An Introduction to Orthogonal Polynomials

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Abstract In this introductory talk, we first revisit with proof for illustration purposes some basic properties of a specific system of orthogonal polynomials, namely the Chebyshev polynomials of the first kind. Then we define the notion of orthogonal polynomials and provide with proof some basic properties such as: The uniqueness of a family of orthogonal polynomials with respect to a weight (up to a multiplicative factor), the matrix representation, the three-term recurrence relation, the Christoffel-Darboux formula and some of its consequences such as the separation of zeros and the Gauss quadrature rules.

Keywords Orthogonal polynomials · Differential equations · Chebyshev polynomials · Zeros

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1 Introduction: An Example of a Family of Orthogonal Polynomials

Univariate orthogonal polynomials (or orthogonal polynomials for short) are systems of polynomials $(p_n)_n$ with $\deg(p_n) = n$, satisfying a certain orthogonality relation. They are very useful in practice in various domains of mathematics, physics, engineering and so on, because of the many properties and relations they satisfy. As examples of areas where orthogonal polynomials play important roles, I could list approximation theory (see [5, 23]) and also numerical analysis (see for example [9, 10]). Among those relations, we can mention the following, with the first seven valid for all families of orthogonal polynomials. The last three are in general valid for some specific families of orthogonal polynomials, the so-called classical orthogonal polynomials (see [1–3, 6, 7, 12, 14]) and the preliminary training given by S. Mboutngam, M. Kenfack Nangho and P. Njionou Sadjang of these proceedings):

- Orthogonality relation
- Matrix representation
- Three-term recurrence relation
- Christoffel-Darboux formula
- Separation of zeros
- Gauss quadrature
- Generating functions
- Second-order holonomic differential, difference or q-difference equation
- Rodrigues formula
- Expansion of functions which are continuous differentiable and square integrable, in terms of Fourier series of OP.

Before going into details and for illustration purposes, let us give a concrete example of a family of orthogonal polynomials, then state and prove some of its properties most of which are common to any family of orthogonal polynomials.

Theorem 1.1 (Chebyshev Polynomials of the First Kind [17, 21]) *The polynomial family $(T_n)_n$ defined by (and called Chebyshev polynomials of the first kind or Chebyshev polynomials for short as we will study only this family in this article)*

$$T_n(x) = \cos(n\theta),\ x = \cos\theta,\ 0 < \theta < \pi,\ n \in \mathbb{N}, \tag{1.1}$$

fulfills the following properties:

1. *T_n is a polynomial of degree n in x with leading coefficient $a_n = 2^{n-1}$, satisfying the following recurrence relation (called three-term recurrence relation)*

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x),\ n \ge 1,\ T_0(x) = 1,\ T_1(x) = x; \tag{1.2}$$

2. *$(T_n)_n$ satisfies the following relation (called orthogonality relation)*

$$\int_0^{\pi} \cos(n\theta)\cos(m\theta)d\theta = k_n\delta_{n,m} = \int_{-1}^{1} T_n(x)\, T_m(x)\frac{dx}{\sqrt{1-x^2}}, \tag{1.3}$$

with $k_0 = \pi,\ k_n = \frac{\pi}{2},\ n \geq 1$.

3. *T_n satisfies the second-order holonomic differential equation:*

$$(1-x^2)\, T_n''(x) - x\, T_n'(x) + n^2\, T_n(x) = 0,\ n \geq 0. \tag{1.4}$$

4. *For any $n \geq 1$, T_n has exactly n zeros, all belonging to the interval of orthogonality $(-1,\ 1)$. Those zeros, ranked in increasing order, are given by:*

$$x_{n,k} = \cos\left(\frac{2(n-k)+1}{2n}\pi\right),\ 1 \leq k \leq n,\ n \geq 1. \tag{1.5}$$

5. *The zeros $x_{n,k}$ of T_n satisfy*

$$x_{n,j} \neq x_{n+1,k},\ \forall n \geq 1,\ 1 \leq j \leq n,\ 1 \leq k \leq n+1; \tag{1.6}$$

$$x_{n+1,k} < x_{n,k} < x_{n+1,k+1},\ 1 \leq k \leq n. \tag{1.7}$$

6. *The monic Chebyshev polynomial of degree $n \geq 1$ is the polynomial deviating least from zero on $[-1, 1]$ among all monic polynomials of degree n:*

$$\min\left\{\max_{-1\leq x\leq 1} |q_n(x)|,\ q_n \in \mathbb{R}[x], q_n(x) = x^n + \ldots\right\} = \max_{-1\leq x\leq 1}\left|\frac{T_n(x)}{2^{n-1}}\right| = \frac{1}{2^{n-1}}, \tag{1.8}$$

where $\mathbb{R}[x]$ is the ring of polynomials with real coefficients.

7. *The following property, which is called the Gauss quadrature formula for the specific case of the Chebyshev polynomials, is valid*

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \frac{\pi}{n}\sum_{k=1}^{n} f(x_{n,k}),\ \forall f \in \mathbb{R}_{2n-1}[x],\ n \geq 1, \tag{1.9}$$

$\mathbb{R}_{2n-1}[x]$ is the ring of polynomials of degree at most $2n-1$, with real coefficients. In addition, the integral of any function continuous on the compact interval $[-1,\ 1]$ can be approximated by the previous formula:

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \lim_{n\to\infty}\frac{\pi}{n}\sum_{k=1}^{n} f(x_{n,k}),\ \forall f \in \mathcal{C}[-1,\ 1], \tag{1.10}$$

where $\mathcal{C}[-1,\ 1]$ is the set of continuous functions on the interval $[-1,\ 1]$.

Proof Let us provide a quick proof of the first six above properties.

Proof of Property 1 Equation (1.2) is obtained by direct computation:

$$T_0(x) = \cos(0) = 1,\ \ T_1(x) = \cos\theta = x,$$

and

$$T_{n+1}(x) + T_{n-1}(x) = \cos(n+1)\theta + \cos(n-1)\theta = 2\,\cos\theta\,\cos(n\theta) = 2\,x\,T_n(x),$$

using the cosine addition formula $\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b$.

Next, we now prove by induction that T_n is a polynomial of degree n in the variable x with 2^{n-1} as leading coefficient, that is

$$T_n(x) = 2^{n-1}\,x^n + \text{lower degree terms},\ n \geq 1. \tag{1.11}$$

For $n = 1$, Eq. (1.11) is satisfied as $T_1(x) = x = 2^{1-1}\,x$ and its degree is 1. By assuming that Eq. (1.11) is satisfied for a fixed integer $n \geq 1$, we can then write T_n as $T_n(x) = 2^{n-1}\,x^n + A_{n-1}(x)$ where A_{n-1} is a polynomial of degree at most $n-1$ in the variable x. We complete the proof by using relation (1.2) to obtain that

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x) = 2x\,(2^{n-1}\,x^n + A_{n-1}(x)) - T_{n-1}(x) = 2^n\,x^{n+1} + \tilde{A}_n(x),$$

where $\tilde{A}_n$ is a polynomial of degree at most n in x. Therefore, T_n is a polynomial of degree n in the variable x with 2^{n-1} as leading coefficient.

From the three-term recurrence relation (1.2), one can generate any T_n; and in particular, the first 10 Chebyshev polynomials are given by:

$$\begin{aligned}
T_0(x) &= 1,\\
T_1(x) &= x;\\
T_2(x) &= 2\,x^2 - 1,\\
T_3(x) &= 4\,x^3 - 3\,x,\\
T_4(x) &= 8\,x^4 - 8\,x^2 + 1,\\
T_5(x) &= 16\,x^5 - 20\,x^3 + 5\,x,\\
T_6(x) &= 32\,x^6 - 48\,x^4 + 18\,x^2 - 1,\\
T_7(x) &= 64\,x^7 - 112\,x^5 + 56\,x^3 - 7\,x,\\
T_8(x) &= 128\,x^8 - 256\,x^6 + 160\,x^4 - 32\,x^2 + 1,\\
T_9(x) &= 256\,x^9 - 576\,x^7 + 432\,x^5 - 120\,x^3 + 9\,x.
\end{aligned} \tag{1.12}$$

Proof of Property 2 Relation (1.3) is proved by direct computation using again the addition formula

$$2\cos(n\theta)\ \cos(m\theta) = \cos(n+m)\theta + \cos(n-m)\theta,$$

and the fact that $x = \cos\theta,\ 0 < \theta < \pi \Longrightarrow dx = -\sin\theta\, d\theta = -\sqrt{1-\cos^2\theta}\, d\theta$.

Proof of Property 3 Relation (1.4) is also proved by direct computation. In fact

$$\begin{aligned}
T_n'(x) &= \frac{d}{dx}T_n(x) = \frac{d\theta}{dx}\frac{d}{d\theta}T_n(x) = \frac{-1}{\sin\theta}\frac{d}{d\theta}\cos(n\theta) = \frac{n\ \sin(n\theta)}{\sin\theta},\ n \ge 1,\\
T_n''(x) &= \frac{d}{dx}\frac{d}{dx}T_n(x)\\
&= \frac{d\theta}{dx}\frac{d}{d\theta}\left(\frac{d\theta}{dx}\frac{d}{d\theta}T_n(x)\right)\\
&= \frac{-1}{\sin\theta}\frac{d}{d\theta}\left(\frac{-1}{\sin\theta}\frac{d}{d\theta}\cos(n\theta)\right)\\
&= \frac{n\cos\theta\ \sin(n\theta)}{\sin\theta\ \sin^2\theta} + \frac{-n^2\cos(n\theta)}{\sin^2\theta}\\
&= \frac{x\ T_n'(x)}{1-x^2} + \frac{-n^2\ T_n(x)}{1-x^2},\ n \ge 1.
\end{aligned}$$

Proof of Property 4 To obtain the zeros of T_n, we solve the following equation for a fixed $n \ge 1,\ x \in (-1,\ 1)$ and $\theta \in (-\pi\ \pi)$.

$$T_n(x) = 0 \Longleftrightarrow \cos(n\theta) = 0 \Longleftrightarrow n\theta = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}.$$

Since $0 < \theta < \pi$, then $0 \le k \le n-1$. Therefore, T_n has exactly n zeros which are $\cos\left(\frac{(2k+1)\pi}{2n}\right),\ 0 \le k \le n-1$. But since those zeros are ranked by decreasing order for the function $\theta \to \cos\theta$ is decreasing on $(-\pi,\ \pi)$ and the sequence $k \to \frac{(2k+1)\pi}{2n}$ is increasing, there is a need to reverse the order. This is done by replacing k by $n-k$. Therefore we obtain the following zeros ranked by increasing order

$$x_{n,k} = \cos\theta_{n,k},\ \text{with}\ \theta_{n,k} = \frac{2(n-k)+1}{2n}\pi,\ 1 \le k \le n.$$

The zeros also belong to the interval of orthogonality $(-1,\ 1)$.

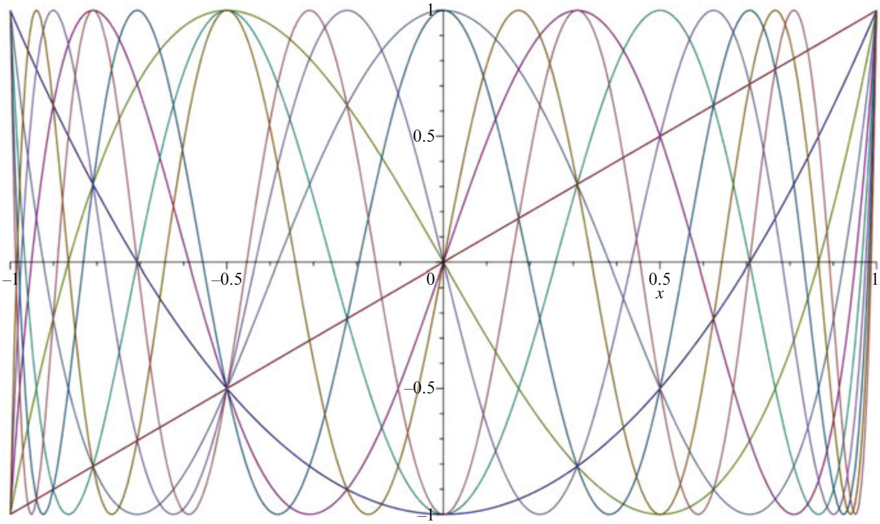

Fig. 1 The first 10 Chebyshev polynomials

Proof of Property 5 Equation (1.6) is satisfied since the cosine function is a bijection from $(0, \pi)$ into $(-1, 1)$ and

$$\theta_{n,j} \neq \theta_{n+1,k}, \; \forall n \geq 1, \; 1 \leq j \leq n, \; 1 \leq k \leq n+1.$$

The inequalities (1.7) are deduced using the fact that the cosine function is strictly decreasing in $(-\pi, \pi)$ combined with the following inequalities which can be obtained by a direct and quick computation

$$\theta_{n+1,k+1} < \theta_{n,k} < \theta_{n+1,k}, \; 1 \leq k \leq n.$$

The interlacing properties of the zeros of the Chebyshev polynomials can be observed on the above graph of the first ten Chebyshev polynomials (Fig. 1).

Proof of Property 6 Let us first denote the monic Chebyshev polynomial of degree n by t_n: $t_n(x) = \frac{T_n(x)}{2^{n-1}}, \; n \geq 1, \; t_0(x) = T_0(x) = 1$. Next, we define the set of monic polynomials of degree n, $\mathcal{P}_n$, the sup-norm $||.||_{max}$ and the subset I of the set of real numbers $\mathbb{R}$, respectively, by

$$\begin{aligned}
\mathcal{P}_n &= \left\{q_n \in \mathbb{R}_n[x], q_n(x) = x^n + \text{lower degree terms}\right\} \\
||p||_{\max} &= \max_{-1 \leq x \leq 1} |p(x)|, \\
I &= \{||p||_{\max}, \; p \in \mathcal{P}_n\}.
\end{aligned}$$

To prove that $\min I = ||t_n||_{\max} = \frac{1}{2^{n-1}}$, for a fixed but arbitrary integer $n \geq 1$, we proceed as follows:

– In the first step, we derive the extrema for the function t_n:

$$t_n'(x) = 0 \Longleftrightarrow \frac{n\sin(n\theta)}{\sin(\theta)} = 0 \Longleftrightarrow \sin(n\theta) = 0,\ \sin\theta \neq 0.$$

Since $0 \leq \theta \leq \pi$, we get $\theta = \frac{k\pi}{n}$, $1 \leq k \leq n-1$. We have excluded $k = 0$ and $k = n$ to make sure that $\sin\theta \neq 0$. The extrema for t_n are therefore

$$z_{n,k} = \cos\left(\frac{k\pi}{n}\right),\ 1 \leq k \leq n-1.$$

– In the second step, we study the sign of $t_n(x)$ on the extrema. Before this, we remark that for $\theta = 0,\ x = 1 := z_{n,0}$ and for $\theta = \pi,\ x = -1 := z_{n,n}$, enabling us to get the following information on the action of t_n on $z_{n,k}$:

$$t_n(-1) = \frac{\cos(n\,\pi)}{2^{n-1}} = \frac{(-1)^n}{2^{n-1}} = t_n(z_{n,n}), \tag{1.13}$$

$$t_n(1) = \frac{\cos(n\,0)}{2^{n-1}} = \frac{1}{2^{n-1}} = t_n(z_{n,0}), \tag{1.14}$$

$$t_n(z_{n,k}) = \frac{(-1)^k}{2^{n-1}},\ 1 \leq k \leq n-1. \tag{1.15}$$

Equations (1.13) and (1.14) confirm that Eq. (1.15) which was initially valid for $1 \leq k \leq n-1$ is also valid for $k = 0,\ n$ and can then be written as

$$t_n(z_{n,k}) = \frac{(-1)^k}{2^{n-1}},\ 0 \leq k \leq n.$$

The previous equation, combined with the fact that $t_n(x) = \frac{\cos(n\theta)}{2^{n-1}}$, allows us to deduce that

$$||t_n||_{\max} = \frac{1}{2^{n-1}}.$$

1. In the third step, we remark that the set I is not empty since it contains $||t_n||_{\max}$. In addition, it has zero as a lower bound. Let us assume that $||t_n||_{\max}$ is not the minimum element of I. Then there exists a polynomial q belonging to $\mathcal{P}_n$ such that

$$-\frac{1}{2^{n-1}} < q(x) < \frac{1}{2^{n-1}},\ -1 \leq x \leq 1.$$

We next set $P_{n-1}(x) = t_n(x) - q(x)$ and observe, taking into account the previous inequalities, that P_{n-1} which is a polynomial of degree at most $n-1$ fulfills the following properties:

$$P_{n-1}(z_{n,2j}) = t_n(z_{n,2j}) - q(z_{n,2j}) = \frac{1}{2^{n-1}} - q(z_{n,2j}) > 0,$$

$$P_{n-1}(z_{n,2j+1}) = t_n(z_{n,2j+1}) - q(z_{n,2j+1}) = \frac{-1}{2^{n-1}} - q(z_{n,2j+1}) < 0,$$

for any integer j such that $0 \leq 2j+1 \leq n$. We obtain a contradiction to the fact that the polynomial P_{n-1} which is of degree at most $n-1$ will have n zeros for it will change its sign n times in the intervals $(z_{n,k},\ z_{n,k+1}),\ k = 0 \ldots n-1$. We therefore conclude that $||t_n||_{\max} = \frac{1}{2^{n-1}}$ is the minimum of I.

Proof Illustration of Property 7 The Gauss formula (1.9) is given in the general case in the paper by A. S. Jooste in these proceedings (see also [3, 6, 12, 22], but one would need to proceed with additional careful computations to verify that the Christoffel number $\lambda_{n,k}$ in the general Gauss quadrature formula is given by $\lambda_{n,k} = \frac{\pi}{n}$ for the specific case of the Chebyshev polynomials T_n (see also [15], Theorem 8.4, where the Christoffel numbers have been given explicitly for Chebyshev polynomials of the first, second, third and fourth kinds). We refer to [6], page 33 and also to [3], page 252 for the proof of relation (1.10) and other approximation formulas.

□

2 Construction of a System of Orthogonal Polynomials

In this section, after having provided a concrete example of a family of orthogonal polynomials with proof of some of its nice properties—some of which are common for any family of orthogonal polynomials—, we will now show how to construct a family of orthogonal polynomials from a scalar product and then relate this with the definition of orthogonal polynomials.

Let us consider a scalar product (,) defined on $\mathbb{R}[x] \times \mathbb{R}[x]$ in terms of a Stieltjes integral as

$$(p, q) = \int_a^b p(x)\, q(x)\, d\alpha(x), \tag{2.1}$$

where $\mathbb{R}[x]$ is the ring of polynomials with a real variable and $d\alpha$ is a non-negative Borel measure supported in the interval $(a,\ b)$. As scalar product, it fulfills the following properties:

$$(p, p) \geq 0,\ \forall p \in \mathbb{R}[x],\ \text{and}\ (p, p) = 0 \Longrightarrow p = 0,$$
$$(p, q) = (q, p),\ \forall p,\ q \in \mathbb{R}[x],$$
$$(\lambda\, p, q) = \lambda\,(p, q),\ \forall \lambda \in \mathbb{R},\ \forall p,\ q \in \mathbb{R}[x],$$
$$(p + q, r) = (p, r) + (q, r),\ \forall p,\ q,\ r \in \mathbb{R}[x].$$

As an example of scalar product on $\mathbb{R}[x]$ with connection to known systems of orthogonal polynomials, we mention:

$$(p, q) = \int_{-1}^{1} p(x)\, q(x) \frac{dx}{\sqrt{1 - x^2}}, \tag{2.2}$$

which yields the Chebyshev orthogonal polynomials.

The following theorem provides a method for construction of a family of polynomials, orthogonal with respect to a given scalar product. It is called Gram-Schmidt orthogonalisation process.

Theorem 2.1 (Gram-Schmidt Orthogonalisation Process [6, 12, 22]) *The polynomial systems $(q_n)_n$ and $(p_n)_n$ defined recurrently by the relations*

$$q_0 = 1,\ q_n = x^n - \sum_{k=0}^{n-1} \frac{(x^n, q_k)}{(q_k, q_k)} q_k,\ n \geq 1,\ p_k = \frac{q_k}{\sqrt{(q_k, q_k)}},\ k \geq 0, \tag{2.3}$$

satisfy the relations

$$\deg(q_n) = \deg(p_n) = n, \forall n \geq 0, \tag{2.4}$$
$$(q_n, q_m) = 0,\ n \neq m,\ (q_n, q_n) \neq 0,\ \forall n \geq 0, \tag{2.5}$$
$$(p_n, p_m) = 0,\ n \neq m,\ (p_n, p_n) = 1,\ \forall n \geq 0. \tag{2.6}$$

The polynomials $(q_n)_n$ and $(p_n)_n$ are said to be orthogonal or orthonormal with respect to the scalar product $(\,,\,)$, respectively. In fact, they represent the same polynomial system with different normalisation: $(q_n)_n$ is monic*—to say the coefficient of the leading monomial is equal to 1; while $(p_n)_n$ is orthonormal—to say $(p_n, p_n) = 1$ or the corresponding norm of p_n is equal to 1.*

Proof Equation (2.4) is obvious while Eq. (2.6) is a direct consequence of Eq. (2.5). We will prove Eq. (2.5) by induction. Because of the properties of the scalar product, we just need to prove the following:

$$(q_n, q_m) = 0,\ \forall n \geq 1,\ 0 \leq m \leq n - 1. \tag{2.7}$$

For $n = 1$, we have, using relations (2.3)

$$(q_1, q_0) = \left(x - \frac{(x, q_0)}{(q_0, q_0)}\, q_0, q_0 \right) = (x, q_0) - \frac{(x, q_0)}{(q_0, q_0)}\, (q_0, q_0) = 0.$$

We now assume that relation (2.7) is satisfied up to a given $n \geq 1$. Let $m \in \mathbb{N}$, $0 \leq m \leq n$.

$$\begin{aligned}(q_{n+1}, q_m) &= \left(x^{n+1} - \sum_{k=0}^{n} \frac{(x^{n+1}, q_k)}{(q_k, q_k)}\, q_k, q_m \right)\\ &= (x^{n+1}, q_m) - \sum_{k=0}^{n} \frac{(x^{n+1}, q_k)}{(q_k, q_k)}\, (q_k, q_m)\\ &= (x^{n+1}, q_m) - \frac{(x^{n+1}, q_m)}{(q_m, q_m)}\, (q_m, q_m) = 0,\end{aligned}$$

since from the induction hypothesis (Eq. (2.7)) and the symmetry of the scalar product, $(q_k, q_m) = 0,\ 0 \leq k \neq m \leq n-1$. □

Definition 2.2 (Orthogonal Polynomials [6]) Any sequence of polynomials $(p_n)_n$ satisfying Eqs. (2.4) and (2.5) (rewritten as follows with q_n replaced by p_n)

$$\deg(p_n) = n, \tag{2.8}$$

$$\int_a^b p_n(x)\, p_m(x)\, d\alpha(x) = 0,\ n \neq m, \tag{2.9}$$

$$\int_a^b p_n(x)\, p_n(x)\, d\alpha(x) \neq 0,\ \forall n \geq 0, \tag{2.10}$$

is said to be orthogonal with respect to the measure $d\alpha$, and called an orthogonal polynomial system or an orthogonal polynomial for short.

Definition 2.3 (Orthogonal Polynomials w.r.t. to a Weight Function [3, 6, 12, 17, 18, 21]) When the measure $d\alpha$ is absolutely continuous, that is $d\alpha(x) = \rho(x)\, dx$ where ρ is an appropriate function—called *weight function*, then relations (2.8)–(2.10) read

$$\deg(p_n) = n, \tag{2.11}$$

$$\int_a^b p_n(x)\, p_m(x)\, \rho(x)\, dx = 0,\ n \neq m, \tag{2.12}$$

$$\int_a^b p_n(x)\, p_n(x)\, \rho(x)\, dx \neq 0,\ \forall n \geq 0. \tag{2.13}$$

The polynomial system $(p_n)_n$ is said to be orthogonal with respect to the *weight function* ρ. Because of the form of the orthogonality relation, the variable here is continuous. We therefore obtain orthogonal polynomials of a continuous variable.

Definition 2.4 (Orthogonal Polynomials of a Discrete Variable [12, 18]) When the measure $d\alpha$ is discrete and supported in $\mathbb{N}$, that is, $\alpha = \rho$ on $\mathbb{N}$, then the relations (2.8), (2.9) and (2.10) become

$$\deg p_n = n,\ n \geq 0,$$

$$\sum_{k=0}^{N} \rho(k)\, p_n(k)\, p_m(k) = 0,\ \forall n,\ m \in \mathbb{N},\ n \neq m,$$

$$\sum_{k=0}^{N} \rho(k)\, p_n(k)\, p_n(k) \neq 0,\ n \geq 0,$$

where the parameter N belongs to $\mathbb{N} \cup \{\infty\}$. $(p_n)_n$ is said to be orthogonal with respect to the discrete weight ρ. It is also called a sequence of orthogonal polynomials of a discrete variable.

Notice that if N is finite, then there exist only a finite number of orthogonal polynomials, this because the bilinear application defined in (2.1) is positive definite not on the entire $\mathbb{R}[x]$ but rather on its linear subspace, $\mathbb{R}_l[x]$, for an appropriate choice of the positive integer l.

Definition 2.5 (Orthogonal Polynomials of a q-Discrete Variable [8, 11, 13, 18]) When the measure $d\alpha$ is q-discrete and supported in $q^{\mathbb{Z}}$, that is, $\alpha = \rho$ on $q^{\mathbb{Z}}$, where $\mathbb{Z}$ is the set of integers, then the relations (2.8), (2.9) and (2.10) become

$$\deg p_n = n,\ n \geq 0,$$

$$\sum_{k=0}^{N} \rho(q^k)\, p_n(q^k)\, p_m(q^k) = 0,\ \forall n,\ m \in \mathbb{N},\ n \neq m,$$

$$\sum_{k=0}^{N} \rho(q^k)\, p_n(q^k)\, p_n(q^k) \neq 0,\ n \geq 0,$$

where the parameter N belongs to $\mathbb{N} \cup \{\infty\}$. $(p_n)_n$ is said to be orthogonal with respect to the q-discrete weight ρ. It is also called a sequence of orthogonal polynomials of a q-discrete variable.

Remark 2.6 When the measure $d\alpha$ is discrete or q-discrete supported on a quadratic or a q-quadratic lattice, this gives the orthogonal polynomials of a quadratic or a q-quadratic variable. As examples of such polynomials, we mention the Wilson and the Askey-Wilson polynomials [4, 8, 12, 13].

3 Basic Properties of Orthogonal Polynomials

3.1 The Uniqueness of a Family of Orthogonal Polynomials

Before stating the result about the uniqueness of a family of orthogonal polynomials, let us start with the following remarks:

1. If $(p_n)_n$ is a family of orthogonal polynomials, then due to the fact that the degree of each p_n is equal to n, any subset of $\{p_n, n \in \mathbb{N}\}$ is a linearly independent subset of the linear space $\mathbb{R}[x]$.
2. Moreover, for any $n \geq 1$, the set $\{p_k,\ 0 \leq k \leq n\}$, like the canonical basis of monomials $\{x^k,\ 0 \leq k \leq n\}$, constitutes a basis of the linear space $\mathbb{R}_n[x]$ of polynomials of degree at most n.

The following result states an equivalent orthogonality relation.

Lemma 3.1 *Let $(p_n)_n$ be a sequence of polynomials with $\deg(p_n) = n$, $n \geq 0$. Then Eqs. (2.9) and (2.10) are equivalent to the two following equations*

$$\int_a^b p_n(x)\, x^m\, d\alpha(x) = 0,\ \forall n \geq 1,\ 0 \leq m \leq n-1, \tag{3.1}$$

$$\int_a^b p_n(x)\, x^n\, d\alpha(x) \neq 0,\ \forall n \geq 0. \tag{3.2}$$

Proof The proof is obtained by combining the orthogonality relations (2.9) and (2.10) or (respectively (3.1) and (3.2)) with the expansion of the polynomial p_m in the canonical basis of monomials (respectively the expansion of x^m in the basis $\{p_k,\ 0 \leq k \leq n\}$). □

The uniqueness of a family of polynomials orthogonal with respect to a measure $d\alpha$ can then be stated as follows.

Theorem 3.2 (Uniqueness of a Family of Orthogonal Polynomials) *To the measure $d\alpha$ corresponds a unique (up to a multiplicative factor) family of orthogonal polynomials. Or equivalently, if $(p_n)_n$ and $(q_n)_n$ are two families of polynomials satisfying relations (2.8)–(2.10), then they are proportional, to say that there exists a sequence $(b_n)_n$ such that $p_n = b_n\, q_n$, $n \geq 0$, with $b_n \neq 0$, $n \geq 0$.*

Proof The proof is obtained by expanding the polynomial q_n in the basis $\{p_k,\ 0 \leq k \leq n\}$ of $\mathbb{R}_n[x]$ and using the orthogonality relations (2.9) and (2.10) to show that the other coefficients, except the leading one, are equal to zero. □

3.2 The Matrix Representation

The following results give information about the Hankel determinant and a matrix representation of a given family of orthogonal polynomials. Before that, let us define what we mean by linear functional and orthogonality with respect to a linear functional.

Definition 3.3 (Linear Functional) Linear functional here means any linear mapping from $\mathbb{R}[x]$ to $\mathbb{R}$.

The sequence of polynomials $(p_n)_n$ will be said to be orthogonal with respect to the linear functional $\mathcal{U}$ if $\deg(p_n) = n$ and

$$\langle \mathcal{U}, x^m p_n \rangle = 0,\ n \geq 0,\ 0 \leq m \leq n, \tag{3.3}$$

$$\langle \mathcal{U}, x^n p_n \rangle \neq 0,\ \forall n \geq 0. \tag{3.4}$$

In this case, the linear functional $\mathcal{U}$ is said to be quasi-definite, to say that there exists a family of polynomials orthogonal with respect to $\mathcal{U}$.

As example, we define a linear functional $\mathcal{L}$ by

$$\langle \mathcal{L}, p \rangle = \int_{-1}^{1} \frac{p(x)}{\sqrt{1-x^2}} dx,$$

corresponding to the Chebyshev orthogonal polynomials $(T_n)_n$. The definition of orthogonality by means of a linear functional is very useful in practice because it enables an elegant proof of equivalent properties of standard orthogonal polynomials [6, 7, 12, 16] in addition to providing the proof of the so-called Favard Theorem [6] stating that any sequence of polynomials satisfying a three-term recurrence relation with some specific restriction on one of its coefficients is orthogonal with respect to a quasi-definite functional.

Theorem 3.4 ([21]) *Let* $(p_n)_n$ *be a sequence of polynomials with* $\deg(p_n) = n$, $n \geq 0$ *and satisfying the orthogonality conditions (2.9) and (2.10).*

1. *Then for any integer* $n \geq 0$*, the following relation holds*

$$\Delta_n > 0,\ n \geq 0, \tag{3.5}$$

where Δ_n *is the Hankel determinant defined by*

$$\Delta_n = \det(\mu_{k+j})_{0\leq k,j\leq n} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & \mu_n \\ \mu_1 & \mu_1 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} & \mu_{2n} \end{vmatrix},\ n \geq 1,\ \Delta_0 := \mu_0. \tag{3.6}$$

The number μ_n which is given by

$$\mu_n = \int_a^b x^n \, d\alpha(x), \;\; n \geq 0,$$

denotes the canonical moment with respect to the measure $d\alpha$.

2. *For any positive integer n, the polynomial p_n has the following matrix representation*

$$p_n(x) = \frac{a_{n,n}}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & \mu_n \\ \mu_1 & \mu_1 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} & \mu_{2n-1} \\ 1 & x & \cdots & x^{n-1} & x^n \end{vmatrix}, \tag{3.7}$$

where $a_{n,n}$ is the leading coefficient of p_n.

3. *Conversely, given any sequence of real numbers $(\mu_n)_n$ satisfying relation (3.5), then since $\Delta_n \neq 0, \; n \geq 0$, there exists a sequence of polynomials orthogonal with respect to the quasi-definite linear functional $\mathcal{U}$ defined on the canonical basis of monomials by*

$$\langle \mathcal{U}, x^n \rangle = \mu_n, \; n \geq 0.$$

In addition, from (3.5) the linear functional $\mathcal{U}$ is positive-definite and as a consequence (see [6]) there exists a positive Borel measure associated with it.

The corresponding family is given explicitly by (3.7).

Proof

1. For the proof of the first property, let $(p_n)_n$ be a sequence of polynomials with $\deg(p_n) = n, \; n \geq 0$ and satisfying the orthogonality conditions (2.9) and (2.10) which are equivalent to orthogonality conditions (3.1) and (3.2). Writing for a fixed integer $n \geq 1$

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} \, x^k,$$

in the orthogonality relation (3.1) for the integers $m = 0 \ldots n$ and then for orthogonality relation (3.2), we obtain the following system of linear equations for the unknowns $(a_{n,k})_k$ whose matrix form is given by

$$\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & \mu_n \\ \mu_1 & \mu_1 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} & \mu_{2n} \end{pmatrix} \begin{pmatrix} a_{n,0} \\ a_{n,1} \\ \vdots \\ a_{n,n-1} \\ a_{n,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ k_n \end{pmatrix}, \tag{3.8}$$

where $k_n = \int_a^b p_n(x)\, x^n d\alpha \neq 0$. Since the polynomial sequence $(p_n)_n$ not only exists and is uniquely determined by fixing k_n, then necessarily, the Hankel determinant is different from zero. The positiveness of the Hankel's determinant will be deduced in the following paragraph.

2. To prove the second property, we first use (3.7) to obtain for $0 \le m \le n$ that

$$\int_a^b p_n(x)\, x^m\, d\alpha(x) = \frac{a_{n,n}}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & \mu_n \\ \mu_1 & \mu_1 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} & \mu_{2n-1} \\ \mu_m & \mu_{m+1} & \cdots & \mu_{m+n-1} & \mu_{m+n} \end{vmatrix}. \tag{3.9}$$

The previous relation reads

$$\int_a^b p_n(x)\, x^m\, d\alpha(x) = 0,\ 0 \le m \le n-1 \tag{3.10}$$

since the $m+1$-th row and the last row of the determinant will be identical. Also, use of (3.9) for $m = n$ taking combined with the following relation

$$\int_a^b p_n(x)\, x^n\, d\alpha(x) = \frac{1}{a_{n,n}} \int_a^b p_n(x)\, p_n(x)\, d\alpha(x) = \frac{d_n^2}{a_{n,n}}$$

obtained using orthogonality, leads to

$$\int_a^b p_n(x)\, x^n\, d\alpha(x) = a_{n,n} \frac{\Delta_n}{\Delta_{n-1}} = \frac{d_n^2}{a_{n,n}} \neq 0,\ n \ge 1. \tag{3.11}$$

We then deduce from Eqs. (3.10) and (3.11) combined with (3.1) and (3.2) that $(p_n)_n$ is orthogonal with respect to $d\alpha(x)$.

The positiveness of Δ_n is seen from the relation

$$\Delta_n = \Delta_0 \prod_{k=1}^{n} \frac{d_k^2}{a_{k,k}^2} = \mu_0 \prod_{k=1}^{n} \frac{d_k^2}{a_{k,k}^2} > 0,$$

deduced from (3.11).

3. The third property is proved by showing, in a similar way as done in the proof of Property 2 above, that the polynomial sequence given by (3.7) satisfies orthogonality relations (3.3) and (3.4).

□

3.3 The Three-Term Recurrence Relation

Theorem 3.5 (Three-Term Recurrence Relation [6, 21, 24]) *Any polynomial sequence* $(p_n)_n$*, orthogonal with respect to the measure* $d\alpha$ *or fulfilling the orthogonality relations (2.8)–(2.10), satisfies the following relation called three-term recurrence relation*

$$x\, p_n(x) = \frac{a_n}{a_{n+1}} p_{n+1} + \left(\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}\right) p_n + \frac{a_{n-1}}{a_n}\,\frac{d_n^2}{d_{n-1}^2} p_{n-1},\ p_{-1} = 0,\ p_0 = 1, \tag{3.12}$$

with

$$p_n = a_n\, x^n + b_n\, x^{n-1} + \textit{ lower degree terms},\quad \textit{and } d_n^2 = (p_n, p_n). \tag{3.13}$$

When (p_n) *is monic (i.e.* $a_n = 1$*) or orthonormal (ie.* $d_n = 1$*), then Eq. (3.12) can be written in the following forms, respectively:*

$$p_{n+1} = (x - \beta_n)\, p_n - \gamma_n\, p_{n-1},\ p_{-1} = 0,\ p_0 = 1, \tag{3.14}$$

with $\beta_n = b_n - b_{n+1},\ \gamma_n = \frac{d_n^2}{d_{n-1}^2}$*, and*

$$x\, p_n = \alpha_{n+1}\, p_{n+1} + \eta_n\, p_n + \alpha_n\, p_{n-1},\ p_{-1} = 0,\ p_0 = 1, \tag{3.15}$$

with $\alpha_n = \frac{a_{n-1}}{a_n},\ \eta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}$.

Also, the recurrence coefficients of the monic and orthonormal forms of the orthogonal polynomial system are connected by

$$\eta_n = \beta_n,\ \gamma_n = \alpha_n^2. \tag{3.16}$$

Proof For fixed $n \geq 0$, we expand $x\, p_n$ in the basis $\{p_0,\ p_1, \ldots,\ p_{n+1}\}$

$$x\, p_n = \sum_{k=0}^{n+1} c_{k,n}\, p_k,$$

and then use orthogonality to obtain

$$c_{k,n} = \frac{\int_a^b x p_n(x)\, p_k(x)\, d\alpha(x)}{\int_a^b p_k(x)\, p_k(x)\, d\alpha(x)} = \frac{\int_a^b p_n(x)\, x p_k(x)\, d\alpha(x)}{\int_a^b p_k(x)\, p_k(x)\, d\alpha(x)} = 0,\ \text{for } 0 \leq k \leq n-2.$$

Hence

$$x\, p_n = c_{n+1,n}\, p_{n+1} + c_{n,n}\, p_n + c_{n-1,n}\, p_{n-1}. \tag{3.17}$$

Substituting (3.13) into (3.12) and identifying the leading coefficients of the monomials x^{n+1} and x^n yields

$$c_{n+1,n} = \frac{a_n}{a_{n+1}}, \ c_{n,n} = \left(\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}\right). \tag{3.18}$$

Using (3.17) twice combined with the orthogonality properties (2.9) and (2.10) gives

$$\begin{aligned} c_{n-1,n}\, d_{n-1}^2 &= \int_a^b p_n(x)(x)\, x p_{n-1}(x)\, d\alpha(x) \\ &= \int_a^b p_n(x)(x)\, [c_{n,n-1}\, p_n + c_{n-1,n-1}\, p_{n-1} + c_{n-2,n-1}\, p_{n-2}] d\alpha(x) \\ &= c_{n,n-1}\, d_n^2, \end{aligned}$$

from which we deduce using (3.18) that

$$c_{n-1,n} = c_{n,n-1} \frac{d_n^2}{d_{n-1}^2} = \frac{a_{n-1}}{a_n} \frac{d_n^2}{d_{n-1}^2}.$$

Equations (3.16) are obtained by identifying the coefficients of Eq. (3.14) with those of the monic form of Eq. (3.15). □

3.4 The Christoffel-Darboux Formula

The following formulas are consequences of the three-term recurrence relation.

Theorem 3.6 (Christoffel-Darboux Formula [6, 21]) *Any system of orthogonal polynomials satisfying the three-term recurrence relation (3.12), satisfies also a so-called Christoffel-Darboux formula given, respectively, in its initial and confluent forms as*

$$\sum_{k=0}^{n} \frac{p_k(x) p_k(y)}{d_k^2} = \frac{a_n}{a_{n+1}} \frac{1}{d_n^2} \frac{p_{n+1}(x)\, p_n(y) - p_{n+1}(y)\, p_n(x)}{x - y}, \ x \neq y, \tag{3.19}$$

$$\sum_{k=0}^{n} \frac{p_k(x) p_k(x)}{d_k^2} = \frac{a_n}{a_{n+1}} \frac{1}{d_n^2} \left(p'_{n+1}(x)\, p_n(x) - p_{n+1}(x)\, p'_n(x)\right). \tag{3.20}$$

Proof For the proof of Eq. (3.19), we multiply by $p_k(y)$, Eq. (3.17) in which n is replaced by k, to obtain

$$x\, p_k(x)\, p_k(y) = c_{k+1,k}\, p_{k+1}(x)\, p_k(y) + c_{k,k}\, p_k(x)\, p_k(y) + c_{k-1,k}\, p_{k-1}(x)\, p_k(y).$$

Interchanging the role of x and y in the previous equation, we obtain

$$y\, p_k(x)\, p_k(y) = c_{k+1,k}\, p_{k+1}(y)\, p_k(x) + c_{k,k}\, p_k(x)\, p_k(y) + c_{k-1,k}\, p_{k-1}(y)\, p_k(x).$$

Subtracting the last two equations from each other, we obtain that

$$\frac{p_k(x)\, p_k(y)}{d_k^2} = \frac{A_k(x, y) - A_{k-1}(x, y)}{x - y},$$

where

$$A_k(x, y) = \frac{c_{k+1,k}}{d_k^2}\,(p_{k+1}(x)\, p_k(y) - p_{k+1}(y)\, p_k(x)),$$

after taking into account the relation $\frac{c_{k+1,k}}{d_k^2} = \frac{a_k}{a_{k+1}}\frac{1}{d_k^2} = \frac{c_{k,k+1}}{d_{k+1}^2}$. Equation (3.19) is obtained by summing the previous equation for k from 0 to n, taking into account that $A_{-1}(x, y) = 0$ as $p_{-1}(x) = 0$. Equation (3.20) is obtained by taking the limit of (3.19) when y tends to x. □

3.5 The Interlacing Properties of the Zeros

The following properties of the zeros of orthogonal polynomials are direct consequences of the confluent form of the Christoffel-Darboux formula (3.20). Their proof is given in the lecture notes of A. Jooste in these proceedings.

Theorem 3.7 (On the Zeros of Orthogonal Polynomials [3, 12, 21, 22]) *If $(p_n)_n$ is a polynomial system, orthogonal with respect to the positive Borel measure $d\alpha$ supported on the interval (a, b), then we have the following properties:*

1. *p_n has n simple real zeros $x_{n,k}$ satisfying $a < x_{n,k} < b,\ 1 \le k \le n$.*
2. *p_n and p_{n+1} have no common zero. The same applies for P_n and P_n';*
3. *if $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ are the n zeros of p_n, then*

$$x_{n+1,k} < x_{n,k} < x_{n+1,k+1},\ \ 1 \le k \le n.$$

Remark 3.8 It should be noticed that the three-term recurrence relation in the current section yields a matrix representation of the multiplication operator, called the Jacobi matrix (see for instance [25]). From this fact one can deduce that the zeros of the n-th orthogonal polynomials are the eigenvalues of the leading principal submatrix of size $n \times n$ of such a Jacobi matrix. This provides a method to find in an efficient numerical way such zeros even in the quasi-definite case.

3.6 Solution to the $L^2(\alpha)$ Extremal Problem

Theorem 3.9 (Minimal Property) *Let $(p_n)_n$ be a sequence of monic polynomials orthogonal with respect to a positive Borel measure $d\alpha(x)$ supported on the real line. For any fixed positive integer n, p_n is the minimal polynomial with respect to the L^2-norm*

$$||p||_\alpha = \sqrt{\int p^2(x)\, d\alpha(x)}$$

associated with the corresponding orthogonality measure:

$$\min\left\{\int q_n^2(x)\, d\alpha(x),\ q_n \in \mathbb{R}[x], q_n(x) = x^n + \text{lower degree terms}\right\} = \int p_n^2(x)\, d\alpha(x) \tag{3.21}$$

where $\mathbb{R}[x]$ is the ring of polynomials with real coefficients.

Proof Let $n \geq 1$ and q_n be a monic polynomial of degree n. Combining the expansion of q_n in terms of the $(p_k)_k$

$$q_n(x) = \sum_{k=0}^{n} a_{n,k}\, p_k(x),$$

with the orthogonality give

$$\int q_n^2(x)\, d\alpha(x) = \sum_{k=0}^{n} a_{n,k}^2\, d_k^2.$$

Therefore,

$$\int q_n^2(x)\, d\alpha(x) \geq a_{n,n}^2\, d_n^2 = d_n^2 = \int p_n^2(x)\, d\alpha(x).$$

In addition, there is equality if and only if $a_{n,k} = 0,\ 0 \leq k \leq n-1$. □

It should be noticed that relation (3.21) which is valid for any sequence of polynomial orthogonal to the positive Borel measure $d\alpha(x)$, is similar to relation (1.8), given for the specific case of the Chebyshev polynomials of the first kind, with the Sup-norm (instead of the corresponding L^2-norm).

3.7 Gauss Quadrature Formula

The following property, which is called the Gauss quadrature formula is valid for any sequence of polynomials orthogonal with respect to the weight function ρ.

Theorem 3.10 ([3, 12, 21, 22]) *Let $(p_n)_n$ be a family of polynomials satisfying orthogonality relations (2.11)–(2.13). Then there exists a sequence of positive real numbers $(\lambda_{n,k})_n$, called Christoffel numbers, such that*

$$\int_a^b \rho(x)\, f(x)dx = \sum_{k=1}^{n} \lambda_{n,k}\, f(x_{n,k}),\ \forall f \in \mathbb{R}_{2n-1}[x],\ n \geq 1, \tag{3.22}$$

where the $x_{n,k}$, $1 \leq k \leq n$ are the zeros of p_n ranked by increasing order. In addition, the integral of any function continuous on the compact interval $[a, b]$ can be approximated by the previous formula:

$$\int_a^b \rho(x)\, f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{n,k}\, f(x_{n,k}),\ \forall f \in \mathcal{C}[a, b]. \tag{3.23}$$

Proof The proof of Eq. (3.22) which generalises property number 7 of the first theorem, is given in the paper by A. Jooste. It can also be found in [3, 12, 21, 22]. The proof of Eq. (3.23) is given in [3, 6]. □

3.8 Concluding Remarks

We would like to complete this paper with the following information and remark which will help to connect this lecture notes with the forthcoming ones, especially with those involved with classical and semi-classical orthogonal polynomials, as well as orthogonal polynomials of the Sobolev type:

1. Among the classes of orthogonal polynomials, we mention the classical orthogonal polynomials and the semi-classical orthogonal polynomials. The first class is contained in the second one.
2. Classical orthogonal polynomials of a continuous, discrete, q-discrete, quadratic and q-quadratic variable, respectively, are those orthogonal with respect to a weight function satisfying a so-called Pearson equation which is a first-order linear homogeneous differential, difference, q-difference, divided-difference or q-divided-difference equation with polynomial coefficients of degree one and at most 2, respectively, with some boundary conditions at the ends of the interval. Depending on the type of the variable, we get classical orthogonal polynomials of continuous, a discrete, a q-discrete, a quadratic and a q-quadratic variable.

   Semi-classical orthogonal polynomial are defined in the same way like the classical ones, but with less restriction on the degree of the polynomial coefficients of the Pearson equation which can take higher values.

3. The properties such as the uniqueness of a family of polynomials orthogonal with respect to a measure, the matrix representation, the three-term recurrence relation, the Christoffel-Darboux formula and its confluent form, the interlacing properties of the zeros and the Gauss quadrature formula are valid for any family of orthogonal polynomials. In addition, it should also be noticed that Theorems 3.4 (Matrix representation), 3.5 (Three-term recurrence relation), and 3.6 (Christoffel-Darboux formula) are also valid if we replace the positive Borel measure by a quasi-definite linear functional. In this case and for Theorem 3.4, the positiveness of the Hankel's determinant is to be replaced by the fact that this determinant does not vanish.
4. The Chebyshev polynomials of the first, second, third and fourth kinds are up to now the only known families of orthogonal polynomials for which the zeros are explicitly known. In addition to the Chebyshev polynomials of the first kind which have been studied here, the three other families are, respectively, given for $z = \cos\theta,\ 0 < \theta < \pi$, by [15, 21]

$$U_n(z) = \frac{\sin((n+1)\theta)}{\sin\theta},\ V_n(z) = \frac{\cos((n+\frac{1}{2})\theta)}{\cos(\frac{\theta}{2})},\ W_n(z) = \frac{\sin((n+\frac{1}{2})\theta)}{\sin(\frac{\theta}{2})}.$$

The zeros of $U_n(z)$, $V_n(z)$ and $W_n(z)$ are given in increasing order, respectively, by

$$z_{n,k} = \cos\theta_{n,k},\ \text{with}\ \theta_{n,k} = \frac{n+1-k}{n+1}\pi,\ k = 1,\ 2, \ldots, n,$$
$$z_{n,k} = \cos\theta_{n,k},\ \text{with}\ \theta_{n,k} = \frac{2(n-k)+1}{2n+1}\pi,\ k = 1, 2, \ldots, n,$$
$$z_{n,k} = \cos\theta_{n,k},\ \text{with}\ \theta_{n,k} = \frac{2(n+1-k)}{2n+1}\pi,\ k = 1, 2, \ldots, n.$$

5. Additional information on general orthogonal polynomials can be found for example in [5, 6, 12, 19–21].

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