

# The Elliptic Sunrise



Luise Adams, Christian Bogner and Stefan Weinzierl

**Abstract** In this talk, we discuss our recent computation of the two-loop sunrise integral with arbitrary non-zero particle masses in the vicinity of the equal mass point. In two space-time dimensions, we arrive at a result in terms of elliptic dilogarithms. Near four space-time dimensions, we obtain a result which furthermore involves elliptic generalizations of Clausen and Glaisher functions.

**Keywords** Feynman integrals · Special functions · Elliptic polylogarithms

## 1 Introduction

In the computation of many Feynman integrals the use of multiple polylogarithms<sup>1</sup> [1]

$$\mathrm{Li}_{(s_1, \dots, s_k)}(z_1, \dots, z_k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{z_1^{n_1} \dots z_k^{n_k}}{n_1^{s_1} \dots n_k^{s_k}}, \quad s_i \geq 1, |z_i| < 1$$

is very advantageous. In particular, these functions, shown as nested sums here, also have representations as iterated integrals, given by the classes of hyperlogarithms [2, 3] or by iterated integrals on moduli spaces of curves of genus zero (see [4]).

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<sup>1</sup>Our summation convention is widely used in the physics literature, including our previous work. Notice that it differs from the convention in [1].

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L. Adams · S. Weinzierl  
PRISMA Cluster of Excellence, Institut für Physik, Johannes Gutenberg-Universität Mainz,  
55099 Mainz, Germany  
e-mail: [ladams@students.uni-mainz.de](mailto:ladams@students.uni-mainz.de)

S. Weinzierl  
e-mail: [weinzierl@uni-mainz.de](mailto:weinzierl@uni-mainz.de)

C. Bogner (✉)  
Institut für Physik, Humboldt-Universität zu Berlin, 10099 Berlin, Germany  
e-mail: [bogner@math.hu-berlin.de](mailto:bogner@math.hu-berlin.de)

Apparently, it is not possible to express every Feynman integral in terms of this framework of functions. This problem is expected to affect an entire class of massive integrals (see e.g. [5]) and was furthermore pointed out for certain massless integrals, arising in  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory [6, 7].

One of the simplest Feynman integrals where multiple polylogarithms are not sufficient to express the result is the massive two-loop sunrise integral

$$S(D, t) = \int \frac{d^D k_1 d^D k_2}{(i\pi^{D/2})^2} \frac{1}{(-k_1^2 + m_1^2)(-k_2^2 + m_2^2)(-(p - k_1 - k_2)^2 + m_3^2)}.$$

In this talk, we consider this integral as a function of the three particle masses satisfying  $0 < m_1 \leq m_2 \leq m_3 < m_1 + m_2$  and of the squared momentum  $t = p^2$ . The condition  $m_3 < m_1 + m_2$  ensures that all pseudo-thresholds are positive. We omit an explicit mass-scale parameter  $\mu$  in our equations. We discuss the computation of this Feynman integral at  $D = 2$  and  $D = 4$  dimensions in terms of the Laurent expansions

$$\begin{aligned} S(2 - 2\varepsilon, t) &= S^{(0)}(2, t) + S^{(1)}(2, t)\varepsilon + \mathcal{O}(\varepsilon^2), \\ S(4 - 2\varepsilon, t) &= S^{(-2)}(4, t)\varepsilon^{-2} + S^{(-1)}(4, t)\varepsilon^{-1} + S^{(0)}(4, t) + \mathcal{O}(\varepsilon). \end{aligned}$$

In the case of  $D = 2$ , the integral is finite and our result is the coefficient  $S^{(0)}(2, t)$ . In the case of  $D = 4$ , we compute the coefficient  $S^{(0)}(4, t)$ . The pole terms were already known and read

$$\begin{aligned} S^{(-2)}(4, t) &= -\frac{1}{2}(m_1^2 + m_2^2 + m_3^2), \\ S^{(-1)}(4, t) &= \frac{1}{4}t - \frac{3}{2}(m_1^2 + m_2^2 + m_3^2) + \sum_{i=1}^3 m_i^2 \ln(m_i^2). \end{aligned}$$

In order to obtain  $S^{(0)}(4, t)$ , we compute the  $\varepsilon$ -coefficient  $S^{(1)}(2, t)$  of the two-dimensional case and relate  $S(2 - 2\varepsilon, t)$  with  $S(4 - 2\varepsilon, t)$  by Tarasov's dimension shift relations [8, 9]. Our work on these integrals is motivated by the search for classes of functions beyond multiple polylogarithms, which are appropriate for the computation of Feynman integrals.

In Sect. 2 we briefly comment on three computational approaches which fail to provide a result in terms of multiple polylogarithms for the massive sunrise integral. We begin our computation with the integral in two dimensions and discuss our first solution of the differential equation for  $S^{(0)}(2, t)$  in Sect. 3. In Sect. 4 we express this result in terms of an elliptic dilogarithm. Section 5 introduces further elliptic generalizations of polylogarithms, understood as elliptic generalizations of Clausen and Glaisher functions, which arise in our results for  $S^{(1)}(2, t)$  and  $S^{(0)}(4, t)$ . Section 6 contains the conclusions of this talk.

## 2 Basic Properties of the Massive Sunrise Integral

The massive sunrise integral was extensively studied in the past [5, 10–28]. Let us recall some important aspects.

Firstly, in [15] the integral  $S(D, t)$  is expressed as a linear combination of generalized hypergeometric functions of Lauricella type C, which are functions of  $t$ , of the squared particle masses and of the dimension  $D$ . While a wide range of generalized hypergeometric functions can be expanded in terms of multiple polylogarithms with today's methods, this has not been achieved for the mentioned result so far.

Secondly, one may attempt to compute the integral by integration over Feynman parameters. In terms of Feynman parameters, the integral in  $D = 2$  dimensions reads

$$S(2, t) = \int_{\sigma} \frac{\omega}{\mathcal{F}},$$

with  $\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2$  and  $\sigma = \{[x_1 : x_2 : x_3] \in \mathbb{P}^2 | x_i \geq 0, i = 1, 2, 3\}$  while the second Symanzik polynomial is given as

$$\mathcal{F} = -x_1 x_2 x_3 t + (x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2) (x_1 x_2 + x_2 x_3 + x_1 x_3).$$

For an attempt to iteratively build up the result in terms of the mentioned iterated integrals which represent the multiple polylogarithms, the polynomial  $\mathcal{F}$  would have to satisfy the criterion of linear reducibility [29]. The latter is a sufficient but not necessary criterion to obtain multiple polylogarithms in the result. However, the polynomial fails this criterion and a change of variables to restore linear reducibility for a new set of integration variables is unknown for this case.

Thirdly, the integral  $S(D, t)$  for generic space-time dimension satisfies an inhomogeneous fourth-order differential equation in  $t$ :

$$\left( P_4 \frac{d^4}{dt^4} + P_3 \frac{d^3}{dt^3} + P_2 \frac{d^2}{dt^2} + P_1 \frac{d}{dt} + P_0 \right) S(D, t) = c_{12} T_{12} + c_{13} T_{13} + c_{23} T_{23} \tag{1}$$

where the  $T_{ij} = T(m_i^2, D) T(m_j^2, D)$  are products of tadpole integrals

$$T(m^2, D) = \int \frac{d^D k}{i\pi^{\frac{D}{2}}} \frac{1}{(-k^2 + m^2)} = \Gamma\left(1 - \frac{D}{2}\right) (m^2)^{\frac{D}{2}-1}.$$

All coefficients  $P_k$  and  $c_{ij}$  are polynomials in  $m_1^2, m_2^2, m_3^2, t, D$ . Their explicit expressions are provided in appendix A of [30]. Each of the functions  $S^{(0)}(2, t)$ ,  $S^{(1)}(2, t)$ ,  $S^{(0)}(4, t)$  satisfies an inhomogeneous differential equation of second or higher order. If any of these operators would factorize into differential operators of first order the corresponding coefficient could be obtained as an iterated integral in a straightforward way (see e.g. Sect. 2 of [31]). However, this is not the case for any of these operators.

All of these points give rise to the expectation, that we need functions beyond multiple polylogarithms to express the integrals  $S^{(0)}(2, t)$ ,  $S^{(1)}(2, t)$ ,  $S^{(0)}(4, t)$ . This expectation is confirmed by our results for these functions.

### 3 The Differential Equation in Two Dimensions

We follow the approach of differential equations and begin with the Feynman integral in  $D = 2$  dimensions. For the case of equal masses  $m_1 = m_2 = m_3$ , a differential equation of second order was already given in [14]. A full solution in terms of integrals over elliptic integrals was obtained in [19].

For the case of arbitrary masses, a differential equation of second order was found later in [23]:

$$L_2 S(2, t) = p_3(t),$$

$$L_2 = p_2(t) \frac{d^2}{dt^2} + p_1(t) \frac{d}{dt} + p_0(t), \quad (2)$$

where  $p_0(t)$ ,  $p_1(t)$ ,  $p_2(t)$  are polynomials in  $t$  and in the  $m_i^2$  and where  $p_3(t)$  furthermore involves  $\ln(m_i^2)$ ,  $i = 1, 2, 3$ . We take this equation as the starting point of our computation and make the classical ansatz

$$S(2, t) = C_1 \psi_1(t) + C_2 \psi_2(t) + \int_0^t dt_1 \frac{p_3(t_1)}{p_0(t_1)W(t_1)} (-\psi_1(t)\psi_2(t_1) + \psi_2(t)\psi_1(t_1)) \quad (3)$$

where  $\psi_1, \psi_2$  are solutions of the homogeneous equation,  $C_1, C_2$  are constants and

$$W(t) = \psi_1(t) \frac{d}{dt} \psi_2(t) - \psi_2(t) \frac{d}{dt} \psi_1(t)$$

is the Wronski determinant.

At this point, it is useful to consider the zero-set of the second Symanzik polynomial  $\mathcal{F}$ . This cubical curve intersects the integration domain  $\sigma$  of the Feynman integral at the three points

$$P_1 = [1 : 0 : 0], \quad P_2 = [0 : 1 : 0], \quad P_3 = [0 : 0 : 1].$$

We choose one of these points  $P_i$  as the origin and transform the curve to Weierstrass normal form

$$y^2 z - x^3 - g_2(t)xz^2 - g_3(t)z^3 = 0. \quad (4)$$

By this transformation, the chosen origin is mapped to the point  $[x : y : z] = [0 : 1 : 0]$ . In this way, we obtain three elliptic curves  $E_{\mathcal{F},i}$  according to the three points  $P_i$ ,  $i = 1, 2, 3$ .

In the chart  $z = 1$  we write Eq. 4 as

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3),$$

which defines the three roots  $e_1, e_2, e_3$  with  $e_1 + e_2 + e_3 = 0$ . These provide the boundaries of the period integrals

$$\psi_1 = 2 \int_{e_2}^{e_3} \frac{dx}{y} = \frac{4}{\tilde{D}^{\frac{1}{4}}} K(k), \quad \psi_2 = 2 \int_{e_1}^{e_3} \frac{dx}{y} = \frac{4i}{\tilde{D}^{\frac{1}{4}}} K(k')$$

of the elliptic curve. Here the polynomial  $\tilde{D}$  is given as

$$\tilde{D} = (t - (m_1 + m_2 - m_3)^2)(t - (m_1 - m_2 + m_3)^2)(t - (-m_1 + m_2 + m_3)^2)(t - (m_1 + m_2 + m_3)^2)$$

and we have obtained the complete elliptic integral of the first kind

$$K(x) = \int_0^1 dt \frac{1}{\sqrt{(1-t^2)(1-x^2t^2)}}$$

with moduli  $k = \sqrt{\frac{e_3 - e_2}{e_1 - e_2}}$ ,  $k' = \sqrt{1 - k^2} = \sqrt{\frac{e_1 - e_3}{e_1 - e_2}}$ . These period integrals  $\psi_1, \psi_2$  are solutions of the homogeneous equation associated to Eq. 3.

We still have to fix the constants. It can be shown that  $C_2$  has to vanish while the other constant  $C_1$  is derived from a known result [32–34] for the zero-mass limit  $S(2, 0)$ . Now all pieces of our ansatz in Eq. 3 are determined. In order to simplify the integrand of the particular solution, we furthermore make use of the remaining two associated period integrals of  $E_{\mathcal{F},i}$ . In conclusion, we obtain a result [31] of the form

$$S(2, t) = S(2, 0) + \frac{\psi_1(t)}{\pi^2} \int_0^t dt_1 \rho(t_1) \quad (5)$$

where the integrand  $\rho$  involves elliptic integrals of the first and second kind.

## 4 The Massive Sunrise Integral in Two Dimensions

The general shape of our result of Eq. 5 has a disadvantage. While the involved elliptic integrals are well-studied functions, nicely related to the underlying elliptic curve of the problem, the integral over these functions is not a known function. This integral might remind us vaguely of an iterated integral, but in this form, it can not be recognized as a generalization of a polylogarithm. However, for the equal-mass

case, it was shown more recently in [24], that the integral can be expressed in terms of an elliptic dilogarithm. Various notions of elliptic polylogarithms were previously introduced in the mathematical literature [35–40].

Before we apply an elliptic generalization of a polylogarithm to the sunrise integral with arbitrary masses, let us briefly recall the basic concept of an elliptic function. With respect to a lattice  $L = \mathbb{Z} + \tau\mathbb{Z}$  with  $\tau \in \mathbb{C}$  and  $\text{Im}(\tau) > 0$ , a function  $f$  is said to be elliptic, if it satisfies  $f(x) = f(x + \lambda)$  for  $\lambda \in L$ . Accordingly, the corresponding function  $\tilde{f}(z)$  of  $z \in \mathbb{C}^*$  defined by  $\tilde{f}(e^{2\pi i x}) = f(x)$  is elliptic, if

$$\tilde{f}(z) = \tilde{f}(z \cdot q_\lambda), \quad q_\lambda \in e^{2\pi i \lambda} \text{ for } \lambda \in L. \tag{6}$$

Recall that a cell of the lattice with  $\tau = \frac{\psi_2}{\psi_1}$  is isomorphic to an elliptic curve with the periods  $\psi_1, \psi_2$ .

A crucial idea for the construction of such elliptic functions is to consider sums of the form  $\sum_{n \in \mathbb{Z}} g(z \cdot q^n)$  over some function  $g$ . If a sum of this type is well-defined, it clearly satisfies the condition of Eq. 6 by construction. This concept can serve for definitions of elliptic generalizations of polylogarithms. For example in [39] it is used to define the class of multiple elliptic polylogarithms. The elliptic dilogarithm in this framework reads

$$\tilde{E}_2(z; u; q) = \sum_{m \in \mathbb{Z}} u^m \text{Li}_2(q^m z)$$

where  $u$  is a sufficiently small damping parameter to guarantee the convergence of the function.

Based on the same basic idea, we define the class of functions [41]

$$\text{ELi}_{n;m}(x; y; q) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^j}{j^n} \frac{y^k}{k^m} q^{jk} = \sum_{k=1}^{\infty} \frac{y^k}{k^m} \text{Li}_n(q^k x),$$

$$E_{n;m}(x; y; q) = \begin{cases} \frac{1}{i} \left( \frac{1}{2} \text{Li}_n(x) - \frac{1}{2} \text{Li}_n(x^{-1}) + \text{ELi}_{n;m}(x; y; q) - \text{ELi}_{n;m}(x^{-1}; y^{-1}; q) \right), \\ \frac{1}{2} \text{Li}_n(x) + \frac{1}{2} \text{Li}_n(x^{-1}) + \text{ELi}_{n;m}(x; y; q) + \text{ELi}_{n;m}(x^{-1}; y^{-1}; q) \end{cases} \tag{7}$$

with the first line for  $n + m$  even and the second line for  $n + m$  odd. Note that our elliptic dilogarithm

$$E_{2;0}(x; y; q) = \frac{1}{i} \left( \frac{1}{2} \text{Li}_2(x) - \frac{1}{2} \text{Li}_2(x^{-1}) + \sum_{i=1}^{\infty} y^i \text{Li}_2(q^i x) - \sum_{j=1}^{\infty} y^{-j} \text{Li}_2(q^j x^{-1}) \right)$$

is closely related to the above function  $\tilde{E}_2$ . We obtain

$$E_{2;0}(x; y; q) = \frac{1}{i} \left( \tilde{E}_2(x; y; q) - \frac{1}{2} \frac{1+y}{1-y} \zeta(2) - \frac{1}{4} \frac{1+y}{1-y} \ln^2(-x) - \frac{y}{(1-y)^2} \ln(-x) \ln(q) - \frac{1}{2} \frac{y(1+y)}{(1-y)^3} \ln^2(q) \right)$$

in the region of parameters given by  $x \in \mathbb{C} \setminus [0, \infty[$ ,  $|y| > 1$  and real-valued  $q$  in the range  $0 \leq q < \min \left( |x|, \frac{1}{|x|}, |y|, \frac{1}{|y|} \right)$ .

Using the function  $E_{2;0}$ , we express our result for the massive sunrise integral in two space-time dimensions in a very compact way as<sup>2</sup>

$$S(2, t) = \frac{\psi_1(q)}{\pi} \sum_{i=1}^3 E_{2;0}(w_i(q); -1; -q) \text{ where } q = e^{\pi i \frac{\psi_2(t)}{\psi_1(t)}}. \tag{8}$$

Note that the dependence on  $t$  is now implicitly expressed in terms of  $q$ , which is defined by the periods of the elliptic curve. The arguments  $w_1, w_2, w_3$  are functions of  $q$  and of the squared particle masses. They are directly obtained from the three intersection points  $P_1, P_2, P_3$  by the consecutive transformations on the elliptic curves  $E_{\mathcal{F},i}, i = 1, 2, 3$ , indicated above. In this sense, every piece of the compact result Eq. 8 is nicely related to the underlying elliptic curves  $E_{\mathcal{F},i}$ .

In the case of equal masses, the result simplifies to

$$S(2, t) = 3 \frac{\psi_1(q)}{\pi} E_{2;0}(\exp(2\pi i/3); -1; -q).$$

## 5 The Massive Sunrise Integral Around Four Dimensions

By use of dimension shift relations [8, 9], we express the coefficient  $S^{(0)}(4, t)$  of the sunrise integral near  $D = 4$  dimensions in terms of coefficients of the  $D = 2$  case [30]. We obtain  $S^{(0)}(4, t)$  as a linear combination of terms  $S^{(0)}(2, t)$ ,  $\frac{\partial}{\partial m_i^2} S^{(0)}(2, t)$ ,  $S^{(1)}(2, t)$ ,  $\frac{\partial}{\partial m_i^2} S^{(1)}(2, t)$ ,  $i = 1, 2, 3$ . Therefore, our remaining task is the computation of  $S^{(1)}(2, t)$ .

From Eq. 1 we obtain the differential equation

$$L_{1,a} L_{1,b} L_2 S^{(1)}(2, t) = I_1(t). \tag{9}$$

Here  $L_{1,a}$  and  $L_{1,b}$  are differential operators of first order,

$$L_{1,a} = p_{1,a} \frac{d}{dt} + p_{0,a} \text{ and } L_{1,b} = p_{1,b} \frac{d}{dt} + p_{0,b},$$

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<sup>2</sup>By a slight abuse of notation, we denote with  $\psi_1$  the above function of  $t$  and the corresponding function of  $q$ .

where  $p_{0,a}, p_{1,a}$  are rational functions of  $t$  and the squared particle masses and  $p_{0,b}, p_{1,b}$  are polynomials in these variables. The homogeneous solutions  $\psi_a, \psi_b$  of these operators, defined by

$$L_{1,a} \psi_a(t) = 0 \text{ and } L_{1,b} \psi_b(t) = 0$$

are easily obtained.

The operator  $L_2$  in Eq. 9 is the one of Eq. 2 which already appeared in the differential equation of the two-dimensional case. The inhomogeneous term  $I_1$  of Eq. 9 is a combination of certain differentiations of our result  $S^{(0)}(2, t)$ , of logarithms in the squared particle masses and of a polynomial in the squared masses and in  $t$ .

Solving Eq. 9 for the combination  $L_2 S^{(1)}(2, t)$ , we obtain

$$L_2 S^{(1)}(2, t) = I_2(t) \tag{10}$$

with

$$I_2(t) = \tilde{C}_1 \psi_b(t) + \tilde{C}_2 \psi_b(t) \int_0^t \frac{\psi_a(t_1) dt_1}{p_{1,b}(t_1) \psi_b(t_1)} + \psi_b(t) \int_0^t \frac{\psi_a(t_1) dt_1}{p_{1,b}(t_1) \psi_b(t_1)} \int_0^{t_1} \frac{I_1(t_2) dt_2}{p_{1,a}(t_2) \psi_a(t_2)}$$

where  $\tilde{C}_1, \tilde{C}_2$  are integration constants.

Now with Eq. 10 we have to solve a similar differential equation as in the two-dimensional case, with the only difference that the inhomogeneous part is more complicated. However, we can make a similar ansatz and we have the same period integrals  $\psi_1, \psi_2$  of  $E_{\mathcal{F},i}$  as solutions of the homogeneous equation. Therefore, it is useful to introduce the variable  $q$  again in the same way as in Eq. 8. In terms of integrals over  $q$ , we obtain

$$S^{(1)}(2, t) = C_3 \psi_1 + C_4 \psi_2 - \frac{\psi_1}{\pi} \int_0^q \frac{dq_1}{q_1} \int_0^{q_1} \frac{dq_2}{q_2} \frac{I_2(q_2) \psi_1(q_2)^3}{\pi p_2(q_2) W(q_2)^2}.$$

The integration constants  $C_3, C_4$  are determined from boundary conditions. Expanding the integrand, we can perform the integrations order by order and obtain a  $q$ -expansion of  $S^{(1)}(2, t)$  to high orders. This step finally allows us to find a result for  $S^{(1)}(2, t)$  in closed form, which can be confirmed to satisfy the differential Eq. 10.

Let us refer to [30] for the explicit result and just highlight some of its properties here. Apart from classical (multiple) polylogarithms, the result involves the functions  $E_{1;0}(x; y; q), E_{2;0}(x; y; q), E_{3;1}(x; y; q)$  as defined in Eq. 7 and furthermore a quadruple sum of the form

$$\Lambda(x_1, x_2; y_1, y_2; -q) = \sum_{j_1=1}^{\infty} \sum_{k_1=1}^{\infty} \sum_{j_2=1}^{\infty} \sum_{k_2=1}^{\infty} \frac{k_1^2 (-q)^{j_1 k_1 + j_2 k_2}}{j_2 (j_1 k_1 + j_2 k_2)^2} \left( x_1^{j_1} y_1^{k_1} - x_1^{-j_1} y_1^{-k_1} \right) \left( x_2^{j_2} y_2^{k_2} + x_2^{-j_2} y_2^{-k_2} \right).$$



For the arguments of these functions, we have  $y, y_1, y_2 \in \{-1, 1\}$  and  $x, x_1, x_2 \in \{w_1, w_2, w_3\}$ , where the  $w_i$  again are the arguments obtained from the intersection points mentioned above.

The appearance of the functions  $E_{1;0}(x; y; q), E_{2;0}(x; y; q), E_{3;1}(x; y; q)$  shows that the framework of Eq. 7, set up for the coefficient  $S^{(0)}(2, t)$ , is also useful for  $S^{(1)}(2, t)$  and hence also for the four-dimensional case. Furthermore, these functions can be viewed as elliptic generalizations of Clausen and Glaisher functions. Recall that the Clausen functions are defined by

$$Cl_n(\varphi) = \begin{cases} \frac{1}{2i} (\text{Li}_n(e^{i\varphi}) - \text{Li}_n(e^{-i\varphi})) & \text{for even } n, \\ \frac{1}{2} (\text{Li}_n(e^{i\varphi}) + \text{Li}_n(e^{-i\varphi})) & \text{for odd } n, \end{cases}$$

and the Glaisher functions are given as

$$Gl_n(\varphi) = \begin{cases} \frac{1}{2} (\text{Li}_n(e^{i\varphi}) + \text{Li}_n(e^{-i\varphi})) & \text{for even } n, \\ \frac{1}{2i} (\text{Li}_n(e^{i\varphi}) - \text{Li}_n(e^{-i\varphi})) & \text{for odd } n. \end{cases}$$

We therefore obtain as ‘non-elliptic limits’ of our functions:

$$\begin{aligned} \lim_{q \rightarrow 0} E_{1;0}(e^{i\varphi}; y; q) &= Cl_1(\varphi), \\ \lim_{q \rightarrow 0} E_{2;0}(e^{i\varphi}; y; q) &= Cl_2(\varphi), \\ \lim_{q \rightarrow 0} E_{3;1}(e^{i\varphi}; y; q) &= Gl_3(\varphi). \end{aligned}$$

As a final remark, let us mention that  $S^{(1)}(2, t)$  is a function of mixed weight. It shares this property with the function  $E_{3;1}(x; y; q)$  which has parts of weight three and of weight four.

## 6 Conclusions

We discussed the computation of the massive sunrise integral in two and around four space-time dimensions. We started with the computation of the  $\mathcal{O}(\varepsilon^0)$ -part of the integral in two dimensions and expressed our result in terms of an elliptic dilogarithm. In this form, the result is very compact and every part of it is nicely related to the underlying elliptic curve, given by the second Symanzik polynomial of the Feynman graph.

We continued with the computation of the  $\mathcal{O}(\varepsilon^1)$ -part in two dimensions. Apart from the elliptic dilogarithm, this result involves further elliptic generalizations of (multiple) polylogarithms, which can be understood as elliptic generalizations of Clausen and Glaisher functions. Due to well-known dimension shift relations, these results provide the  $\mathcal{O}(\varepsilon^0)$ -part of the Feynman integral in four dimensions.

Together with the results of [24, 42], our results give rise to the hope, that elliptic (multiple) polylogarithms may serve as an appropriate class of functions to compute further Feynman integrals beyond multiple polylogarithms. Some of our functions can be related to the functions of [39], where also a framework of iterated integrals, already applied in a different physics context [43], is provided.

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