A Randomized Approximation Algorithm for Metric Triangle Packing

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Abstract. Given an edge-weighted complete graph G on $3n$ vertices, the maximum-weight triangle packing problem (MWTP for short) asks for a collection of n vertex-disjoint triangles in G such that the total weight of edges in these n triangles is maximized. Although MWTP has been extensively studied in the literature, it is surprising that prior to this work, no nontrivial approximation algorithm had been designed and analyzed for its metric case (denoted by MMWTP), where the edge weights in the input graph satisfy the triangle inequality. In this paper, we design the first nontrivial polynomial-time approximation algorithm for MMWTP. Our algorithm is randomized and achieves an expected approximation ratio of $0.66745 - \epsilon$ for any constant $\epsilon > 0$.

Keywords: Triangle packing · Metric · Approximation algorithm · Randomized algorithm · Maximum cycle cover

1 Introduction

An instance of the *maximum-weight triangle packing* problem (MWTP for short) is an edge-weighted complete graph G on $3n$ vertices, where n is a positive integer. Given G, the objective of MWTP is to compute n vertex-disjoint triangles such that the total weight of edges in these n triangles is maximized.

The unweighted (or edge uniformly weighted) variant, denoted MTP for short, is to compute the maximum number of vertex-disjoint triangles in the input graph, which is edge unweighted and is not complete.

In their classic book, Garey and Johnson [8] show that MTP is NP-hard. Kann [14] and van Rooij *et al.* [16] show that MTP is APX-hard even restricted on graphs of maximum degree 4. Chlebik and Chlebikova [5] show that unless

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P = NP, no polynomial-time approximation algorithm for MTP can achieve an approximation ratio of 0.9929. Moreover, Guruswami *et al.* [9] show that MTP remains NP-hard even restricted on chordal, planar, line or total graphs.

MTP can be easily cast as a special case of the *unweighted* 3-*set packing* problem (U3SP for short). Recall that an instance of U3SP is a family $\mathcal{F}$ of sets each of size 3 and the objective is to compute a sub-family $\mathcal{F}' \subset \mathcal{F}$ of the maximum number of disjoint sets. Hurkens and Schrijver [13] (also see Halldorsson [10]) present a nontrivial polynomial-time approximation algorithm for U3SP which achieves an approximation ratio of $\frac{2}{3} - \epsilon$ for any constant $\epsilon > 0$. This ratio has been improved to $\frac{3}{4} - \epsilon$ [6,7]. Manic and Wakabayashi [15] present a polynomial-time approximation algorithm for the special case of MTP on graphs of maximum degree 4; their algorithm achieves an approximation ratio of 0.833.

Analogously, MWTP can be cast as a special case of the *weighted* 3-*set packing* problem (W3SP for short). Two different algorithms both based on local search have been designed for W3SP [1,2] and they happen to achieve the same approximation ratio of $\frac{1}{2} - \epsilon$ for any constant $\epsilon > 0$. For MWTP specifically, Hassin and Rubinstein [11,12] present a better randomized approximation algorithm with an expected approximation ratio of $\frac{43}{83} - \epsilon$ for any constant $\epsilon > 0$. This ratio has been improved to roughly 0.523 by Chen *et al.* [3,4] and Zuylen [17].

This paper focuses on a common special case of MWTP, namely, the *metric* MWTP problem (MMWTP for short), where the edge weights in the input graph satisfy the triangle inequality. One can almost trivially design a polynomial-time approximation algorithm for MMWTP to achieve an approximation ratio of $\frac{2}{3}$; but surprisingly, prior to this work, no nontrivial approximation algorithm had been designed and analyzed. In this paper, we design the first nontrivial polynomial-time approximation algorithm for MMWTP. Our algorithm is randomized and achieves an expected ratio of $0.66745 - \epsilon$ for any constant $\epsilon > 0$. At the high level, given an instance graph G, our algorithm starts by computing the maximum-weight cycle cover $\mathcal{C}$ in G and then uses $\mathcal{C}$ to construct three triangle packings T_1, T_2, and T_3, among which the heaviest one is the output solution. The computation of T_1 and T_2 is deterministic but that of T_3 is randomized.

The details of the algorithm are presented in the next section. We conclude the paper in the last Sect. 3, with some remarks.

2 The Randomized Approximation Algorithm

Hereafter, let G be a given instance of the problem, and we fix an optimal triangle packing B of G for the following argument. Note that there are $3n$ vertices in the input graph G.

The algorithm starts by computing the maximum weight cycle cover $\mathcal{C}$ of G in polynomial time. Obviously, $w(\mathcal{C}) \geq w(B)$, since B is also a cycle cover. Let ϵ be any constant such that $0 < \epsilon < 1$. A cycle C in $\mathcal{C}$ is *short* if its length is at most $\lceil \frac{1}{\epsilon} \rceil$; otherwise, it is *long*. It is easy to transform each long cycle C in $\mathcal{C}$ into two or more short cycles whose total weight is at least $(1 - \epsilon) \cdot w(C)$. So, we hereafter assume that we have modified the long cycles in $\mathcal{C}$ in this way. Then, $\mathcal{C}$ is a collection of short cycles and $w(\mathcal{C}) \geq (1 - \epsilon) \cdot w(B)$.

We will compute three triangle packings T_1, T_2, T_3 in G. The computation of T_1 and T_2 will be deterministic but that of T_3 will be randomized. Our goal is to prove that for a constant ρ with $0 < \rho < 1$, $\max\{w(T_1), w(T_2), \mathcal{E}[w(T_3)]\} \geq (\frac{2}{3} + \rho) \cdot w(B)$, where $\mathcal{E}[X]$ denotes the expected value of a random variable X.

2.1 Computing T_1

We first compute the maximum weight matching M_1 of size n (*i.e.*, n edges) in G. We then construct an auxiliary complete bipartite graph H_1 as follows. One part of $V(H_1)$, denoted as $V \backslash V(M_1)$, consists of the vertices of G that are not endpoints of M_1; the vertices of the other part of $V(H_1)$, denoted as M_1, one-to-one correspond to the edges in M_1. For each edge $\{x, e = \{u, v\}\}$ in the bipartite graph H_1, where $x \in V \backslash V(M_1)$ and $e \in M_1$, its weight is set to $w(u, x) + w(v, x)$. Next, we compute the maximum weight matching M_1' in H_1 and transform it into a triangle packing T_1 with $w(T_1) = w(M_1) + w(M_1')$.

To compare $w(T_1)$ against $w(B)$, we fix a constant δ with $0 \leq \delta < 1$ and classify the triangles in B into two types as follows. A triangle t in B is *balanced* if the minimum weight of an edge in t is at least $1 - \delta$ times the maximum weight of an edge in t; otherwise, it is *unbalanced*.

Lemma 1. *Let $B_{\bar{b}}$ be the set of unbalanced triangles in B, and $\gamma = \frac{w(B_{\bar{b}})}{w(B)}$. Then, $w(T_1) \geq \left(\frac{2}{3} + \frac{2\gamma\delta}{9-3\delta}\right) \cdot w(B)$.*

Proof. For each t in B, let a_t (respectively, b_t) be the maximum (respectively, minimum) weight of an edge in t. Further let $a = \sum_{t \in B} a_t$ and $b = \sum_{t \in B} b_t$. If $t \in B_{\bar{b}}$, then $b_t < (1 - \delta)a_t$ and in turn $(3 - \delta)a_t > w(t)$. Thus, $\sum_{t \in B_{\bar{b}}} a_t \geq \frac{1}{3-\delta} w(B_{\bar{b}}) \geq \frac{\gamma}{3-\delta} w(B)$. Hence, $w(B) \leq 2a + b \leq 3a - \delta \sum_{t \in B_{\bar{b}}} a_t \leq 3a - \frac{\delta\gamma}{3-\delta} w(B)$ and in turn $a \geq \left(\frac{1}{3} + \frac{\delta\gamma}{9-3\delta}\right) w(B)$. Now, since $w(T_1) \geq 2a$, we finally have $w(T_1) \geq \left(\frac{2}{3} + \frac{2\gamma\delta}{9-3\delta}\right) \cdot w(B)$. □

2.2 Computing T_2

Several definitions are in order. A *partial-triangle packing* in a graph is a subgraph P of the graph such that each connected component of P is a vertex, edge, or triangle. A connected component C of P is a *vertex-component* (respectively, *edge-component* or *triangle-component*) of the graph if C is a vertex (respectively, edge or triangle). The *augmented weight* of P, denoted by $\hat{w}(P)$, is $\sum_t w(t) + 2 \sum_e w(e)$, where t (respectively, e) ranges over all triangle-components (respectively, edge-components) of P. Intuitively speaking, if P has at least as many vertex-components as edge-components, then we can trivially augment P into a triangle packing P' (by adding more edges) so that $w(P')$ is no less than the augmented weight of P.

We classify the triangles t in B into three types as follows.

– t is *completely internal* if all its vertices fall on the same cycle in $\mathcal{C}$.
– t is *partially internal* if exactly two of its vertices fall on the same cycle in $\mathcal{C}$.
– t is *external* if no two of its vertices fall on the same cycle in $\mathcal{C}$.

An edge e of B is *external* if the endpoints of e fall on different cycles in $\mathcal{C}$; otherwise, e is *internal*. In particular, an internal edge e of B is *completely* (respectively, *partially*) *internal* if e appears in a completely (respectively, partially) internal triangle in B. A vertex v of G is *external* if it is incident to no internal edges of B. Let $B_{\bar{e}}$ be the partial-triangle packing in G obtained from B by deleting all external edges.

Now, we are ready to explain how to construct T_2 so that $w(T_2) \geq \hat{w}(B_{\bar{e}})$. Let $C_1, \ldots, C_\ell$ be the cycles in $\mathcal{C}$, and $V_1, \ldots, V_\ell$ be their vertex sets. For each $i \in \{1, \ldots, \ell\}$, let $n_i = |V_i|$, p_i be the number of partially internal edges e in B such that both endpoints of e appear in C_i, q_i be the number of external vertices in C_i, and E_i be the set of edges $\{u, v\}$ in G with $\{u, v\} \subseteq V_i$. Obviously, $n_i - 2p_i - q_i$ is a multiple of 3. For each $i \in \{1, \ldots, \ell\}$, let $\tilde{n}_i = \sum_{h=1}^{i} n_h$, $\tilde{p}_i = \sum_{h=1}^{i} p_h$, and $\tilde{q}_i = \sum_{h=1}^{i} q_h$.

Although we do not know p_i and q_i, we easily see that $0 \leq q_i \leq n_i$ and $0 \leq p_i \leq \lfloor \frac{n_i - q_i}{2} \rfloor$. So, for every $j \in \{0, 1, \ldots, n_i\}$ and every $k \in \{0, 1, \ldots, \lfloor \frac{n_i - j}{2} \rfloor\}$, we compute the maximum-weight (under $\hat{w}$) partial-triangle packing $P_i(j, k)$ in the subgraph of G induced by V_i such that $P_i(j, k)$ has exactly j vertex-components and exactly k edge-components. Since $|V_i|$ is bounded by a constant (namely, $\lceil \frac{1}{\epsilon} \rceil$) from above, the computation of $P_i(j, k)$ takes $O(1)$ time.

Although we do not know $\tilde{p}_i$ and $\tilde{q}_i$, we easily see that $0 \leq \tilde{q}_i \leq \tilde{n}_i$ and $0 \leq \tilde{p}_i \leq \lfloor \frac{\tilde{n}_i - \tilde{q}_i}{2} \rfloor$. For every $j \in \{0, 1, \ldots, \tilde{n}_i\}$ and every $k \in \{0, 1, \ldots, \lfloor \frac{\tilde{n}_i - j}{2} \rfloor\}$, we want to compute the maximum-weight (under $\hat{w}$) partial-triangle packing $\tilde{P}_i(j, k)$ in the graph $(\bigcup_{h=1}^{i} V_h, \bigcup_{h=1}^{i} E_h)$ such that $\tilde{P}_i(j, k)$ has exactly j vertex-components and exactly k edge-components. This can be done by dynamic programming in $O(n^3)$ time as follows. Clearly, $\tilde{P}_1(j, k) = P_1(j, k)$ for every $j \in \{0, 1, \ldots, \tilde{n}_1\}$ and every $k \in \{0, 1, \ldots, \lfloor \frac{\tilde{n}_1 - j}{2} \rfloor\}$. Suppose that $1 \leq i < \ell$ and we have computed $\tilde{P}_i(j, k)$ for every $j \in \{0, 1, \ldots, \tilde{n}_i\}$ and every $k \in \{0, 1, \ldots, \lfloor \frac{\tilde{n}_i - j}{2} \rfloor\}$. For every $j \in \{0, 1, \ldots, \tilde{n}_{i+1}\}$ and every $k \in \{0, 1, \ldots, \lfloor \frac{\tilde{n}_{i+1} - j}{2} \rfloor\}$, we can compute $\tilde{P}_{i+1}(j, k)$ by finding a pair (j', k') such that $j' \in \{0, 1, \ldots, n_{i+1}\}$, $k' \in \{0, 1, \ldots, \lfloor \frac{n_{i+1} - j'}{2} \rfloor\}$, and $\hat{w}(P_{i+1}(j', k')) + \hat{w}(\tilde{P}_i(j - j', k - k'))$ is maximized. Obviously, $\tilde{P}_{i+1}(j, k) = P_{i+1}(j', k') \cup \tilde{P}_i(j - j', k - k')$.

Finally, we have $\tilde{P}_\ell(j, k)$ for every $j \in \{0, 1, \ldots, 3n\}$ and every $k \in \{0, 1, \ldots, \lfloor \frac{3n - j}{2} \rfloor\}$. We now find a pair (j', k') such that $j' \in \{0, 1, \ldots, 3n\}$, $k' \in \{0, 1, \ldots, \lfloor \frac{3n - j'}{2} \rfloor\}$, $k' \leq j'$, and $\hat{w}(\tilde{P}_\ell(j', k'))$ is maximized. Obviously, $\hat{w}(\tilde{P}_\ell(j', k')) \geq \hat{w}(B_{\bar{e}})$. Moreover, we can easily transform $\tilde{P}_\ell(j', k')$ into a triangle packing T_2 of G with $w(T_2) \geq \hat{w}(\tilde{P}_\ell(j', k'))$ as follows.

1. Arbitrarily select k' vertex-components of $\tilde{P}_\ell(j', k')$ and connect them to the edge-components of $\tilde{P}_\ell(j', k')$ so that k' vertex-disjoint triangles are formed.
2. Arbitrarily connect the remaining $(j' - k')$ vertex-components of $\tilde{P}_\ell(j', k')$ into $\frac{j' - k'}{3}$ vertex-disjoint triangles.

In summary, we have shown the following lemma:

Lemma 2. *We can construct a triangle packing* T_2 *of* G *with* $w(T_2) \geq \hat{w}(B_{\bar{e}})$ *in* $O(n^3)$ *time.*

2.3 Computing a Random Matching in $\mathcal{C}$

We compute a random matching M in $\mathcal{C}$ as follows.

1. Initialize two sets $L = \emptyset$ and $M = \emptyset$.
2. For each even cycle C_i in $\mathcal{C}$, perform the following three steps:
   (a) Partition $E(C_i)$ into two matchings $M_{i,1}$ and $M_{i,2}$.
   (b) Select a $j_i \in \{1, 2\}$ uniformly at random.
   (c) Add the edges in M_{i,j_i} to L.
3. For each odd cycle C_i in $\mathcal{C}$, perform the following five steps:
   (a) Select an edge $e_i \in E(C_i)$ uniformly at random.
   (b) Partition $E(C_i)\backslash\{e_i\}$ into two matchings $M_{i,1}$ and $M_{i,2}$.
   (c) Select a $j_i \in \{1, 2\}$ uniformly at random.
   (d) Select an edge $e'_i \in M_{i,j_i}$ uniformly at random and add e'_i to M.
   (e) Add the edges in $M_{i,j_i}\backslash\{e'_i\}$ to L.
4. Select two thirds of edges from L uniformly at random and add them to M.

Lemma 3. *Let* c_o *be the number of odd cycles in* $\mathcal{C}$*. Then, immediately before Step 4,* $|L| = \frac{3}{2} \cdot (n - c_o)$*.*

Proof. Immediately before Step 4, $2|L| = 3n - 3c_o$ and hence $|L| = \frac{3}{2} \cdot (n - c_o)$. □

Lemma 4. $|M| = n$*.*

Proof. Immediately before Step 4, $|M| = c_o$. So, by Lemma 3, $|M| = c_o + (n - c_o) = n$ after Step 4. □

Lemma 5. *For every vertex* v *of* G*,* $\Pr[v \notin V(M)] = \frac{1}{3}$*.*

Proof. First consider the case where v appears in an even cycle in $\mathcal{C}$. In this case, $v \in V(M)$ immediately before Step 4. So, after Step 4, $\Pr[v \notin V(M)] = \frac{1}{3}$.

Next consider the case where v appears in an odd cycle C_i in $\mathcal{C}$. There are two subcases, depending on whether or not v is an endpoint of the edge e_i selected in Step 3a. If v is incident to e_i, then $\Pr[v \notin V(M_{i,j_i})] = \frac{1}{2}$ and $\Pr[v \in V(M_{i,j_i}) \wedge v \notin V(e'_i)] = \frac{1}{2} \cdot \left(1 - \frac{2}{n_i - 1}\right)$. Hence, $\Pr[v \notin V(M) \mid v \in V(e_i)] = \frac{1}{2} + \frac{1}{2} \cdot \left(1 - \frac{2}{n_i-1}\right) \cdot \frac{1}{3} = \frac{2n_i - 3}{3(n_i - 1)}$. On the other hand, if v is not an endpoint of e_i, then $\Pr[v \in V(M_{i,j_i})] = 1$ and $\Pr[v \in V(M_{i,j_i}) \wedge v \notin V(e'_i)] = 1 \cdot \left(1 - \frac{2}{n_i - 1}\right) = \frac{n_i - 3}{n_i - 1}$. Thus, $\Pr[v \notin V(M) \mid v \notin V(e_i)] = \frac{n_i - 3}{n_i - 1} \cdot \frac{1}{3} = \frac{n_i - 3}{3(n_i - 1)}$. Therefore, $\Pr[v \notin V(M)] = \frac{2}{n_i} \cdot \frac{2n_i - 3}{3(n_i - 1)} + \left(1 - \frac{2}{n_i}\right) \cdot \frac{n_i - 3}{3(n_i - 1)} = \frac{1}{3}$. □

Lemma 6. *For every edge e of $\mathcal{C}$,* $\Pr[e \in M] = \frac{1}{3}$.

Proof. First consider the case where e appears in an even cycle in $\mathcal{C}$. In this case, $\Pr[e \in M] = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$.

Next consider the case where e appears in an odd cycle C_i in $\mathcal{C}$. There are two subcases, depending on whether or not e is the edge e_i selected in Step 3a. If $e = e_i$, then $\Pr[e \notin M] = 1$. Hence, $\Pr[e \notin M \mid e = e_i] = 1$. On the other hand, if $e \neq e_i$, then $\Pr[e \notin M_{i,j_i}] = \frac{1}{2}$ and $\Pr[e \neq e'_i \mid e \in M_{i,j_i}] = 1 - \frac{2}{n_i-1} = \frac{n_i-3}{n_i-1}$. Thus, $\Pr[e \notin M \mid e \neq e_i] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{n_i-3}{n_i-1} \cdot \frac{1}{3} = \frac{2n_i-3}{3(n_i-1)}$. Therefore, $\Pr[e \notin M] = \frac{1}{n_i} \cdot 1 + \left(1 - \frac{1}{n_i}\right) \cdot \frac{2n_i-3}{3(n_i-1)} = \frac{2}{3}$. □

Lemma 7. *For every vertex v of G and every edge e of $\mathcal{C}$ such that v and e appear in different cycles in $\mathcal{C}$,* $\Pr[e \in M \wedge v \notin V(M)] \geq \frac{1}{9}$.

Proof. Suppose that v and e appear in $C_{i'}$ and $C_{i''}$, respectively. We distinguish four cases as follows.

Case 1: Both $n_{i'}$ and $n_{i''}$ are even. In this case, $\Pr[v \in V(M_{i',j_{i'}})] = 1$ and $\Pr[e \in M_{i'',j_{i''}}] = \frac{1}{2}$. So, $\Pr[v \in V(M_{i',j_{i'}}) \wedge e \in M_{i'',j_{i''}}] = \frac{1}{2}$. Moreover, by Lemma 3, $\Pr[e \in M \wedge v \notin V(M) \mid v \in V(M_{i',j_{i'}}) \wedge e \in M_{i'',j_{i''}}] = \frac{\binom{|L|-2}{\frac{2}{3}|L|-1}}{\binom{|L|}{\frac{2}{3}|L|}} = \frac{(n-c_o)\cdot\frac{1}{2}(n-c_o)}{\frac{3}{2}(n-c_o)\cdot\left(\frac{3}{2}(n-c_o)-1\right)} \geq \frac{2}{9}$. Thus, $\Pr[e \in M \wedge v \notin V(M)] \geq \frac{2}{9} \cdot \frac{1}{2} = \frac{1}{9}$.

Case 2: $n_{i'}$ is even but $n_{i''}$ is odd. In this case, $\Pr[v \in V(M_{i',j_{i'}})] = 1$ and $\Pr[e \in M_{i'',j_{i''}}] = \frac{1}{2} \cdot \frac{n_{i''}-1}{n_{i''}} = \frac{n_{i''}-1}{2n_{i''}}$. Moreover, $\Pr[e = e'_{i''} \mid e \in M_{i'',j_{i''}}] = \frac{2}{n_{i''}-1}$, $\Pr[e = e'_{i''}] = \frac{1}{n_{i''}}$, and $\Pr[e \in M_{i'',j_{i''}} \setminus \{e'_{i''}\}] = \frac{n_{i''}-1}{2n_{i''}} \cdot \left(1 - \frac{2}{n_{i''}-1}\right) = \frac{n_{i''}-3}{2n_{i''}}$. Furthermore, $\Pr[v \notin V(M) \mid e = e'_{i''}] = \frac{1}{3}$ by Lemma 5, and $\Pr[v \notin V(M) \wedge e \in M \mid e \in M_{i'',j_{i''}} \setminus \{e'_{i''}\}] = \frac{\binom{|L|-2}{\frac{2}{3}|L|-1}}{\binom{|L|}{\frac{2}{3}|L|}} = \frac{(n-c_o)\cdot\frac{1}{2}(n-c_o)}{\frac{3}{2}(n-c_o)\cdot\left(\frac{3}{2}(n-c_o)-1\right)} \geq \frac{2}{9}$. Thus, $\Pr[e \in M \wedge v \notin V(M)] \geq \frac{1}{3} \cdot \frac{1}{n_{i''}} + \frac{2}{9} \cdot \frac{n_{i''}-3}{2n_{i''}} = \frac{1}{9}$.

Case 3: $n_{i'}$ is odd but $n_{i''}$ is even. In this case, $\Pr[v \in V(M_{i',j_{i'}})] = \frac{2}{n_{i'}} \cdot \frac{1}{2} + \left(1 - \frac{2}{n_{i'}}\right) \cdot 1 = \frac{n_{i'}-1}{n_{i'}}$ and $\Pr[e \in M_{i'',j_{i''}}] = \frac{1}{2}$. So, $\Pr[v \in V(M_{i',j_{i'}}) \wedge e \in M_{i'',j_{i''}}] = \frac{n_{i'}-1}{2n_{i'}}$ and $\Pr[v \notin V(M_{i',j_{i'}}) \wedge e \in M_{i'',j_{i''}}] = \frac{1}{2n_{i'}}$. Moreover, by Lemma 3, $\Pr[e \in M \wedge v \notin V(M) \mid v \in V(M_{i',j_{i'}}) \wedge e \in M_{i'',j_{i''}}] = \frac{\binom{|L|-2}{\frac{2}{3}|L|-1}}{\binom{|L|}{\frac{2}{3}|L|}} = \frac{(n-c_o)\cdot\frac{1}{2}(n-c_o)}{\frac{3}{2}(n-c_o)\cdot\left(\frac{3}{2}(n-c_o)-1\right)} \geq \frac{2}{9}$ and $\Pr[e \in M \wedge v \notin V(M) \mid v \notin V(M_{i',j_{i'}}) \wedge e \in M_{i'',j_{i''}}] = \frac{2}{3}$. Thus, $\Pr[e \in M \wedge v \notin V(M)] \geq \frac{n_{i'}-1}{2n_{i'}} \cdot \frac{2}{9} + \frac{1}{2n_{i'}} \cdot \frac{2}{3} \geq \frac{1}{9}$.

Case 4: Both $n_{i'}$ and $n_{i''}$ are odd. In this case, $\Pr[v \in V(M_{i',j_{i'}})] = \frac{n_{i'}-1}{n_{i'}}$ and $\Pr[e \in M_{i'',j_{i''}}] = \frac{1}{2} \cdot \frac{n_{i''}-1}{n_{i''}} = \frac{n_{i''}-1}{2n_{i''}}$. Moreover, $\Pr[e = e'_{i''} \mid e \in M_{i'',j_{i''}}] =$

$\frac{2}{n_{i''}-1}$, $\Pr[e = e'_{i''}] = \frac{1}{n_{i''}}$, and $\Pr[e \in M_{i'',j_{i''}} \setminus \{e'_{i''}\}] = \frac{n_{i''}-1}{2n_{i''}} \cdot \left(1 - \frac{2}{n_{i''}-1}\right) = \frac{n_{i''}-3}{2n_{i''}}$. So, $\Pr[v \notin V(M_{i',j_{i'}}) \wedge e = e'_{i'}] = \frac{1}{n_{i'}n_{i''}}$, $\Pr[v \notin V(M_{i',j_{i'}}) \wedge e \in M_{i'',j_{i''}} \setminus \{e'_{i''}\}] = \frac{n_{i''}-3}{2n_{i'}n_{i''}}$, $\Pr[v \in V(M_{i',j_{i'}}) \wedge e = e'_{i'}] = \frac{n_{i'}-1}{n_{i'}n_{i''}}$, $\Pr[v \in V(M_{i',j_{i'}}) \wedge e \in M_{i'',j_{i''}} \setminus \{e'_{i''}\}] = \frac{(n_{i'}-1)(n_{i''}-3)}{2n_{i'}n_{i''}}$. Obviously, $\Pr[e \in M \wedge v \notin V(M) \mid v \in V(M_{i',j_{i'}}) \wedge e = e'_{i''}] = \frac{1}{3}$, $\Pr[e \in M \wedge v \notin V(M) \mid v \notin V(M_{i',j_{i'}}) \wedge e = e'_{i''}] = 1$, and $\Pr[e \in M \wedge v \notin V(M) \mid v \notin V(M_{i',j_{i'}}) \wedge e \in M_{i'',j_{i''}} \setminus \{e'_{i''}\}] = \frac{2}{3}$. Furthermore, by Lemma 3, $\Pr[e \in M \wedge v \notin V(M) \mid v \in V(M_{i',j_{i'}}) \wedge e \in M_{i'',j_{i''}} \setminus \{e'_{i''}\}] = \frac{\binom{|L|-2}{\frac{2}{3}|L|-1}}{\binom{|L|}{\frac{2}{3}|L|}} = \frac{(n-c_o)\cdot\frac{1}{2}(n-c_o)}{\frac{3}{2}(n-c_o)\cdot\left(\frac{3}{2}(n-c_o)-1\right)} \geq \frac{2}{9}$. Thus, $\Pr[e \in M \wedge v \notin V(M)] \geq \frac{1}{3} \cdot \frac{n_{i'}-1}{n_{i'}n_{i''}} + 1 \cdot \frac{1}{n_{i'}n_{i''}} + \frac{2}{3} \cdot \frac{n_{i''}-3}{2n_{i'}n_{i''}} + \frac{2}{9} \cdot \frac{(n_{i'}-1)(n_{i''}-3)}{2n_{i'}n_{i''}} \geq \frac{1}{9}$. □

2.4 Computing T_3

Fix a constant τ with $0 < \tau < 1$. A *good triplet* is a triplet $(x, y; z)$, where $\{x, y\}$ is an edge of some cycle C_i in $\mathcal{C}$ and z is a vertex of some other cycle C_j in $\mathcal{C}$ with $i \neq j$ such that $w(x, y) \leq (1 - \tau) \cdot (w(x, z) + w(y, z))$.

To compute T_3, we initialize $T_3 = \emptyset$ and proceed as follows.

1. Construct an auxiliary edge-weighted and edge-labeled multi-digraph H_3 as follows. The vertex set of H_3 is $V(G)$. For each good triplet $(x, y; z)$, H_3 contains the two arcs (z, x) and (z, y), each of the two arcs has a weight of $w(x, z) + w(y, z)$ in H_3, the label of (z, x) is y, and the label of (z, y) is x.
2. Compute the maximum-weight matching M_3 in H_3 (by ignoring the direction of each arc).
3. Compute a random matching M in $\mathcal{C}$ as in Sect. 2.3.
4. Let N_3 be the set of all arcs $(z, x) \in M_3$ such that $z \notin V(M)$ and $\{x, y\} \in M$, where y is the label of (z, x). (*Comment:* Since both M and N_3 are matchings, no two arcs in N_3 can share a label. Moreover, the endpoints of each edge in M can be the heads of at most two arcs in N_3.)
5. Initialize $N'_3 = N_3$. For every two arcs (z, x) and (z', y) in N'_3 such that $\{x, y\} \in M$, select one of (z, x) and (z', y) uniformly at random and delete it from N'_3.
6. For each $(z, x) \in N'_3$, let T_3 include the triangle t with $V(t) = \{x, y, z\}$, where y is the label of (z, x). (*Comment:* By Step 5 and the comment on Step 4, the triangles included in T_3 in this step are vertex-disjoint.)
7. Let M' be the set of edges (x, y) in M such that neither x nor y is the head or the label of an arc in N'_3. Further let Z be the set of vertices z in G such that $z \notin V(M)$ and z is not the tail of an edge in N'_3. (*Comment:* Since $|M| = n$ by Lemma 4, the comment on Step 6 implies $|Z| = |M'|$.)
8. Select an arbitrary one-to-one correspondence between the edges in M' and the vertices in Z. For each $z \in Z$ and its corresponding edge (x, y) in M', let T_3 include the triangle t with $V(t) = \{x, y, z\}$.

We classify external balanced triangles in B into two types as follows. An external balanced triangle t in B is of *Type 1* if for each vertex v of t, the weight of each edge incident to v in $\mathcal{C}$ is at least $\frac{1}{2}(1-\frac{1}{2}\delta)(1-\tau)w(t)$; otherwise, t is of *Type 2*.

Similarly, we classify partially internal balanced triangles in B into two types as follows. A partially internal balanced triangle t in B is of *Type 1* if the weight of each edge incident to the external vertex of t in $\mathcal{C}$ is at least $\frac{1}{2}(1-\frac{1}{2}\delta)(1-\tau)w(t)$; otherwise, t is of *Type 2*.

Lemma 8. *Let B_1^e be the set of Type-1 external balanced triangles in B. Further let B_1^p be the set of Type-1 partially internal balanced triangles in B. Then, $w(T_1) \geq \frac{2}{3}w(B) + \frac{2-3\delta-6\tau+3\delta\tau}{54}w(B_1^e) + \frac{2-3\delta-6\tau+3\delta\tau}{162}w(B_1^p)$.*

Proof. For the analysis, we use the triangles in $B_1^e \cup B_1^p$ to construct a random matching N in $\mathcal{C}$ as follows.

1. Initialize $N' = \emptyset$. For each triangle t in B, select one edge e_t of t uniformly at random and add it to N'.
2. For each triangle t in B_1^e, choose one neighbor v_t' of v_t in $\mathcal{C}$ uniformly at random, where v_t is the vertex of t not incident to e_t.
3. For each triangle t in B_1^p such that e_t is internal, choose one neighbor v_t' of v_t in $\mathcal{C}$ uniformly at random, where v_t is the external vertex of t.
4. Initialize $X = \emptyset$. For each $t \in B_1^e \cup B_1^p$, if $v_t' \notin V(N')$, then add (v_t, v_t') to X.
5. Let D be the digraph with vertex set $V(G)\setminus V(N')$ and arc set X. Partition X into three matchings X_1, X_2, X_3 in D. (*Comment:* Each connected component of the underlying undirected graph of D is either a cycle of $\mathcal{C}$ or a graph of maximum degree at most 3 whose simplified version is a path. Therefore, the partition in this step can be done.)
6. Select a set Y among X_1, X_2, X_3 uniformly at random.
7. Initialize $N = \{e_t \mid t \in B\setminus(B_1^e \cup B_1^p)\}$. For each $t \in B_1^e$, if $(v_t, v_t') \notin Y$, then add e_t to N; otherwise add $\{v_t, v_t'\}$ to N. Similarly, for each $t \in B_1^p$, if e_t is external or $(v_t, v_t') \notin Y$, then add e_t to N; otherwise add $\{v_t, v_t'\}$ to N.

For each triangle $t \in B_1^e$, let E_t be the set of edges e in $\mathcal{C}$ such that e is incident to a vertex of t. Similarly, for each triangle $t \in B_1^p$, let E_t be the set of edges e in $\mathcal{C}$ such that e is incident to the external vertex of t. Consider a $t \in B_1^e$ and an $e = \{x, y\} \in E_t$ with $x \in V(t)$. Obviously, $\Pr[x = v_t] = \frac{1}{3}$ and $\Pr[y = v_t' \mid x = v_t] = \frac{1}{2}$; hence $\Pr[\{v_t, v_t'\} = e] = \frac{1}{6}$. Moreover, $\Pr[v_t' \notin V(N')] = \frac{1}{3}$ and in turn $\Pr[\{v_t, v_t'\} = e \wedge v_t' \notin V(N')] = \frac{1}{18}$. Furthermore, $\Pr[e \in N \mid \{v_t, v_t'\} = e \wedge v_t' \notin V(N')] = \frac{1}{3}$. So, $\Pr[e \in N] = \frac{1}{3} \cdot \frac{1}{18} = \frac{1}{54}$. Now, if $t \in B_1^e$, then $|E_t| = 6$ and in turn $\Pr[e_t \notin N] = 6 \cdot \frac{1}{54} = \frac{1}{9}$. On the other hand, if $t \in B_1^p$, then $|E_t| = 2$ and in turn $\Pr[e_t \notin N] = 2 \cdot \frac{1}{54} = \frac{1}{27}$.

By the discussions in the last paragraph, $\mathcal{E}[w(N)] \geq \frac{1}{3}\sum_{t \in B\setminus(B_1^e \cup B_1^p)} w(t) + \frac{8}{9} \cdot \frac{1}{3}\sum_{t \in B_1^e} w(t) + \frac{1}{9} \cdot \frac{1}{2}(1-\frac{1}{2}\delta)(1-\tau)\sum_{t \in B_1^e} w(t) + \frac{26}{27} \cdot \frac{1}{3}\sum_{t \in B_1^p} w(t) + \frac{1}{27} \cdot \frac{1}{2}(1-\frac{1}{2}\delta)(1-\tau)\sum_{t \in B_1^p} w(t) = \frac{1}{3}w(B) + \frac{2-3\delta-6\tau+3\delta\tau}{108}w(B_1^e) + \frac{2-3\delta-6\tau+3\delta\tau}{324}w(B_1^p)$. So, $w(T_1) \geq 2 \cdot \mathcal{E}[w(N)] \geq \frac{2}{3}w(B) + \frac{2-3\delta-6\tau+3\delta\tau}{54}w(B_1^e) + \frac{2-3\delta-6\tau+3\delta\tau}{162}w(B_1^p)$. □

Lemma 9. *Let B_2^e be the set of Type-2 external balanced triangles in B to $w(B)$. Further let B_2^p be the set of Type-2 partially internal balanced triangles in B. Then, $\mathcal{E}[w(T_3)] \geq \frac{2(1-\epsilon)}{3}w(B) + \frac{(1-\delta)\tau}{54-18\delta} \cdot w(B_2^e) + \frac{(1-\delta)\tau}{54-18\delta} \cdot w(B_2^p)$.*

Proof. For a set F of edges in H_3, let $\tilde{w}(F)$ denote the total weight of edges of F in H_3. Further let W_2 be the total weight of triangles in $B_2^e \cup B_2^p$.

Consider an arbitrary $t \in B_2^e \cup B_2^p$. Since t is of Type 2, t has a vertex v_t such that some neighbor v'_t of v_t in $\mathcal{C}$ satisfies $w(v_t, v'_t) < \frac{1}{2}(1 - \frac{1}{2}\delta)(1 - \tau)w(t)$. Let z_t and z'_t be the vertices in $V(t)\backslash\{v_t\}$. By the triangle inequality, $w(z_t, v'_t) \geq \frac{1}{2}w(z_t, z'_t)$ or $w(z'_t, v'_t) \geq \frac{1}{2}w(z_t, z'_t)$. Without loss of generality, we may assume that $w(z_t, v'_t) \geq \frac{1}{2}w(z_t, z'_t)$. We claim that $(v_t, v'_t; z_t)$ is a good triplet. To see this, first recall that $(1-\delta)w(z'_t, v_t) \leq w(z_t, v_t)$ because t is balanced. So, $(1-\frac{1}{2}\delta)w(z'_t, v_t) \leq (1+\frac{1}{2}\delta)w(z_t, v_t)+\frac{1}{2}\delta w(z_t, z'_t)$ by the triangle inequality. Thus, $(1 - \frac{1}{2}\delta)\left(w(z_t, v_t) + w(z_t, z'_t) + w(z'_t, v_t)\right) \leq 2w(z_t, v_t) + w(z_t, z'_t) \leq 2w(z_t, v_t) + 2w(z_t, v'_t)$. Hence, $\frac{1}{2}(1 - \frac{1}{2}\delta)w(t) \leq w(z_t, v_t) + w(z_t, v'_t)$. Therefore, $w(v_t, v'_t) < \frac{1}{2}(1 - \frac{1}{2}\delta)(1 - \tau)w(t) \leq (1 - \tau)\left(w(z_t, v_t) + w(z_t, v'_t)\right)$. Consequently, the claim holds.

By the claim in the last paragraph, the set X of all $\{z_t, v_t\}$ with $t \in B_2^e \cup B_2^p$ is a matching in H_3. Moreover, $\tilde{w}(M_3) \geq \tilde{w}(X) = \sum_{t\in B_2^e\cup B_2^p} w(z_t, v_t) \geq \frac{1-\delta}{3-\delta}\sum_{t\in B_2^e\cup B_2^p} w(t) = \frac{1-\delta}{3-\delta}W_2$, where the second inequality holds because t is balanced and in turn $w(z_t, v_t) \geq \frac{1-\delta}{3-\delta}w(t)$. Now, by Lemma 7, $\mathcal{E}[\tilde{w}(N_3)] \geq \frac{1}{9}\tilde{w}(M_3) \geq \frac{1-\delta}{27-9\delta}W_2$ and in turn $\mathcal{E}[\tilde{w}(N'_3)] \geq \frac{1-\delta}{54-18\delta}W_2$. Obviously, $w(T_3) \geq 2w(M)+\tau\cdot\tilde{w}(N'_3)$ by the triangle inequality. Therefore, by Lemma 6, $\mathcal{E}[w(T_3)] \geq \frac{2}{3}\cdot w(\mathcal{C}) + \frac{(1-\delta)\tau}{54-18\delta}W_2 \geq \frac{2(1-\epsilon)}{3}\cdot w(B) + \frac{(1-\delta)\tau}{54-18\delta}W_2$. □

2.5 Analyzing the Approximation Ratio

Let B^i be the set of completely internal balanced triangles in B. For convenience, let $\alpha_1 = \frac{w(B^i)}{w(B)}$, $\alpha_2 = \frac{w(B_1^e)}{w(B)}$, $\alpha_3 = \frac{w(B_2^e)}{w(B)}$, $\alpha_4 = \frac{w(B_1^p)}{w(B)}$, and $\alpha_5 = \frac{w(B_2^p)}{w(B)}$. Then, $\gamma + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1$.

We choose $\delta = 0.08$ and $\tau = 0.22$. Then, by Lemmas 1, 2, 8, and 9, we have the following inequalities:

$$\frac{w(T_1)}{w(B)} \geq \frac{2}{3} + \frac{4}{219}\gamma \tag{1}$$

$$\frac{w(T_2)}{w(B)} \geq \alpha_1 + \frac{2}{3}\alpha_4 + \frac{2}{3}\alpha_5 \tag{2}$$

$$\frac{w(T_1)}{w(B)} \geq \frac{2}{3} + \frac{0.2464}{27}\alpha_2 + \frac{0.2464}{81}\alpha_4 \tag{3}$$

$$\frac{\mathcal{E}[w(T_3)]}{w(B)} \geq \frac{2(1-\epsilon)}{3} + \frac{2.53}{657}\alpha_3 + \frac{2.53}{657}\alpha_5. \tag{4}$$

Suppose that we multiply both sides of Inequalities (1), (2), (3), and (4) by 0.1288, 0.00235, 0.2578, and 0.611, respectively. Then, one can easily verify that the summation of the left-hand sides of the resulting inequalities is

$$0.1288\cdot\frac{w(T_1)}{w(B)} + 0.00235\cdot\frac{w(T_2)}{w(B)} + 0.2578\cdot\frac{w(T_1)}{w(B)} + 0.611\cdot\frac{\mathcal{E}[w(T_3)]}{w(B)},$$

while the summation of the right-hand sides is at least

$$\frac{1.9952}{3} - \frac{1.222}{3}\epsilon + 0.00235(\gamma + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5).$$

Now, using $\gamma + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1$, we finally have

$$(0.3866 + 0.00235 + 0.611) \cdot \max\left\{\frac{w(T_1)}{w(B)}, \frac{w(T_2)}{w(B)}, \frac{\mathcal{E}[w(T_3)]}{w(B)}\right\} \geq \frac{2.00225}{3} - \frac{1.222}{3}\epsilon.$$

That is,

$$\max\{w(T_1), w(T_2), \mathcal{E}[w(T_3)]\} \geq (0.66745 - 0.41\epsilon) \cdot w(B).$$

In summary, we have proven the following theorem, stating that the MMWTP problem admits a better approximation algorithm than the trivial $\frac{2}{3}$-approximation.

Theorem 1. *For any constant* $0 < \epsilon < 0.00078$, *the expected approximation ratio achieved by our randomized approximation algorithm is at least* $0.66745 - \epsilon$.

3 Conclusions

We studied the maximum-weight triangle packing problem on an edge-weighted complete graph G, in which the edge weights satisfy the triangle inequality. Although the non-metric variant has been extensively studied in the literature, it is surprising that prior to our work, no nontrivial approximation algorithm had been designed and analyzed for this common metric case. We designed the first nontrivial polynomial-time approximation algorithm for MMWTP, which is randomized and achieves an expected approximation ratio of $0.66745 - \epsilon$ for any positive constant $\epsilon < 0.00078$. This improves the almost trivial deterministic $\frac{2}{3}$-approximation.

Perhaps more dexterous tuning of the parameters inside our algorithm could lead to certain better worst-case performance ratio, but we doubt it will be significantly better. New ideas are needed for the next major improvement.

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