Let $\mathbb{C}[x_1^{\pm},\ldots,x_n^{\pm}]$ be the ring of Laurent polynomials in n variables with coefficients in $\mathbb{C}$. To every monomial $x^{\boldsymbol{i}}=x_1^{i_1}\cdots x_n^{i_n}$ we associate the point $\boldsymbol{i}=(i_1,\ldots,i_n)\in\mathbb{Z}^n$. The *Newton polytope* of a Laurent polynomial f is the convex hull of the lattice points that correspond to the monomials that appear in f, i.e. if

$f = \sum_{\boldsymbol{i} \in \mathbb{Z}^n} a_{\boldsymbol{i}} x^{\boldsymbol{i}}$ then

$$\mathrm{Newt}(f) = \mathrm{conv}\left\{\boldsymbol{i} \in \mathbb{Z}^n \mid a_{\boldsymbol{i}} \neq 0\right\} \subseteq \mathbb{R}^n.$$

If Q is a lattice polytope in $\mathbb{Z}^n$, we say that a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ is *supported* on Q if every monomial appearing in f corresponds to a lattice point of Q, or equivalently if $\mathrm{Newt}(f) \subseteq Q$.

Given a Fano polytope Q in $\mathbb{Z}^n$, Kasprzyk and Tveiten [21] have introduced and studied a particular class of Laurent polynomials supported on Q; they call them *maximally mutable*, because these behave well with respect to mutations of Fano polytopes [1]. The definition of maximally mutable Laurent polynomials in dimension 2 can be also found in [2]. We are not going to define maximally mutable Laurent polynomials here, we just mention some properties in a particular case.

In dimension 3, when the Fano toric threefold X_Q has Gorenstein singularities (equivalently Q is a reflexive polytope of dimension 3), every maximally mutable Laurent polynomial on Q is such that:

- the coefficient of the monomial 1, corresponding to the origin of $\mathbb{Z}^3$, is 0;
- the monomials corresponding to the vertices of Q have coefficients equal to 1;
- on the edges of Q there are binomial coefficients. (For example, the 4 lattice points of an edge with lattice length 3 have coefficients 1, 3, 3, 1.)

In the case of the polytope P, a Laurent polynomial is supported on P if and only if its monomials are among $1, xz, xyz, yz, x^{-1}z, x^{-1}y^{-1}z, y^{-1}z, z^{-1}$, which correspond to the lattice points of P. The Laurent polynomials on P which satisfy the three properties above form a 1-dimensional family

$$f_a = z\left(x + xy + y + \frac{1}{x} + \frac{1}{xy} + \frac{1}{y} + a\right) + \frac{1}{z}$$

with parameter $a \in \mathbb{C}$. Here a is the coefficient of the centre of the hexagonal facet of P. Kasprzyk and Tveiten [21] show that there are exactly two maximally mutable Laurent polynomials on P, namely f_a with $a = 2$ and $a = 3$. One notices that, in these two cases, the restriction of f_a to the hexagonal facet of P is reducible:

$$f_2 = z(1+x)(1+y)(1+x^{-1}y^{-1}) + z^{-1},$$
$$f_3 = z(1+x^{-1}y^{-1}+y^{-1})(1+xy+y) + z^{-1}.$$

The Newton polytopes of the three factors of $(1+x)(1+y)(1+x^{-1}y^{-1})$ are the three unitary segments appearing in the Minkowski decomposition (3) of the hexagon F. The Newton polytopes of the factors of $(1+x^{-1}y^{-1}+y^{-1})(1+xy+y)$ are the two triangles appearing in the Minkowski decomposition (4) of the hexagon F. The Laurent polynomials f_2 and f_3 are *Minkowski polynomials* in the sense of [1].

3.3 Classical Periods

We now define the classical period of a Laurent polynomial in n variables.

Definition 3.3 ([1, 16]) The *classical period* of $f \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ is the power series

$$\pi_f(t) = \left(\frac{1}{2\pi \mathrm{i}}\right)^n \int_{\Gamma_\varepsilon} \frac{1}{1 - tf(x_1, \dots, x_n)} \frac{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}{x_1 \cdots x_n}$$
$$= \sum_{k=0}^{\infty} \mathrm{coeff}_1(f^k) t^k$$

where in the first formula we are integrating a holomorphic n-form of the torus $(\mathbb{C}^\times)^n = \mathrm{Spec}\, \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ over the real torus $\Gamma_\varepsilon = \{|x_1| = \cdots = |x_n| = \varepsilon\} \subseteq (\mathbb{C}^\times)^n$, for some $0 < \varepsilon \ll 1$, and $\mathrm{coeff}_1(f^k) \in \mathbb{C}$ is the coefficient of the monomial $1 = x_1^0 \cdots x_n^0$ in the Laurent polynomial f^k.

The equality between the two formulae in the definition above comes from applying Cauchy's integral formula n times. The classical period π_f is related to the Hodge theory of the fibres of $f\colon (\mathbb{C}^\times)^n \to \mathbb{A}^1$.

One can see that the classical period of the Laurent polynomial f_a is

$$\pi_{f_a}(t) = 1 + 2at^2 + (6a^2 + 36)t^4 + (20a^3 + 360a + 240)t^6$$
$$+ (70a^4 + 2520a^2 + 3360a + 6300)t^8$$
$$+ (252a^5 + 15120a^3 + 30240a^2 + 113400a + 90720)t^{10} + \cdots$$

for every $a \in \mathbb{C}$. In particular,

$$\pi_{f_2}(t) = 1 + 4t^2 + 60t^4 + 1120t^6 + 24220t^8 + 567504t^{10} + \cdots,$$
$$\pi_{f_3}(t) = 1 + 6t^2 + 90t^4 + 1860t^6 + 44730t^8 + 1172556t^{10} + \cdots.$$

3.4 Fano/Landau–Ginzburg Correspondence

Mirror Symmetry [8, 23] predicts that the mirror of a smooth Fano n-fold V is a pair (Y, w), called *Landau–Ginzburg model*, where Y is an n-fold and $w \in \Gamma(Y, \mathcal{O}_Y)$ is a regular function. The Gromov–Witten theory of V should be related to the Hodge theory of the fibres of $w\colon Y \to \mathbb{A}^1$ as follows: the regularised quantum period

$\widehat{G}_V$ (see Definition 3.1) of V coincides with the period π_w which is defined as

$$\pi_w(t) = \int_\Gamma \frac{\Omega}{1 - tw} \tag{5}$$

where Ω is an appropriate holomorphic n-form on Y and $\Gamma \in \mathrm{H}_n(Y; \mathbb{Z})$ is such that $\int_\Gamma \Omega = 1$.

Under some circumstances (which conjecturally and experimentally should coincide with when there is a toric degeneration of V) there is an open subset of Y that is isomorphic to the torus $(\mathbb{C}^\times)^n = \operatorname{Spec} \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$. In this case the restriction of w to this open subset gives a Laurent polynomial $f \in \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$. In this situation the period π_w in (5), when $Y = (\mathbb{C}^\times)^n$, $\Gamma = \{|x_1| = \cdots = |x_n| = \varepsilon\}$ and $\Omega = (2\pi \mathrm{i})^{-n}(x_1 \cdots x_n)^{-1} \mathrm{d}x_1 \cdots \mathrm{d}x_n$, becomes the classical period of a Laurent polynomial (see Definition 3.3).

Thus, a down-to-earth formulation of Mirror Symmetry between smooth Fano varieties and Laurent polynomials is the following.

Definition 3.4 ([8, 31, 32]) A Laurent polynomial $f \in \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$ is *mirror* to a smooth Fano variety V of dimension n if the classical period of the former coincides with the regularised quantum period of the latter: $\widehat{G}_V = \pi_f$.

The equality $\widehat{G}_V = \pi_f$ is equivalent to G_V being equal to the oscillatory integral

$$\left(\frac{1}{2\pi \mathrm{i}}\right)^n \int_{\Gamma_\varepsilon} \mathrm{e}^{tf} \frac{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}{x_1 \cdots x_n} = \sum_{k=0}^{\infty} \frac{\operatorname{coeff}_1(f^k)}{k!} t^k.$$

Moreover, the equality $\widehat{G}_V = \pi_f$ can be upgraded to an equality between the Gauss–Manin connection on the middle cohomology of the fibres of f and the Dubrovin connection of the quantum D-module of V (see [19]).

Proposition 3.5 ([1, 9]) *The Laurent polynomial f_2 (resp. f_3) is mirror to the smooth Fano threefold X_2 (resp. X_3).*

Proof Set $a = 2$ or $a = 3$. We need to show that the two power series π_{f_a} and $\widehat{G}_{X_a}$ coincide. By comparing the formulae given in Proposition 3.2 and at the end of Sect. 3.3, one can check the equality of finitely many coefficients. In order to prove the equality of all coefficients, one has to show that π_{f_a} and $\widehat{G}_{X_a}$ satisfy the same linear differential equation; this is done in [1] and [9]. □

Now we are ready to illustrate Conjecture 1.1 in the example of the projective cone over the smooth del Pezzo surface of degree 6, which is the running example of this note.

In Proposition 2.2 we saw that the Minkowski decomposition (3) of the hexagonal facet of P into three unitary segments is associated to the smoothing of X to X_2. In Sect. 3.2 we saw that the restriction of f_2 to the hexagonal facet of P is reducible and that the Newton polytopes of its three factors are the three unitary segments appearing in the Minkowski decomposition (3). By Proposition 3.5 we know that f_2 is mirror to X_2. This is an instance of Conjecture 1.1: from the combinatorial input of the Minkowski decomposition of the hexagonal facet of P into three unitary segments we have constructed the smoothing X_2 of X and the Laurent polynomial f_2 which is mirror to X_2.

In a completely analogous manner, we can observe that the Minkowski decomposition (4) of the hexagonal facet of P into two triangles induces the smoothing X_3 of X and the Laurent polynomial f_3. This provides another example for Conjecture 1.1 because f_3 is mirror to X_3.

As mentioned in Sect. 1.3, given a reflexive polytope Q of dimension 3, from the combinatorial datum given by the choice of a Minkowski decomposition of each facet of Q into A-triangles, one constructs an associated Laurent polynomial f supported on Q. From the same combinatorial datum on Q (with a slight additional condition which we do not mention here), by Corti et al. [11] it is possible to construct a smoothing V of the toric Fano threefold X_Q associated to Q. It is conjectured that the smooth Fano threefold V is mirror to the Laurent polynomial f.

This circle of ideas should be considered as an approach to the problem of classifying smooth Fano varieties of dimension ≥ 4. Indeed, computers can classify Fano polytopes; therefore, once one has developed a combinatorial technology for smoothing toric Fano varieties, one should be able to construct all smooth Fano varieties which admit a toric degeneration.

There is another difficulty: a smooth Fano variety may have many toric degenerations, hence may arise from several polytopes. For instance, $X_3 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is itself toric and degenerates to the toric Fano X. Conjecturally, these many toric degenerations of a smooth Fano correspond to many mirror Laurent polynomials; these Laurent polynomials are related via certain birational transformations of the torus $(\mathbb{C}^\times)^n$, which are called *mutations* [1, 13, 16] and preserve the classical periods. Therefore, it is conjectured that deformation families of smooth Fano varieties of dimension n are in one-to-one correspondence with mutation-equivalence classes of some "special" Laurent polynomials in n variables. We are not going to expand on this here because otherwise it would lead us far beyond the scope of this note.

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Chern Numbers of Uniruled Threefolds

Stefan Schreieder and Luca Tasin

Abstract In this paper we show that the Chern numbers of a smooth Mori fibre space in dimension three are bounded in terms of the underlying topological manifold. We also generalise a theorem of Cascini and the second named author on the boundedness of Chern numbers of certain threefolds to the case of negative Kodaira dimension.

Keywords Minimal model program · Mori fibre spaces · Three-folds · Topological properties of complex manifolds · Characteristic classes and numbers

1 Introduction

One of the most basic numerical invariants of a compact complex manifold are its Chern numbers. While these numbers depend only on the topological type of the complex structure of the tangent bundle, they are in general not invariants of the underlying topological manifold, but really depend on the complex structure. In fact, answering a question of Hirzebruch from 1954, all linear combinations of Chern and Hodge numbers which are topological invariants of smooth complex projective varieties have recently been determined in [9–11].

Generalising Hirzebruch's question, Kotschick asked [8] (see also [12]) whether the topology of the underlying smooth manifold determines the Chern numbers of smooth complex projective varieties at least up to finite ambiguity. In [19], we have shown that in dimension at least four, this question has in general a negative answer. That is, there are smooth real manifolds that carry infinitely many complex algebraic structures such that the corresponding Chern numbers are unbounded, except for

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c_n, c_1c_{n-1} and c_2^2 which are known to be bounded (see [14] for the non-trivial one c_1c_{n-1}). This result left however open the case of threefolds, where it remains unknown whether c_1^3 is determined up to finite ambiguity by the underlying smooth manifold.

In [2], Cascini and the second named author started to investigate the boundedness question for Chern numbers via methods from the minimal model program, see also [16, 20] for further developments. In dimension three, the approach in [2] is motivated by the Miyaoka–Yau inequality, which implies that for a minimal smooth complex projective threefold of non-negative Kodaira dimension, c_1^3 can be bounded in terms of the Betti numbers of X, see e.g. [20, Proposition 9]. This observation makes it natural to approach the boundedness of $c_1^3(X)$ by trying to bound the effect on $c_1^3(X)$ of the steps in the minimal model program for X. This leads to a positive answer for the boundedness question for many smooth projective threefolds of non-negative Kodaira dimension whose minimal model program is a composition of blow-downs to points and smooth curves in smooth loci, see [2, Corollary 1.5].

In this paper we focus on the case of threefolds of negative Kodaira dimension. The main difficulty that we face in this case is that the aforementioned Miyaoka–Yau inequality, which was essential for the case of non-negative Kodaira dimension, does not hold any longer. It is also known by examples of LeBrun [13], that the boundedness does not hold in the non-Kähler case. Nonetheless, for any smooth Kähler threefold X we can run a minimal model program thanks to [6, 7]. If X is uniruled then we arrive at a Mori fibre space $Y \to B$, i.e. a Kähler threefold Y with at most terminal singularities together with a morphism of relative Picard rank one with connected fibres to a complex Kähler variety B of smaller dimension whose general fibre is Fano.

The first result of this paper is the following.

Theorem 1 *Let $(X_i)_{i\geq 0}$ be a sequence of Mori fibre spaces, where X_i are smooth Kähler threefolds. If each X_i is homeomorphic to X_0, then the sequence of Chern numbers $c_1^3(X_i)$ is bounded.*

The above result should be compared to the fact that all known examples of sequences of homeomorphic varieties with unbounded Chern numbers are Mori fibre spaces, and in fact projective bundles (see [19]). We therefore believe that together with the aforementioned results from [2], the above theorem puts forward strong evidence for the conjecture that the Chern numbers of smooth projective threefolds are determined up to finite ambiguity by the underlying smooth manifold.

If $X \to B$ is a Mori fibre space and X is a smooth Kähler threefold, then there are three main cases to consider, depending on the dimension of B. If B is a point, then X is a Fano variety and we conclude because Fano varieties of fixed dimension form a bounded family. If B is a curve, then it is smooth projective and X is also projective. Since the Pontryagin classes are up to torsion homeomorphism invariants by Novikov's theorem [17], [20, Proposition 26] proves the above theorem in case all but finitely many of the X_i are Mori fibre spaces over points or curves. Using Novikov's theorem [17] once again, Theorem 1 thus follows from the following more precise result about Mori fibre spaces over surfaces, where we denote by $H^*_{tf}(X, \mathbb{Z})$ the quotient of $H^*(X, \mathbb{Z})$ by the subgroup of all torsion classes.

Theorem 2 *Let $(X_i)_{i\geq 0}$ be a sequence of smooth Kähler threefolds admitting a conic bundle structure $f_i : X_i \longrightarrow S_i$ of relative Picard number 1 over a smooth Kähler surface S_i. If there is an isomorphism of graded rings $H^*_{tf}(X_i, \mathbb{Z}) \simeq H^*_{tf}(X_0, \mathbb{Z})$ which respects the first Pontryagin classes, then the sequence of Chern numbers $c_1^3(X_i)$ is bounded.*

In view of Theorem 1, it is therefore natural to wonder if we can also bound the Chern numbers of certain threefolds of negative Kodaira dimension which are not necessarily Mori fibre spaces themselves. Our next result achieves this by generalising [2, Corollary 1.5] to the case of negative Kodaira dimension. To state it, recall that for any smooth complex projective threefold X, there is a cubic form F_X on $H^2(X, \mathbb{Q})$, given by cup product. For technical reasons, we will assume that the discriminant Δ_{F_X} of the cubic form is non-zero.

Theorem 3 *Let X be a smooth complex projective threefold which is uniruled and let F_X be its associated cubic. Assume that $\Delta_{F_X} \neq 0$ and that there exists a birational morphism $f : X \to Y$ onto a Mori fibre space Y, which is obtained as a composition of divisorial contractions to points and blow-downs to smooth curves in smooth loci.*

Then there exists a constant D depending only on the topology of the 6-manifold underlying X such that

$$|K_X^3| \leq D.$$

A major step in proving Theorem 3 is Proposition 8 (cf. [2, Theorem 1.3(2)]), where we show that in the assumptions of Theorem 3, most of the topological invariants of Y are determined (up to finite ambiguity) a priori by the smooth manifold underlying X. It would be interesting to understand to what extend this is true in general (see [4] for the case of Betti numbers):

Question 4 Let X be a smooth complex projective threefold with cubic form F_X and first Pontryagin class $p_1(X)$. Let $\mathcal{P}$ be the set of pairs $(F_Y, p_1(Y))$, taken up to isomorphism, such that there exists an MMP $X \dashrightarrow Y$. Is the set $\mathcal{P}$ determined by the pair of invariants $(F_X, p_1(X))$ of X up to finite ambiguity?

1.1 Conventions

All manifolds are closed and connected. A Kähler manifold is a complex manifold which admits a Kähler metric. For any (Kähler) manifold X, we denote by $H^*_{tf}(X, \mathbb{Z})$ the quotient $H^*(X, \mathbb{Z})/H^*(X, \mathbb{Z})_{tors}$, where $H^*(X, \mathbb{Z})_{tors}$ denotes the torsion subgroup of $H^*(X, \mathbb{Z})$.

2 Mori Fibre Spaces Over Surfaces

The starting point of our investigation is the following lemma.

Lemma 5 ([21, Sec. 7.1]) *Let $f : X \to S$ be a Mori fibre space such that X is a smooth projective threefold and S is a surface. Then*

(i) $f : X \to S$ *is a standard (i.e. relative Picard number 1) conic bundle and S is smooth;*
(ii) the discriminant $D \subset S$ of f is either empty or a reduced curve with at worst ordinary double points;
(iii) $e(X) = 2(e(S) - p_a(D) + 1)$, $b_1(X) = b_1(S)$ *and* $b_3(X) = 2(b_1(X) + p_a(D) - 1)$*;*
(iv) $D \equiv -f_*K_{X/S}^2$ *and* $-4K_S \equiv f_*K_X^2 + D$.

We will also use the following lemma.

Lemma 6 *Let $f : X \to S$ be a Mori fibre space such that X is a smooth projective threefold and S is a surface. Then,*

$$f_*p_1(X) \equiv -3D,$$

where $p_1(X)$ is the first Pontryagin class of X and $D \subset S$ denotes the discriminant curve of f.

Proof Since S is smooth projective by Lemma 5, the Néron–Severi group of S is generated by very ample curves. Hence, it suffices to compute the intersection product with a general smooth projective curve $C \subset S$. The preimage $R := f^{-1}(C)$ is then the blow-up of a minimal ruled surface over C in $C.D$ many points. The normal bundle of R in X is given by $\mathcal{N}_{R/X} = f^*\mathcal{O}_S(C)|_R$. Since $T_X|_R = T_R \oplus \mathcal{N}_{R/X}$, we get

$$(1 + c_1(X)|_R + c_2(X)|_R)(1 - c_1(\mathcal{N}_{R/X}) + c_1^2(\mathcal{N}_{R/X})) = 1 + c_1(R) + c_2(R).$$

Hence, using $\mathcal{O}_X(R) = f^*\mathcal{O}_S(C)$, we get

$$\begin{aligned}
f_*c_2(X).C &= c_2(X).R \\
&= c_2(X)|_R \\
&= c_2(R) - c_1^2(\mathcal{N}_{R/X}) + c_1(\mathcal{N}_{R/X})c_1(X)|_R \\
&= c_2(R) - f^*\mathcal{O}_S(C)^3 + f^*\mathcal{O}_S(C)^2 c_1(X) \\
&= 2 - 4g(C) + 2 + C.D + C^2 \cdot c_1(\mathbb{P}^1) \\
&= -2K_S.C - 2C^2 + C.D + 2C^2 \\
&= -2K_S.C + C.D.
\end{aligned}$$

By Lemma 5, $f_* c_1^2(X) \equiv -4K_S - D$. Using $p_1 = c_1^2 - 2c_2$, we get

$$\begin{aligned} f_* p_1(X).C &= f_* c_1^2(X).C - 2f_* c_2(X).C \\ &= -4K_S.C - D.C + 4K_S.C - 2D.C \\ &= -3D.C, \end{aligned}$$

which proves the lemma. □

Proof of Theorem 2 By [15, Theorem 1.1], any standard conic bundle $f : X \to S$, where X is a smooth Kähler threefold, has an algebraic deformation. To bound $K_{X_i}^3$ it thus suffices to assume that X_i and S_i are projective for any i.

By assumptions, there is an isomorphism $H^2_{tf}(X_i, \mathbb{Z}) \simeq H^2_{tf}(X_0, \mathbb{Z})$ which respects the trilinear forms given by cup products. We use this isomorphism to identify degree two cohomology classes of X_i with those of X_0 (up to torsion). Using Poincaré duality, we further identify classes of $H^4_{tf}(X_i, \mathbb{Z})$ with linear forms on $H^2_{tf}(X_i, \mathbb{Z}) \simeq H^2_{tf}(X_0, \mathbb{Z})$.

The codimension one linear subspace $f_i{}^*\mathbb{P}(H^2(S_i, \mathbb{Q}))$ of $\mathbb{P}(H^2(X_0, \mathbb{Q}))$ is contained in the cubic hypersurface $\{\alpha \mid \alpha^3 = 0\}$. Passing to a suitable subsequence we can therefore assume that

$$f_i^* H^2(S_i, \mathbb{Q}) \subset H^2(X_0, \mathbb{Q})$$

does not depend on i. Let $\ell_i \in H^4_{tf}(X_0, \mathbb{Z})$ be the class of a fibre of f_i. The linear form determined by this class on $H^2(X_0, \mathbb{Q})$ has kernel $f_i^* H^2(S_i, \mathbb{Q})$, and so $\ell_i \cdot \mathbb{Q}$ is independent of i. Since ℓ_i is an integral class with $K_{X_i}.\ell_i = -2$, we may after possibly passing to another subsequence assume that $\ell_i = \ell$ does not depend on i.

Since the natural cup product pairing on $H^2(S_i, \mathbb{Q})$ can be recovered from the pairing

$$f_i^* H^2(S_i, \mathbb{Q}) \times f_i^* H^2(S_i, \mathbb{Q}) \longrightarrow \ell\mathbb{Q},$$

we get that the pairing on $H^2(S_i, \mathbb{Q})$ is determined by the cubic form on $H^2(X_0, \mathbb{Q})$ and so it does not depend on i.

Since $f_i^* H^2(S_i, \mathbb{Q}) \subset H^2(X_0, \mathbb{Q})$ does not depend on i, the same holds for the homomorphism

$$\psi_i : H^2(S_i, \mathbb{Q}) \longrightarrow \mathbb{Q}, \quad \alpha \longmapsto p_1(X_i).f_i^*\alpha$$

By the projection formula, we have $p_1(X_i).f_i^*\alpha = (f_i)_* p_1(X_i).\alpha$. Lemma 6 thus yields

$$\psi_i(\alpha) = -3D_i.\alpha,$$

where D_i is the discriminant curve of f_i. This shows that the linear form determined by $[D_i] \in H^2(S_i, \mathbb{Q})$ on $H^2(S_i, \mathbb{Q})$ does not depend on i. Since the natural pairing $H^2(S_i, \mathbb{Q}) \times H^2(S_i, \mathbb{Q}) \to \mathbb{Q}$ is perfect by Poincaré duality, we get that the class $[D_i] \in H^2(S_i, \mathbb{Q})$ does not depend on i. Using again the fact that we know the pairing on $H^2(S_i, \mathbb{Q})$, we finally get that the self-intersection D_i^2 does not depend on i.

For any class $y \in H^2(X_0, \mathbb{Q})$, which does not lie in $f_i{}^*H^2(S_i, \mathbb{Q})$, we have

$$H^2(X_0, \mathbb{Q}) = f_i{}^*H^2(S_i, \mathbb{Q}) \oplus y \cdot \mathbb{Q} \quad \text{and} \quad H^4(X_0, \mathbb{Q}) = f_i{}^*H^2(S_i, \mathbb{Q}) \cdot y \oplus \ell \cdot \mathbb{Q}.$$

In particular, $y^2 = uy + \lambda\ell$ for some $\lambda \in \mathbb{Q}$ and $u \in f_i{}^*H^2(S_i, \mathbb{Q})$. Replacing y by a suitable multiple of $y - \frac{1}{2}u$, we may thus assume that

$$y.\ell = -2 \quad \text{and} \quad y^2 \in f_i{}^*H^4(S_i, \mathbb{Q}) = \ell \cdot \mathbb{Q}.$$

For any X_i, we then get

$$K_{X_i} = y + f_i^* z_i$$

for some $z_i \in H^2(S_i, \mathbb{Z})$. Since

$$K_{X_i}^3 = y^3 - 6z_i^2,$$

it suffices to prove the boundedness of z_i^2.

Since $y \cdot \ell = -2$, the pushforward of $2yf_i^* z_i$ via f_i yields $-4z_i$. Lemma 5 therefore implies that

$$(f_i)_* K_{X_i}^2 \equiv -4z_i \equiv -4K_{S_i} - D_i.$$

Hence,

$$16z_i^2 = 16K_{S_i}^2 + 8K_{S_i} D_i + D_i^2.$$

Since D_i^2 does not depend on i and $K_{S_i}^2$ is bounded in terms of the Betti numbers of S_i, the statement follows from the fact that $p_a(D_i)$ is bounded by Lemma 5. □

3 Uniruled Threefolds

Before we turn to the proof of Theorem 3, we state few preliminary facts about terminal $\mathbb{Q}$-factorial threefolds.

3.1 Invariant Triples

Let X be a terminal $\mathbb{Q}$-factorial threefold.

There exists a well-defined class $c_2(X) \in H^2(X,\mathbb{Z})^\vee = \mathrm{Hom}(H^2(Z,\mathbb{Z}),\mathbb{Z})$ obtained in the following way (see page 411 in [22]). For any $\alpha \in H^2(X,\mathbb{Z})$ set

$$c_2(X).\alpha = c_2(\tilde{X}).f^*\alpha,$$

where $f : \tilde{X} \to X$ is a resolution of X.

We then define the Pontryagin class $p_1(X) \in H^2(X,\mathbb{Q})^\vee$ in terms of $c_1(X)$ and $c_2(X)$ in the same way as in the smooth case, where $c_1(X)$ is the class of $-K_X$ in $H^2(X,\mathbb{Q})$:

$$p_1(X) := c_1(X)^2 - 2c_2(X).$$

We also associate to X its cubic form $F_X \in S^3H^2(X,\mathbb{Z})^\vee$, which is induced by the cup product on $H^2(X,\mathbb{Z})$. In this way we can associate to X the triple $(H^2_{t.f.}(X,\mathbb{Z}), F_X, p_1(X))$. When X is smooth, this triple encodes many geometrical properties of the 6-manifold underlying X (see for instance [18] and [1]).

Definition 7 We call $(H^2_{t.f.}(X,\mathbb{Z}), F_X, p_1(X))$ the *invariant triple* of X. Two triples (H, F, p) and (H', F', p'), where H (resp. H') is a free abelian group, $F \in S^3H^\vee$ (resp. $F \in S^3H'^\vee$) is a cubic form and p is a linear form on $H\otimes\mathbb{Q}$ (resp. p' is a linear form on $H'\otimes\mathbb{Q}$) are isomorphic if there exists a linear isomorphism $T : H \to H'$ which identifies F with F' and its $\mathbb{Q}$-extension identifies p with p'.

3.2 Terminal Singularities

We now recall few known facts about terminal singularities in dimension three.

Let (X, p) be the germ of a three-dimensional terminal singularity. The *index* of p is the smallest positive integer r such that rK_X is Cartier. It follows from the classification of terminal singularities, that there exists a deformation of (X, p) into a space with $h \geq 1$ terminal singularities $p_1,\dots,p_h$ which are isolated cyclic quotient singularities of index $r(p_i)$ (for details see [22, Remark 6.4]). The set $\{p_1,\dots,p_h\}$ is called the *basket* $\mathcal{B}(X, p)$ of singularities of X at p. As in [5, Section 2.1], we define

$$\Xi(X, p) = \sum_{i=1}^{h} r(p_i).$$

Thus, if X is a projective variety of dimension 3 with terminal singularities and Sing X denotes the finite set of singular points of X, we may define

$$\Xi(X) = \sum_{p \in \operatorname{Sing} X} \Xi(X, p).$$

3.3 Proof of Theorem 3

The following result is interesting by itself and leads naturally to the problem of understanding what kind of topological invariants are determined up to finite ambiguity during a running of an MMP, see Question 4.

Proposition 8 (Cf. [2, Theorem 1.3(2)]) *Let H be a finitely generated free abelian group of rank $n+1$, $F \in S^3 H^\vee$ be a cubic form such that $\Delta_F \neq 0$ and p a linear form on H. Consider the set $\mathcal{P}$ of invariant triples (H', F', p') up to isomorphism, such that there exist*

(1) a terminal $\mathbb{Q}$-factorial threefold X with associated triple (H, F, p);
(2) a terminal $\mathbb{Q}$-factorial threefold Y with associated triple (H', F', p');
(3) a birational morphism $f : X \to Y$ which is a divisorial contraction to a point or to a smooth curve contained in the smooth locus of Y.

Then the set $\mathcal{P}$ is finite.

Proof Note that the proof of this case works also for $\Delta_F = 0$. Consider the set $\mathcal{A}$ of primitive elements $\alpha \in H$ such that α is proportional to the exceptional divisor E of some divisorial contraction to a point $f : X \to Y$ as in the statement. The elements of $\mathcal{A}$ are points of rank 1 for the Hessian of the cubic form F and so they are finite by Cascini and Tasin [2, Proposition 3.3]. It follows from [2, Proposition 4.7] that for any sub-module $H' = f^* H^2_{t.f.}(Y, \mathbb{Z}) \hookrightarrow H$ there is $\alpha \in \mathcal{A}$ such that $\alpha^2.H' = 0$ and such that the index of $H' \oplus \mathbb{Z}\alpha$ in H is at most r^n, where $r = |\alpha^3|$. This implies that for all possible contractions to points $f: X \to Y$ as in the statement, the inclusion $f^* H^2_{t.f.}(Y, \mathbb{Z}) \hookrightarrow H^2_{t.f.}(X, \mathbb{Z})$ is determined up to finite ambiguity. This determines also F' up to finite ambiguity just restricting F to H'.

To prove the finiteness of p' consider a divisorial contraction to a point $f : X \to Y$ and write

$$c_1(X) = f^* c_1(Y) - cE,$$

where c is the discrepancy of the exceptional divisor E. Since $c_2(X) = f^* c_2(Y)$ we have that

$$p_1(X) = f^* p_1(Y) - 2c f^* c_1(Y).E + c^2 E^2$$

and so $p_1(Y)$ is given by the restriction of $p_1(X)$ to $f^*H^2_{t.f.}(Y, \mathbb{Q})$. This means that also p' is determined up to finite ambiguity and we are done.

We now look at divisorial contractions to curves. Consider $\mathcal{E}$ the set of pairs (E, H') where E is a primitive element in H and $H' \subset H$ is a submodule such that

$$H = \mathbb{Z}[E] \oplus H'$$

and the cubic F assumes the form

$$F = ax_0^3 + \sum_{i=1}^{n} b_i x_0^2 x_i + F'(x_1, \ldots, x_n) \tag{1}$$

with respect to any basis $E, \alpha_1, \ldots, \alpha_n$ with $\alpha_1, \ldots, \alpha_n \in H'$.

By Cascini and Tasin [2, Thm. 3.1] there are only finitely many possible non-equivalent reduced forms for F. In particular, up to finite ambiguity, we can assume that the coefficients of F in the expression (1) are fixed. Since the isotropy group of a cubic with non-zero discriminant is finite ([18, Thm. 4]), we deduce that $\mathcal{E}$ is finite.

If $f: X \to Y$ is a divisorial contraction which contracts a divisor E to a smooth curve C in the smooth locus of Y, then (see [18, Proposition 14] and [2, Proposition 4.8])

$$H^2(X, \mathbb{Z}) = \mathbb{Z}[E] \oplus f^*H^2(Y, \mathbb{Z})$$

and

$$p_1(X) = f^*(p_1(Y)) + E^2 - 2f^*(C).$$

Recalling that $E^2.f^*(\alpha) = -C.\alpha$ for any $\alpha \in H^2(Y, \mathbb{Z})$ we deduce that $p_1(Y)$ is determined by $p_1(X)$, E^2 and by the inclusion $f^*H(Y, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{Z})$ and we conclude using the finiteness of $\mathcal{E}$. □

Proposition 9 *Let $(X_i)_{i \geq 0}$ be a sequence of terminal $\mathbb{Q}$-factorial threefolds admitting a conic bundle structure $f_i : X_i \longrightarrow S_i$ of relative Picard number 1 over a surface S_i. Assume that*

(1) the Euler characteristics $\chi(X_i, \mathcal{O}_{X_i})$ are bounded and $b_2(X_i) = 2$;
(2) the invariant triples of X_0 and X_i are isomorphic for any i;
(3) the sequence $\Xi(X_i)$ is bounded.

Then the sequence of Chern numbers $c_1^3(X_i)$ is bounded.

Proof Let h be an ample generator of $\mathrm{Pic}(S_0)$ and let $x \in H^2(X_0, \mathbb{Z})$ be a primitive class proportional to f_0^*h. Then $x^3 = 0$, $x^2 \neq 0$ and letting $y = c_1(X_0)$ we can write

$$H^2(X_0, \mathbb{Q}) = x \cdot \mathbb{Q} \oplus y \cdot \mathbb{Q}.$$

From now on we will use the isomorphism $H^2(X_i, \mathbb{Q}) \simeq H^2(X_0, \mathbb{Q})$ to think about x and y as basis elements of $H^2(X_i, \mathbb{Q})$. Note that $x^2 y \neq 0$ since already $x^3 = 0$ and $x^2 \neq 0$. Moreover, the space of elements in $H^2(X_0, \mathbb{C})$ with zero cube is a union of three lines (through 0) and so we may assume without loss of generality that for each i, the pullback of the generator of $H^2(S_i, \mathbb{Z})$ to X_i is a multiple of x. In particular, x^2 is a multiple of the class of the general fibre of $X_i \dashrightarrow S_i$ for all i.

We have

$$c_1(X_i) = a_i \cdot x + b_i \cdot y,$$

for some $a_i, b_i \in \mathbb{Q}$. Since $\Xi(X_i)$ is bounded, there is a positive integer r such that rK_{X_i} is Cartier for any i. In particular, $ra_i, rb_i \in \mathbb{Z}$. Since $K_{X_i}.C = -2$ where C is a general fibre, we deduce that the sequence of b_i is bounded.

We are going to bound the sequence of a_i. By the singular version of Riemann–Roch [22, Corollary 10.3] we get

$$48\chi(X_i, \mathcal{O}_{X_i}) = c_1(X_i).p_1(X_i) - c_1(X_i)^3 + T_i$$

where

$$T_i = \sum_{p_\alpha} \left(r(p_\alpha) - \frac{1}{r(p_\alpha)} \right),$$

and the sum runs over all the points of all the baskets of X_i. Note that T_i is a bounded sequence since $\Xi(X_i)$ is bounded. This implies that

$$48\chi(X_i, \mathcal{O}_{X_i}) = -3a_i^2 b_i x^2 y + a_i x(2p_1(X_i) - 3b_i^2 y^2) + b_i^3 y^3 + b_i y p_1(X_i) + T_i$$

and so the a_i are also bounded, since $b_i x^2 y \neq 0$ and $\chi(X_i, \mathcal{O}_{X_i})$ are bounded. □

Proof of Theorem 3 Let $f : X \to Y$ be the birational contraction as in Theorem 3. By the proof of [2, Corollary 1.5], we know that $|K_X^3 - K_Y^3|$ is bounded by a constant depending only on the Betti numbers of X and on the cubic form F_X. To conclude we need to bound K_Y^3 in terms of the topology of X.

Since Y is a Mori fibre space and $\Delta_{F_Y} \neq 0$, we deduce that either Y is a Fano variety or Y has a conic bundle structure over a surface with second Betti number 1 (otherwise there would be an element in $H^2(X, \mathbb{C})$ with square zero, which would imply that $\{F = 0\}$ has a singular point and so $\Delta_{F_Y} = 0$). Since terminal Fano threefolds are bounded, we are left with the conic bundle case.

Proposition 8 assures us that the invariant triple of Y is determined up to finite ambiguity by the invariant triple of X. Moreover, the Euler characteristic $\chi(Y, \mathcal{O}_Y) = \chi(X, \mathcal{O}_X)$ is bounded in terms of the Betti numbers of X and by Cascini and Zhang [3, Prop. 3.3] we also have a bound for $\Xi(Y)$ depending only on $b_2(X)$. The result follows then from Proposition 9. □

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