Boundary Value Problem with Integral Condition for the Mixed Type Equation with a Singular Coefficient

Natalya Vladimirovna Zaitseva

Abstract We study the boundary value problem for the mixed type equation with a singular coefficient and nonlocal integral first-kind condition. We establish the uniqueness criterion and prove the solution existence and stability theorems. The solution of the problem is constructed explicitly and the proof of convergence of the series in the class of regular solutions is derived.

Keywords Mixed type equation · Singular coefficient · Nonlocal integral condition · Uniqueness · Existence · Stability · Fourier–Bessel series

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1 Introduction

Let $D = \{(x, y)|\, 0 < x < l, -\alpha < y < \beta\}$ be a rectangular domain of coordinate plane Oxy, where l, α, β are given positive real numbers. We introduce denotation: $D_+ = D \cap \{y > 0\}$ and $D_- = D \cap \{y < 0\}$.

In the domain D we consider the elliptic-hyperbolic equation

$$Lu \equiv u_{xx} + (\operatorname{sgn} y)u_{yy} + \frac{p}{x}u_x = 0, \tag{1}$$

where $p \geq 1$ is a given positive real number.

Boundary value problems for mixed type equations are one of the most important topics of the modern theory of partial differential equations. Mathematical models of heat transfer in capillary-porous media, formation of a temperature field, movement of a viscous fluid and many others leads to the problems for equations of this type.

N. V. Zaitseva (✉)
N.I. Lobachevskii Institute of Mathematics and Mechanics, Kazan (Volga Region) Federal University, Kazan, Tatarstan, Russia
e-mail: Natalya.Chepanova@kpfu.ru

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Interest in the degenerate equations is caused not only by the need to solve applied problems, but also by the intense development of the theory of mixed type equations. The first boundary value problem for degenerate partial differential equations of elliptic type with variable coefficients was initially studied in [1]. The research of equations which contains the Bessel differential operator holds a special place in this theory. The study of this class of equations was begun by Euler, Poisson, Darboux and was continued in the theory of generalized axisymmetric potential [1–4]. The equations of the three main classes containing the Bessel operator, according to the [5], are called B-elliptic, B-hyperbolic and B-parabolic, respectively. The boundary value problems for parabolic equations with the Bessel operator are studied in [6, 7], a rather complete review of the papers, devoted to boundary value problems for elliptic equations with singular coefficients is given in monograph [8]. An extensive study of B-hyperbolic equations is presented in [9]. The papers [10–16] are also devoted to the study of boundary value problems for singular equations.

In this paper we study the following nonlocal problem with first-kind integral condition when $p \geq 1$ for Eq. (1) in the domain D.

Statement of the Problem Let $p \geq 1$. We need to find function $u(x, y)$ which satisfies the following conditions:

$$u(x, y) \in C^1(\overline{D}) \cap C^2(D_+ \cup D_-), \tag{2}$$

$$Lu(x, y) \equiv 0, \quad (x, y) \in D_+ \cup D_-, \tag{3}$$

$$u(x, \beta) = \varphi(x), \quad u(x, -\alpha) = \psi(x), \quad 0 \leq x \leq l, \tag{4}$$

$$\int_0^l x^p u(x, y)\, dx = A = \text{const}, \quad -\alpha \leq y \leq \beta, \tag{5}$$

where A is a given real number, $\varphi(x)$, $\psi(x)$ are given smooth enough functions, which satisfy conditions

$$\int_0^l x^p \varphi(x)\, dx = \int_0^l x^p \psi(x)\, dx = A. \tag{6}$$

The boundary value problem (2)–(6) has nonlocal boundary conditions on the sides of the rectangle D. When $p \geq 1$ in the domain of ellipticity D_+ of Eq. (1), due to [1], the segment $x = 0$ is free of boundary condition in the class of bounded solutions. By dividing the variables it is easy to show that in the domain of hyperbollicity D_- of the Eq. (1) there is valid equation

$$u_x(0, y) = 0, \quad -\alpha \leq y \leq \beta. \tag{7}$$

Nonlocal problems for different classes of differential equations are studied in the works [17–24]. The integral condition (5) was introduced in [25] for the heat equation. The boundary value problems with (5)-type integral condition have been studied in [26–28].

2 Uniqueness

Let's represent the solution (1) as

$$x^{-p}\frac{\partial}{\partial x}\left(x^{p}\frac{\partial u}{\partial x}\right)+(\operatorname{sgn} y)u_{yy}=0.$$

Let's multiply it by x^p and integrate it over the x variable with fixed $y \in (-\alpha, 0) \cup (0, \beta)$ on interval from ε to $l-\varepsilon$, where $\varepsilon > 0$ is a number small enough. As a result we will get

$$\int_{\varepsilon}^{l-\varepsilon}\frac{\partial}{\partial x}\left(x^{p}\frac{\partial u}{\partial x}\right)dx+(\operatorname{sgn} y)\int_{\varepsilon}^{l-\varepsilon}x^{p}u_{yy}\,dx,$$

or

$$\left(x^{p}\frac{\partial u}{\partial x}\right)\Bigg|_{\varepsilon}^{l-\varepsilon}+(\operatorname{sgn} y)\frac{d^{2}}{dy^{2}}\int_{\varepsilon}^{l-\varepsilon}x^{p}u(x,y)\,dx=0.$$

At $\varepsilon \to 0$, due to the conditions (2) and (5) we will get the local boundary condition

$$u_x(l,y)=0,\quad -\alpha \le y \le \beta. \tag{8}$$

In what follows we will consider the problem (2)–(4), (8) instead of (2)–(6).

We will look for particular solutions of the Eq. (1) which are not equal to zero in the domain $D_+ \cup D_-$ and which satisfy the conditions (2) and (8) in the form $u(x,y) = X(x)Y(y)$. By substituting this product into the Eq. (1) and the condition (8), we will get the following spectral problem with respect to $X(x)$

$$X''(x)+\frac{p}{x}X'(x)+\lambda^{2}X(x)=0,\quad 0<x<l, \tag{9}$$

$$|X(0)|<+\infty,\quad X'(l)=0, \tag{10}$$

where λ^2 is a separation constant.

The general solution of Eq. (9) has the form

$$\widetilde{X}(x) = C_1 x^{\frac{1-p}{2}} J_{\frac{p-1}{2}}(\lambda x) + C_2 x^{\frac{1-p}{2}} Y_{\frac{p-1}{2}}(\lambda x),$$

where $J_\nu(\xi)$, $Y_\nu(\xi)$ are the first-kind and second-kind Bessel functions respectively, $\nu = (p-1)/2$, C_1, C_2 are arbitrary constants.

We put $C_2 = 0$ so the function satisfies the first condition from (10). Since the eigenfunctions of the spectral problem are determined to within a constant factor, we set $C_1 = 1$. Thus, the solution of the Eq. (9), which satisfies the first condition from (10), has the form

$$\widetilde{X}(x) = x^{\frac{1-p}{2}} J_{\frac{p-1}{2}}(\lambda x).$$

Let's note that this function satisfies the condition (7). By substituting the function $\widetilde{X}(x)$ into the second condition from (10) we will get

$$\lambda_0 = 0,$$

$$\widetilde{X}'(l) = \left(x^{\frac{1-p}{2}} J_{\frac{p-1}{2}}(\lambda x) \right)'_x \Big|_{x=l} = -l^{\frac{1-p}{2}} J_{\frac{p+1}{2}}(\lambda l),$$

and now we can obtain

$$J_{\frac{p+1}{2}}(\mu) = 0, \quad \mu = \lambda l. \tag{11}$$

It is known [29, p. 530] that function $J_\nu(\xi)$ with $\nu > -1$ has a countable set of real zeros. We denote the n–th root of the (11) equation by μ_n with given p and find the eigenvalues $\lambda_n = \mu_n / l$ of the problem (9) and (10). According to [30, p. 317] there is valid assimptotic formula for the zeros of the Eq. (11) when n is big enough

$$\mu_n = \lambda_n l = \pi n + \frac{\pi}{4} p + O\left(\frac{1}{n}\right). \tag{12}$$

Let's note that when $\lambda_0 = 0$ the spectral problem (9) and (10) has constant eigenfunction which we will take as one. Thus, the system of eigenfunctions of the problem (9) and (10) has the form

$$\widetilde{X}_0(x) = 1, \quad \lambda_0 = 0, \tag{13}$$

$$\widetilde{X}_n(x) = x^{\frac{1-p}{2}} J_{\frac{p-1}{2}}\left(\frac{\mu_n x}{l}\right) = x^{\frac{1-p}{2}} J_{\frac{p-1}{2}}(\lambda_n x), \quad n \in \mathbb{N}, \tag{14}$$

where eigenvalues λ_n are determined as zeros of the Eq. (11).

Let's note that the system of eigenfunctions (13) and (14) of the problem (9) and (10) is orthogonal in the space $L_2[0, l]$ with a weight x^p and also forms a complete system in this space [31, p. 343].

For further calculations we will use an orthonormal system of functions:

$$X_n(x) = \frac{1}{||\widetilde{X}_n(x)||}\widetilde{X}_n(x), \quad n = 0, 1, 2, \ldots, \tag{15}$$

where

$$||\widetilde{X}_n(x)||^2 = \int_0^l \rho(x)\, \widetilde{X}_n^2(x)\, dx, \quad \rho(x) = x^p. \tag{16}$$

Let $u(x, y)$ be a solution of the problem (2)–(4), (8). Let's introduce the functions

$$u_n(y) = \int_0^l u(x, y)x^p X_n(x)\, dx, \ \ n = 0, 1, 2, \ldots, \tag{17}$$

based on which we consider an auxiliary functions of the form

$$u_{n,\varepsilon}(y) = \int_\varepsilon^{l-\varepsilon} u(x, y)x^p X_n(x)\, dx, \ \ n = 1, 2, \ldots, \tag{18}$$

where $\varepsilon > 0$ is a number small enough. Let's differentiate the Eq. (18) over the y variable twice with $y \in (-\alpha, 0) \cup (0, \beta)$ and with respect to Eq. (1), we will get the equation

$$u''_{n,\varepsilon}(y) = \int_\varepsilon^{l-\varepsilon} u_{yy}(x, y)x^p X_n(x)\, dx = -(\operatorname{sgn} y) \int_\varepsilon^{l-\varepsilon} \left(u_{xx} + \frac{p}{x}u_x\right) x^p X_n(x)\, dx =$$

$$= -(\operatorname{sgn} y) \int_\varepsilon^{l-\varepsilon} \frac{\partial}{\partial x}(x^p u_x) X_n(x)\, dx = -(\operatorname{sgn} y) \left[x^p u_x X_n(x)\Big|_\varepsilon^{l-\varepsilon} - \int_\varepsilon^{l-\varepsilon} x^p u_x X'_n(x)\, dx \right]. \tag{19}$$

From (18), due to Eq. (9), we can obtain

$$u_{n,\varepsilon}(y) = -\frac{1}{\lambda_n^2}\int\limits_{\varepsilon}^{l-\varepsilon} u(x,y)x^p\left[X_n''(x) + \frac{p}{x}X_n'(x)\right]dx =$$

$$= -\frac{1}{\lambda_n^2}\int\limits_{\varepsilon}^{l-\varepsilon} u(x,y)\frac{d}{dx}\left(x^p X_n'(x)\right)dx = -\frac{1}{\lambda_n^2}\left[u(x,y)x^p X_n'(x)\Big|_{\varepsilon}^{l-\varepsilon} - \int\limits_{\varepsilon}^{l-\varepsilon} x^p u_x X_n'(x)\,dx\right],$$

and, thus,

$$\int\limits_{\varepsilon}^{l-\varepsilon} x^p u_x X_n'(x)\,dx = \lambda_n^2 u_{n,\varepsilon}(y) + u(x,y)x^p X_n'(x)\Big|_{\varepsilon}^{l-\varepsilon}.$$

By substituting this expression into (19) we will have

$$u_{n,\varepsilon}''(y) = -(\operatorname{sgn} y)\left[x^p u_x X_n(x)\Big|_{\varepsilon}^{l-\varepsilon} - \lambda_n^2 u_{n,\varepsilon}(y) - u(x,y)x^p X_n'(x)\Big|_{\varepsilon}^{l-\varepsilon}\right].$$

By virtue of (2) in the last equation, we can pass to the limit as $\varepsilon \to 0$, from which, according to the conditions (8) and (10) we obtain the following differential equation that we will use to find the functions (17)

$$u_n''(y) - (\operatorname{sgn} y)\lambda_n^2 u_n(y) = 0, \quad y \in (-\alpha, 0) \cup (0, \beta). \tag{20}$$

It's general solution has the form

$$u_n(y) = \begin{cases} a_n e^{\lambda_n y} + b_n e^{-\lambda_n y}, & y > 0, \\ c_n \cos\lambda_n y + d_n \sin\lambda_n y, & y < 0, \end{cases} \tag{21}$$

where a_n, b_n, c_n, d_n are arbitrary constants which must be defined.

Now we will pick the constants a_n, b_n, c_n and d_n in (21) with respect to (2) such that the conjugation conditions $u_n(0+) = u_n(0-)$, $u_n'(0+) = u_n'(0-)$ are satisfied. Those conditions are satisfied when $a_n = (c_n + d_n)/2$, $b_n = (c_n - d_n)/2$, $n = 1, 2, \ldots$. By substituting the values found in (21) we will have

$$u_n(y) = \begin{cases} c_n \operatorname{ch}\lambda_n y + d_n \operatorname{sh}\lambda_n y, & y > 0, \\ c_n \cos\lambda_n y + d_n \sin\lambda_n y, & y < 0. \end{cases} \tag{22}$$

Now let's substitute (17) into the boundary conditions (4):

$$u_n(\beta) = \int_0^l \varphi(x)x^p X_n(x)\,dx = \varphi_n, \quad u_n(-\alpha) = \int_0^l \psi(x)x^p X_n(x)\,dx = \psi_n. \tag{23}$$

Based on (22) and (23) we can obtain a system for finding the constants c_n and d_n:

$$\begin{cases} c_n \operatorname{ch}\lambda_n\beta + d_n \operatorname{sh}\lambda_n\beta = \varphi_n, \\ c_n \cos\lambda_n\alpha - d_n \sin\lambda_n\alpha = \psi_n, \end{cases} \tag{24}$$

which has the unique solution

$$c_n = \frac{\varphi_n \sin\lambda_n\alpha + \psi_n \operatorname{sh}\lambda_n\beta}{\sin\lambda_n\alpha \operatorname{ch}\lambda_n\beta + \cos\lambda_n\alpha \operatorname{sh}\lambda_n\beta}, \quad d_n = \frac{\varphi_n \cos\lambda_n\alpha - \psi_n \operatorname{ch}\lambda_n\beta}{\sin\lambda_n\alpha \operatorname{ch}\lambda_n\beta + \cos\lambda_n\alpha \operatorname{sh}\lambda_n\beta}, \tag{25}$$

if for all $n \in \mathbb{N}$ the determinant of the system (24) is non-zero:

$$\triangle_n(\alpha, \beta) = \sin\lambda_n\alpha \operatorname{ch}\lambda_n\beta + \cos\lambda_n\alpha \operatorname{sh}\lambda_n\beta \neq 0. \tag{26}$$

By substituting the values we found (25) into (22) we will find the final form of the functions

$$u_n(y) = \begin{cases} \triangle_n^{-1}(\alpha, \beta)\left(\varphi_n \triangle_n(\alpha, y) + \psi_n \operatorname{sh}\lambda_n(\beta - y)\right), & y > 0, \\ \triangle_n^{-1}(\alpha, \beta)\left(\varphi_n \sin\lambda_n(\alpha + y) + \psi_n \triangle_n(-y, \beta)\right), & y < 0. \end{cases} \tag{27}$$

Similarly, we find

$$u_0(y) = \frac{\alpha\varphi_0 + \beta\psi_0}{\alpha + \beta} + \frac{\varphi_0 - \psi_0}{\alpha + \beta}y, \quad y \in (-\alpha, 0) \cup (0, \beta), \tag{28}$$

$$u_0(\beta) = l^{-\frac{p+1}{2}}\sqrt{p+1}\int_0^l \varphi(x)x^p dx = \varphi_0, \quad u_0(-\alpha) = l^{-\frac{p+1}{2}}\sqrt{p+1}\int_0^l \psi(x)x^p dx = \psi_0. \tag{29}$$

When the condition (26) is satisfied, the problem (2)–(4), (8) has the unique solution. Indeed, let $\varphi(x) = \psi(x) \equiv 0$ and $\triangle_n(\alpha, \beta) \neq 0$. Then it follows from (23) and (29) that $\varphi_n = \psi_n \equiv 0$, $n = 0, 1, 2, \ldots$, and it follows from (27) and (28) that $u_n(y) = 0$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Due to (17) we have

$\int_0^l u(x, y)x^p X_n(x)\,dx = 0$. Hence, as the system (15) is complete in the space $L_2[0, l]$ with weight x^p, $u(x, y) = 0$ almost everywhere on the interval $x \in [0, l]$ and for all $y \in [-\alpha, \beta]$. As according to (2) function $u(x, y) \in C(\overline{D})$, then $u(x, y) \equiv 0$ in $\overline{D}$.

Let's suppose that for some values p, l, α, β and some $n = m$ the condition (26) is not satisfied. When $\varphi(x) = \psi(x) \equiv 0$ and $\triangle_m(\alpha, \beta) = 0$ the system (24) is equivalent to one of the equations (let it be the first one)

$$c_m \operatorname{ch}\lambda_m\beta + d_m \operatorname{sh}\lambda_m\beta = 0,$$

which has an infinite set of solutions $\left\{-d_m \dfrac{\operatorname{sh}\lambda_m\beta}{\operatorname{ch}\lambda_m\beta},\ d_m\right\}$. By substituting the values we found into (22) we get

$$u_m(y) = \begin{cases} \widetilde{d_m}\,(\operatorname{sh}\lambda_m y \operatorname{ch}\lambda_m\beta - \operatorname{sh}\lambda_m\beta \operatorname{ch}\lambda_m y)\,, & y \geq 0, \\ \widetilde{d_m}\,(\operatorname{ch}\lambda_m\beta\ \sin\lambda_m y - \operatorname{sh}\lambda_m\beta\ \cos\lambda_m y)\,, & y \leq 0, \end{cases}$$

where $\widetilde{d_m}$ is an arbitrary non-zero constant.

Thus the homogenous problem (2)–(4), (8) has the non-zero solution

$$u_m(x, y) = \begin{cases} \widetilde{d_m}\,(\operatorname{sh}\lambda_m y \operatorname{ch}\lambda_m\beta - \operatorname{sh}\lambda_m\beta \operatorname{ch}\lambda_m y)\,X_m(x), & y \geq 0, \\ \widetilde{d_m}\,(\operatorname{ch}\lambda_m\beta\ \sin\lambda_m y - \operatorname{sh}\lambda_m\beta\ \cos\lambda_m y)\,X_m(x), & y \leq 0, \end{cases} \tag{30}$$

where the functions $X_m(x)$ are determined by (15). It is easy to prove that the built function (30) satisfies all the conditions (2)–(4), (8) when $\varphi(x) = \psi(x) \equiv 0$.

Let's find out for which values of the parameters p, l, α, β the condition (26) is violated. We represent $\triangle_n(\alpha, \beta)$ as

$$\triangle_n(\alpha, \beta) = \sqrt{\operatorname{ch} 2\lambda_n\beta}\ \sin(\mu_n\widetilde{\alpha} + \gamma_n), \tag{31}$$

where $\mu_n = \lambda_n l$, $\widetilde{\alpha} = \alpha/l$, $\gamma_n = \arcsin \dfrac{\operatorname{sh}\lambda_n\beta}{\sqrt{\operatorname{ch} 2\lambda_n\beta}} \to \dfrac{\pi}{4}$ at $n \to +\infty$.

This representation shows that $\triangle_n(\alpha, \beta) = 0$, if $\sin(\mu_n\widetilde{\alpha} + \gamma_n) = 0$, that is, if

$$\widetilde{\alpha} = \frac{\pi k - \gamma_n}{\mu_n}, \quad k = 1, 2, \ldots. \tag{32}$$

Thus we proved

Theorem 1 *If the solution of the problem (2)–(4), (8) exists, then it is unique if and only if the condition (26) is satisfied for all $n \in \mathbb{N}$.*

3 Existence

As according to (31) the expression $\triangle_n(\alpha, \beta)$ has a countable set of zeros, we examine the values of this expression, included in the denominators of the formula (27) when n is big enough.

Lemma 1 *If $\widetilde{\alpha} = a/b$ is a rational number, a, b are mutually prime numbers and $p \neq \frac{1}{a}(4bd - b - 4r)$, $r = \overline{1, b-1}$, $d \in \mathbb{Z}$, then there exists constants $C_0 > 0$, $n_0 \in \mathbb{N}$ such that for all $n > n_0$ there is valid inequality*

$$|\triangle_n(\alpha, \beta)| \geq C_0 e^{\lambda_n \beta}. \tag{33}$$

Proof Let's substitute (14) into (31):

$$\triangle_n(\alpha, \beta) = \sqrt{\operatorname{ch} 2\lambda_n \beta}\, \sin\left(\pi n \widetilde{\alpha} + \frac{\pi}{4} p \widetilde{\alpha} + \gamma_n + O\left(\frac{1}{n}\right)\right).$$

Let $\widetilde{\alpha} = a/b$, $a, b \in \mathbb{N}$, $(a, b) = 1$. Let's divide na by b. According to the division theorem we have

$$na = bq + r, \quad q \in \mathbb{N}_0, \quad 1 \leq r \leq b - 1.$$

Then

$$\triangle_n(\alpha, \beta) = \sqrt{\operatorname{ch} 2\lambda_n \beta}\, (-1)^q \sin\left(\frac{\pi r}{b} + \frac{\pi a}{4b} p + \gamma_n + O\left(\frac{1}{n}\right)\right) =$$

$$= \frac{e^{\lambda_n \beta}}{\sqrt{2}} \sqrt{1 + \operatorname{ch} - 4\lambda_n \beta}\, (-1)^q \sin\left(\frac{\pi r}{b} + \frac{\pi a}{4b} p + \frac{\pi}{4} - \varepsilon_n + O\left(\frac{1}{n}\right)\right),$$

where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ at $n \to +\infty$. Thus there is a number n_0, such that for any $n > n_0$ there is valid inequality

$$|\triangle_n(\alpha, \beta)| \geq \frac{e^{\lambda_n \beta}}{2\sqrt{2}} \left|\sin\left(\frac{\pi r}{b} + \frac{\pi a}{4b} p + \frac{\pi}{4}\right)\right| = C_0 e^{\lambda_n \beta}.$$

In order to get $C_0 > 0$ it is necessary that

$$\frac{\pi r}{b} + \frac{\pi a}{4b} p + \frac{\pi}{4} \neq \pi d, \quad d \in \mathbb{Z},$$

hence

$$p \neq \frac{1}{a}(4bd - b - 4r), \quad d \in \mathbb{Z}. \tag{34}$$

The condition (34) is satisfied for any irrational value $p \geq 1$. □

Lemma 2 *If for $n > n_0$ the condition (33) is satisfied, then there are valid estimates*

$$|u_n(y)| \le C_1(|\varphi_n| + |\psi_n|), \quad y \in [-\alpha, \beta], \tag{35}$$

$$|u_n'(y)| \le C_2 n(|\varphi_n| + |\psi_n|), \quad y \in [-\alpha, \beta], \tag{36}$$

$$|u_n''(y)| \le C_3 n^2(|\varphi_n| + |\psi_n|), \quad y \in [-\alpha, 0), \tag{37}$$

$$|u_n''(y)| \le C_4 n^2(|\varphi_n| + |\psi_n|), \quad y \in (0, \beta], \tag{38}$$

where C_i are positive constants (here and further).

Proof From formula (27) with respect to (33) we can get

$$|u_n(y)| \le \frac{1}{|\triangle_n(\alpha, \beta)|}\,(|\varphi_n|(\operatorname{sh}\lambda_n\beta + \operatorname{ch}\lambda_n\beta) + |\psi_n|\operatorname{sh}\lambda_n\beta) \le$$

$$\le \frac{1}{C_0 e^{\lambda_n\beta}}\,(|\varphi_n|(\operatorname{sh}\lambda_n\beta + \operatorname{ch}\lambda_n\beta) + |\psi_n|\operatorname{sh}\lambda_n\beta) \le \widetilde{C_1}(|\varphi_n| + |\psi_n|), \quad y \ge 0,$$

$$|u_n(y)| \le \frac{1}{C_0 e^{\lambda_n\beta}}\,(|\varphi_n| + |\psi_n|(\operatorname{sh}\lambda_n\beta + \operatorname{ch}\lambda_n\beta)) \le \widetilde{C_2}(|\varphi_n| + |\psi_n|), \quad y \le 0,$$

where $\widetilde{C_i}$ are positive constants (here and further). By denoting $C_1 = \max\{\widetilde{C_1}, \widetilde{C_2}\}$ we get the estimate (35) for all $n > n_0$ and $y \in [-\alpha, \beta]$.

Let's calculate the derivative $u_n'(y)$ based on (27) and with respect to (33) and formula (12):

$$|u_n'(y)| \le \frac{n}{C_0 e^{\lambda_n\beta}}\,(|\varphi_n|(\operatorname{ch}\lambda_n\beta + \operatorname{sh}\lambda_n\beta) - |\psi_n|\operatorname{ch}\lambda_n\beta) \le \widetilde{C_3} n(|\varphi_n|+|\psi_n|), \quad y \ge 0,$$

$$|u_n'(y)| \le \frac{n}{C_0 e^{\lambda_n\beta}}\,(|\varphi_n| - |\psi_n|(\operatorname{sh}\lambda_n\beta + \operatorname{ch}\lambda_n\beta)) \le \widetilde{C_4} n(|\varphi_n| + |\psi_n|), \quad y \le 0.$$

Form those inequalities we can obtain the estimate (36) for all $n > n_0$ and $y \in [-\alpha, \beta]$, where $C_2 = \max\{\widetilde{C_3}, \widetilde{C_4}\}$.

The validity of the estimates (37) and (38) follows from the equalities (12), (20) and the estimate (35). □

Lemma 3 *For n big enough and for all $x \in [0, l]$ there are valid estimates:*

$$|X_n(x)| \le C_5, \quad |X_n'(x)| \le C_6 n, \quad |X_n''(x)| \le C_7 n^2.$$

Proof of this lemma can be found in [32].

Lemma 4 *If functions* $\varphi(x), \psi(x) \in C^2[0,l]$ *and there exists the derivatives* $\varphi'''(x)$, $\psi'''(x)$ *which has finite variation on* $[0,l]$, *and*

$$\varphi'(0) = \varphi''(0) = \psi'(0) = \psi''(0) = \varphi'(l) = \psi'(l) = 0,$$

then there are valid estimates:

$$|\varphi_n| \le C_8/n^4, \quad |\psi_n| \le C_9/n^4.$$

Proof of this lemma can be found in [32].

Based on the found particular solutions (15), (27) and (28), if the conditions (26) and (33) are satisfied, the solution of the problem (2)–(4), (8) is defined as a Fourier–Bessel series

$$u(x,y) = u_0(y)X_0(x) + \sum_{n=1}^{\infty} u_n(y)X_n(x). \tag{39}$$

We will consider the following series together with the series (39):

$$u_y(x,y) = u_0'(y)X_0(x) + \sum_{n=1}^{\infty} u_n'(y)X_n(x), \quad u_x(x,y) = \sum_{n=1}^{\infty} u_n(t)X_n'(x); \tag{40}$$

$$u_{yy}(x,y) = \sum_{n=1}^{\infty} u_n''(y)X_n(x), \quad u_{xx}(x,y) = \sum_{n=1}^{\infty} u_n(y)X_n''(x). \tag{41}$$

According to Lemmas 2 and 3, for any $(x,y) \in \overline{D}$ the series (39) and (40) are majorized, correspondingly, by the series $C_{10}\sum_{n=1}^{\infty}(|\varphi_n| + |\psi_n|)$, $C_{11}\sum_{n=1}^{\infty} n\,(|\varphi_n| + |\psi_n|)$, and the series (41) for any $(x,y) \in \overline{D_+} \cup \overline{D_-}$ are majorized by the series $C_{12}\sum_{n=1}^{\infty} n^2\,(|\varphi_n| + |\psi_n|)$, which, in turn, according to Lemma 4, are estimates by the number series $C_{13}\sum_{n=1}^{\infty} n^{-2}$. Consequently, by virtue of Weierstrass M-test, the series (39) and (40) converges uniformly in the bounded domain $\overline{D}$ and the series (41) converges uniformly in the bounded domains $\overline{D_+}$ and $\overline{D_-}$. Thus we have built the function $u(x,y)$ which is defined by the series (39) and satisfies all the (2)–(4), (8) problem conditions.

If for numbers $\widetilde{\alpha}$ in Lemma 1, for some natural $n = m = m_1, \dots, m_k$, where $1 \le m_1 < \dots < m_k \le n_0$, $k \in \mathbb{N}$, there is $\Delta_m(\alpha,\beta) = 0$ satisfied, then for the solvability of the problem (2)–(4), (8) it is necessary and sufficient to fulfill the conditions

$$\psi_m \operatorname{ch} \lambda_m \beta - \varphi_m \cos \lambda_m \alpha = 0, \quad m = m_1, \ldots, m_k. \tag{42}$$

In this case, the solution of the problem (2)–(4), (8) is determined by the series

$$u(x, y) = \left(\sum_{n=1}^{m_1-1} + \cdots + \sum_{n=m_{k-1}+1}^{m_k-1} + \sum_{n=m_k+1}^{\infty} \right) u_n(y) X_n(x) + \sum_{n=1} u_m(x, y), \tag{43}$$

where m takes the values $m_1, \ldots, m_k$, and the function $u_m(x, y)$ is determined by the formula (30). If the lower limit is greater than the upper limit in some sums, then these sums should be considered equal to zero.

Thus, we proved

Theorem 2 *Let functions $\varphi(x)$ and $\psi(x)$ satisfy the Lemma 4 conditions and the condition (33) is satisfied for $n > n_0$. Then there exists the unique solution $u(x, y)$ of the problem (2)–(4), (8) determined by the series (39), if $\triangle_n(\alpha, \beta) \neq 0$ for all $n = \overline{1, n_0}$; if $\triangle_m(\alpha, \beta) = 0$ with some $m = m_1, \ldots, m_k \leq n_0$, the problem has a solution determined by (43), if and only if the conditions (42) are satisfied.*

Theorem 3 *Let functions $\varphi(x)$ and $\psi(x)$ satisfy the Lemma 4 conditions and the conditions (6) and the inequality (33) is valid for all $n > n_0$. Then there exists the unique solution $u(x, y)$ of the problem (2)–(6) determined by the series (39), if $\triangle_n(\alpha, \beta) \neq 0$ for all $n = \overline{1, n_0}$; if $\triangle_m(\alpha, \beta) = 0$ with some $m = m_1, \ldots, m_k \leq n_0$, the problem has a solution determined by (43), if and only if the conditions (42) are satisfied.*

Proof Let $u(x, y)$ be a solution of the problem (2)–(4), (8) and functions $\varphi(x)$ and $\psi(x)$ satisfies the theorem conditions. Then the Eq. (1) is valid everywhere on set $D_+ \cup D_-$. Let's multiply the Eq. (1) by x^p and integrate it over the x variable with $y \in (-\alpha, 0) \cup (0, \beta)$ fixed on interval from ε to $l - \varepsilon$, where $\varepsilon > 0$ is small enough. As a result we will get

$$\left(x^p \frac{\partial u}{\partial x} \right) \Bigg|_{\varepsilon}^{l-\varepsilon} + (\operatorname{sgn} y) \int_{\varepsilon}^{l-\varepsilon} x^p u_{yy}(x, y)\, dx = 0. \tag{44}$$

By passing to the limit as $\varepsilon \to 0$ and with respect to conditions (2) and (8), we have

$$\int_0^l u_{yy}(x, t) x^p\, dx = 0.$$

By integrating the last equation over the y variable twice we have

$$\int_0^l u(x, y)x^p\, dx = K_1 y + K_2, \quad K_1,\ K_2 = \text{const.} \tag{45}$$

By putting $y = \beta$ and then $y = -\alpha$ in the Eq. (45) and with respect to the conditions (4) and (6) we get

$$\int_0^l u(x, \beta)x^p\, dx = \int_0^l \varphi(x)x^p\, dx = K_1\beta + K_2 = A,$$

$$\int_0^l u(x, -\alpha)x^p\, dx = \int_0^l \psi(x)x^p\, dx = -\alpha K_1 + K_2 = A,$$

and thus we can find the values of the constants $K_1 = 0$ and $K_2 = A$. Then from the formula (45) we have

$$\int_0^l u(x, y)x^p\, dx = A,$$

which means that the condition (5) is satisfied.

Now let $u(x, y)$ be a solution of the problem (2)–(6). Then from the Eq. (44) we can obtain

$$\left(x^p \frac{\partial u}{\partial x}\right)\Big|_{\varepsilon}^{l-\varepsilon} + (\operatorname{sgn} y)\frac{d^2}{dy^2}\int_{\varepsilon}^{l-\varepsilon} x^p u(x, y)\, dx = 0.$$

By passing to limit as $\varepsilon \to 0$ and according to conditions (2) and (5) we obtain the local second-kind boundary condition $u_x(l, y) = 0$.

Thus, we showed that when the conditions (6) are satisfied, the conditions (5) and (8) are equivalent. This means that the problems (2)–(6) and (2)–(4), (8) are also equivalent. □

4 Stability

Theorem 4 *For the solution of the problem (2)–(6) there is valid estimate*

$$||u(x, y)|| \leq C_{14}(||\varphi(x)|| + ||\psi(x)||),$$

where $||f(x)||^2 = \int\limits_0^l \rho(x)|f(x)|^2 dx, \quad \rho(x) = x^p.$

Proof According to the formula (39) with respect to the estimate (35) we can calculate

$$||u||^2 = \int\limits_0^l x^p u^2(x, y)\, dx = \int\limits_0^l x^p \sum_{n=0}^{\infty} u_n(y) X_n(x) \sum_{m=0}^{\infty} u_m(y) X_m(x)\, dx =$$

$$= \sum_{n=0}^{\infty} u_n^2(y) = u_0^2(y) + \sum_{n=1}^{\infty} u_n^2(y) \leq C_{15}(\varphi_0^2 + \psi_0^2) + 2C_1^2 \sum_{n=1}^{\infty} \left(|\varphi_n|^2 + |\psi_n|^2\right) \leq$$

$$\leq C_{15}(\varphi_0^2 + \psi_0^2) + 2C_1^2 \left(\sum_{n=1}^{\infty} \varphi_n^2 + \sum_{n=1}^{\infty} \psi_n^2\right) = C_{14}\left(||\varphi||^2 + ||\psi||^2\right).$$

□

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