Evaluating Generating Functions for Periodic Multiple Polylogarithms via Rational Chen–Fliess Series

Kurusch Ebrahimi-Fard, W. Steven Gray and Dominique Manchon

Abstract The goal of the paper is to give a systematic way to numerically evaluate the generating function of a periodic multiple polylogarithm using a Chen–Fliess series with a rational generating series. The idea is to realize the corresponding Chen–Fliess series as a bilinear dynamical system. A standard form for such a realization is given. The method is also generalized to the case where the multiple polylogarithm has non-periodic components. This allows one, for instance, to numerically validate the Hoffman conjecture. Finally, a setting in terms of dendriform algebras is provided.

Keywords Chen–Fliess series · Dendriform algebra · Hoffman conjecture · Multiple polylogarithms · Rational formal power series

1 Introduction

Given any vector $\mathbf{s} = (s_1, s_2, \dots, s_l) \in \mathbb{N}^l$ with $s_1 \geq 2$ and $s_i \geq 1$ for $i \geq 2$, the associated *multiple polylogarithm* (MPL) of *depth* l and *weight* $|\mathbf{s}| := \sum_{i=1}^{l} s_i$ is taken to be

$$\mathrm{Li}_{\mathbf{s}}(t) := \sum_{k_1 > k_2 > \cdots > k_l \geq 1} \frac{t^{k_1}}{k_1^{s_1} k_2^{s_2} \cdots k_l^{s_l}}, \quad |t| \leq 1, \tag{1}$$

Kurusch Ebrahimi-Fard: (on leave from UHA, Mulhouse, France).

K. Ebrahimi-Fard
Department of Mathematical Sciences, Norwegian University of Science and Technology (NTNU), 7491 Trondheim, Norway
e-mail: kurusch.ebrahimi-fard@ntnu.no; kurusch.ebrahimi-fard@uha.fr

W. Gray (✉)
Old Dominion University, Norfolk, VA 23529, USA
e-mail: sgray@odu.edu

D. Manchon
LMBP, CNRS - Université Clermont-Auvergne, CS 60026, 63178 Aubière Cedex, France
e-mail: Dominique.Manchon@uca.fr

J. I. Burgos Gil et al. (eds.), *Periods in Quantum Field Theory and Arithmetic*, Springer Proceedings in Mathematics & Statistics 314,
https://doi.org/10.1007/978-3-030-37031-2_17

whereupon the *multiple zeta value* (MVZ) of depth l and weight $|\mathbf{s}|$ is the value of (1) at $t = 1$, namely,

$$\zeta(\mathbf{s}) := \mathrm{Li}_{\mathbf{s}}(1).$$

Any such vector $\mathbf{s}$ will be referred to as *admissible*. The MPL in (1) can be represented in terms of iterated Chen integrals with respect to the 1-forms $\omega_j^{(1)} := dt_j/(1 - t_j)$ and $\omega_j^{(0)} := dt_j/t_j$. Indeed, using the standard notation, $|\mathbf{s}_{(j)}| := s_1 + \cdots + s_j$, $j \in \{1, \ldots, l\}$, one can show that

$$\mathrm{Li}_{\mathbf{s}}(t) = \int_0^t \Bigg(\prod_{j=1}^{|\mathbf{s}_{(1)}|-1} \omega_j^{(0)} \Bigg) \omega^{(1)}_{|\mathbf{s}_{(1)}|} \cdots \Bigg(\prod_{j=|\mathbf{s}_{(l-1)}|+1}^{|\mathbf{s}_{(l)}|-1} \omega_j^{(0)} \Bigg) \omega^{(1)}_{|\mathbf{s}_{(l)}|}. \tag{2}$$

For instance,

$$\mathrm{Li}_{(2,1,1)}(t) = \int_0^t \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{1-t_3} \int_0^{t_3} \frac{dt_4}{1-t_4} = \sum_{k_1 > k_2 > k_3 \geq 1} \frac{t^{k_1}}{k_1^2 k_2 k_3}.$$

An MPL of depth l is said to be *periodic* if it can be written in the form $\mathrm{Li}_{\{\mathbf{s}\}^n}(t)$, where $\{\mathbf{s}\}^n$ denotes the n-tuple $(\mathbf{s}, \mathbf{s}, \ldots, \mathbf{s}) \in \mathbb{N}^{nl}$, $n \geq 0$ with $\mathrm{Li}_{\{\mathbf{s}\}^0}(t) := 1$.[1] In this case, the sequence $(\mathrm{Li}_{\{\mathbf{s}\}^n}(t))_{n \in \mathbb{N}_0}$ has the generating function

$$\mathscr{L}_{\mathbf{s}}(t, \theta) := \sum_{n=0}^{\infty} \mathrm{Li}_{\{\mathbf{s}\}^n}(t) \left(\theta^{|\mathbf{s}|}\right)^n. \tag{3}$$

In general, the integral representation (2) implies that $\mathscr{L}_{\mathbf{s}}$ will satisfy a linear ordinary differential equation in t whose solution can be written in terms of a hypergeometric function [1, 4, 5, 28–31]. For example, when $l = 1$ and $\mathbf{s} = (s)$, it follows that

$$\left(\left((1-t)\frac{d}{dt} \right) \left(t \frac{d}{dt} \right)^{s-1} - \theta^s \right) \mathscr{L}_{\mathbf{s}}(t, \theta) = 0, \tag{4}$$

and its solution is the Euler–Gauss hypergeometric function

$$\mathscr{L}_{(s)}(t, \theta) = {}_s F_{s-1} \left(\begin{matrix} -\omega\theta, -\omega^3\theta, \ldots, -\omega^{2s-1}\theta \\ 1, 1, \ldots, 1 \end{matrix} \middle| \, t \right),$$

where $\omega = \mathrm{e}^{\pi i/s}$, a primitive s-th root of -1 [4]. By expanding this solution into a hypergeometric series and equating like powers of θ with those in (3), it is possible to show, for example, when $s = 2$ that

---

[1] Following other authors, $\{\mathbf{s}\}^n = \{(s_1, s_2, \ldots, s_l)\}^n$ will be written more concisely as $\{s_1, s_2, \ldots, s_l\}^n$.

$$\zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!}, \quad n \geq 1. \tag{5}$$

In a similar manner it can be shown that

$$\zeta(\{3,1\}^n) = \frac{2\pi^{4n}}{(4n+2)!}, \quad n \geq 1.$$

This method has yielded a plethora of such MZV identities [3, 4, 6, 32]. The most general case is treated in [31], where it is shown that $\mathscr{L}_{\mathbf{s}}$ satisfies the linear differential equation of Fuchs type

$$(P_{\mathbf{s}} - \theta^{|\mathbf{s}|})\mathscr{L}_{\mathbf{s}}(t,\theta) = 0, \tag{6}$$

where for $\mathbf{s} = (s_1, s_2, \ldots, s_l) \in \mathbb{N}^l$

$$P_{\mathbf{s}} := P_{s_l} P_{s_{l-1}} \cdots P_{s_1}$$

and

$$P_{s_i} := \left((1-t)\frac{d}{dt}\right)\left(t\frac{d}{dt}\right)^{s_i - 1}.$$

(The conventions in [31] are to use $-\theta$ in place of θ and t in place of $1-t$.) In [31] and related work [28–30], the authors develop WKB type asymptotic expansions of these hypergeometric solutions.

The ultimate goal of the present paper is to provide a numerical scheme for estimating $\mathscr{L}_{\mathbf{s}}(t,\theta)$ by in essence mapping the $|\mathbf{s}|$-order linear differential equation (6) to a *system* of $|\mathbf{s}|$ first-order bilinear differential equations which can be solved by standard tools found in software packages like MatLab. Specifically, it will be shown how to construct a dynamical system of the form

$$\dot{z} = N_0 z\, u_0 + N_1 z\, u_1, \quad z(0) = z_0 \tag{7a}$$

$$y = Cz, \tag{7b}$$

which when simulated over the interval (0, 1) has the property that $y(t) = \mathscr{L}_{\mathbf{s}}(t,\theta)$ for any value of θ and $t \in (0,1)$. In this case, the matrices N_0 and N_1 will depend on θ, and the initial condition z_0 and the input functions u_0, u_1 must be suitably chosen. Such a technique could be useful for either disproving certain conjectures involving MZVs or providing additional evidence for the truthfulness of other conjectures. For example, one could validate with a certain level of (numerical) confidence a conjecture of the form

$$\zeta(\{\mathbf{s}_a\}^n) = b^n \zeta(\{\mathbf{s}_b\}^n), \quad n, b \in \mathbb{N},$$

where $\mathbf{s}_a \in \mathbb{N}^{l_a}$, $\mathbf{s}_b \in \mathbb{N}^{l_b}$ with $|\mathbf{s}_a| = |\mathbf{s}_b|$. Take as a specific example the known identity

$$\zeta(\{4\}^n) = 4^n \zeta(\{3,1\}^n) \tag{8}$$

for all $n \geq 1$, so that $\mathbf{s}_a = (4)$, $\mathbf{s}_b = (3,1)$ and $b = 4$ [4]. Note that for $n = 1$ the identity follows immediately from double shuffle relations for MZVs [22]. On the level of generating functions it is evident that

$$\mathscr{L}_{(4)}(1,\theta) = \sum_{n=0}^{\infty} \mathrm{Li}_{\{4\}^n}(1) \left(\theta^4\right)^n = \sum_{n=0}^{\infty} \zeta(\{4\}^n)\, \theta^{4n}$$

$$\mathscr{L}_{(3,1)}(1,\sqrt{2}\theta) = \sum_{n=0}^{\infty} \mathrm{Li}_{\{3,1\}^n}(1) \left((\sqrt{2}\theta)^4\right)^n = \sum_{n=0}^{\infty} 4^n \zeta(\{3,1\}^n)\, \theta^{4n}.$$

Therefore, identity (8) implies that

$$\mathscr{L}_{(4)}(1,\theta) - \mathscr{L}_{(3,1)}(1,\sqrt{2}\theta) = 0, \quad \forall \theta \in \mathbb{R}, \tag{9}$$

a claim that can be tested empirically if these generating functions can be accurately evaluated. The method can also be generalized to address the conjecture of Hoffman that

$$\zeta(\{2\}^n, 2, 2, 2) + 2\zeta(\{2\}^n, 3, 3) = \zeta(2, 1, \{2\}^n, 3), \tag{10}$$

for all integers $n > 0$, which has only been proved for $n \leq 8$ [6]. The idea here is to admit *non-periodic* components in the generating function calculation. For example, $(\{2\}^n, 3, 3)$ can be viewed as having the periodic component $\{2\}^n$ and the non-periodic component $(3,3)$. In the general case, say when $\mathbf{s}_n := (\mathbf{s}_a, \{\mathbf{s}_b\}^n, \mathbf{s}_c)$, $n \geq 0$, the generating function is defined analogously as

$$\mathscr{L}_{(\mathbf{s}_a,\{\mathbf{s}_b\},\mathbf{s}_c)}(t,\theta) := \sum_{n=0}^{\infty} \mathrm{Li}_{\mathbf{s}_n}(t) \left(\theta^{|\mathbf{s}_b|}\right)^n.$$

Therefore, relation (10), if true, would imply that

$$\mathscr{L}_{(\{2\},2,2,2)}(1,\theta) + 2\mathscr{L}_{(\{2\},3,3)}(1,\theta) - \mathscr{L}_{(2,1,\{2\},3)}(1,\theta) = 0, \quad \forall \theta \in \mathbb{R}. \tag{11}$$

The basic approach to estimating $\mathscr{L}_{\mathbf{s}}(t,\theta)$ is to map a periodic multiple polylogarithm to a rational series and then to employ well known concepts from control theory to produce bilinear state space realization (7) of the corresponding rational Chen–Fliess series [2, 16, 17]. The periodic nature of the MPL always ensures that these realizations have a certain built-in recursion/feedback structure. The technique will first be described in general, and then it will be demonstrated by empirically verifying the identities (5), (8), and (10). It should be noted that the connection between polylogarithms and differential equations with singularities at $\{0, 1, \infty\}$ has been well studied by a number of researchers, especially, [8, 10, 19, 20]. (See, in particular, [20, Chapter 4] and the references therein.) In addition, rational series of the type suitable for representing periodic multiple polylogarithms have appeared in [8, 20]. In this regard, the main contribution here is to customize these results specifically and explicitly for periodic multiple polylogarithms and then to actually apply them to problems like the Hoffman conjecture (10).

The paper is organized as follows. In the next section, a brief summary of rational Chen–Fliess series is given to establish the notation and the basic concepts to be employed. Then the general method for evaluating a generating function of a periodic multiple polylogarithm is given in the subsequent section, which also contains in Sect. 3.3 a short digression regarding another way of looking at periodic MPLs in terms of dendriform algebra along the lines of reference [15]. This is followed by several examples in Sect. 4. In particular, the last example shows that the Hoffman conjecture (10) has a high likelihood of being true. The final section gives the paper's conclusions.

2 Preliminaries

2.1 Chen–Fliess Series

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of word η, denoted $|\eta|$, is the number of letters in η. The set of all words with fixed length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. The set $\eta X^* \xi \subseteq X^*$ is the set of all words with prefix η and suffix ξ. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as $(c, \eta) \in \mathbb{R}^\ell$ and called the *coefficient* of the word η in the series c. Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. If the *constant term* $(c, \emptyset) = 0$ then c is said to be *proper*. The collection of all formal power series over the alphabet X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. The subset of polynomials is written as $\mathbb{R}^\ell\langle X\rangle$. Each set forms an associative $\mathbb{R}$-algebra under the catenation product.

Definition 1 Given $\xi \in X^*$, the corresponding ***left-shift operator*** $\xi^{-1} : X^* \to \mathbb{R}\langle X\rangle$ is defined:

$$\eta \mapsto \xi^{-1}(\eta) := \begin{cases} \eta' : \eta = \xi\eta' \\ 0 \ : \text{otherwise.} \end{cases}$$

It is extended linearly to $\mathbb{R}^\ell\langle\langle X\rangle\rangle$.

One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $t_0 < t_1$ be fixed, and consider a class of locally integrable functions $u = (u_1, \ldots, u_m) \in L^m_{1,loc}[t_0, t_1)$ modulo almost-everywhere equality with respect to the Lebesgue measure. For any compact subset

$\Omega = [t_0, s] \subset [t_0, t_1)$, the usual L_1 norm restricted to Ω, denoted here by $\|\cdot\|_{1,\Omega}$, provides a family of seminorms on $L^m_{1,loc}[t_0, t_1)$. Define inductively for each word $\eta \in X^*$ and $u \in L^m_{1,loc}(\Omega)$ an iterated integral by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t) := \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau)\, d\tau, \tag{12}$$

where $x_i \in X$, $\bar{\eta} \in X^*$, $t \in \Omega$, and $u_0 = 1$. The input-output operator corresponding to the series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is the *Fliess operator* or *Chen–Fliess series*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t) \tag{13}$$

[17]. If there exist real numbers $K_c, M_c > 0$ such that the coefficients of the generating series $c = \sum_{\eta \in X^*}(c, \eta)\eta \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ satisfy the growth bound

$$|(c, \eta)| \leq K_c M_c^{|\eta|}\, |\eta|!, \quad \forall \eta \in X^*, \tag{14}$$

then the series (13) converges absolutely and uniformly for every $t \in \Omega$ provided the measure of Ω and $\|u\|_{1,\Omega}$ are sufficiently small [18].

In the case of polylogarithms, it is sufficient to consider the single-input, single-output case $m = \ell = 1$ and to set $t_0 = 0$ and $t_1 = 1$. The convergence situation, however, is a bit different: the underlying iterated integrals (12) involve the locally integrable function $u_1(t) = 1/(1-t)$ on $[0, 1)$, but the function u_0 is now given by $u_0(t) = 1/t$, which is locally integrable only on $(0, 1)$. The growth condition (14) is *not* sufficient to ensure the convergence of a Chen-Fliess series. Even rationality of the generating series c is not sufficient as it can be shown using results from [20, Theorem 4.3.4], for example, that $F_c[\mathrm{Li}_0](t)$ with $c = \sum_{k \geq 0} x_0^k x_1$ and $\mathrm{Li}_0(t) := t/(1-t)$ is divergent. Therefore, the convergence of (13) will have to be addressed for the specific case of interest in the context of polylogarithms.

2.2 Bilinear Realizations of Rational Chen–Fliess Series

A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is called *invertible* if there exists a series $c^{-1} \in \mathbb{R}\langle\langle X\rangle\rangle$ such that $cc^{-1} = c^{-1}c = 1$.[2] In the event that c is not proper, i.e., the coefficient $(c, \emptyset)$ is nonzero, it is always possible to write

$$c = (c, \emptyset)(1 - c'),$$

where $c' \in \mathbb{R}\langle\langle X\rangle\rangle$ is proper. It then follows that

---

[2] The polynomial $1\emptyset$ is abbreviated throughout as 1.

$$c^{-1} = \frac{1}{(c, \emptyset)}(1 - c')^{-1} = \frac{1}{(c, \emptyset)}(c')^*,$$

where the Kleene star of c' is defined by

$$(c')^* := \sum_{i=0}^{\infty} (c')^i.$$

In fact, $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is invertible *if and only if* c is not proper. Now let S be a subalgebra of the $\mathbb{R}$-algebra $\mathbb{R}\langle\langle X\rangle\rangle$ with the catenation product. S is said to be *rationally closed* when every invertible $c \in S$ has $c^{-1} \in S$ (or equivalently, every proper $c' \in S$ has $(c')^* \in S$). The *rational closure* of any subset $E \subset \mathbb{R}\langle\langle X\rangle\rangle$ is the smallest rationally closed subalgebra of $\mathbb{R}\langle\langle X\rangle\rangle$ containing E.

Definition 2 A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is ***rational*** if it belongs to the rational closure of $\mathbb{R}\langle X\rangle$.

Rational series have appeared in a number of different contexts including automata theory [26], control theory [17], formal language theory [25], and polylogarithms [20]. The monograph [2] provides a concise introduction to the area. Of particular importance is an alternative characterization of rationality using the following concept.

Definition 3 A ***linear representation*** of a series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is any triple (μ, γ, λ), where

$$\mu : X^* \to \mathbb{R}^{n \times n}$$

is a monoid morphism, and the vectors $\gamma, \lambda^T \in \mathbb{R}^{n\times 1}$ are such that each coefficient

$$(c, \eta) = \lambda\mu(\eta)\gamma, \quad \forall \eta \in X^*.$$

The integer n is the dimension of the representation.

Definition 4 A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is called ***recognizable*** if it has a linear representation.

Theorem 1 [26] *A formal power series is rational if and only if it is recognizable.*

Returning to (13), Chen–Fliess series F_c is said to be rational when its generating series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is rational. The state space realization (7) is said to *realize F_c on some admissible input set $\mathscr{U}$* when (7a) has a well defined solution, $z(t)$, on the interval $[t_0, t_0 + T]$ for every $T > 0$ with input $u \in \mathscr{U}$ and output

$$y(t) = F_c[u](t) = C(z(t)), \quad t \in [t_0, t_0 + T].$$

Identify with any linear representation (μ, γ, λ) of the series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ the bilinear system

$$(N_0, N_1, z_0, C) := (\mu(x_0), \mu(x_1), \gamma, \lambda).$$

The following result is well known.

Theorem 2 [17, 18] *The statements below are equivalent for a given $c \in \mathbb{R}\langle\langle X\rangle\rangle$:*

i *(μ, γ, λ) is a linear representation of c.*
ii *The bilinear system (N_0, N_1, z_0, C) realizes F_c on the extended space $L_{\mathfrak{p},e}(t_0)$ for any $\mathfrak{p} \geq 1$.*

3 Evaluating Periodic Multiple Polylogarithms

It is first necessary to associate a periodic MPL and its generating function to a rational series. Elements of this idea have appeared in numerous places. The approach taken here is most closely related to the one presented in [21]. The next step is then to find the bilinear realization of the rational Chen–Fliess series in terms of its linear representation (see Theorem 4). The case when non-periodic components are present works similarly but is slightly more complicated (see Theorem 5). Recall that throughout $m = 1$, so that the underlying alphabet is $X := \{x_0, x_1\}$.

3.1 Periodic Multiple Polylogarithms

Given any admissible vector $\mathbf{s} \in \mathbb{N}^l$, there is an associated word $\eta_{\mathbf{s}} \in x_0X^*x_1$ of length $|\mathbf{s}|$

$$\eta_{\mathbf{s}} = x_0^{s_1-1}x_1x_0^{s_2-1}x_1 \cdots x_0^{s_l-1}x_1.$$

In which case, $c_{\mathbf{s}} := (\theta^{|\mathbf{s}|}\eta_{\mathbf{s}})^* = \sum_{n\geq 0}\left(\theta^{|\mathbf{s}|}\eta_{\mathbf{s}}\right)^n$ is a rational series satisfying the identity

$$1 + (\theta^{|\mathbf{s}|}\eta_{\mathbf{s}})c_{\mathbf{s}} = c_{\mathbf{s}}. \tag{15}$$

The idea is to now relate the generating function of the sequence $(\mathrm{Li}_{\{\mathbf{s}\}^n}(t))_{n>0}$ to the Chen–Fliess series with generating series $c_{\mathbf{s}}$. Recall that for any word $x_i\xi' \in X^*$ the iterated integral is defined inductively by

$$E_{x_i\xi'}[u](t) = \int_0^t u_i(\tau)E_{\xi'}[u](\tau)\, d\tau,$$

where $x_i \in X, \xi' \in X^*$. Assume here that the letters x_0 and x_1 correspond to the inputs $u_0(t) := 1/t$ and $u_1(t) := 1/(1-t)$, respectively, and $E_{\emptyset} := 1$. For the formal power series $c_{\mathbf{s}} \in \mathbb{R}\langle\langle X\rangle\rangle$, the corresponding Chen–Fliess series is then taken to be

$$F_{c_{\mathbf{s}}}[u] = \sum_{\xi \in X^*} (c_{\mathbf{s}}, \xi) E_\xi[u].$$

Comparing this to the classical definition (13), the factor $1/t$ can be extracted from u_0 and u_1 so that each integral can be viewed instead as integration with respect to the Haar measure. That is,

$$E_{x_i \xi'}[u](t) = \int_0^t \bar{u}_i(\tau) E_{\xi'}[u](\tau) \frac{d\tau}{\tau},$$

where $\bar{u}_0(t) := 1$ and $\bar{u}_1(t) = t u_1(t)$. The following theorem is central to the paper.

Theorem 3 *For any admissible vector* $\mathbf{s} \in \mathbb{N}^l$,

$$\mathscr{L}_{\mathbf{s}}(t, \theta) = F_{c_{\mathbf{s}}}[\mathrm{Li}_0](t), \quad t \in [0, 1),$$

where $\mathrm{Li}_0(t) := t/(1-t)$, *and the defining series for* $F_{c_{\mathbf{s}}}[\mathrm{Li}_0](t)$ *converges absolutely for any fixed* $t \in [0, 1)$ *provided* $\theta \in \mathbb{R}$ *is sufficiently small.*

Proof First observe that since $c_{\mathbf{s}} = \sum_{n \geq 0} \left(\theta^{|\mathbf{s}|} \eta_{\mathbf{s}}\right)^n$, it follows directly that

$$F_{c_{\mathbf{s}}}[u](t) = \sum_{n=0}^{\infty} F_{\left(\theta^{|\mathbf{s}|}\eta_{\mathbf{s}}\right)^n}[u](t) = \sum_{n=0}^{\infty} E_{\eta_{\mathbf{s}}^n}[u](t) \left(\theta^{|\mathbf{s}|}\right)^n.$$

Comparing this against the definition

$$\mathscr{L}_{\mathbf{s}}(t, \theta) = \sum_{n=0}^{\infty} \mathrm{Li}_{\{\mathbf{s}\}^n}(t) \left(\theta^{|\mathbf{s}|}\right)^n,$$

it is evident that one only needs to verify the identity

$$E_{\eta_{\mathbf{s}}^n}[\mathrm{Li}_0](t) = \mathrm{Li}_{\{\mathbf{s}\}^n}(t), \quad n \geq 0. \tag{16}$$

But this is clear from (2), i.e., for any admissible vector $\mathbf{s} \in \mathbb{N}^l$

$$\mathrm{Li}_{\mathbf{s}}(t) = \int_0^t u_i(\tau) \mathrm{Li}_{\mathbf{s}'}(\tau)\, d\tau,$$

where $\eta_{\mathbf{s}} = x_i \eta_{\mathbf{s}'}$,

$$u_i(t) = \begin{cases} \frac{1}{t} & : i = 0 \\ \frac{t}{1-t}\frac{1}{t} & : i = 1, \end{cases}$$

and $\mathrm{Li}_\emptyset(t) = 1$ [32]. Therefore, it follows directly that $\mathrm{Li}_{\mathbf{s}}(t) = E_{\eta_{\mathbf{s}}}[\mathrm{Li}_0](t)$, from which (16) also follows. To prove the convergence claim, it is sufficient to consider the special case where $\eta_{\mathbf{s}} = x_0^{s_1 - 1} x_1$ so that $c_{\mathbf{s}} = (\theta^{s_1} x_0^{s_1 - 1} x_1)^*$. The general case then follows similarly. Clearly, for any $t \in [0, 1)$

$$E_{x_1}[\mathrm{Li}_0](t, 0) = \ln\left(\frac{1}{1-t}\right) = \sum_{k=1}^{\infty} \frac{t^k}{k}.$$

Hence, for any $s_1 \geq 1$

$$E_{x_0^{s_1-1}x_1}[\mathrm{Li}_0](t, 0) = \sum_{k=1}^{\infty} \frac{t^k}{k^{s_1}} = \mathrm{Li}_{(s_1)}(t) < \infty,$$

and similarly, for any $n \geq 1$

$$E_{\left(x_0^{s_1-1}x_1\right)^n}[\mathrm{Li}_0](t, 0) = \sum_{k_1,k_2,\ldots,k_n=1}^{\infty} \frac{t^{k_1+k_2+\cdots+k_n}}{k_1^{s_1}(k_1+k_2)^{s_1}\cdots(k_1+k_2+\cdots+k_n)^{s_1}}.$$

The convergence claim for the series

$$F_{\left(\theta^{s_1}\,x_0^{s_1-1}x_1\right)^*}[\mathrm{Li}_0](t) = \sum_{n=0}^{\infty} E_{\left(x_0^{s_1-1}x_1\right)^n}[\mathrm{Li}_0](t, 0)\,\theta^{s_1 n}$$

can be verified by the ratio test. Observe

$$\begin{aligned}
&\frac{E_{\left(x_0^{s_1-1}x_1\right)^{n+1}}[\mathrm{Li}_0](t, 0)\,|\theta|^{s_1(n+1)}}{E_{\left(x_0^{s_1-1}x_1\right)^{n}}[\mathrm{Li}_0](t, 0)\,|\theta|^{s_1 n}} \\
&\qquad = \frac{\sum_{k_1,k_2,\ldots,k_{n+1}=1}^{\infty} \frac{t^{k_1+k_2+\cdots+k_{n+1}}}{k_1^{s_1}(k_1+k_2)^{s_1}\cdots(k_1+k_2+\cdots+k_{n+1})^{s_1}}}{\sum_{k_1,k_2,\ldots,k_n=1}^{\infty} \frac{t^{k_1+k_2+\cdots+k_n}}{k_1^{s_1}(k_1+k_2)^{s_1}\cdots(k_1+k_2+\cdots+k_n)^{s_1}}}\,|\theta|^{s_1} \\
&\qquad = \sum_{k_1=1}^{\infty} \frac{t^{k_1}}{k_1^{s_1}} \frac{\sum_{k_2,k_3,\ldots,k_{n+1}=1}^{\infty} \frac{t^{k_2+k_3+\cdots+k_{n+1}}}{(k_1+k_2)^{s_1}(k_1+k_2+k_3)^{s_1}\cdots(k_1+k_2+\cdots+k_{n+1})^{s_1}}}{\sum_{k_2,k_3,\ldots,k_{n+1}=1}^{\infty} \frac{t^{k_2+k_3+\cdots+k_{n+1}}}{k_2^{s_1}(k_2+k_3)^{s_1}\cdots(k_2+k_3+\cdots+k_{n+1})^{s_1}}}\,|\theta|^{s_1} \\
&\qquad < \mathrm{Li}_{(s_1)}(t)|\theta|^{s_1},
\end{aligned}$$

so that ratio is less than one when $|\theta| < (1/\mathrm{Li}_{(s_1)}(t))^{1/s_1}$. □

The key idea now is to apply Theorem 2 and the rational nature of the series $c_{\mathbf{s}}$ in order to build a bilinear realization of the mapping $u \mapsto y = F_{c_{\mathbf{s}}}[u]$ (see [23, 24]) so that $\mathscr{L}_{\mathbf{s}}(t, \theta)$ can be evaluated by numerical simulation of a dynamical system. In principle, one could attempt to ensure that any such realization is minimal in dimension or even canonical in some sense [7, 9, 11, 27]. There is also the potential for lower dimensional realizations to exist if systems other than bilinear realizations are considered. But in the present context these issues are not really essential. In addition, the realizations considered here are in the same general class as those described in [8, 20] for realizing classes of hypergeometric functions and polylogarithms using rational generating series. But in this work they are customized specifically for periodic multiple polylogarithms.

Theorem 4 *For any admissible* $\mathbf{s} \in \mathbb{N}^l$, $\mathscr{L}_{\mathbf{s}}(t, \theta) = F_{c_{\mathbf{s}}}[\mathrm{Li}_0](t)$ *has the bilinear realization*

$$(N_0, N_1, z_0, C) := \big(\mu(x_0), \mu(x_1), \gamma, \lambda\big),$$

where

$$N_0 = \mathrm{diag}\big(N_0(s_1), N_0(s_2), \ldots, N_0(s_l)\big) \tag{17a}$$

$$N_1 = I^+_{|\mathbf{s}|} - N_0 + \theta^{|s|} e_{|\mathbf{s}|} e_1^T \tag{17b}$$

with $N_0(s_i) \in \mathbb{R}^{s_i \times s_i}$ *and* $I^+_{|\mathbf{s}|} \in \mathbb{R}^{|\mathbf{s}| \times |\mathbf{s}|}$ *being matrices of zeros except for a super diagonal of ones,* e_i *is an elementary vector with a one in the* i*-th position, and* $z_0 = C^T = e_1 \in \mathbb{R}^{|\mathbf{s}| \times 1}$.

Proof First recall Definition 1 describing the left-shift operator on X^*, i.e., for any $x_i \in X$, $x_i^{-1}(\cdot)$ is defined by $x_i^{-1}(x_i \eta) = \eta$ with $\eta \in X^*$ and zero otherwise. In which case, $(x_i \xi)^{-1}(\cdot) = \xi^{-1} x_i^{-1}(\cdot)$ for any $\xi \in X^*$. Now assign the first state of the realization to be

$$z_1(t) = F_{c_{\mathbf{s}}}[u](t) = 1 + F_{(\theta^{|\mathbf{s}|}\eta_{\mathbf{s}})c_{\mathbf{s}}}[u](t).$$

In light of the integral representation (2) of MPLs, differentiate z_1 exactly s_1 times so that the input $u_1(t) := \bar{u}_1(t)/t$ appears. Assign a new state at each step along the way. Specifically,

$$\begin{aligned}
\dot{z}_1(t) &= \frac{1}{t} F_{\theta^{|\mathbf{s}|} x_0^{-1}(\eta_{\mathbf{s}}) c_{\mathbf{s}}}[u](t) =: z_2(t)\frac{1}{t} \\
&\;\vdots \\
\dot{z}_{s_1-1}(t) &= \frac{1}{t} F_{\theta^{|\mathbf{s}|}(x_0^{s_1-1})^{-1}(\eta_{\mathbf{s}}) c_{\mathbf{s}}}[u](t) =: z_{s_1}(t)\frac{1}{t} \\
\dot{z}_{s_1}(t) &= \bar{u}_1(t)\frac{1}{t} F_{\theta^{|\mathbf{s}|}(x_0^{s_1-1}x_1)^{-1}(\eta_{\mathbf{s}}) c_{\mathbf{s}}}[u](t) =: z_{s_1+1}(t)\bar{u}_1(t)\frac{1}{t}.
\end{aligned}$$

This produces the first s_1 rows of the matrices in (17) since when $l > 1$

$$\begin{aligned}
\begin{bmatrix} \dot{z}_1(t) \\ \vdots \\ \dot{z}_{s_1-1}(t) \\ \dot{z}_{s_1}(t) \end{bmatrix} &= I^+_{s_1\times(s_1+1)} \begin{bmatrix} z_1(t) \\ \vdots \\ z_{s_1}(t) \\ z_{s_1+1}(t)\bar{u}_1(t) \end{bmatrix} \frac{1}{t} \\
&= \left[\, N_0(s_1) \,\middle|\, 0 \,\right] \left[\begin{array}{c} z_1(t) \\ \vdots \\ z_{s_1}(t) \\ \hline z_{s_1+1}(t) \end{array}\right] \frac{1}{t} + \left[\, \mathbf{0}_{s_1} \,\middle|\, e_{s_1} \,\right] \left[\begin{array}{c} z_1(t) \\ \vdots \\ z_{s_1}(t) \\ \hline z_{s_1+1}(t) \end{array}\right] \bar{u}_1(t)\frac{1}{t}.
\end{aligned}$$

Both $\left[\, N_0(s_1) \,\middle|\, 0 \,\right]$ and $\left[\, \mathbf{0}_{s_1} \,\middle|\, e_{s_1} \,\right]$ denote matrices in $\mathbb{R}^{s_1\times(s_1+1)}$. The pattern is exactly repeated until the final state, then the periodicity of $c_{\mathbf{s}}$ comes into play. Namely,

$$\dot{z}_{|\mathbf{s}|}(t) = \theta^{|\mathbf{s}|}\bar{u}_1(t)\frac{1}{t}F_{(\eta_{\mathbf{s}})^{-1}(\eta_{\mathbf{s}})c_{\mathbf{s}}}[u](t) =: \theta^{|\mathbf{s}|}z_1(t)\bar{u}_1(t)\frac{1}{t},$$

which gives the final rows of N_0 and N_1 in (17). ☐

It is worth pointing out that the validity of (6) is obvious in the present setting. Namely, (6) follows from the fact that (15) implies $\eta_{\mathbf{s}}^{-1}(c_{\mathbf{s}}) - \theta^{|\mathbf{s}|}c_{\mathbf{s}} = 0$, and thus, Theorem 3 gives

$$(P_{\mathbf{s}} - \theta^{|\mathbf{s}|})\mathscr{L}_{\mathbf{s}}(t,\theta) = (P_{\mathbf{s}} - \theta^{|\mathbf{s}|})F_{c_{\mathbf{s}}}[\mathrm{Li}_0](t) = F_{\eta_{\mathbf{s}}^{-1}(c_{\mathbf{s}})-\theta^{|\mathbf{s}|}c_{\mathbf{s}}}[\mathrm{Li}_0](t) = F_{0\cdot c_{\mathbf{s}}}[\mathrm{Li}_0](t) = 0.$$

3.2 Periodic Multiple Polylogarithms with Non-periodic Components

The non-periodic case requires a generalization of the basic set-up. The following lemma links this class of generating functions to the corresponding set of rational Fliess operators.

Lemma 1 *For any admissible* $\mathbf{s} := (\mathbf{s}_a, \{\mathbf{s}_b\}, \mathbf{s}_c)$

$$\mathscr{L}_{\mathbf{s}}(t,\theta) = F_{c_{\mathbf{s}}}[\mathrm{Li}_0](t), \quad t \in [0,1), \quad \theta \in \mathbb{R},$$

where $c_{\mathbf{s}} := \eta_{\mathbf{s}_a}\left(\theta^{|\mathbf{s}_b|}\eta_{\mathbf{s}_b}\right)^* \eta_{\mathbf{s}_c}$.

Proof Similar to the periodic case, $c_{\mathbf{s}} = \sum_{n\geq 0}\eta_{\mathbf{s}_a}\left(\theta^{|\mathbf{s}_b|}\eta_{\mathbf{s}_b}\right)^n \eta_{\mathbf{s}_c}$, and therefore,

$$F_{c_{\mathbf{s}}}[u](t) = \sum_{n=0}^{\infty} F_{\eta_{\mathbf{s}_a}\left(\theta^{|\mathbf{s}_b|}\eta_{\mathbf{s}_b}\right)^n\eta_{\mathbf{s}_c}}[u](t) = \sum_{n=0}^{\infty} E_{\eta_{\mathbf{s}_a}\eta_{\mathbf{s}_b}^n\eta_{\mathbf{s}_c}}[u](t)\left(\theta^{|\mathbf{s}_b|}\right)^n.$$

The same argument used for proving (16) now shows that $E_{\eta_{\mathbf{s}_a}\eta_{\mathbf{s}_b}^n\eta_{\mathbf{s}_c}}[\mathrm{Li}_0](t) = \mathrm{Li}_{\mathbf{s}_n}(t)$, $n \geq 0$. In which case, $F_{c_{\mathbf{s}}}[\mathrm{Li}_0](t) = \mathscr{L}_{\mathbf{s}}(t, \theta)$ as claimed. □

The required generalization of Theorem 4 is a bit more complicated. A simple example is given first to motivate the general approach.

Example 1 Consider the periodic MPL with non-periodic components specified by $\mathbf{s} = (2, 1, \{2\}, 3)$ as appearing in (11). In this case, $c_{\mathbf{s}} = \sum_{n\geq 0} x_0x_1^2(\theta^2 x_0x_1)^n x_0^2x_1 = x_0x_1^2\bar{c}$, where $\bar{c} = x_0^2x_1 + \theta^2x_0x_1\bar{c}$. Assign the first state of the realization to be

$$z_1(t) = F_{c_{\mathbf{s}}}[u](t) = F_{x_0x_1^2\bar{c}}[u](t).$$

The strategy here is to differentiate z_1 exactly $\left|\eta_{\mathbf{s}_a}\right| = \left|x_0x_1^2\right| = 3$ times, assigning new states along the way, in order to remove the prefix $x_0x_1^2$ and isolate $\bar{c}$. At which point, the identity $\bar{c} = x_0^2x_1 + \theta^2x_0x_1\bar{c}$ is used and the process is continued. This will yield a certain block diagonal structure for N_0 and an upper triangular form for N_1. As will be shown shortly, this structure is completely general but possibly redundant. Specifically,

$$\begin{aligned}
\dot{z}_1(t) &= \frac{1}{t}F_{x_1^2\bar{c}}[u](t) =: z_2(t)\frac{1}{t}\\
\dot{z}_2(t) &= \frac{1}{t}\bar{u}_1(t)F_{x_1\bar{c}}[u](t) =: z_3(t)\bar{u}_1(t)\frac{1}{t}\\
\dot{z}_3(t) &= \frac{1}{t}\bar{u}_1(t)F_{\bar{c}}[u](t) = \frac{1}{t}\bar{u}_1(t)F_{x_0^2x_1+\theta^2x_0x_1\bar{c}}[u](t) =: z_4(t)\bar{u}_1(t)\frac{1}{t}\\
\dot{z}_4(t) &= \frac{1}{t}F_{x_0x_1+\theta^2x_1\bar{c}}[u](t) =: z_5(t)\frac{1}{t}\\
\dot{z}_5(t) &= \frac{1}{t}F_{x_1}[u](t) + \frac{\theta^2}{t}\bar{u}_1(t)F_{\bar{c}}[u](t) =: z_6(t)\frac{1}{t} + \theta^2 z_4(t)\bar{u}_1(t)\frac{1}{t}\\
\dot{z}_6(t) &= \bar{u}_1(t)\frac{1}{t}.
\end{aligned}$$

The corresponding realization at this point has the form

$$\begin{aligned}
\dot{z} =& \tilde{N}_0z\bar{u}_0 + \tilde{N}_1z\bar{u}_1 + B_1\bar{u}_1, \quad z(0) = \tilde{z}_0\\
y =& \tilde{C}z,
\end{aligned}$$

which does not have the form of a bilinear realization as defined in (7) since the state equation for z_6 does not depend on z, and thus, the term $B_1\bar{u}_1$ with $B_1 = e_6$ appears. Nevertheless, a permutation of the canonical embedding of Brockett (see [7, Theorem 1]), namely,

$$N_0 = \begin{bmatrix} \tilde{N}_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} \tilde{N}_1 & B_1 \\ 0 & 0 \end{bmatrix}, \quad z_0 = \begin{bmatrix} \tilde{z}_0 \\ 1 \end{bmatrix}, \quad C^T = \begin{bmatrix} \tilde{C}^T \\ 0 \end{bmatrix}, \tag{18}$$

renders an input-output equivalent bilinear realization of the desired form, albeit at the cost of increasing the dimension of the system by one. In this case,

$$N_0 = \left[\begin{array}{ccc|ccc|c} 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ \hline 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0 \end{array}\right], \quad N_1 = \left[\begin{array}{ccc|ccc|c} 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ \hline 0&0&0&0&0&0&0\\ 0&0&0&\theta^2&0&0&0\\ 0&0&0&0&0&0&1\\ \hline 0&0&0&0&0&0&0 \end{array}\right], \quad z(0) = \left[\begin{array}{c} 1\\ 0\\ 0\\ \hline 0\\ 0\\ 0\\ \hline 1 \end{array}\right], \quad C^T = \left[\begin{array}{c} 1\\ 0\\ 0\\ \hline 0\\ 0\\ 0\\ \hline 0 \end{array}\right].$$

Theorem 5 *Consider any admissible* $\mathbf{s} := (\mathbf{s}_a, \{\mathbf{s}_b\}, \mathbf{s}_c)$ *with* $\eta_{\mathbf{s}_a} := x_{i_1} \cdots x_{i_k}$, $k = j_{|\mathbf{s}_a|}$, *and* $|\mathbf{s}_c| > 0$. *Then* $\mathscr{L}_{\mathbf{s}}(t, \theta) = F_{c_{\mathbf{s}}}[\mathrm{Li}_0](t)$ *has the bilinear realization* (N_0, N_1, z_0, C), *where*

$$N_0 = \mathrm{diag}(N_0(\mathbf{s}_a), N_0(\mathbf{s}_b, \mathbf{s}_c), 0), \quad N_1 = \begin{bmatrix} N_1(\mathbf{s}_a) & E_{|\mathbf{s}_a|1} \\ 0 & N_1(\mathbf{s}_b, \mathbf{s}_c) \end{bmatrix}$$

with $N_i(\mathbf{s}_a) \in \mathbb{R}^{|\mathbf{s}_a| \times |\mathbf{s}_a|}$ *being a matrix of zeros and ones depending only on* $\mathbf{s}_a$, $E_{|\mathbf{s}_a|1}$ *is the elementary matrix with a one in position* $(|\mathbf{s}_a|, 1)$, *and* $N_i(\mathbf{s}_b, \mathbf{s}_c) \in \mathbb{R}^{s_{bc} \times s_{bc}}$ *is a matrix of zeros, ones, and the entry* $\theta^{|\mathbf{s}_b|}$. *(Its dimension* s_{bc} *and exact structure depend only on* $\mathbf{s}_b$ *and* $\mathbf{s}_c$.*) Finally,* $z_0 = e_1 + e_{|\mathbf{s}_a| + s_{bc}} \in \mathbb{R}^{(|\mathbf{s}_a| + s_{bc}) \times 1}$ *and* $C = e_1 \in \mathbb{R}^{1 \times (|\mathbf{s}_a| + s_{bc})}$.

Proof Following Example 1, assign the first state of the realization to be

$$z_1(t) = F_{c_{\mathbf{s}}}[u](t) = F_{\eta_{\mathbf{s}_a} \bar{c}}[u](t),$$

where $\bar{c} := \eta_{\mathbf{s}_c} + \theta^{|\eta_{\mathbf{s}_b}|} \eta_{\mathbf{s}_b} \bar{c}$, and differentiate z_1 until the series $\bar{c}$ appears in isolation. Observe

$$\dot{z}_1(t) = \sum_{i=0}^{1} \bar{u}_i(t) \frac{1}{t} F_{x_i^{-1}(\eta_{\mathbf{s}_a})\bar{c}}[u](t) =: e_2^T z(t) \bar{u}_{i_1}(t) \frac{1}{t}.$$

So the first row of N_{i_1} is e_2^T, where x_{i_1} is the first letter of $\eta_{\mathbf{s}_a}$, and the first row of the other realization matrix contains all zeroes. Continuing in this way,

$$\dot{z}_k(t) = \sum_{i=0}^{1} \bar{u}_i(t) \frac{1}{t} F_{\eta_{\mathbf{s}_a}^{-1}(\eta_{\mathbf{s}_a})\bar{c}}[u](t) =: e_{k+1}^T z(t) \bar{u}_{i_k}(t) \frac{1}{t}.$$

Since in general $x_{i_k} = x_1$, the k-th row of N_1 is e_{k+1}^T, and the k-th row of the N_0 contains all zeroes. So far, this is in agreement with the proposed structure of the realization. Next observe that

$$\begin{aligned}\dot{z}_{k+1}(t) &= \sum_{i=0}^{1} \bar{u}_i(t)\frac{1}{t}F_{x_i^{-1}(\bar{c})}[u](t)\\ &= \sum_{i=0}^{1} \bar{u}_i(t)\frac{1}{t}\underbrace{F_{x_i^{-1}(\eta_{\mathbf{s}_c})}[u](t)}_{=:z_{k+2}(t)} + \sum_{j=0}^{1} \bar{u}_j(t)\frac{1}{t}\underbrace{F_{x_j^{-1}(\eta_{\mathbf{s}_b}\bar{c})}[u](t)}_{=:z_{k+3}(t)}.\end{aligned}$$

In this way, new states are created until finally the term $F_{\bar{c}}[u](t) = z_{k+1}(t)$ reappears as it must. This produces an entry $\theta^{|\mathbf{s}_b|}$ in N_1 and preserves the proposed structures of N_0 and N_1. But note, as in Example 1, that the process can continue to create new states, and the state $z_{k+1}(t)$ could reappear if $\eta_{\mathbf{s}_c}$ is a power of $\eta_{\mathbf{s}_b}$, a possibility that has not been excluded. In addition, this realization could produce *copies* of the the first k states if $\eta_{\mathbf{s}_c}$ contains $\eta_{\mathbf{s}_a}$ as a factor. These copies will still preserve the desired structure, but this possibility points out that in general the final realization constructed by this process may not be minimal. Finally, the canonical embedding (18), which is always needed if $|s_c| > 0$, yields the final elements of the proposed structure. $\square$

Clearly, when non-periodic components are present, giving a precise general form of the matrices N_0 and N_1 is not as simple as in the purely periodic case.

3.3 The Dendriform Setting

It is shown in this section that the generating function $\mathscr{L}_{\mathbf{s}}(t, \theta)$ defined in (3), more precisely its t-derivative, is a solution of a higher-order linear dendriform equation in the sense of [15]. The case with non-periodic components can also be considered from that perspective. This provides a purely algebraic setting for the problem and also motivates an interesting generalization in the context of the theory of linear dendriform equations.

Recall that MPLs satisfy shuffle product identities, which are derived from integration by parts for the iterated integrals in (2). For instance,

$$\mathrm{Li}_{(2)}(t)\mathrm{Li}_{(2)}(t) = 4\mathrm{Li}_{(3,1)}(t) + 2\mathrm{Li}_{(2,2)}(t).$$

In slightly more abstract terms this can be formulated using the notion of a dendriform algebra. The reader is referred to [15] for full details. Examples of dendriform algebras include the shuffle algebra as well as associative Rota–Baxter algebras. Indeed, for any $t_0 < t_1$, the space $L_{1,loc}[t_0, t_1)$ is naturally endowed with such a structure consisting of two products:

$$f \succ g := I(f)g \tag{19a}$$

$$f \prec g := fI(g), \tag{19b}$$

where I is the Riemann integral operator defined by $I(f)(t, t_0) := \int_{t_0}^{t} f(s)\, ds$—a Rota–Baxter map of weight zero. It is easily seen to satisfy the axioms of a *dendriform algebra*

$$\begin{aligned} f \succ (g \succ h) &= (f * g) \succ h \\ (f \succ g) \prec h &= f \succ (g \prec h) \\ (f \prec g) \prec h &= f \prec (g * h), \end{aligned}$$

where

$$f * g := f \succ g + f \prec g$$

is an associative product. The example (19) above moreover verifies the extra commutativity property $f \succ g = g \prec f$, making it a commutative dendriform or Zinbiel algebra[3]

$$(f \prec g) \prec h = f \prec (g \prec h + h \prec g).$$

This is another way of saying that Chen's iterated integrals define a shuffle product, which gives rise to the shuffle algebra of MPLs. For more details, including a link between general, i.e., not necessarily commutative, dendriform algebras and Fliess operators, see [13–15].

In the following, the focus is on the commutative dendriform algebra $(C[t_0, t_1), \succ, \prec)$, where $C[t_0, t_1)$ stands for the linear subspace of continuous (hence locally integrable!) functions on $[t_0, t_1)$. The linear operator $R_g^{\succ} : C[t_0, t_1) \to C[t_0, t_1)$ is defined for $g \in C[t_0, t_1)$ by right multiplication using (19a)

$$R_g^{\succ}(f) := f \succ g.$$

Now add the distribution $\delta = \delta_{t_0}$ to the dendriform algebra $C[t_0, t_1)$. In view of the identity $I(\delta) = 1$ on the interval $[t_0, t_1]$, it follows that $R_f^{\succ}(\delta) = \delta \succ f = f$ for any $f \in C[t_0, t_1)$. Consider next the specific functions $u_0(t) = 1/t$ and $u_1(t) = 1/(1 - t)$ which appeared above (with $t_0 = 0$ and $t_1 = 1$ here), and the corresponding linear operators $R_{u_0}^{\succ}$ and $R_{u_1}^{\succ}$. Although u_0 is not locally integrable on $[0, 1)$, the space $C[t_0, t_1)$ is invariant under $R_{u_0}^{\succ}$. For any word $w = x_0^{s_1 - 1} x_1 \cdots x_0^{s_l - 1} x_1 \in x_0 X^* x_1$, the linear operator $R_w^{\succ}$ is defined as the composition of the linear operators associated to its letters, namely,

$$R_w^{\succ} = (R_{u_0}^{\succ})^{s_1 - 1} R_{u_1}^{\succ} \cdots (R_{u_0}^{\succ})^{s_l - 1} R_{u_1}^{\succ}$$

for $w = w_1 \cdots w_{|\mathbf{s}|} = x_0^{s_1 - 1} x_1 \cdots x_0^{s_l - 1} x_1$. Using the shorthand notation $R_w^{\succ} = R_{\mathbf{s}}^{\succ}$ with $\mathbf{s} = (s_1, \ldots, s_l)$, the multiple polylogarithm $\mathrm{Li}_{\mathbf{s}}$ obviously satisfies

---

[3] The space of continuous maps on $[t_0, t_1]$ with values in the algebra $\mathscr{M}_n(\mathbb{R})$ is also a dendriform algebra, with $\prec$ and $\succ$ defined the same way. But it is Zinbiel only for $n = 1$.

$$\frac{d}{dt}\operatorname{Li}_{\mathbf{s}} = R^{\succ}_{\mathbf{s}}(\delta). \tag{21}$$

From (21) it follows immediately that

$$\frac{d}{dt}\mathscr{L}_{\mathbf{s}}(t,\theta) = \sum_{k=0}^{\infty} \theta^{k|\mathbf{s}|}(R^{\succ}_{\mathbf{s}})^{k}(\delta),$$

which in turn yields

$$\frac{d}{dt}\mathscr{L}_{s}(t,\theta) = \delta + \theta^{|\mathbf{s}|}R^{\succ}_{\mathbf{s}}\Big(\frac{d}{dt}\mathscr{L}_{s}(t,\theta)\Big). \tag{22}$$

Equation (22) is a dendriform equation of degree $(|\mathbf{s}|, 0)$ in the sense of [15, Section 7]. The general form of the latter is

$$X = a_{00} + \sum_{q=1}^{|\mathbf{s}|}\theta^{q}\sum_{j=1}^{q}(\cdots(X \succ a_{q1}) \succ a_{q1}\cdots) \succ a_{qq} \tag{23}$$

with $a_{00} := \delta, a_{qj} = 0$ for $q < |\mathbf{s}|$ and $a_{|\mathbf{s}|j} := \widetilde{w}_j$, matching the notations of equation (46) in Ref. [15]. The general solution X of (23) is the first coefficient of a vector Y of length $|\mathbf{s}|$ whose coefficients (discarding the first one) are given by $\theta^{j}R^{\succ}_{w_1\cdots w_j}(X)$ for $j = 1,\ldots,|\mathbf{s}|-1$. This vector satisfies the following matrix dendriform equation of degree $(1, 0)$:

$$Y = (\delta, \underbrace{0,\ldots,0}_{|\mathbf{s}|-1}) + \theta Y \succ N, \tag{24}$$

where the matrix[4] N is given by:

$$N = \begin{bmatrix} 0 & \widetilde{w}_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \widetilde{w}_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \widetilde{w}_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \widetilde{w}_{|\mathbf{s}|-1} \\ \widetilde{w}_{|\mathbf{s}|} & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

First, observe that the $|\mathbf{s}|$-fold product $(\cdots(N \succ N) \succ \cdots) \succ N$ yields a diagonal matrix with the entry $\frac{d}{dt}\operatorname{Li}_{\mathbf{s}}(t)$ in the position $(1, 1)$. Second, matrix N splits into $N = N_0u_0 + N_1u_1$ with N_0, N_1 as in (17). Equation (24) essentially corresponds to the integral equation deduced from (7) giving the state $z(t)$.

---

[4] The size of the matrix can be reduced from $1 + |\mathbf{s}|(|\mathbf{s}| - 1)/2$ to $|\mathbf{s}|$ by eliminating rows and columns of zeroes due to the particular form of (22) compared to equation (46) in [15].

The case with non-periodic components can also be handled in this setting. Observe

$$\frac{d}{dt}\mathscr{L}_{\mathbf{s}_a\{\mathbf{s}_b\}\mathbf{s}_c} = R^{\succ}_{\mathbf{s}_a}\left(\frac{d}{dt}\mathscr{L}_{\{\mathbf{s}_b\}\mathbf{s}_c}\right),$$

and the term $X' = \frac{d}{dt}\mathscr{L}_{\{\mathbf{s}_b\}\mathbf{s}_c}$ satisfies the dendriform equation

$$X' = R^{\succ}_{\mathbf{s}_c}(\delta) + \theta^{|\mathbf{s}_b|}R^{\succ}_{\mathbf{s}_b}(X'). \tag{25}$$

Equation (25) is again a dendriform equation of degree $(|\mathbf{s}_b|, 0)$ with $a_{00} = R^{\succ}_{\mathbf{s}_c}(\delta)$, $a_{qj} = 0$ for $q < |\mathbf{s}_b|$ and $a_{|\mathbf{s}_b|j} = w_j$ using the notation in [15]. The general solution X' of (25) is the first coefficient of a vector Y' of length $|\mathbf{s}_b|$ whose coefficients (discarding the first one) are given by $\theta^j R^{\succ}_{w_1\cdots w_j}(X')$ for $j = 1, \ldots, |\mathbf{s}_b| - 1$. This vector satisfies the following matrix dendriform equation of degree $(1, 0)$

$$Y' = (R^{\succ}_{\mathbf{s}_c}(\delta), \underbrace{0, \ldots, 0}_{|\mathbf{s}_b|-1}) + \theta Y' \succ M',$$

where the matrix M' is given by:

$$M' = \begin{bmatrix} 0 & \widetilde{w}_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \widetilde{w}_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \widetilde{w}_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \widetilde{w}_{|\mathbf{s}_b|-1} \\ \widetilde{w}_{|\mathbf{s}_b|} & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

One can ask the question whether the term $X = \frac{d}{dt}\mathscr{L}_{\mathbf{s}_a\{\mathbf{s}_b\}\mathbf{s}_c}$ itself is a solution of a dendriform equation. In fact, a closer look reveals that the theory of linear dendriform equations presented in [15] has not been sufficiently developed to embrace this more complex setting. In the light of Theorem 5, it is clear that the results in [15] should be adapted in order to address this question. Such a step, however, is beyond the scope of this paper and will thus be postponed to another work. It is worth mentioning that the matrix N needed in the linear dendriform equation

$$Y' = (0, \delta, 0, 0, 0, 0, 0) + \theta Y' \succ N$$

to match the result from Example 1 has the form

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \widetilde{w}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{w}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{w}_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \widetilde{w}_4 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{w}_6 & 0 & \widetilde{w}_5 \\ \widetilde{w}_7 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which reflects the canonical embedding of Brockett. The first component of the vector Y' contains the solution. As indicated earlier, a proper derivation of this result in the context of general dendriform algebras, i.e., extending the results in [15], lies outside the scope of the present paper.

4 Examples

In this section, three examples of the method described above are given corresponding to the generating functions behind the identities (5), (8), and (10).

Example 2 Consider the generating function $\mathscr{L}_{(2)}(t,\theta)$. This example is simple enough that a bilinear realization can be identified directly from (4). For any fixed θ define the first state variable to be $z_1(t) = \mathscr{L}_{(2)}(t,\theta)$, and the second state variable to be $z_2(t) = t\, d\mathscr{L}_{(2)}(t,\theta)/dt$. In which case,

$$\dot{z}_1(t) = z_2(t)\frac{1}{t}, \quad z_1(0) = 1 \tag{26a}$$

$$\dot{z}_2(t) = \theta^2\, z_1(t)\frac{t}{1-t}\,\frac{1}{t}, \quad z_2(0) = 0 \tag{26b}$$

$$y(t) = z_1(t). \tag{26c}$$

Thereupon, system (26) assumes the form of a bilinear system as given by (17), where the inputs are set to be $\bar{u}_0(t) = 1$ and $\bar{u}_1(t) = \mathrm{Li}_0(t) = t/(1-t)$, i.e.,

$$N_0 = N_0(2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad N_1 = N_1(2) = \begin{bmatrix} 0 & 0 \\ \theta^2 & 0 \end{bmatrix}, \quad z(0) = C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(recall the $1/t$ factors in (26) are absorbed into Haar integrators). A simulation diagram for this realization suitable for MatLab's Simulink simulation software is shown Fig. 1. Setting $\theta = 1$ and using Simulink's default integration routine `ode45` (Dormand-Prince method [12]) with a variable step size lower bounded by 10^{-8}, Fig. 2 was generated showing $\mathscr{L}_{(2)}(t,1) = F_{(x_0x_1)^*}[\mathrm{Li}_0](t)$ as a function of t. In particular, it was found numerically that $\mathscr{L}_{(2)}(1,1) \approx 3.6695$, which compares favorably to the theoretical value derived from (5):

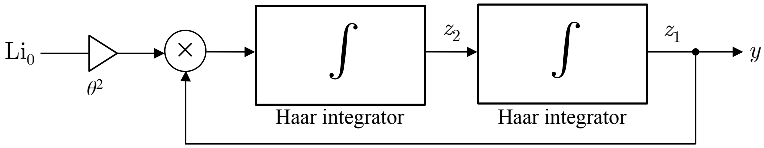
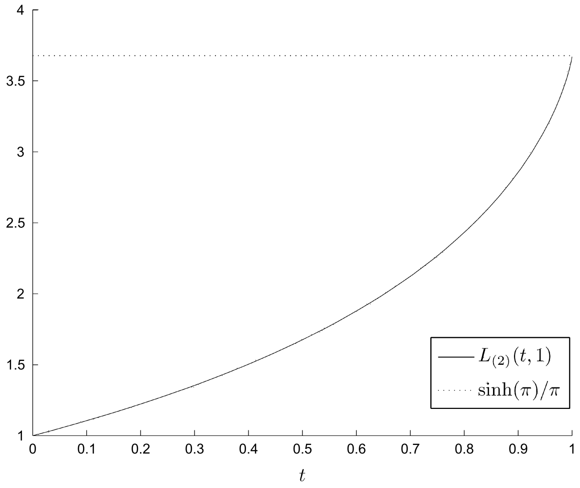


Fig. 1 Unity feedback system realizing $\mathscr{L}_{(2)}(t, 1)$

Fig. 2 Plot of $\mathscr{L}_{(2)}(t, 1)$ versus t



$$\mathscr{L}_{(2)}(1, 1) = \sum_{n=0}^{\infty} \zeta(\{2\}^n) = \sum_{n=0}^{\infty} \frac{\pi^{2n+1}}{(2n+1)^n} = \frac{\sinh(\pi)}{\pi} = 3.6761.$$

Better estimates can be found by more carefully addressing the singularities at the boundary conditions $t = 0$ and $t = 1$ in the Haar integrators.

Example 3 In order to validate (8), the identity (9) is checked numerically. Since the generating functions $\mathscr{L}_{(4)}$ and $\mathscr{L}_{(3,1)}$ are periodic, Theorem 4 applies. For $\mathbf{s} = (4)$ the corresponding bilinear realization is

$$N_0 = N_0(4) = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}, \quad N_1 = N_1(4) = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ \theta^4&0&0&0 \end{bmatrix}, \quad z(0) = C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For $\mathbf{s} = (3, 1)$ the bilinear realization is

$$N_0 = \left[\begin{array}{ccc|c} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ \hline 0&0&0&0 \end{array}\right], \quad N_1 = \left[\begin{array}{c|ccc} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ \hline \theta^4&0&0&0 \end{array}\right], \quad z(0) = C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

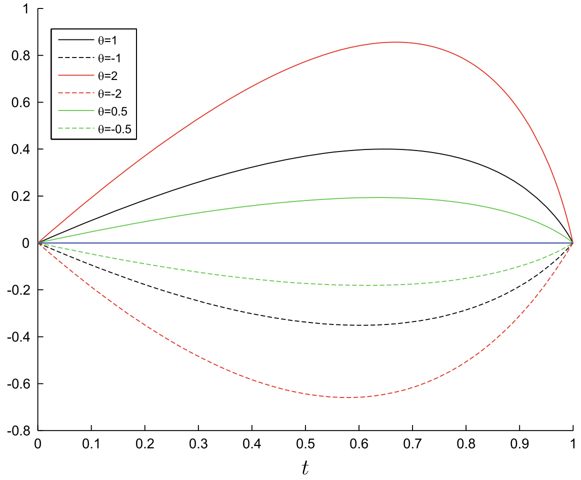


Fig. 3 Plot of $\mathscr{L}_{(4)}(t,\theta) - \mathscr{L}_{(3,1)}(t,\sqrt{2}\theta)$ versus t for different values of θ

These two dynamical systems were simulated using Haar integrators in Simulink and the difference (9) was computed as a function of t as shown in Fig. 3. As expected, this difference is very close to zero when $t = 1$ no matter how the parameter θ is selected. This is pretty convincing numerical evidence supporting (8), which as discussed in the introduction is known to be true.

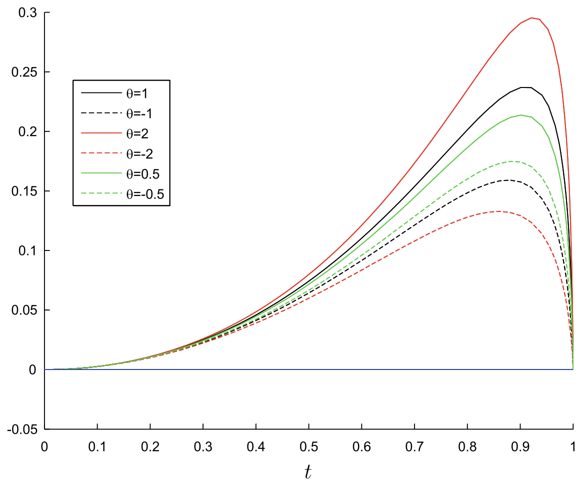
Example 4 Now the method is applied to the generating functions behind the Hoffman conjecture (10). In this case, each multiple polylogarithm has non-periodic components, so Theorem 5 has to be applied three times. The realization for $\mathscr{L}_{(2,1,\{2\},3)}(t,\theta)$ was presented in Example 1. Following a similar approach, the realization for $\mathscr{L}_{(\{2\},2,2,2)}(t,\theta)$ and $\mathscr{L}_{(\{2\},3,3)}(t,\theta)$ are, respectively,

$$N_0 = \left[\begin{array}{cc|cccc|c} 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ \hline 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0 \end{array}\right],\; N_1 = \left[\begin{array}{cc|cccc|c} 0&0&0&0&0&0&0\\ \theta^2&0&1&0&0&0&0\\ \hline 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ \hline 0&0&0&0&0&0&0 \end{array}\right],\; z(0) = \left[\begin{array}{c} 1\\ 0\\ \hline 0\\ 0\\ 0\\ 0\\ \hline 1 \end{array}\right],\; C^T = \left[\begin{array}{c} 1\\ 0\\ \hline 0\\ 0\\ 0\\ 0\\ \hline 0 \end{array}\right]$$

and

$$N_0 = \left[\begin{array}{cc|cccc|c} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ \hline 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0 \end{array}\right],\; N_1 = \left[\begin{array}{cc|cccc|c} 0&0&0&0&0&0&0\\ \theta^2&0&0&0&0&0&0\\ \hline 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ \hline 0&0&0&0&0&0&0 \end{array}\right],\; z(0) = \left[\begin{array}{c} 1\\ 0\\ \hline 0\\ 0\\ 0\\ 0\\ \hline 1 \end{array}\right],\; C^T = \left[\begin{array}{c} 1\\ 0\\ \hline 0\\ 0\\ 0\\ 0\\ \hline 0 \end{array}\right].$$

Fig. 4 Plot of $\mathscr{L}_{(\{2\},2,2,2)}(t,\theta) + 2\mathscr{L}_{(\{2\},3,3)}(t,\theta) - \mathscr{L}_{(2,1,\{2\},3)}(t,\theta)$ versus t for different values of θ



These dynamical systems were simulated to estimate numerically the left-hand side of (11) as shown in Fig. 4. As in the previous example, the case where $t = 1$ is of primary interest. This value is again very close to zero for every choice of θ tested. It is highly likely therefore that the Hoffman conjecture is true.

5 Conclusions

A systematic way was given to numerically evaluate the generating function of periodic multiple polylogarithm using Chen–Fliess series with rational generating series. The method involved mapping the corresponding Chen–Fliess series to a bilinear dynamical system, which could then be simulated numerically using Haar integration. A standard form for such a realization was given, and the method was generalized to the case where the multiple polylogarithm could have non-periodic components. The method was also described in the setting of dendriform algebras. Finally, the technique was used to numerically validate the Hoffman conjecture.

Acknowledgements The second author was supported by grant SEV-2011-0087 from the Severo Ochoa Excellence Program at the Instituto de Ciencias Matemáticas in Madrid, Spain. This research was also supported by a grant from the BBVA Foundation.

References

1. Aoki, T., Kombu, Y., Ohno, Y.: A generating function for sums of multiple zeta values and its applications. Proc. Amer. Math. Soc. **136**, 387–395 (2008)
2. Berstel, J., Reutenauer, C.: Noncommutative Rational Series with Applications. Cambridge University Press, Cambridge, UK (2010)
3. Borwein, J.M., Bradley, D.M., Broadhurst, D.J., Lisonek, P.: Combinatorial aspects of multiple zeta values. Electron. J. Combin. **5** R38, 12 (1998)
4. Borwein, J.M., Bradley, D.M., Broadhurst, D.J., Lisonek, P.: Special values of multidimensional polylogarithms. Trans. Amer. Math. Soc. **353**, 907–941 (2001)
5. Bowman, D., Bradley, D.M.: Multiple polylogarithms: a brief survey. In: Berndt, B.C., Ono, K. (eds.) Q-series with Applications to Combinatorics, Number Theory, and Physics: A Conference on Q-series with Applications to Combinatorics, Number Theory, and Physics, pp. 71–92. AMS, Providence, RI (2001)
6. Borwein, J.M., Zudilin, W.: Math honours: multiple zeta values, available at https://carma.newcastle.edu.au/MZVs/mzv.pdf
7. Brockett, R.W.: On the algebraic structure of bilinear systems. In: Mohler, R., Ruberti, R. (eds.) Theory and Applications of Variable Structure Systems, pp. 153–168. Academic Press, New York (1972)
8. Costermans, C., Minh, H.N.: Some results à l'Abel obtained by use of techniques à la Hopf. In: Proceeding of the Workshop on Global Integrability of Field Theories and Applications, Daresbury UK, pp. 63–83 (2006)
9. D'Alessandro, P., Isidori, A., Ruberti, A.: Realization and structure theory of bilinear dynamical systems. SIAM J. Control **12**, 517–535 (1974)
10. Deneufchâtel, M., Duchamp, G.H.E., Minh, V.H.N., Solomon, A.I.: Independence of hyperlogarithms over function fields via algebraic combinatorics. In: Winkler, F. (ed.) Algebraic Infomatics, Lecture Notes in Computer Science, vol. 6742, pp. 127–139. Springer, Berlin, Heidelberg (2011)
11. Dorissen, H.T.: Canonical forms for bilinear systems. Syst. Control Lett. **13**, 153–160 (1989)
12. Dormand, J.R., Prince, P.J.: A family of embedded Runge-Kutta formulae. J. Comput. Appl. Math. **6**, 19–26 (1980)
13. Duffaut Espinosa, L.A., Gray, W.S., Ebrahimi-Fard, K.: Dendriform-tree setting for fully non-commutative Fliess operators. In: Proceedings 53rd IEEE Conference on Decision and Control, Los Angeles, CA, pp. 4814–4819 (2014), arXiv:1409.0059
14. Duffaut Espinosa, L.A., Gray, W.S., Ebrahimi-Fard, K.: Dendriform-tree setting for fully non-commutative Fliess operators. IMA J. Math. Control Inform. **35**, 491–521 (2018)
15. Ebrahimi-Fard, K., Manchon, D.: Dendriform equations. J. Algebra **322**, 4053–4079 (2009)
16. Elliott, D.L.: Bilinear Control Systems. Springer, Dordrecht (2009)
17. Fliess, M.: Fonctionnelles causales non linéaires et indéterminées non commutatives. Bull. Soc. Math. France **109**, 3–40 (1981)
18. Gray, W.S., Wang, Y.: Fliess operators on L_p spaces: convergence and continuity. Syst. Control Lett. **46**, 67–74 (2002)
19. Minh, H.N.: Finite polyzêtas, Poly-Bernoulli numbers, identities of polyzêtas and noncommutative rational power series. In: Proceedings of WORDS'03, 4th International Conference on Combinatorics on Words, Turku, Finland, pp. 232–250 (2003)
20. Minh, V.H.N.: Calcul Symbolique Non Commutatif. Presses Académiques Francophones, Saarbrücken, Germany (2014)
21. Hoseaux, V., Jacob, G., Oussous, N.E., Petitot, M.: A complete Maple package for noncommutative rational power series. In: Li, Z., Sit, W.Y. (eds.) Computer Mathematics: Proceedings of the Sixth Asian Symposium, Beijing, China. World Scientific, pp. 174–188 (2003)
22. Ihara, K., Kaneko, M., Zagier, D.: Derivation and double shuffle relations for multiple zeta values. Compositio Math. **142**, 307–338 (2006)
23. Isidori, A.: Nonlinear Control Systems, 3rd edn. Springer, London (1995)

24. Jacob, G.: Réalisation des systèmes réguliers (ou bilinéaires) et séries génératrices non commutatives. In: Landau, I.D. (ed.) Outils et Modèles Mathématiques pour L'automatique, L'analyse de Systèmes et Le Traitement du Signal. CNRS-RCP 567, CNRS, Paris, pp. 325–357 (1981)
25. Nivat, M.: Transductions des languages de Chomsky. Ann. Inst. Fourier **18**, 339–455 (1968)
26. Schützenberger, M.P.: On the definition of a family of automata. Inform. Control **4**, 245–270 (1961)
27. Sussmann, H.J.: Minimal realizations and canonical forms for bilinear systems. J. Franklin Inst. **301**, 593–604 (1976)
28. Zakrzewski, M., Żoładek, H.: Linear differential equations and multiple zeta-values. I. Zeta(2). Fund. Math. **210**, 207–242 (2010)
29. Zakrzewski, M., Żoładek, H.: Linear differential equations and multiple zeta-values. II. A generalization of the WKB method. J. Math. Anal. Appl. **383**, 55–70 (2011)
30. Zakrzewski, M., Żoładek, H.: Linear differential equations and multiple zeta-values. III. Zeta(3). J. Math. Phys. **53** 013507 (2012)
31. Żoładek, H.: Note on multiple zeta-values. Bul. Acad. Ştiinţe Repub. Mold. Mat. **43**, 78–82 (2003)
32. Zudilin, W.: Algebraic relations for multiple zeta values. Russian Math. Surveys **58**, 1–29 (2003)