

# Uniform Approach to Double Shuffle and Duality Relations of Various $q$ -Analogues of Multiple Zeta Values via Rota–Baxter Algebras



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**Abstract** The multiple zeta values (MZVs) have been studied extensively in recent years. Currently there exist a few different types of  $q$ -analogues of the MZVs ( $q$ -MZVs) defined and studied by mathematicians and physicists. In this paper, we give a uniform treatment of these  $q$ -MZVs by considering their double shuffle relations (DBSFs) and duality relations. The main idea is a modification and generalization of the one used by Castillo Medina et al. to a few other types of  $q$ -MZVs including the one defined by the author in 2003. With different approach, Takeyama already studied this type by “regularization” and observed that there exist a new family of  $\mathbb{Q}$ -linear relations which are not consequences of the DBSFs. We call these duality relations in this paper and generalize them to all other types of  $q$ -MZVs. Since there are still some missing relations we further define the most general type of  $q$ -MZVs together with a new kind of relations called **P-R** relations which are used to lower the deficiencies further. As an application, we will confirm a conjecture of Okounkov on the dimensions of certain  $q$ -MZV spaces, either theoretically or numerically, for the weight up to 12. Some relevant numerical data are provided at the end.

**Keywords** Multiple zeta values ·  $q$ -analogue of multiple zeta values · Double shuffle relations · Duality relations · Rota-Baxter algebras

## 1 Introduction

The multiple zeta values are iterated generalizations of the Riemann zeta values to the multiple variable setting. Euler [8] first studied the double zeta values in the 18th century. Hoffman [13] and Zagier [28] independently considered systematically the following more general form in the early 1990’s. Let  $\mathbb{N}$  be the set of positive integers. For any  $d \in \mathbb{N}$  and  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$  with  $s_1 \geq 2$  one defines the *multiple zeta values* (MZVs) as the  $d$ -fold sum

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$$\zeta(\mathbf{s}) = \sum_{k_1 > \dots > k_d > 0} \frac{1}{k_1^{s_1} \dots k_d^{s_d}}.$$

In 1980s, Ecalle studied some quite general mathematical objects called “moulds” (functions with variable number of variables) of which the MZVs are one of the examples [9, p. 429]. He even mentioned their iterated integral representation [9, p. 431, Remark 4] without explicitly producing it.

A lot of important and sometimes surprising applications of MZVs have been found in many areas in mathematics and theoretical physics in recent years; see [4, 5, 10, 18, 19]. One of the most powerful ideas is to consider the so-called double shuffle relations (DBSFs). The stuffle relations are obtained directly by using the above series definition when multiplying two MZVs. The other, the shuffle relations, can be produced by multiplying their integral representations and using Chen’s theory of iterated integrals [6]. The interested reader is referred to the seminal paper [16] for more details.

Lagging behind the above development for about a decade, a few  $q$ -analogs were proposed and studied by different mathematicians and physicists. All of these  $q$ -analogs enjoy the property that when  $q \rightarrow 1$  one can recover the ordinary MZVs defined in the above if no divergence occurs. In this paper, by modifying and generalizing an idea in [7] we give a uniform treatment of these  $q$ -analogs by using some suitable Rota–Baxter algebras which reflect the properties of Jackson’s iterated integral representations of these  $q$ -analogs.

Recall that for any fixed  $q$  with  $0 < q < 1$  one can define the  $q$ -analog of positive integers by setting  $[k] = [k]_q := 1 + q + \dots + q^{k-1} = (1 - q^k)/(1 - q)$  for all  $k \in \mathbb{N}$ . To summarize the various versions of  $q$ -analog of MZVs ( $q$ -MZVs for abbreviation), we first define a general type of  $q$ -MZV of  $2d$  variables  $s_1, \dots, s_d, a_1, \dots, a_d \in \mathbb{Z}$  by

$$\zeta_q^{\mathbf{a}}[\mathbf{s}] := \sum_{k_1 > \dots > k_d > 0} \frac{q^{k_1 a_1 + \dots + k_d a_d}}{[k_1]^{s_1} \dots [k_d]^{s_d}} = (1 - q)^{|\mathbf{s}|} \sum_{k_1 > \dots > k_d > 0} \frac{q^{k_1 a_1 + \dots + k_d a_d}}{(1 - q^{k_1})^{s_1} \dots (1 - q^{k_d})^{s_d}}, \quad (1)$$

where  $|\mathbf{s}| = s_1 + \dots + s_d$  is called the *weight* and  $d$  the *depth*. The variables of  $\mathbf{a}$  are called *auxiliary variables*. Also, it is often convenient to study its modified form by dropping the power of  $1 - q$ , i.e.,

$$\mathfrak{z}_q^{\mathbf{a}}[\mathbf{s}] := \sum_{k_1 > \dots > k_d > 0} \frac{q^{k_1 a_1 + \dots + k_d a_d}}{(1 - q^{k_1})^{s_1} \dots (1 - q^{k_d})^{s_d}},$$

In the following table, we list a few different versions of  $q$ -MZVs that have been studied so far by different authors, except for one new type (type IV in the table). We only write down their modified form although sometimes the original authors considered only  $\zeta_q$ . We note that in 2004, Bradley [3] apparently defined  $\zeta_q^{(s_1-1, \dots, s_d-1)}[s_1, \dots, s_d]$  independently, and later, Okuda and Takeyama also studied some of the relations among this type of  $q$ -MZVs in [22]. Additionally, it is not hard

**Table 1** A time line of different versions of  $q$ -MZVs. ★=this paper

Type	Year	Authors	$q$ -MZV	DBSF
	2001	Schlesinger [23]	$\mathfrak{z}_q^{(0,\dots,0)}[s_1, \dots, s_d]$	See (2)
I	2002	Kaneko et al. [17]	$\mathfrak{z}_q^{(s-1)}[s]$ (depth=1)	N/A
I	2003	Zhao [29]	$\mathfrak{z}_q^{(s_1-1,\dots,s_d-1)}[s_1, \dots, s_d]$	[25], [26], ★
II	2003	Zudilin [30]	$\mathfrak{z}_q^{(s_1,\dots,s_d)}[s_1, \dots, s_d]$	[24], ★
III	2012	Ohno et al. [20]	$\mathfrak{z}_q^{(1,0,\dots,0)}[s_1, \dots, s_d]$	[7], ★
IV	2014	Zhao ★	$\mathfrak{z}_q^{(s_1-1,s_2,\dots,s_d)}[s_1, \dots, s_d]$	★
BK	2013	Bachmann & Kühn [1]	$\mathfrak{z}_q^{\text{BK}}[s_1, \dots, s_d]$	[31]
O	2014	Okounkov [21]	$\mathfrak{z}_q^{\text{O}}[s_1, \dots, s_d], s_j \geq 2$	★
G	2003	Zhao [29]	$\mathfrak{z}_q^{(a_1,\dots,a_d)}[s_1, \dots, s_d]$	★

to see that Schlesinger’s version diverges when  $|q| < 1$  but can converge if  $|q| > 1$ . In fact, for  $\mathbf{s} \in \mathbb{Z}^d$

$$\mathfrak{z}_{q^{-1}}^{(0,\dots,0)}[s_1, \dots, s_d] = (-1)^{s_1+\dots+s_d} \mathfrak{z}_q^{(s_1,\dots,s_d)}[s_1, \dots, s_d] = (-1)^{s_1+\dots+s_d} \mathfrak{z}_q^{\text{II}}[s_1, \dots, s_d]. \quad (2)$$

So it suffices to consider type II in order to understand Schlesinger’s  $q$ -MZVs. The last column of Table 1 provides the references where DBSFs are considered systematically (not only the stuffle), some of which are apparently different from our approach in this paper.

In this paper, we will use suitable Rota–Baxter algebras to study types I-IV  $q$ -MZVs listed in Table 1 in details. We also briefly consider the general type G and Okounkov’s type O  $q$ -MZVs. Note that the numerators inside the summands of  $\zeta_q^{\text{BK}}$  and  $\zeta_q^{\text{O}}$  are not exact powers of  $q$ , but some polynomials of  $q$  enjoying nice properties. Further, for  $\zeta_q^{\text{O}}$  the polynomial numerator is at worst a sum of two  $q$ -powers so our method can still work. See Corollary 6.6. It may be difficult to use the approach here to study the Bachmann and Kühn type since the numerators are much more complicated.

In the classical setting, the so-called regularized DBSFs play extremely important roles in discovering and proving  $\mathbb{Q}$ -linear relations among the MZVs. The first serious attempt to discover the DBSFs among  $q$ -MZVs was carried out by the author in [29] by using Jackson’s iterated  $q$ -integrals. However, the computation there was too complicated so only very few relations were found successfully. The real breakthrough came with Takeyama’s successful application of Hoffman’s algebras to study type I  $q$ -MZVs in [26]. However, his approach to the shuffle relations relies on some auxiliary multiple polylogarithm functions and consequently it is very hard to see why these relations should hold.

The situation looks much better with the appearance of the recent paper [7] by Castillo Medina, Ebrahimi-Fard and Manchon who generalized Chen’s iterated integrals to Jackson’s iterated  $q$ -integrals to study type III  $q$ -MZVs by using Rota–Baxter

algebra techniques. Later, Singer [24, 25] adopted the algebraic setup of the DBSFs to study type I and type II  $q$ -MZVs. Motivated by this new idea, in this paper we will consider all the  $q$ -MZVs of type I, II, III and IV by finding/using their correct realizations in terms of Jackson's iterated  $q$ -integrals. Then by combining the Rota–Baxter algebra technique and Hoffman's algebra of words we are able to study the DBSFs of all of these  $q$ -MZVs.

When one considers the  $\mathbb{Q}$ -linear relations among the ordinary MZVs, the main difficulty lies in the insufficiency of DBSFs produced by only admissible arguments. In the  $q$ -analog setting, the situation is only partially similar and sometimes much more complicated.

For type I  $q$ -MZVs, our computation shows that the DBSFs CAN provide all the  $\mathbb{Q}$ -linear relations. However, in order to study these relations, as Takeyama noticed first, one has to enlarge the set of type I  $q$ -MZVs to something we call type  $\tilde{I}$   $q$ -MZVs which are a kind of “regularized”  $q$ -MZVs in the sense that one needs to consider some convergent versions of  $q$ -MZVs when  $s_1 = 1$  by modifying the auxiliary variables of  $\mathbf{a}$ . But for these type  $\tilde{I}$   $q$ -MZVs themselves, DBSFs are insufficient to provide all the  $\mathbb{Q}$ -linear relations and a certain “Resummation Identity” defined by Takeyama is required. In this paper, we will adopt the term “duality” due to its similarity to the duality relations of the ordinary MZVs. Moreover, for type  $\tilde{I}$   $q$ -MZVs of weight bounded by  $w$  there are often still missing relations even after we consider both DBSFs and duality relations within the same weight and depth range. These missing relations can be recovered only after we increase the weight and depth. This phenomenon is not unique to type  $\tilde{I}$   $q$ -MZVs. We have recorded this fact by using the “deficiency” numbers listed in the tables in the last section of this paper.

Similar to type I, we find that type IV  $q$ -MZVs also need to be “regularized” when  $s_1 = 1$ . Again, we achieve this by introducing some convergent versions of the  $q$ -MZVs by modifying the auxiliary variables in  $\mathbf{a}$ .

It turns out that type II  $q$ -MZVs behave the most regularly and enjoy some properties closest to those of the classical MZVs. For example, their duality relations (see Theorem 8.4) have the cleanest form. Moreover, every other type of  $q$ -MZVs considered in this paper can be converted to type II. But still, there are relations that cannot be proved by DBSFs and dualities, at least when one is confined within the same weight and depth range. In fact, we find three independent  $\mathbb{Q}$ -linear relations in weight 4 that can only be proved when we consider weight 5 DBSFs and dualities.

All type III  $q$ -MZVs are convergent, even for negative arguments. For simplicity, in this paper we consider only those with nonnegative arguments  $s_1, \dots, s_d$  with  $s_1 \geq 1$ . In this case, the DBSFs are still insufficient. In the last section, we will see that in weight 3 there is already a missing relation which can be recovered by the duality. Essentially because of the need to apply the duality relations, we have to modify the original Jackson's iterated integral representation given in [7]. See the remarks after Theorem 6.1. In contrast to the other types of  $q$ -MZVs, we cannot suppress the deficiency for type III even if we consider more DBSFs and duality relations by increasing the weight and depth. This might be caused by our restriction of only nonnegative arguments and thus further investigations are called for.

On the other hand, we can improve the above situation by considering the more general type G values. All the missing relations are thus proved up to and including weight 4 and at the same time both deficiencies are decreased in weight 5 and 6. The key idea here is to convert all type G values to type II values by using a new kind of relations called **P-R** relations.

We point out that our method can be easily adapted to study  $q$ -MZVs of the following general forms:

$$\mathfrak{z}_q^{(s_1-a_1, \dots, s_d-a_d)}[s_1, \dots, s_d], \quad \mathfrak{z}_q^{(a_1, \dots, a_d)}[s_1, \dots, s_d],$$

where  $a_1 \geq a_2 \geq \dots \geq a_d \geq 0$  are all fixed integers. Furthermore, when the weight is not too large, our method can be programmed to compute all the relations among  $q$ -MZVs of the general form  $\mathfrak{z}_q^{\mathbf{a}}[s]$  when  $\mathbf{a}$  is taken within a certain range. This will be carried out in Sect. 9.

As an application, for small weight cases it is possible to confirm Okounkov’s conjecture [21] on the dimension of the  $q$ -MZVs  $\mathfrak{z}_q^{\mathbf{0}}[s]$  using Corollary 6.6. We do this numerically up to weight 12 and give rigorous proof up to weight 6 (both inclusive).

Throughout the paper we will use the modified form  $\mathfrak{z}_q$  most of the time. All the results can be translated into the standard form  $\zeta_q$  by inserting the correct powers of  $(1 - q)^w$ , where  $w$  is the corresponding weight, into the formulas.

## 2 Convergence Domain for $q$ -MZVs

We need the following result to find the convergence domain for different types of  $q$ -MZVs. It is Proposition 2.2 of [29] where the order of the indices in the definition of  $\zeta_q^{(a_1, \dots, a_d)}[s_1, \dots, s_d]$  (denoted by  $f_q(s_d, \dots, s_1; a_d, \dots, a_1)$  in loc. cit.) is opposite to this paper.

**Proposition 2.1** *The function  $\zeta_q^{(a_1, \dots, a_d)}[s_1, \dots, s_d]$  converges if  $\text{Re}(a_1 + \dots + a_j) > 0$  for all  $j = 1, \dots, d$ . It can be analytically continued to a meromorphic function over  $\mathbb{C}^{2d}$  via the series expansion*

$$\zeta_q^{(a_1, \dots, a_d)}[s_1, \dots, s_d] = (1 - q)^{|\mathbf{s}|} \sum_{b_1, \dots, b_d=0}^{\infty} \prod_{j=1}^d \left[ \binom{s_j + b_j - 1}{b_j} \frac{q^{(d+1-j)(b_j+a_j)}}{1 - q^{b_1+a_1+\dots+b_j+a_j}} \right]. \quad (3)$$

*It has the following (simple) poles:  $a_1 + \dots + a_j \in \mathbb{Z}_{\leq 0} + \frac{2\pi i}{\log q} \mathbb{Z}$  for  $j = 1, \dots, d$ .*

**Corollary 2.2** *Let  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{Z}^d$ . Then*

- (i)  $\zeta_q^{\mathbf{I}}[\mathbf{s}]$  converges if  $s_1 + \dots + s_j > j$  for all  $j = 1, \dots, d$ .
- (ii)  $\zeta_q^{\mathbf{II}}[\mathbf{s}]$  converges if  $s_1 + \dots + s_j > 0$  for all  $j = 1, \dots, d$ .
- (iii)  $\zeta_q^{\mathbf{III}}[\mathbf{s}]$  always converges.
- (iv)  $\zeta_q^{\mathbf{IV}}[\mathbf{s}]$  converges if  $s_1 + \dots + s_j > 1$  for all  $j = 1, \dots, d$ .

**Definition 2.3** For convenience, a composition  $\mathbf{s} \in \mathbb{Z}_{\geq 0}$  is said to be *type  $\tau$ -admissible* if  $\mathbf{s}$  satisfies the condition for type  $\tau$   $q$ -MZVs in the corollary. Here and in what follows,  $\tau = \text{I, II, III, or IV}$ .

### 3 Rota–Baxter Algebra

In this section we briefly review some fundamental facts of Rota–Baxter algebras which will be crucial in the study of the  $q$ -analog of *shuffle* relations for all of  $q$ -MZVs considered in this paper. For a good introduction to the Rota–Baxter algebras, see [11].

**Definition 3.1** Fix an algebra  $A$  over a commutative ring  $R$  and an element  $\lambda \in R$ . We call  $A$  a Rota–Baxter  $R$ -algebra and  $\mathcal{P}$  a Rota–Baxter operator of weight  $\lambda$  if the operator  $\mathcal{P}$  satisfies the following Rota–Baxter relation of weight  $\lambda$ :

$$\mathcal{P}(x)\mathcal{P}(y) = \mathcal{P}(\mathcal{P}(x)y) + \mathcal{P}(x\mathcal{P}(y)) + \lambda\mathcal{P}(xy) \quad \forall x, y \in A. \quad (4)$$

Recall that for any continuous function  $f(x)$  on  $[\alpha, \beta]$ , Jackson’s  $q$ -integral is defined by

$$\int_{\alpha}^{\beta} f(x) d_q x := \sum_{k \geq 0} f(\alpha + q^k(\beta - \alpha))(q^k - q^{k+1})(\beta - \alpha). \quad (5)$$

Taking  $\alpha = 0$  and  $\beta = t$  in (5), we set

$$\mathbf{J}[f](t) := (1 - q) \sum_{k \geq 0} f(q^k t) q^k t = (1 - q) \sum_{k \geq 0} \mathbf{E}^k[\mathbf{M}[f]](t) = (1 - q)\mathbf{P}[\mathbf{M}[f]](t) \quad (6)$$

where the multiplication operator  $\mathbf{M}[f](t) := tf(t)$ ,

$$\mathbf{E}[f](t) := \mathbf{E}_q[f](t) := f(qt), \text{ and } \mathbf{P}[f](t) := \mathbf{P}_q[f](t) := f(t) + f(qt) + f(q^2t) + \dots$$

are the  $q$ -expanding and the (principle)  $q$ -summation operators, respectively. We also need to define the (remainder)  $q$ -summation operator

$$\mathbf{R}[f](t) := \mathbf{R}_q[f](t) := f(qt) + f(q^2t) + \dots = (\mathbf{P}[f] - f)(t).$$

So,  $\mathbf{P}$  is the principle part (i.e. the whole thing) while  $\mathbf{R}$  is the remainder (i.e., without the first term). Clearly,  $\mathbf{P} = \mathbf{R} + \mathbf{I}$  where, as an operator,  $\mathbf{I}[f] = f$ . This implies  $\mathbf{PR} = \mathbf{RP}$ .

Let  $t\mathbb{Q}[[t, q]]$  be the ring of formal series in two variables with  $t > 0$ . Then  $\mathbf{J}$ ,  $\mathbf{E}$ ,  $\mathbf{P}$  and  $\mathbf{R}$  are all  $\mathbb{Q}[[q]]$ -linear endomorphisms of  $t\mathbb{Q}[[t, q]]$ . We can further define the inverse to  $\mathbf{P}$  which is called the  $q$ -difference operator:

$$\mathbf{D} := \mathbf{I} - \mathbf{E}. \tag{7}$$

The following results extend those of [7, (21)–(23)]. In the final computation we will not need  $\mathbf{D}$  since we will consider only nonnegative arguments in all the  $q$ -MZVs. But in the theoretical part of this paper we do need to use  $\mathbf{D}$  for type III  $q$ -MZVs.

**Proposition 3.2** *For any  $f, g \in t\mathbb{Q}[[t, q]]$  we have*

$$\mathbf{P}[f]\mathbf{P}[g] = \mathbf{P}[\mathbf{P}[f]g] + \mathbf{P}[f\mathbf{P}[g]] - \mathbf{P}[fg], \tag{8}$$

$$\mathbf{R}[f]\mathbf{R}[g] = \mathbf{R}[\mathbf{R}[f]g] + \mathbf{R}[f\mathbf{R}[g]] + \mathbf{R}[fg], \tag{9}$$

$$\mathbf{R}[f]\mathbf{P}[g] = \mathbf{R}[\mathbf{R}[f]g] + \mathbf{R}[f\mathbf{R}[g]] + \mathbf{R}[f]g + \mathbf{R}[fg], \tag{10}$$

$$\mathbf{J}[f]\mathbf{J}[g] = \mathbf{J}[\mathbf{J}[f]g] + \mathbf{J}[f\mathbf{J}[g]] - (1 - q)\mathbf{J}[\mathbf{I}fg], \tag{11}$$

$$= \mathbf{J}[f\mathbf{J}[g]] + q\mathbf{J}[\mathbf{J}[E[f]]g], \tag{12}$$

$$\mathbf{D}[f]\mathbf{D}[g] = \mathbf{D}[f]g + f\mathbf{D}[g] - \mathbf{D}[fg], \tag{13}$$

$$\mathbf{D}[f]\mathbf{P}[g] = \mathbf{D}[f\mathbf{P}[g]] + \mathbf{D}[f]g - fg, \tag{14}$$

$$\mathbf{D}[f]\mathbf{R}[g] = \mathbf{D}[f\mathbf{R}[g]] + \mathbf{D}[fg] - fg, \tag{15}$$

$$\mathbf{D}\mathbf{P} = \mathbf{P}\mathbf{D} = \mathbf{I}, \quad \mathbf{P}\mathbf{R} = \mathbf{R}\mathbf{P}. \tag{16}$$

**Proof** The identities (8), (13) and (14) are just (21), (23) and (26) of [7], respectively. All the others follow from  $\mathbf{R} = \mathbf{P} - \mathbf{I}$  easily.

By Proposition 3.2 we see that  $\mathbf{P}$  and  $\mathbf{R}$  are both Rota–Baxter operators on  $t\mathbb{Q}[[t, q]]$  (of weight  $-1$  and  $1$ , respectively) but  $\mathbf{D}$  is not. In fact,  $\mathbf{D}$  satisfies the condition (13) of a differential Rota–Baxter operator [12]. Moreover, it is *invertible* in the sense that Rota–Baxter operator  $\mathbf{P}$  and the differential  $\mathbf{D}$  are mutually inverse by (16).

We end this section with an identity which will be used to interpret Takeyama’s Resummation Identity in [26]. For any  $n \in \mathbb{N}$ , set

$$\mathbf{P}^n = \underbrace{\mathbf{P} \circ \dots \circ \mathbf{P}}_{n \text{ times}} \quad \text{and} \quad \mathbf{R}^n = \underbrace{\mathbf{R} \circ \dots \circ \mathbf{R}}_{n \text{ times}}.$$

**Theorem 3.3** *Let  $d \in \mathbb{N}$  and  $\alpha_j, \beta_j \in \mathbb{N}$  for all  $j = 1, \dots, d$ . Let  $\mathbf{y}(t) = \frac{t}{1-t}$ . Then we have*

$$\mathbf{R}^{\alpha_1} \mathbf{y}^{\beta_1} \dots \mathbf{R}^{\alpha_\ell} \mathbf{y}^{\beta_\ell}(t) = \sum_{\substack{j_1 \geq \beta_1, \dots, j_\ell \geq \beta_\ell \\ k_1 \geq \alpha_1, \dots, k_\ell \geq \alpha_\ell}} \prod_{r=1}^{\ell} \left[ \binom{j_r - 1}{\beta_r - 1} \binom{k_r - 1}{\alpha_r - 1} q^{k_r \sum_{s=r}^{\ell} j_s t^{j_r}} \right]. \tag{17}$$

**Proof** First we show that

$$\mathbf{R}^\alpha(t^j) = \frac{q^{\alpha j} t^j}{(1 - q^j)^\alpha}. \tag{18}$$

Indeed, if  $\alpha = 1$  then

$$\mathbf{R}(t^j) = \sum_{k \geq 1} q^{kj} t^j = \frac{q^j t^j}{1 - q^j}.$$

So (18) can be proved easily by induction.

Now we proceed to prove that for any integer  $m \geq 0$

$$\mathbf{R}^{\alpha_1} \mathbf{y}^{\beta_1} \dots \mathbf{R}^{\alpha_\ell} (\mathbf{y}^{\beta_\ell}(t) \cdot t^m) = \sum_{\substack{j_1 \geq \beta_1, \dots, j_\ell \geq \beta_\ell \\ k_1 \geq \alpha_1, \dots, k_\ell \geq \alpha_\ell}} t^m \prod_{r=1}^\ell \left[ \binom{j_r - 1}{\beta_r - 1} \binom{k_r - 1}{\alpha_r - 1} q^{k_r(m + \sum_{s=r}^\ell j_s)} t^{j_r} \right]. \tag{19}$$

If  $\ell = 1$  then we have

$$\begin{aligned} \mathbf{R}^\alpha (\mathbf{y}^\beta(t) \cdot t^m) &= \mathbf{R}^\alpha \left( \left( \frac{t}{1-t} \right)^\beta t^m \right) = \mathbf{R}^\alpha \sum_{j \geq 0} \binom{\beta + j - 1}{j} t^{m+\beta+j} \\ &= \mathbf{R}^\alpha \sum_{j \geq \beta} \binom{j - 1}{\beta - 1} t^{m+j} \\ &= \sum_{j \geq \beta} \binom{j - 1}{\beta - 1} \frac{q^{\alpha(m+j)} t^{m+j}}{(1 - q^{m+j})^\alpha} \quad (\text{by (18)}) \\ &= \sum_{j \geq \beta} \binom{j - 1}{\beta - 1} \sum_{k \geq 0} \binom{\alpha + k - 1}{k} q^{(\alpha+k)(m+j)} t^{m+j} \\ &= \sum_{j \geq \beta} \sum_{k \geq \alpha} \binom{j - 1}{\beta - 1} \binom{k - 1}{\alpha - 1} q^{k(m+j)} t^{m+j}. \end{aligned}$$

This proves (19) when  $\ell = 1$ . In general

$$\begin{aligned} &\mathbf{R}^{\alpha_1} \mathbf{y}^{\beta_1} \dots \mathbf{R}^{\alpha_{\ell-1}} (\mathbf{y}^{\beta_{\ell-1}}(t) (\mathbf{R}^{\alpha_\ell} (\mathbf{y}^{\beta_\ell}(t) \cdot t^m))) \\ &= \sum_{j_\ell \geq \beta_\ell} \sum_{k_\ell \geq \alpha_\ell} \binom{j_\ell - 1}{\beta_\ell - 1} \binom{k_\ell - 1}{\alpha_\ell - 1} q^{k_\ell(m+j_\ell)} \mathbf{R}^{\alpha_1} \mathbf{y}^{\beta_1} \dots \mathbf{R}^{\alpha_{\ell-1}} (\mathbf{y}^{\beta_{\ell-1}}(t) \cdot t^{m+j_\ell}). \end{aligned}$$

So (19) follows immediately by induction. We can now finish the proof of the theorem by taking  $m = 0$ .

**Corollary 3.4** *Let  $d \in \mathbb{N}$  and  $\alpha_j, \beta_j \in \mathbb{N}$  for all  $j = 1, \dots, d$ . Then we have*

$$\mathbf{R}^{\alpha_1} \mathbf{y}^{\beta_1} \dots \mathbf{R}^{\alpha_\ell} \mathbf{y}^{\beta_\ell} (1) = \mathbf{R}^{\beta_\ell} \mathbf{y}^{\alpha_\ell} \dots \mathbf{R}^{\beta_1} \mathbf{y}^{\alpha_1} (1). \tag{20}$$



**Proof** In (17) we use the substitutions  $j_r \leftrightarrow k_{\ell+1-r}$  for all  $r = 1, \dots, \ell$ . Then the power of  $q$  in the term of (17) indexed by  $(j_1, \dots, j_\ell, k_1, \dots, k_\ell)$  is equal to

$$\begin{aligned} \sum_{r=1}^{\ell} \sum_{s=r}^{\ell} j_s k_r &\longrightarrow \sum_{r=1}^{\ell} \sum_{s=r}^{\ell} j_{\ell+1-r} k_{\ell+1-s} = \sum_{s=1}^{\ell} \sum_{r=1}^s j_{\ell+1-r} k_{\ell+1-s} \\ &= \sum_{s=1}^{\ell} \sum_{r=1}^{\ell+1-s} j_{\ell+1-r} k_s = \sum_{s=1}^{\ell} \sum_{r=s}^{\ell} j_r k_s = \sum_{r=1}^{\ell} k_r \sum_{s=r}^{\ell} j_s. \end{aligned}$$

which follows from the substitution  $s \leftrightarrow \ell + 1 - s$  followed by  $r \leftrightarrow \ell + 1 - r$  and  $r \leftrightarrow s$ . This proves the corollary.

### 4 $q$ -Analogues of Hoffman algebras

We know that (regularized) DBSFs lead to many (and conjecturally all)  $\mathbb{Q}$ -linear relations among the MZVs. The key idea here was first suggested by Hoffman [14] who used some suitable algebra of words to codify both the stuffle (also called harmonic shuffle [27] or quasi-shuffle [15]) relations coming from the series representation of MZVs and the shuffle relations coming from the iterated integral expressions of MZVs. The detailed regularization process can be found in [16]. To study similar relations of the  $q$ -MZVs we should modify the Hoffman algebras in the  $q$ -analog setting.

The following definition for type I  $q$ -MZVs was first proposed by Takeyama [26]. We adopt different notations here in hoping to give a uniform and more transparent presentation for all types of  $q$ -MZVs.

First we consider some algebras which will be used to define the stuffle relations later.

**Definition 4.1** Let  $X_\theta^*$  be the set of words on the alphabet  $X_\theta = \{a, a^{-1}, b, \theta\}$ . Denote by  $\mathfrak{A}_\theta = \mathbb{Q}\langle a, a^{-1}, b, \theta \rangle$  the noncommutative polynomial  $\mathbb{Q}$ -algebra of words from  $X_\theta^*$ . Set

$$\gamma := b - \theta, \quad z_s := a^{s-1}b, \quad z'_s := a^{s-1}\theta, \quad s \in \mathbb{Z}.$$

Let  $Y_I := \{\theta\} \cup \{z_k\}_{k \geq 1}$ ,  $Y_{II} := \{z'_k\}_{k \geq 0}$ ,  $Y_{III} := \{z_k\}_{k \in \mathbb{Z}}$  and  $Y_{IV} := Y_{II}$ . We point out that  $z_0, z'_0 \neq \mathbf{1}$  where  $\mathbf{1}$  is the empty word. We put a tilde on top of both I and IV since we need to consider some kind of regularization due to convergence issues involved in type I and IV  $q$ -MZVs. This is realized by the introduction of the letter  $\theta$ . Again, we use  $Y_\tau^*$  to denote the set of words generated on  $Y_\tau$  for any type  $\tau$ .

Let  $\mathfrak{A}_I^1, \mathfrak{A}_{II}^1, \mathfrak{A}_{III}$  and  $\mathfrak{A}_{IV}$  be the subalgebras of  $\mathfrak{A}_\theta$  freely generated by the sets  $Y_I, Y_{II}, Y_{III}$  and  $Y_{IV}$ , respectively. Set

$$\mathfrak{A}_{\mathbb{I}}^1 := \mathbb{Q}\mathbf{1} + \sum_{k \in \mathbb{Z}} z'_k \mathfrak{A}_{\mathbb{I}} \not\subseteq \mathfrak{A}_{\mathbb{I}}, \quad \mathfrak{A}_{\tilde{\mathbb{N}}}^1 := \mathbb{Q}\mathbf{1} + \theta \mathfrak{A}_{\tilde{\mathbb{N}}} + \sum_{k \geq 1} z_k \mathfrak{A}_{\tilde{\mathbb{N}}} \not\subseteq \mathfrak{A}_{\tilde{\mathbb{N}}}.$$

Here, all integer subscripts are allowed in  $Y_{\mathbb{I}}$  because type  $\mathbb{I}$   $q$ -MZVs converge for all integer arguments. Further, we define the following subalgebras corresponding to the convergent values:

$$\begin{aligned} \mathfrak{A}_{\mathbb{I}}^0 &:= \mathbb{Q}\mathbf{1} + \sum_{k \geq 2} z_k \mathfrak{A}_{\mathbb{I}}^1, & \mathfrak{A}_{\tilde{\mathbb{N}}}^0 &:= \mathbb{Q}\mathbf{1} + \theta \mathfrak{A}_{\tilde{\mathbb{N}}}^1 + \sum_{k \geq 2} z_k \mathfrak{A}_{\tilde{\mathbb{N}}}^1 \not\subseteq \mathfrak{A}_{\tilde{\mathbb{N}}}^1, \\ \mathfrak{A}_{\mathbb{I}}^0 &:= \mathbb{Q}\mathbf{1} + \sum_{k \geq 1} z'_k \mathfrak{A}_{\mathbb{I}}^1 \not\subseteq \mathfrak{A}_{\mathbb{I}}^1, & \mathfrak{A}_{\mathbb{I}}^0 &:= \mathfrak{A}_{\mathbb{I}}^1, \\ \mathfrak{A}_{\tilde{\mathbb{N}}}^0 &:= \mathbb{Q}\mathbf{1} + \sum_{k \geq 2} z_k \mathfrak{A}_{\tilde{\mathbb{N}}}, & \mathfrak{A}_{\tilde{\mathbb{N}}}^0 &:= \mathbb{Q}\mathbf{1} + \theta \mathfrak{A}_{\tilde{\mathbb{N}}} + \sum_{k \geq 2} z_k \mathfrak{A}_{\tilde{\mathbb{N}}} \not\subseteq \mathfrak{A}_{\tilde{\mathbb{N}}}^1. \end{aligned}$$

For each type  $\tau$  the words in  $\mathfrak{A}_{\tau}^0$  are called *type  $\tau$ -admissible*. This is consistent with Definition 2.3 since we consider only non-negative compositions  $\mathbf{s}$ .

**Definition 4.2** To define the stuffle product for type  $\tau = \tilde{\mathbb{I}}$  and  $\mathbb{I}$ , similar to the MZV case, we define a commutative product  $[-, -]_{\tau}$  first:

$$[z_k, z_l]_{\tilde{\mathbb{I}}} = z_{k+l} + z_{k+l-1}, \quad [\theta, z_k]_{\tilde{\mathbb{I}}} = z_{k+1}, \quad [\theta, \theta]_{\tilde{\mathbb{I}}} = z_2 - \theta, \quad [z'_k, z'_l]_{\mathbb{I}} = z'_{k+l} \tag{21}$$

for all  $k, l \geq 1$ . Now we define the stuffle product  $*_{\tau}$  on  $\mathfrak{A}_{\tau}^1$  inductively as follows. For any words  $\mathbf{u}, \mathbf{v} \in \mathfrak{A}_{\tau}^1$  and letters  $\alpha, \beta \in Y_{\tau}$ , we set  $\mathbf{1} *_{\tau} \mathbf{u} = \mathbf{u} = \mathbf{u} *_{\tau} \mathbf{1}$  and

$$(\alpha \mathbf{u}) *_{\tau} (\beta \mathbf{v}) = \alpha(\mathbf{u} *_{\tau} \beta \mathbf{v}) + \beta(\alpha \mathbf{u} *_{\tau} \mathbf{v}) + [\alpha, \beta]_{\tau}(\mathbf{u} *_{\tau} \mathbf{v}). \tag{22}$$

*Remark 4.3* (i). The definition for  $*_{\tilde{\mathbb{I}}}$  is the same as in [26].

(ii). One can check that  $*_{\tau}$  is well-defined for  $\tau = \tilde{\mathbb{I}}$  and  $\mathbb{I}$ . Namely,  $\mathbf{u} *_{\tau} \mathbf{v} \in \mathfrak{A}_{\tau}^1$  if  $\mathbf{u}, \mathbf{v} \in \mathfrak{A}_{\tau}^1$ .

(iii). It is not hard to check that for  $\tau = \tilde{\mathbb{I}}$  and  $\mathbb{I}$ ,  $(\mathfrak{A}_{\tau}^0, *_{\tau}) \subset (\mathfrak{A}_{\tau}^1, *_{\tau})$  as subalgebras.

(iv). In the following, we will need to define stuffle product  $*_{\text{ord}}$  on  $\mathfrak{A}_{\mathbb{I}}$  by setting  $\tau = \text{ord}$  and  $[z_r, z_s]_{\text{ord}} = z_{r+s}$  for all  $r, s \in \mathbb{Z}$  in (22).

In [7], the stuffle product  $\sqcup$  for type  $\mathbb{I}$   $q$ -MZVs is defined. We will modify this in the following way (see the remarks after Theorem 6.1). Our modified stuffle product for type  $\mathbb{I}$   $q$ -MZVs will be denoted by  $*_{\mathbb{I}}$ .

**Definition 4.4** We now define a stuffle product  $*_{\mathbb{I}}$  on  $\mathfrak{A}_{\mathbb{I}}^1$ . First, we define an injective shifting operator  $\mathcal{S}_{-}$  on any word of  $\mathfrak{A}_{\mathbb{I}}^1$  by acting on the first letter:

$$\mathcal{S}_{-}(z'_n \mathbf{w}) := z_n \mathbf{w} - z_{n-1} \mathbf{w} \in \mathfrak{A}_{\mathbb{I}} \quad \text{for all } n \in \mathbb{Z} \text{ and } \mathbf{w} \in Y_{\mathbb{I}}^*. \tag{23}$$

For any  $k, l \in \mathbb{Z}$  and any  $\mathbf{u}, \mathbf{v} \in Y_{\mathbb{I}}^*$ , define the stuffle product  $*_{\mathbb{I}}$  by

$$\mathbf{1} *_{\mathbb{I}} \mathbf{1} = \mathbf{1}, \quad \mathbf{1} *_{\mathbb{I}} z'_k \mathbf{u} = z'_k \mathbf{u} *_{\mathbb{I}} \mathbf{1} = z'_k \mathbf{u},$$

$$z'_k \mathbf{u} *_{\mathbb{I}} z'_l \mathbf{v} = z'_k (\mathbf{u} *_{\text{ord}} \mathcal{S}_-(z'_l \mathbf{v})) + z'_l (\mathcal{S}_-(z'_k \mathbf{u}) *_{\text{ord}} \mathbf{v}) + (z'_{k+l} - z'_{k+l-1})(\mathbf{u} *_{\text{ord}} \mathbf{v}).$$

Here  $*_{\text{ord}}$  is the ordinary stuffle defined in Remark 4.3 (iv).

For type  $\tilde{\mathbb{V}}$ , we provide a definition similar to type  $\mathbb{I}$ .

**Definition 4.5** We now define a stuffle product  $*_{\tilde{\mathbb{V}}}$  on  $\mathfrak{A}_{\tilde{\mathbb{V}}}^1$ . First, we define a shifting operator  $\mathcal{S}_+$  similar to (23) by

$$\mathcal{S}_+(z_n \mathbf{w}) := z'_n \mathbf{w} + z'_{n-1} \mathbf{w} \in \mathfrak{A}_{\tilde{\mathbb{V}}} \quad \text{for all } n \in \mathbb{N} \text{ and } \mathbf{w} \in Y_{\tilde{\mathbb{V}}}^*.$$

Then, for any  $k, l \geq 1$  and any  $\mathbf{u}, \mathbf{v} \in Y_{\tilde{\mathbb{V}}}^*$  we set

$$\mathbf{1} *_{\tilde{\mathbb{V}}} \mathbf{1} = \mathbf{1}, \quad \mathbf{1} *_{\tilde{\mathbb{V}}} z_k \mathbf{u} = z_k \mathbf{u} *_{\tilde{\mathbb{V}}} \mathbf{1} = z_k \mathbf{u},$$

$$z_k \mathbf{u} *_{\tilde{\mathbb{V}}} z_l \mathbf{v} = z_k (\mathbf{u} *_{\mathbb{I}} \mathcal{S}_+(z_l \mathbf{v})) + z_l (\mathcal{S}_+(z_k \mathbf{u}) *_{\mathbb{I}} \mathbf{v}) + (z_{k+l} + z_{k+l-1})(\mathbf{u} *_{\mathbb{I}} \mathbf{v}),$$

$$z_k \mathbf{u} *_{\tilde{\mathbb{V}}} \theta \mathbf{v} = \theta \mathbf{v} *_{\tilde{\mathbb{V}}} z_k \mathbf{u} = z_k (\mathbf{u} *_{\mathbb{I}} \theta \mathbf{v}) + \theta (\mathcal{S}_+(z_k \mathbf{u}) *_{\mathbb{I}} \mathbf{v}) + z_{k+1} (\mathbf{u} *_{\mathbb{I}} \mathbf{v}),$$

$$\theta \mathbf{u} *_{\tilde{\mathbb{V}}} \theta \mathbf{v} = \theta (\mathbf{u} *_{\mathbb{I}} \theta \mathbf{v}) + \theta (\theta \mathbf{u} *_{\mathbb{I}} \mathbf{v}) + (z_2 - \theta)(\mathbf{u} *_{\mathbb{I}} \mathbf{v}),$$

where  $*_{\mathbb{I}}$  is the stuffle product on  $\mathfrak{A}_{\tilde{\mathbb{V}}}^1 = \mathfrak{A}_{\tilde{\mathbb{V}}}$  defined in Definition 4.2.

**Lemma 4.6** *The stuffle products  $*_{\mathbb{I}}$  and  $*_{\tilde{\mathbb{V}}}$  are both well-defined. Namely, if  $\mathbf{u}, \mathbf{v} \in \mathfrak{A}_{\tau}^1$  then  $\mathbf{u} *_{\tau} \mathbf{v} \in \mathfrak{A}_{\tau}^1$  for  $\tau = \mathbb{I}$  or  $\tilde{\mathbb{V}}$ .*

*Proof* We prove the lemma for type  $\tilde{\mathbb{V}}$  only. Type  $\mathbb{I}$  is similar but simpler.

First we note that  $k + l - 1 \geq 1$  if  $k, l \geq 1$ . So the first letter of each of the terms of  $\mathbf{u} *_{\tilde{\mathbb{V}}} \mathbf{v}$  has the right form, i.e., either  $\theta$  or  $z_k$  for  $k \geq 1$ . We need to show that after truncating the first letter each term lies in  $\mathfrak{A}_{\tilde{\mathbb{V}}}$ . Notice that  $\mathcal{S}_+(z_l \mathbf{v}), \mathcal{S}_+(z_k \mathbf{u}) \in \mathfrak{A}_{\tilde{\mathbb{V}}}$  and  $*_{\mathbb{I}}$  does not decrease the size the subscripts (which are all non-negative). The lemma is now proved.

**Proposition 4.7** *Let  $\tau = \tilde{\mathbb{I}}, \mathbb{I}, \mathbb{I}$  or  $\tilde{\mathbb{V}}$ . Then the stuffle algebras  $(\mathfrak{A}_{\tau}^1, *_{\tau})$  are all commutative and associative.*

*Proof* This follows from the fact that the product  $[-, -]_{\tau}$  are all commutative and associative which can be verified easily.

We now turn to the shuffle algebra which is an analog of the corresponding algebra for MZVs reflecting the properties of their representations using iterated integrals.

**Definition 4.8** Let  $X_{\pi} = \{\pi, \delta, y\}$  be an alphabet and  $X_{\pi}^*$  be the set of words generated by  $X_{\pi}$ . Define  $\mathfrak{A}_{\pi} = \mathbb{Q}\langle \pi, \delta, y \rangle$  to be the noncommutative polynomial  $\mathbb{Q}$ -algebra of words of  $X_{\pi}^*$ . We may embed  $\mathfrak{A}_{\theta}$  defined by Definition 4.1 as a subalgebra of  $\mathfrak{A}_{\pi}$  in two different ways: put  $\rho = \pi - \mathbf{1}$  and let

- (A)  $a := \pi, \quad a^{-1} := \delta, \quad b := \pi y, \quad \theta = \rho y \implies \gamma := y,$
- (B)  $a := \rho, \quad a^{-1} := -, \quad b := \pi y, \quad \theta = \rho y \implies \gamma := y.$

We denote the image of the embedding by  $\mathfrak{A}_\theta^{(A)}$  and  $\mathfrak{A}_\theta^{(B)}$ , respectively. The dash – for the image of  $a^{-1}$  in (B) means it does not matter what image we choose since  $a^{-1}$  appears only when we consider type III  $q$ -MZVs using (A). We will use embedding (B) for the other three types for which  $a^{-1}$  will not be utilized essentially because of convergence issues.

### 5 $q$ -Stuffle relations

First we define the  $\mathbb{Q}$ -linear realization maps  $\mathfrak{z}_q : \mathfrak{A}_\tau^0 \rightarrow \mathbb{C}$  ( $\tau = \tilde{\text{I}}, \text{II}$ ) by  $\mathfrak{z}_q[\mathbf{1}] = 1$  and

$$\mathfrak{z}_q[y_1^\tau \dots y_d^\tau] := \sum_{k_1 > \dots > k_d > 0} M_{k_1}^\tau(y_1^\tau) \dots M_{k_d}^\tau(y_d^\tau),$$

for every admissible word  $y_1^\tau \dots y_d^\tau \in \mathfrak{A}_\tau^0$  where the  $\mathbb{Q}$ -linear maps

$$M_m^{\tilde{\text{I}}}(\theta) := \frac{q^m}{1 - q^m}, \quad M_m^{\tilde{\text{I}}}(z_s) := \frac{q^{(s-1)m}}{(1 - q^m)^s}, \quad M_m^{\text{II}}(z'_s) := \frac{q^{sm}}{(1 - q^m)^s},$$

for all  $m \in \mathbb{N}$ . Note that  $M_k^{\tilde{\text{I}}}(\gamma) = M_k^{\tilde{\text{I}}}(z_1 - \theta) = 1$ . For example, we have

$$\mathfrak{z}_q[z_2 z_5 \gamma^2 z_1] = \mathfrak{z}_q^{(1,4,0,0,0)}[2, 5, 0, 0, 1], \quad \mathfrak{z}_q[\theta z_7 \theta z_4] = \mathfrak{z}_q^{(1,6,1,3)}[1, 7, 1, 4],$$

which are not  $q$ -MZVs of type I.

For type  $\tau = \text{III}$  or  $\tilde{\text{IV}}$ , we similarly define the  $\mathbb{Q}$ -linear realization maps  $\mathfrak{z}_q : \mathfrak{A}_\tau^0 \rightarrow \mathbb{C}$  by  $\mathfrak{z}_q[\mathbf{1}] = 1$  and

$$\mathfrak{z}_q[y_1^\tau \dots y_d^\tau] := \sum_{k_1 > \dots > k_d > 0} M_{k_1}^{1,\tau}(y_1^\tau) M_{k_2}^\tau(y_2^\tau) \dots M_{k_d}^\tau(y_d^\tau),$$

for every admissible word  $y_1^\tau \dots y_d^\tau \in \mathfrak{A}_\tau^0$  where the  $\mathbb{Q}$ -linear maps

$$M_m^{1,\text{III}}(z'_s) := \frac{q^m}{(1 - q^m)^s}, \quad M_m^{\text{III}}(z_s) := \frac{1}{(1 - q^m)^s},$$

$$M_m^{\tilde{\text{IV}}}(\theta) = M_m^{1,\tilde{\text{IV}}}(\theta) := \frac{q^m}{1 - q^m}, \quad M_m^{1,\tilde{\text{IV}}}(z_s) := \frac{q^{m(s-1)}}{(1 - q^m)^s}, \quad M_m^{\tilde{\text{IV}}}(z'_s) := \frac{q^{sm}}{(1 - q^m)^s},$$

for all  $m \in \mathbb{N}$ .

The following theorem is parallel to [7, Proposition 9] and includes [26, Theorem 1].

**Theorem 5.1** *Let  $\tau = \tilde{\text{I}}, \text{II}, \text{III}$  or  $\tilde{\text{IV}}$ . For any  $\mathbf{u}_\tau, \mathbf{v}_\tau \in \mathfrak{A}_\tau^0$  we have*

$$\mathfrak{z}_q[\mathbf{u}_\tau *_\tau \mathbf{v}_\tau] = \mathfrak{z}_q[\mathbf{u}_\tau] \mathfrak{z}_q[\mathbf{v}_\tau]. \tag{24}$$

**Proof** Since type  $\tilde{\text{I}}$  case is just [26, Theorem 1], we only need to consider the other three types. The proof is basically the same as that of [26, Theorem 1]. In fact, it suffices to observe that

$$\begin{aligned} M_m^{\text{II}}(z'_k)M_m^{\text{II}}(z'_l) &= M_m^{\text{II}}(z'_{k+l}), & M_m^{1,\text{III}}(z'_k)M_m^{\text{III}}(z_l) &= M_m^{\text{III}}(z_{k+l} - z_{k+l-1}), \\ M_m^{\text{III}}(z_k)M_m^{\text{III}}(z_l) &= M_m^{\text{III}}(z_{k+l}), & M_m^{1,\text{III}}(z'_k)M_m^{1,\text{III}}(z'_l) &= M_m^{1,\text{III}}(z'_{k+l} - z'_{k+l-1}), \\ M_m^{\tilde{\text{IV}}}(z'_k)M_m^{\tilde{\text{IV}}}(z'_l) &= M_m^{\tilde{\text{IV}}}(z'_{k+l}), & M_m^{1,\tilde{\text{IV}}}(z_k)M_m^{\tilde{\text{IV}}}(z'_l) &= M_m^{\tilde{\text{IV}}}(z'_{k+l} + z'_{k+l-1}), \\ M_m^{\tilde{\text{IV}}}(\theta)M_m^{\tilde{\text{IV}}}(z'_k) &= M_m^{\tilde{\text{IV}}}(z'_{k+1}), & M_m^{1,\tilde{\text{IV}}}(z_k)M_m^{1,\tilde{\text{IV}}}(z_l) &= M_m^{1,\tilde{\text{IV}}}(z_{k+l} + z_{k+l-1}), \\ M_m^{1,\tilde{\text{IV}}}(\theta)M_m^{\tilde{\text{IV}}}(z'_k) &= M_m^{\tilde{\text{IV}}}(z'_{k+1}), & M_m^{\tilde{\text{IV}}}(\theta)M_m^{\tilde{\text{IV}}}(\theta) &= M_m^{1,\tilde{\text{IV}}}(\theta)M_m^{\tilde{\text{IV}}}(\theta) = M_m^{\tilde{\text{IV}}}(z'_2), \\ M_m^{1,\tilde{\text{IV}}}(\theta)M_m^{1,\tilde{\text{IV}}}(z_k) &= M_m^{1,\tilde{\text{IV}}}(z_{k+1}), & M_m^{1,\tilde{\text{IV}}}(\theta)M_m^{1,\tilde{\text{IV}}}(\theta) &= M_m^{1,\tilde{\text{IV}}}(z_2 - \theta), \end{aligned}$$

for all  $k, l \geq 0, m \geq 1$ . Of course, we need to assume  $k, l \geq 2$  for  $M_m^{1,\tilde{\text{IV}}}(z_k)$  and  $M_m^{1,\tilde{\text{IV}}}(z_l)$ .

## 6 Jackson’s Iterated $q$ -Integrals

Set

$$x_0 := x_0(t) = \frac{1}{t}, \quad x_1 := x_1(t) = \frac{1}{1-t}, \quad \mathbf{y} := \mathbf{y}(t) = \frac{t}{1-t}.$$

Recall that for  $a = x_0(t)dt$  and  $b = x_1(t)dt$ , we can express MZVs by Chen’s iterated integrals:

$$\zeta(s_1, \dots, s_d) = \int_0^1 a^{s_1-1} b \dots a^{s_d-1} b.$$

Replacing the Riemann integrals by Jackson’s  $q$ -integrals (6) one gets

**Theorem 6.1** ([7, (29)]) *For  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$  set  $w = |\mathbf{s}|$  and*

$$\tilde{\zeta}_q^{\text{III}}[\mathbf{s}; t] := \mathbf{J}\left[c_1 \mathbf{J}\left[c_2 \dots \mathbf{J}\left[c_w \dots\right]\right]\right](t),$$

where  $c_i = x_1$  if  $i \in \{u_1, u_2, \dots, u_d\}$ ,  $u_j := s_1 + s_2 + \dots + s_j$ , and  $c_i = x_0$  otherwise. Or, equivalently, set  $\mathbf{w} = \pi^{s_1} y \pi^{s_2} y \dots \pi^{s_d} y$  and

$$\tilde{\zeta}_q^{\text{III}}[\mathbf{w}; t] := \mathbf{P}^{s_1}[\mathbf{y} \dots \mathbf{P}^{s_d}[\mathbf{y}] \dots](t).$$

Then

$$\mathfrak{z}_q^{\mathbb{I}}[\mathbf{s}] = \tilde{\mathfrak{z}}_q^{\mathbb{I}}[\mathbf{w}; q]$$

However, the representation of  $\zeta_q^{\mathbb{I}}[\mathbf{s}]$  using  $\tilde{\mathfrak{z}}_q^{\mathbb{I}}$  in Theorem 6.1 is not ideal in the sense that one has to evaluate  $t$  at  $q$ . We would like to use Corollary 3.4 so we need to set  $t = 1$ . This leads to the idea of replacing the first factor  $\mathbf{P}^{s_1}$  by  $\mathbf{P}^{s_1-1}\mathbf{R}$  and, more generally, the following two generalizations. For any  $\mathbf{a} = (a_1, \dots, a_d) \in (\mathbb{Z}_{\geq 0})^d$ , define

$$\mathfrak{z}_q[\rho^{a_1}y \dots \rho^{a_d}y; t] := \mathbf{R}^{a_1}[\mathbf{y}\mathbf{R}^{a_2}[\mathbf{y} \dots \mathbf{R}^{a_d}[\mathbf{y}] \dots ]](t).$$

**Theorem 6.2** Let  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$  and  $\mathbf{a} = (a_1, \dots, a_d) \in (\mathbb{Z}_{\geq 0})^d$ . Put  $w = |\mathbf{s}|$  and  $\mathbf{w} = \mathbf{w}^{\mathbf{a}}(\mathbf{s}) = \rho^{a_1}\pi^{s_1-a_1}y \dots \rho^{a_d}\pi^{s_d-a_d}y$ . Then

$$\mathfrak{z}_q[\mathbf{w}^{\mathbf{a}}(\mathbf{s}); t] := \mathbf{R}^{a_1}[\mathbf{P}^{s_1-a_1}[\mathbf{y}\mathbf{R}^{a_2}[\mathbf{P}^{s_2-a_2}[\mathbf{y} \dots \mathbf{R}^{a_d}[\mathbf{P}^{s_d-a_d}[\mathbf{y}]] \dots ]]]](t). \tag{25}$$

Suppose  $a_1 + \dots + a_j > 0$  for all  $j = 1, \dots, d$ . Then we have

$$\zeta_q^{\mathbf{a}}[\mathbf{s}] = (1 - q)^w \mathfrak{z}_q[\mathbf{w}^{\mathbf{a}}(\mathbf{s}); 1], \quad \mathfrak{z}_q^{\mathbf{a}}[\mathbf{s}] = \mathfrak{z}_q[\mathbf{w}^{\mathbf{a}}(\mathbf{s})] := \mathfrak{z}_q[\mathbf{w}^{\mathbf{a}}(\mathbf{s}); 1]. \tag{26}$$

**Proof** First we observe three important facts: for any  $k \geq 1$  we have

$$\mathbf{P}(t^k) = \sum_{j \geq 0} q^{kj} t^k = \frac{t^k}{1 - q^k}, \quad \mathbf{D}(t^k) = t^k(1 - q^k), \quad \text{and} \quad \mathbf{R}(t^k) = \sum_{j \geq 1} q^{kj} t^k = \frac{q^k t^k}{1 - q^k}$$

by the definition of the two summation operators and the difference operator. Repeatedly applying this we get

$$\mathbf{P}^m(t^k) = \mathbf{P}\left(\frac{t^k}{(1 - q^k)^{m-1}}\right) = \frac{t^k}{(1 - q^k)^m}, \quad \forall m \in \mathbb{Z}, \tag{27}$$

$$\mathbf{R}^m(t^k) = \mathbf{R}\left(\left(\frac{q^k}{1 - q^k}\right)^{m-1} t^k\right) = \frac{q^{mk} t^k}{(1 - q^k)^m} \quad \forall m \in \mathbb{Z}_{\geq 0}. \tag{28}$$

Thus

$$\mathbf{P}(\mathbf{y}(t) \cdot t^k) = \sum_{j \geq 0} \frac{q^{j(k+1)} t^{k+1}}{1 - q^j t} = \sum_{j \geq 0} \sum_{\ell \geq 0} q^{j(k+\ell+1)} t^{k+\ell+1} = \sum_{\ell > k} \frac{t^\ell}{1 - q^\ell}.$$

Similarly, we have

$$\mathbf{D}(\mathbf{y}(t) \cdot t^k) = \frac{t^{k+1}}{1 - t} - \frac{q^{k+1} t^{k+1}}{1 - qt} = \sum_{\ell \geq 0} (1 - q^{k+\ell+1}) t^{k+\ell+1} = \sum_{\ell > k} (1 - q^\ell) t^\ell,$$

and

$$\mathbf{R}(\mathbf{y}(t) \cdot t^k) = \sum_{j \geq 1} \frac{q^{j(k+1)} t^{k+1}}{1 - q^j t} = \sum_{j \geq 1} \sum_{\ell \geq 0} q^{j(k+\ell+1)} t^{k+\ell+1} = \sum_{\ell > k} \frac{q^\ell t^\ell}{1 - q^\ell}.$$

It follows from (27) and (28) that

$$\mathbf{P}^m(\mathbf{y}(t) \cdot t^k) = \sum_{\ell > k} \frac{t^\ell}{(1 - q^\ell)^m} \quad \forall m \in \mathbb{Z}, \tag{29}$$

$$\mathbf{R}^m(\mathbf{y}(t) \cdot t^k) = \sum_{\ell > k} \frac{q^{m\ell} t^\ell}{(1 - q^\ell)^m} \quad \forall m \in \mathbb{Z}_{\geq 0}. \tag{30}$$

We now prove by induction on the the depth  $d$  that for all  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ ,

$$\mathfrak{z}_q[\mathbf{w}^{\mathbf{a}}(\mathbf{s}); t] = \sum_{k_1 > \dots > k_d > 0} \frac{t^{k_1} q^{k_1 a_1} \dots q^{k_d a_d}}{(1 - q^{k_1})^{s_1} \dots (1 - q^{k_d})^{s_d}}. \tag{31}$$

When  $d = 1$ , i.e.,  $\mathbf{s} = s$ , then by (29) followed by (28)

$$\mathfrak{z}_q[\mathbf{w}^a(s); t] = \mathbf{R}^a \mathbf{P}^{s-a}[\mathbf{y}](t) = \sum_{k > 0} \frac{\mathbf{R}^a(t^k)}{(1 - q^k)^{s-a}} = \sum_{k > 0} \frac{q^{ak} t^k}{(1 - q^k)^s}.$$

This proof works even when  $s = a$  because of (30) (take  $k = 0$  and  $m = a$  there).

Turning to the general case, we let  $d \geq 2$  and assume (31) holds for smaller depths. Then by the inductive assumption

$$\begin{aligned} \mathfrak{z}_q[\mathbf{w}^{\mathbf{a}}(\mathbf{s}); t] &= \mathbf{R}^{a_1} \mathbf{P}^{s_1 - a_1} [\mathbf{y} \mathbf{R}^{a_2} \mathbf{P}^{s_2 - a_2} [\mathbf{y} \dots \mathbf{R}^{a_d} \mathbf{P}^{s_d - a_d} [\mathbf{y}] \dots]](t) \\ &= \sum_{k_2 > \dots > k_d > 0} \frac{\mathbf{R}^{a_1} \mathbf{P}^{s_1 - a_1} (\mathbf{y}(t) \cdot t^{k_2}) q^{k_2 a_2} \dots q^{k_d a_d}}{(1 - q^{k_2})^{s_2} \dots (1 - q^{k_d})^{s_d}} \\ &= \sum_{k_1 > \dots > k_d > 0} \frac{\mathbf{R}^{a_1}(t^{k_1}) q^{k_2 a_2} \dots q^{k_d a_d}}{(1 - q^{k_1})^{s_1 - a_1} (1 - q^{k_2})^{s_2} \dots (1 - q^{k_d})^{s_d}} \quad (\text{by (29)}) \\ &= \sum_{k_1 > \dots > k_d > 0} \frac{t^{k_1} q^{k_1 a_1} \dots q^{k_d a_d}}{(1 - q^{k_1})^{s_1} \dots (1 - q^{k_d})^{s_d}} \end{aligned}$$

by (28). Again, if  $s_1 = a_1$  the proof is still valid. This completes the proof of (31). Setting  $t = 1$  we arrive at (26).

By change of variables  $a_j \rightarrow s_j - a_j$  for all  $j = 1, \dots, d$  we immediately obtain the next result. Observe that by (25), we have

$$\mathfrak{z}_q[\mathbf{w}^{\mathbf{s}-\mathbf{a}}(\mathbf{s}); t] := \mathbf{R}^{s_1 - a_1} [\mathbf{P}^{a_1} [\mathbf{y} \mathbf{R}^{s_2 - a_2} [\mathbf{P}^{a_2} [\mathbf{y} \dots \mathbf{R}^{s_d - a_d} [\mathbf{P}^{a_d} [\mathbf{y}]] \dots]]]](t),$$

where  $\mathbf{s} - \mathbf{a} = (s_1 - a_1, \dots, s_d - a_d)$ .

**Theorem 6.3** Let  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ ,  $\mathbf{a} = (a_1, \dots, a_d) \in (\mathbb{Z}_{\geq 0})^d$ , and  $w = |\mathbf{s}|$ . Suppose  $s_1 + \dots + s_j > a_1 + \dots + a_j$  for all  $j = 1, \dots, d$ . Then we have

$$\zeta_q^{s-\mathbf{a}}[\mathbf{s}] = (1 - q)^w \mathfrak{z}_q[\mathbf{w}^{s-\mathbf{a}}(\mathbf{s}); 1], \quad \mathfrak{z}_q^{s-\mathbf{a}}[\mathbf{s}] = \mathfrak{z}_q[\mathbf{w}^{s-\mathbf{a}}(\mathbf{s})] := \mathfrak{z}_q[\mathbf{w}^{s-\mathbf{a}}(\mathbf{s}); 1]. \tag{32}$$

By specializing the preceding two theorems to the four types of  $q$ -MZVs in Table 1 we quickly find the following corollary. For future reference, we will say  $\mathbf{w}_\tau$  has the typical type  $\tau$  form for each type  $\tau$ .

**Corollary 6.4** For  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ , we set

$$\begin{aligned} \mathbf{w}_I(\mathbf{s}) &:= \rho^{s_1-1} \pi y \dots \rho^{s_d-1} \pi y = z_{s_1} \dots z_{s_d} \in \mathfrak{A}_\theta^{(B)} \subset \mathfrak{A}_\pi y \quad (s_1 \geq 2), \\ \mathbf{w}_{II}(\mathbf{s}) &:= \rho^{s_1} y \dots \rho^{s_d} y = z'_{s_1} \dots z'_{s_d} \in \mathfrak{A}_\theta^{(B)} \subset \mathfrak{A}_\pi y, \\ \mathbf{w}_{III}(\mathbf{s}) &:= \pi^{s_1-1} \rho y \pi^{s_2} y \dots \pi^{s_d} y = z'_{s_1} z_{s_2} \dots z_{s_d} \in \mathfrak{A}_\theta^{(A)} \subset \mathfrak{A}_\pi y, \\ \mathbf{w}_{IV}(\mathbf{s}) &:= \rho^{s_1-1} \pi y \rho^{s_2} y \dots \rho^{s_d} y = z_{s_1} z'_{s_2} \dots z'_{s_d} \in \mathfrak{A}_\theta^{(B)} \subset \mathfrak{A}_\pi y \quad (s_1 \geq 2), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{z}_q[\mathbf{w}_I(\mathbf{s}); t] &:= \mathbf{R}^{s_1-1} [\mathbf{P}[y \mathbf{R}^{s_2-1} [\mathbf{P}[y \dots \mathbf{R}^{s_d-1} [\mathbf{P}[y]] \dots ]]](t), \\ \mathfrak{z}_q[\mathbf{w}_{II}(\mathbf{s}); t] &:= \mathbf{R}^{s_1} [y \mathbf{R}^{s_2} [y \dots \mathbf{R}^{s_d} [y] \dots ]](t), \\ \mathfrak{z}_q[\mathbf{w}_{III}(\mathbf{s}); t] &:= \mathbf{P}^{s_1-1} [\mathbf{R}[y [\mathbf{P}^{s_2} [y [\mathbf{P}^{s_3} [y \dots \mathbf{P}^{s_d} [y] \dots ]]]]](t), \\ \mathfrak{z}_q[\mathbf{w}_{IV}(\mathbf{s}); t] &:= \mathbf{R}^{s_1-1} [\mathbf{P}[y \mathbf{R}^{s_2} [y \mathbf{R}^{s_3} [y \dots \mathbf{R}^{s_d} [y] \dots ]]](t). \end{aligned}$$

Then for all the types  $\tau = I, II, III$  and  $IV$ , we have

$$\zeta_q^\tau[\mathbf{s}] = (1 - q)^w \mathfrak{z}_q[\mathbf{w}_\tau(\mathbf{s}); 1], \quad \mathfrak{z}_q^\tau[\mathbf{s}] = \mathfrak{z}_q[\mathbf{w}_\tau(\mathbf{s})] := \mathfrak{z}_q[\mathbf{w}_\tau(\mathbf{s}); 1].$$

Moreover, similar results hold for type  $\tilde{I}$  and  $\tilde{IV}$   $q$ -MZVs. We may replace any of the consecutive strings  $\rho^{s_j-1} \pi$  by a single  $\rho$  in  $\mathbf{w}_{\tilde{I}}(\mathbf{s})$  and  $\mathbf{w}_{\tilde{IV}}(\mathbf{s})$ , and replace the corresponding operator string  $\mathbf{P}^{s_j-1} \mathbf{R}$  by a single  $\mathbf{R}$ .

We now apply the above to Okounkov’s  $q$ -MZVs. For any  $n \in \mathbb{N}$  we let  $n^-$  and  $n^+$  be the two nonnegative integers such that

$$\frac{n-1}{2} \leq n^- \leq \frac{n}{2} \leq n^+ \leq \frac{n+1}{2}.$$

Clearly we have  $n^+ + n^- = n$  always,  $n^+ = n^-$  if  $n$  is even, and  $n^+ = n^- + 1$  if  $n$  is odd. We can now define a variation of Okounkov’s  $q$ -MZVs. Let  $\mathbf{s} \in (\mathbb{Z}_{\geq 2})^d$ . Then

$$\zeta_q^O[\mathbf{s}] := \sum_{k_1 > \dots > k_d > 0} \prod_{j=1}^d \frac{q^{k_j s_j^+} + q^{k_j s_j^-}}{[k_j]^{s_j}} = (1 - q)^{|\mathbf{s}|} \sum_{k_1 > \dots > k_d > 0} \prod_{j=1}^d \frac{q^{k_j s_j^+} + q^{k_j s_j^-}}{(1 - q^{k_j})^{s_j}}.$$



Again, its modified form is:

$$\mathfrak{z}_q^{\text{O}}[\mathbf{s}] := \sum_{k_1 > \dots > k_d > 0} \prod_{j=1}^d \frac{q^{k_j s_j^+} + q^{k_j s_j^-}}{(1 - q^{k_j})^{s_j}}.$$

*Remark 6.5* The above variation is equal to Okounkov’s original  $q$ -MZVs up to a suitable 2-power. More precisely, the power is given by the number of even arguments in  $\mathbf{s}$ .

**Corollary 6.6** For  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ , we set

$$\mathbf{w}_0(\mathbf{s}) = (\rho^{s_1^-} \pi^{s_1^+} + \rho^{s_1^+} \pi^{s_1^-}) y \dots (\rho^{s_d^-} \pi^{s_d^+} + \rho^{s_d^+} \pi^{s_d^-}) y \in \mathfrak{A}_\theta^{(B)} \subset \mathfrak{A}_\pi y$$

and

$$\mathfrak{z}_q[\mathbf{w}_0(\mathbf{s}); t] = (\mathbf{R}^{s_1^-} \mathbf{P}^{s_1^+} + \mathbf{R}^{s_1^+} \mathbf{P}^{s_1^-}) [\mathbf{y} \dots (\mathbf{R}^{s_d^-} \mathbf{P}^{s_d^+} + \mathbf{R}^{s_d^+} \mathbf{P}^{s_d^-}) [\mathbf{y}] \dots ](t).$$

Then we have

$$\zeta_q^{\text{O}}[\mathbf{s}] = (1 - q)^w \mathfrak{z}_q[\mathbf{w}_0(\mathbf{s}); 1], \quad \mathfrak{z}_q^{\text{O}}[\mathbf{s}] = \mathfrak{z}_q[\mathbf{w}_0(\mathbf{s}); 1].$$

It is possible to obtain the shuffle relations among  $\mathfrak{z}_q^{\text{O}}[\mathbf{s}]$ -values using Corollary 6.6. The stuffle relations among  $\mathfrak{z}_q^{\text{O}}[\mathbf{s}]$  is mentioned implicitly in Okounkov’s original paper. For our modified version, they can be derived from the following fact (cf. Proposition 2.2 (ii) of [2]). Let  $F_n^{\text{O}}(t) = (t^{n^+} + t^{n^-}) / (1 - t)^n$  for all  $n \geq 2$ . Then for all  $r, s \in \mathbb{Z}_{\geq 2}$ , we have

$$F_r^{\text{O}}(t) \cdot F_s^{\text{O}}(t) = \begin{cases} 2F_{r+s}^{\text{O}}(t), & \text{if } r \text{ or } s \text{ is even;} \\ 2F_{r+s}^{\text{O}}(t) + \frac{1}{2}F_{r+s-2}^{\text{O}}(t), & \text{if } r \text{ and } s \text{ are odd.} \end{cases}$$

For example,

$$\begin{aligned} \mathfrak{z}_q^{\text{O}}[2, 3] \mathfrak{z}_q^{\text{O}}[2] &= 2\mathfrak{z}_q^{\text{O}}[2, 2, 3] + \mathfrak{z}_q^{\text{O}}[2, 3, 2] + 2\mathfrak{z}_q^{\text{O}}[4, 3] + 2\mathfrak{z}_q^{\text{O}}[2, 5], \\ \mathfrak{z}_q^{\text{O}}[2, 3] \mathfrak{z}_q^{\text{O}}[3] &= 2\mathfrak{z}_q^{\text{O}}[2, 3, 3] + \mathfrak{z}_q^{\text{O}}[3, 2, 3] + 2\mathfrak{z}_q^{\text{O}}[5, 3] + 2\mathfrak{z}_q^{\text{O}}[2, 6] + \frac{1}{2}\mathfrak{z}_q^{\text{O}}[2, 4]. \end{aligned}$$

## 7 $q$ -Shuffle Relations

In contrast to the MZV case, the  $q$ -shuffle product is much more difficult to define than the  $q$ -stuffle product. In this section we will use the Rota–Baxter algebra approach to define this for type  $\text{I}$ ,  $\text{II}$ ,  $\text{III}$ , and  $\widetilde{\text{IV}}$   $q$ -MZVs. Note that this has been done for type  $\text{III}$   $q$ -MZVs in [7] which we recall first.

The  $q$ -shuffle product on  $\mathfrak{A}_\pi$  (see Definition 4.8) is defined recursively as follows: for any words  $\mathbf{u}, \mathbf{v} \in X_\pi^*$  we define  $\mathbf{1} \sqcup \mathbf{u} = \mathbf{u} \sqcup \mathbf{1} = \mathbf{u}$  and

$$(y\mathbf{u}) \sqcup \mathbf{v} = \mathbf{u} \sqcup (y\mathbf{v}) = y(\mathbf{u} \sqcup \mathbf{v}), \tag{33}$$

$$\pi \mathbf{u} \sqcup \pi \mathbf{v} = \pi(\mathbf{u} \sqcup \pi \mathbf{v}) + \pi(\pi \mathbf{u} \sqcup \mathbf{v}) - \pi(\mathbf{u} \sqcup \mathbf{v}), \tag{34}$$

$$\delta \mathbf{u} \sqcup \delta \mathbf{v} = \mathbf{u} \sqcup \delta \mathbf{v} + \delta \mathbf{u} \sqcup \mathbf{v} - \delta(\mathbf{u} \sqcup \mathbf{v}), \tag{35}$$

$$\delta \mathbf{u} \sqcup \pi \mathbf{v} = \pi \mathbf{v} \sqcup \delta \mathbf{u} = \delta(\mathbf{u} \sqcup \pi \mathbf{v}) + \delta \mathbf{u} \sqcup \mathbf{v} - \mathbf{u} \sqcup \mathbf{v} \tag{36}$$

for any words  $\mathbf{u}, \mathbf{v} \in X_\pi^*$ . Equation (33) reflects the fact that when  $\mathbf{y}(t)$  is multiplied in front of either of the two factors in a product, it can be multiplied after taking the product. Equations (34)–(36) formalize (8), (13) and (14), respectively.

**Corollary 7.1** For any words  $\mathbf{u}, \mathbf{v} \in X_\pi^*$ , we have

$$\rho \mathbf{u} \sqcup \rho \mathbf{v} = \rho(\mathbf{u} \sqcup \rho \mathbf{v}) + \rho(\rho \mathbf{u} \sqcup \mathbf{v}) + \rho(\mathbf{u} \sqcup \mathbf{v}), \tag{37}$$

$$\rho \mathbf{u} \sqcup \pi \mathbf{v} = \pi \mathbf{v} \sqcup \rho \mathbf{u} = \rho(\rho \mathbf{u} \sqcup \mathbf{v}) + \rho(\mathbf{u} \sqcup \rho \mathbf{v}) + \rho \mathbf{u} \sqcup \mathbf{v} + \rho(\mathbf{u} \sqcup \mathbf{v}), \tag{38}$$

$$\delta \mathbf{u} \sqcup \rho \mathbf{v} = \rho \mathbf{v} \sqcup \delta \mathbf{u} = \delta(\mathbf{u} \sqcup \pi \mathbf{v}) - \mathbf{u} \sqcup \mathbf{v} = \delta(\mathbf{u} \sqcup \rho \mathbf{v}) + \delta(\mathbf{u} \sqcup \mathbf{v}) - \mathbf{u} \sqcup \mathbf{v}. \tag{39}$$

*Proof* These follows easily from (33)–(36) and the relation  $\rho = \pi - \mathbf{1}$ .

**Corollary 7.2** For  $j = 1, 2$  let  $X_\theta^{(j)}$  and  $X_\theta^{(j),*}$  be the embedding of  $X_\theta$  and  $X_\theta^*$  into  $X_\pi^*$ , respectively, by Definition 4.8. For any  $\alpha, \beta \in X_\theta^{(j)}$  and  $\mathbf{u}, \mathbf{v} \in X_\theta^{(j),*}$ , we have  $\mathbf{1} \sqcup \mathbf{u} = \mathbf{u} \sqcup \mathbf{1} = \mathbf{u}$  and

$$\alpha \mathbf{u} \sqcup \beta \mathbf{v} = \alpha(\mathbf{u} \sqcup \beta \mathbf{v}) + \beta(\alpha \mathbf{u} \sqcup \mathbf{v}) + [\alpha, \beta]_j(\mathbf{u} \sqcup \mathbf{v}),$$

where  $[\alpha, \beta]_j$  is determined by  $[a, b]_1 = [b, a]_1 = -b$ ,  $[a, b]_2 = [b, a]_2 = 0$  and

$$[a, a]_j = (-1)^j a, \quad [b, b]_j = -b\gamma, \quad [\alpha, \gamma]_j = [\gamma, \alpha]_j = -\alpha\gamma. \tag{40}$$

*Proof* All of these identities follow from straightforward computation using (33)–(39). For example,

$$\begin{aligned} b\mathbf{u} \sqcup b\mathbf{v} &= \pi y\mathbf{u} \sqcup \pi y\mathbf{v} = \pi(y\mathbf{u} \sqcup \pi y\mathbf{v}) + \pi(\pi y\mathbf{u} \sqcup y\mathbf{v}) - \pi(y\mathbf{u} \sqcup y\mathbf{v}) \\ &= \pi y(\mathbf{u} \sqcup \pi y\mathbf{v}) + \pi y(\pi y\mathbf{u} \sqcup \mathbf{v}) - \pi y y(\mathbf{u} \sqcup \mathbf{v}) \\ &= b(\mathbf{u} \sqcup b\mathbf{v}) + b(b\mathbf{u} \sqcup \mathbf{v}) - b\gamma(\mathbf{u} \sqcup \mathbf{v}). \end{aligned} \tag{41}$$

Similarly,

$$\begin{aligned} \theta \mathbf{u} \sqcup \theta \mathbf{v} &= \rho y\mathbf{u} \sqcup \rho y\mathbf{v} = \rho(y\mathbf{u} \sqcup \rho y\mathbf{v}) + \rho(\rho y\mathbf{u} \sqcup y\mathbf{v}) + \rho(y\mathbf{u} \sqcup y\mathbf{v}) \\ &= \rho y(\mathbf{u} \sqcup \rho y\mathbf{v}) + \rho y(\rho y\mathbf{u} \sqcup \mathbf{v}) + \rho y y(\mathbf{u} \sqcup \mathbf{v}) \\ &= \theta(\mathbf{u} \sqcup \theta \mathbf{v}) + \theta(\theta \mathbf{u} \sqcup \mathbf{v}) + \theta\gamma(\mathbf{u} \sqcup \mathbf{v}). \end{aligned} \tag{42}$$

The rest of the proof is left to the interested reader.

**Proposition 7.3** *The algebra  $(\mathfrak{A}_\pi, \sqcup)$  is commutative and associative.*

*Proof* See [7, Theorem 7].

The following corollary generalizes [26, Proposition 1].

**Corollary 7.4** *For  $j = 1$  or  $2$  the algebras  $(\mathfrak{A}_\theta^{(j)}, \sqcup)$  are commutative and associative.*

*Proof* This follows immediately from Proposition 7.3 since  $(\mathfrak{A}_\theta^{(j)}, \sqcup)$  are sub-algebras of  $(\mathfrak{A}_\pi, \sqcup)$  if  $\sqcup$  for  $\mathfrak{A}_\theta^{(j)}$  is defined as in Corollary 7.2.

Our next theorem shows that we may use the shuffle algebra structure defined above to describe the  $q$ -shuffle relations among different types of  $q$ -MZVs. Before doing so, we need to show that the  $q$ -shuffle products really make sense for all the types.

**Proposition 7.5** *Embed  $\mathfrak{A}_I^0, \mathfrak{A}_{II}^0, \mathfrak{A}_{IV}^0 \subset \mathfrak{A}_\theta^{(B)}$  and  $\mathfrak{A}_{III}^0 \subset \mathfrak{A}_\theta^{(A)}$ . Then for each type  $\tau$ , if the two words  $\mathbf{u}, \mathbf{v} \in \mathfrak{A}_\tau^0$  have the typical type  $\tau$  form listed in Corollary 6.4 then there is an algorithm to express  $\mathbf{u} \sqcup \mathbf{v}$  using only those words in the same form.*

*Proof* The case for type  $\tilde{I}$  is proved by [26, Proposition 2].

Type II is in fact the easiest since we can restrict ourselves to use only (33) and (37) to compute the shuffle and therefore  $\pi$  never comes into the picture. Clearly all such words must start with  $\rho$  and end with  $y$ .

For type III let's assume  $\mathbf{u} = \pi^{s_1-1} \rho y \pi^{s_2} y \dots \pi^{s_d} y$  and  $\mathbf{v} = \pi^{a_1-1} \rho y \pi^{a_2} y \dots \pi^{a_d} y$ . If we use the definition (34) repeatedly then in each word appearing in  $\mathbf{u} \sqcup \mathbf{v}$  the first  $\rho$  always appears before all the  $y$ 's. Such a word can be written in the form  $\pi^s \rho^r y \dots$  for some  $s \in \mathbb{Z}$  and  $r \geq 1$  (notice that if  $\rho$  and  $\pi$  are commutative). Now we can rewrite this as  $\pi^s (\pi - \mathbf{1})^{r-1} \rho y \dots$  and replace all the  $\rho$ 's after the first  $y$  by  $\pi - \mathbf{1}$ . This produces a word of typical type III form.

Type  $\tilde{IV}$  is similar to type III except that we need to take  $\theta$  into account. Notice that by definition if  $\mathbf{w} \in \mathfrak{A}_{IV}^0$  then it can be written as  $\theta \mathbf{w}'$ , or  $z_k \mathbf{w}'$  ( $k \geq 2, \mathbf{w}' \in Y_{IV}^*$ ) or a finite linear combination of these. So we have three cases to check. First, we prove that for all  $k, l \geq 2$  and  $\mathbf{u}, \mathbf{v} \in Y_{IV}^*$

$$z_k \mathbf{u} \sqcup z_l \mathbf{v} \in \mathfrak{A}_{IV}^0. \tag{43}$$

Indeed, putting  $k = r + 1$  and  $l = s + 1$  we have

$$\begin{aligned} & \rho^r \pi y \mathbf{u} \sqcup \rho^s \pi y \mathbf{v} \\ &= \rho(\rho^{r-1} \pi y \mathbf{u} \sqcup \rho^s \pi y \mathbf{v}) + \rho(\rho^r \pi y \mathbf{u} \sqcup \rho^{s-1} \pi y \mathbf{v}) + \rho(\rho^{r-1} \pi y \sqcup \rho^{s-1} \pi y \mathbf{v}). \end{aligned}$$

Now inside each of the three parentheses we replace every  $\pi$  by  $\rho + \mathbf{1}$  and use only (33) and (37) to expand (recall that  $\theta = \rho y$ ). We see that every term in the

expansion has the form  $\rho^n y \mathbf{w}$  for some  $n \geq 1$  and  $\mathbf{w} \in Y_{\tilde{N}}^*$ . If  $n = 1$  then we have  $\rho^n y \mathbf{w} = \theta \mathbf{w} \in \mathfrak{A}_{\tilde{N}}^0$ . If  $n \geq 2$  we can write it as

$$\rho^{n-1}(\pi - \mathbf{1})y\mathbf{w} = \sum_{j=1}^{n-1} (-1)^{j-1} \rho^{n-j} \pi y \mathbf{w} + (-1)^{n-1} \theta \mathbf{w} \in \mathfrak{A}_{\tilde{N}}^0 \tag{44}$$

with each word of typical type  $\tilde{IV}$  form.

Now we assume  $k = r + 1 \geq 2$  and  $\mathbf{u}, \mathbf{v} \in Y_{\tilde{N}}^*$ . Then

$$\begin{aligned} z_k \mathbf{u} \sqcup \theta \mathbf{v} &= \rho^r \pi y \mathbf{u} \sqcup \rho y \mathbf{v} \\ &= \rho(\rho^{r-1} \pi y \mathbf{u} \sqcup \rho y \mathbf{v}) + \rho y(\rho^r \pi y \mathbf{u} \sqcup \mathbf{v}) + \rho y(\rho^{r-1} \pi y \sqcup \mathbf{v}) \in \mathfrak{A}_{\tilde{N}}^0 \end{aligned}$$

since  $\rho y = \theta$  and the first term can be dealt with as in the proof of (43).

Finally,

$$\theta \mathbf{u} \sqcup \theta \mathbf{v} \in \mathfrak{A}_{\tilde{N}}^0$$

follows from (42) immediately. This completes the proof of the proposition.

The following theorem generalizes [26, Theorem 2] but it does not contain [7, Theorem 7] since our word representation of type III  $q$ -MZVs is different from that given in [7].

**Theorem 7.6** *Embed  $\mathfrak{A}_{\tilde{I}}^0, \mathfrak{A}_{\tilde{II}}^0, \mathfrak{A}_{\tilde{IV}}^0 \subset \mathfrak{A}_{\theta}^{(B)}$  and  $\mathfrak{A}_{\tilde{III}}^0 \subset \mathfrak{A}_{\theta}^{(A)}$ . Then for each type  $\tau$  and for any  $\mathbf{u}_{\tau}, \mathbf{v}_{\tau} \in \mathfrak{A}_{\tau}^0$ , we have*

$$\mathfrak{z}_q[\mathbf{u}_{\tau}] \mathfrak{z}_q[\mathbf{v}_{\tau}] = \mathfrak{z}_q[\mathbf{u}_{\tau} \sqcup \mathbf{v}_{\tau}]. \tag{45}$$

**Proof** For each type  $\tau$  we observe that  $\mathfrak{z}_q[\mathbf{w}_{\tau}; t]$  satisfies (45) because of the identities in Proposition 3.2. Then the theorem follows from the fact that  $\mathfrak{z}_q[\mathbf{w}_{\tau}] = \mathfrak{z}_q[\mathbf{w}_{\tau}; 1]$  for any word  $\mathbf{w}_{\tau} \in \mathfrak{A}_{\tau}^0$  by Corollary 6.4.

## 8 Duality Relations

The DBSFs do not contain all linear relations among the various types of  $q$ -MZVs. In [26], Takeyama discovered the following relations which provides some of the missing relations for type  $\tilde{I}$   $q$ -MZVs, at least in the small weight cases. He called them *Resummation Identities*. We would rather call them “duality” relations because of their similarity to the duality relations for the classical MZVs.

**Theorem 8.1** ([26, Theorem 4]) *For a positive integer  $k$ , set*

$$\varphi_k := (-1)^k \left( \sum_{j=2}^k (-1)^j z_j - \theta \right),$$

where  $\varphi_1 = \theta = \rho y \in \mathfrak{A}_\theta^{(B)}$ . Let  $\ell \in \mathbb{N}$  and  $\alpha_j, \beta_j \in \mathbb{Z}_{\geq 0}$  for all  $j = 1, \dots, \ell$ . Then we have

$$\mathfrak{z}_q^{\text{I}}[\varphi_{\alpha_1+1} \gamma^{\beta_1} \cdots \varphi_{\alpha_\ell+1} \gamma^{\beta_\ell}] = \mathfrak{z}_q^{\text{I}}[\varphi_{\beta_\ell+1} \gamma^{\alpha_\ell} \cdots \varphi_{\beta_1+1} \gamma^{\alpha_1}]. \tag{46}$$

We can use the Rota–Baxter algebra approach to give a new proof of this result.

**Proof** Notice that  $\gamma = y, z_j = \rho^{j-1} \pi y$  and  $\theta = \rho y$  with the embedding  $\mathfrak{A}_1^0 \subset \mathfrak{A}_\theta^{(B)}$ . Since  $\pi = \rho + \mathbf{1}$ , for all  $k \geq 1$ , we have (cf. (44))

$$\begin{aligned} \varphi_k &= (-1)^k \left( \sum_{j=2}^k (-1)^j \rho^{j-1} (\rho + \mathbf{1}) y - \rho y \right) \\ &= (-1)^k \left( \sum_{j=2}^k (-1)^j \rho^j y + \sum_{j=2}^k (-1)^j \rho^{j-1} y - \rho y \right) = \rho^k y. \end{aligned} \tag{47}$$

Thus the theorem follows from Corollaries 3.4 and 6.4 easily.

*Remark 8.2* Although not mentioned explicitly in [26], there is a subtle point in applying Theorem 8.1. Notice that in the expression of  $\varphi_k$  the letter  $\theta$  appears. However,  $q$ -MZVs of the form such as  $\zeta_q^{\text{I}}[\theta \gamma z_2 \gamma] = \zeta_q^{\text{I}}[\rho y^2 \rho^2 \pi y]$  is not really defined. In fact, it should be denoted by  $\zeta_q^{\text{I}}[\theta \gamma z_2 \gamma] = \zeta_q^{(1,0,1,0)}[1, 0, 2, 0]$  (and such values always converge by Proposition 2.1 because of the leading 1 in the auxiliary variable  $\mathbf{a}$ ). But, suitable  $\mathbb{Q}$ -linear combinations of (46) may lead to identities in which only  $z_k$ 's appear. Then all terms can be written as honest  $\zeta_q^{\text{I}}$ -values. This explains the use of two admissible structures  $\widehat{\mathfrak{H}}^0$  and  $\mathfrak{H}^0$  in [26]. For an illuminating example, see the proof of Proposition 7 of op. cit. This remark also applies to Theorem 8.5 for the duality of type  $\widetilde{\text{IV}}$   $q$ -MZVs.

Similar relations for type II  $q$ -MZVs have the most aesthetic appeal and is the primary reason why we prefer to call it by the name ‘‘duality’’.

**Theorem 8.3** Let  $\ell \in \mathbb{N}$  and  $\alpha_j, \beta_j \in \mathbb{N}$  for all  $j = 1, \dots, \ell$ . Then we have

$$\mathfrak{z}_q^{\text{II}}[\rho^{\alpha_1} y^{\beta_1} \cdots \rho^{\alpha_\ell} y^{\beta_\ell}] = \mathfrak{z}_q^{\text{II}}[\rho^{\beta_\ell} y^{\alpha_\ell} \cdots \rho^{\beta_1} y^{\alpha_1}].$$

**Proof** This follows from Corollaries 3.4 and 6.4 immediately.

Of course we may apply the same idea to type III and  $\widetilde{\text{IV}}$   $q$ -MZVs.

**Theorem 8.4** Let  $\ell \in \mathbb{N}$  and  $\alpha_j, \beta_j \in \mathbb{N}$  for all  $j = 1, \dots, \ell$ . Then we have

$$\begin{aligned} \mathfrak{z}_q^{\mathbb{I}}[(\pi - \mathbf{1})^{\alpha_1-1} \rho y^{\beta_1} (\pi - \mathbf{1})^{\alpha_2} y^{\beta_2} \dots (\pi - \mathbf{1})^{\alpha_\ell} y^{\beta_\ell}] \\ = \mathfrak{z}_q^{\mathbb{I}}[(\pi - \mathbf{1})^{\beta_\ell-1} \rho y^{\alpha_\ell} (\pi - \mathbf{1})^{\beta_{\ell-1}} y^{\alpha_{\ell-1}} \dots (\pi - \mathbf{1})^{\beta_1} y^{\alpha_1}]. \end{aligned}$$

**Proof** Since  $\rho = \pi - \mathbf{1}$  this follows from Corollaries 3.4 and 6.4 easily.

**Theorem 8.5** Let  $\ell \in \mathbb{N}$  and  $\alpha_j, \beta_j \in \mathbb{N}$  for all  $j = 1, \dots, \ell$ . Then we have

$$\mathfrak{z}_q^{\tilde{\mathbb{I}}}[\varphi_{\alpha_1} y^{\beta_1-1} \rho^{\alpha_2} y^{\beta_2} \dots \rho^{\alpha_\ell} y^{\beta_\ell}] = \mathfrak{z}_q^{\tilde{\mathbb{I}}}[\varphi_{\beta_\ell} y^{\alpha_\ell-1} \rho^{\beta_{\ell-1}} y^{\alpha_{\ell-1}} \dots \rho^{\beta_1} y^{\alpha_1}].$$

Here  $\varphi_1 = \theta = \rho y \in \mathfrak{A}_\theta^{(B)}$ .

**Proof** This follows from (47), Corollaries 3.4 and 6.4.

### 9 The General Type G $q$ -MZVs

All of the  $q$ -MZVs of type  $\tilde{\mathbb{I}}, \mathbb{II}, \mathbb{III}$  and  $\tilde{\mathbb{IV}}$  considered in the above are some special forms of the  $q$ -MZVs  $\mathfrak{z}_q^{(a_1, \dots, a_d)}[s_1, \dots, s_d]$  where  $1 \leq a_1 \leq s_1, 0 \leq a_j \leq s_j$  for all  $j \geq 2$ , all of which are convergent by Proposition 2.1. We call these *type G  $q$ -MZVs*. Similar to the first four types, we may use words to encode these values according to Theorem 6.2. Namely, we can define

$$\mathfrak{z}_q[\rho^{a_1} y \rho^{a_2} y \dots \rho^{a_d} y; t] := \mathbf{R}^{a_1} \mathbf{y}[\mathbf{R}^{a_2} \mathbf{y}[\dots [\mathbf{R}^{a_d} \mathbf{y}] \dots ]](t).$$

By the relation  $\pi = \rho + \mathbf{1}$  and  $\mathbf{P} = \mathbf{R} + \mathbf{I}$ , we get

$$\mathfrak{z}_q[\rho^{a_1} \pi^{b_1} y \rho^{a_2} \pi^{b_2} y \dots \rho^{a_d} \pi^{b_d} y; t] = \mathbf{R}^{a_1} [\mathbf{P}^{b_1} [\mathbf{y} \mathbf{R}^{a_2} [\mathbf{P}^{b_2} [\mathbf{y} \dots \mathbf{R}^{a_d} [\mathbf{P}^{b_d} [\mathbf{y}]] \dots ]]]](t).$$

Thus we have

$$\mathfrak{z}_q[\mathbf{w}^{\mathbf{a}}(\mathbf{s})] := \mathfrak{z}_q[\mathbf{w}^{\mathbf{a}}(\mathbf{s}); 1] = \mathfrak{z}_q^{(a_1, \dots, a_d)}[s_1, \dots, s_d],$$

where  $\mathbf{w}^{\mathbf{a}}(\mathbf{s}) = \rho^{a_1} \pi^{s_1-a_1} y \dots \rho^{a_d} \pi^{s_d-a_d} y \in X_\pi^*$ . The shuffle product structure is reflected by  $(X_\pi^*, \sqcup)$  where the  $\sqcup$  is defined by (33), (34), (37) and (38).

We observe that there is often more than one way to express a type G  $q$ -MZV using words because of the relation  $\pi = \rho + \mathbf{1}$ . For example, using the relations

$$\pi^2 \rho y = \pi \rho^2 y + \pi \rho y = \rho^3 y + 2\rho^2 y + \rho y$$

we get immediately the relations

$$\mathfrak{z}_q^{(1)}[3] = \mathfrak{z}_q^{(2)}[3] + \mathfrak{z}_q^{(1)}[2] = \mathfrak{z}_q^{(3)}[3] + 2\mathfrak{z}_q^{(2)}[2] + \mathfrak{z}_q^{(1)}[1].$$

We call all such relations **P-R** relations.

**Proposition 9.1** *For all  $\mathbf{u}, \mathbf{v} \in \mathfrak{A}_G^0$ , we have  $\mathbf{u} \sqcup \mathbf{v} \in \mathfrak{A}_G^0$ .*

*Proof* Notice that admissible words in  $\mathfrak{A}_G^0$  must end with  $y$  and have at least one  $\rho$  before the first  $y$ . Moreover, the converse is also true. This is rather straightforward if we use the **P-R** relations repeatedly to get rid of all the  $\pi$ 's.

Now, by using the definition of  $\sqcup$  it is not hard to see that  $\mathbf{u} \sqcup \mathbf{v}$  ends with  $y$  and has at least one  $\rho$  before the first  $y$  if both  $\mathbf{u}$  and  $\mathbf{v}$  are admissible. So  $\mathbf{u} \sqcup \mathbf{v} \in \mathfrak{A}_G^0$  and the proposition is proved.

To define the stuffle product we let

$$Y_G = \{z_{a,s} \mid a, s \in \mathbb{Z}_{\geq 0}, a \leq s\},$$

and let  $\mathfrak{A}_G$  be the the noncommutative polynomial  $\mathbb{Q}$ -algebra of words of  $Y_G^*$  built on the alphabet  $Y_G$ . Define the type  $G$ -admissible words as those in

$$\mathfrak{A}_G^0 = \bigcup_{1 \leq a \leq s} z_{a,s} \mathfrak{A}_G.$$

We can regard  $\mathfrak{A}_G$  as a subalgebra of  $X_\pi^*$  by setting  $z_{ta,s} = \rho^a \pi^{s-a} y$ . Then stuffle product  $*_G$  on  $\mathfrak{A}_G^0$  can be defined inductively as follows. For any words  $\mathbf{u}, \mathbf{v} \in \mathfrak{A}_G^0$  and letters  $z_{a,s}, z_{a',s'} \in Y_G$  with  $1 \leq a \leq s$  and  $1 \leq a' \leq s'$  we set  $\mathbf{1} *_G \mathbf{u} = \mathbf{u} = \mathbf{u} *_G \mathbf{1}$  and

$$(z_{a,s} \mathbf{u}) *_G (z_{a',s'} \mathbf{v}) = z_{a,s} (\mathbf{u} *_G z_{a',s'} \mathbf{v}) + z_{a',s'} (z_{a,s} \mathbf{u} *_G \mathbf{v}) + z_{a+a',s+s'} (\mathbf{u} *_G \mathbf{v}).$$

It is easy to show that  $(\mathfrak{A}_G^0, *_G)$  is a commutative and associative algebra.

We leave the proof of the following theorems to the interested readers. The first result clearly provides the DBSFs of type  $G$   $q$ -MZVs.

**Theorem 9.2** *For any  $\mathbf{u}, \mathbf{v} \in \mathfrak{A}_G^0 \subset X_\pi^*$  we have*

$$\mathfrak{z}_q[\mathbf{u} *_G \mathbf{v}] = \mathfrak{z}_q[\mathbf{u} \sqcup \mathbf{v}] = \mathfrak{z}_q[\mathbf{u}] \mathfrak{z}_q[\mathbf{v}]. \tag{48}$$

The duality relations are given in the cleanest form by Theorem 8.4 which can be translated into the following.

**Theorem 9.3** *Let  $\ell \in \mathbb{N}$  and  $\alpha_j, \beta_j \in \mathbb{N}$  for all  $j = 1, \dots, \ell$ . Set*

$$\begin{aligned} \mathbf{s} &= (\alpha_1, 0^{\beta_1-1}, \alpha_2, 0^{\beta_2-1}, \dots, \alpha_\ell, 0^{\beta_\ell-1}), \\ \mathbf{s}^\vee &= (\beta_\ell, 0^{\alpha_\ell-1}, \beta_{\ell-1}, 0^{\alpha_{\ell-1}-1}, \dots, \beta_1, 0^{\alpha_1-1}). \end{aligned}$$

*Then we have*

$$\mathfrak{z}_q^s[\mathbf{s}] = \mathfrak{z}_q^{s^\vee}[\mathbf{s}^\vee].$$

*Remark 9.4* We point out that the duality in Theorem 9.3 cannot be used to derive any relation among MZVs. For example, when  $\ell = 1$  the MZV  $\zeta(\alpha, 0^{\beta-1})$  converges if  $\alpha > \beta$  while the MZV  $\zeta(\beta, 0^{\alpha-1})$  converges if  $\beta > \alpha$ .

## 10 Numerical Data

In this last section, we compute the  $\mathbb{Q}$ -linear relations among various types of  $q$ -MZVs of small weight by using the DBSFs and the duality relations. Most of the computation is carried out with the computer algebra system MAPLE, version 16. My laptop has Intel Core i7 with CPU speed at 2.4 GHz and 16 GB RAM.

For each type  $\tau$  we will define the set of type  $\tau$ -admissible words  $W_{\leq w}^\tau$  with both weight and depth bounded by  $w$ . This is necessary since we allow 0 in some types of  $q$ -MZVs. We have to control the number of 0's occurring as arguments in  $q$ -MZVs since otherwise the dimensions to be considered becomes infinite. Another reason that the depth has to be bounded is because the duality essentially swaps the depth and the weight.

We denote by  $Z_{\leq w}^\tau$  the  $\mathbb{Q}$ -space generated by  $q$ -MZVs of type  $\tau$  corresponding to the type  $\tau$ -admissible words  $W_{\leq w}^\tau$ ,  $DS_{\leq w}^\tau$  the space generated by the DBSFs, and  $DU_{\leq w}^\tau$  the space generated by the duality relations. Hence  $DU_{\leq w}^\tau \setminus DS_{\leq w}^\tau$  gives the duality relations that are not contained in  $DS_{\leq w}^\tau$ .

**Type I.** We have seen that it is necessary to consider  $q$ -MZVs of the form  $\mathfrak{z}_q^a(\mathbf{s})$  with  $(a_j, s_j) = (s_j - 1, s_j)$  or  $(a_j, s_j) = (1, 1)$ . The latter case corresponds to the words containing the letter  $\theta$ . We have called all of these values *type I  $q$ -MZVs*.

**Proposition 10.1** *Let  $F_{-1} = 0, F_0 = 1, F_1 = 1, \dots$  be the Fibonacci sequence. Then for all  $w \geq 1$  we have*

$$\#W_{\leq w}^I = 2^{w-1} - 1 \quad \text{and} \quad \#W_{\leq w}^{\tilde{I}} = F_{2w} - 1.$$

*Proof* The first equation follows from the same argument as that for MZVs. It is given by the number of integer solutions to the inequality

$$s_1 + \dots + s_d \leq w, \quad d \geq 1, s_1 \geq 2, s_2, \dots, s_d \geq 1.$$

Or, more directly and perhaps much easier, we can count the corresponding admissible words. Clearly, there are  $2^{w-1}$  ways to form a word consisting of  $w - 1$  letters where the letters can be either  $\rho$  or  $\pi$ . Let  $S_w$  be the set of such words. We now show that there is a one-to-one correspondence between  $S_w$  and the set  $A_w$  of admissible word of weight  $w$ . First, from each word  $\mathbf{w} \in S_w$  we can obtain a word in  $A_w$  by inserting a letter  $y$  after each  $\pi$  in  $\mathbf{w}$  and attach  $\pi y$  at the end. On the other hand, for each word in  $A_w$  we may chop off the ending  $\pi y$  and removing all the  $y$ 's to get a word in  $\mathbf{w}$ . This establishes the one-to-one correspondence.



**Table 2** Dimension of  $q$ -MZVs of type  $\tilde{I}$

$w$	2	3	4	5	6	7
$\#\mathcal{W}_{\leq w}^{\tilde{I}}$	4	12	33	88	232	609
Lower bound of $\dim \mathcal{Z}_{\leq w}^{\tilde{I}}$	3	7	14	27	50	91
$\dim \mathcal{DS}_{\leq w}^{\tilde{I}}$	1	4	17	56	171	497
$\dim \mathcal{DU}_{\leq w}^{\tilde{I}} \setminus \mathcal{DS}_{\leq w}^{\tilde{I}}$	0	0	1	2	3	6
Deficiency	0	1,0	1,0	3	8	15

We now prove the second equation. Let  $a_n$  (resp.  $b_n$ ) be the number of type  $\tilde{I}$   $q$ -MZVs of weight  $n$  beginning with  $(a_j, s_j) = (1, 1)$  (resp.  $(a_j, s_j) = (s_j - 1, s_j)$ ). Let's call the two different beginnings 1-initial and 2-initial, respectively. Then  $a_1 = 1$  and  $b_1 = 0$ . Now to produce weight  $n + 1$  1-initials one can attach  $(t, s) = (1, 1)$  to the beginning of any weight  $n$  type  $\tilde{I}$   $q$ -MZVs. Moreover, one can change the beginning of any weight  $n$  1-initial to  $(t, s) = (0, 1)$  and then attach  $(t, s) = (1, 1)$ . Thus  $a_{n+1} = 2a_n + b_n$ . To obtain 2-initials of weight  $n + 1$  one either changes a 1-initial of weight  $n$  to begin with  $(t, s) = (1, 2)$  or changes a 2-initial value of weight  $n$  to begin with  $(s, s + 1)$  from  $(s - 1, s)$  (i.e., increases the first argument by 1). Hence  $b_{n+1} = a_n + b_n$ . Thus it is easy to see that  $a_n = F_{2n-2}$  and  $b_n = F_{2n-3}$  for all  $n \geq 1$ . Therefore

$$\#\mathcal{W}_{\leq w}^{\tilde{I}} = \sum_{n=0}^{2w-2} F_n = F_{2w} - 1$$

which can be proved easily by induction.

We find up to weight 3 the following identity (49) cannot be proved by DBSFs and dualities up to weight 3. Let  $1_n$  denote the string where 1 is repeated  $n$  times. Then

$$\mathfrak{z}_q^{(1,1)}[2, 1] = \mathfrak{z}_q^{(1,1)}[1, 1] - \mathfrak{z}_q^{(1_3)}[1_3] + \mathfrak{z}_q^{(1,1,0)}[1_3]. \tag{49}$$

Interestingly, (49) can be proved using weight 4 DBSFs and dualities. This is why we put  $\mathbf{0}$  as the final deficiency (Table 2).

Having proved (49), we find, up to weight 4, the only one missing relation is

$$\begin{aligned} \mathfrak{z}_q^{(2,1)}[3, 1] = & \mathfrak{z}_q^{(1,0,1)}[1_3] - 2\mathfrak{z}_q^{(1_3)}[1, 2, 1] + \mathfrak{z}_q^{(1_2,0)}[1, 2, 1] \\ & + \mathfrak{z}_q^{(1_3)}[2, 1_2] - \mathfrak{z}_q^{(1,0,1)}[2, 1_2] - \mathfrak{z}_q^{(1_3,0)}[1_4] + \mathfrak{z}_q^{(1_2,0_2)}[1_4]. \end{aligned} \tag{50}$$

In weight 5, there are three missing relations:

$$\begin{aligned} \mathfrak{z}_q^{(1_4)}[1_3, 2] = & \mathfrak{z}_q^{t_1}[1_4] - \mathfrak{z}_q^{t_2}[1_4] - \mathfrak{z}_q^{(1_4)}[s_1] - \mathfrak{z}_q^{t_1}[1_3, 2] - 2\mathfrak{z}_q^{t_3}[1_5] - 2\mathfrak{z}_q^{t_4}[s_2] + 2\mathfrak{z}_q^{t_5}[1_5], \\ \mathfrak{z}_q^{t_4}[1_4] = & \mathfrak{z}_q^{(1_3)}[2, 1_2] - \mathfrak{z}_q^{t_2}[s_1] - 2\mathfrak{z}_q^{(1_4)}[s_1] - \mathfrak{z}_q^{t_6}[s_2] + \mathfrak{z}_q^{(1_4)}[s_2] + \mathfrak{z}_q^{t_7}[1_5] - \mathfrak{z}_q^{t_1}[1_4] \\ & - \mathfrak{z}_q^{(1_3)}[2, 1, 2] - \mathfrak{z}_q^{t_5}[1_5] + \mathfrak{z}_q^{t_8}[1_4] + 2\mathfrak{z}_q^{t_2}[1_4] - \mathfrak{z}_q^{t_2}[2, 1_3] - \mathfrak{z}_q^{t_8}[2, 1_3], \end{aligned}$$

**Table 3** Dimension of  $q$ -MZVs of type I

$w$	2	3	4	5	6	7	8	9
$\sharp W^I_{\leq w}$	1	3	7	15	31	63	127	255
Lower bound of $\dim Z^I_{\leq w}$	1	2	4	7	11	18	27	42
$\dim DS^I_{\leq w}$	0	1	3	8	20	45		
$\dim DU^I_{\leq w} \setminus DS^I_{\leq w}$	0	0	0	0	0	0		
Deficiency	0	0	0	0	0	0		

$$\begin{aligned} \mathfrak{z}_q^{(14)}[1_3, 2] &= 3\mathfrak{z}_q^{(13)}[2, 1_2] - 3\mathfrak{z}_q^{(13)}[2, 1, 2] - 3\mathfrak{z}_q^{\mathbf{t}_4}[1_4] + \mathfrak{z}_q^{\mathbf{t}_2}[1_4] - \mathfrak{z}_q^{\mathbf{t}_3}[1_5] \\ &\quad - \mathfrak{z}_q^{(14)}[\mathbf{s}_1] - \mathfrak{z}_q^{\mathbf{t}_4}[\mathbf{s}_1] - 2\mathfrak{z}_q^{\mathbf{t}_2}[\mathbf{s}_1] - \mathfrak{z}_q^{(1_3, 0_2)}[1_5] - 2\mathfrak{z}_q^{(14)}[\mathbf{s}_2] + \mathfrak{z}_q^{\mathbf{t}_4}[\mathbf{s}_2] \\ &\quad + \mathfrak{z}_q^{\mathbf{t}_4}[2, 1_3] + 2\mathfrak{z}_q^{\mathbf{t}_1}[1_4] + \mathfrak{z}_q^{\mathbf{t}_1}[1_3, 2] + 2\mathfrak{z}_q^{(1_2, 0_2, 1)}[1_5], \end{aligned}$$

where  $\mathbf{s}_1 = (1, 2, 1_2)$ ,  $\mathbf{s}_2 = (1_2, 2, 1)$ ,  $\mathbf{t}_1 = (1, 0, 1_2)$ ,  $\mathbf{t}_2 = (1_2, 0, 1)$ ,  $\mathbf{s}_3 = (\mathbf{t}_2, 0)$ ,  $\mathbf{t}_4 = (1_3, 0)$ ,  $\mathbf{t}_5 = (1, 0_2, 1_2)$ ,  $\mathbf{t}_6 = (1, 0, 1, 0)$ ,  $\mathbf{t}_7 = (1, 0_3, 1)$ , and  $\mathbf{t}_8 = (1, 0_2, 1)$ .

Equation (50) was initially verified numerically. Even with all the DBSFs and dualities from weight 5 and 6 this still would not follow. Fortunately, we will see in a moment that this relation can be proved using type G  $q$ -MZVs. However, the three missing relations in weight 5 are only proved numerically, since, unfortunately, there are too many type G  $q$ -MZVs of weight 5 so the computer computation requires too much memory to provide a solution at the moment.

Using the relations obtained above for type  $\tilde{I}$   $q$ -MZVs we can compute the following data for type I  $q$ -MZVs (Table 3).

It is consistent with Takeyama’s computation at the end of [26]. However, our computation shows that the DBSFs from type  $\tilde{I}$   $q$ -MZVs already imply all the relations among type I  $q$ -MZVs, at least when the weight is less than 8. We thus can think these type  $\tilde{I}$  DBSFs as “regularized” DBSFs for type I  $q$ -MZVs.

**Conjecture 10.2** *All the  $\mathbb{Q}$ -linear relations of type I  $q$ -MZVs can be derived by the regularized DBSFs, i.e., by the DBSFs for type  $\tilde{I}$   $q$ -MZVs.*

**Type II.** For each fixed weight  $w \geq 1$  we collect all the type II-admissible words of the following form since we want to use the duality relations to its maximal utility. Such admissible words must consist of letters  $\rho$  and  $y$  only, begin with  $\rho$ , end with  $y$ , and the occurrence of  $\rho$  and  $y$  is at most  $w$  each. For example, we have the duality

$$\mathfrak{z}_q^{\text{II}}(\rho^3 y^2 \rho y^4) = \mathfrak{z}_q^{\text{II}}(\rho^4 y \rho^2 y^3) \implies \mathfrak{z}_q^{\text{II}}(3, 0, 1, 0_3) = \mathfrak{z}_q^{\text{II}}(4, 2, 0_2)$$

when we consider weight 6.

**Proposition 10.3** *For all  $w \geq 1$ , the number of type II-admissible words is*

**Table 4** Dimension of  $q$ -MZVs of type II

$w$	1	2	3	4	5	6
$\sharp W_{\leq w}^{\mathbb{I}}$	1	5	19	69	251	923
Lower bound of $\dim Z_{\leq w}^{\mathbb{I}}$	1	3	12	30	73	173
$\dim DS_{\leq w}^{\mathbb{I}}$	0	1	5	28	124	536
$\dim DU_{\leq w}^{\mathbb{I}} \setminus DS_{\leq w}^{\mathbb{I}}$	0	1	2	8	35	127
Deficiency	0	0	0	3,0	19,6	87

$$\sharp W_{\leq w}^{\mathbb{I}} = \sum_{i=0}^{w-1} \sum_{j=0}^{w-1} \binom{i+j}{j} = \binom{2w}{w} - 1.$$

*Remark 10.4* This is the sequence A030662 according to the On-Line Encyclopedia of Integer Sequences <http://oeis.org>.

**Proof** For the first equality, note that if  $i + 1$  (resp.  $j + 1$ ) is the number of occurrence of  $\rho$  (resp.  $y$ ) in an admissible word of  $W_{\leq w}^{\mathbb{I}}$  then we can put one  $\rho$  at the beginning and one  $y$  at the end, then put  $i$  of the other  $\rho$ 's and  $j$  of the other  $y$ 's in between in arbitrary order. Thus, by a well-known binomial identity

$$1 + \sharp W_{\leq w}^{\mathbb{I}} = 1 + \sum_{i=0}^{w-1} \sum_{j=0}^{w-1} \binom{i+j}{j} = 1 + \sum_{i=0}^{w-1} \binom{w+i}{w-1} = \sum_{i=0}^w \binom{w+i-1}{i} = \binom{2w}{w}.$$

This completes the proof of the proposition.

Up to weight 4, the following three independent relations cannot be proved using DBSFs and dualities up to weight 4 (Table 4).

$$\begin{aligned} \mathfrak{z}_q^{\mathbb{I}}[1, 0, 3] &= \mathfrak{z}_q^{\mathbb{I}}[2, 2] + 3\mathfrak{z}_q^{\mathbb{I}}[1_2, 2] + 2\mathfrak{z}_q^{\mathbb{I}}[1, 0, 2, 0] - 2\mathfrak{z}_q^{\mathbb{I}}[1_2, 0, 1] \\ &\quad + \mathfrak{z}_q^{\mathbb{I}}[1_2, 0, 2] + \mathfrak{z}_q^{\mathbb{I}}[1_2, 1, 0] - \mathfrak{z}_q^{\mathbb{I}}[1, 2, 0, 1] + 2\mathfrak{z}_q^{\mathbb{I}}[2, 0, 1_2], \\ \mathfrak{z}_q^{\mathbb{I}}[3, 0] &= \mathfrak{z}_q^{\mathbb{I}}[2, 2] - 2\mathfrak{z}_q^{\mathbb{I}}[3, 1] + \mathfrak{z}_q^{\mathbb{I}}[1, 0, 2, 0] - 2\mathfrak{z}_q^{\mathbb{I}}[1_2, 0, 1] + 2\mathfrak{z}_q^{\mathbb{I}}[1_2, 1, 0] \\ &\quad - \mathfrak{z}_q^{\mathbb{I}}[2, 0, 2, 0] + \mathfrak{z}_q^{\mathbb{I}}[3, 0_2, 0] + 2\mathfrak{z}_q^{\mathbb{I}}[3, 0_2, 1] - \mathfrak{z}_q^{\mathbb{I}}[3, 0, 1, 0] + 2\mathfrak{z}_q^{\mathbb{I}}[3, 1, 0_2], \\ \mathfrak{z}_q^{\mathbb{I}}[1, 0, 3] &= \mathfrak{z}_q^{\mathbb{I}}[2, 2] + 2\mathfrak{z}_q^{\mathbb{I}}[3, 1] + \mathfrak{z}_q^{\mathbb{I}}[1_2, 2] + 4\mathfrak{z}_q^{\mathbb{I}}[1_2, 0, 1] + \mathfrak{z}_q^{\mathbb{I}}[1_2, 0, 2] \\ &\quad + \mathfrak{z}_q^{\mathbb{I}}[1_2, 1, 0] + \mathfrak{z}_q^{\mathbb{I}}[1, 2, 0, 1] + 4\mathfrak{z}_q^{\mathbb{I}}[2, 0, 1, 0] + 2\mathfrak{z}_q^{\mathbb{I}}[2, 1, 0, 1] + 2\mathfrak{z}_q^{\mathbb{I}}[2, 1_2, 0]. \end{aligned}$$

But using DBSFs and dualities in weight 5, these can all be verified. In weight 5, we have to use the relations from weight 6 to push the deficiency from 19 down to 6. It is very likely that relations from weight 7 (or even higher) can reduce this further down to 0. But our computer runs out of memories so this is not proved.

**Type III.** The set of type III-admissible words  $W_{\leq w}^{\mathbb{III}}$  up to weight  $w$  consist of those of the form  $\pi^{s_1-1} \rho y \pi^{s_2} y \cdots \pi^{s_d} y$  with  $d \leq w$ ,  $|s| \leq w$ ,  $s_1 \geq 1$  and  $s_2, \dots, s_d \geq 0$ . First we have

**Table 5** Dimension of  $q$ -MZVs of type III

$w$	1	2	3	4	5	6
$\sharp W_{\leq w}^{\text{III}}$	1	5	19	69	251	923
Lower bound of $\dim Z_{\leq w}^{\text{III}}$	1	4	12	30	73	173
$\dim DS_{\leq w}^{\text{III}}$	0	1	5	28	124	536
$\dim DU_{\leq w}^{\text{III}} \setminus DS_{\leq w}^{\text{III}}$	0	0	1	1	5	4
Deficiency	0	0	1,0	10,0	49,6	210,87

**Proposition 10.5** For all  $w \geq 1$ , we have

$$\sharp W_{\leq w}^{\text{III}} = \binom{2w}{w} - 1.$$

*Proof* Notice there is an onto map from  $W_{\leq w}^{\text{III}}$  to  $W_{\leq w}^{\text{II}}$  by changing the all the  $\pi$ 's to  $\rho$ . For the inverse map, we can change all the  $\rho$ 's to  $\pi$  except for the one immediately before the first  $y$ . Thus this is a one-to-one correspondence and therefore the proposition follows from Proposition 10.3.

We find that the deficiency is not zero when the weight  $w = 3, 4, 5, 6$ . Moreover, none of these missing  $\mathbb{Q}$ -linear relations can be recovered even if we consider all the DBSFs and dualities of weight up to 6.

The only missing relation in weight 3 that cannot be proved is

$$\mathfrak{z}_q^{\text{III}}[1, 0, 1] = 2\mathfrak{z}_q^{\text{III}}[1, 1, 0] - \mathfrak{z}_q^{\text{III}}[1, 2, 0] - \mathfrak{z}_q^{\text{III}}[2, 0, 0] + \mathfrak{z}_q^{\text{III}}[2, 0, 1]. \tag{51}$$

Up to weight 4 there are 10 missing, up to weight 5, 49, and up to weight 6, 210. Below, we will see that all of the 10 missing relations up to weight 4 including (51) can be proved using type G  $q$ -MZVs. Similarly, the deficiency up to weight 5 and 6 can be reduced to 6 and 87, respectively.

**Type IV.** To study type IV  $q$ -MZVs  $\mathfrak{z}_q^{(s_1-1, s_2, \dots, s_d)}[s_1, \dots, s_d]$  we have used the special type II values  $\mathfrak{z}_q^{\text{II}}[1, s_2, \dots, s_d]$  to facilitate us (which can be thought as a kind of regularization). Type IV  $q$ -MZVs together with these values have been called type  $\tilde{\text{IV}}$   $q$ -MZVs.

**Proposition 10.6** For all  $w \geq 1$ , we have

$$\sharp W_{\leq w}^{\text{IV}} = \binom{2w-1}{w} - 1, \quad \sharp W_{\leq w}^{\tilde{\text{IV}}} = \binom{2w}{w} - 1.$$

*Remark 10.7* The first number gives the sequence A010763 according to the On-Line Encyclopedia of Integer Sequences <http://oeis.org>.

*Proof* Notice that type IV-admissible  $q$ -MZVs are in one-to-one correspondence to the set  $\{(x_1, \dots, x_l) \in (\mathbb{Z}_{\geq 0})^l \mid x_1 + \dots + x_l = j, 0 \leq j \leq w-2, 1 \leq l \leq w\}$ . For

**Table 6** Dimension of  $q$ -MZVs of type  $\tilde{IV}$

$w$	1	2	3	4	5	6
$\#W_{\leq w}^{IV}$	1	5	19	69	251	923
Lower bound of $\dim Z_{\leq w}^{IV}$	1	4	12	30	73	173
$\dim DS_{\leq w}^{IV}$	0	1	5	28	124	536
$\dim DU_{\leq w}^{IV} \setminus DS_{\leq w}^{IV}$	0	0	1	1	4	4
Deficiency	0	0	1,0	10,0	50,6	210,87

each fixed  $j$  we see that the number of nonnegative integer solutions of  $x_1 + \dots + x_l = j$  is given by  $\binom{l+j-1}{l-1}$ . But

$$\sum_{l=1}^w \binom{l+j-1}{l-1} = \binom{w+j}{w-1}$$

by a well-known binomial identity. By the proof similar to that of Proposition 10.3 we see that

$$\#W_{\leq w}^{IV} = \sum_{j=0}^{w-2} \binom{w+j}{w-1} = \binom{2w-1}{w} - 1.$$

For the second equation, we note that in the word form we have the additional contribution of the following words:  $\rho y$  and  $\rho y \rho^{s_1} y \dots \rho^{s_d} y$ ,  $|s| < w$ ,  $1 \leq d < w$ . The number of such words is given by ( $i$  = number of  $\rho$ 's,  $j$  = number of  $y$ 's)

$$1 + \sum_{j=0}^{w-2} \sum_{i=0}^{w-1} \binom{i+j}{i} = 1 + \sum_{j=0}^{w-2} \binom{w+j}{w-1} = 1 + \#W_{\leq w}^{IV}.$$

Therefore

$$\#W_{\leq w}^{\tilde{IV}} = 1 + 2\#W_{\leq w}^{IV} = 2\binom{2w-1}{w} - 1 = \binom{2w}{w} - 1.$$

The proposition is now proved.

Type  $\tilde{IV}$   $q$ -MZVs are similar to type II and III in the sense that the deficiency is often nonzero, at least when the weight is less than 6. For example, in weight 3 we have the following identity which cannot be proved using the DBSFs and dualities if we only restrict to type  $\tilde{IV}$   $q$ -MZVs of weight and depth no greater than 3.

$$\mathfrak{z}_q^{IV}[2, 0, 1] = \mathfrak{z}_q^{II}[1, 0, 1] + \mathfrak{z}_q^{III}[1, 2, 0]$$

However this identity follows from weight 4 DBSFs and dualities.

**Table 7** Dimension of  $q$ -MZVs of type IV

$w$	2	3	4	5	6
$\#W_{\leq w}^{IV}$	2	9	34	125	461
Lower bound of $\dim Z_{\leq w}^{IV}$	2	7	20	55	141
$\dim DS_{\leq w}^{IV}$	0	7	9	51	205
$\dim DU_{\leq w}^{IV} \setminus DS_{\leq w}^{IV}$	0	0	0	2	24
Deficiency	0	0	5,0	17,0	91,56

Comparing Tables 5 and 6 we observe that there should be some hidden relations between type III and IV  $q$ -MZVs. Although the dimensions seem to be the same, at least for lower weight, the deficiencies are very different. But using the most general type G values to be considered in a moment, we can make all the deficiencies smaller.

We can now use all of the relations among type IV  $q$ -MZVs to deduce those for type IV and collect the data in Table 7. Furthermore, by converting all the missing relations using type II values we can reduce all the deficiencies up to weight 5 to 0. For weight 6, using type II values we can only reduce the deficiency from 91 to 56. It is possible that this can be further reduced to 0 using weight 7 relations of type II values.

**Type G.** To study the general type G  $q$ -MZVs  $\mathfrak{z}_q^{(a_1, \dots, a_d)}[s_1, \dots, s_d]$  we need all of the following relations we have defined so far: DBSFs, P-R and duality relations.

**Proposition 10.8** For all  $w \geq 1$ , we have

$$\#W_{\leq w}^G = \sum_{1 \leq d \leq k \leq w} \sum_{\substack{x_1 + \dots + x_d = d+k-1 \\ x_1, \dots, x_d \geq 1}} x_1 x_2 \cdots x_d.$$

**Proof** For each fixed depth  $d$  and weight  $k \leq w$ , let  $\mathfrak{z}_q^{(a_1, \dots, a_d)}[s_1, \dots, s_d]$  be a type G-admissible  $q$ -MZV satisfying  $s_1 + \dots + s_d = k, 1 \leq a_1 \leq s_1, 0 \leq a_j \leq s_j$  for all  $j \geq 2$ . When  $s_1, \dots, s_d$  are fixed and  $a_1, \dots, a_d$  vary, the number of such values is given by

$$s_1(s_2 + 1)(s_3 + 1) \cdots (s_d + 1).$$

Hence the proposition follows by setting  $x_1 = s_1, x_2 = s_2 + 1, \dots, x_d = s_d + 1$ .

Let  $\mathbf{P-R}_{\leq w}^G$  be the space generated by all the P-R relations of weight bounded by  $w$ . Then we see that DBSFs are far from enough and both P-R relations and duality relations contribute non-trivially. Table 8 provides our computational data for the lower weight cases. One can see that the number of admissible words increases very fast so that it is very difficult to prove relations of other type  $q$ -MZVs by first finding all the relations for type G  $q$ -MZVs. This is possible theoretically, but not feasible with our current computer powers.

**Table 8** Dimension of  $q$ -MZVs of type G

$w$	1	2	3	4	5	6
$\sharp W_{\leq w}^G$	1	8	49	294	1791	11087
Lower bound of $\dim Z_{\leq w}^G$	1	4	12	30	73	173
$\dim DS_{\leq w}^G$	0	1	8	76	$\leq 608$	
$\dim \mathbf{P}\text{-}\mathbf{R}_{\leq w}^G \setminus (DS_{\leq w}^G \cup DU_{\leq w}^G)$	0	3	27	177	$\leq 1540$	
$\dim DU_{\leq w}^G \setminus (\mathbf{P}\text{-}\mathbf{R}_{\leq w}^G \cup DS_{\leq w}^G)$	0	0	2	8	$\leq 219$	
Deficiency	0	0	0	3, 0		

**Table 9** Dimension of type O  $q$ -MZVs, proved rigorously for  $w \leq 6$  and numerically for  $w \leq 12$

$w$	2	3	4	5	6	7	8	9	10	11	12
$\sharp W_{\leq w}^O$	1	2	4	7	12	20	33	54	88	143	232
Lower bound of $\dim Z_{\leq w}^O$	1	2	4	7	11	18	27	42	63	95	142
$\dim DS_{\leq w}^O \cup DU_{\leq w}^O$	0	0	0	0	1	2	6	12	25	48	90

Fortunately, by using **P-R** relations, all the type G  $q$ -MZVs can be converted to  $\mathbb{Q}$ -linear combinations of type II values. Therefore, the three missing relations in weight 4 must be provable using weight 5 DBSFs, **P-R** and duality relations.

Hence, as we expected, the missing relation (51) for type III  $q$ -MZVs of weight 3 and the 9 missing relations of weight 4 can now be proved. And furthermore, the only one missing relation (50) for type  $\tilde{I}$   $q$ -MZVs of weight 4 can now be proved. We can also obtain the lower bound of  $\dim Z_{\leq w}^G$  from that of type II  $q$ -MZVs.

**Type O.** Using Corollary 6.6 we may regard Okounkov’s  $q$ -MZVs as  $\mathbb{Q}$ -linear combinations of the  $q$ -MZVs  $z_q^a[s]$  for suitable auxiliary variable  $\mathbf{a}$ . Further by using the **P-R** relations we may further reduce this to type II  $q$ -MZVs where we don’t need the letter  $\pi$  (Table 9).

Applying the same idea as above it is possible to verify the following Okounkov’s dimension conjecture, at least when the weight is small.

**Conjecture 10.9** Let  $Z_w^O$  be the  $\mathbb{Q}$ -vector space generated by  $z_q^O[s]$ ,  $|s| \leq w$ . Then

$$\sum_{w=0}^{\infty} t^w \dim Z_{\leq w}^O = \frac{1}{1-t-t^2+t^6+t^8-t^{13}} - \frac{1}{1-t}$$

$$= t^2 + 2t^3 + 4t^4 + 7t^5 + 11t^6 + 18t^7 + 27t^8 + 42t^9 + 63t^{10} + 95t^{11} + 142t^{12} + O(t^{13}).$$

For example, we have verified all of the following  $\mathbb{Q}$ -linearly independent relations in the lower weight cases up to  $q^{100}$ , and we can rigorously prove the first identity (52) involving only weight 4 and 6 values by using the relations we have found for type II  $q$ -MZVs: ( $z = z_q^O$ )

$$\begin{aligned}
4_3[6] &= 3[2, 2] + 12_3[3, 3] - 6_3[4, 2], \\
4_3[7] &= 3[2, 3] + 3[3, 2] + 8_3[3, 4] + 6_3[4, 3] - 4_3[5, 2], \\
3[8] &= 3[2, 4] - 3[6] + 2_3[3, 3] + 6_3[4, 4], \\
9_3[8] &= 3[6] - 6_3[3, 3] + 3_3[4, 2] + 20_3[3, 5] + 16_3[5, 3] - 10_3[6, 2], \\
3[8] &= 2_3[2, 6] - 3[6] + 2_3[3, 3] + 4_3[3, 5] - 16_3[5, 3] \\
&\quad - 6_3[2, 3, 3] + 3_3[2, 4, 2] - 6_3[3, 2, 3] - 3_3[4, 2, 2], \\
4_3[3, 6] &= 3[2, 5] + 4_3[5, 2] + 3_3[3, 4] + 6_3[4, 5] + 8_3[5, 4] + 2_3[7, 2], \\
8_3[9] &= 3[3, 4] - 5_3[2, 5] - 8_3[5, 2] - 30_3[4, 5] - 2_3[4, 3] - 36_3[5, 4] - 10_3[6, 3], \\
6_3[4, 2] &= 10_3[6] + 42_3[8] - 60_3[2, 6] - 12_3[3, 3] - 120_3[3, 5] + 312_3[5, 3] \\
&\quad - 15_3[2, 2, 2] + 180_3[2, 3, 3] - 90_3[2, 4, 2] + 180_3[3, 2, 3] + 60_3[3, 3, 2], \\
72_3[9] &= 62_3[5, 2] + 40_3[2, 5] - 4_3[3, 4] + 40_3[3, 6] - 2_3[4, 3] + 240_3[4, 5] \\
&\quad + 264_3[5, 4] - 5_3[2, 2, 3] - 60_3[3, 3, 3] - 30_3[4, 2, 3], \\
16_3[9] &= 2_3[3, 4] - 10_3[2, 5] - 12_3[2, 7] - 8_3[5, 2] - 60_3[4, 5] - 24_3[5, 4] \\
&\quad + 4_3[2, 3, 2] + 4_3[3, 2, 2] + 3_3[2, 2, 3] + 24_3[2, 3, 4] + 18_3[2, 4, 3] \\
&\quad + 12_3[3, 3, 3] - 12_3[2, 5, 2] + 24_3[3, 2, 4] + 6_3[4, 3, 2], \\
64_3[9] &= 40_3[2, 5] + 20_3[2, 7] - 8_3[3, 4] + 44_3[5, 2] + 20_3[3, 6] - 4_3[4, 3] \\
&\quad + 240_3[4, 5] + 168_3[5, 4] - 5_3[2, 3, 2] - 5_3[2, 2, 3] - 40_3[2, 3, 4] - 30_3[2, 4, 3] \\
&\quad + 20_3[2, 5, 2] - 5_3[3, 2, 2] - 40_3[3, 2, 4] - 100_3[3, 3, 3] + 10_3[3, 4, 2], \\
56_3[9] &= 30_3[2, 5] + 20_3[2, 7] + 26_3[5, 2] - 3[3, 4] + 40_3[3, 6] - 6_3[4, 3] \\
&\quad + 180_3[4, 5] + 112_3[5, 4] - 5_3[2, 2, 3] - 5_3[2, 3, 2] - 5_3[3, 2, 2] - 40_3[2, 3, 4] \\
&\quad + 20_3[5, 2, 2] - 40_3[3, 2, 4] - 30_3[2, 4, 3] + 20_3[2, 5, 2] - 140_3[3, 3, 3].
\end{aligned} \tag{52}$$

Therefore, Conjecture 10.9 is proved rigorously up to weight 6 (inclusive), and verified numerically up to weight 12 (inclusive). The list of relations for weight 10 to 12 is too long to be presented here.

## 11 Conclusions

We have studied various  $q$ -analogs of MZVs in this paper using the uniform method of Rota–Baxter algebras. Among these  $q$ -MZVs, there are many  $\mathbb{Q}$ -linear relations, most of which can be proved using DBSFs, **P-R** and duality relations.

From the data collected in Sect. 10, we have seen that for all of the type  $\tilde{\text{I}}$ ,  $\text{II}$ ,  $\text{III}$  and  $\tilde{\text{IV}}$   $q$ -MZVs duality relations are necessary to generate some  $\mathbb{Q}$ -linear relations among  $q$ -MZVs that are missed by the DBSFs, at least when the weight is large enough. However, the combination of all the DBSFs and dualities are often not exhaustive yet. Sometimes, this difficulty can be overcome by increasing the weight and depth. But this seems to fail in some other cases, for example, for type  $\tilde{\text{I}}$   $q$ -MZVs of weight 4.



We can improve the above situation by considering the more general type G values. The advantage is that we have the new **P-R** relations which provide a lot of new relations between type G  $q$ -MZVs, much more than the DBSFs and dualities combined. The disadvantage is that there are too many type G values so that even when the weight is 5 our computer power is too weak to produce all the necessary relations. However, by using **P-R** relations all type G values can be converted to  $\mathbb{Q}$ -linear combinations of type II values which can be handled by computer a lot easier.

As we mentioned in the introduction our method can be easily adapted to study the  $q$ -MZVs of the following forms:

$$\mathfrak{z}_q^{(a_1, \dots, a_d)}[s_1, \dots, s_d], \quad \mathfrak{z}_q^{(s_1 - a_1, \dots, s_d - a_d)}[s_1, \dots, s_d],$$

where  $a_1 \geq a_2 \geq \dots \geq a_d \geq 0$  are all integers. The monotonicity guarantees that a good stuffle structure can be defined. For  $\mathfrak{z}_q^{(a_1, \dots, a_d)}[s_1, \dots, s_d]$ , we need to use embedding (A) together with shifting operator  $\mathcal{S}_-$  in defining the stuffle and, for  $\mathfrak{z}_q^{(s_1 - a_1, \dots, s_d - a_d)}[s_1, \dots, s_d]$ , we need (B) together with  $\mathcal{S}_+$ .

As an application, we are able to prove Okounkov’s Conjecture 10.9 rigorously up to weight 6 (inclusive), and verify it numerically up to weight 12 (inclusive). It would be more effective if one can define a shuffle structure for type O values themselves and find a relation to the differential operator  $q \frac{d}{dq}$  which should play an important role in the study of these values.

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