Discrete-Time Insurance Models. Optimization of Their Performance by Reinsurance and Bank Loans

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Abstract. The popularity of discrete-time models in applied probability is explained as follows. They are more precise in some situations. In other cases they can be used as approximation of the corresponding continuous-time models. So, we consider two discrete-time insurance models and study the quality of their performance. The company reliability or the expected discounted costs incurred by its control can be chosen as an objective function (target or risk measure). It is possible to consider a finite or infinite planning horizon. The control includes reinsurance treaties and/or bank loans. The optimal control (maximizing the reliability or minimizing the costs) is established for the finite planning horizon. Its asymptotic behavior, as the horizon tends to infinity, is also investigated.

Keywords: Discrete time · Reinsurance · Bank loans · Optimization · Costs

1 Introduction

It is well known, see, e.g., [5], that almost all applied probability models arising in insurance, finance, queuing and reliability theory, dams and inventory theory, communications and population dynamics are of input-output type. Therefore such a model can be described by specifying the input, output, control, planning horizon $T \le \infty$, as well as functional Ψ, reflecting the system configuration and operation mode, and objective function evaluating the quality of system performance. Input, output and control are some processes (deterministic or stochastic) defined for $t \le T$, their dimensions are not necessarily the same. So, the system state (defined by application of functional Ψ to input, output and control) is also a process, maybe multi-dimensional. It is possible to control input, output or the system configuration and operation mode. An objective function (valuation criterium, target or risk measure) can be chosen in different

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ways. The most popular are reliability and cost approaches. In the first case, one wishes to minimize the ruin probability or maximize the uninterrupted working period. In the second case, the control is named optimal if it minimizes the (expected) costs associated with the system control or maximizes the (expected) profit obtained by the control application. For certainty, the further discussion will be conducted in terms of insurance models. However it is always possible to give another interpretation to the input and output processes to get a model arising in other applications. Thus, why insurance models were chosen? Insurance can be considered as risk management or decision making under uncertainty, see, e.g., [13,20]. It is necessary to protect the capital or other interests of citizens, enterprizes and organizations, i.e. individuals and legal entities. Moreover, insurance has the longest history among the above mentioned applied probability domains, see, e.g., [2]. In fact, methods for transferring or distributing risk were practiced by Chinese and Babylonian traders already 2–3 thousands years BC. Thus, Code of Hammurabi, c. 1750 BC, contained the laws for maritime insurance. Mutual societies, run by their members with no external shareholders to pay were first to appear. Next step is joint stock companies. A modern insurance company has a two-fold nature. Its primary task is indemnification of policyholders claims. The secondary, but very important, task is dividend payments to shareholders.

New problems have arisen in actuarial sciences during the last twenty years, see, e.g., [5]. This period is characterized by interplay of insurance and finance, unification of reliability and cost approaches, see, e.g., [6], as well as, consideration of complex systems. Sophisticated mathematical tools are used for analysis and optimization of insurance systems including dividends, reinsurance, capital injections and investment.

Nowadays, discrete-time models became popular in applied probability (see, e.g., [5]), since they are more precise in some situations. Thus, the reinsurance treaties are usually negotiated at the end of year, the decision on dividends payment is also based on the results of financial year. In other cases, the discrete-time models can be used as approximation of the corresponding continuous-time models when analytical results are unavailable, see, e.g., [12]. In such domains as inventories or population dynamics discrete-time models have arisen from the beginning. In insurance such models appeared later (see, e.g., [3] and [11]). The first review on discrete-time insurance models [14] was published in 2009. Below we develop the models proposed in [3–10]. Namely, we study the optimal policies for performance of insurance company. As control we choose reinsurance and bank loans.

2 Main Results

We consider the following discrete-time insurance model. Let $\{X_n\}_{n\geq 1}$ be a sequence of non-negative independent identically distributed (i.i.d.) random variables (r.v.'s) with a known distribution function F, possessing a density $\varphi(s) > 0$ for $s > 0$ and a finite expectation. Here X_k is the claim size during the k-th period. The initial capital (or surplus) of the company is equal to x. The premium paid at the beginning of the first period is included in the initial capital. The other premiums will be mentioned explicitly.

Our aim is to establish a strategy of bank loans minimizing the additional costs associated with claims indemnification. In the same time a company can use reinsurance. Below we suppose that the same reinsurance treaty is applied each period. The next step is to choose the optimal treaty or let it vary from one period to another.

2.1 Proportional Reinsurance

Suppose that we use a proportional reinsurance. More precisely, we apply a quota share treaty with coefficient $\beta > 0$. That means, the direct insurer retains $Z = \beta X$ if the initial demand for indemnification is X. Accordingly, the insurance company retains only part of premiums received from its clients. We denote this amount by M.

One-period Case. Let us begin by treating a one-period case. It is possible to get a bank loan at the beginning of the period, the interest rate being c, or after the claim arrival, with the interest rate $r > c$. Let $f_1(x)$ be the minimal expected costs associated with a loan.

Lemma 1. *The following statement is valid.*

$$f_1(x) = -cx + \min_{y \geq x} \underbrace{(cy + \frac{r}{\beta}\int_y^{\infty}(s-y)\varphi(\frac{s}{\beta})\,ds)}_{G_1(y)}. \tag{1}$$

Moreover, there exists a constant $\overline{y_1} = \beta F^{-1}(1 - \frac{c}{r})$ *such that*

$$f_1(x) = \begin{cases} -cx + G_1(\overline{y_1}) & \text{if } x \leq \overline{y_1}, \\ -cx + G_1(x) & \text{otherwise.} \end{cases} \tag{2}$$

Proof. Obviously, the interests for a loan up to level y at the beginning of the period are $c(y-x)$ and the company has to pay $rE(Z-y)^+ = \frac{r}{\beta}\int_y^{\infty}(s-y)\varphi(\frac{s}{\beta})\,ds$ for additional loan. So, relation (1) holds. Using the formula

$$\Big(\int_{\alpha(y)}^{\beta(y)} \gamma(y,x)\,dx\Big)'_y = \int_{\alpha(y)}^{\beta(y)} \gamma'_y(y,x)\,dx + \gamma(y,\beta(y))\beta'(y) - \gamma(y,\alpha(y))\alpha'(y),$$

we get immediately

$$G_1'(y) = c - r\int_{\frac{y}{\beta}}^{\infty}\varphi(s)\,ds.$$

Now the equation $G_1'(y) = 0$ can be rewritten as $r - c = rF(\frac{y}{\beta})$, whence it follows that there exists a unique solution of this equation $\overline{y_1} = \beta F^{-1}(1 - \frac{c}{r})$. It is due to the fact that $G_1''(y) = \frac{r}{\beta}\varphi(\frac{y}{\beta}) > 0$. Thus, the optimal level for bank loan is

$$y_1(x) = \begin{cases} \overline{y_1} & \text{if } x \le \overline{y_1}, \\ x & \text{otherwise,} \end{cases}$$

providing the desired result (2). □

Multi-period Case. Now, turn to multi-period case. Let α be the discount factor and $f_k(x)$ denote the minimal expected discounted k-periods costs incurred by loans. The following result is valid.

Theorem 1. *The relation*

$$f_k(x) = -cx + \min_{y \ge x} \underbrace{(cy + \frac{r}{\beta}\int_y^\infty (s - y)\varphi(\frac{s}{\beta})\,ds + \alpha \int_0^\infty f_{k-1}(y + M - \beta s)\varphi(s)\,ds)}_{G_k(y)}$$

takes place. Moreover, there exists a constant $\overline{y_k}$ such that the optimal bank loan $y_k(x)$ at the beginning of the k-step process, $k \ge 2$, is defined as follows

$$y_k(x) = \begin{cases} \overline{y_k} & \textit{if } x \le \overline{y_k}, \\ x & \textit{otherwise.} \end{cases}$$

Hence,

$$f_k(x) = \begin{cases} -cx + G_k(\overline{y_k}) & \textit{if } x \le \overline{y_k}, \\ -cx + G_k(x) & \textit{otherwise.} \end{cases} \tag{3}$$

The sequence $\{\overline{y_k}\}$ is non-decreasing in k.

Proof. We use the dynamic programming (see, e.g., [1]) and carry out the proof by induction. For $k = 2$ we easily obtain the relation

$$f_2(x) = -cx + \min_{y \ge x} \underbrace{(cy + \frac{r}{\beta}\int_y^\infty (s - y)\varphi(\frac{s}{\beta})\,ds + \alpha \int_0^\infty f_1(y + M - \beta s)\varphi(s)\,ds)}_{G_2(y)}$$

called the Bellman equation. It can be rewritten as

$$f_2(x) = -cx + \min_{y \ge x}(G_1(y) + \alpha \int_0^\infty f_1(y + M - \beta s)\varphi(s)\,ds).$$

Hence, due to (2),

$$G_2'(y) = G_1'(y) + \alpha \int_0^{\frac{y+M-\overline{y_1}}{\beta}} G_1'(y+M-\beta s)\varphi(s)\,ds - \alpha c$$

and

$$G_2''(y) = G_1''(y) + \alpha \int_0^{\frac{y+M-\overline{y_1}}{\beta}} G_1''(y+M-\beta s)\varphi(s)\,ds > 0.$$

Since the function $G_2'(y)$ is increasing, we establish inequality $\overline{y_1} < \overline{y_2}$ verifying that $G_2'(\overline{y_1}) < 0$. In fact,

$$G_2'(\overline{y_1}) = \alpha[\int_0^{\frac{M}{\beta}} (c - r \int_{\frac{\overline{y_1}+M-\beta s}{\beta}}^{\infty} \varphi(t)\,dt)\varphi(s)\,ds - c].$$

The right-hand expression can be written in the form

$$\alpha[c(F(\frac{M}{\beta}) - 1) - r \int_0^{\frac{M}{\beta}} (1 - F(\frac{\overline{y_1}+M-\beta s}{\beta})\varphi(s)\,ds],$$

where both terms are negative.

It is also clear that

$$\lim_{y\to\infty} G_2'(y) = c - \alpha c + \lim_{y\to\infty} (-r \int_{\frac{y}{\beta}}^{\infty} \varphi(s)\,ds)$$

$$+\alpha \lim_{y\to\infty} \int_0^{\frac{y+M-\overline{y_1}}{\beta}} (c - r \int_{\frac{y+M-\beta s}{\beta}}^{\infty} \varphi(t)\,dt)\varphi(s)\,ds > 0$$

and equation $G_2'(y) = 0$ has a solution $\overline{y_2} < \infty$.

Thus,

$$f_2(x) = \begin{cases} -cx + G_2(\overline{y_2}) & \text{if } x \le \overline{y_2}, \\ -cx + G_2(x) & \text{otherwise,} \end{cases}$$

and the first step of induction is completed. In the same way, for $k > 2$,

$$f_k(x) = -cx + \min_{y\ge x} \underbrace{(cy + \frac{r}{\beta}\int_y^{\infty} (s-y)\varphi(\frac{s}{\beta})\,ds + \int_0^{\infty} f_{k-1}(y+M-\beta s)\varphi(s)\,ds)}_{G_k(y)}.$$

Assuming that for $k \le n$ there exist $\overline{y_k}$ satisfying the relation $G_k'(y) = 0$ and

$$f_k(x) = \begin{cases} -cx + G_k(\overline{y_k}) & \text{if } x \le \overline{y_k}, \\ -cx + G_k(x) & \text{otherwise,} \end{cases}$$

we have to prove that the same is true for $k = n + 1$ and $\overline{y_{n+1}} > \overline{y_n}$.

Clearly, for $k \le n$,

$$f_k'(x) = \begin{cases} -c & \text{if } x \le \overline{y_k}, \\ -c + G_k'(x) & \text{otherwise.} \end{cases}$$

Recall that $G_n'(y)$ has the form

$$G_n'(y) = c - r \int_{\frac{y}{\beta}}^{\infty} \varphi(s)\,ds + \alpha \int_0^{\infty} f_{n-1}'(y + M - \beta s)\varphi(s)\,ds \tag{4}$$

and

$$G_n''(y) = \frac{r}{\beta}\varphi(\frac{y}{\beta}) + \alpha \int_0^{\infty} f_{n-1}''(y + M - \beta s)\varphi(s)\,ds$$

is nonnegative.

Obviously,

$$G_{n+1}'(y) = G_n'(y) + \alpha \int_0^{\infty} (f_n'(y + M - \beta s) - f_{n-1}'(y + M - \beta s))\varphi(s)\,ds,$$

therefore $G_{n+1}'(\overline{y_n})$ is equal to

$$\alpha \int_0^{\infty} (f_n'(\overline{y_n} + M - \beta s) - f_{n-1}'(\overline{y_n} + M - \beta s))\varphi(s)\,ds.$$

Now, we use the expression

$$f_n'(x) - f_{n-1}'(x) = \begin{cases} 0, & \text{if } x \le \overline{y_{n-1}}, \\ -G_{n-1}'(x), & \text{if } x \in (\overline{y_{n-1}}, \overline{y_n}], \\ G_n'(x) - G_{n-1}'(x) & \text{otherwise.} \end{cases} \tag{5}$$

That means,

$$G_{n+1}'(\overline{y_n}) = \alpha \int_0^{\frac{M}{\beta}} (G_n'(\overline{y_n} + M - \beta s) - G_{n-1}'(\overline{y_n} + M - \beta s))\varphi(s)\,ds$$

$$-\alpha \int_{\frac{M}{\beta}}^{\frac{\overline{y_n}-\overline{y_{n-1}}+M}{\beta}} G'_{n-1}(\overline{y_n}+M-\beta s)\varphi(s)\,ds. \tag{6}$$

Since $G'_{n-1}(y) > 0$ for $y > \overline{y_{n-1}}$, the second integral is positive (however it is taken with sign minus). The first integral can be transformed as follows

$$\int_0^{\frac{M}{\beta}} (G'_n(\overline{y_n}+M-\beta s) - G'_{n-1}(\overline{y_n}+M-\beta s))\varphi(s)\,ds$$

$$= \alpha \int_0^{\frac{M}{\beta}} (\int_0^{\infty} (f'_{n-1}(\overline{y_n}+2M-\beta(t+s)) - f'_{n-2}(\overline{y_n}+2M-\beta(t+s)))\varphi(t)\,dt)\varphi(s)\,ds.$$

Using the induction assumption the integral under consideration can be reduced, step by step, to the multiple integral with integrand depending on $G'_2(\cdot) - G'_1(\cdot)$ multiplied by α^{n-1} and the sum of negative terms similar to the second integral in (6) multiplied by α^k, $k = 2, \ldots, n-1$. We omit the explicit expression due to its bulkiness only mentioning that it is negative. Hence, it follows immediately that $f_{n+1}(x)$ has also the desired form. □

Theorem 1 enables us to treat infinite planning horizon and establish the optimal strategy of bank loans determined by one critical level $\overline{y}$ as shows

Corollary 1. *For $\alpha < 1$ there exists $\overline{y} = \lim_{n\to\infty} \overline{y_n}$.*

Proof. It follows from (3) and (4) that

$$G'_n(y) = V(y) + \alpha \int_0^{\frac{y-\overline{y_{n-1}}+M}{\beta}} G'_{n-1}(y+M-\beta s)\varphi(s)\,ds$$

where $V(y) = G'_1(y) - \alpha c$.

Thus, $G'_n(y) = V(y)$ for $y + M \le \overline{y_{n-1}}$ and $G'_n(y) \ge V(y)$ otherwise, since $G'_{n-1}(y+M-\beta s) \ge 0$ if $0 \le s \le (y - \overline{y_{n-1}} + M)\beta^{-1}$.

Let $V(\overline{z}) = 0$, in other words, $\overline{F}(\overline{z}) = \frac{c}{r}(1-\alpha)$. Obviously, $G'_n(\overline{z}) \ge V(\overline{z}) = 0$, that is, $\overline{y_n} \le \overline{z} < \infty$. Hence, there exists $\overline{y} = \lim_{n\to\infty} \overline{y_n}$. □

Theorem 2. *For $\alpha < 1$, functions $f_n(x)$ converge uniformly, as $n \to \infty$, to the solution of the functional equation*

$$f(x) = -cx + \min_{y\ge x}[G_1(y) + \alpha \int_0^{\infty} f(x+M-\beta s)\varphi(s)\,ds].$$

Proof. Let $y_n(x)$ be the optimal decision at the first step of the n-period process. Then

$$f_n(x) = -cx + \min_{y \geq x} G_n(y) = -cx + G_n(y_n(x)) \leq -cx + G_n(y_{n+1}(x)).$$

Therefore

$$G_{n+1}(y_{n+1}(x)) - G_n(y_{n+1}(x)) \leq f_{n+1}(x) - f_n(x) \leq G_{n+1}(y_n(x)) - G_n(y_n(x)).$$

That means,

$$|f_{n+1}(x) - f_n(x)| \leq \max(|G_{n+1}(y_n(x)) - G_n(y_n(x))|, |G_{n+1}(y_{n+1}(x)) - G_n(y_{n+1}(x))|).$$

The right-hand side of this inequality is bounded by $\max\limits_y |G_{n+1}(y) - G_n(y)|$.

Put $u_n = \max\limits_x |f_{n+1}(x) - f_n(x)|$. Since

$$G_{n+1}(y) - G_n(y) = \alpha \int_0^\infty (f_n(y + M - \beta s) - f_{n-1}(y + M - \beta s))\varphi(s)\, ds,$$

one gets immediately

$$u_n \leq \alpha u_{n-1} \leq \alpha^2 u_{n-2} \leq \ldots \leq \alpha^{n-1} u_1.$$

Thus, if we show that $u_1 < \infty$, then the uniform convergence of $f_n(x)$ to a limit $f(x)$ is obvious. It is due to the fact that, for any x, the function $|f_n(x) - f_1(x)| \leq |f_2(x) - f_1(x)| + |f_3(x) - f_2(x)| + \ldots + |f_n(x) - f_{n-1}(x)|$ is majorized by the partial sum of a geometric progression $\{u_1 \alpha^k\}_{k \geq 1}$.

Recall that using (5) we have

$$u_1 \leq \max[G_2(\overline{y_2}) + G_1(\overline{y_2}), \alpha \cdot \max_{x \geq \overline{y_1}} \int_0^\infty (f_1(x + M - \beta s)\varphi(s)\, ds].$$

Clearly, $G_2(\overline{y_2}) + G_1(\overline{y_2}) < \infty$. Accordingly to (2) it is possible to write

$$\int_0^\infty f_1(x + M - \beta s)\varphi(s)\, ds = \beta r \int_0^{\frac{x+M-\overline{y_1}}{\beta}} \varphi(s) \int_{\frac{x+M-\beta s}{\beta}}^\infty s_1 \varphi(s_1)\, ds_1\, ds$$

$$-r \int_0^{\frac{x+M-\overline{y_1}}{\beta}} (x + M - \beta s)\overline{F}(\frac{x + M - \beta s}{\beta})\varphi(s)\, ds + G_1(\overline{y_1})\overline{F}(\frac{x + M - \overline{y_1}}{\beta})$$

$$-c(x + M)\overline{F}(\frac{x + M - \overline{y_1}}{\beta}) + c\beta \int_{\frac{x+M-\overline{y_1}}{\beta}}^\infty s\varphi(s)\, ds.$$

Since we treat only the domain $\{x \geq \overline{y_1}\}$, all the summands of the above equality are bounded. This is due to existence of $EX_1 = \mu < \infty$.

Thus, uniform convergence of $f_n(x)$ to $f(x)$ is established. Obviously, $f(x)$ satisfies the functional equation stated in the theorem. □

2.2 Non-proportional Reinsurance

Now we turn to the case of non-proportional reinsurance, namely, suppose that a stop-loss treaty with retention a is applied each period. That means, instead of the claim X_k, $k \geq 1$, insurer has to pay $Z_k = \min(X_k, a)$ during the k-th period. Since the claims are supposed to be i.i.d. r.v.'s, the premiums obtained by the insurance company are equal to

$$M = (1 + \gamma_1)EX - (1 + \gamma_2)E(X - a)^+.$$

Here X has the same distribution function F as all X_k, $k \geq 1$, whereas γ_1 and γ_2 are the safety loadings of insurer and reinsurer, respectively. We assume that F has a density $\varphi(x) > 0$, for $x > 0$, and a finite mean value.

We would like to establish the optimal policy of bank loans minimizing the associated expected costs.

One-period Case. As previously, we begin by consideration of one-step process establishing the form of $f_1(x)$. If insurer decides to take a loan to raise the surplus up to level y, the expected costs are equal to $c(y - x) + rE(Z_1 - y)^+$. It is clear that

$$f_1(x) = -cx + \min_{y \geq x} G_1(y)$$

where

$$G_1(y) = cy + D(y) \quad \text{with } D(y) = r \int_0^\infty [\min(s, a) - y]^+ \, dF(s).$$

Now we can prove the following result.

Lemma 2. *Optimal loan level is given by* $\overline{y_1} = \min(a, F^{-1}(1 - \frac{c}{r}))$.

Proof. It is clear that under the stop-loss treaty with retention a the insurer never has to pay more than a. Thus, it is unreasonable to take a loan up to level greater than a.

Rewriting $G_1(y)$, for $y \leq a$, in the form

$$G_1(y) = cy + r \int_y^a (s - y)\varphi(s)\, ds + r(a - y)P(X > a)]$$

we get

$$G_1'(y) = c - r\overline{F}(y) \quad \text{and } G_1''(y) = r\varphi(y) > 0. \tag{7}$$

Therefore $G_1'(y) < 0$ for $y < F^{-1}(1 - \frac{c}{r})$. Hence, the desired result is obvious. □

Multi-period Case. For the multi-step case we have the following Bellman equation

$$f_k(x) = -cx + \min_{y \ge x} G_k(y)$$

with $G_k(y)$ given by

$$G_1(y) + \alpha \int_0^\infty f_{k-1}(y + M - \min(s,a))\, dF(s).$$

The following result is valid.

Theorem 3. *The optimal levels $\overline{y_k}$, $k \ge 1$, for the bank loans strategy at the beginning of k-step process form an increasing sequence.*

Proof. We begin by treating the case $\overline{F}(a) \le \frac{c}{r}$, in other words, $\overline{y_1} \le a$. Thus,

$$f_1(x) = -cx + \begin{cases} G_1(\overline{y_1}) & \text{if } x \le \overline{y_1}, \\ G_1(x) & \text{otherwise.} \end{cases}$$

Therefore,

$$f_1'(x) = -c + \begin{cases} 0 & \text{if } x \le \overline{y_1}, \\ G_1'(x) & \text{otherwise} \end{cases}$$

and, according to (7), $f_1'(x) \le 0$ for all x. It follows immediately, that

$$G_2'(y) = G_1'(y) + \alpha \int_0^\infty f_1'(y + M - \min(s,a))\, dF(s) \le G_1'(y).$$

Clearly,

$$G_2'(\overline{y_1}) \le G_1'(\overline{y_1}) = 0.$$

Hence, $\overline{y_2} \ge \overline{y_1}$ and initial step of induction is proved.

Now suppose that for $k \le n$ we have established that there exist the critical levels $\overline{y_k}$ given by $G_k'(\overline{y_k}) = 0$. They form a non-decreasing sequence and

$$f_k'(x) = -c + \begin{cases} 0 & \text{if } x \le \overline{y_k}, \\ G_k'(x) & \text{otherwise.} \end{cases}$$

Moreover, we assume that $G_k'(x) - G_{k-1}'(x) \le 0$ for all x.

Obviously,

$$f_k'(x) - f_{k-1}'(x) = \begin{cases} 0 & \text{if } x \le \overline{y_{k-1}}, \\ -G_{k-1}'(x) & \text{if } x \in (\overline{y_{k-1}}, \overline{y_k}), \\ G_k'(x) - G_{k-1}'(x) & \text{if } x \ge \overline{y_k}. \end{cases} \tag{8}$$

Consider

$$G'_{n+1}(x) = G'_n(x) + \alpha \int_0^\infty (f'_n(y+M-\min(s,a)) - f'_{n-1}(y+M-\min(s,a)))\, dF(s).$$

Due to (8) and other induction assumptions, $f'_n(x) - f'_{n-1}(x) \le 0$ for all x. That means $G'_{n+1}(x) \le G'_n(x)$ for all x. Taking $x = \overline{y_n}$ we get the desired result $\overline{y_n} \le \overline{y_{n+1}}$. The case $\overline{F}(a) < \frac{c}{r}$ can be treated similarly. □

It is possible to consider the infinite planning horizon proving

Corollary 2. *A functional equation*

$$f(x) = -cx + \min_{y \ge x}[cy + \int_y^a r(s-y)\varphi(s)\, ds + r(a-y)P(X > a)$$

$$+\alpha \int_0^\infty f(y + M - \min(s,a))\, dF(s)]$$

has a unique solution for $\alpha < 1$.

The proof is omitted because the methods employed are similar to those used in Theorem 2.

3 Conclusion

We considered two discrete-time models with bank loans and proportional (or non-proportional) reinsurance for a finite planning horizon. It is established that the optimal loans policy is determined by a sequence of critical levels $\overline{y_n}$, $n \ge 1$. That means, if the surplus at the beginning of the n-step process $x < \overline{y_n}$, $n \ge 1$, then optimal decision is to raise it up to level $\overline{y_n}$, otherwise the loan is not necessary. The sequence of critical levels $\{\overline{y_n}\}$ is bounded non-decreasing, so it converges, as $n \to \infty$, to the limit $\overline{y}$ if $\alpha < 1$. Moreover, it is proved that the minimal costs $f_n(x)$ converge uniformly in x to a function $f(x)$ which is the unique solution of a functional equation.

For $\alpha = 1$ we have to choose another risk measure introducing the notion of asymptotically optimal policy. It is possible as well to calculate the company ruin probability for any α under the optimal loans strategy.

The next step is investigation of models stability with respect to small fluctuations of system parameters and perturbations of the underlying distributions. The books [16,17] and [18] are useful for this purpose, as well as the results obtained in [8,15,19].

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