The Gamma Function

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Abstract After the so-called elementary functions as the exponential and the trigonometric functions and their inverses, the Gamma function is the most important special function of classical analysis. In this note, we present the definition and properties of the Gamma and the Beta functions.

Keyword Gamma function

Mathematics Subject Classification (2000) 33E50

1 Definition

The Gamma function $\Gamma(z)$ developed by Euler (1707–1783) is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \ \text{Re}(z) > 0. \tag{1.1}$$

If we consider the integral (1.1), it is known that at infinity the behaviour of the exponential function dominates the behaviour of any power function, so that $t^{z-1}e^{-t} \to 0$ as $t \to \infty$ for any value of z, and hence no problem is expected from the upper limit of the integral. When $t \to 0$, we have $e^{-t} \simeq 1$ and then for $c > 0$ very small and $z = x \in \mathbb{R}$, we may write (1.1) as

$$\begin{aligned}\Gamma(x) &\simeq \int_0^c t^{x-1} dt + \int_c^\infty e^{-t} t^{x-1} dt \\ &= \left[\frac{1}{x} t^x\right]_0^c + \int_c^\infty e^{-t} t^{x-1} dt.\end{aligned}$$

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M. Foupouagnigni, W. Koepf (eds.), *Orthogonal Polynomials*,
Tutorials, Schools, and Workshops in the Mathematical Sciences,
https://doi.org/10.1007/978-3-030-36744-2_9

For the first term to remain finite as $t \to 0$, we must have $x > 0$. The main references are [1–5].

2 Properties of the Gamma and Beta Functions

$\Gamma(x) > 0$ for all $x \in (0, \infty)$ and for $x = 1$, we have

$$\Gamma(1) = \int_0^\infty e^{-t} dt = \left[-e^{-t}\right]_0^\infty = 1.$$

Using integration by parts, it follows that for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$,

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt = \left[-e^{-t} t^z\right]_0^\infty + z \int_0^\infty t^{z-1} e^{-t} dt = z\Gamma(z).$$

Using the latter recurrence relation $\Gamma(z + 1) = z\Gamma(z)$ and the initial condition $\Gamma(1) = 1$, it follows that for $z = n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, one gets

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\cdots 2 \cdot 1 \cdot \Gamma(1) = n!.$$

The Gamma function therefore can be seen as an extension of the factorial function to real and complex arguments.

From the recurrence relation $\Gamma(z + 1) = z\Gamma(z)$ we have

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1). \tag{2.1}$$

Since $\Gamma(1) = 1$, we deduce from (2.1) that $\Gamma(0) = \infty$. With this result, we get

$$\Gamma(-1) = \frac{1}{-1}\Gamma(0) = \infty, \;\; \Gamma(-2) = \frac{1}{-2}\Gamma(-1) = \infty, \;\; \text{etc.}$$

That means $\Gamma(n) = \infty$ if n is zero or a negative integer.

The right-hand side of (2.1) is defined for $\mathrm{Re}(z + 1) > 0$, i.e., for $\mathrm{Re}(z) > -1$. By iteration, we get

$$\Gamma(z) = \frac{1}{z(z+1)\cdots(z+n-1)}\Gamma(z+n) \;\; (n \in \mathbb{N}). \tag{2.2}$$

Equation (2.2) enables us to define $\Gamma(z)$ for $\mathrm{Re}(z) > -n$ as an analytic function except for $z = 0, -1, -2, \ldots, -n + 1$. Thus, $\Gamma(z)$ can be continued analytically to the whole complex z-plane except for simple poles at $z = 0, -1, -2, \ldots$. It is now possible to draw a graph of $\Gamma(x)$ $(x \in \mathbb{R})$ as shown in Fig. 1.

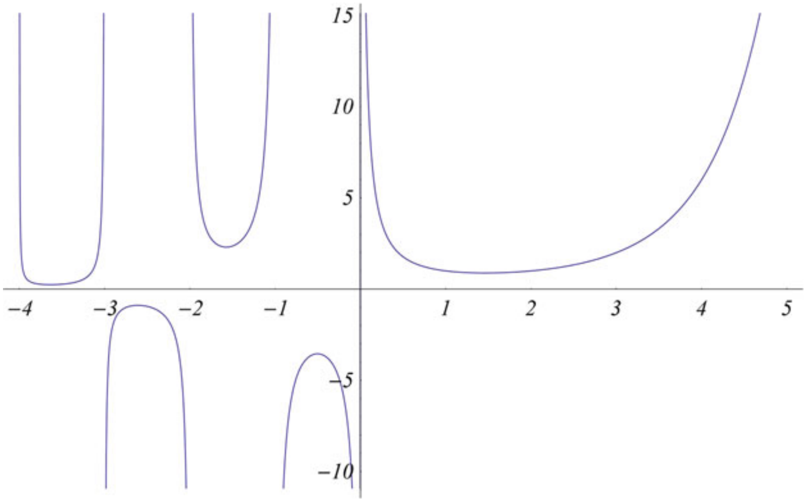

Fig. 1 The Gamma function on the real axis

The shifted factorial

$$(z)_n := z(z+1)\cdots(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)} \ (n \in \mathbb{N}_0), \tag{2.3}$$

which occurs in (2.2), is called the Pochhammer symbol.

At the poles $-n$ $(n \in \mathbb{N}_0)$ of the Gamma function, we get

$$\begin{aligned}\lim_{z\to -n}(z+n)\Gamma(z) &= \lim_{z\to -n}\frac{(z+n)\Gamma(z+n)}{z(z+1)\cdots(z+n-1)} \\ &= \lim_{z\to -n}\frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)} = \frac{(-1)^n}{n!}.\end{aligned}$$

This result may be interpreted as the residue of $\Gamma(z)$ at the simple poles $z=-n$.

We have the identity

$$\begin{aligned}(z)_n &= z(z+1)\cdots(z+n-2)(z+n-1) \\ &= (-1)^n(-z)(-z-1)\cdots(1-z-n+1)(1-z-n) \\ &= (-1)^n(1-z-n)(1-z-n+1)\cdots(1-z-n+n-2)(1-z-n+n-1) \\ &= (-1)^n(1-z-n)_n.\end{aligned}$$

Using the definition (2.3), we can rewrite this as

$$\frac{\Gamma(z+n)}{\Gamma(z)} = (-1)^n\frac{\Gamma(1-z)}{\Gamma(1-z-n)}$$

or equivalently

$$\Gamma(z)\Gamma(1-z) = (-1)^n\Gamma(z+n)\Gamma(1-(z+n)).$$

Since $z = 0, -1, -2, \ldots$ are the poles of $\Gamma(z)$, we deduce that $1/\Gamma(z)$ is analytic in the entire complex plane with zeros $0, -1, -2, \ldots$. It follows that the zeros of $1/\Gamma(1-z)$ are $1, 2, \ldots$. This means that

$$\frac{1}{\Gamma(z)\Gamma(1-z)}$$

is analytic in the entire complex plane with zeros $\ldots, -2, -1, 0, 1, 2, \ldots$ similar as the function $\sin(\pi z)$. It can be shown that the following relationship between the Gamma and circular functions is valid, where the last statement is the Euler product for the sine function:

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{\sin(\pi z)}{\pi} = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right). \tag{2.4}$$

One similarly has

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{(z)_{n+1}} = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{\frac{z}{n}},$$

where

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right) \approx 0.57721\,56649\,01532\,86060\,65120\,90082$$

denotes the Euler–Mascheroni constant. Equation (2.4) is called the reflection formula of the Gamma function.

Equation (2.4) with $z = \frac{1}{2}$ yields immediately

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{2.5}$$

which, in view of (1.1), implies

$$\int_0^{\infty}\frac{e^{-t}}{\sqrt{t}}dt = \sqrt{\pi}.$$

Equation (2.2) combined with (2.5) yields

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!},\ \Gamma\left(-n+\frac{1}{2}\right) = (-1)^n\sqrt{\pi}\frac{2^{2n}n!}{(2n)!},\ n\in\mathbb{N}_0.$$

If we set $t = u^2$ in the definition (1.1) so that $dt = 2udu$, we get

$$\Gamma(z) = 2\int_0^\infty e^{-u^2}(u^2)^{z-1}udu = 2\int_0^\infty e^{-u^2}u^{2z-1}du.$$

This means that

$$\Gamma(z) = 2\int_0^\infty e^{-t^2}t^{2z-1}dt, \tag{2.6}$$

and for $z = \frac{1}{2}$, we derive the following result

$$\int_0^\infty e^{-t^2}dt = \frac{1}{2}\sqrt{\pi}.$$

The binomial coefficients can be expressed as

$$\binom{z}{n} = \frac{z(z-1)\cdots(z-n+1)}{n!} = (-1)^n\frac{(-z)_n}{n!},$$

or, equivalently, as

$$\binom{z}{n} = \frac{\Gamma(z+1)}{n!\Gamma(z-n+1)},$$

for arbitrary $z \in \mathbb{C}$, $z+1 \neq 0, -1, \ldots,$ and $z-n+1 \neq 0, -1, \ldots$. Since $\Gamma(-k) = \infty$ for $k \in \mathbb{N}_0$, we may set $1/\Gamma(-k) = 0$ which reads again as $1/k! = 0$ for $k = -1, -2, \ldots$. We deduce that for $k, n \in \mathbb{N}_0$, we have

$$\binom{n}{k} = 0 \text{ for } k < 0 \text{ and } k > n.$$

3 The Beta Function

The Beta function is defined by the integral

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, \ \mathrm{Re}(z) > 0, \ \mathrm{Re}(w) > 0. \tag{3.1}$$

The substitution $t = 1-u$ shows that

$$B(z, w) = \int_0^1 u^{w-1}(1-u)^{z-1}du = B(w, z). \tag{3.2}$$

By setting $t = \cos^2\theta$ so that $dt = -2\cos\theta\sin\theta d\theta$, we get

$$\begin{aligned} B(z, w) &= \int_{\pi/2}^{0} (\cos^2\theta)^{z-1}(\sin^2\theta)^{w-1}(-2\cos\theta\sin\theta)d\theta \\ &= 2\int_0^{\pi/2} \cos^{2z-1}\theta\,\sin^{2w-1}\theta d\theta. \end{aligned} \tag{3.3}$$

Now we want to show that

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{3.4}$$

We first consider the product

$$\Gamma(z)\,\Gamma(w) = \int_0^\infty t^{z-1}\,e^{-t}\,dt \cdot \int_0^\infty u^{w-1}\,e^{-u}\,du$$

and use the substitutions $t = x^2$ and $u = y^2$ to obtain

$$\begin{aligned} \Gamma(z)\,\Gamma(w) &= 4\int_0^\infty e^{-x^2}\,x^{2z-1}\,dx \int_0^\infty e^{-y^2}\,y^{2w-1}\,dy \\ &= 4\int_0^\infty\int_0^\infty e^{-x^2-y^2}\,x^{2z-1}\,y^{2w-1}\,dx\,dy. \end{aligned}$$

Applying polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ to this double integral, we get

$$\begin{aligned} \Gamma(z)\,\Gamma(w) &= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r^{2z+2w-2}\,\cos^{2z-1}\theta\cdot\sin^{2w-1}\theta\cdot r\,dr\,d\theta \\ &= 2\int_0^\infty e^{-r^2}\,r^{2z+2w-1}\,dr\cdot 2\int_0^{\pi/2}\cos^{2z-1}\theta\cdot\sin^{2w-1}\theta\,d\theta \\ &= \Gamma(z+w)\,B(w, z) = \Gamma(z+w)\,B(z, w) \end{aligned}$$

where Eqs. (2.6) and (3.3) are utilized. This proves (3.4).

Relation (3.4) not only confirms the symmetry property in (3.2), but also continues the Beta function analytically for all complex values of z and w, except when $z, w \in \{0, -1, -2, \ldots\}$. Thus we may write

$$B(z, w) = \begin{cases} \int_0^1 t^{z-1}(1-t)^{w-1}dt & (\mathrm{Re}(z) > 0,\ \mathrm{Re}(w) > 0) \\ \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} & (\mathrm{Re}(z) < 0,\ \mathrm{Re}(w) < 0,\ z, w \notin \{0, -1, -2, \ldots\}). \end{cases}$$

The following relations are valid:

$$B(z+1, w) = \frac{z}{z+w} B(z, w),$$
$$B(z, w+1) = \frac{w}{z+w} B(z, w).$$

Indeed, we have

$$\begin{aligned} B(z+1, w) &= \frac{\Gamma(z+1)\Gamma(w)}{\Gamma(z+w+1)} \\ &= \frac{z\Gamma(z)\Gamma(w)}{(z+w)\Gamma(z+w)} \\ &= \frac{z}{z+w} B(z, w). \end{aligned}$$

For further reading on the Gamma and Beta functions, one might go through the following books. This article presents the most important part of [3, Chap. 1].

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