Chapter 2
1930–2010: Nonsmooth Dynamics' Linear Age

By a nonsmooth system, we typically mean a system of ordinary differential equations in $\mathbf{x} \in \mathbb{R}^n$, dependent on some discontinuous parameter ν,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}; \nu) , \qquad \nu = \text{step}(\sigma(\mathbf{x})) , \tag{2.1}$$

or defined on disjoint regions,

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{f}^1(\mathbf{x}) & \text{if } \sigma(\mathbf{x}) > 0 , \\ \mathbf{f}^0(\mathbf{x}) & \text{if } \sigma(\mathbf{x}) < 0 . \end{cases} \tag{2.2}$$

The first form is often associated with the application to electronic controllers by V. I. Utkin [146, 147], while the latter was tackled in a more general way by A. F. Filippov [50, 51] using differential inclusions. In fact Utkin and Filippov's works both mainly concern the same situation, as Utkin himself discussed recently in [149], namely when $\mathbf{F}$ is linear in ν such that

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}; \nu) = \nu\mathbf{f}^1(\mathbf{x}) + (1 - \nu)\mathbf{f}^0(\mathbf{x}) . \tag{2.3}$$

A discontinuity may also be introduced by applying some map

$$\mathbf{x} \mapsto \mathbf{R}(\mathbf{x}) \qquad \text{on } \ \sigma(\mathbf{x}) = 0 , \tag{2.4}$$

at the discontinuity, for example, a restitution law $\dot{x} \mapsto -r\dot{x}$ during an impact (see, e.g., [35]), or a drop in mass $m \mapsto m/2$ during cellular mitosis (see, e.g., [43]).

With an explicit expression (2.3) in terms of a discontinuous term ν, Filippov showed that a very substantial theory of qualitative dynamics was possible. The partial solutions in $\sigma > 0$ or $\sigma < 0$ are described by standard dynamical systems theory, and the task of nonsmooth dynamics is to study the effect of concatenating those solutions at $\sigma = 0$, and to find any solutions that may evolve along $\sigma = 0$.

For solutions that traverse the discontinuity transversally there has been considerable progress in extending notions of stability from differentiable systems. To

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characterize the stability of orbits as they cross a discontinuity one may use the saltation matrix (see, e.g., [35, 97]), to describe bifurcations in an orbit's intersection with the discontinuity one has the discontinuity mappings (see [35, 110]), and to study the separation of orbits to establish connections between manifolds or existence of limit cycles one may extend the idea of Melnikov functions (see, e.g., [12, 62, 91], though at present there are large and increasing number of papers investigating this line of study). There is also work ongoing to extend other powerful tools, such as those of inverse integrating factors [23].

A particular area of interest has been the study of oscillations in the presence of impact or other discontinuous control actions. Two key problems concern the number of co-existing limit cycles that a given class of systems may contain, or studying how this number changes via global bifurcations.

The question of the number of limit cycles in a system follows in the spirit of Hilbert's 16th problem [69]. For nonsmooth systems this was dealt a decisive blow with the demonstration that even a piecewise-linear system can have arbitrarily many limit cycles, depending on the shape of the discontinuity threshold [99, 112]. Say $(\dot{x}, \dot{y}) = (-y, \nu)$, where $\nu = \text{sign}(\sigma(x, y))$. For $\sigma = x$ this system is a *fused centre*, similar to the phase portraits in Fig. 1.3 except every orbit is a closed cycle, so this has infinitely many periodic orbits, but they are not *limit* cycles. If we tilt this to $\sigma = x - y$, there are no closed cycles at all, similar to Fig. 1.3(i). But if we oscillate the discontinuity threshold by letting, say, $\sigma = x - \frac{1}{2}\sin(y)$, then every point $(x, y) = (0, n\pi)$ on $\sigma = 0$ for any integer n generates a limit cycle, in an alternating attracting (even n) repelling (odd n) sequence, and therefore infinitely many in number. A considerable literature exists restricting to simpler discontinuity thresholds, allowing multiple thresholds, or (to a lesser extent) allowing the vector fields to be nonlinear, with applications to electronic controllers and impact oscillators among the most studied, see, e.g., [29, 35, 150] for a more detailed review.

The bifurcation of limit cycles has a similarly rich literature. The most important realization has been that the 'hard' character of the discontinuity can actually be exploited to characterize *global* bifurcations by their *local* geometry near $\sigma = 0$. The study began in earnest with Feigin's study of C-bifurcations (the 'C' coming from the Russian word сшивать for solutions that 'sew' through a discontinuity threshold in [44, 45, 46, 47, 48]).

Figure 2.1(i) illustrates two distinct types of limit cycle, one which exists as a smooth cycle in the subsystem $\dot{\mathbf{x}} = \mathbf{f}^0(\mathbf{x})$, the other which passes transversally ('sews') through a discontinuity threshold $\sigma = 0$, with portions passing through both $\dot{\mathbf{x}} = \mathbf{f}^0(\mathbf{x})$ and $\dot{\mathbf{x}} = \mathbf{f}^1(\mathbf{x})$. The intermediary case is a smooth cycle in the subsystem $\dot{\mathbf{x}} = \mathbf{f}^0(\mathbf{x})$ that *grazes* the surface tangentially. Although the dynamics is global, the only substantive change occurs in the local geometry near grazing, Fig. 2.1(middle), which stretches trajectories apart according to a square root scaling. This creates a square root in the global return map to some section Π taken through the flow, Fig. 2.1(right). If an orbit gains or loses segments of sliding in such a bifurcation, then one obtains return maps that are locally piecewise linear or of power $3/2$. These local forms are described by *discontinuity mappings*, and have been derived generally for the lowest codimension grazing bifurcations involving impact or sliding [21, 34, 35, 60, 110].

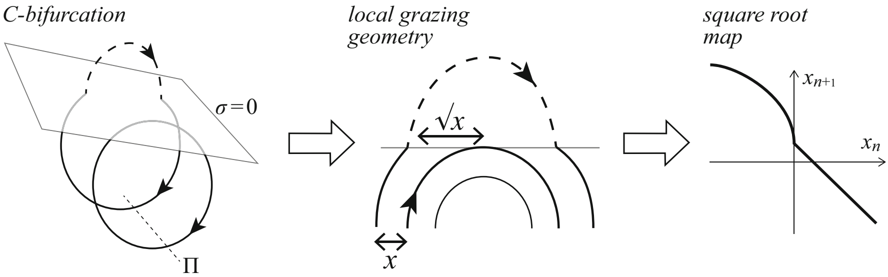

Fig. 2.1 A sewing or 'C' bifurcation, the local geometry near the discontinuity threshold, and the map induced on the return surface *Π*

The most well-studied nonsmooth maps are the square root map like that in Fig. 2.1(iii), piecewise-linear continuous maps $x \mapsto a + x(b + c\,\text{step}(x))$, and maps with a gap $x \mapsto a + bx + (1 + cx)\,\text{step}(x)$. The bifurcations undergone by their fixed points and periodic orbits exhibit considerable complexity, some key results of which can be traced through the papers [8, 11, 53, 56, 57, 73, 89, 100, 110, 114, 130, 133]. Most notable perhaps is the typical form taken by sequences of bifurcations of periodic orbits. For differentiable maps, an important result is the universality of the period doubling sequence by which a limit cycle bifurcates into eventual chaos, Fig. 2.2(i). By contrast, nonsmooth maps are almost unrestricted in the kinds of sequences they can exhibit, with the potential for periodic orbits to be appear of almost arbitrary period, or to jump suddenly to chaos. Figure 2.2(ii) shows a period incrementing sequence, interrupted by jumps to chaos, in a square root map.

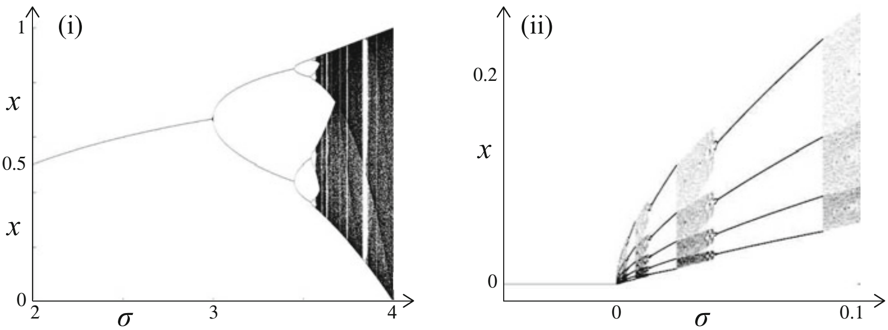

Fig. 2.2 (i) Period doubling cascade to chaos in a differentiable map, showing periodic attractors of the logistic map $x \mapsto \sigma x(1 - x)$. (ii) Period incrementing with windows of chaos in a nonsmooth map, showing periodic attractors of the square root map $x \mapsto \frac{3}{5}x + \sqrt{\sigma - x}\,\text{step}(\sigma - x)$ (from [35])

It remains an open problem to establish in general how nonsmooth flows and their global return maps are related, in particular how singularities and bifurcations of a grazing flow are related to the gradients, gaps, or power laws of return maps, especially in higher dimensions. As well as the power laws associated with grazing, it is known that sliding regions create horizontal or vertical branches in a map, as is illustrated in Fig. 2.3. If the discontinuity set is attracting, then many initial

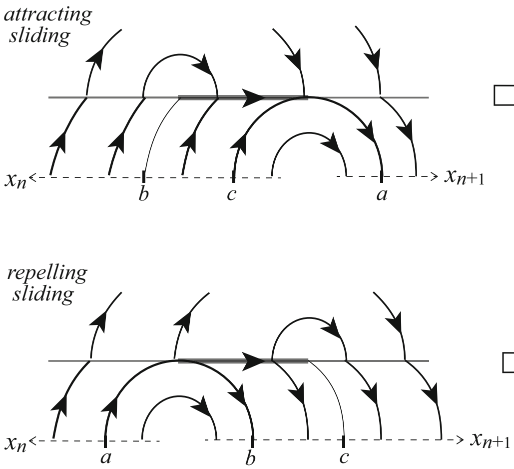

Fig. 2.3 Attracting or repelling sliding in a nonsmooth flow results in horizontal or vertical branches in return maps. Attracting: all points entering the depicted region with a coordinate $x_n \in [b, c]$ maps to an outgoing coordinate $x_{n+1} = a$. Repelling: a point entering the depicted region with a coordinate $x_n = a$ maps to a set of outgoing coordinates $x_{n+1} \in [b, c]$

conditions collapse onto the same sliding trajectory, resulting in a horizontal branch in a return map. Conversely if the discontinuity set is repelling, then any one initial condition in sliding explodes into a continuous family of trajectories outside sliding, resulting in a vertical branch in a return map.

The connection of nonsmooth maps to grazing flows has re-enlivened their study, after they received much attention in the 1970s for their connection to homoclinic bifurcations in differentiable flows, in generating robust chaos, and for their importance in ergodic theory, leading to such prototypes as the tent map, the doubling map or dyadic transformation, the border collision normal form, and the Lozi map. A survey of this topic would be too large to include here, but as a starting point and for some connections to more recent theory the reader may begin with [54, 59, 63].

The main tools necessary to study purely 'sewing' type behaviour—which evolves transversally through a discontinuity threshold—are given by the various methods mentioned above: a saltation matrix to describe how an orbit crosses a discontinuity, a Poincaré map describing how orbits return to a discontinuity, Melnikov methods to describe splitting between such returning orbits, and the discontinuity mappings associated with grazing. Henceforth we will be concerned mainly with far more troublesome issues related to the phenomenon of sliding, that is, evolution along discontinuities.

Filippov proposed in [51] that we study systems like (2.1) or (2.2) by forming a differential inclusion, making the system continuous but set-valued at $\sigma(\mathbf{x}) = 0$, by saying that $\dot{\mathbf{x}}$ lies in a connected set that contains the values of both $\mathbf{f}^0(\mathbf{x})$ and $\mathbf{f}^1(\mathbf{x})$ when $\sigma(\mathbf{x}) = 0$ (assuming the domains of $\mathbf{f}^0$ and $\mathbf{f}^1$ can be extended to such a point on the discontinuity threshold). By placing further assumptions of convexity and continuity on this set (section 4 of [51]), Filippov was able to prove that solutions existed, and even that they vary continuously with parameters and initial conditions.

Filippov's solutions, however, belonged to sets of *possible* solutions, with no criteria to select one solution over any other. He did offer one argument that provides unique solutions under certain conditions of stability, given by forming the differential inclusion from the smallest possible convex set that contains $\mathbf{f}^0(\mathbf{x})$ and $\mathbf{f}^1(\mathbf{x})$ (see section 4 part 1a of [51]). This is simply the family of vector fields generated by the right-hand side of (2.3) as ν varies over $\nu \in [0, 1]$, and constitutes the *linear* formulation of the nonsmooth problem.

Having a definite form (2.3) for the vector field allows us to solve for motion on $\sigma = 0$ as follows. At any $\mathbf{x}$ on $\sigma = 0$, if there exists some ν such that $\dot{\sigma} = 0$, then this gives *sliding* motion along $\sigma = 0$. Substituting that ν back into (2.3) gives the *sliding vector field*. If there exists no such ν, then crossing (or 'sewing') must occur. We already found such dynamics for the boat and genes of Sects. 1.2 and 1.3.

Being differentiable, the subsystems on $\sigma > 0$ and $\sigma < 0$ can exhibit equilibria within their respective domains. The sliding vector field on $\sigma = 0$ can also possess its own *sliding equilibria*. These have been given various names in different contexts, but I find them misleading in the wrong contexts. The term 'switched equilibria' is sometimes found, but we wish here to distinguish equilibria on the switching threshold that do or do not involve sliding. The term 'pseudo-equilibria' is very common, but 'pseudo' may suggest that they are somehow artificial, which they are not. Sliding equilibria are not, in particular, to be confused with 'virtual equilibria', which are points where $\mathbf{f}^1 = 0$ in $\sigma < 0$ or $\mathbf{f}^0 = 0$ in $\sigma > 0$, outside the domains of those vector fields as defined by (2.2)—these can influence dynamics via their drag on the nearby flow, but they do not exist as states of the system (2.2), and therefore are truly artificial. We shall use the term 'sliding equilibria' which more accurately identifies them with the sliding dynamics.

Bifurcations can occur in which equilibria become sliding equilibria or vice versa, or pairs of equilibria and sliding equilibria co-annihilate, known as boundary equilibrium bifurcations [35]. Partial classifications exist for planar systems [35, 51], but one must be careful, because in attempting to form such classifications it is easy to miss less intuitive cases like that in Fig. 2.4, see [74]. Boundary equilibrium bifurcations allow steady states of the system to be created on or off the discontinuity set, or to change stability, and in doing so create periodic orbits as we saw in Fig. 1.3.

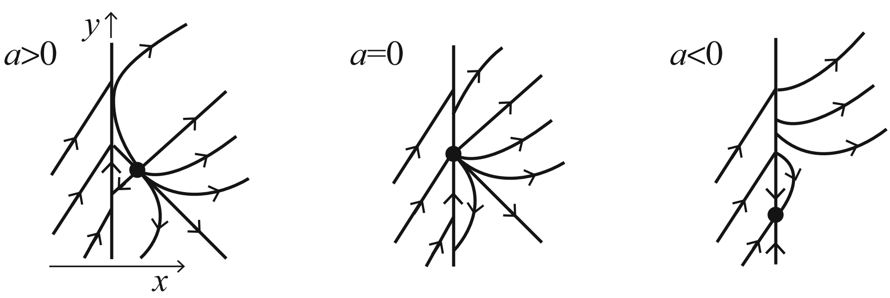

Fig. 2.4 A boundary equilibrium bifurcation in the system $\dot{x} = x + y - a - \nu$, $\dot{y} = y - 2\nu$, $\nu = \text{step}(x)$. A repelling node in $x > 0$ becomes an attracting sliding node on $x = 0$ as a changes sign

There is much to be done in such systems classifying both local and global bifurcations, particularly in systems with multiple switches or more than two dimensions. For more detailed reviews of the state of the art in recent years the reader may also see, e.g., [29, 150]. But bifurcations, oscillations, and chaos are not the whole story. Like the dynamics of smooth systems around the middle of the twentieth century, nonsmooth dynamics is embarking upon its nonlinear age, with all the novel phenomena made possible by nonlinear dependence on discontinuous quantities like ν, and the difference between terms like $\nu, \nu^2, \nu^3, \ldots$ when $\nu = \text{step}(\sigma)$.