Chapter 9
Closing Remarks: Open Challenges

We have proposed here that piecewise-smooth dynamics can move forward, both in theory and application, by embracing a few less idealized notions, chiefly: *switching layers* as a blurring of ideal discontinuity thresholds, *sliding attractors* as a generalization of sliding modes, and *nonlinear* switching as a more general model of dynamics at a discontinuity. These elements allow us to distinguish between ideal piecewise-smooth analysis, and the results of implementing the discontinuity by a range of practically motivated processes. By bringing such non-idealities together as perturbations we may at last begin moving beyond Filippov's 'linear' theory and begin to fulfil his own larger vision.

Nonsmooth models are popular because they:

1. permit an intuitive geometric description of abrupt change;
2. reduce unsolvable problems to piecewise-solvable sub-problems;
3. give an idealized expression of switching.

Despite their simplicity and idealization, however, they:

1′. are sometimes numerically unstable in ways not fully understood;
2′. are often difficult and messy to analyze, case specific, and subject to a curse of dimensionality (see [58]);
3′. suffer from non-uniqueness.

In short, what nonsmooth systems gain in qualitative simplicity, they lose in analytic simplicity, generality, and determinacy. Nevertheless they have become ever more popular because of their wide applications and seeming ease of implementation. We have highlighted phenomena that reveal some unappreciated dangers of such models, while at the same time facilitating more detailed modelling and analysis.

The uncertainties of nonsmooth models present much greater modelling freedom than is currently taken advantage of. They are an admission of a lack of knowledge of the precise processes involved in a transition, leaving space in our equations to avoid over-modelling of aspects we cannot fix based on available data. They offer a ground-level model, the leading order of some more sophisticated approximation, to

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which we can add nonlinearities and implementations in a manner consistent with empirical data.

In this way, nonsmooth differential equations occupy a middle ground between differentiable equations on the one hand, and stochastic differential equations on the other hand, one fully deterministic and the other entirely non-deterministic. Instead, piecewise-smooth systems are *almost* deterministic, being differentiable in the neighbourhoods of almost all points in space, except at the discontinuity, where indeterminacy enters the picture, with many outcomes that can be studied via switching layers and hidden dynamics.

The peculiar outcome of the investment game says something fundamental about the mathematics of dynamic choice. The interaction of two players' decisions leads to volatile behaviour in the overall system, and this arises despite, and actually directly because, the investor's own dynamics fall onto an attractor. That volatility is breathed life by the incremental details of how each player's decisions are implemented.

The phenomenon of jitter reveals a surprising aspect of sliding dynamics, that despite providing a robust state of motion on a discontinuity threshold, it can nevertheless be responsible for very irregular behaviour. *Jitter* should not be confused with *chatter*, which is an entirely different phenomenon, its less technical usage meaning just the irregular jumping to-and-fro across the discontinuity threshold seen in Figs. 4.2, 4.3, and 7.1, and its more technical usage meaning a Zeno convergence of infinite impacts in finite time (see, e.g., [75]).

The definition of an implementation (5.3) of the system (5.1) is a simplification of Seidman's solution concept. In [124] Seidman loosens Filippov's definition of the convex set $\mathcal{F}(\mathbf{x})$ in a way that permits less idealized discontinuity thresholds, proving the existence of solutions when the set $\mathcal{F}^{\varepsilon}(\mathbf{x})$ contains all values that the vector field attains in an ε-neighbourhood of $\mathbf{x}$. This is a subtle change to Filippov's definition, but means that the discontinuity need only be defined as taking place in some ε-neighbourhood of a threshold, and $\dot{\mathbf{x}}$ need not have a unique, or indeed *any*, value at every point $\mathbf{x}$ in the neighbourhood, provided it has a value at *some* points in the neighbourhood. This is suggestive of a more general interpretation of nonsmooth dynamics as a modelling methodology, and suggests how to more formally define solutions through the *switching layers of implementation* introduced here. (The definition also has the advantage of extending to infinite dimensional systems, see [124]). The definition of switching layers bears some resemblance to boundary layers, to 'inflation' of differential equations (see [64]), and perhaps to other similar analytical methods or implementation techniques, any of which may bring welcome insights to aid in our further understanding of modelling with nonsmooth dynamics.

We summarize our main conclusion as follows. Given a piecewise-defined system (5.1), the inclusion (5.2) can be formed to prove the existence of solutions (using Filippov's theory [51]), but to specify or simulate them we need a more explicit formulation. Many applications are naturally described by (5.10) because they contain certain discontinuous parameters, such as currents being activated, or physical constants changing across material interfaces. Smoothing implementations as we discussed in Sect. 6.3 are exactly described by (5.10) and give dynamics similar to the piecewise-smooth system. Hybrid implementations of switching, of the types discussed in Sect. 6.2, turn out to be best described by the convex hull (5.5a), and if multiple switches are involved can exhibit great variation inside the hull, though noise tends to push the system towards the piecewise-smooth ideal. Switching layers provide a starting point to analyze such dynamics in detail.