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Vladislav V. Kravchenko
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Transmutation Operators and Applications

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Editors
Vladislav V. Kravchenko
Department of Mathematics
Center for Research and Advanced Studies
National Polytechnic Institute
Queretaro, Mexico

Sergei M. Sitnik
Belgorod State National Research
University (BelGU)
Belgorod, Russia

ISSN 2297-0215 ISSN 2297-024X (electronic)
Trends in Mathematics
ISBN 978-3-030-35913-3 ISBN 978-3-030-35914-0 (eBook)
https://doi.org/10.1007/978-3-030-35914-0

Mathematics Subject Classification (2010): 26A33, 31A10, 31A35, 33Cxx, 34A05, 34A08, 34B24, 34B30, 34L25, 34L40, 35A20, 35A22, 35C15, 35J75, 35L81, 44Axx

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The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This volume *Transmutation Operators and Applications* consists of invited papers gathered in the following three parts:

- Part I. Transmutations, Integral Equations, and Special Functions.
- Part II. Transmutations in ODEs, Forward and Inverse Problems.
- Part III. Transmutations for Partial and Fractional Differential Equations.

The papers in the volume are contributed by experts in transmutation theory and related topics and demonstrate the vitality and importance of this theory and its rich connections with applications in pure mathematics and applied sciences.

Given below is the list of all contributions followed by short abstracts to each paper.

Part I: **Transmutations, Integral Equations, and Special Functions**

- *Vladislav V. Kravchenko (Mexico), Sergey M. Sitnik (Russia). Some recent developments in the transmutation operator approach.*

  This is an editorial introduction paper. It introduces basic notions and results of transmutation theory and gives a brief historical survey with some important references.
- *Amin Boumenir, Vu Kim Tuan (USA). Transmutation operators and their applications.*

  The authors approach the subject of transmutations from the operator theoretic point of view and use them to compare general differential operators and Krein's type of strings. They also examine their existence, construction, and various applications to inverse and computational spectral theory.
- *Lyubov Britvina (Russia). Hankel generalized convolutions with the associated Legendre functions in the kernel and their applications.*

  This investigation is devoted to finding the existence conditions, boundary properties, and applications of convolution operators for the ν-th order

Hankel transform

$$\mathrm{H}_\nu[f](x) = \int_0^\infty f(t) J_\nu(xt)\, t\, dt\,, \quad x \in \mathbb{R}_+\,.$$

The generalized convolutions defined by the Parseval type equalities

$$\mathrm{H}_\nu[h_1](x) = x^{-\nu}\mathrm{H}_\mu[f](x)\mathrm{H}_\mu[g](x)\,,$$
$$\mathrm{H}_\mu[h_2](x) = x^{-\nu}\mathrm{H}_\nu[f](x)\mathrm{H}_\mu[g](x)$$

are considered in spaces $L_1(\mathbb{R}_+, \sqrt{t}dt)$ and $L_2(\mathbb{R}_+, tdt)$. Properties and estimates for the convolution kernel are investigated. Also integral operators are considered related to generalized convolutions for the Hankel transform $\mathrm{H}_\nu[f](x)$. Watson's type theorems for convolution operators are proved, and integral operators with nonsymmetric kernels are studied. Some applications to solving integral equations are given.

- *Djurdje Cvijović (Serbia), Tibor K. Pogány (Croatia). Second type Neumann series related to Nicholson's and to Dixon–Ferrar formula.*

  The second type Neumann series are considered whose building blocks are Nicholson's and to the Dixon–Ferrar formulae for $J_\nu^2(x) + Y_\nu^2(x)$. Related closed-form double definite integral expressions are established by using the associated Dirichlet's series Cahen's Laplace integral for the Nicholson's case. However, using Dixon–Ferrar formula a double definite integral expression is again obtained. Certain open problems are posed in the last section of the chapter.
- *Sh. T. Karimov (Uzbekistan), S. M. Sitnik (Russia). On some generalizations of multidimensional generalized Erdélyi–Kober operators and their applications.*

  The authors investigate the composition of a multidimensional generalized Erdélyi–Kober operator with differential operators of high order. In particular, with powers of the differential Bessel operator. Applications of proved properties to solving the Cauchy problem for a multidimensional polycaloric equation with a Bessel operator are shown. An explicit formula for solving the formulated problem is constructed. In the appendix, we briefly describe a general context for transmutations and integral transforms used in this paper. Such a general context is formed by integral transforms composition method (ITCM).
- *D. B. Karp (Vietnam, Russia), E. G. Prilepkina (Russia). Alternative approach to Miller–Paris transformations and their extensions.*

  The paper deals with Miller–Paris transformations which are extensions of Euler's transformations for the Gauss hypergeometric functions to generalized hypergeometric functions of higher order having integral parameter differences (IPD). In our recent work, we computed the degenerate versions of these transformations corresponding to the case when one parameter difference is equal to a negative integer. The purpose of this paper is to present an independent new derivation of both the general and the degenerate forms of Miller–Paris transformations. In doing so, we employ the generalized Stieltjes transform representation of the generalized hypergeometric functions and some partial fraction expansions. This approach leads to different forms of the characteristic polynomials; one of them appears noticeably simpler than the original form due to Miller and Paris. Two extensions are further presented of the degenerate transformations to the generalized hypergeometric functions with additional free parameters and additional parameters with negative integral differences.

- *S. P. Khekalo, V. V. Meshcheryakov, K. O. Politov (Russia). Transmutation operators for ordinary Dunkl–Darboux operators.*

  The study is developed of transmutation operators for differential–difference operators, analogous to Dunkl operator. The basis for the study of operators' properties is the intertwining operator and Darboux transformations theories.
- *A. A. Larin (Russia). Theorems on restriction of Fourier–Bessel and multidimensional Bessel transforms to spherical surfaces.*

  The paper deals with problems of L_q-summability with a weight over spherical surface of Fourier–Bessel and n-dimensional Bessel transforms for functions from some weighted spaces. The results have applications to PDE theory. Results of this paper may be applied in transmutation theory, for example, for estimating solutions of singular B-elliptic PDEs.
- *V. I. Makovetsky, S. M. Sitnik, (Russia). Necessary condition for the existence of an intertwining operator and classification of transmutations on its basis.*

  The authors study second-order ordinary differential operators with functional coefficients for all derivatives and the Volterra integral operator with a definite kernel. Results of the paper establish a hyperbolic equation and additional conditions that allow one to construct a kernel according to the ODE. The statements of the paper show the possibility of splitting the ODE into classes according to the type of the kernel of the Volterra operator. Examples are considered related to ODE with Pöschl–Teller type potentials, Bessel functions with complex arguments, and Euler's relation for hypergeometric functions.
- *V. F. Molchanov (Russia). Polynomial quantization on line bundles.*

  We expand polynomial quantization on G/H to the case when a representation of the group G on functions on G/H is induced by a character of the subgroup H. As it is well known, the main content of the representation theory is based on intertwining operators—intertwining transforms, transmutations. In this paper, we focus on the Berezin transform. It connects symbols of different types.
- *A. B. Muravnik (Russia). Fourier–Bessel transforms of measures and qualitative properties of solutions of singular differential equations.*

  In this paper, we review a number of results about the Fourier–Bessel transformation of nonnegative functions. For the specified case, weighted L_∞-norms of the spherical mean of $|\hat{f}|^2$ are estimated by its weighted L_1-norms; note that such a phenomenon does not take place in the general case, i.e., without the requirement of the nonnegativity of f. Moreover, unlike the classical case of the Fourier transform, this phenomenon takes place for one-variable functions as well: weighted L_∞-norms of the Fourier–Bessel transform are estimated by its weighted L_2-norms. Those results are applied to the investigation of singular differential equations containing Bessel operators acting with respect to selected spatial variables (the so-called *special variables*); equations of such kind arise in models of mathematical physics with degenerative heterogeneities and in axially symmetric problems. The proposed approach provides a priori estimates for weighted L_∞-norms of the solutions (for ordinary differential equations) and of weighted spherical means of the squared solutions (for partial differential equations).

- *E. L. Shishkina (Poland). Inversion of hyperbolic B-potentials.*

  The paper is devoted to the study of the fractional integral operator which is a negative real power of the singular wave operator generated by Bessel operator and its inverse using weighted generalized functions. Such operators are called hyperbolic B-potentials. Boundedness, Green, and inversion formulas were proved for hyperbolic B-potentials here.
- *S. M. Sitnik (Russia), O. V. Skoromnik (Belorussia). One-dimensional and multi-dimensional integral transforms of Buschman–Erdélyi type with Legendre functions in kernels.*

  This paper consists of two parts. In the first part we give a brief survey of results on Buschman–Erdélyi operators, which are transmutations for the Bessel singular operator. Main properties and applications of Buschman–Erdélyi operators are outlined. In the second part of the paper we consider multi-dimensional integral transforms of Buschman–Erdélyi type with Legendre functions in kernels. Complete proofs are given in this part, main tools are based on Mellin transform properties and usage of Fox H-functions.
- *Vladimir B. Vasilyev (Russia). Distributions, non-smooth manifolds, transmutations, and boundary value problems.*

  The author discusses the problem of constructing the theory of pseudo-differential equations on manifolds with a non-smooth boundary. Using special factorization principle and transmutation operators, we consider some general boundary value problems for elliptic pseudo-differential equations in canonical non-smooth manifolds.

Part II: **Transmutations in ODEs, Forward and Inverse Problems**

- *Sergey Buterin (Russia). On a transformation operator approach in the inverse spectral theory of integral and integro-differential operators.*

  A brief survey is given on using transformation operators in the inverse spectral theory of integral and integro-differential operators possessing a convolutional term to be recovered. The central place of this approach is occupied by reducing the inverse problem to solving some nonlinear equation, which can be solved globally. We illustrate this scheme on several examples, among which there are: one-dimensional perturbation of the convolution operator, Sturm–Liouville type integro-differential operators and an integro-differential Dirac system.
- *Ahmed Fitouhi, Wafa Binous (Tunisia). Expansion in terms of appropriate functions and transmutation.*

This work presents and summarizes the main steps of the work of Fitouhi et al. on the expansions in series of appropriate functions, namely the Bessel functions of the first kind for second-order differential Bessel perturbed operators. By changing functions or variables, we can reduce the operators associated with certain polynomials and special functions to the operators considered like the Jacobi polynomials and the Whittaker functions. Taking into account that the principal part of these operators is closely related to the function of Bessel and that the latter verify recursive relations, we show that their eigenfunctions can be developed in series of Bessel functions which induce two integral representations of Mehler and Sonine type. These representations suggest to define transmutation operators with the second derivative operator for the first one and with the Bessel operator for the second. This new approach is different from that studied by Levitan, Marchenko, Sitnik, and many other authors. It allows in particular to give a series development of the kernels of the transmutation operator and its inverse. In the same direction, further work on the expansion in polynomials of Laguerre and Gegenbauer concerning the perturbed operators with discrete spectrum operators has been the subject of other works but the study of related transmutations is not up to date.

- *A.V. Glushak (Russia). Transmutation operators as a solvability concept of abstract singular equations.*

One of the methods of studying differential equations is the transmutation operators method. Detailed study of the theory of transmutation operators with applications may be found in the literature. Application of transmutation operators establishes many important results for different classes of differential equations including singular differential equations with the Bessel operator

$$B_k = \frac{d^2}{dt^2} + \frac{k}{t}\frac{d}{dt}, \quad k \in \mathbb{R}.$$

For example, singular PDE named Euler–Poisson–Darboux equation (EPD) has the form

$$\frac{\partial^2 u(t,x)}{\partial t^2} + \frac{k}{t}\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x), \quad k > 0, \quad x \in \mathbb{R}^n,$$

where Δ is the space-variable Laplace operator. In previous papers, singular EPD equation was reduced to a simpler wave equation (with $k = 0$) using the appropriate transmutation operator. In this case, the formulas for the solution are written using spherical means acting by spatial variables.

In this paper, transmutation operators are used in more general case when in EPD equation the space-variable Laplace operator is replaced by some abstract operator acting in Banach space. Also some other abstract singular equations will be studied by this method.

- *Ilyes Karoui, Wafa Binous, Ahmed Fitouhi (Tunisia). On the Bessel–Wright operator and transmutation with applications.*

In this paper, we summarize and complete the study of the Bessel–Wright operator and the transmutation operator recently introduced by A. Fitouhi with coauthors. Special motivation is given for the translation operator and the wavelet transform and for the resolution of the associated wave and heat equation.

- *L. A. Khvostchinskaya (Belorussia). On a method of solving integral equations of Carleman type on the pair of segments.*

  The method is considered to solve integral equations of Carleman type on the pair of adjacent and disjoint segments. The problem is reduced to boundary problem of Riemann with piecewise constant matrix and four and five singular points. The solution is expressed via the solution of a differential equation of the Fuchs class in which it was possible to define all the parameters.
- *S. M. Sitnik, O. Yaremko, N. Yaremko (Russia). Transmutation operators and boundary value problems in mechanics.*

  Transmutation operators method is used to solve and study boundary value problems. In this paper, several ways to obtain transformation operators are considered: the finite integral transforms, Neumann series, the Fourier transforms, and reflection techniques. The finite integral transform technique leads to solution in the form of a composition of the Fourier sine transform and inverse finite integral transform. The Neumann series technique implies decomposition of the solution in power series of the shift operator. The Fourier transform technique provides transition to the Fourier images and comparison with the model boundary value problem. Reflection technique involves a consistent approach to the solution as a reflection from the borders. In all cases, the solution of the boundary value problem is obtained as an expansion in the solutions of the model boundary value problem. In some cases, the sum of a series can be calculated in elementary functions. New formulas have been found for solving the Dirichlet problem in a three-dimensional layer.
- *V. A. Yurko (Russia). Solution of inverse problems for differential operators with delay.*

  Non-self-adjoint second-order differential operators with a constant delay are studied. We establish properties of the spectral characteristics and investigate the inverse problem of recovering operators from their spectra. For this nonlinear inverse problem, the uniqueness theorem is proved and an algorithm for constructing the global solution is provided.

Part III: **Transmutations, Integral Equations, and Special Functions**

- *M. Al-Kandari (Kuwait), L. A.-M. Hannaa, Yu. F. Luchko (Germany). Transmutations of the composed Erdélyi-Kober fractional operators and their applications.*

  This chapter provides a survey of an important class of transmutations for the composed Erdélyi–Kober fractional operators and some of their applications. The transmutations are given in a closed form as the generalized Obrechkoff–Stiltjes integral transforms. They translate the composed Erdélyi–Kober fractional operators to multiplication with a power function. These transmutations can be applied for treating the linear fractional integro-differential equations containing both the right- and the left-hand side Erdélyi–Kober fractional derivatives. The equations of this type are subject of active research in fractional calculus of variations and by determination of the scale-invariant solutions of the partial differential equations of fractional order to mention only a few of many relevant research areas.

- *V. E. Fedorov, Aliya A. Abdrakhmanova (Russia). Distributed order equations in Banach spaces with sectorial operators.*

  We study the Cauchy problem for a class of solved with respect to the distributed Gerasimov–Caputo derivative inhomogeneous equations in Banach spaces with a linear unbounded operator, generating an analytic in a sector resolving family of operators. The unique solvability theorem for the Cauchy problem was proved; the form of the solution is found. These results were applied to the research of the Cauchy problem and the Showalter–Sidorov problem for linear inhomogeneous equations in Banach spaces with a degenerate operator at the distributed order derivative. In the case of the generation by the pair of operators (at unknown function and its distributed order derivative) of an analytic resolving family of the corresponding degenerate homogeneous equation, we obtain the theorems of the existence of a unique solution to such problems and derive the form of the solution. Abstract results for the degenerate equation are used for research of initial boundary value problems as to their unique solvability for a class of distributed order in time equations with polynomials of self-adjoint elliptic differential operator with respect to the spatial variables.

- *Mark M. Malamud (Russia). Transformation operators for fractional order ordinary differential equations and their applications.*

  The survey is concerned with triangular transformation operators for fractional order $\alpha = n - \varepsilon$ ordinary differential equations. We discuss the existence of transformation operators in the case of holomorphic coefficients. Similarity between such operators and the simplest fractional differentiation D_0^α is discussed too.

  Applications to the unique determination of the operator from n spectra of boundary value problems are given. Applications to the completeness property of certain boundary value problems for such equations is discussed too.

- *Marina V. Plekhanova, Guzel D. Baybulatova (Russia). Strong solutions of semilinear equations with lower fractional derivatives.*

  We find conditions of a unique strong solution existence for the Cauchy problem to solved with respect to the highest fractional Gerasimov–Caputo derivative semilinear fractional order equation in a Banach space with nonlinear operator, depending on the lower Gerasimov–Caputo derivatives. Then the generalized Showalter–Sidorov problem for semilinear fractional order equation in a Banach space with a degenerate linear operator at the highest order fractional derivative is researched in the sense of strong solution. The nonlinear operator in this equation depends on time and on lower fractional derivatives. The corresponding unique solvability theorem was applied to study of linear degenerate fractional order equation with depending on time linear operators at lower fractional derivatives. Applications of the abstract results are demonstrated on examples of initial-boundary value problems to partial differential equations with time-fractional derivatives.

- *I. P. Polovinkin, M. V. Polovinkina (Russia). Mean value theorems and properties of solutions of linear differential equations.*

  This paper describes an accompanying distributions technique that allows to obtain mean value formulas for linear homogeneous partial differential equations. One of these formulas can be interpreted as a generalization of the Asgeirsson principle for the string vibration equation into the case of an arbitrary natural order. In addition, this mean value formula is an exact difference scheme for a two-dimensional linear homogeneous equation with a symbol factorized up to linear factors.
- *Arsen Pskhu (Russia). Transmutations for multi-term fractional operators.*

  In this paper, we construct a transmutation operator for fractional multi-term differential operators. The constructed operator intertwines multi-term differential operators and the operator of first-order differentiation and allows us to find explicit representations of solutions for initial and boundary value problems for fractional multi-term evolution type differential equations. As an example, we find solutions to a boundary value problem for the multi-term fractional diffusion equation in an unbounded domain.
- *E. L. Shishkina (Poland), S. M. Sitnik (Russia). Fractional Bessel integrals and derivatives on semi-axes.*

  In this paper, we study fractional powers of the Bessel differential operator. The fractional powers are defined explicitly in the integral form without the use of integral transforms in its definitions. Some general properties of the fractional powers of the Bessel differential operator are proved and some are listed. Among them are different variations of definitions, relations with the Mellin and Hankel transforms, group property, evaluation of resolvent integral operator in terms of the Wright, or generalized Mittag–Leffler functions. At the end, some topics are indicated for further study and possible generalizations. Also the aim of the paper is to attract attention and give references to not widely known results on fractional powers of the Bessel differential operator. This class of fractional operators is in close connection with transmutation theory and classic transmutational operators. We also study connections of Bessel fractional operators with different kinds of integral transforms.
- *Marina V. Shitikova (Russia). The fractional derivative expansion method in nonlinear dynamics of structures: a memorial essay.*

  The history of formulation of the efficient method for studying the nonlinear dynamic response of structures, damping features of which depend on natural frequencies of vibrations, is presented. This technique is the modified version of the method of multiple scales. This memorial essay is dedicated to the bright memory of two great scientists, Ali Hasan Nayfeh and Yury Rossikhin, who had gone away one after another in 2 days, March 27 and 29, 2017.
- *N. V. Zaitseva (Russia). Boundary value problem with integral condition for the mixed type equation with a singular coefficient.*

We study the boundary value problem for the mixed type equation with a singular coefficient and nonlocal integral first-kind condition. We establish the uniqueness criterion and prove the solution existence and stability theorems. The solution of the problem is constructed explicitly, and the proof of convergence of the series in the class of regular solutions is derived.

Queretaro, Mexico Vladislav V. Kravchenko
Belgorod, Russia Sergei M. Sitnik

Contents

Part I
Transmutations, Integral Equations and Special Functions

Some Recent Developments in the Transmutation Operator Approach

Vladislav V. Kravchenko and Sergei M. Sitnik

Abstract This is a brief overview of some recent developments in the transmutation operator approach to practical solution of mathematical physics problems. It introduces basic notions and results of transmutation theory, and gives a brief historical survey with some important references. Mainly applications to linear ordinary and partial differential equations and to related boundary value and spectral problems are discussed.

Linear second order differential equations arise in innumerable models and problems of mathematics, physics, engineering, chemistry, biology and even social sciences. While linear ordinary differential equations of first order are easily solved, and the method of their solution is taught to students even of specialities not particularly close to mathematics, the situation of linear ordinary second order differential equations with variable coefficients is pretty much different. No general method for their solution in a closed form is known. On one hand this resembles the situation that had been occurring throughout centuries that separated the full understanding by the antique mathematicians of the algebraic quadratic equations from the epoch of N. Tartaglia, G. Cardano and L. Ferrari when finally algebraic equations of third and fourth orders succumbed to the efforts of mathematicians. On the other hand, the problem of a closed form solution of linear ordinary second order differential equations with variable coefficients is not even contemplated among the most important mathematical problems (of the century or millennium), perhaps because it is not expected to be solved ever.

One of the approaches used at all times is to reduce the difficult problem to a simpler one. Since linear second order equations with constant coefficients admit

V. V. Kravchenko (✉)
Department of Mathematics, Center for Research and Advanced Studies, National Polytechnic Institute, Queretaro, Mexico
e-mail: vkravchenko@math.cinvestav.edu.mx

S. M. Sitnik
Belgorod State National Research University (BelGU), Belgorod, Russia
e-mail: sitnik@bsu.edu.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_1

such a closed-form solution, a natural idea is to relate solutions of the equation with constant coefficients to solutions of the equation with variable coefficients via an operator which is called a transmutation operator. Consider the second order linear differential expression

$$L := -\frac{d^2}{dx^2} + q(x) \tag{1}$$

with q being an L_2-function defined on a finite interval. The equation

$$Ly(x) = \lambda y(x), \quad \lambda \in \mathbb{C} \tag{2}$$

is called the one-dimensional Schrödinger equation or very often the Sturm–Liouville equation, taking into account that a large variety of linear ordinary second order equations reduce to this form by a Liouville transformation.

A transmutation operator is sought to relate L to the simplest linear second order expression $B := -\frac{d^2}{dx^2}$ by the formula

$$LT = TB.$$

If T is linear and invertible its knowledge allows one to solve (2) at least formally. Indeed, one can look for a solution of (2) in the form $y = Tv$, where v is a solution of the equation $Bv = \lambda v$ (whose general solution is of course $v(x) = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x$). Then $Ly = LTv = TBv = \lambda Tv = \lambda y$, thus y is a solution of (2).

This idea in the theory of linear differential equations appeared in 1938 in the work [18] by J. Delsarte and later on it was developed in [1, 8–11, 19, 29, 47–51, 60, 62] and in many other publications. In particular, for Eq. (2) with the Sturm–Liouville operator (1) in [53] it was proved that such an operator T exists and even possesses some wonderful properties. Namely, it can be realized in the form of a linear Volterra integral operator of second kind with a continuous integral kernel. Hence T is invertible and its inverse T^{-1} admits the same form of a linear Volterra integral operator of second kind. Additionally, such T can be chosen to preserve the initial conditions fulfilled by the solutions. In [53] also applications to generalized positive definite functions were proposed. Similarly in [46] such transmutations were constructed on semi-axis with applications to inverse and scattering problems. Also transmutations for the Bessel operator

$$B_c := \frac{d^2}{dx^2} + \frac{c}{x}\frac{d}{dx} \tag{3}$$

of Sonine and Poisson types were introduced into the theory (cf. [8–11, 29, 33, 49, 62] together with transmutations for the permuted Bessel operator

$$L := \frac{d^2}{dx^2} + \frac{c}{x}\frac{d}{dx} + q(x) \tag{4}$$

which were widely applied, cf. [62–64]. A new class of Buschman-Erdélyi transmutations was studied in [29, 59, 61, 62]. For applications to special radial Schrödinger equation and construction of Jost solutions cf. [26, 27]. A general method for constructing transmutations from basic integral transforms called Integral Transform Composition Method (ITCM) was developed in [22, 28, 29, 62]. Transmutations for problems with Stark potentials were considered in [25] and with quantum oscillator potential in [52]. Interesting problems in transmutation theory in connection with fractional powers of Bessel operators were studied in [58]. In papers of E. Shishkina transmutations were applied to Euler–Poisson–Darboux equations [24, 57] and to the potential theory [54–56]. Applications of transmutations to problems in mechanics were considered in [68]. Connections of transmutation theory and generalized analytic functions were studied in [3, 35, 67]. Starting from the paper of V. Stashevskaya [64] a line of studying transmutations based on Paley–Wiener theory was developed in [13, 65, 66]. Applications of Sonine and Poisson type transmutations to pseudo differential and PDE equations were considered in [29, 33]. Applications to hyper-Bessel equations based on Obreshkov transform were studied in [20, 34]. Special representations of transmutation kernels via Bessel function series were developed in [14].

An important property of the Volterra-type transmutation operator related to (1) or (4) consists in the fact that the coefficient q often called the potential, can be easily found whenever the integral kernel of the transmutation operator T happens to be known. This together with other attractive properties converted the transmutation operators into one of the main theoretical tools of spectral theory and especially of the theory of inverse spectral problems developed in the works of V. A. Marchenko, I. M. Gel'fand and B. M. Levitan and of many other mathematicians. During that classical period in transmutation theory many famous problems were studied with the aid of this technique, among them: the inverse problem by a spectral function data via the Marchenko equation, the inverse scattering problem by a scattering data via the Gelfand–Levitan equation, Gelfand–Levitan trace formulas and many other. We refer to the books [1, 8–10, 23, 47, 48, 50, 51, 69] presenting this important and extremely beautiful piece of modern mathematics.

Attempts to convert the transmutation operators of this kind into practical tools for solving different problems of mathematical physics have been made for decades. Many applications to problems of mathematical physics were considered in [8–11]. We mention a series of publications of R. Gilbert with coauthors (referring to [2] and references therein) in which transmutation operators were used for solving acoustic wave propagation problems in inhomogeneous media, the work of D. Colton (see [15]) in which with the aid of transmutation operators complete systems of solutions for parabolic PDEs with variable coefficients were introduced and applied to solution of initial-boundary value problems. In those works the integral kernels of the transmutation operators were computed numerically by the successive approximation method whose implementation complicates since the iterations involve two-dimensional integrals. In [4] the transmutation operator kernel was approximated by a partial sum of its trigonometric series, however based on this method solution of linear ODEs does not seem practical.

In a series of recent publications [5–7, 12, 30, 32, 41, 44, 45] the idea from [2] and [15] to obtain complete systems of solutions of PDEs with variable coefficients as images of complete systems of solutions of PDEs with constant coefficients under the action of an appropriate transmutation operator was further developed based on the observation known since the work of M. K. Fage (see the book [21]) and called in [7] the mapping property of the transmutation operator, which indicates what are the images of integer nonnegative powers of the independent variable under the action of the transmutation operator. They result to be so-called formal powers arising from spectral parameter power series (SPPS) representations of solutions of linear ODEs (see [31, 38]) and for their computation an efficient recurrent integration procedure is developed. Thus, some complete systems of solutions for classes of PDEs can be constructed without knowledge of the transmutation operator itself but simply computing the formal powers.

Another advancement in the efficient construction of the integral transmutation kernels was reported in [39, 40, 42, 43] for transmutation operators with boundary conditions in the origin (according to the terminology used by B. M. Levitan), and in [17, 37] for the transmutation operators with boundary conditions at infinity. Based on the proposed representations for the integral transmutation kernels new practical and efficient methods were developed for solution of forward [17, 39, 40, 43] and inverse spectral and scattering problems [16, 36, 37]. In the case of the forward problems large sets of spectral data can be computed with a nondeteriorating accuracy due to the possibility of convenient uniform estimates for the approximate solutions. Meanwhile the approach developed for solving the inverse problems leads to a direct reduction of the problem to a corresponding system of linear algebraic equations. This new and promising area of the transmutation operator theory and applications is still in its beginning, attracting attention of researchers from different applied fields.

In general, the diversity of the topics associated with the transmutation operator theory and, in particular, of those considered in the present volume reveals the importance of the transmutation operators in a large number of fields as well as their intrinsic interconnections and applications.

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Transmutation Operators and Their Applications

Amin Boumenir and Vu Kim Tuan

Abstract We approach the subject of transmutations from the operator theoretic point of view and use them to compare general differential operators, and Krein's type of strings. We also examine their existence, construction, and their various applications to inverse and computational spectral theory.

Keywords Transmutation · Transformation operators · Krein strings · Sampling

Mathematics Subject Classification (2010) 34B25, 47A68

1 Introduction

The idea of transmutation operators or transformation operators V such that

$$L_2 V = V L_1, \tag{1}$$

where L_i are differential operators goes back to Delsartes, Gelfand, Levitan, Marchenko, Faddeev, et al. in the early 1950s, who established some fundamental ideas. In this survey we recall the main results obtained by the authors around this subject, with a focus on the operator and spectral theory point of view and with applications to inverse spectral problems as well as computational spectral theory. Note that relation (1) does not mean that the operators L_1 and L_2 are similar as their spectra may be totally different.

In all that follows, we denote by L_i, for $i = 1, 2$, self-adjoint operators acting in the separable Hilbert spaces H_i, and usually L_i are differential operators. Assume that their spectra σ_i are simple, and denote their "eigenfunctions" by $y_i(\lambda)$, i.e.

$$L_i y_i(\lambda) = \lambda y_i(\lambda) \quad \text{for} \quad \lambda \in \sigma_i. \tag{2}$$

A. Boumenir (✉) · V. K. Tuan
Department of Mathematics, University of West Georgia, Carrollton, GA, USA
e-mail: boumenir@westga.edu; vu@westga.edu

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_2

One easy way to define a transmutation V, is by pairing their eigenfunctions

$$y_2(\lambda) = V y_1(\lambda) \qquad \text{for } \lambda \in \sigma_1 \cap \sigma_2. \tag{3}$$

In what space would (3) hold will be clarified below, as eigenfunctionals $y_i(\lambda)$ would exist only when $\lambda \in \sigma_i$, and $y_i(\lambda) \in H_i$ only when λ is an eigenvalue, and $y_i(\lambda) \notin H_i$ if λ is in the continuous part of the spectrum.

We shall adopt the following definition for a transmutation

Definition 1 We say that V is a transmutation $L_2 \to L_1$ if

[i] $V : H_1 \longrightarrow H_2$ and $\overline{\mathrm{Dom}(V)} = H_1$
[ii] The set $\Omega := \{f \in \mathrm{Dom}(V) \text{ and } L_1 f \in \mathrm{Dom}(V)\}$ is dense in H_1
[iii] $L_2 V(f) = V L_1(f)$ holds for any $f \in \Omega$.

The above definition agrees with the definition of a transformation operator as given in [60], except for its boundedness. Below we examine the questions of existence, reconstruction, and domains of these transmutations. When a section is dealing with one operator only we shall use L instead of L_i.

If the operator V is invertible, then $L_2 = V L_1 V^{-1}$ and this helps reconstruct the operator L_2 from the knowledge of both L_1 and V. This idea became an essential tool in the solution of the inverse spectral problem by the Gelfand Levitan theory, see [46, 61, 64]. Further concepts and applications of transmutations can be found in the books by Begehr and Gilbert, Carroll, Katrakhov and Sitnik, Levitan, Marchenko, and Trimeche, to name a few, see [4, 38–41, 52, 59, 60, 63, 64, 70, 72].

We briefly outline the main sections in this survey. Section 2 is about the existence and reconstruction of transmutations, Sect. 3 is about transmutations between two Krein strings, and finally Sect. 4 is devoted to their applications in the area of differential equations and spectral theory.

2 Existence and Construction of Transmutations

2.1 Classical Transmutations

It is well known that when dealing with the Sturm Liouville operators

$$L_2(f)(x) = \begin{cases} -f''(x) + q(x) f(x), & x \geq 0, \\ f'(0) - h f(0) = 0, \end{cases} \quad \text{and} \quad L_1(f) = \begin{cases} -f''(x), & x \geq 0, \\ f'(0) = 0, \end{cases} \tag{4}$$

then, for $q \in L^{1,loc}[0,\infty)$, and $q(x)$, $h \in \mathbb{R}$, there exists a Volterra type transmutation, $V = 1 + \mathbf{K}$, that maps their eigensolutions

$$y_2(x,\lambda) = \cos\left(x\sqrt{\lambda}\right) + \int_0^x K(x,t)\cos\left(t\sqrt{\lambda}\right)dt. \tag{5}$$

In case we change the boundary conditions in (4) to $f(0) = 0$, then the pairing between the normalized eigensolutions is also a Volterra type integral operator

$$y_2(x,\lambda) = \frac{\sin\left(x\sqrt{\lambda}\right)}{\sqrt{\lambda}} + \int_0^x L(x,t)\frac{\sin\left(t\sqrt{\lambda}\right)}{\sqrt{\lambda}}dt. \tag{6}$$

Recall that when Fadeev condition holds, $\int_0^\infty (1+x)\,|q(x)|\,dx < \infty$, see [42, 75], we have Jost solutions

$$y_2(x,\lambda) = \exp\left(ix\sqrt{\lambda}\right) + \int_x^\infty H(x,t)\exp\left(it\sqrt{\lambda}\right)dt. \tag{7}$$

If it is also known that when the kernels K and L are C^2 smooth, they additionally satisfy a system of partial differential equations, for example

$$\begin{cases} K_{xx}(x,t) - K_{tt}(x,t) = q(x)K(x,t), & 0 < t < x, \\ K(x,x) = h + \frac{1}{2}\int_0^x q(t)dt, \\ K_t\,(x,0) = 0. \end{cases} \tag{8}$$

We recall the following proposition that can be found in [60, 61, 64, 65].

Proposition 2 *Assume that $K(x,t) \in C^2$, then K is the kernel of the transmutation (5) if and only if it is a solution of (8).*

However the smoothness adds more restrictions on the potential q. Below we look for alternative ways to show the existence of the kernels K and L in (5) and (6).

2.2 Transmutations by Paley–Wiener Theorem

One can prove the existence of the kernel L in (6) by using the Paley–Wiener theorem, instead of solving the hyperbolic system such as (8) which is much more difficult and also requires smoothness. Recall that

$$PW_x = \left\{F \text{ entire: } \int_{-\infty}^{\infty} |F(\lambda)|^2\,d\lambda < \infty \text{ and } F(\lambda) = O\left(e^{|\mathrm{Im}\lambda|x}\right)\right\}, \quad x > 0, \tag{9}$$

then the Paley–Wiener theorem states, [78]

$$F \in PW_x \quad \Leftrightarrow \quad F(\lambda) = \int_{-x}^{x} f(t) e^{-it\lambda} dt, \quad \text{where } f \in L^2(-x,\ x). \tag{10}$$

Let the normalized eigensolution $y(x, \lambda)$ of (4) with $y(0, \lambda) = 0$, i.e. $h = \infty$, be the solution of the IVP

$$\begin{cases} -y''(x, \lambda) + q(x) y(x, \lambda) = \lambda y(x, \lambda), \\ y(0, \lambda) = 0 \ \text{ and } \ y'(0, \lambda) = 1, \end{cases} \tag{11}$$

and so y is also a solution of the integral equation

$$y(x, \lambda^2) = \frac{\sin(x\lambda)}{\lambda} + \int_0^x \frac{\sin((x-t)\lambda)}{\lambda} q(t) y(t, \lambda^2)\, dt. \tag{12}$$

Define the Picard iterations by

$$f_0(x, \lambda) = \frac{\sin(x\lambda)}{\lambda},$$
$$f_n(x, \lambda) = \int_0^x \frac{\sin((x-t)\lambda)}{\lambda} q(t)\ f_{n-1}(t, \lambda) dt \quad \text{for } n \geq 1,$$

to obtain, [42], that for each $x > 0$,

$$|f_n(x, \lambda)| \leq C \frac{x}{1 + |\lambda|\, x} e^{|\mathrm{Im}\lambda| x} \frac{1}{n!} \left(\int_0^x \frac{t}{1 + |\lambda|\, t} |q(t)|\, dt \right)^n,$$

which means that the solution of (12), and also (11), is given by the sum

$$y(x, \lambda^2) = \sum_{n \geq 0} f_n(x, \lambda), \tag{13}$$

which converges absolutely and uniformly in any compact domain of the complex plane provided $xq(x) \in L^{1,loc}[0, \infty)$. It is readily seen that from (13) we have

$$\lambda y(x, \lambda^2) - \sin(x\lambda) = O\left(\frac{1}{|\lambda|}\right) \quad \text{and} \quad \lambda y(x, \lambda^2) - \sin(x\lambda) = O\left(e^{|\mathrm{Im}\lambda| x}\right),$$

which, by (9), implies that $\lambda y(x, \lambda^2) - \sin(x\lambda) \in PW_x$ and by (10), and the fact that $\lambda y(x, \lambda^2) - \sin(x\lambda)$ is an odd function of λ, we have the existence of $L(x, .) \in L^2(0, x)$ such that

$$\lambda y(x, \lambda^2) - \sin(x\lambda) = \int_0^x L(x, t) \sin(t\lambda)\, dt \quad \text{for } x > 0 \text{ and } \lambda \in \mathbb{C},$$

which is (6).

Proposition 3 *Let $y(x,\lambda)$ be a solution to (11) where $xq(x) \in L^{1,loc}[0,\infty)$, then there exists $L(x,.) \in L^2(0,\ x)$ such that*

$$y(x,\lambda) = \frac{\sin\left(x\sqrt{\lambda}\right)}{\sqrt{\lambda}} + \int_0^x L(x,t)\frac{\sin\left(t\sqrt{\lambda}\right)}{\sqrt{\lambda}}dt \qquad \textit{for } x > 0. \tag{14}$$

We now examine the spaces that contain those eigenfunctionals, so that for example the mapping generated by (14) makes sense.

2.3 Rigged Hilbert Spaces

How to find the domain of a transmutation, for example say V defined by (5)? First observe that $\cos(x\sqrt{\lambda}) \notin H = L^2(0,\infty)$ because the spectrum $\sigma_1 = [0,\infty)$ is continuous. It is shown in [47], when $\lambda \in \sigma$, then eigenfunctionals would grow slowly, $y(x,\lambda) = O\left(x^{3/2+\varepsilon}\right)$, whereas when $\lambda \notin \sigma$, then the eigensolutions would grow faster. In general, if $\lambda \in \sigma$, then there exists a Weyl sequence $\xi_n \in H$ such that $\|\xi_n\| = 1$ and $L\xi_n - \lambda\xi_n \to 0$. Obviously if $\{\xi_n\}$ happens to be compact in H then λ belongs to the discrete spectrum, i.e. is an eigenvalue, while if $\{\xi_n\}$ is not compact in H, then λ belongs to the continuous spectrum. Since we are using self-adjoint operators, there is no residual spectrum.

In 1955, Gelfand and Kostuychenko came up with Gelfand's Rigged spaces, to show that eigenfunctions are generalized functions in the case of continuous spectrum. To this end assume that a subspace Φ_i is dense in H_i and compactly embedded in H_i, i.e. $\Phi_i \hookrightarrow H_i$, such as for example Sobolev spaces, and is also invariant under L_i, i.e. $L_i : \Phi_i \to \Phi_i$. Then the identification of the dual of the Hilbert space $H_i' = H_i$ leads to the triplet

$$\Phi_i \hookrightarrow H_i \hookrightarrow \Phi_i', \tag{15}$$

and so a Weyl sequence $\{\xi_n\}$ would either converge in H_i and if not, then certainly in Φ_i', see [47]. Therefore it follows that the transmutation V in (3) in fact maps

$$V : \Phi_1' \to \Phi_2'.$$

Note that V is also densely defined, since $\{y_1(\lambda)\}_{\lambda\in\sigma_1}$ is a complete set of eigenfunctionals in Φ_1'. Using the duality of the spaces and embedding (15) for the two spaces we obtain the following

$$\begin{array}{ccc} \Phi_1' & \xrightarrow{V} & \Phi_2' \\ \cup & & \cup \\ H_1 & & H_2 \\ \cup & & \cup \\ \Phi_1 & \xleftarrow[V']{} & \Phi_2 \end{array}$$

Note that due to the densities $\overline{\Phi_i}^{H_i} = H_i$, the operator V' can be extended by closure, as an operator $\overline{V'} : H_2 \longrightarrow H_1$. Also the operators L_i can be extended to $\widetilde{L}_i : \Phi_i' \longrightarrow \Phi_i'$, which then allows to see the transmutation relation $V\,\widetilde{L}_1 = \widetilde{L}_2\,V$, to hold in the dual spaces. Below we shall show that a dual relation also holds in H_i, namely $L_1 V' = V' L_2$, where V' is extended to an operator acting $H_2 \longrightarrow H_1$.

We can easily construct the spaces Φ_i in case $\sigma_i = \mathbb{R}$. Use the rigged spaces $S \hookrightarrow L^2_{\Gamma_i} \hookrightarrow S'$, where S is the Schwartz space of rapidly decreasing functions, to define $\Phi_i = \mathcal{F}_i(S)$.

The first application of the above diagram using Gelfand's rigged spaces for transmutations is the Gelfand-Levitan theory, [5, 47], see Sect. 4.1.

2.4 Transmutation with Distinct Spectra

We now examine (3) in the case σ_1 and σ_2 are distinct, with the possibility that $\sigma_1 \cap \sigma_2 = \emptyset$. The question is then: How can we still generate a transmutation V and still make sense of (3)? Recall that in case when A and B are finite matrices satisfying $VA = BV$, and if $\lambda \in \sigma_A$, then there exists $w \neq 0$, such that $Aw = \lambda w$ and so $\lambda V w = BVw$ which means $\sigma_A \subset \sigma_B$ which contradicts the fact that $\sigma_A \cap \sigma_B = \emptyset$. To avoid these finite matrices counterexamples, which are possible only when σ_i are finite, we shall consider σ_1 to be an infinite set with a finite accumulation point, see [20, 23–25, 47].
In this section we assume

$$L_i \text{ is a self-adjoint operator acting in } H_i : \Phi_i \hookrightarrow H_i \hookrightarrow \Phi_i',$$
$$\text{with } \Phi_i \subset Dom\,(L_i)\,, \text{ and } L_i\Phi_i \subset \Phi_i. \tag{16}$$

Let $\mathcal{O}$ be an open connected domain containing the real line. Then $\sigma_1 \cup \sigma_2 \subset \mathcal{O}$. Consider the space of analytic functions in $\mathcal{O}$

$$\mathcal{C} = \left\{ F \text{ analytic in } \mathcal{O} : \ (1 + |\lambda|)\, F(\lambda) \in L^2_{\Gamma_1} \cap L^2_{\Gamma_2} \right\}. \tag{17}$$

We can define an operator $E := L^2_{\Gamma_1} \cap \mathcal{C} \to L^2_{\Gamma_2} \cap \mathcal{C}$, since analytic functions defined on σ_1 can be extended uniquely over σ_2. It remains to define the sets $\mathcal{B}_i = \{f \in H_i : \mathcal{F}_i(f) \in \mathcal{C}\}$. The following proposition can be found in [24, Theorem 3.1]

Proposition 4 *Let (16) hold, with $\mathcal{C}$ given by (17), $\Gamma_i = O\left(\lambda^{\beta_i}\right)$, with $\beta_i \in \mathbb{R}$, and let σ_1 be an infinite set with a finite accumulation point. Then there exists an operator $W : \mathcal{B}_1 \to \mathcal{B}_2$, defined by*

$$E\mathcal{F}_1(f)(\lambda) = \mathcal{F}_2(Wf)(\lambda) \qquad \textit{for } f \in \mathcal{B}_1,$$

which transmutes L_i,

$$L_2 W = W L_1 \quad \text{in } \mathcal{B}_2. \tag{18}$$

Proof We have the following diagram connecting the various operators and defining the transmutation W

$$\begin{array}{ccc} \mathcal{F}_1(f) \in \mathcal{C} \subset L^2_{\Gamma_1} & \xrightarrow{E} & \mathcal{F}_1(f) \in \mathcal{C} \subset L^2_{\Gamma_2} \\ \mathcal{F}_1 \uparrow & & \downarrow \mathcal{F}_2^{-1} \\ f \in \mathcal{B}_1 & \underset{W}{\longrightarrow} & Wf \in \mathcal{B}_2 \end{array}.$$

To see (18) use, for $f \in \mathcal{B}_1$,

$$\begin{aligned} \mathcal{F}_2(L_2 W f)(\lambda) &= \lambda \mathcal{F}_2(Wf)(\lambda) = \lambda E \mathcal{F}_1(f)(\lambda) = E \lambda \mathcal{F}_1(f)(\lambda) \\ &= E \mathcal{F}_1(L_1 f)(\lambda) = \mathcal{F}_2(W L_1 f)(\lambda), \end{aligned}$$

which implies that

$$W L_1 f = L_2 W f \quad \text{for } f \in \mathcal{B}_1.$$

2.5 Transmutation with Disjoint Spectra

In the previous section we saw how to construct a transmutation in case σ_1 had a finite accumulation point, which was sufficient to imply the uniqueness of the analytic extension by the operator E. Observe that (18) can also be seen as the homogeneous part of an operator equation in X

$$L_2 X - X L_1 = Y, \tag{19}$$

where Y, L_1 and L_2 are given operators. When L_1 and L_2 are bounded operators, one can prove the existence and uniqueness of a solution X, see [1, 3, 66],

$$X = \frac{1}{2\pi i} \int_{\Gamma} (L_2 - \lambda I)^{-1} Y (L_1 - \lambda I)^{-1} d\lambda,$$

and (19) has a unique solution if and only if (18) has the trivial solution only. Observe that Eq. (18), in the simple case when L_1 and L_2 are finite matrices with disjoint spectra, has the trivial solution only $W = 0$, see also the Sylvester-Rosenblum theorem [3]. It is also known that if L_1 and L_2 are unbounded operators, then uniqueness may not hold, see also examples using the shift operator in[3]. If we define the linear operator τ_{12} by

$$\tau_{12}(X) := L_2 X - X L_1,$$

then (19) becomes

$$\tau_{12}(X) = Y.$$

Thus the existence and uniqueness of a solution X to (17) is equivalent to the invertibility of the operator τ_{12}. It turns out that the spectrum of τ_{12} always contains the direct sum $\overline{\sigma_2 - \sigma_1}$, [1], and so if $\sigma_1 \cap \sigma_2 \neq \emptyset$ then τ_{12} is not invertible. In other words, any nontrivial bounded operator solution W for (18) must belong to the null space of the operator τ_{12}. We now define the interpolation operator which connects both transforms $\mathcal{F}_2\,(f)\,(\lambda)$ and $\mathcal{F}_1\,(f)\,(\lambda)$, [20].

Definition 5 J is an interpolation operator, (I.O.) if

[1] J is a densely closed linear operator $L^2_{\Gamma_1} \xrightarrow{J} L^2_{\Gamma_2}$.
[2] The set $S := \{F \in \mathrm{Dom}(J) \text{ and } \lambda F(\lambda) \in \mathrm{Dom}(J)\}$ is dense in $L^2_{\Gamma_1}$.
[3] For any $F \in S$ we have $\lambda J\,(F)\,(\lambda) = J\,(\lambda F)\,(\lambda)$.

If J is a sampling operator in the classical sense then condition [3] $\lambda J\,(F)\,(\lambda) = J\,(\lambda F)\,(\lambda)$ is obvious; as shown by the following simple example of an interpolation operator.

Example: Let $\sigma_1 = \mathbb{Z}$ where $\mathbb{Z}$ is the set of integers and $\sigma_2 = \{\lambda_n\}$ where $\lambda_n \notin \mathbb{Z}$ and thus $\sigma_1 \cap \sigma_2 = \emptyset$. The Shannon-Whittaker-Kotelnikov sampling theorem, [78], allows us to write down a mapping explicitly for $F \in PW_\pi$

$$F(\mu) := \sum_{n\in\mathbb{Z}} F(n)\frac{\sin(\pi(\mu-n))}{\pi(\mu-n)} \qquad \text{for} \qquad \sum_{n\in\mathbb{Z}} |F(n)|^2 < \infty. \tag{20}$$

Thus take the space $L^2_{\Gamma_1}$ where the measure $\Gamma_1(\lambda) = [\lambda]$ represents the greatest integer function in λ. If $\{F(n)\}_{n\in\mathbb{Z}}$ is given then $\{F(\lambda_n)\}_{n\in\mathbb{Z}}$ can be obtained explicitly, from

$$J(F)(\lambda_n) := \sum_{k\in Z} F(k)\frac{\sin(\pi(\lambda_n-k))}{\pi(\lambda_n-k)}. \tag{21}$$

A mapping $L^2_{\Gamma_1} \xrightarrow{J} L^2_{\Gamma_2}$ can now be defined by the operation in (21) and by (20) we in fact have $J(F)(\lambda_n) = F(\lambda_n)$. It remains to see that condition *[3]* then holds since for $\lambda F(.) \in L^2_{\Gamma_1}$ we have

$$J\,(\lambda F(.))\,(\lambda_n) = \lambda_n F(\lambda_n) = \lambda_n J(F)(\lambda_n).$$

We have the following main result

Proposition 6 *Assume that L_i are unbounded self adjoint operators acting in H_i with spectral functions Γ_i for $i = 1, 2$, let J be a linear operator $L^2_{\Gamma_1} \overset{J}{\to} L^2_{\Gamma_2}$ and define*

$$W = \mathcal{F}_1^{-1}\, J\, \mathcal{F}_2. \tag{22}$$

Then W is a transmutation operator if and only if J is an I.O.

We end this section by recalling that A. Zayed posed the problem of sampling at shifted integers, which was solved by constructing a transmutation between Laguerre operators, see [10].

3 Transmutation for Strings

3.1 Transmutation for Strings

We are concerned now with the existence and representation of transmutation operators between two Krein strings $\mathbb{S}_1$ and $\mathbb{S}_2$, which are respectively defined by

$$\begin{cases} \mathbb{S}_i(f) = -\frac{d}{dM_i(x)}\frac{d^+}{dx^+}f(x), & 0 < x < L, \\ bf'(0) - af(0) = 0, \end{cases} \tag{23}$$

where $dM_i(x)$, for $i = 1, 2$, are Stieltjes measures, i.e. $M_i(x)$ is a real valued function, continuous from the right, nondecreasing and normalized by $M_i(0+) = 0$, [45, 57]. The string $\mathbb{S}_i$ models the vibration of a string and $M_i(x)$ can be seen as its mass between 0 and x, while L is its total length. The constants $a,\ b$ are real with $a^2 + b^2 \neq 0$, and describe how the strings are tied down at the origin. Observe that M_i can include jumps and $\frac{d^+}{dx^+}f(x)$ denotes the usual right derivative at a point x. Recall that $\mathbb{S}_i$, defined by (23) is a symmetric operator, acting in the Hilbert spaces, see [50, 57]

$$L^2_{M_i} = \left\{ f \text{ measurable: } ||f||^2_{M_i} = \int_0^L |f(x)|^2\, dM_i(x) < \infty \right\}.$$

Let us denote by y_i the normalized eigensolutions of the initial value problems

$$\begin{cases} \mathbb{S}_i(y_i(x, \lambda)) = \lambda y_i(x, \lambda), \\ y_i(0, \lambda) = b,\ \ y_i'(0, \lambda) = a. \end{cases} \tag{24}$$

Note also that in general, a string such as $\mathbb{S}_1$ cannot be reduced to a Sturm–Liouville equation such as (4), [50]. Also the Liouville transformation cannot be used unless M_i is C^3 and is strictly increasing. For applications and numerical methods of the string we refer to [37, 41, 42, 45, 58, 60, 67, 73]. To avoid any ambiguity about the division by zero, M.G. Krein interpreted the initial value problem $\frac{-d}{dM_i(x)}\frac{d^+}{dx^+}y(x) = f(x),\ \ y(0) = b,\ \ y'(0) = a$, when $f \in L^2_{M_i}$, as an integral equation

$$y(x) = ax + b - \int_0^x \int_0^t f(\xi)dM_i(\xi)dt. \tag{25}$$

For a self-adjoint extension, we need to examine the right end point. In case the length is infinite, $L = \infty$, it is well known that operator $\mathbb{S}_i$ is in the limit point case at $x = \infty$ if and only if $\int_0^\infty x^2 dM_i(x) = \infty$, see [57, p. 70]. In all that follows we assume that we are in the limit point case, otherwise we must add a boundary condition at $x = \infty$ to make $\mathbb{S}_i$ in (23) self-adjoint. In case the length is finite, $L < \infty$, the type of a boundary condition to be added at $x = L$ depends on the presence of a jump of the mass at $x = L$, which is called "heavy mass", see [45].

When $\mathbb{S}_i$ is self-adjoint, its eigensolutions, (24), form the kernel of the transform associated with $\mathbb{S}_i$

$$L^2_{M_i} \overset{\mathcal{F}_i}{\to} L^2_{\Gamma_i},$$

where

$$\mathcal{F}_i(f)(\lambda) = \int_0^\infty f(x)y_i(x,\lambda)dM_i(x) \qquad \text{and} \qquad f(x) = \int \mathcal{F}_i(f)(\lambda)y_i(x,\lambda)d\Gamma_i(\lambda),$$

and the spectral function Γ_i is non decreasing, right continuous, $\sigma_i =$supp $d\Gamma_i$ is the spectrum of $\mathbb{S}_i$ and the Parseval relation, for any $f,\ g \in L^2_{M_i}$ yields

$$\int_0^\infty f(x)\overline{g(x)}dM_1(x) = \int \mathcal{F}_i(f)(\lambda)\overline{\mathcal{F}_i(g)(\lambda)}d\Gamma_i(\lambda).$$

We now introduce a notation used to compare Stieltjes measures, see [35]

$$d\Gamma_1(\lambda) = O\left(d\Gamma_2\left(\lambda\right)\right) \text{ as } \ \lambda \to \infty,$$

if for all measurable functions with respect to $d\Gamma_1$, and $d\Gamma_2$

$$\int_N^\infty |f(\lambda)|\, d\Gamma_1(\lambda) \le c\int_N^\infty |f(\lambda)|\, d\Gamma_2(\lambda) \text{ holds for large } N.$$

The fact that $d\Gamma_1$ is absolutely continuous with respect to $d\Gamma_2$, is denoted by $d\Gamma_1 \ll d\Gamma_2$ and means there exists $g \in L^{1,loc}_{\Gamma_2}$ such that $d\Gamma_1(\lambda) = g(\lambda)d\Gamma_2(\lambda)$. Similarly $d\Gamma_1 \ll^{2,loc} d\Gamma_2$ means that $g \in L^{2,loc}_{\Gamma_2}$ while $d\Gamma_1 \ll^{\infty} d\Gamma_2$

means $\operatorname{esssup}_{\lambda \in \operatorname{supp} d\Gamma_2} g(\lambda) < \infty$ and finally the cut-off function is defined by $x_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$.

When integrating functions of two variables with respect to one of the variable, we shall indicate it by labeling the measure. For example $f(x,t) \in L^2_{M_1(t)}$ means

$$||f(x,t)||^2_{M_1(t)} = \int_0^\infty |f(x,t)|^2 \, dM_1(t) < \infty.$$

In all that follows we assume that the strings in (23) have infinite lengths, $M_i(0+) = 0$, $L = \infty$, and are self-adjoint. To this end we need either

$$\int_0^\infty x^2 dM_i(x) = \infty \quad \text{for } i = 1, 2 \text{ (LP case at } x = \infty),$$

or $\int_0^\infty x^2 dM_i(x) < \infty$, limit circle case at $x = \infty$, but then we must add a boundary condition there.

The normalized eigenfunctions of $\mathbb{S}_i$, see (23) and (25), satisfy the integral equation

$$y_i(x,\lambda) - ax - b = -\lambda \int_0^\infty (x-t)_+ y_i(t,\lambda) dM_i(t).$$

For any fixed x, we have $(x-t)_+ \in L^2_{M_i(t)}$, and so $-\frac{1}{\lambda}(y_i(x,\lambda) - ax - b)$, as its $\mathcal{F}_i$ transform, belongs to $L^2_{\Gamma_i}$. Therefore, by the Parseval relation we get

$$\int \frac{1}{\lambda^2} |y_i(x,\lambda) - ax - b|^2 \, d\Gamma_i(\lambda) = \int_0^x (x-t)^2 dM_i(t) \quad \text{for } x \geq 0. \tag{26}$$

Similar relations hold for the transform $\mathcal{F}_2$ associated with operator $\mathbb{S}_2$ and its spectral function Γ_2. Using the above relation we have

Proposition 7 *For all* $x \geq 0$ *we deduce*

(i) $\frac{1}{\lambda}(y_i(x,\lambda) - ax - b) \in L^2_{\Gamma_i}$.
(ii) $\int \frac{1}{\lambda^2} |y_i(x,\lambda) - ax - b|^2 \, d\Gamma_i(\lambda) = \int_0^x (x-t)^2 dM_i(t)$.
(iii) The set $\frac{1}{\lambda}(y_i(x,\lambda) - ax - b)$ is complete in $L^2_{\Gamma_i}$.

We now prove the existence of a transmutation by pairing between two eigensolutions of $\mathbb{S}_1$ and $\mathbb{S}_2$

Proposition 8 *Assume that* $d\Gamma_1(\lambda) = O(d\Gamma_2(\lambda))$ *as* $\lambda \to \infty$, *then for each* $x > 0$ *there exists* $H(x,.) \in L^2_{M_1}$ *such that*

$$y_2(x,\lambda) = y_1(x,\lambda) + \lambda \int_0^\infty H(x,t) y_1(t,\lambda) dM_1(t). \tag{27}$$

To this end use the fact that

$$dy_1^+(x,\lambda) = -\lambda y_1(x,\lambda) dM_1(x)$$

to recast (27) into an operator form

$$y_2(x,\lambda) = y_1(x,\lambda) - \int_0^\infty H(x,t) dy_1^+(t,\lambda). \tag{28}$$

To find the domain of the integral operator in (28) that maps $y_1(.,\lambda) \to y_2(.,\lambda)$, we need to examine the integrability of the kernel H. For that purpose we have the following proposition, which by itself is of independent interest.

Theorem 9 *Let* $d\Gamma_1 \ll^\infty d\Gamma_2$*, then*

$$\|H(x,t)\|_{M_1(t)} \le c \left\| (x-t)_+ \right\|_{M_1(t)+M_2(t)} . \tag{29}$$

In all that follows by c we denote a universal constant, that can be distinct in different places.
In terms of integrals (29) means that

$$\int_0^\infty |H(x,t)|^2 \, dM_1(t) \le c \int_0^x (x-t)^2 \, d\left[M_1 + M_2\right](t).$$

Corollary 10 *If* $d\Gamma_1 \ll^\infty d\Gamma_2$ *and* $dM_1 \ll^\infty dM_2$ *then*

$$\|H(x,t)\|_{M_1(t)} \le c \, \|(x-t)_+\|_{M_2(t)} . \tag{30}$$

Thus the norm of $H(x,\cdot)$ satisfies the inequality

$$\|H(x,t)\|_{M_1(t)} \le cx\sqrt{M_2(x)}.$$

We now state the converse of Theorem 9.

Theorem 11 *Assume that*

(i) $y_2(x,\lambda) = y_1(x,\lambda) + \lambda \int_0^\infty H(x,t) y_1(t,\lambda) dM_1(t)$,
(ii) $\|H(x,t)\|_{M_1(t)} \le c \, \|(x-t)_+\|_{M_2(t)}$,
(iii) $dM_1 \ll^\infty dM_2$,

then $$d\Gamma_1 \ll^\infty d\Gamma_2.$$

Combining Proposition 8, Theorems 9, 11 and Corollary 10 we arrive at

Theorem 12 *Let* $dM_1 \ll^\infty dM_2$ *then*

$$y_2(x,\lambda) = y_1(x,\lambda) + \lambda \int_0^\infty H(x,t) y_1(t,\lambda) \, dM_1(t),$$

$$\text{with} \quad \|H(x,t)\|_{M_1(t)} \le c \|(x-t)_+\|_{M_2(t)},$$

if and only if $d\Gamma_1 \ll^\infty d\Gamma_2$.

Under the assumption that Γ_1 grows slowly at $\lambda = 0$, we can prove the square integrability of $H(\cdot, t)$ with respect to $dM_1(x)$.

Proposition 13 *Let* $d\Gamma_1 \ll^{\infty} d\Gamma_2$ *and* $dM_1 \ll^{\infty} dM_2$. *If moreover,* $\int_0^{\varepsilon} \frac{1}{\lambda^2} d\Gamma_1(\lambda) < \infty$ *for some* $\epsilon > 0$, *then* $\int_0^{\infty} |H(x,t)|^2 \, dM_1(t) \in L^2_{M_1(x)}$.

Proposition 14 *Assume that* $d\Gamma_1 = O(d\Gamma_2)$ *as* $\lambda \to \infty$, *then for each fixed* $x > 0$, $f \to \int_0^{\infty} H(x,t) f(t) dM_1(t)$ *defines a bounded functional on* $L^2_{M_1}$.

We now show that (27) can be used to define an integral operator in $L^2_{M_1}$. More precisely we have

Proposition 15 *Assume* $d\Gamma_1 = O(d\Gamma_2)$ *as* $\lambda \to \infty$, $\frac{d\Gamma_1}{d\Gamma_2} \in L^{2,loc}_{\Gamma_2}$, $dM_1 = O(dM_2)$ *as* $x \to \infty$, *then the operator* $L^2_{M_1} \to L^2_{M_1}$

$$g \to \int_0^{\infty} H(x,t) \mathbb{S}_1 g(t) dM_1(t)$$

is densely defined in $L^2_{M_1}$.

We now obtain a sufficient condition for the integral operator in (27), which we denote by $\mathbb{H}$, to be compact

Proposition 16 *Assume that* $d\Gamma_1 \ll^{\infty} d\Gamma_2$, *then* $\mathbb{H}$ *is a compact operator from* $L^2_{M_1}$ *into* $L^2_{M_1}$ *if*

$$\int_0^{\infty} \int_0^{x} (x-t)^2 \, dM_2(t) dM_2(x) < \infty \text{ and } \int_0^{\infty} \int_0^{x} (x-t)^2 \, dM_1(t) dM_2(x) < \infty. \tag{31}$$

Proposition 17 *Assume that* $dM_1 \ll^{2,loc} dM_2$ *and* $d\Gamma_1(\lambda) = O(d\Gamma_2)$ *as* $\lambda \to \infty$. *Then* $\mathbb{H}^* : L^2_{M_1} \to L^2_{M_1}$, *which is defined by*

$$\mathbb{H}^*(f)(t) = \int_0^{\infty} H(x,t) f(x) dM_1(x),$$

is densely defined, and for any $f \in C_o$.

3.2 Adding a Potential

We now extend the above construction to include operators, for $i = 1, 2$, such as

$$\begin{cases} \mathbb{S}_{iq}(y_i)(x) := -\frac{d}{dM_i(x)} \frac{d^+}{dx^+} y_i(x,\lambda) + q_i(x) y_i(x,\lambda) = \lambda y_i(x,\lambda), & 0 < x < \infty, \\ y_i(0,\lambda) = b, \; y_i'(0,\lambda) = a, \end{cases} \tag{32}$$

where the potential $q_i \in L^{2,loc}_{M_i}(0,\infty)$. The classical Sturm–Liouville problem corresponds to particular case when $M_1(x) = x$, i.e. $\mathbb{S}_{1q}(y_i)(x) := -\frac{d^2}{dx^2} y_i(x,\lambda) + q(x)y_i(x,\lambda) = \lambda y_i(x,\lambda)$.
Equation (32) is equivalent to

$$y_i(x,\lambda) = ax + b + \int_0^x (x-t)\, q_i(t) y_i(t,\lambda) dM_i(t) + \lambda \int_0^x (x-t)\, y_i(t,\lambda) dM_i(t). \tag{33}$$

The addition of a potential changes dramatically the spectrum from being mainly positive to possibly covering the whole real line. Thus the support of the spectral function is a subset of the real line.

We now show that a transmutation between the strings $\mathbb{S}_{1q}$ and $\mathbb{S}_{2q}$ exists under minimal conditions $d\Gamma_1(\lambda) = O(d\Gamma_2(\lambda))$ as $\lambda \to \pm\infty$.

Proposition 18 *Assume that* $d\Gamma_1(\lambda) = O(d\Gamma_2(\lambda))$ *as* $\lambda \to \pm\infty$, *then there exists* $H_q(x,t) \in L^2_{M_1(t)}$ *such that for* $x \geq 0$

$$y_2(x,\lambda) = y_1(x,\lambda) + \lambda \int_0^\infty H_q(x,t) y_1(t,\lambda) dM_1(t).$$

Proposition 19 *Assume that* $\lim_{x\to 0} \frac{DM_1}{Dx^{\alpha_1}} = k_1 \neq 0$, $\lim_{x\to 0} \frac{DM_2}{Dx^{\alpha_2}} = k_2 \neq 0$, *and spectrum of* $\mathbb{S}_1$ *is bounded from below. If* $0 < \alpha_1 \leq \alpha_2$ *then there exists* $H_q(x,\cdot) \in L^2_{M_1}(0,x)$ *such that for* $x \geq 0$

$$y_2(x,\lambda) = y_1(x,\lambda) + \lambda \int_0^\infty H_q(x,t) y_1(t,\lambda) dM_1(t).$$

Proof We only need to show that conditions of Proposition 18 hold. First when $\lambda \to -\infty$, $d\Gamma_1(\lambda) = 0$ and so the condition $d\Gamma_1(\lambda) = O(d\Gamma_2(\lambda))$ is trivially verified. However when $\lambda \to \infty$, we have two separate cases, see [51]: if

$$a \neq 0 \quad \text{then} \quad \Gamma_i(\lambda) = c_1 \lambda^{\frac{\alpha_i}{1+\alpha_i}} + o\left(\lambda^{\frac{\alpha_i}{1+\alpha_i}}\right) \quad \text{as} \quad \lambda \to \infty,$$

$$a = 0 \quad \text{then} \quad \Gamma_i(\lambda) = c_1 \lambda^{\frac{2+\alpha_i}{1+\alpha_i}} + o\left(\lambda^{\frac{2+\alpha_i}{1+\alpha_i}}\right) \quad \text{as} \quad \lambda \to \infty.$$

Since $0 < \alpha_1 \leq \alpha_2$ implies $\frac{\alpha_1}{1+\alpha_1} \leq \frac{\alpha_2}{1+\alpha_2}$ i.e. $\frac{d\Gamma_1}{d\Gamma_2} \approx c\lambda^{\frac{\alpha_1-\alpha_2}{(1+\alpha_1)(1+\alpha_2)}} < \infty$ as $\lambda \to \infty$. □

We now present explicit examples which show that the representation of the transmutations cannot be in general triangular or close to unity, see [48].

3.3 Examples

Many examples of all kinds of spectra, for various potentials, can be found in [45, 71]. The purpose of the following examples is to illustrate two essential facts of transmutations for strings, which makes all the difference from the usual transmutations for Sturm–Liouville operators (5). The first is that one should have the parameter λ in the integral. Secondly the upper bound may not be just x as in the Gelfand–Levitan theory.

Example 1 Let $M_1(x) = x$ and $M_2(x) = \rho^2 x,\ a = 1,\ b = 0,\ \rho > 1$. Then for $x \geq 0$, we have

$$[\mathbb{S}_1] \begin{cases} -\frac{d}{dx}\frac{d^+}{dx^+} y_1(x,\lambda) = \lambda y_1(x,\lambda), \\ y_1(0,\lambda) = 1,\ y_1'(0,\lambda) = 0, \end{cases} \quad \text{and} \quad [\mathbb{S}_2] \begin{cases} -\frac{d}{\rho^2 dx}\frac{d_t^+}{dx^+} y_2(x,\lambda) = \lambda y_2(x,\lambda), \\ y_2(0,\lambda) = 1,\ y_2'(0,\lambda) = 0. \end{cases}$$

The eigenfunctionals of the strings are

$$y_1(x,\lambda) = \cos(x\sqrt{\lambda}) \ \text{ and } \ y_2(x,\lambda) = \cos(\rho x\sqrt{\lambda}) \qquad \text{for } x \geq 0,$$

and their spectral functions, [64], are given by

$$d\Gamma_1(\lambda) = \frac{1}{\pi}\frac{d\lambda}{\sqrt{\lambda_+}} \qquad \text{and } \ d\Gamma_2(\lambda) = \frac{\rho}{\pi}\frac{d\lambda}{\sqrt{\lambda_+}}.$$

Thus by Proposition 18, the transmutation exists

$$\cos(\rho x\sqrt{\lambda}) = \cos(x\sqrt{\lambda}) + \lambda\int_0^\infty H(x,t)\cos(t\sqrt{\lambda})dt.$$

Computing the kernel H, we obtain

$$\begin{aligned} H(x,t) &= \frac{1}{\pi}\int_0^\infty \frac{1}{\lambda}\left(\cos\left(\rho x\sqrt{\lambda}\right) - \cos(x\sqrt{\lambda})\right)\cos\left(t\sqrt{\lambda}\right)\frac{d\lambda}{\sqrt{\lambda}} \\ &= \min\left\{\frac{\rho+1}{2}x,\ \left|\frac{\rho-1}{2}x+t\right|\right\}\operatorname{sign}\left[\frac{\rho-1}{2}x+t\right] \\ &\quad + \min\left\{\frac{\rho+1}{2}x,\ \left|\frac{\rho-1}{2}x-t\right|\right\}\operatorname{sign}\left[\frac{\rho-1}{2}x-t\right]. \end{aligned}$$

One can verify that $H(x,t) = 0$ if $t > \rho x$, but $H(x,t) = \rho x - t \neq 0$ if $t < \rho x$, but close to ρx. Therefore we deduce an explicit form of the type

$$\cos(\rho x\sqrt{\lambda}) = \cos(x\sqrt{\lambda}) + \lambda\int_0^{\rho x} H(x,t)\cos(t\sqrt{\lambda})dt. \tag{34}$$

Here we notice that the multiplier λ is needed because for any fixed $x > 0$ we have $\cos(\rho x\sqrt{\lambda}) - \cos(x\sqrt{\lambda}) \notin L^2_{\sqrt{\lambda+}}$ while the integral on the right hand side

$$\int_0^{\rho x} H(x,t)\cos(t\sqrt{\lambda})dt \in L^2_{\sqrt{\lambda_+}}.$$

Thus the role of λ is to ensure that $\frac{1}{\lambda}\left(\cos(\rho x\sqrt{\lambda}) - \cos(x\sqrt{\lambda})\right) \in L^2_{\sqrt{\lambda_+}}$.

As for the upper bound in the integral (34), it cannot be just x as in the transmutation used by Gelfand-Levitan. It is easily seen that since the growth type of $\cos(\rho x\sqrt{\lambda}) - \cos(x\sqrt{\lambda})$ as a function of $\sqrt{\lambda}$ is ρx, when $\rho > 1$ the Paley–Wiener theorem implies that the support of the transform must be included in $[-\rho x, \rho x]$ and the fact that the transform is even, reduces it to $[0, \rho x]$.

Example 2 Consider transmuting the strings when $\alpha, \beta > 0$,

$$[\mathbb{S}_1]\begin{cases} -x^{-\alpha}y_1''(x,\lambda) = \lambda y_1(x,\lambda), & x > 0, \\ y_1(0,\lambda) = 1 \quad y_1'(0,\lambda) = 0, \end{cases}$$

$$[\mathbb{S}_2]\begin{cases} -x^{-\beta}y_2''(x,\lambda) = \lambda y_2(x,\lambda), & x > 0, \\ y_2(0,\lambda) = 1 \quad y_2'(0,\lambda) = 0. \end{cases}$$

Their eigensolutions are the well known Bessel functions

$$y_1(x,\lambda) = c_1(\lambda)\sqrt{x}Y_{\frac{1}{2+\alpha}}\left(\frac{2\sqrt{\lambda}}{2+\alpha}x^{\frac{2+\alpha}{2}}\right) \quad \text{and} \quad y_2(x,\lambda) = c_2(\lambda)\sqrt{x}Y_{\frac{1}{2+\beta}}\left(\frac{2\sqrt{\lambda}}{2+\beta}x^{\frac{2+\beta}{2}}\right).$$

The spectral functions are [51],

$$\Gamma_1(\lambda) \approx C_1\lambda^{\frac{\alpha+1}{\alpha+2}} \text{ and } \Gamma_2(\lambda) \approx C_2\lambda^{\frac{\beta+1}{\beta+2}} \quad \text{as } \lambda \to \infty.$$

This leads to the following conclusion.

Corollary 20 *If $\beta \geq \alpha > 0$ then there exists a transmutation such that*

$$y_2(x,\lambda) = y_1(x,\lambda) + \lambda\int_0^\infty H(x,t)y_1(t,\lambda)t^\alpha dt.$$

In terms of Bessel functions the above relation becomes

$$c_2(\lambda)\sqrt{x}Y_{\frac{1}{2+\beta}}\left(\frac{2\sqrt{\lambda}}{2+\beta}x^{\frac{2+\beta}{2}}\right) = c_1(\lambda)\sqrt{x}Y_{\frac{1}{2+\alpha}}\left(\frac{2\sqrt{\lambda}}{2+\alpha}x^{\frac{2+\alpha}{2}}\right) + \lambda c_1(\lambda)\int_0^\infty H(x,t)\sqrt{t}Y_{\frac{1}{2+\alpha}}\left(\frac{2\sqrt{\lambda}}{2+\alpha}t^{\frac{2+\alpha}{2}}\right)t^\alpha dt.$$

An interesting particular case is when $\alpha = 1$, i.e. $\varphi(x,\lambda) = \cos\left(x\sqrt{\lambda}\right)$, thus for $\beta \geq 1$ we have

$$c_2(\lambda)\sqrt{x}Y_{\frac{1}{2+\beta}}\left(\frac{2\sqrt{\lambda}}{2+\beta}x^{\frac{2+\beta}{2}}\right) = \cos\left(x\sqrt{\lambda}\right) + \lambda\int_0^{\infty} H(x,t)\cos\left(t\sqrt{\lambda}\right)dt.$$

4 Applications

4.1 The Gelfand-Levitan Theory

Since transforms are defined by the eigenfunctionals $y_i(\lambda)$, we need to use duality bracket $<,>_{\Phi\times\Phi'}$, thus for L_i we write

$$\mathcal{F}_i(f)(\lambda) =< f,\ y_i(\lambda) >_{\Phi_i\times\Phi_i'} \qquad \text{and} \qquad f = \int \mathcal{F}_i(f)(\lambda) y_i(\lambda) d\Gamma_i(\lambda),$$

where Γ_i is the spectral measure associated with L_i, for $i = 1, 2$. Recall that Γ_i is a right continuous nondecreasing function, i.e. a Stieljes measure, $\sigma_i = \operatorname{supp} d\Gamma_i$, with jumps at the eigenvalues λ_n defined by

$$\Gamma_i(\lambda_n) - \Gamma_i\left(\lambda_n^-\right) = \frac{1}{\|y_i(\lambda_n)\|_i^2}.$$

As the transmutation maps eigenfunctionals, it also induces a map on the transforms

$$\begin{aligned}\mathcal{F}_2(f)(\lambda) &=< f,\ y_2(\lambda) >_{\Phi_2\times\Phi_2'} =< f,\ Vy_1(\lambda) >_{\Phi_2\times\Phi_2'} =< V'f,\ y_1(\lambda) >_{\Phi_1\times\Phi_1'}\\ &= \mathcal{F}_1\left(V'f\right)(\lambda) \quad \text{for } f \in \Phi_2.\end{aligned} \tag{35}$$

This leads to the first Gelfand-Levitan theory. Assume that $d\Gamma_1$ is a locally absolutely continuous measure with respect to $d\Gamma_2$, i.e. $\sigma_1 \subset \sigma_2$, and there exists a $g \in L^{1,loc}_{d\Gamma_2}$ such that $d\Gamma_1 = g(\lambda)d\Gamma_2$, then we have the following proposition, see [5].

Proposition 21 *Assume that V is a transmutation between L_1 and L_2, defined by (3), then it satisfies*

$$\frac{d\Gamma_1}{d\Gamma_2}(L_2) = \overline{V}V' \quad in\ \Phi_2'. \tag{36}$$

Proof Use Parseval identity to write, for $f,\ g \in \Phi_2$, where $(.,.)_i$ is the inner product in H_i,

$$\int \mathcal{F}_1\left(V'f\right)(\lambda)\overline{\mathcal{F}_1\left(V'g\right)(\lambda)}d\Gamma_1(\lambda) = \left(V'f,\ V'g\right)_1 =< f,\ \overline{V}V'g >_{\Phi_2\times\Phi_2'} .$$

On the other hand we have

$$\begin{aligned}\int \mathcal{F}_1\left(V'f\right)(\lambda)\overline{\mathcal{F}_1\left(V'g\right)(\lambda)}d\Gamma_1(\lambda) &= \int \mathcal{F}_2(f)(\lambda)\overline{\mathcal{F}_2(g)(\lambda)}d\Gamma_1(\lambda)\\ &= \int \mathcal{F}_2(f)(\lambda)\overline{\frac{d\Gamma_1}{d\Gamma_2}(\lambda)\,\mathcal{F}_2(g)(\lambda)}d\Gamma_2(\lambda)\\ &= \int \mathcal{F}_2(f)(\lambda)\overline{\mathcal{F}_2\left(\frac{d\Gamma_1}{d\Gamma_2}(L_2)\,g\right)(\lambda)}d\Gamma_2(\lambda)\\ &= \left(f,\ \frac{d\Gamma_1}{d\Gamma_2}(L_2)\,g\right) =< f,\ \frac{d\Gamma_1}{d\Gamma_2}(L_2)\,g >_{\Phi_2\times\Phi_2'} .\end{aligned}$$

Since $f \in \Phi_2$ is arbitrary we deduce (36) which is the nonlinear integral equation in the Gelfand-Levitan theory. Observe that (36) makes it obvious that V' is a bounded operator $H_2 \longrightarrow H_1$ if and only if $\sup\frac{d\Gamma_1}{d\Gamma_2}(\lambda)$ is bounded. This follows from the fact that V' is unitary equivalent to the multiplication operator by $\sqrt{\frac{d\Gamma_1}{d\Gamma_2}(\lambda)}$ in the space of transforms.

Note that the operator can also be written as

$$\frac{d\Gamma_1}{d\Gamma_2}(L_2)\,f(x) = \int \mathcal{F}_2(f)(\lambda)\ y_2(\lambda)d\Gamma_1(\lambda) = \langle F(x),\,f\rangle ,$$

which covers the general case when $\mathrm{d}\Gamma_1$ is not abs-$\mathrm{d}\Gamma_2$.

For example when $\sigma_2 \subset \sigma_1$, we can define the kernel $F(x) = \int y_2(x,\lambda)\,\overline{y_2(.,\lambda)}\,d\Gamma_1(\lambda) \in \Phi_2'$ and the operator

$$\begin{aligned}\Phi_2 &\longrightarrow \Phi_2'\\ f &\longrightarrow \mathbb{F}f(x) =< f,\ F(x) > .\end{aligned}$$

Proposition 22 *Assume that $y_2(\lambda)$ in (3) is defined for $\lambda \in \sigma_1$, and the kernel $\int y_2(x,\lambda)\,\overline{y_2(.,\lambda)}d\Gamma_1(\lambda) \in \Phi_2'$, then we have the factorization*

$$\mathbb{F} = \overline{V}V' \quad in\ \Phi_2'$$

This factorization is nothing else than the non linear equation in [46, 61, 64, 65]. Given $\mathbb{F}$, the existence of a triangular or Volterra type operator V, simply follows from how close is $\mathbb{F}$ from the identity operator, see the topic of factorization of operators close to identity in [48].

4.2 Gelfand-Levitan Revisited

We are now concerned with the conditions for the solvability of the inverse spectral problem for the singular Sturm–Liouville (S–L) operator

$$\begin{cases} L\,(y) := -y''(x,\lambda) + q(x)y(x,\lambda) = \lambda y(x,\lambda), & x \in [0,\infty), \\ y'(0,\lambda) - hy(0,\lambda) = 0, \end{cases} \tag{37}$$

where $q \in L^{1,loc}[0,\infty)$, $q(x)$ and h are real. Recall that in the celebrated 1951 paper by Gelfand and Levitan [46], the necessary and sufficient conditions were stated separately. To close the gap, in 1953, [55], M.G. Krein announced two necessary and sufficient conditions for ρ to be a spectral function which he then revised by adding a third condition in 1957, [56]. Few years later, Gasymov and Levitan in 1964 closed the gap of the 1951 result by showing that two conditions only are necessary and sufficient for the solvability of the inverse spectral problem. To state these conditions denote by $\sigma\,(\lambda) := \rho(\lambda) - \frac{2}{\pi}\sqrt{\lambda_+}$, where $\lambda_+ = \max\,(0,\lambda)$ and $\mathcal{F}_1\,(f)\,(\lambda) = \int_0^\infty f(x)\cos\left(x\sqrt{\lambda}\right)dx$ the classical Fourier cosine transform.

Theorem 23 (Gelfand-Levitan-Gasymov) *(G-L-G) For a monotone increasing function ρ to be a spectral function of (37) where q has m locally integrable derivatives it is necessary and sufficient that*

[A] ***Existence:*** *For any $f \in L^2(0,\infty)$ with compact support*

$$\int |\mathcal{F}_1(f)(\lambda)|^2\,d\rho(\lambda) = 0 \;\Rightarrow\; f = 0 \quad a.e. \tag{38}$$

[B] ***Smoothness:*** *The sequence of functions $\Phi_N\,(x) := \int_{-\infty}^N \cos\left(x\sqrt{\lambda}\right)d\sigma\,(\lambda)$, converges boundedly in every finite interval to a function Φ that has $m+1$ locally integrable derivatives and $\Phi(0) = -h$.*

The 1957 M.G.Krein's result is also stated below.

Theorem 24 (M. G. Krein) *In order for ρ to be the spectral function of*

$$\begin{cases} L\,(y) := -y''(x,\lambda) + q(x)y(x,\lambda) = \lambda y(x,\lambda), & 0 < x < l, \\ y'(0,\lambda) - hy(0,\lambda) = 0, \end{cases}$$

for a given l it is necessary and sufficient that

(1) *The function $\Pi(t) = \int_{-\infty}^{\infty} \frac{1-\cos\left(t\sqrt{\lambda}\right)}{\lambda}d\rho(\lambda)$, where $0 \le t \le 2l$, is finite and has two absolute continuous derivatives on every interval $[0, r]$, where $r < 2l$.*
(2) $\Pi'(0) = 1$.
(3) $\liminf_{R\to\infty} sup\ N\,(R)\,/\sqrt{R} \ge l/\pi$, *where N(R) represents the number of points in the spectrum that are also contained in the interval $[0, R]$.*

The 1953 result included only the first two conditions and the third condition was added in 1957 as a correction. The issue of whether Krein's type result needed two or three conditions was settled down by Yavryan, [76], in 1992. He re-examined Krein's 1957 result and showed, by using directional functionals, that the third condition follows from the first two. Thus as in the Gelfand-Levitan-Gasymov theory only two conditions are needed in Krein's result. The major differences, in both theorem is in the required smoothness and whether the measure used is ρ or σ. We need also to mention that in his book, [64, Theorem 2.3.1, p. 142], Marchenko has a similar theorem that falls in between Gelfand-Levitan and Krein theorems, where the smoothness condition is: $\Psi(x) = \left(\frac{1-\cos(\lambda x)}{\lambda^2},\ R\right)$ should be at least three times continuously differentiable. Note here that Ψ uses λ instead of $\sqrt{\lambda}$ while R is a distribution, and the reconstructed potential is only continuous. It is clear that the Gelfand-Levitan-Gasymov paper gives the best smoothness, namely $q \in L^{1,loc}(0, \infty)$. The authors revisited the Gelfand-Levitan-Gasymov theorem and showed that in fact only one, namely the second condition is needed, see [31].

Theorem 25 (Gelfand-Levitan-Gasymov Revisited) *For a monotone increasing function ρ to be the spectral function of a problem (37) where q has m locally integrable derivatives it is necessary and sufficient that the sequence of functions Φ_N converges boundedly to a function Φ that has $m + 1$ locally integrable derivatives.*

Observe that Yavryan's proof dealt with Krein's approach only and covered the regular case while ours applied the Gelfand-Levitan approach which was for the singular case. At the end [76], Yavryan pointed out that the proof can be extended to the half line case but gave no details.

4.3 Transmutation Between Orthogonal Polynomials

Is it possible to transmute an orthogonal polynomial system, o.p.s. for short, into another o.p.s? In other words, under what conditions can we find a transmutation operator V such that

$$q_n(x) = V\left(p_n(x)\right), \qquad n \geq 0,$$

where $q_n(x)$ and $p_n(x)$ form o.p.s. Stirling and Tchebyshev may have been the first ones who addressed a similar question, where recurrence relations and connection coefficients were sought between systems of polynomials, see [2]. For example can we transmute Legendre polynomials into Laguerre or Hermite polynomials, and what shape would the transmutation V have?

In all that follows **P** and **Q** denote two self-adjoint operators acting respectively in the separable Hilbert spaces H_P and H_Q. Let us assume that $\sigma_P = \{\lambda_n\}_{n\geq 1}$ and $\sigma_Q = \{\mu_n\}_{n\geq 1}$ are simple spectra, that is, $dim(N(P - \lambda_n I_d)) = dim(N(Q -$

$\mu_n I_d)) = 1$ and where $\mathbf{P}(p_n) = \lambda_n p_n$ and $\mathbf{Q}q_n = \mu_n q_n$ denote the eigenfunctions of $\mathbf{P}$ and $\mathbf{Q}$ for $n \geq 0$.

Eigenfunctions expansion for $f \in H_Q$ and $\phi \in H_P$ leads to

$$f = \sum_{n\geq 0} \alpha(f,n) q_n, \qquad \phi = \sum_{n\geq 0} \beta(\phi,n) p_n,$$

where $\alpha(f,n) := (f, q_n)\frac{1}{||q_n||^2}$, $\beta(\phi,n) = (f, p_n)\frac{1}{||p_n||^2}$ and $(.,.)$ is the inner product on the Hilbert spaces.

We now focus on the case when the spectra are different and have no finite accumulation points, see [28].

There are several instances where we can still find a relation between o.p.s. and yet their spectra are disjoint. The Mehler formula exhibits a such situation

$$q_n(\theta) := \frac{2}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})\eta}{\sqrt{2\cos\eta - 2\cos\theta}} d\eta, \qquad 0 \leq \theta \leq \pi.$$

Indeed, we have a transmutation defined by

$$\mathbf{F}\,(f)\,(\theta) := \frac{2}{\pi}\int_0^\theta \frac{1}{\sqrt{2\cos\eta - 2\cos\theta}} f(\eta) d\eta.$$

The $q_n(\theta) := P_n(\cos(\theta))$, where $P_n(x)$ is the Legendre polynomial of degree n, are the eigenfunctions of the self-adjoint operator

$$\mathbf{Q}(u) := \frac{-1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{du}{d\theta}\right], \qquad 0 < \theta < \pi,$$

acting in $L^2[(0,\pi), \sin(\theta)d\theta]$. The second system of eigenfunctions is defined by $p_n(x) := \cos\left(n + \frac{1}{2}\right)x$ and is associated with the operator

$$\begin{cases} \mathbf{P} := \frac{-d^2}{dx^2}, & 0 < x < \pi, \\ f'(0) = f(\pi) = 0, \end{cases}$$

which is self-adjoint in $L^2(0,\pi)$. The spectra are given by

$$\sigma_P := \{n(n+1)\}_{n\geq 0}\,, \qquad \sigma_Q := \left\{ \left(n + \frac{1}{2}\right)^2 \right\}_{n\geq 0},$$

and are obviously disjoint and have no accumulation points.

Thus let us assume, in general, that we are given two o.p.s generated by two self-adjoint operators $\mathbf{P}$, $\mathbf{Q}$ such that $\sigma_P \neq \sigma_Q$, and related by an operator V

$$V(p_n) = q_n, \quad \text{where } n \geq 0.$$

We further assume that we can interpolate both spectra, i.e. we can find mappings a and b such that

$$\begin{array}{ll} a : \sigma_P \to \mathbb{N} & \qquad b : \sigma_Q \to \mathbb{N} \\ a(\lambda_n) = n & \qquad b(\mu_n) = n. \end{array}$$

In other words a is the inverse of mapping $n \to \lambda_n$, and similarly for b, see [28]

Proposition 26 *Assume that $a(\lambda_n) = b(\mu_n) = n$ for $n \geq 0$, and $Vp_n = q_n$, then there exists a transmutation V such that*

$$Va(\mathbf{P}) = b(\mathbf{Q})V. \tag{39}$$

Remark The functions a, and b are not always needed. Sometimes we can find a direct relation between the eigenvalues

$$\delta(\lambda_i) = \mu_i,$$

and consequently (39) reduces to $V\delta(\mathbf{P}) = \mathbf{Q}V$.

Let us denote the linear operator acting in H_Q, defined through the Fourier coefficients

$$\mathbf{S}(\mathbf{Q})f := \sum_{n\geq 0} \|q_n\|^2 \|p_n\|^{-2} \alpha(f, n) q_n,$$

where

$$f := \sum_{n\geq 0} \alpha(f, n) q_n.$$

It is clear that

$$||\mathbf{S}(\mathbf{Q})|| = \sup_{n\geq 0} \|q_n\|^2 \|p_n\|^{-2}.$$

We now have a factorization result

Proposition 27 *Let V be such that $q_n = Vp_n$, then*

$$\mathbf{S}(\mathbf{Q}) = \overline{V}V^*, \tag{40}$$

where V^* is the adjoint of V.

Remark $\mathbf{S}(\mathbf{Q})$ is bounded if and only if V is also bounded.

4.3.1 Example

Consider the singular differential operator defined by

$$x(1-x)y''(x) + [c - (a+b+1)x]\, y'(x) - aby(x) = 0, \qquad 0 < x < 1.$$

If $c > 2$, then we are in the limit point case, and its solution is then given by

$$y(x) = {}_2F_1\ (a, b; c; x).$$

Thus if we choose a or b to be a nonpositive integer then the solution is a polynomial in x. We now introduce two self-adjoint operators, defined by

$$\mathbf{Q}y(x) := \left[(1-x)^{k_1-c}x^{c-1}\right]^{-1}\left((1-x)^{1+k_1-c}x^c y'(x)\right)', \qquad 0 < x < 1,$$

$$\mathbf{P}y(x) := \left[(1-x)^{k_2-c}x^{c-1}\right]^{-1}\left((1-x)^{1+k_2-c}x^c y'(x)\right)', \qquad 0 < x < 1,$$

and acting respectively in $L^2((0,1), (1-x)^{k_1-c}x^{c-1}\ dx)$ and $L^2((0,1),\ (1-x)^{k_2-c}x^{c-1}dx)$, $k_1 \neq k_2$, and where

$$L^2((0,1), w(x)dx) : \left\{ f \text{ measurable} : \int_0^1 |f(x)|^2\, w(x)dx < \infty \right\}.$$

It is easily seen that

$$q_n(x) = {}_2F_1\ (-n, k_1 + n; c; x)$$

satisfy $x(1-x)y'' + [c - (k_1+1)x]y + n(k_1+n)y = 0$ and consequently

$$\mathbf{Q}q_n(x) = \mu_n q_n(x),$$

where

$$\mu_n = -n(n + k_1), \qquad n \geq 0.$$

Similarly

$$p_n(x) = {}_2F_1\ (-n, k_2 + n; c; x)$$

are eigenfunctions of **P**, i.e.

$$\mathbf{P}p_n(x) = \lambda_n p_n(x),$$

where

$$\lambda_n = -n(n + k_2), \qquad n \geq 0.$$

We now work out a transmutation. What is the simplest operator V mapping $p_n \to q_n$? that is

$${}_2F_1(-n,\ k_2 + n; c; x) \to\ {}_2F_1(-n,\ k_1 + n; c; x).$$

Observe that a hypergeometric function ${}_2F_1\ (a, b; c; x)$ is an analytic functions of its arguments, and since it is a finite sum then

$${}_2F_1\ (-n, k_1 + n; c; x) = \exp\left[(k_1 - k_2)\frac{\partial}{\partial b}\right]\ {}_2F_1(-n,\ k_2 + n; c; x).$$

Thus we have the transmutation defined by

$$V := \exp\left[(k_1 - k_2)\frac{\partial}{\partial b}\right],$$

which is a differential operator of infinite order. Next we need to define V for at least a dense subspace of $L^2((0, 1),\ (1 - x)^{k_2 - c}x^{c-1}dx)$. To this end observe that we need only to define the action of V on polynomials since they are dense in $L^2((0, 1),\ (1 - x)^{k_2 - c}x^{c-1}dx)$. We recall that

$$f = \sum_{k=0}^{n} c_k p_k \tag{41}$$

form a dense subspace in $L^2((0, 1),\ (1 - x)^{k_2 - c}x^{c-1}dx)$ and

$$Vf = \sum_{k=0}^{n} c_k q_k.$$

Thus V is densely defined. The interpolating functions are as follows

$$a(\lambda_n) = \frac{-k_2 + \sqrt{k_2^2 + 4\lambda_n}}{2} = n,$$

and

$$b(\mu_n) = \frac{-k_1 + \sqrt{k_1^2 + 4\mu_n}}{2} = n.$$

Then the transmutation condition

$$Va(\mathbf{P}) = b(\mathbf{Q})V$$

is translated into the following relation

$$\exp\left[(k_2 - k_1)\frac{\partial}{\partial b}\right]\frac{-k_2 + \sqrt{k_2^2 + 4\mathbf{P}}}{2} = \frac{-k_1 + \sqrt{k_1^2 + 4\mathbf{Q}}}{2}\exp\left[(k_2 - k_1)\frac{\partial}{\partial b}\right].$$

Remark In case $k_2 > c - 1$, then $x^n \in L^2((0,1),\ (1-x)^{k_2 - c}x^{c-1}dx)$, and we could define V directly on polynomials. For example, we can use generating functions to express x^n in terms of ${}_2F_1(-n,\ k_2 + n; c; x)$.

4.4 Direct Reconstruction of the Spectral Function

We now show how we can reconstruct these kernels and also the spectral function, see [7, 8, 19, 30]. Let y be a solution of

$$\begin{cases} -y''(x,\lambda) + q(x)y(x,\lambda) = \lambda y(x,\lambda), \\ y(0,\lambda) = 1 \text{ and } y'(0,\lambda) = h, \end{cases} \tag{42}$$

where $q \in L^{1,loc}[0,\infty)$ and $q(x), h \in \mathbb{R}$. We then have the classical transmutation and its inverse

$$y(x,\lambda) = \cos\left(x\sqrt{\lambda}\right) + \int_0^x k(x,t)\cos\left(t\sqrt{\lambda}\right)dt,$$

$$\cos\left(x\sqrt{\lambda}\right) = y(x,\lambda) + \int_0^x H(x,t)y(t,\lambda)dt.$$

If we assume that the spectral function $d\Gamma$ is abs- $d\sqrt{\lambda}$ continuous then we have

$$\frac{\pi}{2}\frac{d\Gamma}{d\sqrt{\lambda}}(\lambda) = 1 + \int_0^\infty H(t,0)\cos\left(t\sqrt{\lambda}\right)dt. \tag{43}$$

Thus to reconstruct $d\Gamma$, we only need a method to reconstruct H. To this end recall that it follows from (42), that we also have

$$y(x,\lambda) = \cos\left(x\sqrt{\lambda}\right) + \int_0^x \frac{\sin\left((x-t)\sqrt{\lambda}\right)}{\sqrt{\lambda}} q(t)y(t,\lambda)dt,$$

and thus we have a direct connection between H and q,

$$\int_0^x H(x,t)y(t,\lambda)dt = -\int_0^x \frac{\sin\left((x-t)\sqrt{\lambda}\right)}{\sqrt{\lambda}} q(t)y(t,\lambda)dt \quad \text{for} \quad \lambda \in \mathbb{C}. \tag{44}$$

In order to find a formula for H we need to remove the λ from the right hand side. To this end denote by $\mathcal{L} = -D^2 + q(x)$ the differential expression, and so we have

$$\mathcal{L}y(x,\lambda) = \lambda y(x,\lambda),$$

and let

$$a_n(x-t) = \frac{(-1)^n}{(2n+1)!}(x-t)_+^{2n+1},$$

where $x_+ = x$ if $x > 0$ and 0 otherwise. Thus if $q \in C^\infty[0,\infty)$, and $q^{(k)}(0) = 0$ for all $k \geq 0$, we can recast (44) as

$$\begin{aligned}\int_0^x H(x,t)y(t,\lambda)dt &= -\sum_{n\geq 0}\int_0^x q(t)a_n(x-t)\ \lambda^n\ y(t,\lambda)dt \\ &= -\sum_{n\geq 0}\int_0^x q(t)a_n(x-t)\ \ \mathcal{L}^n y(t,\lambda)dt \\ &= -\sum_{n\geq 0}\int_0^x \mathcal{L}^n\left[q(t)a_n(x-t)\right]\ \ y(t,\lambda)dt.\end{aligned}$$

The partial sums do collapse as it is seen from

$$\sum_{k=0}^n \mathcal{L}^k\left[q(t)a_k(x-t)\right] = \mathcal{L}^{n+1}a_n(x-t),$$

and so we deduce that

$$\mathcal{L}^{n+1} a_n (x-t) \longrightarrow H(x,t).$$

Proposition 28 *Assume that* $q \in C^{\infty}[0,\infty)$, *and* $q^{(k)}(0) = 0$ *for all* $k \geq 0$, *then*

$$H(x,t) = \lim_{n\to\infty} -\left(-D^2 + q(t)\right)^{n+1} a_n (x-t) \qquad in \ L^2_{dt}(0,x). \tag{45}$$

On the other hand we have

$$\int \cos\left(x\sqrt{\lambda}\right)\cos\left(t\sqrt{\lambda}\right)\left(\frac{\pi}{2}\frac{d\Gamma(\lambda)}{d\sqrt{\lambda}} - 1\right) d\left(\frac{2}{\pi}\sqrt{\lambda}\right) = H(x,t) + H(t,x) + \int_0^x H(x,\eta)H(\eta,t)d\eta.$$

Taking the inverse cosine transform yields

$$\cos\left(x\sqrt{\lambda}\right)\left[\frac{\pi}{2}\frac{d\Gamma(\lambda)}{d\sqrt{\lambda}} - 1\right] = \int_0^x H(x,t)\cos\left(t\sqrt{\lambda}\right)dt + \int_x^{\infty} H(t,x)\cos\left(t\sqrt{\lambda}\right)dt + \int_0^x H(x,\eta)\int_t^{\infty} H(\eta,t)\cos\left(t\sqrt{\lambda}\right)dt\,d\eta.$$

Letting $x \to 0^+$, we get (43), where H now can be computed by taking the limit in (45).

4.5 The Lieb and Thirring Constant

Lieb and Thirring [62] have shown that the sum of the moments of the negative eigenvalues $-\lambda_1 \leq -\lambda_2 \leq \cdots \leq 0$ (if any) of the Schrödinger operator $-\Delta - V$ on $L^2\left(\mathbb{R}^d\right)$ is bounded by

$$\sum \lambda_i^{\gamma} \leq L_{\gamma,d}\int_{\mathbb{R}^d} (V_-(x))^{\gamma+d/2}dx, \tag{46}$$

where $V_-(x) := \max\{V(x),0\}$. One of the challenges is to find the smallest possible constant $L_{\gamma,d}$, known as the sharp constant in (46). For the sake of simplicity we shall restrict ourselves to eigenvalue inequalities in the case $d=1$.

It is well known that if $d=1$ then (46) cannot hold for $\gamma < 1/2$. For the limit case $\gamma = 1/2$, Hundertmark, Lieb, and Thomas [49] have shown that if $V_- \in L^1(\mathbb{R})$, then

$$\sum \sqrt{\lambda_i} \leq L_{1/2,1}\int_{-\infty}^{\infty} V_-(x)\,dx, \tag{47}$$

where $L_{1/2,1} = \frac{1}{2}$ is a sharp constant. The main tool used in deriving (47) is the Birman-Schwinger principle which relates negative eigenvalues of the Schrödinger operator with eigenvalues of a certain integral operator. If V is continuous, $V(x) \to 0$ as $|x| \to \infty$ and $\int_{\mathbb{R}} V(x)dx$ exists (possibly conditionally), Schmincke [68] uses the commutation method to obtain the lower bound for the sum of the negative eigenvalues

$$\frac{1}{4}\int_{-\infty}^{\infty} V(x)\,dx \le \sum \sqrt{\lambda_i}, \tag{48}$$

and here $\frac{1}{4}$ is a sharp constant.
If we also assume that $(1+|x|)V(x) \in L^1(\mathbb{R})$, then Schmincke's inequality (48) follows at once from the Faddeev–Zakharov trace formula [77]

$$\int_{-\infty}^{\infty} V(x)\,dx = 4\sum \sqrt{\lambda_i} + \frac{1}{\pi}\int_{-\infty}^{\infty} \ln(1-|R(k)|^2)dk, \tag{49}$$

since the reflection coefficient of the operator H satisfies $R(k) \in [0, 1]$.

A well known fact in the spectral theory of operators is that negative eigenvalues depend on the self-adjoint extensions. Also if a Lieb-Thirring inequality holds, it must do so for all isospectral operators, since it does not make use of the energy of the bound states. Thus to shed some light on these hidden connections we shall study the sum of the negative eigenvalues, $\gamma = 1/2$, of the Schrödinger operator on the half-line and under the sole condition that $q \in L^1(0,\infty)$. Thus consider the one-dimensional self-adjoint Schrödinger operator on the half line

$$\begin{cases} H(y) := -y''(x,\lambda) - q(x)y(x,\lambda) = -\lambda y(x,\lambda), & x \in (0,\infty), \\ y'(0,\lambda) - hy(0,\lambda) = 0, \text{ where } & h, q(x) \in \mathbb{R}. \end{cases} \tag{50}$$

Let $\alpha_j = 1/\int_0^\infty |y(x,-\lambda_i)|^2\,dx$ be the norming constant, which represent the jump size of the spectral function [60] at $-\lambda_j$. One of the main results of [] is the identity

$$\int_0^\infty q_N(x)dx = \int_0^\infty q_0(x)dx - 2\sum_{j=1}^N \alpha_j + 4\sum_{j=1}^N \sqrt{\lambda_j}, \tag{51}$$

where q_0 is obtained from q_N by removing all N negative eigenvalues. The appearance of the norming constants α_j in the formula distinguishes (51) from the Faddeev–Zakharov trace formula (49) and brings out a new relation between isospectral operators. We then prove that if q_0 generates no negative eigenvalues, then the estimate

$$\int_0^\infty q_0(x)dx \le h_0 \tag{52}$$

holds, which yields the Schmincke inequality for the half-line

$$\frac{1}{4}\int_0^\infty q(x)dx - \frac{h}{4} < \sum \sqrt{\lambda_j}. \tag{53}$$

Theorem 29 *Assume that $q_N \in L^1(0,\infty)$, and q_0 is obtained from q_N by removing all negative eigenvalues of H_N. Then $q_0 \in L^1(0,\infty)$, and we have the identity*

$$\int_0^\infty q_N(x)dx = \int_0^\infty q_0(x)dx - 2\sum_{j=1}^N \alpha_j + 4\sum_{j=1}^N \sqrt{\lambda_j}. \tag{54}$$

Proposition 30 *Assume that $q_N \in L^1(0,\infty)$, then formula (54) becomes*

$$\frac{1}{4}\int_0^\infty q_N(x)dx - \frac{1}{4}h_N \le -\frac{1}{4}\sum_{j=1}^N \alpha_j + \sum_{j=1}^N \sqrt{\lambda_j} < \sum_{j=1}^N \sqrt{\lambda_j}. \tag{55}$$

Assume additionally that the Faddeev condition holds we obtain a formula for product of eigenvalues

Proposition 31 *Assume that $(1+x)q_N(x) \in L^1(0,\infty)$ then*

$$\int_0^\infty x\,(q_N(x) - q_0(x))\,dx = \ln\left(\frac{4\prod_{j=1}^N \lambda_j}{\prod_{j=1}^N \alpha_j^2}\right).$$

For more details see [36, 62, 68, 74].

4.6 Gelfand-Levitan for the String

In the 1950s, [57], M.G. Krein proposed a method based on the theory of functions, to recover the mass of a string from its vibrating frequencies. Recall that if $M(x)$ represents the mass of the string in the interval $[0, x)$, then M is a nondecreasing function and the oscillations of the string are described by the eigensolutions of the symmetric operator

$$L := \frac{-d}{dM(x)}\frac{d^+}{dx^+}, \qquad x > 0, \tag{56}$$

where $\frac{d^+}{dx^+}$ is the right derivative. Krein could recover the function M, [57, Theorem 11.1], from the knowledge of its spectral function ρ, which is also a nondecreasing, right continuous function, if

$$\int_0^\infty \frac{1}{1+\lambda} d\rho(\lambda) < \infty. \tag{57}$$

He first established that if ρ is a continued fraction then M is a step function and most importantly he found an algorithm to explicitly compute the location and size of the jumps in M from those fractions. These special strings are the so called Stieltjes strings. He then shows that if ρ satisfies (57), then the string can be approximated by a sequence of Stieltjes strings, and the corresponding step functions of the mass also converge to the original mass. Not only was the condition (57) easy to verify, continued fractions lead to an algebraic set of rules, which allowed for an effective and explicit recovery of the mass $M(x)$ in certain cases. This approach had many far reaching applications in function theory, moments problem, integral equations, and prediction theory, see [45] and the references therein. There are many striking differences between both methods, that lead to following basic question:

Is it possible to recover the mass of the string in (56) by using a Gelfand-Levitan theory?

The answer would be a first step towards bridging both methods, and would help clarify many questions in inverse spectral problems. Recall that the G-L theory compares two close operators, with identical principle part, i.e. $-D^2 \to -D^2 + q(x)$, which explains why their spectral functions are close as $\lambda \to \infty$. On the other hand, a key idea in the spectral theory of the string, is the behavior of the spectral function $\rho(\lambda)$ as $\lambda \to \infty$ depends of the behavior of the mass $M(x)$ as $x \to 0$. Observe that the spectral function for $-\frac{1}{x^\alpha}\frac{d^2}{dx^2}$ on $[0, \infty)$ is precisely $\rho = c_\alpha \lambda^{1\pm\frac{1}{\alpha+2}}$, where the $\pm$ accounts for either the Dirichlet or Neumann boundary conditions at $x = 0$. Thus, in the spirit of the G-L theory if we are given $\rho \sim c_\alpha \lambda^{1\pm\frac{1}{\alpha+2}}$, then the principal part of the operator must be $-\frac{1}{x^\alpha}\frac{d^2}{dx^2}$ which leads to following question:

Statement of the Problem Given a nondecreasing, right continuous function, $\rho(\lambda)$ subject to

$$\rho(\lambda) \sim c_\alpha \lambda^{1\pm\frac{1}{\alpha+2}} \quad \text{as} \quad \lambda \to \infty, \quad \alpha > -1,$$

find a function q such that $\rho(\lambda)$ is the spectral function associated with a self-adjoint extension of an operator defined by

$$Lf := \frac{-1}{x^\alpha}\frac{d^2 f}{dx^2} + q(x)f, \qquad x > 0. \tag{58}$$

Clearly the eigensolutions of the unperturbed operator, i.e. $y'' + \lambda x^\alpha y = 0$ can be expressed by Bessel functions, which in turn, help provide explicit conditions on the spectral function of (58). This means a direct extension of the G-L theory, to more general operators defined by (58), since their analysis corresponds precisely to the special case $\alpha = 0$. Recall that the analysis in [46] is based on the properties of a solution of a certain linear integral equation, where the kernel

$$F(x,t) = \int_{-\infty}^{\infty} \cos x\sqrt{\lambda}\,\cos t\sqrt{\lambda}\,d\sigma(\lambda), \qquad \sigma(\lambda) = \rho(\lambda) - \frac{2}{\pi}\sqrt{\lambda_+}.$$

The Bessel functions allow growth $\rho(\lambda) \sim c_\alpha \lambda_+^{1\pm\frac{1}{\alpha+2}}$, $\alpha > -1$, that is $\lim_{\lambda\to\infty} \ln\rho(\lambda)/\ln(\lambda) \in (0,2)$, whereas the original G-L theory, which corresponds to $\alpha = 0$, means that $\lim\limits_{\lambda\to\infty} \ln\rho(\lambda)/\ln(\lambda) \in \{1/2,\ 3/2\}$.

We now outline the algorithm. Given $\rho \sim c\lambda_+^\tau$, where $\tau \in (0,2)$ we solve $\tau = 1\pm\frac{1}{\alpha+2}$ for $\alpha > -1$, and the sign $\pm$ would indicate the boundary condition, say Dirichlet or Neuman. G-L theory would recover a potential q such that ρ matches

$$L_2 := \frac{-1}{x^\alpha}\frac{d^2}{dx^2} \to L_1 := \frac{-1}{x^\alpha}\frac{d^2}{dx^2} + q(x),$$

the spectral function of L_1 and then by using a special transformation operator

$$L_1 := \frac{-1}{x^\alpha}\frac{d^2}{dx^2} + q(x) \to L_3 := \frac{-1}{w(x)}\frac{d^2}{dx^2}.$$

The operator L_3 represents a string with mass $M(x) = \int_0^x w(\eta)d\eta$. We feel that our method is not only closer in spirit to the original G-L theory but also provides a generalization, see [6, 9, 22].

4.6.1 The Transformation Operator

For our purpose we only need to express the eigenfunctionals $\varphi(x,\lambda)$ in terms of the eigenfunctionals $y(x,\lambda)$ and this is achieved with the help of the transformation operator. As in [46], we shall use a Volterra operator to connect solutions, i.e.

$$\varphi(x,\lambda) = y_N(x,\lambda) + \int_0^x K(x,t)y_N(t,\lambda)t^\alpha\,dt, \qquad x > 0, \tag{59}$$

$$y_N(x,\lambda) = \varphi(x,\lambda) + \int_0^x H(x,t)\varphi(t,\lambda)t^\alpha\,dt.$$

The kernels of the above transformation operators are defined in the following sector

$$\Omega := \left\{(x,t) \in \mathbb{R}^2 : 0 < t < x,\ \ 0 < x < \infty\right\}.$$

Let us try to find some conditions on K in the Neumann case.

Proposition 32 *Assume that* $K \in C^2(\Omega)$, $q \in C[0,\infty)$, $\alpha \neq 0$, $\alpha > -1$, *then*

$$\varphi(x,\lambda) = y_N(x,\lambda) + \int_0^x K(x,t) y_N(t,\lambda) t^\alpha dt \tag{60}$$

is a solution of $L_1\varphi(x,\lambda) = \lambda\varphi(x,\lambda)$ *if and only if*

$$\begin{cases} \frac{1}{x^\alpha} K_{xx}(x,t) - \frac{1}{t^\alpha} K_{tt}(x,t) = q(x) K(x,t), & 0 < t < x, \\ q(x) = 2x^{-\frac{\alpha}{2}} \frac{d}{dx}(x^{\frac{\alpha}{2}} K(x,x)), \\ K_t(x,0) = 0. \end{cases} \tag{61}$$

For further details we refer to [22, 43–45, 50, 51, 53, 54, 57, 58, 73].

4.7 Sampling and Transmutation

Shannon sampling formula helps reconstruct entire functions in PW_π from their sampled values over $\mathbb{Z}$

$$F(\lambda) = \sum_{n\in\mathbb{Z}} F(n) \frac{\sin(\pi\lambda - n\pi)}{(\pi\lambda - n\pi)} \tag{62}$$

Using Kramer's theorem, [78], you can generate a sampling theorem whose sampling points are the eigenvalues of a Sturm–Liouville problem. It remains to see that given any sequence $\{\mu_n\}_{n\geq 0}$ such that

$$\mu_n^2 \text{ are distincts, } \mu_n = n + \frac{\gamma}{n} + o\left(\frac{1}{n}\right) \text{ and } \alpha_n > 0, \ \alpha_n = \frac{\pi}{2} + o\left(\frac{1}{n}\right) \tag{63}$$

then μ_n^2 are the eigenvalues of the Sturm–Liouville problem

$$\begin{cases} -y''(x,\mu) + q(x)y(x,\mu) = \mu^2 y(x,\lambda), & 0 < x < \pi, \\ y'(0,\mu) - hy(0,\mu) = 0, \quad y'(\pi,\mu) + Hy(\pi,\mu) = 0. \end{cases} \tag{64}$$

The transmutation representation of the solution is given by

$$y(x,\mu) = \cos(x\mu) + \int_0^x K(x,t)\cos(t\mu)\, dt. \tag{65}$$

which could be reconstructed by the Gelfand-Levitan theory from the sequence $\left\{\mu_n^2,\ \alpha_n\right\}_{n\geq 0}$, where α_n are the norming constants. The following irregular sampling theorem can be found in [12, Proposition 4], where PW_π^e is the Paley–Wiener space of even functions.

Proposition 33 *Assume that* $\{\mu_n, \alpha_n\}_{n\geq 0}$ *satisfy (63), then for* $F \in PW_\pi^e$, *we have*

$$F(\mu) = \sum_{n\geq 0} F(\mu_n) S_n(\mu)$$

where $S_n(\mu) = \frac{1}{\alpha_n}\int_0^\pi y(x,\mu_n)y(x,\mu)dx$

It is shown that the sampling functions S_n do not depend of the norming constants, $\{\alpha_n\}$ which are used only in the Gelfand-Levitan construction. For further detail we refer to [12, 21, 29, 32–34, 37, 69].

4.8 Computational Spectral Theory

Transmutations allow the use of the sampling theorem to compute eigenvalues of Sturm–Liouville operators. Basically, it helps represent the characteristic function explicitly by a sampling series, which can then be approximated for computational purposes. For the sake of simplicity, consider the eigenvalues of (64), where $q \in L(0,\pi)$. The characteristic function then is

$$\begin{aligned}\Delta(\mu) &= y'(\pi,\mu) + Hy(\pi,\mu)\\ &= -\mu\sin(\pi\mu) + (H + K(\pi,\pi))\cos(\pi\mu) + \int_0^\pi (HK(\pi,t) + K_x(\pi,t))\cos(t\mu)\,dt\\ &= G(\mu) + S(\mu)\end{aligned}$$

where G is a known function given by

$$G(\mu) = -\mu\sin(\pi\mu) + \left\{H + h + \frac{1}{2}\int_0^\pi q(x)dx\right\}\cos(\pi\mu)$$

while

$$S(\mu) = \int_0^\pi (HK(\pi,t) + K_x(\pi,t))\cos(t\mu)\,dt$$

is unknown. Observe that $S \in PW_\pi^e$ and so by Shannon's sampling theorem, (21), we have

$$S(\lambda) = \sum_{n\in\mathbb{Z}} S(n)\frac{\sin(\pi\lambda - n\pi)}{(\pi\lambda - n\pi)}$$

To compute the samples $S(n)$, we use the fact that

$$S(n) = \Delta(n) - G(n)$$

where the values of $\{\Delta(n)\}_{n\geq 0}$ are computed numerically by integrating the initial value problem from (64), while $G(n)$ is given a known formula. Thus we have

$$\Delta(\mu) = G(\mu) + \sum_{n\in\mathbb{Z}} [\Delta(n) - G(n)] \frac{\sin(\pi\lambda - n\pi)}{(\pi\lambda - n\pi)}. \tag{66}$$

One can truncate the series to obtain guaranteed error bounds and to approximate the roots of Δ, see [13]. For further detail we refer to [11, 13–18, 26, 27, 34].

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Hankel Generalized Convolutions with the Associated Legendre Functions in the Kernel and Their Applications

Lyubov Britvina

Abstract This investigation is devoted to finding the existence conditions, boundary properties and applications of convolution operators for the ν-th order Hankel transform

$$\mathrm{H}_\nu[f](x) = \int_0^\infty f(t) J_\nu(xt)\, t\, dt\,, \quad x \in \mathbb{R}_+\,.$$

The generalized convolutions defined by the Parseval type equalities

$$\mathrm{H}_\nu[h_1](x) = x^{-\nu}\mathrm{H}_\mu[f](x)\mathrm{H}_\mu[g](x)\,,$$
$$\mathrm{H}_\mu[h_2](x) = x^{-\nu}\mathrm{H}_\nu[f](x)\mathrm{H}_\mu[g](x)$$

are considered in spaces $L_1(\mathbb{R}_+, \sqrt{t}dt)$ and $L_2(\mathbb{R}_+, tdt)$. Properties and estimates for the convolution kernel are investigated. Also integral operators are considered related to generalized convolutions for the Hankel transform $\mathrm{H}_\nu[f](x)$. Watson's type theorems for convolution operators are proved and integral operators with nonsymmetric kernels are studied. Some applications to solving integral equations are given.

Keywords Hankel transform · Convolution · Convolution transform · Watson's theorem · Integral equations

Mathematics Subject Classification (2010) Primary 44A35; Secondary 42B10

L. Britvina (✉)
Novgorod State University, Veliky Novgorod, Russia
e-mail: Lyubov.Britvina@novsu.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_3

1 Introduction

The Hankel transform is the most extensively studied area of the theory of Bessel transforms. When we are dealing with problems that show circular symmetry, Hankel transform may be very useful (see, for, example, [1, 2]). Laplace's partial differential equation in cylindrical coordinates can be transformed into an ordinary differential equation by using the Hankel transform. Because the Hankel transform is the two-dimensional Fourier transform of a circularly symmetric function, it plays an important role in optical data processing. Also it is known (cf. [3–5]) the transform (1.1) is a particular case of Mellin‘s convolution type transform.

In this investigation we will consider existence conditions, boundary properties and applications of convolution operators for the Hankel transform.

Let $f(t)$ be a function defined for $t \in \mathbb{R}_+$. The ν-th order Hankel transform of $f(t)$ is defined as [3, 6]

$$\mathrm{H}_\nu[f](x) = \int_0^\infty f(t) J_\nu(xt)\, t\, dt\,, \quad x \in \mathbb{R}_+\,, \tag{1.1}$$

where $J_\nu(z)$ is the Bessel function [6, 7] of the first kind of order ν, Re $\nu > -1/2$. The most important special cases of the Hankel transform correspond to $\nu = 0$ and $\nu = 1$.

Here we will consider transform (1.1) in weight Lebesgue spaces $L_1(\mathbb{R}_+, \sqrt{t}dt)$ and $L_2(\mathbb{R}_+, tdt)$ with the norms

$$||f||_{L_1(\mathbb{R}_+,\sqrt{t}dt)} = \int_0^\infty |f(t)|\sqrt{t}dt < \infty,$$

$$||f||_{L_2(\mathbb{R}_+,tdt)} = \left(\int_0^\infty |f(t)|^2 tdt\right)^{1/2} < \infty.$$

As known [3] the Hankel transform $\mathrm{H}_\nu[f](x)$ of the function $f(t) \in L_1(\mathbb{R}_+, \sqrt{t}dt)$ multiplied by $\sqrt{x}$ belongs to the space $C_0(\mathbb{R}_+)$ of bounded continuous functions vanishing at infinity. Under some additional conditions the inversion formula holds. For instance, it does if $f(t)$ is a function of bounded variation on any finite interval $(0, R)$.

In the case of $L_2(\mathbb{R}_+, tdt)$-space we should define the Hankel transform in the mean-square convergence sense, namely

$$F_\nu(x) = \mathrm{H}_\nu[f](x) = \underset{N\to\infty}{\mathrm{l.i.m.}} \int_{1/N}^{N} f(t) J_\nu(xt)\, t\, dt,$$

and familiar Plancherel's theorem [3] says that $\mathrm{H}_\nu\colon L_2(\mathbb{R}_+, t dt) \to L_2(\mathbb{R}_+, x dx)$ is an isometric isomorphism with the reciprocal formula

$$f(t) = \underset{N\to\infty}{\mathrm{l.i.m.}} \int_{1/N}^{N} \mathrm{H}_\nu[f](x) J_\nu(xt)\, x\, dx,$$

and Parseval's equality

$$||\mathrm{H}_\nu f||_{L_2(\mathbb{R}_+, xdx)} = ||f||_{L_2(\mathbb{R}_+, tdt)}. \tag{1.2}$$

Various generalized convolutions generated by the Hankel transform and other integral transforms can be constructed by using the definition of generalized convolution or polyconvolution introduced by V.A. Kakichev [8, 9]. The corresponding results can be found, for example, in [8–14]

Let A_1, A_2 and A_3 be linear operators, $A_j\colon M_j \to N_j$, $j = 1, 2$ and $A_3\colon M_3 \leftrightarrow N_3$. Assume that some weight function $\alpha(x)$ exists such that for all functions $(A_1 f)(x) \in N_1$ and $(A_2 k)(x) \in N_2$ the product $\alpha(x)(A_1 f)(x)(A_2 k)(x)$ belongs to the space N_3.

Definition 1.1 The generalized convolution, or polyconvolution, of functions $f(t) \in M_1$ and $k(t) \in M_2$, under A_1, A_2, A_3, with weight function $\alpha(x)$, is the function $h(t) \in M_3$ denoted by $\left(f_{A_1} \overset{\alpha}{*} k_{A_2}\right)_{A_3}(t)$ for which the factorization property

$$(A_3 h)(x) = A_3\left[\left(f_{A_1} \overset{\alpha}{*} k_{A_2}\right)_{A_3}\right](x) = \alpha(x)(A_1 f)(x)(A_2 k)(x)$$

is valid.

The classical convolution for the Hankel transform was first introduced by Ya.I. Zhitomirskii [15] in 1955. He constructed the convolution using the translation operator which was first introduced and studied by B.M. Levitan in 1949 [16] (see also [17]). Also this classical convolution and corresponding translation operator were investigated by I.I. Hirschman, D.T. Haimo, F.M. Cholowinski [18–20]. In this context, it is important to note a large amount of research by I.A. Kipriyanov, L.N. Lyakhov, S.M. Sitnik, E.L. Shishkina, S.S. Platonov and others authors (see, for example, [21–24]).

In 1967 V.A. Kakichev [8] constructed this convolution by using Definition 1.1.

The explicit expression of this convolution is

$$\begin{aligned}\left(f * k\right)(t) = \frac{t^\nu}{2^\nu \sqrt{\pi}\,\Gamma(\nu + 1/2)} \int_0^\pi \sin^{2\nu} s \\ \times \int_0^\infty f(\tau) \frac{k\left(\sqrt{t^2 + \tau^2 - 2t\tau \cos s}\right)}{\left(t^2 + \tau^2 - 2t\tau \cos s\right)^{\nu/2}} \tau^{\nu+1}\, d\tau\, ds.\end{aligned} \tag{1.3}$$

If $f(t),\ k(t) \in L_1(\mathbb{R}_+, \sqrt{t}dt)$, Re $\nu > 1/2$ then polyconvolution (1.3) of functions $f(t)$ and $k(t)$ with the weight function $\alpha(x) = x^{-\nu}$ exists [14].

A number of convolution constructions involving the Hankel transform was derived by N.X. Thao and N.T. Hai [14]. Some polyconvolutions obtained by the author were exhibited in [10–13]. The results presented in these papers are based on the Kakichev approach to the notion of the polyconvolution.

If one of the functions in the convolution $\left(f_{A_1} \overset{\alpha}{*} k_{A_2}\right)_{A_3}(t)$, say the function $k(t)$, is fixed, then one can study the transform of convolution type:

$$A : f \to \mathcal{L}\left(f_{A_1} \overset{\alpha}{*} k_{A_2}\right)_{A_3},$$

where $\mathcal{L}$ is an operator. The function $k(t)$ is called the kernel of the transform A.

Integral transforms related to various convolution constructions was considered in papers [25–30]

This paper is a continuation of the investigation of convolution operators and their applications given in [10]. Here we consider two generalized convolutions defined by the Parseval type equalities

$$h_1(t) = \left(f_\mu \overset{-\nu}{*} k_\mu\right)_\nu(t) = \mathrm{H}_\nu^{-1}\Big[x^{-\nu}\mathrm{H}_\mu[f](x)\mathrm{H}_\mu[k](x)\Big](t) \tag{1.4}$$

$$h_2(t) = \left(f_\nu \overset{-\nu}{*} k_\mu\right)_\mu(t) = \mathrm{H}_\mu^{-1}\Big[x^{-\nu}\mathrm{H}_\nu[f](x)\mathrm{H}_\mu[k](x)\Big](t) \tag{1.5}$$

Note that the first convolution $h_1(t)$ is commutative and the second convolution $h_2(t)$ is not commutative.

We study their mapping properties. Integral operators related to these convolutions are constructed and their existence and boundary properties are found. Also we give some applications to the corresponding class of convolution equations.

2 Properties and Estimates for the Convolution's Kernel

Let us consider the function

$$\Omega_{\mu;\nu}(u, v; t) = \int_0^\infty x^{1-\nu} J_\mu(xu) J_\mu(xv) J_\nu(xt)\, dx. \tag{2.1}$$

This function defines both convolutions (1.4)–(1.5). We notice that the function $\Omega_{\mu;\nu}(u, v; t)$ is symmetrical relative to permutations of variables u and v, that is, $\Omega_{\mu;\nu}(u, v; t) = \Omega_{\mu;\nu}(v, u; t)$. Therefore, the estimates and formulas below are valid when these variables are rotated.

Using the asymptotic expansion of Bessel functions of the first kind (see, for example, [7])

$$J_\nu(y) = \sqrt{\frac{2}{\pi y}} \cos\left(y - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O(y^{-3/2}),\ y \to +\infty,$$
$$J_\nu(y) = O(y^\nu),\ y \to 0+,$$

we can easily show that a positive number C_1 which is independent of $y \in (0, \infty)$ such that

$$|\sqrt{y} J_\nu(y)| < C_1, \qquad \forall y \in (0, \infty)$$

exists. The function $y^{-\nu} J_\mu(y) \in L_1(\mathbb{R}_+)$ for $\mathrm{Re}\,\nu > 1/2$ and $\mathrm{Re}\,\mu > \mathrm{Re}\,\nu - 1$ and bounded function for $\mathrm{Re}\,\mu \geq \mathrm{Re}\,\nu \geq -1/2$, i.e.

$$|y^{-\nu} J_\mu(y)| < C_2, \qquad \forall y \in (0, \infty) \tag{2.2}$$

Therefore, for $\mathrm{Re}\,\nu > 1/2$

$$|\Omega^N_{\mu;\nu}(u, v; t)| = \left| \int_0^N x^{1-\nu} J_\mu(xu) J_\mu(xv) J_\nu(xt)\, dx \right| \leq \frac{C_1^2}{\sqrt{uv}} \int_0^N \left| x^{-\nu} J_\nu(xt) \right| dx$$
$$\leq \frac{C_1^2\, t^{\mathrm{Re}\,\nu - 1}}{\sqrt{uv}} \int_0^\infty \left| x^{-\nu} J_\nu(x) \right| dx \leq \frac{C\, t^{\mathrm{Re}\,\nu - 1}}{\sqrt{uv}},$$

where C is independent of t, u, v and N.

Similarly, for $\mathrm{Re}\,\nu > 1/2$ and $\mathrm{Re}\,\mu > \mathrm{Re}\,\nu - 1$ we obtain the estimate

$$|\Omega^N_{\mu;\nu}(u, v; t)| \leq \frac{C\, u^{\mathrm{Re}\,\nu - 1}}{\sqrt{tv}},$$

where C is also independent of t, u, v and N.

Thus, the function $\Omega_{\mu;\nu}(u, v; t)$ defined by expression (2.1) exists. Moreover, this function can be presented as [31], formula 2.12.42.11

$$\begin{aligned}
\Omega_{\mu;\nu}(u, v; t) &= 0\,, && 0 < t < |u - v|\,; \\
&= \frac{(uv)^{\nu-1}}{\sqrt{2\pi}\, t^\nu} P^{1/2-\nu}_{\mu-1/2}(\cos s) \sin^{\nu-1/2} s\,, && |u - v| < t < u + v\,; \\
&= -\frac{\sqrt{2}(uv)^{\nu-1}}{\pi^{3/2} t^\nu} \sin\Big[(\mu - \nu)\pi \Big] \mathbf{e}^{(2\nu-1)\pi i/2} \\
&\times v^\nu\, Q^{1/2-\nu}_{\mu-1/2}(\cosh r) \sinh^{\nu-1/2} r\,, && t > u + v\,,
\end{aligned}$$

where $P_\mu^\nu(x)$, $Q_\mu^\nu(x)$ are the associated Legendre functions of the first and second kind, respectively, Re $\mu > -1$, Re $\nu > -1/2$, $t, u, v > 0$, and

$$\cos s = \frac{u^2 + v^2 - t^2}{2uv}, \quad \cosh r = \frac{t^2 - u^2 - v^2}{2uv},$$

On the other hand, the function $\Omega_{\mu;\nu}(u, v; t)$ is the Hankel transform of the product of the Bessel functions, i.e.

$$\Omega_{\mu;\nu}(u, v; t) = \mathrm{H}_\nu \left[x^{-\nu} J_\mu(xu) J_\mu(xv) \right](t) \tag{2.3}$$

$$= \mathrm{H}_\mu \left[x^{-\nu} J_\mu(xv) J_\nu(xt) \right](u). \tag{2.4}$$

Therefore, $\sqrt{t}\Omega_{\mu;\nu}(u, v; t)$ belongs to the space $C_0(\mathbb{R}_+)$ of bounded continuous functions vanishing at $t \to \infty$ as when Re $\nu > 1/2$, Re $\mu >$ Re $\nu - 1$

$$||x^{-\nu} J_\mu(xu) J_\mu(xv)||_{L_1(\mathbb{R}_+, \sqrt{x}dx)} \le C \frac{u^{\mathrm{Re}\,\nu - 1}}{\sqrt{v}}.$$

Similarly, when Re $\nu > 1/2$, Re $\mu >$ Re $\nu - 1$ we have the estimate

$$||x^{-\nu} J_\mu(xv) J_\nu(xt)||_{L_1(\mathbb{R}_+, \sqrt{x}dx)} \le C \frac{v^{\mathrm{Re}\,\nu - 1}}{\sqrt{t}}$$

and $\sqrt{u}\Omega_{\mu;\nu}(u, v; t)$ belongs to the space $C_0(\mathbb{R}_+)$ as function of variable u. This statement can also be obtained using a similar estimate for Re $\nu > 1/2$

$$||x^{-\nu} J_\mu(xv) J_\nu(xt)||_{L_1(\mathbb{R}_+, \sqrt{x}dx)} \le C \frac{t^{\mathrm{Re}\,\nu - 1}}{\sqrt{v}}.$$

From the equalities (2.3)–(2.4) we obtain

$$x^{-\nu} J_\mu(xu) J_\mu(xv) = \mathrm{H}_\nu^{-1} \left[\Omega_{\mu;\nu}(u, v; t) \right](x)$$
$$= \int_0^\infty t J_\nu(xt)\, \Omega_{\mu;\nu}(u, v; t)\, dt,$$
$$x^{-\nu} J_\mu(xu) J_\nu(xt) = \mathrm{H}_\mu^{-1} \left[\Omega_{\mu;\nu}(u, v; t) \right](x)$$
$$= \int_0^\infty v J_\mu(xv)\, \Omega_{\mu;\nu}(u, v; t)\, dv.$$

For $\nu > 1/2$, $\nu - \mu < 1/2$ we get accordingly the following estimates using the formula 2.12.31.2 from [31]

$$\begin{aligned}
||\Omega_{\mu;\nu}(u, v; t)||^2_{L_2(\mathbb{R}_+, tdt)} &= ||x^{-\nu} J_\mu(xu) J_\mu(xv)||^2_{L_2(\mathbb{R}_+, xdx)} \\
&\leq C_{\mu,\nu} \frac{u^{2\nu-1}}{v}, \\
||\Omega_{\mu;\nu}(u, v; t)||^2_{L_2(\mathbb{R}_+, tdt)} &= ||x^{-\nu} J_\mu(xu) J_\nu(xt)||^2_{L_2(\mathbb{R}_+, xdx)} \\
&\leq C_{\mu,\nu} \frac{u^{2\nu-1}}{t}, \\
||\Omega_{\mu;\nu}(u, v; t)||^2_{L_2(\mathbb{R}_+, tdt)} &= ||x^{-\nu} J_\mu(xu) J_\mu(xt)||^2_{L_2(\mathbb{R}_+, xdx)} \\
&\leq C_\nu \frac{t^{2\nu-1}}{u},
\end{aligned}$$

where $C_{\mu,\nu}$ and C_ν are independent of u and t, the parameters ν and μ are real.

The function $\Omega_{\mu;\nu}(u, v; t)$ define two polyconvolutions (1.4)–(1.5), which can be presented by form

$$h_1(t) = \int_0^\infty \int_0^\infty uvf(u)g(v)\Omega_{\mu;\nu}(u, v; t)\, du\, dv, \tag{2.5}$$

$$h_2(t) = \int_0^\infty \int_0^\infty uvf(u)g(v)\Omega_{\mu;\nu}(t, v; u)\, du\, dv. \tag{2.6}$$

3 Mapping Properties of the Generalized Convolutions

The following theorems gives the existence conditions and mapping properties of polyconvolutions (2.5) and (2.6).

In space $L_1(\mathbb{R}_+, \sqrt{t}dt)$ the corresponding convolutions we investigate in[10–13]

Theorem 3.1 *Suppose that* $f(t)$, $k(t) \in L_1(\mathbb{R}_+, \sqrt{t}dt)$ *and* $\text{Re}\, \nu > 1/2$, $\text{Re}\, \mu > (2\text{Re}\, \nu - 3)/4$. *Then the function* $h_1(t)$ *exists and the following factorization relation is valid*

$$\text{H}_\nu[h_1](x) = x^{-\nu}\text{H}_\mu[f](x)\text{H}_\mu[k](x) \in L_1(\mathbb{R}_+, \sqrt{x}dx).$$

Theorem 3.2 *Suppose that* $f(t)\,,k(t)\in L_1(\mathbb{R}_+,\sqrt{t}dt)$ *and* $\operatorname{Re}\nu>1/2$, $\operatorname{Re}\mu>\operatorname{Re}\nu-1$. *Then the function* $h_2(t)$ *exists and the factorization relation*

$$\mathrm{H}_\mu[h_2](x)=x^{-\nu}\mathrm{H}_\nu[f](x)\mathrm{H}_\mu[k](x)\in L_1(\mathbb{R}_+,\sqrt{x}dx)$$

is valid.

Now we find the existence conditions in other weight Lebesgue spaces.

Theorem 3.3 *Let* $f(t)\in L_1(\mathbb{R}_+,\sqrt{t}dt)$, $k(t)\in L_2(\mathbb{R}_+,tdt)$ *and* $\nu>0$, $\mu>-1/2$, *then the generalized convolution (1.4) of the functions* $f(t)$ *and* $k(t)$ *exists.*

Proof Using the definition of generalized convolution (1.4) and the definition of the Hankel transform (1.1), we obtain

$$\begin{aligned}h_1(t)&=\int_0^\infty \mathrm{H}_\mu[f](x)\mathrm{H}_\mu[k](x)J_\nu(xt)\,x^{1-\nu}\,dx \qquad (3.1)\\&=\int_0^\infty x^{1-\nu}\,dx\int_0^\infty\int_0^\infty f(u)k(v)J_\nu(xt)J_\mu(xu)J_\mu(xv)\,uv\,dudv\,.\end{aligned}$$

Let us prove the existence of the polyconvolution $h_1(t)$. Applying Schwarz's inequality we get

$$\begin{aligned}|h_1(t)|^2&\le\int_0^\infty|\sqrt{x}\mathrm{H}_\mu[f](x)|\,|\mathrm{H}_\mu[k](x)|^2xdx\int_0^\infty x^{-2\nu}|\sqrt{x}\mathrm{H}_\mu[f](x)|\,J_\nu^2(xt)dx\\&\le C\int_0^\infty|\mathrm{H}_\mu[k](x)|^2xdx\int_0^\infty x^{-2\nu}\,J_\nu^2(xt)dx\\&\le Ct^{2\nu-1}||k||^2_{L_2(\mathbb{R}_+,tdt)}. \qquad (3.2)\end{aligned}$$

Here we used Parseval's equality (1.2) and the formula 2.12.31.2 from [31].

Therefore, the convolution $h_1(t)$ exists for all fixed $t\in\mathbb{R}_+$ and the function $h_1(t)$ is bounded continuous on $\mathbb{R}_+$.

Changing the order of integration in (3.1) by virtue of (3.2) and using definition (2.1) we obtain

$$\begin{aligned}h_1(t)&=\int_0^\infty\int_0^\infty f(u)k(v)uv\,dudv\int_0^\infty x^{1-\nu}J_\mu(xu)J_\mu(xv)J_\nu(xt)\,dx\\&=\int_0^\infty\int_0^\infty uvf(u)g(v)\Omega_{\mu;\nu}(u,v;t)\,du\,dv.\end{aligned}$$

We found the explicit form (2.5) for the polyconvolution $h_1(t)$. Theorem 3.3 is proved. □

The following assertion can be proved in a similar way.

Theorem 3.4 *Let* $f(t) \in L_1(\mathbb{R}_+, \sqrt{t}dt)$, $k(t) \in L_2(\mathbb{R}_+, tdt)$ *and* $\nu > 0,\ \nu - \mu < 1/2$, *then the generalized convolution (1.5) of the functions* $f(t)$ *and* $k(t)$ *exists. The explicit form for the polyconvolution* $h_2(t)$ *is defined by (2.6).*

Theorem 3.5 *Let* $f(t), k(t) \in L_2(\mathbb{R}_+, tdt)$ *and* $\mathrm{Re}\mu \geq \mathrm{Re}\nu > -1/2$, *then the generalized convolution (1.4) of the functions* $f(t)$ *and* $k(t)$ *exists.*

Proof Indeed, calling again Schwarz's inequality and using (2.2) we find

$$|h_2(t)|^2 \leq \int_0^\infty \left|\mathrm{H}_\nu[f](x)\right|^2 \left|x^{-\nu} J_\mu(xt)\right| x\,dx \int_0^\infty \left|\mathrm{H}_\mu[k](x)\right|^2 \left|x^{-\nu} J_\mu(xt)\right| x\,dx$$

$$\leq \mathrm{Const}\, t^{2\mathrm{Re}\nu} \int_0^\infty \left|\mathrm{H}_\nu[f](x)\right|^2 x\,dx \int_0^\infty \left|\mathrm{H}_\mu[k](x)\right|^2 x\,dx$$

$$= \mathrm{Const}\, t^{2\mathrm{Re}\nu} ||f||^2_{L_2(\mathbb{R}_+,tdt)} ||k||^2_{L_2(\mathbb{R}_+,tdt)}.$$

Appealing to Fubini's theorem we represent convolution (1.4) by the equality (2.5) completing the proof of Theorem 3.5. □

The following theorem is proving in same manner.

Theorem 3.6 *Let* $f(t), k(t) \in L_2(\mathbb{R}_+, tdt)$ *and* $\mathrm{Re}\nu > -1/2,\ \mathrm{Re}\mu > -1/2$, *then the generalized convolution (1.5) of the functions* $f(t)$ *and* $k(t)$ *exists.*

In next section we construct integral transforms related to the generalized convolutions (2.5)–(2.6).

4 Integral Transforms Related to the Hankel Polyconvolution

We involve the following differential operators (cf. [32])

$$N_{m,\nu} = t^\nu \left(\frac{d}{t\,dt}\right)^m t^{m-\nu}, \qquad S^k_{m,\nu} = \left[N_{m,-\nu} N_{m,\nu+m}\right]^k, \tag{4.1}$$

which possess the properties:

(a) $S^k_{m,\nu} = S^k_{m,-\nu} = S_{km,\nu}$,

where $S_{n,\nu} = S^1_{n,\nu} = S^n_{1,\nu} = \left[\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{\nu^2}{t^2}\right]^n$.

(b) $N_{m,\,\pm\nu+km} N_{m,\,\pm\nu+(k-1)m} \ldots N_{m,\,\pm\nu+m} N_{m,\,\pm\nu} = N_{m(k+1),\,\pm\nu+km}$.

It should be noted that some special cases of these operators are involved, for instance, in various equations of elasticity theory.

Using properties (a) and (b), it is readily verified that all well-known differential operators related to Hankel transform can be expressed in terms of $N_{m,\,\pm\nu}$ and $S_{m,\nu}$. Therefore, we restrict ourselves to operators (4.1).

Further, we assume that a function is differentiable enough times at all $t \in \mathbb{R}_+$ when the differential operators (4.1) are applied to it.

The main result is presented in the following theorems.

Theorem 4.1 *If* $f(t), k(t) \in L_2(\mathbb{R}_+, tdt)$ *and* $\mathrm{Re}\mu \geq \mathrm{Re}\nu > -1/2$*, and*

$$\left(k_\mu \overset{-\nu}{*} \overline{k}_\mu\right)_\nu (t) = \mathrm{H}_\nu\left[\frac{x^\nu}{r_n^2(x)}\right], \quad r_n(x) = \sum_{m=0}^{n} a_m x^{2m}, \tag{4.2}$$

where $x \in \mathbb{R}_+$; $a_m \in \mathbb{R}$, $\forall m = 0, 1, \ldots, n$, a_0, $a_n \neq 0$*; and we assume that* $r_n^{-2}(x) \in L_2(\mathbb{R}_+, tdt)$; $n \geq 1$. *Then the formula*

$$g(t) = \sum_{m=0}^{n} (-1)^m a_m S_{m,\nu} \left(k_\mu \overset{-\nu}{*} f_\mu\right)_\nu (t), \quad t \in \mathbb{R}_+ \tag{4.3}$$

defines almost everywhere a function $g(t) \in L_2(\mathbb{R}_+, tdt)$*, such that*

$$||g||_{L_2(\mathbb{R}_+,tdt)} = ||f||_{L_2(\mathbb{R}_+,tdt)}. \tag{4.4}$$

Moreover, the inversion formula

$$f(t) = \sum_{m=0}^{n} (-1)^m a_m S_{m,\mu} \left(\overline{k}_\mu \overset{-\nu}{*} g_\nu\right)_\mu (t), \quad t \in \mathbb{R}_+ \tag{4.5}$$

holds almost everywhere.

Proof Since $k(t) \in L_2(\mathbb{R}_+, tdt)$, $\mathrm{Re}\mu \geq \mathrm{Re}\nu > -1/2$ then we have

$$\left(k_\mu \overset{-\nu}{*} \overline{k}_\mu\right)_\nu (t) = \int_0^\infty x^{1-\nu} \mathrm{H}_\mu[k](x)\, \overline{\mathrm{H}_\mu[k](x)} J_\nu(xt)\, dx\,, \quad t \in \mathbb{R}_+ .$$

Hence, condition (4.2) can be written in the form

$$\int_0^\infty x^{1-\nu}\Big|\mathrm{H}_\mu[k](x)\Big|^2 J_\nu(xt)\,dx \;= \mathrm{H}_\nu\left[\frac{x^\nu}{r_n^2(x)}\right](t)\,, \quad t\in\mathbb{R}_+\,.$$

From the uniqueness property of the Hankel transform it follows that

$$x^{-\nu}\Big|\mathrm{H}_\mu[k](x)\Big|^2 = \frac{x^\nu}{r_n^2(x)}\,. \tag{4.6}$$

Consequently,

$$\Big|\mathrm{H}_\mu[k](x)\Big| = \frac{x^\nu}{\Big|r_n(x)\Big|}\,, \quad \forall x\in\mathbb{R}_+\,. \tag{4.7}$$

Conditions (4.2) and (4.7) are equivalent in the space $L_2(\mathbb{R}_+, tdt)$.
If $t^l s(t)\in L_2(\mathbb{R}_+, tdt)$, $l=0,1,\ldots,2m$ then [32]

$$S_{m,\nu}\mathrm{H}_\nu[s](x) = \mathrm{H}_\nu\left[(-1)^m t^{2m}s(t)\right](x)\,.$$

Therefore, for $r_n(t)s(t)\in L_2(\mathbb{R}_+, tdt)$ we obtain

$$\sum_{m=0}^{n}(-1)^m a_m S_{m,\nu}\mathrm{H}_\nu[s](x) = \mathrm{H}_\nu\left[r_n(t)s(t)\right](x)\,. \tag{4.8}$$

Formula (4.7) shows that $x^{-\nu}r_n(x)\mathrm{H}_\mu[k](x)$ is bounded. Thus, we have $x^{-\nu}r_n(x)\mathrm{H}_\mu[k](x)\mathrm{H}_\mu[f](x) \in L_2(\mathbb{R}_+, tdt)$. We apply formula (4.8) with $s(x) = x^{-\nu}\mathrm{H}_\mu[k](x)\mathrm{H}_\mu[f](x)$ then

$$\begin{aligned} g(t) &= \sum_{m=0}^{n}(-1)^m a_m S_{m,\nu}\int_0^\infty x^{1-\nu}\mathrm{H}_\mu[k](x)\mathrm{H}_\mu[f](x)\,J_\nu(xt)\,dx \\ &= \int_0^\infty x^{1-\nu}r_n(x)\mathrm{H}_\mu[k](x)\mathrm{H}_\mu[f](x)\,J_\nu(xt)\,dx\,, \quad t\in\mathbb{R}_+ \end{aligned} \tag{4.9}$$

is defines almost everywhere. Moreover, $g(t)\in L_2(\mathbb{R}_+, tdt)$.

Now the Parseval identity (1.2) for the Hankel transform along with Eq. (4.7) gives

$$\begin{aligned} ||g(t)||_{L_2(\mathbb{R}_+,tdt)} &= ||x^{-\nu}r_n(x)\mathrm{H}_\mu[k](x)\mathrm{H}_\mu[f](x)||_{L_2(\mathbb{R}_+,xdx)} \\ &= ||\mathrm{H}_\mu[f](x)||_{L_2(\mathbb{R}_+,xdx)} = ||f(t)||_{L_2(\mathbb{R}_+,tdt)}. \end{aligned}$$

Therefore, formula (4.4) is proved.

On the other hand, formula (4.9) is equivalent to the following relation

$$\mathrm{H}_\nu[g](x) = x^{-\nu} r_n(x) \mathrm{H}_\mu[k](x) \mathrm{H}_\mu[f](x)\,, \quad x \in \mathbb{R}_+\,.$$

Combining with (4.6) we obtain

$$\mathrm{H}_\mu[f](x) = x^{-\nu} r_n(x) \overline{\mathrm{H}_\mu[k](x)} \mathrm{H}_\nu[g](x)\,, \quad x \in \mathbb{R}_+\,.$$

Consequently, we arrive at the inversion formula for transform (4.3)

$$\begin{aligned} f(t) &= \int_0^\infty x^{1-\nu} r_n(x) \overline{\mathrm{H}_\mu[k](x)} \mathrm{H}_\nu[g](x)\, J_\mu(xt)\, dx \\ &= \sum_{m=0}^n (-1)^m a_m S_{m,\mu} \int_0^\infty x^{1-\nu} \overline{\mathrm{H}_\mu[k](x)} \mathrm{H}_\nu[g](x)\, J_\mu(xt)\, dx \\ &= \sum_{m=0}^n (-1)^m a_m S_{m,\mu} \left(\overline{k}_\mu \overset{-\nu}{*} g_\nu \right)_\mu (t)\,. \end{aligned}$$

Theorem 4.1 is proved. □

The following theorem is proving in same manner.

Theorem 4.2 *If* $f(t), k(t) \in L_2(\mathbb{R}_+, tdt)$ *and* $\mathrm{Re}\nu > -1/2,\ \mathrm{Re}\mu > -1/2$ *and*

$$\left(k_\nu \overset{-\nu}{*} \overline{k}_\nu \right)_\nu (t) = \mathrm{H}_\nu \left[\frac{x^\nu}{r_n^2(x)} \right], \quad r_n(x) = \sum_{m=0}^n a_m x^{2m}\,,$$

where $x \in \mathbb{R}_+\,;\ a_m \in \mathbb{R}\,,\ \ \forall m = 0, 1, \ldots, n\,, a_0,\ a_n \neq 0$*; and we assume that* $r_n^{-2}(x) \in L_2(\mathbb{R}_+, tdt)\,;\ \ n \geq 1$. *Then*

$$g(t) = \sum_{m=0}^n (-1)^m a_m S_{m,\mu} \left(k_\nu \overset{-\nu}{*} f_\mu \right)_\mu (t), \quad t \in \mathbb{R}_+$$

defines almost everywhere a function $g(t) \in L_2(\mathbb{R}_+, tdt)$*, such that*

$$||g||_{L_2(\mathbb{R}_+, tdt)} = ||f||_{L_2(\mathbb{R}_+, tdt)}.$$

Moreover, the inversion formula

$$f(t) = \sum_{m=0}^n (-1)^m a_m S_{m,\mu} \left(\overline{k}_\nu \overset{-\nu}{*} g_\mu \right)_\mu (t)$$

holds.

We note, that integral transforms with unsymmetrical kernels can be constructed in the same manner. The results are presented in following theorems.

Theorem 4.3 *Let $k_1(x)$ and $h_1(x)$ be bounded functions on $\mathbb{R}_+$ such that $k_1(x)h_1(x) \equiv 1$.*

$$\tilde{k}(t) = \mathrm{H}_\mu\left[\frac{x^\nu k_1(x)}{r_n(x)}\right], \quad r_n(x) = \sum_{m=0}^{n} a_m x^{2m},$$

$$\tilde{h}(t) = \mathrm{H}_\mu\left[\frac{x^\nu h_1(x)}{\rho_l(x)}\right], \quad \rho_l(x) = \sum_{m=0}^{l} b_m x^{2m},$$

where $x \in \mathbb{R}_+$; $a_m,\ b_m \in \mathbb{R}$, $\forall m$; $a_0,\ a_n,\ b_0,\ b_l \neq 0$; $n,\ l \geq 1$.

If $\tilde{k}(t),\ \tilde{h}(t),\ f(t) \in L_2(\mathbb{R}_+, tdt)$, $\mathrm{Re}\mu \geq \mathrm{Re}\nu > -1/2$ *Then the formula*

$$g(t) = \sum_{m=0}^{n} (-1)^m a_m S_{m,\nu}\left(\tilde{k}_\mu \overset{-\nu}{*} f_\mu\right)_\nu (t), \quad t \in \mathbb{R}_+$$

defines almost everywhere a function $g(t) \in L_2(\mathbb{R}_+, tdt)$. Moreover, the inversion formula

$$f(t) = \sum_{m=0}^{l} (-1)^m b_m S_{m,\mu}\left(\tilde{h}_\mu \overset{-\nu}{*} g_\nu\right)_\mu (t), \quad t \in \mathbb{R}_+$$

holds almost everywhere.

The proof of Theorem 4.3 is similar to the proof Theorem 4.1, where condition (4.6) is replaced by

$$\mathrm{H}_\mu[\tilde{k}](x)\mathrm{H}_\mu[\tilde{h}](x) = \frac{x^{2\nu} k_1(x) h_1(x)}{r_n(x)\rho_l(x)}. \tag{4.10}$$

Also we can construct transforms with unsymmetrical kernels related to polyconvolution (2.6).

Theorem 4.4 *Let $k_1(x)$ and $h_1(x)$ be bounded functions on $\mathbb{R}_+$ such that $k_1(x)h_1(x) \equiv 1$.*

$$\tilde{k}(t) = \mathrm{H}_\nu\left[\frac{x^\nu k_1(x)}{r_n(x)}\right], \quad r_n(x) = \sum_{m=0}^{n} a_m x^{2m},$$

$$\tilde{h}(t) = \mathrm{H}_\nu\left[\frac{x^\nu h_1(x)}{\rho_l(x)}\right], \quad \rho_l(x) = \sum_{m=0}^{l} b_m x^{2m},$$

where $x \in \mathbb{R}_+$; $a_m,\ b_m \in \mathbb{R}$, $\forall m$; $a_0,\ a_n,\ b_0,\ b_l \neq 0$; $n,\ l \geq 1$.

If $\tilde{k}(t),\ \tilde{h}(t),\ f(t)\ \in\ L_2(\mathbb{R}_+, tdt)$*,* $\mathrm{Re}\nu > -1/2$, $\mathrm{Re}\mu > -1/2$*. Then the formula*

$$g(t) = \sum_{m=0}^{n} (-1)^m a_m S_{m,\mu} \left(\tilde{k}_\nu \overset{-\nu}{*} f_\mu \right)_\mu (t), \quad t \in \mathbb{R}_+$$

defines almost everywhere a function $g(t) \in L_2(\mathbb{R}_+, tdt)$*. Moreover, the inversion formula*

$$f(t) = \sum_{m=0}^{l} (-1)^m b_m S_{m,\mu} \left(\tilde{h}_\nu \overset{-\nu}{*} g_\mu \right)_\mu (t), \quad t \in \mathbb{R}_+$$

holds almost everywhere.

The following two convolution transforms can be constructed in the same manner. The convergence of integrals follows from the equality of the orders for the polynomials $r_n(x)$ and $\rho_n(x)$.

Theorem 4.5 *Let* $k_1(x)$ *and* $h_1(x)$ *be bounded functions on* $\mathbb{R}_+$ *such that* $k_1(x)h_1(x) \equiv 1$.

$$\tilde{k}(t) = \mathrm{H}_\mu \left[\frac{x^\nu k_1(x)}{r_n(x)} \right], \quad r_n(x) = \sum_{m=0}^{n} a_m x^{2m},$$

$$\tilde{h}(t) = \mathrm{H}_\mu \left[\frac{x^\nu h_1(x)}{\rho_n(x)} \right], \quad \rho_n(x) = \sum_{m=0}^{n} b_m x^{2m},$$

where $x \in \mathbb{R}_+;\ a_m,\ b_m \in \mathbb{R},\ \forall m;\ a_0,\ a_n,\ b_0,\ b_n \neq 0;\ n \geq 1.$

If $\tilde{k}(t),\ \tilde{h}(t),\ f(t) \in L_2(\mathbb{R}_+, tdt)$*,* $\mathrm{Re}\mu \geq \mathrm{Re}\nu > -1/2$*. Then the formula*

$$g(t) = \sum_{m=0}^{n} (-1)^m b_m S_{m,\nu} \left(\tilde{k}_\mu \overset{-\nu}{*} f_\mu \right)_\nu (t), \quad t \in \mathbb{R}_+$$

defines almost everywhere a function $g(t) \in L_2(\mathbb{R}_+, tdt)$*. Moreover, the inversion formula*

$$f(t) = \sum_{m=0}^{n} (-1)^m a_m S_{m,\mu} \left(\tilde{h}_\mu \overset{-\nu}{*} g_\nu \right)_\mu (t), \quad t \in \mathbb{R}_+$$

holds almost everywhere.

Theorem 4.6 *Let $k_1(x)$ and $h_1(x)$ be bounded functions on $\mathbb{R}_+$ such that $k_1(x)h_1(x) \equiv 1$.*

$$\tilde{k}(t) = \mathrm{H}_\nu\left[\frac{x^\nu k_1(x)}{r_n(x)}\right], \quad r_n(x) = \sum_{m=0}^{n} a_m x^{2m},$$

$$\tilde{h}(t) = \mathrm{H}_\nu\left[\frac{x^\nu h_1(x)}{\rho_n(x)}\right], \quad \rho_n(x) = \sum_{m=0}^{n} b_m x^{2m},$$

where $x \in \mathbb{R}_+$; $a_m,\ b_m \in \mathbb{R}$, $\forall m$; $a_0,\ a_n,\ b_0,\ b_n \neq 0$; $n \geq 1$.

If $\tilde{k}(t),\ \tilde{h}(t),\ f(t) \in L_2(\mathbb{R}_+, tdt)$, $\mathrm{Re}\nu > -1/2$, $\mathrm{Re}\mu > -1/2$. *Then the formula*

$$g(t) = \sum_{m=0}^{n} (-1)^m b_m S_{m,\mu}\left(\tilde{k}_\nu \overset{-\nu}{*} f_\mu\right)_\mu(t), \quad t \in \mathbb{R}_+$$

defines almost everywhere a function $g(t) \in L_2(\mathbb{R}_+, tdt)$. Moreover, the inversion formula

$$f(t) = \sum_{m=0}^{n} (-1)^m a_m S_{m,\mu}\left(\tilde{h}_\nu \overset{-\nu}{*} g_\mu\right)_\mu(t), \quad t \in \mathbb{R}_+$$

holds almost everywhere.

Now we consider some examples of integral transforms with unsymmetrical kernels related to polyconvolutions (2.5)–(2.6). Each of these transforms is a linear integral equation and the inversion formula is the solution of this equation.

5 Examples

Here $t \in \mathbb{R}_+$, the operator $S_{m,\nu}$ has been introduced above (see formula (4.1)),

$$S_{1,\nu} = \frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{\nu^2}{t^2}.$$

A function $f(t)$ satisfies conditions of Theorems 4.1–4.6.

We use the following denotations for Bessel functions [6, 7]:

- $J_\nu(z)$ is the Bessel function of the first kind of order ν.
- $Y_\nu(z)$ is the Bessel function of the second kind of order ν, also called the Neumann function.
- $I_\nu(z)$ is the modified Bessel function of the first kind of order ν.

- $K_\nu(z)$ is the modified Bessel function of the second kind or the Macdonald function of order ν.
- $H_\nu^{(1)}(z)$ is the Bessel function of the third kind of order ν, also known as the Hankel function of the first kind.

In first examples we consider the case $n = 1$ and we use the denotations $r_1(x) = r(x)$ and $\rho_1(x) = \rho(x)$.

Example Let $k_1 \equiv 1$, $h_1 \equiv 1$, $r(x) = a^2 + x^2$ and $\rho = b^2 + x^2$ then (see [31], formula 2.12.4.28)

$$\tilde{k} = a^\nu K_\nu(at), \quad \tilde{h} = b^\nu K_\nu(bt).$$

The integral transforms with these kernels are written as

$$g(t) = b^2 \left(\tilde{k}_\nu \overset{-\nu}{*} f_\mu\right)_\mu (t) - S_{1,\mu} \left(\tilde{k}_\nu \overset{-\nu}{*} f_\mu\right)_\mu (t)$$
$$= f(t) + (b^2 - a^2) \left[K_\mu(at) \int_0^t f(\tau) I_\mu(a\tau)\tau d\tau + I_\mu(at) \int_t^\infty f(\tau) K_\mu(a\tau)\tau d\tau \right],$$

$$f(t) = a^2 \left(\tilde{h}_\nu \overset{-\nu}{*} g_\mu\right)_\mu (t) - S_{1,\mu} \left(\tilde{h}_\nu \overset{-\nu}{*} g_\mu\right)_\mu (t)$$
$$= g(t) + (a^2 - b^2) \left[K_\mu(bt) \int_0^t g(\tau) I_\mu(b\tau)\tau d\tau + I_\mu(bt) \int_t^\infty g(\tau) K_\mu(b\tau)\tau d\tau \right].$$

Here $a \neq b$.

Analogously we obtain reciprocal formulas in following

Example Let $k_1 \equiv 1$, $h_1 \equiv 1$, $r(x) = x^2 - a^2$ and $\rho = x^2 - b^2$. Then (see [6], chap. XIII, par. 13.53, formula 4)

$$\tilde{k} = -\frac{\pi}{2} a^\nu Y_\nu(at), \quad \tilde{h} = -\frac{\pi}{2} b^\nu Y_\nu(bt).$$

And the integral transform can be written in form ($a \neq b$)

$$g(t) = -b^2 \left(\tilde{k}_\nu \overset{-\nu}{*} f_\mu\right)_\mu (t) - S_{1,\mu} \left(\tilde{k}_\nu \overset{-\nu}{*} f_\mu\right)_\mu (t)$$
$$= f(t) + \frac{\pi i}{2}(b^2 - a^2) \left[H_\mu^{(1)}(at) \int_0^t f(\tau) J_\mu(a\tau)\tau d\tau + J_\mu(at) \int_t^\infty f(\tau) H_\mu^{(1)}(a\tau)\tau d\tau \right],$$

$$f(t) = -a^2 \left(\tilde{h}_\nu \overset{-\nu}{*} g_\mu\right)_\mu (t) - S_{1,\mu} \left(\tilde{h}_\nu \overset{-\nu}{*} g_\mu\right)_\mu (t)$$

$$= g(t) + \frac{\pi i}{2}(a^2 - b^2) \left[H_\mu^{(1)}(bt) \int_0^t g(\tau) J_\mu(b\tau)\tau d\tau + J_\mu(bt) \int_t^\infty g(\tau) H_\mu^{(1)}(b\tau)\tau d\tau \right].$$

Finally, we consider

Example Let $k_1 \equiv 1, h_1 \equiv 1, r(x) = x^2 + a^2$ and $\rho = x^2 - b^2$. Then we obtain

$$\tilde{k} = a^\nu K_\nu(at), \quad \tilde{h} = -\frac{\pi}{2} b^\nu Y_\nu(bt)$$

and $(a \neq b)$

$$g(t) = -b^2 \left(\tilde{k}_\nu \overset{-\nu}{*} f_\mu\right)_\mu (t) - S_{1,\mu} \left(\tilde{k}_\nu \overset{-\nu}{*} f_\mu\right)_\mu (t)$$

$$= f(t) - (a^2 + b^2) \left[K_\mu(at) \int_0^t f(\tau) I_\mu(a\tau)\tau d\tau + I_\mu(at) \int_t^\infty f(\tau) K_\mu(a\tau)\tau d\tau \right],$$

$$f(t) = a^2 \left(\tilde{h}_\nu \overset{-\nu}{*} g_\mu\right)_\mu (t) - S_{1,\mu} \left(\tilde{h}_\nu \overset{-\nu}{*} g_\mu\right)_\mu (t)$$

$$= g(t) + \frac{\pi i}{2}(a^2 + b^2) \left[H_\mu^{(1)}(bt) \int_0^t g(\tau) J_\mu(b\tau)\tau d\tau + J_\mu(bt) \int_t^\infty g(\tau) H_\mu^{(1)}(b\tau)\tau d\tau \right].$$

In the similar manner we can construct the other convolution transforms by using Theorems 4.3–4.6 and presented examples.

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Second Type Neumann Series Related to Nicholson's and to Dixon–Ferrar Formula

Djurdje Cvijović and Tibor K. Pogány

Dedicated to Gradimir V. Milovanović to his 70th birthday anniversary

Abstract The second type Neumann series are considered which building blocks are Nicholson's and the Dixon–Ferrar formulae for $J_\nu^2(x) + Y_\nu^2(x)$. Related closed form double definite integral expressions are established by using the associated Dirichlet's series Cahen's Laplace integral for the Nicholson's case. However, using Dixon–Ferrar formula a double definite integral expression is again obtained. Certain Open Problems are posed in the last section of the chapter.

Keywords Bessel function of the first kind J_ν · Bessel function of the second kind Y_ν · Modified Bessel function of the second kind K_ν · Integral representation formula · Nicholson's formula · Dixon–Ferrar formula · Neumann series of Bessel functions · Cahen's Laplace integral formula

MSC 2010 Primary 33C10, 33E20, 40C10; Secondary 33E30, 40H05

1 Introduction to Nicholson's Formula

Special functions play an important role among others in the theory of transmutations; they are also very useful in many applications, combining special functions and transmutation theory [10, 17, 31]. In turn, this work is a contribution to the use

D. Cvijović
Atomic Physics Laboratory, Vinča Institute of Nuclear Sciences, Belgrade, Serbia
e-mail: djurdje@vin.bg.ac.rs

T. K. Pogány (✉)
Faculty of Maritime Studies, University of Rijeka, Rijeka, Croatia

Institute of Applied Mathematics, Óbuda University, Budapest, Hungary
e-mail: poganj@pfri.hr

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_4

of special functions in an important area of the classical analysis which deals with infinite series of Neumann type, see for instance the monumental monograph's parts [35, Chapters XVI–XIX] and the recently published book [7] devoted completely to here discussed topic.

One of the most celebrated mathematical formula from ancient times is the Pythagoras' theorem $a^2 + b^2 = c^2$ which trigonometrical form is

$$\sin^2 x + \cos^2 x = 1.$$

On the other hand the famous Nicholson's formula for $\Re(x) > 0$ reads [20, p. 234]

$$B_\nu^2(x) := J_\nu^2(x) + Y_\nu^2(x) = \frac{8}{\pi^2} \int_0^\infty K_0(2x\ \sinh t)\ \cosh(2\nu t)\,\mathrm{d}t\,, \tag{1}$$

where J_ν, Y_ν stand for the Bessel functions of the first and second kind, respectively, of the orders ν, while K_η denotes the modified Bessel function of the second kind of the order η; specially, regarding (1) there holds (among other numerous equivalent expressions for this function) [8, p. 19]:

$$K_0(z) = \int_0^\infty \frac{\cos(zy)}{\sqrt{1+y^2}}\,\mathrm{d}y, \qquad z > 0\,. \tag{2}$$

The formula (1) is the elegant generalization of both previously listed formulae. Indeed, having in mind that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi\,x}}\,\sin x \qquad \text{and} \qquad Y_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi\,x}}\,\cos x\,,$$

we have

$$\begin{aligned} B_{\frac{1}{2}}^2(x) &= \frac{\pi\,x}{2}\cdot\frac{8}{\pi^2}\int_0^\infty K_0(2x\sinh t)\ \cosh t\,\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^\infty K_0(s)\,\mathrm{d}s = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1, \end{aligned}$$

by which we arrive at the 'sine-squared + cosine-squared = 1' identity.

The next result about $B_\nu^2(x)$ is the Dixon–Ferrar formula [11, p. 142]

$$B_\nu^2(x) = \frac{8\,\cos(\nu\pi)}{\pi^2}\int_0^\infty K_{2\nu}(2x\ \sinh t)\,\mathrm{d}t, \qquad \Re(x) > 0,\ |\Re(\nu)| < \frac{1}{2}. \tag{3}$$

It is worth to be mention that this integral is in fact a corollary of the more general Watson's determinant[1] relation [11, p. 141, Eq. (60)]

$$W_{\nu,\mu}(x) := \begin{vmatrix} J_\nu(x) & J_\mu(x) \\ Y_\nu(x) & Y_\mu(x) \end{vmatrix} = \frac{4\,\sin(\nu-\mu)\pi}{\pi^2} \int_0^\infty K_{\nu-\mu}(2x\,\sinh t)\, \mathrm{e}^{-(\mu+\nu)t}\, \mathrm{d}t\,.$$

which holds true for all $\Re(x) > 0,\ |\Re(\mu-\nu)| < 1$.

The Bessel functions J_ν, Y_ν and the modified Bessel function K_ν play inevitable roles in physics, pure and applied mathematics and engineering sciences. For instance, the product $P_\nu(x) := I_\nu(x)K_\nu(x)$ is part of certain applications, e.g. consult the articles [29, 30] concerning the hydrodynamic and hydromagnetic instability of cylindrical models; the paper of Hasan [15] discussed the electrogravitational instability of non–oscillating streaming fluid cylinder under the influence of selfgravitating, capillary and electrodynamic forces. Links to different kind proofs and use of the monotonicity of $P_\nu(x)$ are given by Baricz and Pogány in [4], where an exhaustive references list is given therein. Also compare [1, 13, 21, 23]. Moreover, Klimek and McBride [18] prove that a Dirac operator, regarding to Atiyah–Patodi–Singer–like boundary conditions on the solid torus, has a bounded inverse, which is in fact compact operator. In [32, 33] van Heijster et al. investigated the existence, stability and interaction of localized structures in a one–dimensional generalized FitzHugh–Nagumo type model. Further, van Heijster and Sandstede [34] study existence and stability issues of radially symmetric solutions in the planar variant of this model. Finally, Baricz and Pogány [4] have focused to closed integral representations for the first and second type Neumann series for the modified Bessel functions I_ν and K_μ which appear in related Chebyshev–type discrete inequalities in the manner of such results established together with Jankov and Süli in a set of articles [3, 5, 6, 27]. However, a thorough overview of the related results can be found in the monograph by Baricz, Jankov Maširević and Pogány [7], see chapter 2.

The *first type Neumann series* was introduced in [27] (also see [5, 7]) as

$$\mathfrak{N}_\nu^\mu(x) := \sum_{n\geq 1} a_n\, Z_{\nu+\mu n}(x)\,,$$

where Z denotes a cylinder function. The *second type Neumann–series* was introduced by Baricz and Pogány in the book chapter [4] in the form

$$\mathfrak{S}_{\nu,\eta}^{\mu,\tau}\big[Z^{(1)}, Z^{(2)}\big](x) := \sum_{n\geq 1} a_n\, Z_{\nu+\mu n}^{(1)}(x)\, Z_{\eta+\tau n}^{(2)}(x)\,,$$

which is discussed in detail in [4] for the case $\mathfrak{S}_\nu^1[I, K]$ (we quote the identical parameters here, and in what follows, only once). The rule that the Neumann series

[1] Instead of the determinant form, we will use the most popular cross–product expression $J_\nu(x)Y_\mu(x) - J_\mu(x)Y_\nu(x)$ throughout.

of cylinder functions of the first type are built by *one* cylinder function, while in the case when products of two (or more) special functions are the building blocks we deal with second type series. This nomenclature has been followed in the above mentioned monograph [7] too.

Our main goal in this work is to establish several integral representation formulae for the second type Neumann series which terms are Nicholson or Dixon–Ferrar input squared sums; in other words

$$\mathscr{N}_\nu^\mu\big[J,\,Y\big](x) := \sum_{n\geq 1} a_n B_{\nu+\mu n}^2(x) = \sum_{n\geq 1} a_n\big[J_{\nu+\mu n}^2(x) + Y_{\nu+\mu n}^2(x)\big].$$

The main derivation tools consist from Cahen's Laplace integral formula for Dirichlet series [9, p. 97] (see also Perron's memoir [22]) and the Euler–Maclaurin summation formula [27, p. 2365] adapted to our considerations.

Here and in what follows, $[a]$ and $\{a\} = a - [a]$ denote the integer and fractional part of some a, respectively.

2 Preparation: Euler–Maclaurin Summation Formula, Dirichlet Series and Cahen's Formula

Consider the real-valued function $x \mapsto a_x = a(x)$ and suppose that $a \in \mathrm{C}^1[k, m]$, $k, m \in \mathbb{Z}$, $k < m$. The classical Euler–Maclaurin summation formula states that [19, p. 539, **296**]

$$\sum_{j=k}^m a_j = \int_k^m a(x)\mathrm{d}x + \frac{1}{2}\big(a_k + a_m\big) + \int_k^m \Big(x - [x] - \frac{1}{2}\Big)a'(x)\mathrm{d}x.$$

On introducing the kind of a differential operator

$$\mathfrak{d}_x := 1 + \{x\}\frac{\mathrm{d}}{\mathrm{d}x},$$

obvious transformations yield the following condensed form of the Euler–Maclaurin formula[2] [27, p. 2365, Eq. (3)]

$$\sum_{j=k+1}^m a_j = \int_k^m \big(a(x) + \{x\}a'(x)\big)\mathrm{d}x = \int_k^m \mathfrak{d}_x\, a(x)\,\mathrm{d}x. \tag{4}$$

This formula has been used in another purposes for instance in [28].

[2]The formula was discovered independently by Leonhard Euler and Colin Maclaurin around 1735. Euler needed it to compute slowly converging infinite series, while Maclaurin used it to calculate integrals.

Our next main mathematical tool is the *Dirichlet series of the* λ_n*–type*

$$\mathcal{D}_{\boldsymbol{a}}(s) := \sum_{n\ge 1} a_n\, \mathrm{e}^{-\lambda_n s}\,, \qquad \Re(s) > 0\,, \tag{5}$$

where

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty \qquad \text{as} \qquad n \to \infty\,.$$

For $\lambda_n = n$, (5) becomes the power series

$$\mathcal{D}_{\boldsymbol{a}}(s) := \sum_{n\ge 1} a_n\, \mathrm{e}^{-ns}\,, \qquad \Re(s) > 0\,,$$

and for $\lambda_n = \ln n$, we have series of the form

$$\mathcal{D}_{\boldsymbol{a}}(s) := \sum_{n\ge 1} a_n\, n^{-s}\,, \qquad \Re(s) > 0\,,$$

which is called *ordinary Dirichlet series*.

In this note we deal with series of the form (5) where s is real variable. We also need a variant of closed integral form representation of Dirichlet series, see [14], [16, **C. §V**]. The result is the widely known Cahen's formula [9, p. 97], [14]

$$\mathcal{D}_{\boldsymbol{a}}(s) = s \int_0^\infty \mathrm{e}^{-s\,t} \mathcal{A}_{\boldsymbol{a}}(t)\, \mathrm{d}t\,, \qquad s > 0\,. \tag{6}$$

The articles [24–26] contain certain special cases of (6) specifying among others $a_n = 1$. However, the so–called counting sum

$$\mathcal{A}_{\boldsymbol{a}}(t) = \sum_{n:\, \lambda_n \le t} a_n$$

we find by the Euler–Maclaurin summation formula (4) (see [24, 26, 27]), assuming that $\boldsymbol{a} := a(x)\big|_{\mathbb{N}}$, $a \in C^1[0,\infty)$ we sum up $\mathcal{A}_{\boldsymbol{a}}(t)$ completing the closed form integral representation of Dirichlet series $\mathcal{D}_{\boldsymbol{a}}(s)$ without any sums. Namely

$$\mathcal{A}_{\boldsymbol{a}}(t) = \sum_{n=1}^{[\lambda^{-1}(t)]} a_n = \int_0^{[\lambda^{-1}(t)]} \mathfrak{d}_u\, a(u)\, \mathrm{d}u\,,$$

as by assumption λ is monotone with an unique inverse λ^{-1} being $\lambda|_{\mathbb{N}} = (\lambda_n)$.

3 Main Results: *Accessum per Definitionem*

Bearing in mind the previously declared nomenclature it follows that

$$\mathscr{N}_\nu^\mu\big[J,\ Y\big](x) = \mathfrak{G}_\nu^\mu[J, J](x) + \mathfrak{G}_\nu^\mu[Y, Y](x)$$
$$= \sum_{n\geq 1} a_n J_{\mu n+\nu}^2(x) + \sum_{n\geq 1} a_n Y_{\mu n+\nu}^2(x), \tag{7}$$

therefore we can approach the integral expression for $\mathscr{N}_\nu^\mu\big[J,\ Y\big](x)$ in at least two main directions:

(i) finding the integral form for $\mathfrak{G}_\nu^\mu[Z, Z](x)$, where $Z \in \{J, Y\}$ and then apply (7), or
(ii) to exploit either Nicholson's or the Dixon–Ferrar formulae in expressing B_ν^2.

The reason why we cannot apply Nicholson's formula (1) in the case (i) directly is that the term

$$\cosh 2(\mu n + \nu)t = \mathscr{O}(\mathrm{e}^{2|\mu|tn}), \qquad t > 0,\ n \to \infty$$

in the integrand does not permit to transform $\mathscr{N}_\nu^\mu\big[J,\ Y\big](x)$ into an absolutely convergent Dirichlet series.

In turn, choosing the termwise consideration of $B_\nu^2(x) = J_\nu^2(x) + Y_\nu^2(x)$ we can achieve our first main integral representation result.

Theorem 1 *Let $a \in \mathrm{C}^1(\mathbb{R}_+)$ and let $a|_{\mathbb{N}} = (a_n)_{n\in\mathbb{N}}$. For all $\mu, \nu > -1/2$ and for all*

$$|x| < \frac{2}{\mathrm{e}} \min\left(\mathrm{e}, \frac{\mu}{\mathfrak{l}}\right), \qquad \mathfrak{l} := \left(\limsup_{n\to\infty} \frac{\sqrt[n]{|a_n|}}{n^{2\mu}}\right)^{1/(2\mu)}.$$

we have the integral representation

$$\mathfrak{G}_\nu^\mu[J, J](x) = -\mu \int_1^\infty \int_0^{[w]} \frac{\partial}{\partial w}\Big[\Gamma^2(\mu w + \nu + 1/2)\, J_{\mu w+\nu}^2(x)\Big]$$
$$\times\, \mathfrak{d}_u \frac{a(u)}{\Gamma^2(\mu u + \nu + 1/2)}\, \mathrm{d}w\mathrm{d}u. \tag{8}$$

Proof Consider the integral representation formula [12, **8.411** Eq.(10)]

$$J_\nu(z) = \frac{\left(z/2\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu + 1/2)} \int_{-1}^1 \cos(zt)(1 - t^2)^{\nu-1/2}\mathrm{d}t,$$

which holds true for all $z \in \mathbb{C}$, $\Re\{\nu\} > -1/2$. Applying this integral form of J_ν to (7) we get

$$\mathfrak{G}_\nu^\mu[J, J](x) = \sum_{n \geq 1} a_n J_{\mu n+\nu}^2(x)\,. \tag{9}$$

Precisely,

$$\mathfrak{G}_\nu^\mu[J, J](x) = \frac{4}{\pi}\Big(\frac{x}{2}\Big)^{2\nu} \int_0^1 \int_0^1 \frac{\cos(xt)\cos(xs)}{[(1-t^2)(1-s^2)]^{1/2-\nu}}\,\mathcal{D}_1(t, s)\,\mathrm{d}t\,\mathrm{d}s,$$

with the Dirichlet series

$$\mathcal{D}_1(t, s) = \sum_{n\geq 1} \frac{a_n}{\Gamma^2(\mu n+\nu+1/2)} \exp\left\{-n \ln\left(\frac{4}{x^2(1-t^2)(1-s^2)}\right)^\mu\right\}.$$

Because $\Gamma(z) = \sqrt{2\pi}\, z^{z-1/2} \mathrm{e}^{-z}\big(1+\mathscr{O}(z^{-1})\big)$, $|z| \to \infty$, we see that $\mathcal{D}_1(t, s)$ is absolutely convergent for all $x \in \mathbb{R}$ and $t, s \in (-1, 1)$ such that

$$|x|(1-t^2)(1-s^2) \leq |x| < \frac{2\mu}{\mathrm{e}}\left(\limsup_{n\to\infty} \frac{\sqrt[n]{|a_n|}}{n^{2\mu}}\right)^{-1/(2\mu)} = \frac{2\mu}{\mathrm{e}\,\mathfrak{l}}\,. \tag{10}$$

Furthermore, for $\mathcal{D}_1(t)$ there holds the Cahen's Laplace integral when the Dirichlet exponent

$$\ln\left(\frac{4}{x^2(1-t^2)(1-s^2)}\right)^\mu > 0\,.$$

In this case we can take $|x| < 2$ and $t, s \in (-1, 1)$, since the required positivity condition is satisfied when

$$\frac{4}{x^2(1-t^2)(1-s^2)} \geq \frac{4}{x^2} > 1.$$

Hence, the x–domain becomes the stated

$$|x| < \frac{2}{\mathrm{e}} \min\left(\mathrm{e}, \mu\, \mathfrak{l}^{-1}\right)\,.$$

Thus, for all such x we deduce by the Cahen's formula (6) that

$$\mathcal{D}_1(t, s) = \mu \ln \frac{4}{x^2(1-t^2)(1-s^2)} \int_0^\infty \left(\frac{x^2(1-t^2)(1-s^2)}{4}\right)^{\mu w} \mathcal{A}_1(w)\,\mathrm{d}w; \tag{11}$$

where

$$\mathcal{A}_1(w) := \sum_{j=1}^{[w]} \frac{a_j}{\Gamma^2(\mu j + \nu + 1/2)}$$

denotes the counting function, see [16, **V**] or [28, §4, §6]. Now, it remains to sum the $\mathcal{A}_1(w)$ by the Euler–Maclaurin formula (4) cf. [28, Lemma 1]:

$$\mathcal{A}_1(w) = \int_0^{[w]} \mathfrak{d}_u \, \frac{a(u)}{\Gamma^2(\mu u + \nu + 1/2)} \, \mathrm{d}u. \tag{12}$$

Putting $\mathcal{A}_1(w)$ and $\mathcal{D}_1(t, s)$ from (12) and (11) into (9), we get

$$\begin{aligned} \mathfrak{G}_\nu^\mu[J, J](x) = -\frac{2\mu x}{\pi} \int_0^\infty \int_0^{[w]} \mathfrak{d}_u \, \frac{a(u)}{\Gamma^2(\mu u + \nu + 1/2)} \\ \times \left\{ \int_0^1 \int_0^1 \cos(xt)\cos(xs) \Big[\frac{x^2}{4}(1-t^2)(1-s^2)\Big]^{\nu+\mu w - 1/2} \right. \\ \left. \times \ln\Big[\frac{x^2}{4}(1-t^2)(1-s^2)\Big] \, \mathrm{d}t\mathrm{d}s \right\} \mathrm{d}w \, \mathrm{d}u. \end{aligned} \tag{13}$$

For the inner ts–integral $\mathcal{I}_x(\rho)$, say, where $\rho = \mu w + \nu - 1/2$, there holds

$$\begin{aligned} \int \mathcal{I}_x(\rho) \, \mathrm{d}\rho &= \Big(\frac{x^2}{4}\Big)^\rho \left[\int_0^1 \cos(tx)(1-t^2)^\rho \, \mathrm{d}t\right]^2 \\ &= \frac{\pi}{2x}\Gamma^2(\rho+1) \, \big[J_{\rho+1/2}(x)\big]^2 , \end{aligned} \tag{14}$$

provided the Fourier cosine transform of $(1-t^2)^\rho \, \mathbf{1}_{[0,1]}(t)$[3]:

$$\int_0^1 \cos(tx)(1-t^2)^\rho \, \mathrm{d}t = \frac{\sqrt{\pi}}{2}\Gamma(\rho+1)\Big(\frac{2}{x}\Big)^{\rho+1/2} J_{\rho+1/2}(x) .$$

By (14) we finally have

$$\mathcal{I}_x = \frac{\pi}{2x} \frac{\partial}{\partial w}\Big[\Gamma(\mu w + \nu + 1/2) \, J_{\mu w + \nu}(x)\Big]^2. \tag{15}$$

Substituting (15) into (13) we arrive at the stated integral expression (8), remarking that the integration domain reduces to $[1, \infty)$ as $[w]$ vanishes for all $w \in [0, 1)$. □

[3]Here, and in what follows $\mathbf{1}_A$ denotes the indicator function of the set A.

Theorem 2 *Let $a \in \mathrm{C}^1(\mathbb{R}_+)$ and $a|_{\mathbb{N}} = \{a_n\}_{n\in\mathbb{N}}$. Then for all $x \in (0,2) \cap \mathcal{I}_a$, where*

$$\mathcal{I}_a := \begin{cases} \left(0, \frac{2\mu}{\mathrm{e}\mathfrak{l}}\right), & -1/2 < \nu \le 1/2 \\ \left(\frac{2\mu}{\mathrm{e}}\mathfrak{L}, \frac{2\mu\mathrm{e}}{\mathfrak{l}}\right), & 1/2 < \nu \le 3/2 \\ \left(\frac{4\mu}{\mathrm{e}}\mathfrak{L}, \frac{\mu\mathrm{e}}{\mathfrak{l}}\right), & \nu > 3/2 \end{cases}, \tag{16}$$

and

$$\mathfrak{l} := \left(\limsup_{n\to\infty} |a_n|^{1/n}\, n^{-2\mu}\right)^{1/(2\mu)},$$
$$\mathfrak{L} := \left(\limsup_{n\to\infty} |a_n|^{1/n}\, n^{2\mu}\right)^{1/(2\mu)},$$

there holds

$$\mathfrak{G}_\nu^\mu[Y,Y](x) = \mu \int_1^\infty \int_0^{[w]} \mathfrak{d}_u \frac{a(u)}{\Gamma^2(\mu u + \nu + 1/2)} \times \frac{\partial}{\partial w}\Big[\Gamma^2(\mu w + \nu + 1/2)\, Y^2_{\mu w+\nu}(x)\Big]\, \mathrm{d}w\, \mathrm{d}u\,, \tag{17}$$

when $(a_n)_{n\in\mathbb{N}}$ satisfies

$$\begin{cases} \mathfrak{l}\cdot\mathfrak{L} < 1, & \nu \in (-1/2, 3/2] \\ \mathfrak{l}\cdot\mathfrak{L} < 4^{-\mu}, & \nu > 3/2 \end{cases}. \tag{18}$$

Proof We focus to $\mathfrak{G}_\nu^\mu[Y,Y](x)$, built by Bessel $Y_{\nu+\mu n}(x)$ terms. Firstly, let us establish the x–region of convergence and the related parameter constraints upon ν. The Gubler–Weber formula reads [35, p. 165]

$$Y_\nu(z) = \frac{2\,(z/2)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\int_0^1 \frac{\sin(zt)\,\mathrm{d}t}{(1-t^2)^{1/2-\nu}} - \int_0^\infty \frac{\mathrm{e}^{-zt}\,\mathrm{d}t}{(1+t^2)^{1/2-\nu}}\right), \tag{19}$$

from which evidently follows that

$$Y_\nu(x) \le \frac{2\,(x/2)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\int_0^1 \frac{\sin(xt)\,\mathrm{d}t}{(1-t^2)^{1/2-\nu}} + \int_0^\infty \frac{\mathrm{e}^{-xt}\,\mathrm{d}t}{(1+t^2)^{1/2-\nu}}\right),$$

where $x > 0$ and $\nu > -1/2$.

We have the following estimates [5, pp. 957–958]

$$Y_\nu(x) - \frac{(x/2)^\nu}{\Gamma(\nu+1)} \le \begin{cases} \dfrac{(x/2)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+1/2)}, & -1/2 < \nu \le 1/2 \\ \dfrac{(x/2)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+1/2)} + \dfrac{2^\nu \Gamma(\nu)}{\pi\, x^\nu}, & 1/2 < \nu \le 3/2 \\ \dfrac{x^{\nu-1}}{\sqrt{2\pi}\,\Gamma(\nu+1/2)} + \dfrac{2^{2\nu-3/2}\Gamma(\nu)}{\pi x^\nu}, & \nu > 3/2 \end{cases} .$$

Hence, by the Arithmetic Mean—Quadratic Mean inequality

$$Y_\nu^2(x) \le \begin{cases} \dfrac{2(x/2)^{2\nu}}{\Gamma^2(\nu+1)} + \dfrac{2(x/2)^{2\nu-2}}{\pi\,\Gamma^2(\nu+1/2)}, & -1/2 < \nu \le 1/2 \\ \dfrac{3(x/2)^{2\nu}}{\Gamma^2(\nu+1)} + \dfrac{3(x/2)^{2\nu-2}}{\pi\,\Gamma^2(\nu+1/2)} + \dfrac{3\cdot 2^{2\nu}\Gamma^2(\nu)}{\pi^2 x^{2\nu}}, & 1/2 < \nu \le 3/2 \\ \dfrac{3(x/2)^{2\nu}}{\Gamma^2(\nu+1)} + \dfrac{3x^{2\nu-2}}{2\pi\,\Gamma^2(\nu+1/2)} + \dfrac{3\cdot 4^{2\nu}\Gamma^2(\nu)}{8\pi^2 x^{2\nu}}, & \nu > 3/2 \end{cases} ; \tag{20}$$

also compare [2].

Firstly, the case $-1/2 < \nu \le 1/2$ results in the estimate

$$|\mathfrak{G}_\nu^\mu[Y,Y](x)| \le 2\Big(\frac{x}{2}\Big)^{2\nu} \sum_{n\ge1} \frac{|a_n|}{\Gamma^2(\mu n+\nu+1)}\Big(\frac{x}{2}\Big)^{2\mu n} + \frac{2}{\pi}\Big(\frac{x}{2}\Big)^{2\nu-2} \sum_{n\ge1} \frac{|a_n|}{\Gamma^2(\mu n+\nu+1/2)}\Big(\frac{x}{2}\Big)^{2\mu n}.$$

By the Cauchy–Hadamard theorem the uniform convergence occurs when $0 < x < 2\mu/(\mathrm{e}\,\mathfrak{l})$ since the positivity of the argument x.

Next, for $1/2 < \nu \le 3/2$ we have

$$\begin{aligned} |\mathfrak{G}_\nu^\mu[Y,Y](x)| &\le 3\Big(\frac{x}{2}\Big)^{2\nu} \sum_{n\ge1} \frac{|a_n|}{\Gamma^2(\mu n+\nu+1)}\Big(\frac{x}{2}\Big)^{2\mu n} \\ &\quad + \frac{3}{\pi}\Big(\frac{x}{2}\Big)^{2\nu-2} \sum_{n\ge1} \frac{|a_n|}{\Gamma^2(\mu n+\nu+1/2)}\Big(\frac{x}{2}\Big)^{2\mu n} \\ &\quad + \frac{3}{\pi^2}\Big(\frac{2}{x}\Big)^{2\nu} \sum_{n\ge1} |a_n|\Gamma^2(\mu n+\nu)\Big(\frac{2}{x}\Big)^{2\mu n}. \end{aligned} \tag{21}$$

Obviously the first and the second power series in (21) converge uniformly for $0 < x < 2\mu/(\mathrm{e}\,\mathfrak{l})$, while the third series is uniformly convergent for all $x > (2\mu/\mathrm{e})\,\mathfrak{L}$.

Consequently, the interval of uniform convergence becomes

$$\mathcal{I}_a' = \left(\frac{2\mu}{\mathrm{e}\,\mathfrak{L}}, \frac{2\mu}{\mathrm{e}\,\mathfrak{l}}\right),$$

provided $\mathfrak{l} \cdot \mathfrak{L} < 1$. This implies that the necessary condition for convergence of $\mathscr{N}_{\nu,\mu}[J, Y](x)$ is $\limsup_{n\to\infty} |a_n|^{1/n} < 1$.

In the case $\nu > 3/2$ we have by (20) that

$$\begin{aligned} \left|\mathfrak{G}_\nu^\mu[Y, Y](x)\right| &\le 3\Big(\frac{x}{2}\Big)^{2\nu} \sum_{n\ge1} \frac{|a_n|}{\Gamma^2(\mu n+\nu+1)} \Big(\frac{x}{2}\Big)^{2\mu n} \\ &\quad + \frac{3\,x^{2\nu-2}}{2\pi} \sum_{n\ge1} \frac{|a_n|\,x^{2\mu n}}{\Gamma^2(\mu n+\nu+1/2)} \\ &\quad + \frac{3}{8\pi^2}\Big(\frac{4}{x}\Big)^{2\nu} \sum_{n\ge1} |a_n|\Gamma^2(\mu n+\nu)\Big(\frac{4}{x}\Big)^{2\mu n}\,; \end{aligned}$$

in fact our present right–hand–side expression only modestly differs from the previously discussed bound (21). So, by virtue of the same approach we show that the first two series converge in $0 < x < 2\mu/(\mathrm{e}\,\mathfrak{l})$ and $0 < x < \mu/(\mathrm{e}\,\mathfrak{l})$ respectively, while the third series converges uniformly for all $x > (4\mu/\mathrm{e})\,\mathfrak{L}$. This yields the x–domain of uniform convergence

$$\mathcal{I}_a'' = \left(\frac{4\mu}{\mathrm{e}\,\mathfrak{L}}, \frac{\mu}{\mathrm{e}\,\mathfrak{l}}\right).$$

The domain $\mathcal{I}_a''$ is not empty if it necessarily holds $4\,\mathfrak{l} \cdot \mathfrak{L} < 1$. Then the coefficients a_n satisfy

$$\limsup_{n\to\infty} |a_n|^{1/n} < 4^{-\mu}$$

for convergence of $\mathfrak{G}_\nu^\mu[Y, Y](x)$. Collecting these cases we get (16) and (18).

Now, let us focus on the integral representation for $\mathfrak{G}_\nu^\mu[Y, Y](x)$, where $x \in \mathcal{I}_a''$. By the Gubler–Weber formula (19) we have

$$\begin{aligned} \mathfrak{G}_\nu^\mu[Y, Y](x) &= \frac{4}{\pi}\Big(\frac{x}{2}\Big)^{2\nu} \sum_{n\ge1} \frac{a_n}{\Gamma^2(\mu n+\nu+1/2)} \Big(\frac{x}{2}\Big)^{2\mu n} \\ &\quad \times \left(\int_0^1 \frac{\sin(xt)\,\mathrm{d}t}{(1-t^2)^{1/2-\mu n-\nu}} - \int_0^\infty \frac{\mathrm{e}^{-xt}\,\mathrm{d}t}{(1+t^2)^{1/2-\mu n-\nu}}\right)^2, \end{aligned} \tag{22}$$

which can be presented in the form of a linear combination of sums $N_j(x)$, $j = 1, 2, 3$ say, (up to the multiplicative prefix terms):

$$\begin{aligned}
N_1(x) &= \sum_{n\geq 1} \frac{a_n (x/2)^{2\mu n}}{\Gamma^2(\mu n+\nu+1/2)} \int_0^1\int_0^1 \frac{\sin(xt)\sin(xs)\,\mathrm{d}t\mathrm{d}s}{[(1-t^2)(1-s^2)]^{1/2-\mu n-\nu}} \\
&= \int_0^1\int_0^1 \frac{\sin(xt)\sin(xs)}{[(1-t^2)(1-s^2)]^{1/2-\nu}}\, \mathcal{D}_{11}(t,s)\,\mathrm{d}t\mathrm{d}s \\
N_2(x) &= \sum_{n\geq 1} \frac{a_n (x/2)^{2\mu n}}{\Gamma^2(\mu n+\nu+1/2)} \int_0^1\int_0^\infty \frac{\sin(xt)\mathrm{e}^{-sx}\,\mathrm{d}t\mathrm{d}s}{[(1-t^2)(1+s^2)]^{1/2-\mu n-\nu}} \\
&= \int_0^1\int_0^\infty \frac{\sin(xt)\mathrm{e}^{-sx}}{[(1-t^2)(1+s^2)]^{1/2-\nu}}\, \mathcal{D}_{12}(t,s)\,\mathrm{d}t\mathrm{d}s \\
N_3(x) &= \sum_{n\geq 1} \frac{a_n (x/2)^{2\mu n}}{\Gamma^2(\mu n+\nu+1/2)} \int_0^\infty\int_0^\infty \frac{\mathrm{e}^{-(t+s)x}\,\mathrm{d}t\mathrm{d}s}{[(1+t^2)(1+s^2)]^{1/2-\mu n-\nu}} \\
&= \int_0^\infty\int_0^\infty \frac{\mathrm{e}^{-(t+s)x}}{[(1+t^2)(1+s^2)]^{1/2-\nu}}\, \mathcal{D}_{22}(t,s)\,\mathrm{d}t\mathrm{d}s;
\end{aligned}$$

with the associated Dirichlet series

$$\begin{aligned}
\mathcal{D}_{11}(t,s) &= \sum_{n\geq 1} \frac{a_n}{\Gamma^2(\mu n+\nu+1/2)} \exp\left\{-n\mu \ln\frac{4}{x^2(1-t^2)(1-s^2)}\right\}, \\
\mathcal{D}_{12}(t,s) &= \sum_{n\geq 1} \frac{a_n}{\Gamma^2(\mu n+\nu+1/2)} \exp\left\{-n\mu \ln\frac{4}{x^2(1-t^2)(1+s^2)}\right\}, \qquad (23) \\
\mathcal{D}_{22}(t,s) &= \sum_{n\geq 1} \frac{a_n}{\Gamma^2(\mu n+\nu+1/2)} \exp\left\{-n\mu \ln\frac{4}{x^2(1+t^2)(1+s^2)}\right\},
\end{aligned}$$

respectively. The first Dirichlet series $\mathcal{D}_{11}(t,s)$ we treat in the usual way. The Dirichlet exponent should be positive of the whole domain, that means

$$\frac{x^2(1-t^2)(1-s^2)}{4} \leq \frac{x^2}{4} < 1\,,$$

which holds for all $|x| < 2$ and all $(t,s) \in [0,1]^2$, consequently $x \in (0,2)$. Next, the Dirichlet series converges for all $|x| < 2\mu/(\mathrm{e}\,\mathfrak{l})$, compare (10). *Ergo*, the convergence domain for x turns out to be $x \in (0, 2\mu/(\mathrm{e}\,\mathfrak{l}))$. Now, with the aid of the Cahen's formula (6), we have

$$\mathcal{D}_{11}(t,s) = \mu \ln\frac{4}{x^2(1-t^2)(1-s^2)} \int_0^\infty \left[\frac{x^2}{4}(1-t^2)(1-s^2)\right]^{\mu w} \mathcal{A}_1(w)\,\mathrm{d}w,$$

where the counting function is solved around (11). Collecting these expressions we deduce that

$$N_1(x) = -\mu \left(\frac{2}{x}\right)^{2\nu-1} \int_0^\infty \int_0^{[w]} \mathfrak{d}_u \frac{a(u)}{\Gamma^2(\mu u+\nu+1/2)} \times \left(\int_0^1 \int_0^1 \sin(xt)\sin(xs) \left[\frac{x^2(1-t^2)(1-s^2)}{4}\right]^{\mu w+\nu-1/2} \times \ln \frac{x^2(1-t^2)(1-s^2)}{4} \mathrm{d}t\, \mathrm{d}s\right) \mathrm{d}w\, \mathrm{d}u\,.$$

Integrating with respect to $\rho := \mu w + \nu - 1/2$ the ts-integral $\mathcal{I}_{11}(w)$, say, we get the squared Fourier sine transform of $(1-t^2)^\rho\, \mathbf{1}_{[0,1]}(t)$, *viz.*

$$\int \mathcal{I}_{11}(\rho)\,\mathrm{d}\rho = \left(\frac{x^2}{4}\right)^\rho \left[\int_0^1 \sin(xt)\,(1-t^2)^\rho\,\mathrm{d}t\right]^2 = \frac{\pi}{x}\,\Gamma^2(\rho+1/2)\,\mathbf{H}^2_{\rho+1/2}(x)\,,$$

where $\mathbf{H}_\alpha(x)$ stands for the Struve function of the order α. Thus,

$$\mathcal{I}_{11}(x) = \frac{\pi}{x}\frac{\partial}{\partial w}\Gamma^2(\mu w+\nu+1/2)\,\mathbf{H}^2_{\mu w+\nu}(x)\,.$$

Finally, we infer

$$N_1(x) = -\frac{\mu\pi}{4}\left(\frac{2}{x}\right)^{2\nu} \int_0^\infty \int_0^{[w]} \mathfrak{d}_u \frac{a(u)}{\Gamma^2(\mu u+\nu+1/2)} \times \frac{\partial}{\partial w}\Big[\Gamma^2(\mu w+\nu+1/2)\,\mathbf{H}^2_{\mu w+\nu}(x)\Big]\,\mathrm{d}w\,\mathrm{d}u\,. \tag{24}$$

In the continuation we repeat the previous steps in obtaining the integral expressions for $N_2(x)$ and $N_3(x)$. By the way, *mutatis mutandis* $\mathcal{D}_{12}(t,s) = \mathcal{D}_{11}(t,\mathrm{i}s)$ and $\mathcal{D}_{22}(t,s) = \mathcal{D}_{11}(\mathrm{i}t,\mathrm{i}s)$, but the lines of the consideration are the same as in the previous case. However, in both remaining cases exactly the same counting function $\mathcal{A}_1(w)$ occurs, see the structure of (23). Similarly to the procedure applied in proof of [5, Theorem 2.4.] we conclude that

$$N_2(x) = -\frac{\mu\pi}{4}\left(\frac{2}{x}\right)^{2\nu} \int_0^\infty \int_0^{[w]} \mathfrak{d}_u \frac{a(u)}{\Gamma^2(\mu u+\nu+1/2)} \times \frac{\partial}{\partial w}\Big[\Gamma^2(\mu w+\nu+1/2)\mathbf{H}_{\mu w+\nu}(x)\big(\mathbf{H}_{\mu w+\nu}(x) - Y_{\mu w+\nu}(x)\big)\Big]\,\mathrm{d}w\mathrm{d}u\,. \tag{25}$$

By the double-use of the Laplace transform we achieve the integral form

$$N_3(x) = -\frac{\mu\pi}{4}\left(\frac{2}{x}\right)^{2\nu}\int_0^\infty\int_0^{[w]} \mathfrak{d}_u\, \frac{a(u)}{\Gamma^2(\mu u+\nu+1/2)} \times \frac{\partial}{\partial w}\left[\Gamma^2(\mu w+\nu+1/2)\big(\mathbf{H}_{\mu w+\nu}(x) - Y_{\mu w+\nu}(x)\big)^2\right] \mathrm{d}w\mathrm{d}u\,. \tag{26}$$

Building now the desired linear combination from all three integral representations (24)–(26), associated with (22) we arrive at the stated result (17). □

The relation (7) connects the results of Theorem 1 and Theorem 2. So, the final form of the second type Neumann series which consists from Nicholson–type building blocks is

Theorem 3 *Consider the same parameter space as in Theorem 1. and Theorem 2. Then we have*

$$\mathscr{N}_\nu^\mu\big[J,\, Y\big](x) = \sum_{n\geq 1} a_n\big[J^2_{\mu n+\nu}(x) + Y^2_{\mu n+\nu}(x)\big] = -\mu\int_1^\infty\int_0^{[w]} \frac{\partial}{\partial w}\Big[\Gamma^2(\mu w+\nu+1/2)\big(J^2_{\mu w+\nu}(x) + Y^2_{\mu w+\nu}(x)\big)\Big] \times \mathfrak{d}_u\, \frac{a(u)}{\Gamma^2(\mu u+\nu+1/2)}\,\mathrm{d}w\,\mathrm{d}u\,. \tag{27}$$

4 Main Results: The Dixon–Ferrar Formula

The second type Neumann series of building functions coming from the Bessel functions family contain two ore more special functions product(s) in the general term, which summing indices occur exclusively in the parameter(s) of these functions. This time the Nicholson type $B_\nu^2(x) = J_\nu^2(x) + Y_\nu^2(x)$ functions are considered.

A different approach in finding the integral representation for the Neumann series $\mathscr{N}_\nu^\mu\big[J,\, Y\big](x)$ would enabled by the Dixon–Ferrar formula (3) which reads

$$B_\nu^2(x) = \frac{8\,\cos(\nu\pi)}{\pi^2}\int_0^\infty K_{2\nu}(2x\,\sinh t)\,\mathrm{d}t, \qquad \Re(x) > 0,\ |\Re(\nu)| < \frac{1}{2}. \tag{28}$$

Unfortunately the Dixon–Ferrar formula is useless for our purposes, being the orders of the Bessel functions J_ν, Y_ν included into Neumann series

$$\mathscr{N}_\nu^\mu\big[J,\, Y\big](x) = \sum_{n\geq 1} a_n\big[J^2_{\mu n+\nu}(x) + Y^2_{\mu n+\nu}(x)\big]$$

outside of the parametric space, which is evidently $|\Re(\nu)| < 1/2$, for any positive integer $n \geq n^* = n^*(\mu, \nu) = [(1-2\nu)/(2\mu)]$ where $[\cdot]$ notices the integer part operator. However, in we consider the partial sum of a Neumann series of the form

$$\mathscr{N}^*[J, Y](x) := \sum_{n=1}^{\left[\frac{1-2\nu}{2\mu}\right]} a_n \big[J^2_{\mu n+\nu}(x) + Y^2_{\mu n+\nu}(x)\big]\,,$$

we could express this *finite* sum in the form of an integral *via* the more convenient tool, which is the Dixon–Ferrar formula (3).

Theorem 4 *For all* $x > 0$ *and* $\mu, \nu > 0$; $0 \leq n^* = (1-2\nu)/(2\mu) < 1$, *we have*

$$\mathscr{N}^*[J, Y](x) = \frac{8}{\pi^{3/2}} \int_0^\infty \int_0^{\left[\frac{1-2\nu}{2\mu}\right]} \frac{K_0(2xs)}{(1+s^2)^{\nu-1/2}} \times \mathfrak{d}_u \frac{a(u)\cos\pi(\mu u+\nu)\Gamma(\mu u+\nu+1/2)}{(1+s^2)^{\mu u}}\, \mathrm{d}s\, \mathrm{d}u\,. \tag{29}$$

Proof Detecting that the partial sum $\mathscr{N}^*[J, Y](x)$ is defined for any positive x, and for the quoted parameter space the Dixon–Ferrar formula (3) there holds, we clearly deduce

$$\begin{aligned}\mathscr{N}^*[J, Y](x) &= \frac{8}{\pi} \int_0^\infty \sum_{n=1}^{n^*} a_n \cos\pi(\mu n+\nu) K_{2(\mu n+\nu)}(2x \sinh t)\, \mathrm{d}t \\ &= \frac{8}{\pi^{3/2}} \left(\frac{2}{x}\right)^{2\nu} \int_0^\infty \int_0^\infty \frac{\cos(2xs\sinh t)}{(1+s^2)^{\nu-1/2}} \mathcal{A}_2^*(s)\, \mathrm{d}t\, \mathrm{d}s,\end{aligned}$$

where the counting function

$$\mathcal{A}_2^*(s) = \sum_{n=1}^{n^*} a_n \frac{\cos\pi(\mu n+\nu)\,\Gamma(\mu n+\nu+1/2)}{(1+s^2)^{\mu n}}$$

takes, by the Euler–Maclaurin summation formula (4), the following integral form:

$$\mathcal{A}_2^*(s) = \int_0^{n^*} \mathfrak{d}_u \frac{a(u)\cos\pi(\mu u+\nu)\,\Gamma(\mu u+\nu+1/2)}{(1+s^2)^{\mu u}}\, \mathrm{d}u\,.$$

In turn, we point out that here all summation–integration–integration order exchanges are legitimate and by the special case of the Basset formula (2), see e.g. [35, p. 172, Eq. (1)]

$$K_0(z) = \int_0^\infty \cos(z\sinh t)\, \mathrm{d}t = \int_0^\infty \frac{\cos(zy)}{\sqrt{1+y^2}}\, \mathrm{d}y, \qquad z > 0,$$

obvious steps lead to the asserted expression (29). □

5 Discussion: Open Problems

The methodology used in getting integral representations for such series when the included special functions *except one* are incorporated into the coefficients belongs to Baricz and Pogány who treated such kind series in the book chapter [4]. However, in this work we have not used this derivation method, we give the advantage to the several times previously applied 'classical' approach developed in a set of articles, see the corresponding references list in [7, pp. 27–86, Chapter 2].

The proving procedure of our main result includes integral representations for Bessel functions of the first and second kind J_ν, Y_ν, respectively, modified Bessel function of the second kind K_ν which occurs in the famous Dixon–Ferrar formula, Struve function $\mathbf{H}_\nu$ (which 'mysteriously' disappeared from the final result in Theorem 3), the Cahen's Laplace integral formula for the Laplace integral form of the Dirichlet series and finally, the Euler–Maclaurin summation method were used.

Finally, we mention some of plethora of possible further related research directions which could include

(iii) Schlömilch series;
(iv) Kapteyn series;
(v) Dini series of the Nicholson and/or Dixon–Ferrar type functions and their counterparts from the Bessel and alike members form these functions class; and
(vi) the inverse problem: assuming that there holds the integral representation (27) of the second type Neumann series, describe the class of functions consisting from $\boldsymbol{a} = a(x)$ which restriction $a(x)\big|_{n\in\mathbb{N}} = (a_n)$.

Acknowledgements T.K. Pogány acknowledges the support given by the NAWA project PROM PPI/PRO/2018/1/00008 and thanks to the Department of Mathematical Physics, The Henryk Niewodniczański Institute of Nuclear Physics of Polish Academy of Sciences, Kraków, Poland for the warm hospitality and the excellent working atmosphere during his stay there during February 2019.

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On Some Generalizations of the Properties of the Multidimensional Generalized Erdélyi–Kober Operators and Their Applications

Sh. T. Karimov and S. M. Sitnik

Abstract In this paper we investigate the composition of a multidimensional generalized Erdélyi–Kober operator with differential operators of high order. In particular, with powers of the differential Bessel operator. Applications of proved properties to solving the Cauchy problem for a multidimensional polycaloric equation with a Bessel operator are shown. An explicit formula for solving the formulated problem is constructed. In the appendix we briefly describe a general context for transmutations and integral transforms used in this paper. Such a general context is formed by integral transforms composition method (ITCM).

Keywords Fractional integrals and derivatives · Multidimensional Erdélyi–Kober operators · Bessel differential operator · Multidimensional polycaloric equation · Cauchy problem

MSC 26A33, 35K50, 35K67

1 Introduction

Various modifications and generalizations of the classical Riemann–Liouville operators of fractional integration and differentiation are widely used in theory and applications. Such modifications include, particularly, the Erdélyi–Kober operators [1, 2]. These operators turned to be very useful in application to integral and differential equations as well as in other issues of science and technology [3, 4]. Their various modifications, generalizations, and applications can be found in works of Erdélyi [5, 6], Sneddon [7, 8], and Kyriakova [9].

Sh. T. Karimov
Fergana State University, Fergana, Uzbekistan

S. M. Sitnik (✉)
Belgorod State National Research University (BelGU), Belgorod, Russia
e-mail: sitnik@bsu.edu.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_5

One-dimensional generalized Erdélyi–Kober operator with the Bessel function in the kernel and its applications were considered by Lowndes [10, 11].

Modifications of fractional integration of Erdélyi–Kober type for two and many variables have been studied in [12–18] and others. A survey of some studies on this topic can be found in [3, 4, 9].

In [18] a multidimensional generalized Erdélyi–Kober operator was introduced in the form

$$J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = J_{\lambda_1,\lambda_2,\dots,\lambda_n} \begin{pmatrix} \alpha_1, \ \alpha_2, \ \dots, \ \alpha_n \\ \eta_1, \ \eta_2, \ \dots, \ \eta_n \end{pmatrix} f(x)$$

$$= J_{\lambda_1}^{x_1}(\eta_1, \alpha_1) J_{\lambda_2}^{x_2}(\eta_2, \alpha_2) \cdots J_{\lambda_n}^{x_n}(\eta_n, \alpha_n) f(x)$$

$$= \left[\prod_{k=1}^{n} \frac{2x_k^{-2(\alpha_k+\eta_k)}}{\Gamma(\alpha_k)} \right] \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} \prod_{k=1}^{n} \left[t_k^{2\eta_k+1} \left(x_k^2 - t_k^2 \right)^{\alpha_k - 1} \right.$$

$$\left. \times \ \bar{J}_{\alpha_k - 1} \left(\lambda \sqrt{x_k^2 - t_k^2} \right) \right] f(t_1, \ t_2, \ \dots, \ t_n) dt_1 dt_2 \dots dt_n, \qquad (1.1)$$

where $\lambda, \alpha, \eta \in R^n$, $\alpha_k > 0$, $\eta_k \geq -1/2$, $k = \overline{1, n}$; $\Gamma(\alpha)$ is the Euler gamma function; $\bar{J}_\nu(z)$ is Bessel-Clifford function expressed through the Bessel function $J_\nu(z)$, using the formula $\bar{J}_\nu(z) = \Gamma(\nu + 1)(z/2)^{-\nu} J_\nu(z)$ and $J_{\lambda_k}^{x_k}(\eta_k, \alpha_k)$ is a particular Erdélyi–Kober integral of α_k-order of kth variable

$$J_{\lambda_k}^{x_k}(\eta_k, \alpha_k) f(x) = \frac{2x_k^{-2(\alpha_k+\eta_k)}}{\Gamma(\alpha_k)} \int_0^{x_k} (x_k^2 - t^2)^{\alpha_k - 1} \bar{J}_{\alpha_k - 1} \left(\lambda_k \sqrt{x_k^2 - t^2} \right)$$

$$\times t^{2\eta_k+1} f(x_1, x_2, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n) dt.$$

In this paper we also study the basic properties of the operator (1.1) and show that the inverse operator has the form

$$J_\lambda^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = J_{i\lambda} \begin{pmatrix} -\alpha \\ \eta + \alpha \end{pmatrix} f(x)$$

$$= 2^{n-|m|} \left[\prod_{k=1}^{n} \frac{x_k^{-2\eta_k}}{\Gamma(m_k - \alpha_k)} \left(\frac{1}{x_k} \frac{\partial}{\partial x_k} \right)^{m_k} \right] \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} \prod_{k=1}^{n} \left[t_k^{2(\eta_k+\alpha_k)+1} \right.$$

$$\left. \times \left(x_k^2 - t_k^2 \right)^{m_k - 1 - \alpha_k} \bar{I}_{m_k - 1 - \alpha_k} \left(\lambda \sqrt{x_k^2 - t_k^2} \right) \right] f(t_1, \ t_2, \ \dots, \ t_n) dt_1 dt_2 \dots dt_n, \qquad (1.2)$$

where $\alpha_k > 0$, $m_k = [\alpha_k] + 1$, $\eta_k \geq -1/2$, $k = \overline{1, n}$, $\bar{I}_\nu(z) = \Gamma(\nu + 1)(z/2)^{-\nu} I_\nu(z)$, $I_\nu(z)$ is the Bessel function of the imaginary argument. $m = (m_1, m_2, \ldots, m_n)$ is a multi-index, and $|m| = m_1 + m_2 + \ldots + m_n$ is its length.

Taking into account $\bar{J}_\nu(0) = 1$, in the limit for $\lambda_k \to 0$, $k = \overline{1, n}$, we obtain

$$J_0 \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = J_{0,0,\ldots,0} \begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_n \\ \eta_1, \eta_2, \ldots, \eta_n \end{pmatrix} f(x)$$

$$= \prod_{k=1}^{n} \left[\frac{2x_k^{-2(\alpha_k+\eta_k)}}{\Gamma(\alpha_k)} \right] \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} \prod_{k=1}^{n} \left[t_k^{2\eta_k+1} \left(x_k^2 - t_k^2 \right)^{\alpha_k - 1} \right] f(t) dt_1 \ldots dt_n, \tag{1.3}$$

This operator is a multidimensional analog of the ordinary (not generalized) Erdélyi–Kober operator. In this case, the inverse operator (1.2) takes the following form:

$$J_0^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = 2^{n-|m|} \left[\prod_{k=1}^{n} \frac{x_k^{-2\eta_k}}{\Gamma(m_k - \alpha_k)} \left(\frac{1}{x_k} \frac{\partial}{\partial x_k} \right)^{m_k} \right]$$

$$\int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} \prod_{k=1}^{n} \left[t_k^{2(\eta_k+\alpha_k)+1} \left(x_k^2 - t_k^2 \right)^{m_k - 1 - \alpha_k} \right] f(t_1, t_2, \ldots, t_n) dt_1 dt_2 \ldots dt_n. \tag{1.4}$$

In addition, in [18] the following theorem is proved:

Theorem 1.1 *Let* $\alpha_k > 0, \eta_k \geq -1/2$; $f(x) \in C^2(\Omega^n)$; $\lim\limits_{x_k \to 0} x_k^{2\eta_k+1} f_{x_k}(x) = 0$, $k = \overline{1, n}$. *Then the transmutation formula holds:*

$$(B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2) J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} B_{\eta_k}^{x_k} f(x), \ k = \overline{1, n},$$

in particular, if $\lambda_k = 0$, $k = \overline{1, n}$, *then*

$$B_{\eta_k+\alpha_k}^{x_k} J_0 \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = J_0 \begin{pmatrix} \alpha \\ \eta \end{pmatrix} B_{\eta_k}^{x_k} f(x), \ k = \overline{1, n},$$

where $\Omega^n = \prod\limits_{k=1}^{n} (0, b_k) = (0, b_1) \times (0, b_2) \times \ldots \times (0, b_n)$ *be the Cartesian product,* $b_k > 0$, $k = \overline{1, n}$.

This Theorem implies

Corollary 1.1 *Suppose that the conditions of Theorem 1.1 are satisfied. Then*

$$\sum_{k=1}^{n}\left[B^{x_k}_{\eta_k+\alpha_k}+\lambda_k^2\right]J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x)=J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\sum_{k=1}^{n}\left[B^{x_k}_{\eta_k}\right]f(x),$$

in particular, if $\eta_k=-1/2,\ k=\overline{1,n}$, *then*

$$\sum_{k=1}^{n}\left[B^{x_k}_{\alpha_k-1/2}+\lambda_k^2\right]J_\lambda\begin{pmatrix}\alpha\\ -1/2\end{pmatrix}f(x)=J_\lambda\begin{pmatrix}\alpha\\ -1/2\end{pmatrix}\Delta f(x),$$

where $\Delta f(x)\equiv\sum\limits_{k=1}^{n}\left[\partial^2 f(x)/\partial x_k^2\right]$ *is the multidimensional Laplace operator.*

Similarly, we can prove the validity of the following theorem.

Theorem 1.2 *Let* $\alpha_k>0,\ \eta_k\geq-1/2,\ k=\overline{1,n},\ f(x)\in C^{2n}(\Omega^n)$, $x_k^{2\eta_k+1}B^{x_k}_{\eta_k}f(x)$ *are integrable in a neighborhood of* $x_k=0$ *and* $\lim\limits_{x_k\to 0}x_k^{2\eta_k+1}f_{x_k}(x)=0,\ k=\overline{1,\ n}$. *Then*

$$\prod_{k=1}^{n}(B^{x_k}_{\eta_k+\alpha_k}+\lambda_k^2)J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x)=J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\prod_{k=1}^{n}B^{x_k}_{\eta_k}f(x),$$

in particular, if $\eta_k=-1/2,\ k=\overline{1,n}$, *then*

$$\prod_{k=1}^{n}\left[B^{x_k}_{\alpha_k-(1/2)}+\lambda_k^2\right]J_\lambda\begin{pmatrix}\alpha\\ -1/2\end{pmatrix}f(x)=J_\lambda\begin{pmatrix}\alpha\\ -1/2\end{pmatrix}\frac{\partial^{2n}f(x)}{\partial x_1^2\partial x^2\ldots\partial x_n^2}.$$

The proof of Theorem 1.2 is analogous to the proof of Theorem 1.1.

In this paper, these properties are generalized for an iterated Bessel differential operator of high order. The results obtained are applied to the investigation of problems for higher-order multi-dimensional partial differential equations with singular coefficients.

In the appendix we briefly describe a general context for transmutations and integral transforms used in this paper. Such a general context is formed by integral transforms composition method (ITCM), cf. [19–21].

2 Generalization of the Properties of the Generalized Erdelyi–Kober Operator

Let $\left[B_{\eta_k}^{x_k}\right]^0 = E$, where E is the unit operator, $\left[B_{\eta_k}^{x_k}\right]^{m_k} = \left[B_{\eta_k}^{x_k}\right]^{m_k-1}\left[B_{\eta_k}^{x_k}\right]$ is the m_k th power of the operator $B_{\eta_k}^{x_k}$, $k = \overline{0, n}$.

Theorem 2.1 *Let* $\alpha_k > 0$, $\eta_k \geq -1/2$; $f(x) \in C^{2m_0}(\Omega^n)$; $x_k^{2\eta_k+1}\left[B_{\eta_k}^{x_k}\right]^{p_k+1} f(x)$ *functions are integrable in a neighborhood of the origin and* $\lim\limits_{x_k\to 0} x_k^{2\eta_k+1}(\partial/\partial x_k)$ $\left[B_{\eta_k}^{x_k}\right]^{p_k} f(x) = 0$, $p_k = \overline{0, m_k - 1}$, $k = \overline{1, n}$. *Then*

$$\left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right]^{m_k} J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} \left[B_{\eta_k}^{x_k}\right]^{m_k} f(x),\ k = \overline{1, n}, \tag{2.1}$$

where $m_0 = \max\{m_1, m_2, \ldots, m_n\}$.

We note that Theorem 2.1 is also true in the case when some or all of the $\lambda_k = 0$, $k = \overline{1, n}$.

Proof Theorem 2.1 can be proved by the method of mathematical induction on m_k, $k = \overline{1, n}$. We arbitrarily fix $k \in N$, where N is the set of natural numbers. The proof of (2.1) for a fixed k and $m_k = 1$ is given in Theorem 1.1. Assume that equality (2.1) holds for $m_k = l_k$ and prove that it holds for $m_k = l_k + 1$.

From equality

$$\left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right]^{l_k+1} J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = \left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right]\left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right]^{l_k} J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x)$$

by the induction hypothesis, if the conditions of Theorem 2.1 are satisfied, we have

$$\left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right]\left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right]^{l_k} J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x)$$

$$= \left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right] J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} \left[B_{\eta_k}^{x_k}\right]^{l_k} f(x).$$

In the last equality, applying Theorem 1.1 to the functions $\left[B_{\eta_k}^{x_k}\right]^{l_k} f(x)$, under the conditions $\lim\limits_{x_k\to 0} x_k^{2\eta_k+1}(\partial/\partial x_k)\left[B_{\eta_k}^{x_k}\right]^{l_k} f(x) = 0$, $k = \overline{1, n}$, we obtain the validity of formula (2.1). □

Corollary 2.1 *Suppose that the conditions of Theorem 2.1 are satisfied. Then*

$$\sum_{k=1}^{n}\left[B^{x_k}_{\eta_k+\alpha_k}+\lambda_k^2\right]^{m_k} J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x) = J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\sum_{k=1}^{n}\left[B^{x_k}_{\eta_k}\right]^{m_k} f(x),$$

in addition, if $f(x)\in C^{2|m|}(\Omega^n)$, *then*

$$\prod_{k=1}^{n}\left[B^{x_k}_{\eta_k+\alpha_k}+\lambda_k^2\right]^{m_k} J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x) = J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\prod_{k=1}^{n}\left[B^{x_k}_{\eta_k}\right]^{m_k} f(x). \tag{2.2}$$

Theorem 2.2 *Let* $\alpha_k>0,\ \ \eta_k\geq -1/2,\ \ \ q\in N;\ \ \ f(x)\in C^{2q}(\Omega^n);$ *the functions* $x_k^{2\eta_k+1}\left[B^{x_k}_{\eta_k}\right]^{l+1} f(x)$ *are integrable in a neighborhood of the origin and*

$$\lim_{x_k\to 0} x_k^{2\eta_k+1}\frac{\partial}{\partial x_k}\left[B^{x_k}_{\eta_k}\right]^{l} f(x)=0,\, l=\overline{0,q-1},\, k=\overline{1,\ n}.$$

Then

$$\left[\sum_{k=1}^{n}\left(B^{x_k}_{\eta_k+\alpha_k}+\lambda_k^2\right)\right]^{q} J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x) = J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\left[\sum_{k=1}^{n}B^{x_k}_{\eta_k}\right]^{q} f(x).$$

This Theorem is proved using the polynomial formula

$$\left[\sum_{k=1}^{n}\left(B^{x_k}_{\eta_k+\alpha_k}+\lambda_k^2\right)\right]^{q} = \sum_{|m|=q}\frac{q!}{m!}\prod_{k=1}^{n}\left(B^{x_k}_{\eta_k+\alpha_k}+\lambda_k^2\right)^{m_k}$$

and with the use of equality (2.2), where $m!=m_1!m_2!\dots m_n!$

Corollary 2.2 *Suppose that the conditions of Theorem 2.1 are satisfied. Then for* $\eta_k=-1/2,\ \ k=\overline{1,n}$,

$$\left[\sum_{k=1}^{n}\left(B^{x_k}_{\alpha_k-1/2}+\lambda_k^2\right)\right]^{q} J_\lambda\begin{pmatrix}\alpha\\ -1/2\end{pmatrix} f(x) = J_\lambda\begin{pmatrix}\alpha\\ -1/2\end{pmatrix}\Delta^q f(x),$$

in particular, for $\lambda_k=0$, *we have the equality*

$$\Delta_B^q J_0\begin{pmatrix}\alpha\\ -1/2\end{pmatrix} f(x) = J_0\begin{pmatrix}\alpha\\ -1/2\end{pmatrix}\Delta^q f(x).$$

Let $L^{(y)}$ be a linear differential operator of order $l\in N$ independent of variable $x=(x_1,\ x_2,\ \dots,\ x_n)$ in the variable $y=(y_1,y_2,\ \dots,\ y_s)\in R^s$.

Theorem 2.3 *Let* $\alpha_k > 0,\ \eta_k \geq -1/2,\ k = \overline{1,n},\ q \in N;\ f(x,y) \in C^{2q,lq}_{x,y}(\Omega^n \times \Omega^s)$, *the functions* $x_k^{2\eta_k+1}\left[B^{x_k}_{\eta_k}\right]^{j+1} f(x,y)$ *are integrable in a neighborhood of the origin and* $\lim\limits_{x_k\to 0} x_k^{2\eta_k+1}(\partial/\partial x_k)\left[B^{x_k}_{\eta_k}\right]^j f(x,y) = 0,\ j = \overline{0,q-1},\ k = \overline{1,\ n}$. *Then*

$$\left[L^{(y)} \pm \sum_{k=1}^{n}\left(B^{x_k}_{\eta_k+\alpha_k} + \lambda_k^2\right)\right]^q J_\lambda^{(x)}\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x,y)$$

$$= J_\lambda^{(x)}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\left[L^{(y)} \pm \sum_{k=1}^{n} B^{x_k}_{\eta_k}\right]^q f(x,y),$$

where the superscripts in the operators mean the variables by which these operators operate.

Proof Using the binomial formula, we obtain

$$\left[L^{(y)} \pm \sum_{k=1}^{n}\left(B^{x_k}_{\eta_k+\alpha_k} + \lambda_k^2\right)\right]^q J_\lambda^{(x)}\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x,y)$$

$$= \sum_{j=0}^{q} C_q^j (\pm 1)^j \left(L^{(y)}\right)^{q-j}\left[\sum_{k=1}^{n}\left(B^{x_k}_{\eta_k+\alpha_k} + \lambda_k^2\right)\right]^j J_\lambda^{(x)}\begin{pmatrix}\alpha\\ \eta\end{pmatrix} f(x,y),$$

$C_k^j = k!/[j!(k-j)!]$ is binomial coefficients.

Next, applying Theorem 2.2, we have

$$J_\lambda^{(x)}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\sum_{j=0}^{q} C_q^j (\pm 1)^j \left(L^{(y)}\right)^{q-j}\left[\sum_{k=1}^{n} B^{x_k}_{\eta_k}\right]^j f(x,y)$$

$$= J_\lambda^{(x)}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\left[L^{(y)} \pm \sum_{k=1}^{n} B^{x_k}_{\eta_k}\right]^q f(x,y).$$

□

Corollary 2.3 *Suppose that the conditions of Theorem 2.3 are satisfied. If*

$$L^{(y)} = -\sum_{k=\omega+1}^{\omega+\sigma}\left(B^{x_k}_{\eta_k+\alpha_k} + \lambda_k^2\right),$$

where $x=(x_1,x_2,\dots,x_\omega)$, $y=(x_{\omega+1},x_{\omega+2},\dots,x_{\omega+\sigma})$, $\omega+\sigma=n$, *then*

$$\left[\sum_{k=1}^{\omega}\left(B^{x_k}_{\eta_k+\alpha_k}+\lambda_k^2\right)-\sum_{k=\omega+1}^{\omega+\sigma}\left(B^{x_k}_{\eta_k+\alpha_k}+\lambda_k^2\right)\right]^q J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x)$$

$$=J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\left[\sum_{k=1}^{\omega}B^{x_k}_{\eta_k}-\sum_{k=\omega+1}^{\omega+\sigma}B^{x_k}_{\eta_k}\right]^q f(x),\quad \omega+\sigma=n.$$

Let $[D^{x_k}_{\eta_k}]^0=E$, $D^{x_k}_{\eta_k}\equiv x_k^{-2\eta_k}\left(\frac{1}{x_k}\frac{\partial}{\partial x_k}\right)x_k^{2\eta_k}$ and $[D^{x_k}_{\eta_k}]^{m_k}=[D^{x_k}_{\eta_k}]^{m_k-1}D^{x_k}_{\eta_k}$ the degree of an operator $D^{x_k}_{\eta_k}$ that is representable in the form $[D^{x_k}_{\eta_k}]^{m_k}=x_k^{-2\eta_k}\left(\frac{1}{x_k}\frac{\partial}{\partial x_k}\right)^{m_k}x_k^{2\eta_k}$, m_k be nonnegative integers, $k=\overline{1,n}$.

Theorem 2.4 *If* $\alpha_k>0$, $\eta_k\geq-(1/2)$, $k=\overline{1,n}$, $f(x)\in C^{m_0}(\Omega^n)$, *the functions* $x_{x_k}^{2\eta_k+1}[D^{x_k}_{\eta_k}]^{l_k+1}f(x)$ *are integrable in a neighborhood of the origin and* $\lim\limits_{x_k\to0}x_k^{2\eta_k}[D^{x_k}_{\eta_k}]^{l_k}f(x)=0$, $l_k=\overline{0,m_k-1}$, $k=\overline{1,n}$, *then*

$$[D^{x_k}_{\eta_k+\alpha_k}]^{m_k}J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x)=J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}[D^{x_k}_{\eta_k}]^{m_k}f(x),\quad k=\overline{1,n},\tag{2.3}$$

where $m_0=\max\{m_1,m_2,\dots,m_n\}$.

Proof This Theorem is also proved using the method of mathematical induction on m_k, $k=\overline{1,n}$. Arbitrarily fix $k\in N$. The proof of formula (2.3) for $m_k=1$, $k=\overline{1,n}$ is given in [22, 23], according to which we have

$$[D^{x_k}_{\eta_k+\alpha_k}]J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x)=J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}[D^{x_k}_{\eta_k}]f(x),\quad k=\overline{1,n}.\tag{2.4}$$

Suppose that (2.3) holds for $m_k=l_k$ and we prove that it holds for $m_k=l_k+1$.

$$[D^{x_k}_{\eta_k+\alpha_k}]^{l_k+1}J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x)=[D^{x_k}_{\eta_k+\alpha_k}][D^{x_k}_{\eta_k+\alpha_k}]^{l_k}J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x).\tag{2.5}$$

By the induction hypothesis, if the conditions of Theorem 2.4 are satisfied, we have

$$[D^{x_k}_{\eta_k+\alpha_k}]^{l_k}J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x)=J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}[D^{x_k}_{\eta_k}]^{l_k}f(x).$$

Then the equality (2.5) takes the form

$$[D_{\eta_k+\alpha_k}^{x_k}]^{l_k+1} J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = [D_{\eta_k+\alpha_k}^{x_k}] J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} [D_{\eta_k}^{x_k}]^{l_k} f(x).$$

Further, applying formula (2.4) to the functions $[D_{\eta_k}^{x_k}]^{l_k} f(x)$, under the conditions $\lim\limits_{x_k \to 0} x_k^{2\eta_k} [D_{\eta_k}^{x_k}]^{l_k} f(x) = 0$, we obtain the validity of formula (2.3). □

Corollary 2.4 *Suppose that the conditions of Theorem 2.4 are satisfied, then*

$$\prod_{k=1}^{n} [D_{\eta_k+\alpha_k}^{x_k}]^{m_k} J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} f(x) = J_\lambda \begin{pmatrix} \alpha \\ \eta \end{pmatrix} \prod_{k=1}^{n} [D_{\eta_k}^{x_k}]^{m_k} f(x).$$

Theorem 2.5 *Let* $0 < \alpha_k < 1$, $\eta_k \geq -1/2$; $g(x) \in C^{2p}(\Omega^n)$; $\dfrac{\partial}{\partial x_k} \left[B_{\eta_k+\alpha_k}^{x_k}\right]^l g(x)$ *are integrable in a neighborhood of the origin and* $\lim\limits_{x_k \to 0} x_k^{2(\eta_k+\alpha_k)+1} \dfrac{\partial}{\partial x_k} \left[B_{\eta_k+\alpha_k}^{x_k}\right]^l g(x) = 0$, $l = \overline{0, p-1}$, $p \in N$, $k = \overline{1, n}$. *Then*

$$\left[B_{\eta_k}^{x_k} - \lambda_k^2\right]^p J_\lambda^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} g(x) = J_\lambda^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} \left[B_{\eta_k+\alpha_k}^{x_k}\right]^p g(x), \ k = \overline{1, n},$$

or

$$\left[B_{\eta_k}^{x_k}\right]^p J_\lambda^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} g(x) = J_\lambda^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} \left[B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right]^p g(x), \ \ k = \overline{1, n},$$

in particular, if $\lambda_k = 0$, *then*

$$\left[B_{\eta_k}^{x_k}\right]^p J_0^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} g(x) = J_0^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} \left[B_{\eta_k+\alpha_k}^{x_k}\right]^p g(x), \ \ k = \overline{1, n}.$$

The proof of the Theorem 2.5 is analogous to the proof of Theorem 2.1.

Corollary 2.5 *Suppose that the conditions of Theorem 2.5 are satisfied, then*

$$\left[\sum_{k=1}^{n} \left(B_{\eta_k}^{x_k} - \lambda_k^2\right)\right]^p J_\lambda^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} g(x) = J_\lambda^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} \left[\sum_{k=1}^{n} B_{\eta_k+\alpha_k}^{x_k}\right]^p g(x),$$

or

$$\left[\sum_{k=1}^{n} B_{\eta_k}^{x_k}\right]^p J_\lambda^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} g(x) = J_\lambda^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} \left[\sum_{k=1}^{n} \left(B_{\eta_k+\alpha_k}^{x_k} + \lambda_k^2\right)\right]^p g(x).$$

If the conditions $\lim\limits_{x_k\to 0} x_k^{2\alpha_k}\dfrac{\partial}{\partial x_k}\left[B^{x_k}_{\alpha_k-(1/2)}\right]^l g(x)=0,\ l=\overline{0,p-1},\ k=\overline{1,\ n}$, are satisfied, then the last equality for $\lambda_k=0,\quad \eta_k=-(1/2),\ k=\overline{0,m-1}$, implies the validity of equality

$$\Delta^p J_0^{-1}\begin{pmatrix}\alpha\\ -1/2\end{pmatrix} g(x)=J_0^{-1}\begin{pmatrix}\alpha\\ -1/2\end{pmatrix}\Delta_B^p g(x) \tag{2.6}$$

where $\Delta^p=\left[\sum\limits_{k=1}^{n}\dfrac{\partial^2}{\partial x_k^2}\right]^p$ is pth power of the multidimensional Laplace operator, and

$$\Delta_B^p=\left[\sum_{k=1}^{n}\left(B^{x_k}_{\alpha_k-(1/2)}\right)\right]^p=\left[\sum_{k=1}^{n}\left(\frac{\partial^2}{\partial x_k^2}+\frac{2\alpha_k}{x_k}\frac{\partial}{\partial x_k}\right)\right]^p.$$

We note that the Theorems proved allow us to reduce higher-order equations with singular coefficients to polyharmonic, polycaloric, and polywave equations, and thereby to establish and investigate the correct initial and boundary value problems for such equations.

3 Applications

The results obtained are applicable to the construction of the solution of the analogue of the Cauchy problem for a multidimensional polycaloric equation with the Bessel operator.

Singular parabolic equations with Bessel operator belong to the class of equations degenerating on the boundary of the domain with respect to the space variables. These equations are often encountered in applications. Thus, in the mathematical simulation of numerous problems of heat transfer in immobile media (solids), the problems of diffusion boundary layer [24], and the problems of propagation of heat in process of injection of hot liquids in oil pools [25], we get singular parabolic equations with Bessel operator.

Degenerating equations and equations with singular coefficients form an important field of the contemporary theory of partial differential equations. Numerous works are devoted to the study of these equations. In this field, an important place is occupied by the initial and boundary-value problems for parabolic equations with Bessel operator. The theory of classical solutions to the Cauchy problem for singular parabolic equations of the second order was developed in [26–30]. The Cauchy problem for singular parabolic equations in the classes of distributions and in the classes of generalized functions of the type S' was studied in [31, 32]. However, the initial and boundary value problems for equations with Bessel operators of high orders are studied quite poorly.

In the domain $\Omega = \{(x,t) : x \in R_+^n,\ t \in R_+^1\}$, where $x = (x_1, x_2, \dots, x_n)$ be a point of the n-dimensional Euclidean space R^n, $R_+^n = \left\{x \in R^n : x_k > 0,\ k = \overline{1,n}\right\}$, we consider the problem of finding the solution $u(x,t)$ of the equation

$$L_\gamma^m(u) \equiv \left(\frac{\partial}{\partial t} - \Delta_B\right)^m u(x,t) = 0, \quad (x,t) \in \Omega, \tag{3.1}$$

satisfying the initial conditions

$$\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \varphi_k(x),\ x \in R_+^n,\ k = \overline{0, m-1} \tag{3.2}$$

and homogeneous boundary conditions

$$\left.\frac{\partial^{2k+1} u}{\partial x_j^{2k+1}}\right|_{x_j=0} = 0, \quad t > 0, \quad j = \overline{1,n}, \quad k = \overline{0, m-1}, \tag{3.3}$$

where $\Delta_B = \sum\limits_{k=1}^{n} B_{\gamma_k}^{x_k}$, $B_{\gamma_k}^{x_k} = \partial^2/\partial x_k^2 + [(2\gamma_k + 1)/x_k](\partial/\partial x_k)$ is the Bessel operator acting on variable x_k; $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n) \in R^n$, $\gamma_k \in R$, $\gamma_k > -1/2$, $k = \overline{1,n}$, m is a natural number; $\varphi_k(x)$, $k = \overline{0, m-1}$ given differentiable functions.

We note that in the problems of the general theory of partial differential equations containing the Bessel operator with one or more variables, the main investigation apparatus is the corresponding integral Fourier–Bessel transform. Unlike traditional methods, here we apply the properties of the multidimensional Erdélyi–Kober operator to solve the problem.

Suppose that the solution of the Eq. (3.1) satisfying conditions (3.2) and (3.3) exists. We seek this solution in the form

$$u(x,t) = J_0^{(x)}\begin{pmatrix}\alpha\\ \eta\end{pmatrix} U(x,t), \tag{3.4}$$

where $\alpha, \eta \in R^n$, $\alpha_k = \gamma_k + (1/2) > 0$, $\eta_k = -(1/2)$, $k = \overline{1,n}$, and $U(x,t)$ is an unknown function differentiable sufficiently many times, and $J_0^{(x)}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}$ is a multidimensional Erdélyi–Kober operator of fractional order (1.3) acting on a variable $x \in R^n$.

Substituting (3.4) into the boundary conditions (3.3), and then into Eq. (3.1) and the initial conditions (3.2), and using Theorem 2.3 for $L^{(t)} \equiv \partial/\partial t$, we arrive at the problem of determination of the solution $U(x,t)$ of the equation

$$\left(\frac{\partial}{\partial t}-\Delta\right)^m U(x,t)=0,\ \ (x,t)\in\Omega, \tag{3.5}$$

satisfying the initial conditions

$$\left.\frac{\partial^k U}{\partial t^k}\right|_{t=0}=\Phi_k(x),\ \ x\in R^n,\ \ k=\overline{0,m-1}, \tag{3.6}$$

and the homogeneous boundary conditions

$$\left.\frac{\partial^{2k+1}U}{\partial x_j^{2k+1}}\right|_{x_j=0}=0,t>0,\quad j=\overline{1,n},\quad k=\overline{0,m-1}, \tag{3.7}$$

where $\Phi_k(x)=J_0^{-1}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}\varphi_k(x)$, $\eta_k=-(1/2)$, $(k=\overline{0,m-1})$, $J_0^{-1}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}$ is the inverse operator (1.4).

By using the boundary conditions (3.7), we extend the functions $\Phi_k(x)$ evenly to $x_k<0$, $(k=\overline{0,m-1})$ and denote the extended functions by $\tilde{\Phi}_k(x)$. Then in the domain $\tilde{\Omega}=\{(x,y):\ x\in R^n,\ \ t>0\}$ we obtain the problem of finding a solution of Eq. (3.5) satisfying the initial conditions

$$\left.\frac{\partial^k U}{\partial t^k}\right|_{t=0}=\tilde{\Phi}_k(x),\ \ x\in R,\ \ k=\overline{0,m-1}, \tag{3.8}$$

We introduce the notation $W_0(x,t)=U(x,t)$ and $W_k(x,t)=\left(\frac{\partial}{\partial t}-\Delta\right)^k W_0$. In this notation, the problem (3.5) and (3.8) is equivalent to the problem of determination the functions $W_k(x,t)$, $k=\overline{0,m-1}$, satisfying the system of equations

$$\begin{cases}\dfrac{\partial W_k}{\partial t}-\Delta W_k=W_{k+1},\ \ (x,t)\in\tilde{\Omega},\ \ k=\overline{0,m-2},\\[2mm] \dfrac{\partial W_{m-1}}{\partial t}-\Delta W_{m-1}=0,\ \ (x,t)\in\tilde{\Omega}\end{cases} \tag{3.9}$$

with the initial conditions

$$W_k(x,0)=F_k(x),\ \ x\in R^n,\quad k=\overline{0,m-1}, \tag{3.10}$$

where

$$F_k(x) = \sum_{j=0}^{k} (-1)^{k-j} C_k^j \Delta^{k-j} \tilde{\Phi}_j(x), \;\; k = \overline{0, m-1}. \tag{3.11}$$

For the solution of problem (3.9) and (3.10), we use the following lemma.

Lemma 3.1 *If $g(x) \in L_1(R^n)$, then the equality*

$$\int_0^t \frac{d\tau}{\left(2\sqrt{\pi(t-\tau)}\right)^n} \int_{R^n} \exp\left[-\frac{|x-y|^2}{4(t-\tau)}\right] \left\{ \frac{1}{\left(2\sqrt{\pi\tau}\right)^n} \int_{R^n} g(\eta) \exp\left[-\frac{(y-\eta)^2}{4\tau}\right] d\eta \right\} dy$$
$$= \frac{t}{\left(2\sqrt{\pi t}\right)^n} \int_{R^n} g(\eta) \exp\left[-\frac{|\eta - x|^2}{4t}\right] dy. \tag{3.12}$$

Proof In view of the uniform convergence of the improper integrals on the left-hand side of equality (3.12), we can change the order of integration with respect to η and y. Then we take the inner integral by the formula [33]

$$\int_{-\infty}^{+\infty} \exp\left[-p\xi^2 - q\xi\right] d\xi = \sqrt{\frac{\pi}{p}} \exp\left(\frac{q^2}{4p}\right), \quad \operatorname{Re} p > 0,$$

we obtain

$$\prod_{j=1}^{n} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x_j - y_j)^2}{4(t-\tau)} - \frac{(y_j - \eta_j)^2}{4\tau}\right] dy_j$$
$$= \left[2\frac{\sqrt{\pi}}{\sqrt{t}} \sqrt{\tau(t-\tau)}\right]^n \exp\left[-\frac{|\eta - x|^2}{4t}\right]. \tag{3.13}$$

Substituting (3.13) into the left-hand side of (3.12), after reducing such terms, we obtain the assertion of Lemma 3.1. □

We now successively solve each equation in system (3.9) starting from the last equation. By using the initial conditions (3.10) and Lemma 3.1, we determine the solution of problem (3.9) and (3.10). In view of the relation $W_0(x,t) = U(x,t)$, we obtain the solution of problem (3.5)–(3.7) in the form

$$U(x,t) = \left(2\sqrt{\pi t}\right)^{-n} \sum_{k=0}^{m-1} \frac{t^k}{k!} \int_{R^n} F_k(s) \exp\left[-\frac{|s-x|^2}{4t}\right] ds, \tag{3.14}$$

where $F_k(x)$ $(k = \overline{0, m-1})$ are known functions given by equalities (3.11).

In view of the evenness of the functions $F_k(x)$, $k = \overline{0, m-1}$, we can rewrite equality (3.14) in the form

$$U(x,t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} U_k(x,t), \tag{3.15}$$

where

$$U_k(x,t) = \int\limits_{R_+^n} F_k(s) G(x,t,s) ds, \tag{3.16}$$

$$G(x,s,t) = \prod_{j=1}^{n} G_0(x_j, s_j, t),$$

$$G_0(x_j, s_j, t) = \frac{1}{2\sqrt{\pi t}} \left\{ \exp\left[-\frac{(s_j - x_j)^2}{4t} \right] + \exp\left[-\frac{(s_j + x_j)^2}{4t} \right] \right\}.$$

To analyze the behavior of the functions $F_k(x)$, $k = \overline{0, m-1}$, it is necessary to perform certain transformations. To this end, we prove the following lemma:

Lemma 3.2 *Suppose that the functions $\varphi_j(x) \in C^{2(m-j)-1}(R_+^n)$, $j = \overline{0, m-1}$, are continuous and bounded, and that all derivatives of the functions $\varphi_j(x)$, up to the order $2(m-j)-1$, $j = \overline{0, m-1}$ inclusively, are equal to zero for $x_k = 0$, $k = \overline{1, n}$. Then the equalities*

$$\lim_{x_k \to 0} x_k^{2\alpha_k} \frac{\partial}{\partial x_k} \left[B^{x_k}_{\alpha_k - (1/2)} \right]^l \varphi_j(x) = 0, k = \overline{1, n}, l = \overline{0, m-1}, j = \overline{0, m-1}, \tag{3.17}$$

$$\lim_{x_k \to 0} [B^{x_k}_{\gamma_k}]^i \varphi_{j-i}(x) = 0, \;\; k = \overline{1, n}, \;\; i = \overline{0, j}, \;\; j = \overline{0, m-1}, \tag{3.18}$$

are true.

Proof By induction, we can prove the following equality:

$$\left(\frac{1}{x} \frac{d}{dx} \right)^p h(x) = \sum_{j=1}^{p} (-1)^{j+1} A_{pj} \frac{h^{(p-j+1)}(x)}{x^{p+j-1}}, \tag{3.19}$$

where A_{pj} are constants given by the recurrence relations

$$A_{(p+1)1} = A_{p1} = 1, \;\; p \geq 1, \;\; A_{(p+1)j} = (p+j-1) A_{p(j-1)} + A_{pj}, \;\; p \geq 2, \;\; j = \overline{2, p},$$

$$A_{(p+1)(p+1)} = (2p-1) A_{pp} = (2p-1)!!, \;\; p \geq 1.$$

We rewrite (3.17) in the form

$$\lim_{x_k\to 0} H(x) = \lim_{x_k\to 0} x_k^{2\alpha_k} \frac{\partial}{\partial x_k}\left[B^{x_k}_{\alpha_k-(1/2)}\right]^l \varphi_j(x)$$

$$= \lim_{x_k\to 0} x_k^{1+2\alpha_k} \sum_{q=0}^{l} C_l^q (2\alpha_k)^{l-q} \left(\frac{1}{x_k}\frac{\partial}{\partial x_k}\right)^{l-q+1} \frac{\partial^{2q}\varphi_j(x)}{\partial x_k^{2q}}$$

Taking (3.19) into account, we have

$$\lim_{x_k\to 0} H(x)$$

$$= \sum_{q=0}^{l} C_l^q (2\alpha_k)^{l-q} \sum_{j=1}^{l-q+1} (-1)^{j+1} A_{(l-q+1)j} \lim_{x_k\to 0} \frac{\partial^{l-q-j+2}\varphi_j(x)/\partial x_k^{l-q-j+2}}{x_k^{l-q+j-1-2\alpha_k}}$$

Applying the L'Hospital rule $l-q+j$ times [34] to the last equality and taking into account the condition of the Lemma 3.2, we obtain

$$\lim_{x_k\to 0} \frac{\partial^{l-q-j+2}\varphi_j(x)/\partial x_k^{l-q-j+2}}{x_k^{l-q+j-1-2\alpha_k}} = \frac{\lim_{x_k\to 0} x_k^{1+2\alpha_k}[\partial^{2l+2q}\varphi_j(x)/\partial x_k^{2l+2q}]}{(l-q+j)!} = 0.$$

This proves of (3.17). Equality (3.18) is proved similarly. □

We now transform the functions $F_k(x)$, $k=\overline{0,m-1}$. By virtue of Lemma 3.2, the functions $\Phi_k(x)$ satisfy all conditions of Theorem 2.5. Therefore, taking into account formula (2.6), equality (3.11) for $x_k > 0,\ k=\overline{1,n}$ can be represented in the form

$$F_k(x) = J_0^{-1}\begin{pmatrix}\alpha\\ -1/2\end{pmatrix} f_k(x),\ \ k=\overline{0,m-1}, \tag{3.20}$$

where

$$f_k(x) = \sum_{j=0}^{k} (-1)^j C_k^j \Delta_B^j \varphi_{k-j}(x),\ \ k=\overline{0,m-1}. \tag{3.21}$$

Taking into account the form of the inverse operator (1.4) for $m_j=1,\ \ j=\overline{1,n}$, equality (3.20) can be represented as $F_k(x) = [\partial^n/(\partial x_1\partial x_2\ldots\partial x_n)]\bar{F}_k(x)$, where

$$\bar{F}_k(x) = \prod_{j=1}^{n}\left[\frac{1}{\Gamma(1-\alpha_j)}\right]\int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_n}\prod_{j=1}^{n}\left[(x_j^2-s_j^2)^{-\alpha_j}s_j^{2\alpha_j}\right]f_k(s)ds_1ds_2\ldots ds_n.$$

We note that, by Lemma 3.2, it follows from (3.21) that the functions $f_k(x)$, $k = \overline{0, m-1}$ for $x_j \geq 0$, are continuous, bounded, and $f_k(x)|_{x_j=0} = 0$, so that from the last equality we have

$$\bar{F}_k(x)\big|_{x_j=0} = 0, \quad j = \overline{1, n}, \ \ k = \overline{0, m-1}. \tag{3.22}$$

Taking (3.22) into account in (3.16), we integrate by parts. Then, substituting in this equality the value of the functions $\bar{F}_k(x)$, we obtain

$$U_k(x,t) = -\prod_{j=1}^{n}\left[\frac{1}{\Gamma(1-\alpha_j)}\right]\int\limits_{R_+^n} f_k(s)\prod_{j=1}^{n}\left[s_j^{2\alpha_j} G_1(x_j, s_j, t)\right] ds, \tag{3.23}$$

where

$$G_1(x_j, s_j, t) = \int\limits_{s_j}^{+\infty} (y_j^2 - s_j^2)^{-\alpha_j} \frac{\partial}{\partial y_j} G_0(x_j, y_j, t) dy_j. \tag{3.24}$$

Let us calculate the integral (3.24). Applying the formula [33, p. 451]

$$\int\limits_0^{+\infty} e^{-a\lambda^2}\cos(b\lambda)d\lambda = \sqrt{\frac{\pi}{4a}}\exp\left[-\frac{b^2}{4a}\right], \quad \operatorname{Re} a > 0,$$

function $G_0(x_j, y_j, t)$ can be represented in the form

$$G_0(x_j, y_j, t) = \frac{2}{\pi}\int\limits_0^{+\infty} e^{-t\lambda^2}\cos(x_j\lambda)\cos(y_j\lambda)d\lambda.$$

We find the derivative with respect to y_j and substitute the obtained expression for the function G_{0y} in (3.24). Then we use the uniform convergence of integrals and change the order of integration. Taking the inner integral with the help of the Mehler–Sonine formula [35, p. 93], we get

$$G_1(x_j, s_j, t)$$

$$= -\frac{2^{(1/2)-\alpha_j}}{\sqrt{\pi}}\Gamma(1-\alpha_j)s_j^{(1/2)-\alpha}\int\limits_0^{+\infty} e^{-t\lambda^2}\lambda^{\alpha_j+(1/2)} J_{\alpha_j-(1/2)}(\lambda s_j)\cos(x_j\lambda)d\lambda. \tag{3.25}$$

Now, substituting (3.23) into (3.15), and its in (3.4), after changing the order of integration, we obtain

$$u(x,t) = -\prod_{j=1}^{n}\left[\frac{2x_j^{1-2\alpha_j}}{\Gamma(\alpha_j)\Gamma(1-\alpha_j)}\right]\sum_{k=0}^{m-1}\frac{t^k}{k!}\int\limits_{R_+^n} f_k(s)\prod_{j=1}^{n}s_j^{2\alpha_j}G_2(x_j,s_j,t)ds \tag{3.26}$$

where

$$G_2(x_j,s_j,t) = \int\limits_0^{x_j}(x_j^2-\xi_j^2)^{\alpha_j-1}\,G_1(\xi_j,s_j,t)d\xi_j \tag{3.27}$$

We substitute the expression (3.25) for the function G_1 in (3.27) and change the order of integration. Then, using the Poisson formula [35, p. 93], we compute the inner integral. As a result, we find

$$G_2(x_j,s_j,t)$$
$$= -\frac{1}{2}\Gamma(\alpha_j)\Gamma(1-\alpha_j)\left(\frac{s_j}{x_j}\right)^{(1/2)-\alpha_j}\int\limits_0^{\infty}e^{-t\lambda^2}J_{\alpha_j-(1/2)}(s_j\lambda)J_{\alpha_j-(1/2)}(x_j\lambda)\lambda d\lambda.$$

Further, taking into account the following formula [35, p. 60]

$$\int\limits_0^{\infty}e^{-t\lambda^2}J_\nu(s\lambda)J_\nu(x\lambda)\lambda d\lambda = \frac{1}{2t}e^{-\frac{x^2+s^2}{4t}}I_\nu\left(\frac{xs}{2t}\right),$$

$\operatorname{Re}\nu > -1,\ \operatorname{Re}t > 0$, we have

$$G_2(x_j,s_j,t) = -\frac{1}{4t}\Gamma(\alpha_j)\Gamma(1-\alpha_j)\left(\frac{s_j}{x_j}\right)^{(1/2)-\alpha_j}e^{-\frac{x_j^2+s_j^2}{4t}}I_{\alpha_j-(1/2)}\left(\frac{x_js_j}{2t}\right). \tag{3.28}$$

Substituting (3.28) into (3.26) and taking into account both and $\alpha_j = \gamma_j + 1/2 < 1$, $\gamma_j > -1/2$, $j = \overline{1,n}$, we find the final form of the solution of the Eq. (3.1) for $|\gamma_j| < 1/2$, $j = \overline{1,n}$, satisfying conditions (2.5) and (2.6):

$$u(x,t) = \frac{1}{(2t)^n}\prod_{j=1}^{n}x_j^{-\gamma_j}\sum_{k=0}^{m-1}\frac{t^k}{k!}\int\limits_{R_+^n}f_k(s)G(x,s,t)ds, \tag{3.29}$$

where $f_k(x) = \sum\limits_{j=0}^{k} (-1)^j C_k^j \Delta_B^{k-j} \varphi_j(x)$,

$$G(x, s, t) = \prod_{j=1}^{n} \left\{ s_j^{\gamma_j+1} \exp\left[-\frac{x_j^2 + s_j^2}{4t} \right] I_{\gamma_j} \left(\frac{x_j s_j}{2t} \right) \right\}$$

$$= \prod_{j=1}^{n} \left[s_j^{\gamma_j+1} I_{\gamma_j} \left(\frac{x_j s_j}{2t} \right) \right] \exp\left[-\frac{|x|^2 + |s|^2}{4t} \right], \quad |x|^2 = \sum_{j=1}^{n} x_j^2. \tag{3.30}$$

A direct verification shows that the following theorem holds.

Theorem 3.1 *Let* $|\gamma_j| < 1/2$, $j = \overline{1, n}$, *and the functions* $\varphi_j(x) \in C^{2(m-j)-1}(R_+^n)$, $j = \overline{0, m-1}$ *are continuous, bounded, and all derivatives of the functions* $\varphi_j(x)$, *up to the order* $2(m-j)-1$, $j = \overline{0, m-1}$ *inclusively, are equal to zero for* $x_k = 0$, $k = \overline{1, n}$. *Then the function* $u(x, t)$, *defined by* (3.29)*, is a classical solution of equation* $L_\gamma^m(u) = 0$, *satisfying conditions* (3.2) *and* (3.3).

Appendix: Integral Transform Composition Method (ITCM) in Transmutation Theory: How It Works

In the appendix we briefly describe a general context for transmutations and integral transforms used in this paper. Such a general context is formed by integral transforms composition method (ITCM).

Below we give a brief survey and outline some applications of the integral transforms composition method (ITCM) for obtaining transmutations via integral transforms. It is possible to derive wide range of transmutation operators by this method. Classical integral transforms are involved in the integral transforms composition method (ITCM) as basic blocks, among them are Fourier, sine and cosine-Fourier, Hankel, Mellin, Laplace and some generalized transforms. The ITCM and transmutations obtaining by it are applied to deriving connection formulas for solutions of singular differential equations and more simple non-singular ones. We consider well-known classes of singular differential equations with Bessel operators, such as classical and generalized Euler–Poisson–Darboux equation and the generalized radiation problem of A. Weinstein. Methods of this paper are applied to more general linear partial differential equations with Bessel operators, such as multivariate Bessel-type equations, GASPT (Generalized Axially Symmetric Potential Theory) equations of Weinstein, Bessel-type generalized wave equations with variable coefficients,ultra B-hyperbolic equations and others. So with many results and examples the main conclusion of this paper is illustrated: the integral transforms composition method (ITCM) of constructing transmutations is very important and effective tool also for obtaining connection formulas and explicit representations of solutions to a wide class of singular differential equations, including ones with Bessel operators.

What is ITCM and How It Works?

In transmutation theory explicit operators were derived based on different ideas and methods, often not connecting altogether. So there is an urgent need in transmutation theory to develop a general method for obtaining known and new classes of transmutations.

In this section we give such general method for constructing transmutation operators. We call this method *integral transform composition method* or shortly ITCM. The method is based on the representation of transmutation operators as compositions of basic integral transforms. The integral transform composition method (ITCM) gives the algorithm not only for constructing new transmutation operators, but also for all now explicitly known classes of transmutations, including Poisson, Sonine, Vekua-Erdelyi-Lowndes, Buschman-Erdelyi, Sonin-Katrakhov and Poisson-Katrakhov ones, cf. [36–45, 63–65] as well as the classes of elliptic, hyperbolic and parabolic transmutation operators introduced by Carroll [37–39].

The formal algorithm of ITCM is the next. Let us take as input a pair of arbitrary operators A, B, and also connecting with them generalized Fourier transforms F_A, F_B, which are invertible and act by the formulas

$$F_A A = g(t) F_A, \quad F_B B = g(t) F_B, \tag{A.1}$$

where t is a dual variable, g is an arbitrary function with suitable properties. It is often convenient to choose $g(t) = -t^2$ or $g(t) = -t^\alpha$, $\alpha \in \mathbb{R}$.

Then the essence of ITCM is to obtain formally a pair of transmutation operators P and S as the method output by the next formulas:

$$S = F_B^{-1} \frac{1}{w(t)} F_A, \quad P = F_A^{-1} w(t) F_B \tag{A.2}$$

with arbitrary function $w(t)$. When P and S are transmutation operators intertwining A and B:

$$SA = BS, \quad PB = AP. \tag{A.3}$$

A formal checking of (A.3) can be obtained by direct substitution. The main difficulty is the calculation of compositions (A.2) in an explicit integral form, as well as the choice of domains of operators P and S.

Let us list the main advantages of Integral Transform Composition Method (ITCM).

- Simplicity—many classes of transmutations are obtained by explicit formulas from elementary basic blocks, which are classical integral transforms.
- ITCM gives by a unified approach all previously explicitly known classes of transmutations.
- ITCM gives by a unified approach many new classes of transmutations for different operators.
- ITCM gives a unified approach to obtain both direct and inverse transmutations in the same composition form.
- ITCM directly leads to estimates of norms of direct and inverse transmutations using known norm estimates for classical integral transforms on different functional spaces.
- ITCM directly leads to connection formulas for solutions to perturbed and unperturbed differential equations.

An obstacle for applying ITCM is the next one: we know acting of classical integral transforms usually on standard spaces like L_2, L_p, C^k, variable exponent Lebesgue spaces [46] and so on. But for application of transmutations to differential equations we usually need some more conditions hold, say at zero or at infinity. For these problems we may first construct a transmutation by ITCM and then expand it to the needed functional classes.

Let us stress that formulas of the type (A.2) of course are not new for integral transforms and its applications to differential equations. But ITCM is new when applied to transmutation theory! In other fields of integral transforms and connected differential equations theory compositions (A.2) for the choice of classical Fourier transform leads to famous pseudo-differential operators with symbol function $w(t)$. For the choice of the classical Fourier transform and the function $w(t) = (\pm it)^{-s}$ we obtain fractional integrals on the whole real axis, for $w(t) = |x|^{-s}$ we obtain M.Riesz potential, for $w(t) = (1 + t^2)^{-s}$ in formulas (A.2) we obtain Bessel potential and for $w(t) = (1 \pm it)^{-s}$ - modified Bessel potentials [3].

The next choice for ITCM algorithm,

$$A = B = B_\nu, \quad F_A = F_B = H_\nu, \quad g(t) = -t^2, \quad w(t) = j_\nu(st) \tag{A.4}$$

leads to generalized translation operators of Delsart [47–49], for this case we have to choose in ITCM algorithm defined by (A.1)–(A.2) the above values (A.4) in which B_ν is the Bessel operator, H_ν is the Hankel transform, j_ν is the normalized (or "small") Bessel function. In the same manner other families of operators commuting with a given one may be obtained by ITCM for the choice $A = B$, $F_A = F_B$ with arbitrary functions $g(t), w(t)$ (generalized translation commutes with the Bessel operator). In case of the choice of differential operator A as quantum oscillator and connected integral transform F_A as fractional or quadratic Fourier transform [50] we may obtain by ITCM transmutations also for this case [43]. It is possible to apply ITCM instead of classical approaches for obtaining fractional powers of Bessel operators [43, 51–54].

Direct applications of ITCM to multidimensional differential operators are obvious, in this case t is a vector and $g(t), w(t)$ are vector functions in (A.1)–(A.2). Unfortunately for this case we know and may derive some new explicit transmutations just for simple special cases. But among them are well-known and interesting classes of potentials. In case of using ITCM by (A.1)–(A.2) with Fourier transform and $w(t)$—positive definite quadratic form we come to elliptic Riesz potentials [3, 55]; with $w(t)$—indefinite quadratic form we come to hyperbolic Riesz potentials [3, 55, 56]; with $w(x,t) = (|x|^2 - it)^{-\alpha/2}$ we come to parabolic potentials [3]. In case of using ITCM by (A.1)–(A.2) with Hankel transform and $w(t)$ - quadratic form we come to elliptic Riesz B-potentials [57, 58] or hyperbolic Riesz B-potentials [59]. For all above mentioned potentials we need to use distribution theory and consider for ITCM convolutions of distributions, for inversion of such potentials we need some cutting and approximation procedures, cf. [56, 59]. For this class of problems it is appropriate to use Schwartz or/and Lizorkin spaces for probe functions and dual spaces for distributions.

So we may conclude that the method we consider in the paper for obtaining transmutations—ITCM is effective, it is connected to many known methods and problems, it gives all known classes of explicit transmutations and works as a tool to construct new classes of transmutations. Application of ITCM needs the next three steps.

Step 1. For a given pair of operators A, B and connected generalized Fourier transforms F_A, F_B define and calculate a pair of transmutations P, S by basic formulas (A.1)–(A.2).

Step 2. Derive exact conditions and find classes of functions for which transmutations obtained by step 1 satisfy proper intertwining properties.

Step 3. Apply now correctly defined transmutations by steps 1 and 2 on proper classes of functions to deriving connection formulas for solutions of differential equations.

The next part of this article is organized as follows. First we illustrate step 1 of the above plan and apply ITCM for obtaining some new and known transmutations. For step 2 we prove a general theorem for the case of Bessel operators, it is enough to solve problems to complete strict definition of transmutations. And after that we give an example to illustrate step 3 of applying obtained by ITCM transmutations to derive formulas for solutions of a model differential equation.

Application of ITCM to Index Shift B–Hyperbolic Transmutations

In this section we apply ITCM to obtain integral representations for index shift B-hyperbolic transmutations. It corresponds to step 1 of the above plan for ITCM algorithm.

Let us look for the operator T transmuting the Bessel operator B_ν into the same operator but with another parameter B_μ. To find such a transmutation we use ITCM with Hankel transform. Applying ITCM we obtain an interesting and important family of transmutations, including index shift B-hyperbolic transmutations, "descent" operators, classical Sonine and Poisson-type transmutations, explicit integral representations for fractional powers of the Bessel operator, generalized translations of Delsart and others.

So we are looking for an operator $T_{\nu,\mu}^{(\varphi)}$ such that

$$T_{\nu,\mu}^{(\varphi)} B_\nu = B_\mu T_{\nu,\mu}^{(\varphi)} \tag{A.5}$$

in the factorized due to ITCM form

$$T_{\nu,\mu}^{(\varphi)} = H_\mu^{-1}\Big(\varphi(t) H_\nu\Big), \tag{A.6}$$

where H_ν is a Hankel transform. Assuming $\varphi(t) = Ct^\alpha$, $C \in \mathbb{R}$ does not depend on t and $T_{\nu,\mu}^{(\varphi)} = T_{\nu,\mu}^{(\alpha)}$ we can derive the following theorem.

Theorem A.1 *Let f be a proper function for which the composition (A.6) is correctly defined,*

$$\operatorname{Re}(\alpha + \mu + 1) > 0, \quad \operatorname{Re}\left(\alpha + \frac{\mu - \nu}{2}\right) < 0.$$

Then for transmutation operator $T_{\nu,\mu}^{(\alpha)}$ obtained by ITCM and such that

$$T_{\nu,\mu}^{(\alpha)} B_\nu = B_\mu T_{\nu,\mu}^{(\alpha)}$$

the next integral representation is true

$$\begin{aligned}
&\left(T_{\nu,\mu}^{(\alpha)} f\right)(x)\\
&= C\frac{2^{\alpha+3}\Gamma(\frac{\alpha+\mu+1}{2})}{\Gamma(\frac{\mu+1}{2})}\Big[\frac{x^{-1-\mu-\alpha}}{\Gamma(-\frac{\alpha}{2})}\\
&\quad\times \int_0^x f(y)\,{}_2F_1\Big(\frac{\alpha+\mu+1}{2}, \frac{\alpha}{2}+1; \frac{\nu+1}{2}; \frac{y^2}{x^2}\Big) y^\nu dy + \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\mu+1}{2})\Gamma(\frac{\nu-\mu-\alpha}{2})}\\
&\quad\times \int_x^\infty f(y)\,{}_2F_1\Big(\frac{\alpha+\mu+1}{2}, \frac{\alpha+\mu-\nu}{2}+1; \frac{\mu+1}{2}; \frac{x^2}{y^2}\Big) y^{\nu-\mu-\alpha-1} dy\Big].
\end{aligned} \tag{A.7}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Corollary A.1 *Let* $f \in L^2(0,\infty)$, $\alpha = -\mu$; $\nu = 0$. *For* $\mu > 0$ *we obtain the operator*

$$\left(T_{0,\mu}^{(-\mu)} f\right)(x) = \frac{2\Gamma(\frac{\mu+1}{2})}{\sqrt{\pi}\Gamma(\mu/2)} x^{1-\mu} \int_0^x f(y)(x^2-y^2)^{\frac{\mu}{2}-1} dy, \tag{A.8}$$

such that

$$T_{0,\mu}^{(-\mu)} D^2 = B_\mu T_{0,\mu}^{(-\mu)} \tag{A.9}$$

and $T_{0,\mu}^{(-\mu)} 1 = 1$,

The operator (A.8) is the well-known Poisson operator (see [47]). We will use conventional symbol

$$\mathcal{P}_x^\mu f(x) = C(\mu) x^{1-\mu} \int_0^x f(y)(x^2-y^2)^{\frac{\mu}{2}-1} dy, \tag{A.10}$$

$$\mathcal{P}_x^\mu 1 = 1, \quad C(\mu) = \frac{2\Gamma(\frac{\mu+1}{2})}{\sqrt{\pi}\Gamma(\frac{\mu}{2})}.$$

We remark that if $u = u(x,t)$, $x, t \in \mathbb{R}$, $u(x,0) = f(x)$ and $u_t(x,0) = 0$, then

$$\mathcal{P}_t^\mu u(x,t)|_{t=0} = f(x), \quad \frac{\partial}{\partial t}\mathcal{P}_t^\mu u(x,t)\Big|_{t=0} = 0. \tag{A.11}$$

Indeed, we have

$$\begin{aligned}\mathcal{P}_t^\mu u(x,t)|_{t=0} &= C(\mu) t^{1-\mu} \int_0^t u(x,y)(t^2-y^2)^{\frac{\mu}{2}-1} dy\Big|_{t=0} \\ &= C(\mu) \int_0^1 u(x,ty)|_{t=0}(1-y^2)^{\frac{\mu}{2}-1} dy = f(x)\end{aligned}$$

and

$$\frac{\partial}{\partial t}\mathcal{P}_t^\mu u(x,t)\Big|_{t=0} = C(\mu) \int_0^1 u_t(x,ty)|_{t=0}(1-y^2)^{\frac{\mu}{2}-1} dy = 0.$$

Corollary A.2 *For* $f \in L^2(0,\infty)$, $\alpha=\nu-\mu$; $-1 < \operatorname{Re}\nu < \operatorname{Re}\mu$ *we obtain the first "descent" operator*

$$\left(T_{\nu,\mu}^{(\nu-\mu)} f\right)(x) = \frac{2\Gamma(\frac{\mu+1}{2})}{\Gamma(\frac{\mu-\nu}{2})\Gamma(\frac{\nu+1}{2})} x^{1-\mu} \int_0^x f(y)(x^2-y^2)^{\frac{\mu-\nu}{2}-1} y^\nu dy. \tag{A.12}$$

such that

$$T_{\nu,\mu}^{(\nu-\mu)} B_\nu = B_\mu T_{\nu,\mu}^{(\nu-\mu)}, \quad T_{\nu,\mu}^{(\nu-\mu)} 1 = 1.$$

Corollary A.3 *Let $f \in L_{1,w}$ with the weight function $w(y) = |y|^{\operatorname{Re}\nu - \operatorname{Re}\mu}$, $\alpha = 0$, $-1 < \operatorname{Re}\mu < \operatorname{Re}\nu$. In this case we obtain the second "descent" operator:*

$$\left(T_{\nu,\mu}^{(0)} f\right)(x) = \frac{2\Gamma(\nu-\mu)}{\Gamma^2(\frac{\nu-\mu}{2})} \int_x^\infty f(y)(y^2 - x^2)^{\frac{\nu-\mu}{2}-1} y\, dy. \tag{A.13}$$

In [44] the formula (A.13) was obtained as a particular case of Buschman-Erdelyi operator of the third kind but with different constant:

$$\left(T_{\nu,\mu}^{(0)} f\right)(x) = \frac{2^{1-\frac{\nu-\mu}{2}}}{\Gamma(\frac{\nu-\mu}{2})} \int_x^\infty f(y) y \left(y^2 - x^2\right)^{\frac{\nu-\mu}{2}-1} dy. \tag{A.14}$$

As might be seen in the form (A.13) as well as (A.14) the operator $T_{\nu,\mu}^{(0)}$ does not depend on the values ν and μ but only on the difference between ν and μ.

Corollary A.4 *Let $f \in L^2(0,\infty)$, $\operatorname{Re}(\alpha+\nu+1) > 0$, $\operatorname{Re}\alpha < 0$. If we take $\mu = \nu$ in (A.7) we obtain the operator*

$$\begin{aligned}\left(T_{\nu,\nu}^{(\alpha)} f\right)(x) = \frac{2^{\alpha+3}\Gamma(\frac{\alpha+\nu+1}{2})}{\Gamma(-\frac{\alpha}{2})\Gamma(\frac{\nu+1}{2})} \Big[& x^{-1-\nu-\alpha} \int_0^x f(y) \\ & \times {}_2F_1\Big(\frac{\alpha+\nu+1}{2}, \frac{\alpha}{2}+1; \frac{\nu+1}{2}; \frac{y^2}{x^2}\Big) y^\nu dy \\ & + \int_x^\infty f(y) {}_2F_1\Big(\frac{\alpha+\nu+1}{2}, \frac{\alpha}{2}+1; \frac{\nu+1}{2}; \frac{x^2}{y^2}\Big) y^{-\alpha-1} dy \Big]\end{aligned} \tag{A.15}$$

which is an explicit integral representation of the negative fractional power α of the Bessel operator: B_ν^α.

So it is possible and easy to obtain fractional powers of the Bessel operator by ITCM. For different approaches to fractional powers of the Bessel operator and its explicit integral representations cf. [9, 43, 51–54, 60–62].

Theorem A.2 *If we apply ITCM with* $\varphi(t) = j_{\frac{\nu-1}{2}}(zt)$ *in* (A.6) *and with* $\mu = \nu$ *then the operator*

$$\left(T_{\nu,\nu}^{(\varphi)} f\right)(x)$$
$$= {}^{\nu}T_x^z f(x) = H_\nu^{-1}\left[j_{\frac{\nu-1}{2}}(zt) H_\nu[f](t)\right](x)$$
$$= \frac{2^\nu \Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}(4xz)^{\nu-1}\Gamma(\frac{\nu}{2})} \int_{|x-z|}^{x+z} f(y) y[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\nu}{2}-1} dy \tag{A.16}$$

coincides with the generalized translation operator (see [47–49]), for which the next properties are valid

$${}^{\nu}T_x^z (B_\nu)_x = (B_\nu)_z \, {}^{\nu}T_x^z, \tag{A.17}$$

$${}^{\nu}T_x^z f(x)|_{z=0} = f(x), \quad \frac{\partial}{\partial z} \, {}^{\nu}T_x^z f(x)\Big|_{z=0} = 0. \tag{A.18}$$

More frequently used representation of generalized translation operator ${}^{\nu}T_z^x$ is (see [47–49])

$${}^{\nu}T_x^z f(x) = C(\nu) \int_0^\pi f(\sqrt{x^2+z^2-2xz\cos\varphi}) \sin^{\nu-1}\varphi d\varphi, \tag{A.19}$$

$$C(\nu) = \left(\int_0^\pi \sin^{\nu-1}\varphi d\varphi\right)^{-1} = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}\ \Gamma(\frac{\nu}{2})}.$$

It is easy to see that it is the same as ours.

So it is possible and easy to obtain generalized translation operators by ITCM, and its basic properties follows immediately from ITCM integral representation.

Application of Transmutations Obtained by ITCM to Integral Representations of Solutions to Hyperbolic Equations with Bessel Operators

Let us solve the problem of obtaining transmutations by ITCM (step 1) and justify integral representation and proper function classes for it (step 2). Now consider applications of these transmutations to integral representations of solutions to hyperbolic equations with Bessel operators (step 3). For simplicity we consider model equations, for them integral representations of solutions are mostly known. More complex problems need more detailed and spacious calculations. But even for these model problems considered below application of the transmutation method based on ITCM is new, it allows more unified and simplified approach to hyperbolic equations with Bessel operators of EPD/GEPD types.

Standard approach for solving differential equations is to find its general solution first, and then substitute given functions to find particular solutions. Here we will show how to obtain general solution of EPD type equation using transmutation operators.

Proposition A.3 *A general solution of the equation*

$$\frac{\partial^2 u}{\partial x^2} = (B_\mu)_t u, \quad u = u(x, t; \mu) \tag{A.20}$$

for $0 < \mu < 1$ *is represented in the form*

$$u = \int_0^1 \frac{\Phi(x + t(2p - 1))}{(p(1 - p))^{1-\frac{\mu}{2}}}\, dp + t^{1-\mu} \int_0^1 \frac{\Psi(x + t(2p - 1))}{(p(1 - p))^{\mu/2}}\, dp, \tag{A.21}$$

with a pair of arbitrary functions Φ, Ψ.

Proof First, we consider the wave equation when $a = 1$,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}. \tag{A.22}$$

A general solution to this equation has the form

$$F(x + t) + G(x - t), \tag{A.23}$$

where F and G are arbitrary functions. Applying operator (A.10) (obtained by ITCM) by variable t we obtain that one solution to the Eq. (A.20) is

$$u_1 = 2C(\mu) \frac{1}{t^{\mu-1}} \int_0^t [F(x + z) + G(x - z)](t^2 - z^2)^{\frac{\mu}{2}-1}\, dz.$$

Let us transform the resulting general solution as follows

$$u_1 = \frac{C(\mu)}{t^{\mu-1}} \int_{-t}^t \frac{F(x + z) + F(x - z) + G(x + z) + G(x - z)}{(t^2 - z^2)^{1-\frac{\mu}{2}}}\, dz.$$

Introducing a new variable p by formula $z = t(2p - 1)$ we obtain

$$u_1 = \int_0^1 \frac{\Phi(x + t(2p - 1))}{(p(1 - p))^{1-\frac{\mu}{2}}}\, dp,$$

where

$$\Phi(x+z)=[F(x+z)+F(x-z)+G(x+z)+G(x-z)]$$

is an arbitrary function.

It is easy to see that if $u(x,t;\mu)$ is a solution of (A.20) then $t^{1-\mu}u(x,t;2-\mu)$ is also a solution. Therefore the second solution to (A.20) is

$$u_2 = t^{1-\mu}\int_0^1 \frac{\Psi(x+t(2p-1))}{(p(1-p))^{\mu/2}}\,dp,$$

where Ψ is an arbitrary function, not coinciding with Φ. Summing u_1 and u_2 we obtain general solution to (A.20) of the form (A.21). From the (A.21) we can see that for summable functions Φ and Ψ such a solution exists for $0<\mu<1$. □

Now we derive a general solution to GEPD type equation by transmutation method.

Proposition A.4 *A general solution to the equation*

$$(B_\nu)_x u = (B_\mu)_t u, \quad u = u(x,t;\nu,\mu) \tag{A.24}$$

for $0<\mu<1$, $0<\nu<1$ *is*

$$\begin{aligned} u = \frac{2\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2})}\Big(& x^{1-\nu}\int_0^x (x^2-y^2)^{\frac{\nu}{2}-1}dy\int_0^1 \frac{\Phi(y+t(2p-1))}{(p(1-p))^{1-\frac{\mu}{2}}}\,dp \\ &+ t^{1-\mu}x^{1-\nu}\int_0^x (x^2-y^2)^{\frac{\nu}{2}-1}dy\int_0^1 \frac{\Psi(y+t(2p-1))}{(p(1-p))^{\mu/2}}\,dp.\Big) \end{aligned} \tag{A.25}$$

Proof Applying the Poisson operator (A.10) (again obtained by ITCM) with index ν by variable x to the (A.21) we derive general solution (A.25) to the Eq. (A.24). □

Now let apply transmutations for finding general solution to GEPD type equation with spectral parameter.

Proposition A.5 *A general solution to the equation*

$$(B_\nu)_x u = (B_\mu)_t u + b^2 u, \quad u = u(x,t;\nu,\mu) \tag{A.26}$$

for $0 < \mu < 1$, $0 < \nu < 1$ *is*

$$
\begin{aligned}
u = & \frac{2\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2})}\Big(x^{1-\nu}\int_0^x (x^2-y^2)^{\frac{\nu}{2}-1}dy \\
& \times \int_0^1 \frac{\Phi(y+t(2p-1))}{(p(1-p))^{1-\frac{\mu}{2}}} j_{\frac{\mu}{2}-1}(2bt\sqrt{p(1-p)})\,dp \\
& + t^{1-\mu}x^{1-\nu}\int_0^x (x^2-y^2)^{\frac{\nu}{2}-1}dy \\
& \times \int_0^1 \frac{\Psi(y+t(2p-1))}{(p(1-p))^{\mu/2}} j_{-\frac{\mu}{2}}(2bt\sqrt{p(1-p)})\,dp.\Big)
\end{aligned}
\tag{A.27}
$$

Proof A general solution to the equation

$$
\frac{\partial^2 u}{\partial x^2} = (B_\mu)_t u + b^2 u, \quad u = u(x,t;\mu), \quad 0 < \mu < 1
$$

is (see [24, p. 328])

$$
\begin{aligned}
u = & \int_0^1 \frac{\Phi(x+t(2p-1))}{(p(1-p))^{1-\frac{\mu}{2}}} j_{\frac{\mu}{2}-1}(2bt\sqrt{p(1-p)})\,dp \\
& + t^{1-\mu}\int_0^1 \frac{\Psi(x+t(2p-1))}{(p(1-p))^{\mu/2}} j_{-\frac{\mu}{2}}(2bt\sqrt{p(1-p)})\,dp.
\end{aligned}
$$

Applying Poisson operator (A.10) (again obtained by ITCM) with index ν by variable x to the (A.21) we derive general solution (A.25) to the Eq. (A.24). □

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Alternative Approach to Miller-Paris Transformations and Their Extensions

D. B. Karp and E. G. Prilepkina

Abstract The paper deals with Miller-Paris transformations which are extensions of Euler's transformations for the Gauss hypergeometric functions to generalized hypergeometric functions of higher-order having integral parameter differences (IPD). In our recent work we computed the degenerate versions of these transformations corresponding to the case when one parameter difference is equal to a negative integer. The purpose of this paper is to present an independent new derivation of both the general and the degenerate forms of Miller-Paris transformations. In doing so we employ the generalized Stieltjes transform representation of the generalized hypergeometric functions and some partial fraction expansions. This approach leads to different forms of the characteristic polynomials, one of them appears noticeably simpler than the original form due to Miller and Paris. Two extensions are further presented of the degenerate transformations to the generalized hypergeometric functions with additional free parameters and additional parameters with negative integral differences.

Keywords Generalized hypergeometric function · Miller-Paris transformation · Karlsson-Minton formula · Integral parameter differences (IPD)

MSC 2010 33C20

1 Introduction and Preliminaries

Transformation, reduction and summation formulas for hypergeometric functions is a vast subject with rich history dating back to Leonard Euler. Among important applications of such formulas (let alone hypergeometric functions in general) are

D. B. Karp (✉) · E. G. Prilepkina
Far Eastern Federal University, Vladivostok, Russia

Institute of Applied Mathematics, Vladivostok, Russia
e-mail: karp.db@dvfu.ru; prilepkina.eg@dvfu.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_6

quantum physics [28], non-equilibrium statistical physics [14] and many other fields [33]. The main developments up to the end of twentieth century can be found, for instance, in the books [1, 2, 18]. Hypergeometric functions also play an important role in the theory of transmutations appearing in the kernels of the integral representations of many transmutation operators [3, 15]. In this way they are also very useful in many applications, combining special functions and transmutation theory [3, 15, 34].

One particular example useful for simplifying sums that arise in theoretical physics (such as Racah coefficients) is the following summation formula established by Minton [24] in 1970:

$$ {}_{r+2}F_{r+1}\left(\begin{matrix}-k, b, f_1+m_1, \dots, f_r+m_r\\ b+1, f_1, \dots, f_r\end{matrix}\middle|\,1\right) = \frac{k!}{(b+1)_k}\frac{(f_1-b)_{m_1}\cdots(f_r-b)_{m_r}}{(f_1)_{m_1}\cdots(f_r)_{m_r}}, \quad k \ge m,\ k \in \mathbb{N}, \tag{1} $$

and slightly generalized by Karlsson [7] who replaced $-k$ by an arbitrary complex number a satisfying $\Re(1-a-m) > 0$ to get

$$ {}_{r+2}F_{r+1}\left(\begin{matrix}a, b, f_1+m_1, \dots, f_r+m_r\\ b+1, f_1, \dots, f_r\end{matrix}\middle|\,1\right) = \frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(b+1-a)}\frac{(f_1-b)_{m_1}\cdots(f_r-b)_{m_r}}{(f_1)_{m_1}\cdots(f_r)_{m_r}}. \tag{2} $$

Here and throughout the paper ${}_pF_q$ stands for the generalized hypergeometric function (see [1, Section 2.1], [18, Section 5.1], [26, Sections 16.2–16.12] or [2, Chapter 12]), $(a)_k = \Gamma(a+k)/\Gamma(a)$ is rising factorial and $\Gamma(z)$ is Euler's gamma function. These formulas attracted attention to generalized hypergeometric function with integral parameter differences, for which Michael Schlosser subsequently introduced the acronym IPD, motivated by the title of Karlsson's paper [7]. These summation formulas were generalized and extended in many directions: Gasper [6] deduced a q-analogue and a generalization of Minton's and Karlsson's formulas; Chu [4, 5] found extensions to bilateral hypergeometric and q-hypergeometric series; their results were re-derived by simpler means and further generalized by Schlosser [32], who also found multidimensional extensions to hypergeometric functions associated with root systems [31]. For further developments in this directions, see also [29, 30]. We also mention an interesting work [17] by Letessier, Valent and Wimp, where an order reduction for the differential equation satisfied by the generalized hypergeometric functions with some integral parameter differences was established.

Another surge in interest to IPD-type hypergeometric functions is related with transformation formulas for such functions, generalizing the classical Euler's transformations for the Gauss function ${}_2F_1$ and Kummer's transformation for the confluent hypergeometric functions ${}_1F_1$. Unlike Minton-Karlsson formulas dealing the generalized hypergeometric functions evaluated at 1 these transformations are certain identities for these functions evaluated at an arbitrary value of the argument. They were developed in a series of papers published over last 15 years, the most general form was presented in a seminal paper [22] by Miller and Paris. For a vector of positive integers $\mathbf{m} = (m_1, \ldots, m_r)$, $m = m_1 + m_2 + \ldots + m_r$, and a complex vector $\mathbf{f} = (f_1, \ldots, f_r)$ these transformations are given by [16, Theorem 1]

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, \mathbf{f}+\mathbf{m} \\ c, \mathbf{f} \end{matrix}\middle| x\right) = (1-x)^{-a} {}_{m+2}F_{m+1}\left(\begin{matrix} a, c-b-m, \boldsymbol{\zeta}+1 \\ c, \boldsymbol{\zeta} \end{matrix}\middle| \frac{x}{x-1}\right) \tag{3} $$

if $(c-b-m)_m \neq 0$, and, if also $(c-a-m)_m \neq 0$ and $(1+a+b-c)_m \neq 0$, then

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, \mathbf{f}+\mathbf{m} \\ c, \mathbf{f} \end{matrix}\middle| x\right) = (1-x)^{c-a-b-m} {}_{m+2}F_{m+1}\left(\begin{matrix} c-a-m, c-b-m, \boldsymbol{\eta}+1 \\ c, \boldsymbol{\eta} \end{matrix}\middle| x\right). \tag{4} $$

Here the vector $\boldsymbol{\zeta} = \boldsymbol{\zeta}(c, b, \mathbf{f}) = (\zeta_1, \ldots, \zeta_m)$ comprises the roots of the polynomial

$$ Q_m(t) = Q(b, c, \mathbf{f}, \mathbf{m}; t) = \frac{1}{(c-b-m)_m} \sum_{k=0}^{m} (b)_k C_{k,r}(t)_k (c-b-m-t)_{m-k}, \tag{5} $$

where $C_{0,r} = 1$, $C_{m,r} = 1/(\mathbf{f})_{\mathbf{m}}$, $(\mathbf{f})_{\mathbf{m}} = (f_1)_{m_1} \cdots (f_r)_{m_r}$, and

$$ C_{k,r} = C_{k,r}(\mathbf{f}, \mathbf{m}) = \frac{1}{(\mathbf{f})_{\mathbf{m}}} \sum_{j=k}^{m} \sigma_j \mathbf{S}_j^{(k)} = \frac{(-1)^k}{k!} {}_{r+1}F_r\left(\begin{matrix} -k, \mathbf{f}+\mathbf{m} \\ \mathbf{f} \end{matrix}\right). \tag{6} $$

In this formula and below we routinely omit the argument 1 from the generalized hypergeometric function: ${}_pF_q(\mathbf{a}; \mathbf{b}) := {}_pF_q(\mathbf{a}; \mathbf{b}; 1)$. The numbers σ_j $(0 \le j \le m)$ are defined via the generating function

$$ (f_1 + x)_{m_1} \ldots (f_r + x)_{m_r} = \sum_{j=0}^{m} \sigma_j x^j, $$

and $\mathbf{S}_j^{(k)}$ stands for the Stirling's number of the second kind. A simple rearrangement of Pochhammer's symbols leads to an alternative form of the polynomial $Q_m(t)$ as given in [13, (3.7)]:

$$ Q(b, c, \mathbf{f}, \mathbf{m}; t) = \frac{(c-b-t-m)_m}{(c-b-m)_m} \sum_{k=0}^{m} {}_{r+1}F_r\left(\begin{matrix} -k, \mathbf{f}+\mathbf{m} \\ \mathbf{f} \end{matrix}\right) \frac{(t)_k (b)_k}{(1+t+b-c)_k k!}. \tag{7} $$

Further, $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_m)$ in (4) are the roots of

$$ \hat{Q}_m(t) = \sum_{k=0}^{m} \frac{(-1)^k C_{k,r} (a)_k (b)_k (t)_k}{(c-a-m)_k (c-b-m)_k} {}_3F_2\left(\begin{matrix} -m+k, t+k, c-a-b-m \\ c-a-m+k, c-b-m+k \end{matrix}\right). \tag{8} $$

See [16, 19–21, 23] and references therein for further details.

Both formulas (3) and (4) fail when $c = b + p$, $p \in \{1, \ldots, m\}$. In our recent paper [12] we used a careful limit transition to derive the degenerate forms of the transformations (3) and (4) for such values of c. When evaluated at $x = 1$ these degenerate transformations lead to extensions of Minton-Karlsson summation formulas (1), (2) which we also investigated in [13] using several different techniques. The main purpose of this paper is to present an alternative derivation of both the general transformations (3), (4) and their degenerate forms found in [12]. Our derivation of the general case is presented in Sect. 2 and is based on the representation of the generalized hypergeometric function by the generalized Stieltjes transform of a particular type of Meijer's G function, namely $G^{p,0}_{p,p}$, which for historical reasons we prefer to call the Meijer-Nørlund function. Details regarding this representation, its history and numerous applications can be found in [8, 9, 11]. Our approach leads to different forms of the characteristic polynomials (5), (8). Comparison with those new forms yields an identity for finite hypergeometric sums which may be difficult to obtain directly. The degenerate forms and their extensions are presented in Sect. 3. Their derivation hinges on certain simple partial fraction decompositions. The results differ from those in [12]: here the negative parameter difference $-p$ (recall that $b - c = -p$) may take any (negative) integer value regardless of whether degeneration happens or not in the corresponding general Miller-Paris transformation. We further present two extensions: to several negative parameters differences instead of one and to a pair of additional unrestricted parameters on top and bottom of the generalized hypergeometric function.

2 Miller-Paris Transformations: General Case

Before proceeding to the main results let us introduce some notation. Let $\mathbb{N}$ and $\mathbb{C}$ denote the natural and complex numbers, respectively; further, put

$$\Gamma(\mathbf{a}) = \Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p), \quad (\mathbf{a})_n = (a_1)_n(a_2)_n\cdots(a_p)_n,$$
$$\mathbf{a} + \mu = (a_1 + \mu, a_2 + \mu, \ldots, a_p + \mu).$$

Inequalities like $\Re(\mathbf{a}) > 0$ and properties like $-\mathbf{a} \notin \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ will be understood element-wise. The symbol $\mathbf{a}_{[k]}$ will stand for the vector $\mathbf{a}$ with omitted k-th component. The function $G^{m,n}_{p,q}$ is Meijer's G function (see [18, section 5.2], [26, 16.17], [27, 8.2] or [2, Chapter 12]).

We begin with a lemma expressing the Meijer-Nørlund function $G^{p,0}_{p,p}$ with integral parameter differences in terms of beta density times a rational function.

Lemma 1 *Let* $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$, $m = m_1 + m_2 + \ldots + m_r$, $\mathbf{f} = (f_1, \ldots, f_r) \in \mathbb{C}^r$ *and* $b, c \in \mathbb{C}$. *Then*

$$G^{r+1,0}_{r+1,r+1}\left(t\,\middle|\,\begin{matrix}c, \mathbf{f}\\ b, \mathbf{f}+\mathbf{m}\end{matrix}\right) = \frac{t^b(1-t)^{c-b-1}}{\Gamma(c-b)} \sum_{k=0}^{m} D_k (c-b-k)_k \frac{t^k}{(t-1)^k}, \tag{9}$$

where

$$D_k = D_k(\mathbf{f}, \mathbf{m}, b) = \sum_{j=k}^{m} \alpha_j \mathbf{S}_j^{(k)} = \frac{(-1)^k (\mathbf{f}-b)_{\mathbf{m}}}{k!} {}_{r+1}F_r\left(\begin{matrix}-k, 1-\mathbf{f}+b\\ 1-\mathbf{f}+b-\mathbf{m}\end{matrix}\right). \tag{10}$$

The numbers α_j *are defined via the generating function*

$$(\mathbf{f}-b-t)_{\mathbf{m}} = \sum_{j=0}^{m} \alpha_j t^j, \tag{11}$$

and $\mathbf{S}_j^{(k)}$ *stands for the Stirling number of the second kind.*

Proof By the well-known expansion of the Meijer-Nørlund function $G^{r+1,0}_{r+1,r+1}$, see, for instance [10, (2.4)], if $f_i - f_j \notin \mathbb{Z}$, $f_i - c \notin \mathbb{Z}$ we have

$$G^{r+1,0}_{r+1,r+1}\left(t\,\middle|\,\begin{matrix}c, \mathbf{f}\\ b, \mathbf{f}+\mathbf{m}\end{matrix}\right) = \frac{t^b \Gamma(\mathbf{f}+\mathbf{m}-b)}{\Gamma(c-b)\Gamma(\mathbf{f}-b)} {}_{r+1}F_r\left(\begin{matrix}1-c+b, 1-\mathbf{f}+b\\ 1-\mathbf{f}-\mathbf{m}+b\end{matrix}\,\middle|\, t\right)$$
$$= \frac{t^b \Gamma(\mathbf{f}+\mathbf{m}-b)}{\Gamma(c-b)\Gamma(\mathbf{f}-b)} \sum_{n=0}^{\infty} \frac{(1-c+b)_n (1-\mathbf{f}+b)_n}{(1-\mathbf{f}-\mathbf{m}+b)_n n!} t^n.$$

Next, using $\Gamma(\mathbf{f}-b)(\mathbf{f}-b)_{\mathbf{m}} = \Gamma(\mathbf{f}+\mathbf{m}-b)$ and

$$\frac{(1-\mathbf{f}+b)_n}{(1-\mathbf{f}-\mathbf{m}+b)_n} = \frac{(\mathbf{f}-b-n)_{\mathbf{m}}}{(\mathbf{f}-b)_{\mathbf{m}}} = \frac{1}{(\mathbf{f}-b)_{\mathbf{m}}} \sum_{k=0}^{m} \alpha_k n^k,$$

where α_k is defined in (11), we obtain:

$$G^{r+1,0}_{r+1,r+1}\left(t\,\middle|\,\begin{matrix}c, \mathbf{f}\\ b, \mathbf{f}+\mathbf{m}\end{matrix}\right) = \frac{t^b \Gamma(\mathbf{f}+\mathbf{m}-b)}{\Gamma(c-b)\Gamma(\mathbf{f}-b)(\mathbf{f}-b)_{\mathbf{m}}} \sum_{n=0}^{\infty} \frac{(1-c+b)_n}{n!} t^n \sum_{k=0}^{m} \alpha_k n^k$$
$$= \frac{t^b}{\Gamma(c-b)} \sum_{k=0}^{m} \alpha_k \sum_{n=0}^{\infty} \frac{(1-c+b)_n}{n!} t^n n^k$$

$$
= \frac{t^b}{\Gamma(c-b)} \sum_{k=0}^{m} \alpha_k \sum_{l=0}^{k} \mathbf{S}_k^{(l)} \frac{(1-c+b)_l t^l}{(1-t)^{1-c+b+l}}
$$

$$
= \frac{t^b(1-t)^{c-b-1}}{\Gamma(c-b)} \sum_{l=0}^{m} \frac{(-1)^l (c-b-l)_l t^l}{(1-t)^l} \sum_{k=l}^{m} \alpha_k \mathbf{S}_k^{(l)}.
$$

To get the pre-ultimate equality we applied the definition of the Stirling numbers of the second kind via $n^k = \sum_{l=0}^{k} \mathbf{S}_k^{(l)}[n]_l$, $[n]_l = n(n-1)\dots(n-l+1)$ and the next relation (with $\delta = 1-c+b$):

$$
\sum_{n=0}^{\infty} \frac{(\delta)_n t^n}{n!} n^k = \sum_{n=0}^{\infty} \frac{(\delta)_n t^n}{n!} \sum_{l=0}^{k} \mathbf{S}_k^{(l)}[n]_l = \sum_{l=0}^{k} \mathbf{S}_k^{(l)} \sum_{n=l}^{\infty} \frac{(\delta)_n t^n}{(n-l)!} =
$$

$$
\sum_{l=0}^{k} \mathbf{S}_k^{(l)} \sum_{n=0}^{\infty} \frac{(\delta)_{n+l} t^{n+l}}{n!} = \sum_{l=0}^{k} \mathbf{S}_k^{(l)} \sum_{n=0}^{\infty} \frac{(\delta)_l (\delta+l)_n t^{n+l}}{n!} = \sum_{l=0}^{k} \mathbf{S}_k^{(l)} \frac{(\delta)_l t^l}{(1-t)^{\delta+l}}.
$$

Thus, we have proved formula (9) with the expression for D_k given by the first equality in (10). It remains to prove the second equality in (10). To this end we will borrow the technique from [21, Theorem 2]. By Taylor's theorem:

$$
R(t) = (\mathbf{f}-b-t)_{\mathbf{m}} = \sum_{j=0}^{m} \alpha_j t^j = \sum_{j=0}^{m} R^{(j)}(0) \frac{t^j}{j!}.
$$

On the other hand, from the theory of finite differences:

$$
\Delta^k R(t) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} R(t+j) = k! \sum_{j=k}^{m} \mathbf{S}_j^{(k)} \frac{R^{(j)}(t)}{j!}.
$$

Comparing with the first formula in (10) we get:

$$
D_k = \sum_{j=k}^{m} \mathbf{S}_j^{(k)} \frac{R^{(j)}(0)}{j!} = \frac{1}{k!} \Delta^k R(0) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} R(j)
$$

$$
= \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (\mathbf{f}-b-j)_{\mathbf{m}}.
$$

Substituting the identity

$$
(\mathbf{f}-b-j)_{\mathbf{m}} = (\mathbf{f}-b)_{\mathbf{m}} \frac{(1-\mathbf{f}+b)_j}{(1-f+b-\mathbf{m})_j}
$$

into the above expression after simple rearrangement and in view of the formula

$$\frac{1}{(k-j)!} = (-1)^j \frac{(-k)_j}{k!}.$$

we get the second equality in (10). □

As a corollary we get an expansion of ${}_{r+2}F_{r+1}$ with r positive integer parameter differences.

Corollary 1 *The following expansion holds:*

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, \mathbf{f}+\mathbf{m} \\ c, \mathbf{f} \end{matrix} \middle| x\right) = \frac{1}{(\mathbf{f})_{\mathbf{m}}} \sum_{k=0}^{m} (-1)^k D_k (b)_k {}_2F_1\left(\begin{matrix} a, b+k \\ c \end{matrix} \middle| x\right), \tag{12}$$

where $D_k = D_k(\mathbf{f}, \mathbf{m}, b)$ *is defined* (10).

Proof Indeed, by [9, (2)] and (9)

$$\frac{\Gamma(\mathbf{f}+\mathbf{m})}{\Gamma(\mathbf{f})} {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, \mathbf{f}+\mathbf{m} \\ c, \mathbf{f} \end{matrix} \middle| x\right) = \frac{\Gamma(c)}{\Gamma(b)} \int_0^1 G^{r+1,0}_{r+1,r+1}\left(t \middle| \begin{matrix} c, \mathbf{f} \\ b, \mathbf{f}+\mathbf{m} \end{matrix}\right) \frac{dt}{t(1-xt)^a}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{l=0}^{m} \alpha_l \sum_{k=0}^{l} \mathbf{S}_l^{(k)} (1-c+b)_k \int_0^1 \frac{t^{b+k-1}(1-t)^{c-b-k-1}}{(1-xt)^a} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{l=0}^{m} \alpha_l \sum_{k=0}^{l} \mathbf{S}_l^{(k)} (1-c+b)_k \frac{\Gamma(b+k)\Gamma(c-b-k)}{\Gamma(c)} {}_2F_1\left(\begin{matrix} a, b+k \\ c \end{matrix} \middle| x\right)$$

$$= \sum_{l=0}^{m} \alpha_l \sum_{k=0}^{l} \mathbf{S}_l^{(k)} (-1)^k (b)_k {}_2F_1\left(\begin{matrix} a, b+k \\ c \end{matrix} \middle| x\right) = \sum_{k=0}^{m} (-1)^k (b)_k {}_2F_1\left(\begin{matrix} a, b+k \\ c \end{matrix} \middle| x\right) \sum_{l=k}^{m} \mathbf{S}_l^{(k)} \alpha_l.$$

It remains to apply the first equality in (10). □

Using Corollary 1 we can easily recover the first Miller-Paris transformation (3) (see [22, (1.3)], [16, Theorem 1]) in the following form.

Theorem 1 *Let* $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$, $m = m_1 + m_2 + \ldots + m_r$, $\mathbf{f} = (f_1, \ldots, f_r) \in \mathbb{C}^r$ *and* $a, b, c \in \mathbb{C}$ *be such that* $(c-b-m)_m \neq 0$. *Then for all* $x \in \mathbb{C}\backslash[1, \infty)$ *we have:*

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, \mathbf{f}+\mathbf{m} \\ c, \mathbf{f} \end{matrix} \middle| x\right) = (1-x)^{-a} {}_{m+2}F_{m+1}\left(\begin{matrix} a, c-b-m, \boldsymbol{\zeta}+1 \\ c, \boldsymbol{\zeta} \end{matrix} \middle| \frac{x}{x-1}\right), \tag{13}$$

where $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_m)$ *are the roots of the polynomial*

$$P_m(x) = \frac{1}{(c-b-m)_m}\sum_{k=0}^{m}(b)_k(1-c+b)_k D_k(c-b-m-x)_{m-k}$$

with D_k *defined in* (10).

Proof Apply Euler-Pfaff transformation [1, (2.2.6)] to the Gauss function ${}_2F_1$ on the right hand side of (12) and calculate:

$$\frac{\Gamma(\mathbf{f}+\mathbf{m})}{\Gamma(\mathbf{f})}{}_{r+2}F_{r+1}\left(\begin{matrix}a, b, \mathbf{f}+\mathbf{m}\\ c, \mathbf{f}\end{matrix}\middle|\, x\right) = (1-x)^{-a}\sum_{l=0}^{m}(-1)^l D_l(b)_l\, {}_2F_1\left(\begin{matrix}a, c-b-l\\ c\end{matrix}\middle|\, \frac{x}{x-1}\right)$$

$$= (1-x)^{-a}\sum_{l=0}^{m}(-1)^l D_l(b)_l\sum_{j=0}^{\infty}\frac{(a)_j(c-b-l)_j}{(c)_j\, j!}\frac{x^j}{(x-1)^j}$$

$$= (1-x)^{-a}\sum_{j=0}^{\infty}\frac{(a)_j x^j}{(c)_j\, j!(x-1)^j}\sum_{l=0}^{m}(-1)^l D_l(b)_l(c-b-l)_j$$

$$= (1-x)^{-a}\sum_{j=0}^{\infty}\frac{(a)_j x^j}{(c)_j\, j!(x-1)^j}\sum_{l=0}^{m}(-1)^l D_l(b)_l\frac{(c-b-m)_j(c-b-m+j)_{m-l}}{(c-b-m)_{m-l}}$$

$$= (1-x)^{-a}\sum_{j=0}^{\infty}\frac{(a)_j(c-b-m)_j x^j}{(c)_j\, j!(x-1)^j}\tilde{P}_m(j)$$

$$= (1-x)^{-a}\sum_{j=0}^{\infty}\frac{(a)_j(c-b-m)_j x^j}{(c)_j\, j!(x-1)^j}A_m(j-\zeta_1)\cdots(j-\zeta_m)$$

$$= (1-x)^{-a}\sum_{j=0}^{\infty}\frac{(a)_j(c-b-m)_j x^j}{(c)_j\, j!(x-1)^j}A_m(-\zeta_1)(-\zeta_2)\cdots(-\zeta_m)\frac{(1-\zeta_1)_j\cdots(1-\zeta_m)_j}{(-\zeta_1)_j\cdots(-\zeta_m)_j},$$

$$= (1-x)^{-a}\tilde{P}_m(0)\, {}_{m+2}F_{m+1}\left(\begin{matrix}a, c-b-m, -\boldsymbol{\zeta}+1\\ c, -\boldsymbol{\zeta}\end{matrix}\middle|\, \frac{x}{x-1}\right),$$

where

$$\tilde{P}_m(x) = \sum_{l=0}^{m}(-1)^l D_l(b)_l\frac{(c-b-m+x)_{m-l}}{(c-b-m)_{m-l}} = A_m(x-\zeta_1)\cdots(x-\zeta_m),$$

and the identity

$$(c-b-l)_j = \frac{(c-b-m)_j(c-b-m+j)_{m-l}}{(c-b-m)_{m-l}}$$

has been used. It remains to define $P_m(x) = \tilde{P}_m(-x)$ and note that $\tilde{P}_m(0)$ must equal $(\mathbf{f})_\mathbf{m}$ by taking $x = 0$ in the resulting identity. □

By comparing the Miller-Paris formula (3) with (13) it is clear that $P_m(x)$ must be a constant multiple of $Q_m(x)$ defined in (5).

We give a direct proof of this fact below.

Lemma 2 *We have*

$$P_m(x) = (\mathbf{f})_{\mathbf{m}} Q(b, c, \mathbf{f}, \mathbf{m}; x).$$

Proof A combination of [13, Theorem 2.3] with [13, Theorem 3.2] gives the following identity for Q_m:

$$\frac{(\mathbf{f}-b)_{\mathbf{m}}(1-c+x)_m}{(\mathbf{f})_{\mathbf{m}}(1-c+b)_m} Q(1-c+b, 1-x+b, 1-\mathbf{f}+b-\mathbf{m}, \mathbf{m}; b) = Q(b, c, \mathbf{f}, \mathbf{m}; x).$$

Using this identity and definition (5) we obtain

$$P_m(x) = \frac{1}{(c-b-m)_m} \sum_{k=0}^{m} (b)_k (1-c+b)_k D_k (c-b-m-x)_{m-k}$$

$$= \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(c-b-m)_m} \sum_{k=0}^{m} (b)_k (1-c+b)_k \frac{(-1)^k}{k!} {}_{r+1}F_r\left(\begin{matrix}-k, 1-\mathbf{f}+b\\ 1-\mathbf{f}+b-\mathbf{m}\end{matrix}\right)(c-b-m-x)_{m-k}$$

$$= \frac{(\mathbf{f}-b)_{\mathbf{m}}(1-c+x)_m}{(1-c+b)_m} Q(1-c+b, 1-x+b, 1-\mathbf{f}+b-\mathbf{m}, \mathbf{m}; b) = (\mathbf{f})_{\mathbf{m}} Q(b, c, \mathbf{f}, \mathbf{m}; x). \quad \square$$

Our version of the second Miller-Paris transformation (4) (see [22, Theorem 4],[16, Theorem 1]) is the following theorem.

Theorem 2 *Let* $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$, $m = m_1 + m_2 + \ldots + m_r$, $\mathbf{f} = (f_1, \ldots, f_r) \in \mathbb{C}^r$ *and* $a, b, c \in \mathbb{C}$ *be such that* $(c-a-m)_m \neq 0$, $(c-b-m)_m \neq 0$, $(1+a+b-c)_m \neq 0$. *Then for all* $x \in \mathbb{C}\backslash[1, \infty)$ *we have:*

$${}_{r+2}F_{r+1}\left(\begin{matrix}a, b, \mathbf{f}+\mathbf{m}\\ c, \mathbf{f}\end{matrix}\middle| x\right) = (1-x)^{c-a-b-m} {}_{m+2}F_{m+1}\left(\begin{matrix}c-a-m, c-b-m, \boldsymbol{\eta}+1\\ c, \boldsymbol{\eta}\end{matrix}\middle| x\right), \tag{14}$$

where $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_m)$ *are the roots of the polynomial*

$$\hat{P}_m(t) = \sum_{k=0}^{m} \frac{(-1)^k (a)_k (-b-m)_k (t)_k (c-a-m-t)_{m-k}}{(c-a-m)_m (c-b-m)_k k!} {}_{r+2}F_{r+1}\left(\begin{matrix}-k, b, \mathbf{f}+\mathbf{m}\\ b+m-k+1, \mathbf{f}\end{matrix}\right). \tag{15}$$

The proof of this theorem will require the following two lemmas which might be of independent interest.

Lemma 3 *For any nonnegative integers* $0 \le i \le k \le m$ *the following summation formula holds*:

$$\sum_{j=i}^{k}(-k)_j(\alpha-m+j)_{m-i}\frac{(-j)_i}{j!}=(-1)^i(-k)_i\frac{(-m)_k(\alpha-m)_m}{(-m)_i(\alpha-m)_k}. \tag{16}$$

Proof Writing S for the left hand side of (16) and using the straightforward identities

$$(-j)_i=\frac{(-1)^i j!}{(j-i)!},\quad (\alpha-m+j)_{m-i}=\frac{(\alpha-i)_j(\alpha-m)_{m-i}}{(\alpha-m)_j},\quad (\beta)_{k+r}=(\beta)_k(\beta+k)_r,$$

we compute by changing the summation index to $n=j-i$:

$$S=(-1)^i(\alpha-m)_{m-i}\sum_{j=i}^{k}\frac{(-k)_j(\alpha-i)_j}{(\alpha-m)_j(j-i)!}=(-1)^i(\alpha-m)_{m-i}\sum_{n=0}^{k-i}\frac{(-k)_{n+i}(\alpha-i)_{n+i}}{(\alpha-m)_{n+i}n!}$$

$$=(-1)^i(\alpha-m)_{m-i}\sum_{n=0}^{k-i}\frac{(-k)_i(-k+i)_n(\alpha-i)_i(\alpha)_n}{(\alpha-m)_i(\alpha-m+i)_n n!}$$

$$=\frac{(-1)^i(-k)_i(\alpha-i)_i(\alpha-m)_{m-i}}{(\alpha-m)_i}\sum_{n=0}^{k-i}\frac{(-k+i)_n(\alpha)_n}{(\alpha-m+i)_n n!}$$

$$=\frac{(-1)^i(-k)_i(\alpha-i)_i(\alpha-m)_{m-i}}{(\alpha-m)_i}\frac{(-m+i)_{k-i}}{(\alpha-m+i)_{k-i}}=(-1)^i(-k)_i(i-m)_{k-i}\frac{(\alpha-m)_m}{(\alpha-m)_k}.$$

The pre-ultimate equality here is the celebrated Chu-Vandermonde identity [1, Corollary 2.2.3]. It remains to apply $(i-m)_{k-i}=(-m)_k/(-m)_i$. □

Lemma 4 *For any nonnegative integers* $0 \le k \le m$ *the following summation formula holds*:

$$\sum_{i=0}^{k}\frac{(-k)_i(b)_i}{(-m)_i i!}{}_{r+1}F_r\left(\begin{matrix}-i,\mathbf{f}+\mathbf{m}\\ \mathbf{f}\end{matrix}\right)=\frac{(-b-m)_k}{(-m)_k}{}_{r+2}F_{r+1}\left(\begin{matrix}-k,b,\mathbf{f}+\mathbf{m}\\ b+m-k+1,\mathbf{f}\end{matrix}\right). \tag{17}$$

Proof Let ϵ be a small positive number. According to (7) with $t=-k$, $c=b+m-k+1+\epsilon$,

$$\sum_{i=0}^{k}\frac{(-k)_i(b)_i}{(-m-\epsilon)_i i!}{}_{r+1}F_r\left(\begin{matrix}-i,\mathbf{f}+\mathbf{m}\\ \mathbf{f}\end{matrix}\right)=\frac{(1-k+\epsilon)_m}{(1+\epsilon)_m}Q_m(b;b+m-k+1+\epsilon;\mathbf{f};\mathbf{m};-k)$$

Further, by [13, Theorem 3.2],

$$\frac{(1-k+\epsilon)_m}{(1+\epsilon)_m}Q_m(b;b+m-k+1+\epsilon;\mathbf{f};\mathbf{m};-k)$$
$$=\frac{(1-k+\epsilon)_m}{(1+\epsilon)_m}\frac{\Gamma(b+m+1+\epsilon)\Gamma(1-k+\epsilon)}{\Gamma(b+m-k+1+\epsilon)\Gamma(1+\epsilon)}{}_{r+2}F_{r+1}\left(\begin{matrix}-k,b,\mathbf{f}+\mathbf{m}\\ b+m-k+1+\epsilon,\mathbf{f}\end{matrix}\right)$$
$$=\frac{\Gamma(1-k+\epsilon+m)}{\Gamma(1+\epsilon+m)}\frac{\Gamma(b+m+1+\epsilon)}{\Gamma(b+m-k+1+\epsilon)}{}_{r+2}F_{r+1}\left(\begin{matrix}-k,b,\mathbf{f}+\mathbf{m}\\ b+m-k+1+\epsilon,\mathbf{f}\end{matrix}\right).$$

Letting $\epsilon\to 0$ and using $m!/(m-k)!=(-1)^k(-m)_k$ and $\Gamma(b+m+1)/\Gamma(b+m-k+1)=(-1)^k(-b-m)_k$ we arrive at (17). □

Remark This lemma could also be derived from [21, Theorem 4] but as conditions of this theorem are violated here, some special treatment would still be needed. We also prefer to give an independent proof based entirely on our results.

Proof of Theorem 2 Applying (3) with a and b interchanged to the right hand side of the same formula we immediately get (14) with the characteristic polynomial given by

$$\hat{P}_m(t)=\frac{1}{(c-a-m)_m}\sum_{k=0}^{m}(a)_k(c-a-m-t)_{m-k}(t)_k\frac{(-1)^k}{k!}{}_{r+1}F_r\left(\begin{matrix}-k,\boldsymbol{\zeta}+1\\ \boldsymbol{\zeta}\end{matrix}\right),$$

where $\boldsymbol{\zeta}=\boldsymbol{\zeta}(b,c,\mathbf{f})=(\zeta_1,\ldots,\zeta_m)$ are the roots of the polynomial $Q_m(b,c,\mathbf{f};t)$ defined in (5). Next, we calculate using $Q_m(0)=1$ and (16):

$${}_{r+1}F_r\left(\begin{matrix}-k,\boldsymbol{\zeta}+1\\ \boldsymbol{\zeta}\end{matrix}\right)=1+\frac{(-k)(\boldsymbol{\zeta}+1)}{(\boldsymbol{\zeta})1!}+\cdots+\frac{(-k)_k(\boldsymbol{\zeta}+1)_k}{(\boldsymbol{\zeta})_k k!}$$
$$1+\frac{(-k)(\boldsymbol{\zeta}+1)}{(\boldsymbol{\zeta})1!}+\cdots+\frac{(-k)_k(\boldsymbol{\zeta}+k)}{(\boldsymbol{\zeta})k!}=1+\frac{(-k)Q_m(-1)}{Q_m(0)1!}+\cdots+\frac{(-k)_kQ_m(-k)}{Q_m(0)k!}$$
$$=\frac{1}{(c-b-m)_m}\sum_{j=0}^{k}\frac{(-k)_j}{j!}\sum_{i=0}^{m}(b)_i(-j)_i(c-b-m+j)_{m-i}\frac{(-1)^i}{i!}{}_{r+1}F_r\left(\begin{matrix}-i,\mathbf{f}+\mathbf{m}\\ \mathbf{f}\end{matrix}\right)$$
$$=\frac{1}{(c-b-m)_m}\sum_{i=0}^{k}\frac{(-1)^i(b)_i}{i!}{}_{r+1}F_r\left(\begin{matrix}-i,\mathbf{f}+\mathbf{m}\\ \mathbf{f}\end{matrix}\right)\sum_{j=i}^{k}\frac{(-k)_j}{j!}(-j)_i(c-b-m+j)_{m-i}$$
$$=\frac{(-m)_k}{(c-b-m)_k}\sum_{i=0}^{k}\frac{(-k)_i(b)_i}{(-m)_i i!}{}_{r+1}F_r\left(\begin{matrix}-i,\mathbf{f}+\mathbf{m}\\ \mathbf{f}\end{matrix}\right)=\frac{(-b-m)_k}{(c-b-m)_k}{}_{r+2}F_{r+1}\left(\begin{matrix}-k,b,\mathbf{f}+\mathbf{m}\\ b+m-k+1,\mathbf{f}\end{matrix}\right),$$

where we applied (17) in the last equality. Substituting this formula into the above expression for $\hat{P}_m$ we arrive at (15). □

Corollary 2 *The following identity is true:* $\hat{Q}_m(t)=\hat{P}_m(t)$*, where* $\hat{Q}_m$ *is defined in* (8) *and* $\hat{P}_m$ *is defined in* (15).

Proof Indeed, comparing (4) and (14) we see that $\hat{Q}_m(t)$ and $\hat{P}_m(t)$ have the same roots and thus may only differ by a nonzero multiplicative constant. However, it is straightforward that $\hat{Q}_m(0) = \hat{P}_m(0) = 1$ and the claim follows. □

Note that the identity $\hat{Q}_m(t) = \hat{P}_m(t)$ represents a non-trivial hypergeometric transformation which seems to be hard to obtain directly.

3 Miller-Paris Transformations: Degenerate Case

As we mentioned in the introduction and the statements of Theorems 1 and 2, formulas (13) and (14) fail when $c = b + p$, $p \in \{1, \ldots, m\}$. The purpose of this section is to present two transformations valid when $c = b + p$ with arbitrary $p \in \mathbb{N}$. Hence, they cover both degenerate and non-degenerate cases. Some of the coefficients appearing in these transformations can be expressed in terms of Nørlund's coefficients $g_n(\mathbf{a}; \mathbf{b})$ which were introduced by Nørlund in [25, (1.33)] and investigated in our papers [10, section 2.2], [9, Property 6] and [13, section 2]. For completeness we also give a short and slightly different account here. The functions $g_n(\mathbf{a}; \mathbf{b})$, $n \in \mathbb{N}_0$, are polynomials separately symmetric in the components of the vectors $\mathbf{a} = (a_1, \ldots, a_{p-1})$ and $\mathbf{b} = (b_1, \ldots, b_p)$. They can be defined either via the power series generating function [25, (1.33)], [9, (11)]

$$G^{p,0}_{p,p}\left(1 - z \middle| \begin{matrix} \mathbf{b} \\ \mathbf{a}, 0 \end{matrix}\right) = \frac{z^{\nu_p - 1}}{\Gamma(\nu_p)} \sum_{n=0}^{\infty} \frac{g_n(\mathbf{a}; \mathbf{b})}{(\nu_p)_n} z^n, \tag{18}$$

where $\nu_p = \nu_p(\mathbf{a}; \mathbf{b}) = \sum_{j=1}^{p} b_j - \sum_{j=1}^{p-1} a_j$, or via the inverse factorial generating function [25, (2.21)]

$$\frac{\Gamma(z + \nu_p)\Gamma(z + \mathbf{a})}{\Gamma(z + \mathbf{b})} = \sum_{n=0}^{\infty} \frac{g_n(\mathbf{a}; \mathbf{b})}{(z + \nu_p)_n}.$$

As, clearly, $\nu_p(\mathbf{a} + \alpha; \mathbf{b} + \alpha) = \nu_p(\mathbf{a}; \mathbf{b}) + \alpha$, we have (by changing $z \to z + \alpha$)

$$\frac{\Gamma(z + \alpha + \nu_p)\Gamma(z + \alpha + \mathbf{a})}{\Gamma(z + \alpha + \mathbf{b})} = \sum_{n=0}^{\infty} \frac{g_n(\mathbf{a} + \alpha; \mathbf{b} + \alpha)}{(z + \nu_p + \alpha)_n} = \sum_{n=0}^{\infty} \frac{g_n(\mathbf{a}; \mathbf{b})}{(z + \alpha + \nu_p)_n}.$$

Hence, $g_n(\mathbf{a} + \alpha; \mathbf{b} + \alpha) = g_n(\mathbf{a}; \mathbf{b})$ for any α. Nørlund found two different recurrence relations for $g_n(\mathbf{a}; \mathbf{b})$ (one in p and one in n). The simplest of them reads [25, (2.7)]

$$g_n(\mathbf{a}, \alpha; \mathbf{b}, \beta) = \sum_{s=0}^{n} \frac{(\beta - \alpha)_{n-s}}{(n - s)!} (\nu_p - \alpha + s)_{n-s} g_s(\mathbf{a}; \mathbf{b}), \quad p = 1, 2, \ldots, \tag{19}$$

with the initial values $g_0(-;b_1)=1$, $g_n(-;b_1)=0$, $n\geq 1$. This recurrence was solved by Nørlund [25, (2.11)] as follows:

$$g_n(\mathbf{a};\mathbf{b})=\sum_{0\leq j_1\leq j_2\leq\cdots\leq j_{p-2}\leq n}\prod_{m=1}^{p-1}\frac{(\psi_m+j_{m-1})_{j_m-j_{m-1}}}{(j_m-j_{m-1})!}(b_{m+1}-a_m)_{j_m-j_{m-1}}, \tag{20}$$

where $\psi_m=\sum_{i=1}^{m}(b_i-a_i)$, $j_0=0$, $j_{p-1}=n$. Another recurrence relation for $g_n(\mathbf{a};\mathbf{b})$ discovered by Nørlund [25, (1.28)] has order p in the variable n and coefficients polynomial in n. Details can be found in [10, section 2.2]. The first three coefficients are given by [10, Theorem 2]:

$$g_0(\mathbf{a};\mathbf{b})=1,\quad g_1(\mathbf{a};\mathbf{b})=\sum_{m=1}^{p-1}(b_{m+1}-a_m)\psi_m,$$

$$g_2(\mathbf{a};\mathbf{b})=\frac{1}{2}\sum_{m=1}^{p-1}(b_{m+1}-a_m)_2(\psi_m)_2+\sum_{k=2}^{p-1}(b_{k+1}-a_k)(\psi_k+1)\sum_{m=1}^{k-1}(b_{m+1}-a_m)\psi_m.$$

For $p=2$ and $p=3$ and arbitrary $n\in\mathbb{N}_0$ explicit expressions for $g_n(\mathbf{a};\mathbf{b})$ discovered by Nørlund [25, eq.(2.10)] are:

$$\begin{aligned} g_n(a;\mathbf{b})&=\frac{(b_1-a)_n(b_2-a)_n}{n!}\quad\text{for } p=2;\\ g_n(\mathbf{a};\mathbf{b})&=\frac{(\nu_3-b_2)_n(\nu_3-b_3)_n}{n!}{}_3F_2\left(\begin{matrix}-n,b_1-a_1,b_1-a_2\\ \nu_3-b_2,\nu_3-b_3\end{matrix}\right)\quad\text{for } p=3,\end{aligned} \tag{21}$$

where $\nu_m:=\sum_{j=1}^{m}b_j-\sum_{j=1}^{m-1}a_j$. The right hand side here is invariant with respect to the permutation of the elements of $\mathbf{b}$. Finally, for $p=4$ we have [10, p.12]

$$g_n(\mathbf{a};\mathbf{b})=\frac{(\nu_4-b_3)_n(\nu_4-b_4)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(\nu_2-a_2)_k(\nu_2-a_3)_k}{(\nu_4-b_3)_k(\nu_4-b_4)_k}{}_3F_2\left(\begin{matrix}-k,b_1-a_1,b_2-a_1\\ \nu_2-a_2,\nu_2-a_3\end{matrix}\right).$$

Next, define $W_{m-1}(n)=W_{m-1}(b,\mathbf{f},\mathbf{m};n)=\sum_{k=0}^{m-1}\delta_k n^k$ to be the polynomial of degree $m-1$ given by

$$W_{m-1}(n)=W_{m-1}(b,\mathbf{f},\mathbf{m};n)=\left(\frac{(\mathbf{f}+\mathbf{m})_n}{(\mathbf{f})_n}-\frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}\right)\frac{(b)_n}{(b+1)_n}=\frac{b((\mathbf{f}+n)_{\mathbf{m}}-(\mathbf{f}-b)_{\mathbf{m}})}{(b+n)(\mathbf{f})_{\mathbf{m}}}. \tag{22}$$

The following theorem gives two extensions of the Karlsson's formula (2) to arbitrary argument.

Theorem 3 *The following transformation formulas hold:*

$$(1-x)^a{}_{r+2}F_{r+1}\left(\begin{matrix}a, b, \mathbf{f}+\mathbf{m}\\ b+1, \mathbf{f}\end{matrix}\middle| x\right) = \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}{}_2F_1\left(\begin{matrix}1, a\\ b+1\end{matrix}\middle| \frac{x}{x-1}\right) + \sum_{l=0}^{m-1} Y_l \frac{(a)_l x^l}{(1-x)^l} \tag{23}$$

and

$$(1-x)^{a-1}{}_{r+2}F_{r+1}\left(\begin{matrix}a, b, \mathbf{f}+\mathbf{m}\\ b+1, \mathbf{f}\end{matrix}\middle| x\right) = \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}{}_2F_1\left(\begin{matrix}1, b+1-a\\ b+1\end{matrix}\middle| x\right) + \sum_{l=0}^{m-1} Y_l \frac{(a)_l x^l}{(1-x)^{l+1}}. \tag{24}$$

Here

$$Y_l = Y_l(b, \mathbf{f}, \mathbf{m}) = \sum_{k=l}^{m-1} \delta_k \mathbf{S}_k^{(l)} = \frac{(-1)^l}{l!}{}_{r+2}F_{r+1}\left(\begin{matrix}-l, b, \mathbf{f}+\mathbf{m}\\ b+1, \mathbf{f}\end{matrix}\right) - \frac{(-1)^l(\mathbf{f}-b)_{\mathbf{m}}}{(b+1)_l(\mathbf{f})_{\mathbf{m}}}$$

$$= \frac{(-1)^{m-l-1}b}{(\mathbf{f})_{\mathbf{m}}} \sum_{i=0}^{m-1-l} (-1)^i g_{m-1-l-i}(-\mathbf{f}; -\mathbf{f}-\mathbf{m}, l)(1-b)_i, \tag{25}$$

where δ_k are the coefficients of the polynomial $W_{m-1}(x)$ defined in (22) and $g_n(\cdot;\cdot)$ are Nørlund's coefficients given in (20).

Proof First, we show that for $|x| < 1$ the following equality holds:

$${}_{r+2}F_{r+1}\left(\begin{matrix}a, b, \mathbf{f}+\mathbf{m}\\ b+1, \mathbf{f}\end{matrix}\middle| x\right) = \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}{}_2F_1\left(\begin{matrix}a, b\\ b+1\end{matrix}\middle| x\right) + \sum_{l=0}^{m-1} \frac{(a)_l x^l}{(1-x)^{a+l}} \sum_{k=l}^{m-1} \delta_k \mathbf{S}_k^{(l)}. \tag{26}$$

Indeed, in view of (22),

$${}_{r+2}F_{r+1}\left(\begin{matrix}a, b, \mathbf{f}+\mathbf{m}\\ b+1, \mathbf{f}\end{matrix}\middle| x\right) = \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{n!(b+1)_n} + \sum_{n=0}^{\infty} \frac{(a)_n x^n W_{m-1}(n)}{n!}$$

$$= \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}{}_2F_1\left(\begin{matrix}a, b\\ b+1\end{matrix}\middle| x\right) + \sum_{k=0}^{m-1} \delta_k \sum_{n=0}^{\infty} \frac{(a)_n x^n}{n!} n^k.$$

Using the definition of the Stirling numbers $n^k = \sum_{l=0}^{k} \mathbf{S}_k^{(l)}[n]_l$ in terms of falling factorials $[n]_l = n(n-1)\ldots(n-l+1)$, we get

$$\sum_{n=0}^{\infty} \frac{(a)_n x^n}{n!} n^k = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{n!} \sum_{l=0}^{k} \mathbf{S}_k^{(l)}[n]_l = \sum_{l=0}^{k} \mathbf{S}_k^{(l)} \sum_{n=l}^{\infty} \frac{(a)_n x^n}{(n-l)!}$$

$$= \sum_{l=0}^{k} \mathbf{S}_k^{(l)} \sum_{n=0}^{\infty} \frac{(a)_{n+l} x^{n+l}}{n!} = \sum_{l=0}^{k} \mathbf{S}_k^{(l)} \sum_{n=0}^{\infty} \frac{(a)_l (a+l)_n x^{n+l}}{n!} = \sum_{l=0}^{k} \mathbf{S}_k^{(l)} (a)_l \frac{x^l}{(1-x)^{a+l}},$$

which implies (26) after exchanging the order of summations. It remains to apply Euler's transformations to the ${}_2F_1$ on the right hand side of (26) to get (23) and (24) with Y_l given by the first formula in (25).

To obtain the second expression for Y_l recall that

$$W_{m-1}(x) = \frac{b[(\mathbf{f}+x)_{\mathbf{m}} - (\mathbf{f}-b)_{\mathbf{m}}]}{(b+x)(\mathbf{f})_{\mathbf{m}}} = \sum_{k=0}^{m-1} \frac{W_{m-1}^{(k)}(0)}{k!} x^k,$$

yielding $\delta_k = W_{m-1}^{(k)}(0)/k!$. Using the technique from [21, Theorem 2] we now apply the following formula from the theory of finite differences:

$$\Delta^l W_{m-1}(x) = \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} W_{m-1}(x+j) = l! \sum_{j=l}^{m-1} \mathbf{S}_j^{(k)} \frac{W_{m-1}^{(j)}(x)}{j!}.$$

In view of (22) we have

$$\frac{1}{l!} \Delta^l W_{m-1}(0) = \sum_{k=l}^{m-1} \delta_k \mathbf{S}_k^{(l)} = \frac{1}{l!} \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} W_{m-1}(j)$$

$$= \frac{1}{l!} \sum_{j=0}^{l} \frac{(-1)^{l-j} (b)_j}{(b+1)_j} \binom{l}{j} \left[\frac{(\mathbf{f}+\mathbf{m})_j}{(\mathbf{f})_j} - \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}} \right]$$

$$= \frac{(-1)^l}{l!} \sum_{j=0}^{l} \frac{(-l)_j (b)_j}{(b+1)_j j!} \left[\frac{(\mathbf{f}+\mathbf{m})_j}{(\mathbf{f})_j} - \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}} \right]$$

$$= \frac{(-1)^l}{l!} {}_{r+2}F_{r+1}\left(\begin{matrix} -l, b, \mathbf{f}+\mathbf{m} \\ b+1, \mathbf{f} \end{matrix} \right) - \frac{(-1)^l}{l!} \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}} {}_2F_1\left(\begin{matrix} -l, b \\ b+1 \end{matrix} \right)$$

$$= \frac{(-1)^l}{l!} {}_{r+2}F_{r+1}\left(\begin{matrix} -l, b, \mathbf{f}+\mathbf{m} \\ b+1, \mathbf{f} \end{matrix} \right) - \frac{(-1)^l (\mathbf{f}-b)_{\mathbf{m}}}{(b+1)_l (\mathbf{f})_{\mathbf{m}}},$$

where we employed the relation

$$\binom{l}{j} = (-1)^j \frac{(-l)_j}{j!}$$

in the second line and the Chu-Vandermonde identity in the last equality. Further, according to [13, Theorem 2.1]

$$ {}_{r+2}F_{r+1}\left(\begin{matrix}-l, b, \mathbf{f}+\mathbf{m}\\ b+1, \mathbf{f}\end{matrix}\right) = \frac{l!}{(b+1)_l}\frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}} - \frac{(-1)^m l! b}{(\mathbf{f})_{\mathbf{m}}} q_l, $$

where $q_l = \sum_{i=0}^{m-l-1} g_{m-l-i-1}(b-\mathbf{f}; b-\mathbf{f}-\mathbf{m}, b+l)(b-i)_i$, and Nørlund's coefficient $g_n(\cdot;\cdot)$ is defined in (20). Substituting and using the shifting property $g_n(\mathbf{a}+\alpha; \mathbf{b}+\alpha) = g_n(\mathbf{a}; \mathbf{b})$ of Nørlund's coefficients, we get:

$$ \sum_{k=l}^{m-1} \delta_k \mathbf{S}_k^{(l)} = \frac{(-1)^{m-l-1} b}{(\mathbf{f})_{\mathbf{m}}} \sum_{i=0}^{m-l-1} g_{m-l-i-1}(-\mathbf{f}; -\mathbf{f}-\mathbf{m}, l)(b-i)_i $$

which is equivalent to the second formula in (25). □

Remark If $\Re(1-a-m) > 0$ an application of the Gauss summation formula to ${}_2F_1$ on the right hand side of (26) results in Karlsson's formula (2).

Theorem 3 can be generalized as follows. Suppose $\mathbf{b} = (b_1, \dots, b_l)$ is a complex vector, $\mathbf{p} = (p_1, \dots, p_l)$ is a vector of positive integers, $p = p_1 + p_2 + \dots + p_l$, and all elements of the vector $\boldsymbol{\beta} = (b_1, b_1+1, \dots, b_1+p_1-1, \dots, b_l, b_l+1, \dots, b_l+p_l-1) = (\beta_1, \beta_2, \dots, \beta_p)$ are distinct. It is straightforward to verify the partial fraction decomposition

$$ \prod_{j=1}^{p} \frac{1}{\beta_j + x} = \frac{1}{(\boldsymbol{\beta}+x)_1} = \sum_{q=1}^{p} \frac{1}{B_q(\beta_q + x)}, \quad \text{where } B_q = \prod_{\substack{v=1\\ v\neq q}}^{p} (\beta_v - \beta_q). $$

Then

$$ \frac{(\mathbf{b})_n}{(\mathbf{b}+\mathbf{p})_n} = \frac{(\mathbf{b})_{\mathbf{p}}}{(\mathbf{b}+n)_{\mathbf{p}}} = \frac{(\mathbf{b})_{\mathbf{p}}}{(\boldsymbol{\beta}+n)_1} = (\mathbf{b})_{\mathbf{p}} \sum_{q=1}^{p} \frac{(\beta_q)_n}{\beta_q B_q (\beta_q+1)_n}. $$

Applying the definition of the generalized hypergeometric function and Theorem 3, we obtain

$$ (1-x)^a {}_{r+p+2}F_{r+p+1}\left(\left.\begin{matrix}a, \mathbf{b}, \mathbf{f}+\mathbf{m}\\ \mathbf{b}+\mathbf{p}, \mathbf{f}\end{matrix}\right| x\right) = (1-x)^a \sum_{q=1}^{p} \frac{(\mathbf{b})_{\mathbf{p}}}{\beta_q B_q} {}_{r+2}F_{r+1}\left(\left.\begin{matrix}a, \beta_q, \mathbf{f}+\mathbf{m}\\ \beta_q+1, \mathbf{f}\end{matrix}\right| x\right) $$

$$ = \frac{(\mathbf{b})_{\mathbf{p}}}{(\mathbf{f})_{\mathbf{m}}} \sum_{q=1}^{p} \frac{(\mathbf{f}-\beta_q)_{\mathbf{m}}}{\beta_q B_q} {}_2F_1\left(\left.\begin{matrix}a, 1\\ \beta_q+1\end{matrix}\right| \frac{x}{x-1}\right) + (\mathbf{b})_{\mathbf{p}} \sum_{q=1}^{p} \frac{1}{\beta_q B_q} \sum_{k=0}^{m-1} \delta_{kq} \sum_{l=0}^{k} \mathbf{S}_k^{(l)} \frac{(a)_l x^l}{(1-x)^l} $$

$$ = \frac{(\mathbf{b})_{\mathbf{p}}}{(\mathbf{f})_{\mathbf{m}}} \sum_{q=1}^{p} \frac{(\mathbf{f}-\beta_q)_{\mathbf{m}}}{\beta_q B_q} {}_2F_1\left(\left.\begin{matrix}a, 1\\ \beta_q+1\end{matrix}\right| \frac{x}{x-1}\right) + (\mathbf{b})_{\mathbf{p}} \sum_{q=1}^{p} \frac{1}{\beta_q B_q} \sum_{l=0}^{m-1} Y_l(\beta_q, \mathbf{f}, \mathbf{m}) \frac{(a)_l x^l}{(1-x)^l}, \tag{27} $$

where δ_{kq} are the coefficients of the polynomial $W_{m-1}(\beta_q, \mathbf{f}, \mathbf{m}; x) = \sum_{k=0}^{m-1} \delta_{kq} x^k$, Y_l is defined in (25) and $\beta_q = b+q-1$ is q-th component of the vector $\boldsymbol{\beta}$. Similarly, applying the second transformation yields:

$$(1-x)^{a-1}{}_{r+p+2}F_{r+p+1}\left(\begin{matrix}a, \mathbf{b}, \mathbf{f}+\mathbf{m}\\ \mathbf{b}+\mathbf{p}, \mathbf{f}\end{matrix}\middle| x\right) = (1-x)^{a-1}\sum_{q=1}^{p}\frac{(\mathbf{b})_\mathbf{p}}{\beta_q B_q}{}_{r+2}F_{r+1}\left(\begin{matrix}a, \beta_q, \mathbf{f}+\mathbf{m}\\ \beta_q+1, \mathbf{f}\end{matrix}\middle| x\right)$$
$$= \frac{(\mathbf{b})_\mathbf{p}}{(\mathbf{f})_\mathbf{m}}\sum_{q=1}^{p}\frac{(\mathbf{f}-\beta_q)_\mathbf{m}}{\beta_q B_q}{}_2F_1\left(\begin{matrix}1, \beta_q+1-a\\ \beta_q+1\end{matrix}\middle| x\right) + \sum_{q=1}^{p}\frac{(\mathbf{b})_\mathbf{p}}{\beta_q B_q}\sum_{l=0}^{m-1} Y_l(\beta_q, \mathbf{f}, \mathbf{m})\frac{(a)_l x^l}{(1-x)^{l+1}}. \tag{28}$$

In both formulas the sum of the Gauss functions ${}_2F_1$ does not seem to collapse into a single hypergeometric function. However, it does happen when **b** only contains one component. We formulate this result in the form of the following theorem.

Theorem 4 *Suppose $p \in \mathbb{N}$. Then the following identity hold true*:

$$(1-x)^a{}_{r+2}F_{r+1}\left(\begin{matrix}a, b, \mathbf{f}+\mathbf{m}\\ b+p, \mathbf{f}\end{matrix}\middle| x\right) = \frac{T_{p-1}(0)}{\Gamma(b)(\mathbf{f})_\mathbf{m}}{}_{p+1}F_p\left(\begin{matrix}a, 1, -\boldsymbol{\lambda}+1\\ b+p, -\boldsymbol{\lambda}\end{matrix}\middle| \frac{x}{x-1}\right)$$
$$+\sum_{q=1}^{p}\frac{(-1)^{q-1}(b)_p}{(b+q-1)(q-1)!(p-q)!}\sum_{l=0}^{m-1} Y_l(b+q-1, \mathbf{f}, \mathbf{m})\frac{(a)_l x^l}{(1-x)^l}, \tag{29}$$

where Y_l is defined in (25), $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{p-1})$ *are the roots of the polynomial*

$$T_{p-1}(z) = \sum_{q=1}^{p}\frac{(-1)^{q-1}(\mathbf{f}-b-q+1)_\mathbf{m}\Gamma(b+q-1)}{(q-1)!(p-q)!}(b+q+z)_{p-q} \tag{30}$$

of degree $p-1$. Furthermore,

$$(1-x)^{a-1}{}_{r+2}F_{r+1}\left(\begin{matrix}a, b, \mathbf{f}+\mathbf{m}\\ b+p, \mathbf{f}\end{matrix}\middle| x\right) = \frac{\Gamma(b-a+1)T^*_{p-1}(0)}{\Gamma(b)(\mathbf{f})_\mathbf{m}}{}_{p+1}F_p\left(\begin{matrix}1, b+1-a, -\boldsymbol{\lambda}^*+1\\ b+p, -\boldsymbol{\lambda}^*\end{matrix}\middle| x\right)$$
$$+\sum_{q=1}^{p}\frac{(-1)^{q-1}(b)_p}{(b+q-1)(q-1)!(p-q)!}\sum_{l=0}^{m-1} Y_l(b+q-1, \mathbf{f}, \mathbf{m})\frac{(a)_l x^l}{(1-x)^{l+1}}, \tag{31}$$

where $\boldsymbol{\lambda}^* = (\lambda^*_1, \ldots, \lambda^*_{p-1})$ are the roots of the polynomial

$$T^*_{p-1}(z) = \sum_{q=1}^{p}\frac{(-1)^{q-1}(\mathbf{f}-b-q+1)_\mathbf{m}\Gamma(b+q-1)}{\Gamma(b+q-a)(q-1)!(p-q)!}(b+q+z)_{p-q}(b+1-a+z)_{q-1}$$

of degree $p-1$.

Remark It is instructive to compare identities (29) and (31) with the degenerate Miller-Paris transformations derived in [12, Theorems 1 and 3]. One important difference is that the above theorem holds for any $p \in \mathbb{N}$, while in [12] p is restricted to the set $\{1, \ldots, m\}$. Nevertheless, with a little effort one can make sure that (29) and [12, (16)] are related by a rather simple rearrangement and the polynomial T_{p-1} is a constant multiple of the polynomial R_p from [12]. The same, however, does not hold for (31), and the polynomial T^*_{p-1} is not a constant multiple of $\hat{R}_p$ from [12].

Proof If $\mathbf{b} = (b)$ we can represent the sum of hypergeometric functions in (27) as a single hypergeometric function of a higher order as follows. Using the definition of the hypergeometric function

$$\sum_{q=1}^{p} \frac{(\mathbf{f}-\beta_q)_{\mathbf{m}}}{\beta_q B_q} {}_2F_1\left(\begin{matrix} a, 1 \\ \beta_q + 1 \end{matrix}\middle| t\right) = \sum_{n=0}^{\infty} (a)_n t^n \sum_{q=1}^{p} \frac{(\mathbf{f}-\beta_q)_{\mathbf{m}}}{\beta_q B_q (b+q)_n}. \tag{32}$$

Applying the formula

$$(b+q)_n = \frac{\Gamma(b+p)(b+p)_n}{\Gamma(b+q)(b+n+q)_{p-q}}, \tag{33}$$

we get

$$\sum_{n=0}^{\infty} (a)_n t^n \sum_{q=1}^{p} \frac{(\mathbf{f}-\beta_q)_{\mathbf{m}}}{\beta_q B_q (b+q)_n} = \frac{1}{\Gamma(b+p)} \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(b+p)_n} \sum_{q=1}^{p} \frac{(\mathbf{f}-\beta_q)_{\mathbf{m}}}{\beta_q B_q} \Gamma(b+q)(b+n+q)_{p-q}$$
$$= \frac{1}{\Gamma(b+p)} \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(b+p)_n} T_{p-1}(n), \tag{34}$$

where $T_{p-1}(n)$ is the polynomial of degree $p-1$ defined in (30) in view of $\boldsymbol{\beta} = (b, \ldots, b+p-1)$, $\beta_q = b+q-1$ and $B_q = (-1)^{q-1}(q-1)!(p-q)!$. Setting $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{p-1})$ to be the roots of this polynomial, we can write

$$T_{p-1}(n) = \frac{\Gamma(b+1)}{b(p-1)!} (\mathbf{f}-b)_{\mathbf{m}} (n-\boldsymbol{\lambda})_1 = \frac{\Gamma(b)}{(p-1)!} (\mathbf{f}-b)_{\mathbf{m}} (-\boldsymbol{\lambda})_1 \frac{(-\boldsymbol{\lambda}+1)_n}{(-\boldsymbol{\lambda})_n}.$$

Hence, it follows from (34) that

$$\sum_{n=0}^{\infty} (a)_n t^n \sum_{q=1}^{p} \frac{(\mathbf{f}-b-q+1)_{\mathbf{m}}}{\beta_q B_q (b+q)_n} = \frac{\Gamma(b)}{\Gamma(b+p)} \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(b+p)_n} \frac{(\mathbf{f}-b)_{\mathbf{m}} (-\boldsymbol{\lambda})_1 (-\boldsymbol{\lambda}+1)_n}{(-\boldsymbol{\lambda})_n (p-1)!}$$
$$= \frac{(\mathbf{f}-b)_{\mathbf{m}} (-\boldsymbol{\lambda})_1}{(b)_p (p-1)!} {}_{p+1}F_p\left(\begin{matrix} a, 1, -\boldsymbol{\lambda}+1 \\ b+p, -\boldsymbol{\lambda} \end{matrix}\middle| t\right). \tag{35}$$

Substituting this result into (27) with $t = x/(x-1)$ yields (29).

The proof of the second transformation is similar. We transform the first term in (28) using formula (33) and the identity

$$(b+n+q-a)_{p-q}=\frac{(b-a+n+1)_{p-1}}{(b-a+n+1)_{q-1}}$$

as follows (keeping in mind that $\beta_q=b+q-1$, $B_q=(-1)^{q-1}(q-1)!(p-q)!$):

$$\frac{(b)_p}{(\mathbf{f})_\mathbf{m}}\sum_{q=1}^{p}\frac{(\mathbf{f}-\beta_q)_\mathbf{m}}{\beta_q B_q}{}_2F_1\left(\left.\begin{matrix}\beta_q-a+1,1\\ \beta_q+1\end{matrix}\right|x\right)=\frac{(b)_\mathbf{p}}{(\mathbf{f})_\mathbf{m}}\sum_{n=0}^{\infty}x^n\sum_{q=1}^{p}\frac{(\mathbf{f}-\beta_q)_\mathbf{m}}{\beta_q B_q}\frac{(b+q-a)_n}{(b+q)_n}$$

$$=\frac{(b)_p}{(\mathbf{f})_\mathbf{m}}\sum_{n=0}^{\infty}x^n\sum_{q=1}^{p}\frac{(\mathbf{f}-\beta_q)_\mathbf{m}}{\beta_q B_q}\frac{\Gamma(b+q)(b+n+q)_{p-q}\Gamma(b+p-a)(b+p-a)_n}{\Gamma(b+p)(b+p)_n\Gamma(b+q-a)(b+n+q-a)_{p-q}}$$

$$=\frac{\Gamma(b+p-a)}{\Gamma(b)(\mathbf{f})_\mathbf{m}}\sum_{n=0}^{\infty}\frac{(b+p-a)_n x^n}{(b+p)_n}\sum_{q=1}^{p}\frac{(\mathbf{f}-\beta_q)_\mathbf{m}}{\beta_q B_q}\frac{\Gamma(b+q)(b+n+q)_{p-q}}{\Gamma(b+q-a)(b+n+q-a)_{p-q}}$$

$$=\frac{\Gamma(b+p-a)}{\Gamma(b)(\mathbf{f})_\mathbf{m}}\sum_{n=0}^{\infty}\frac{(b+p-a)_n x^n}{(b+p)_n(b+n+1-a)_{p-1}}T^*_{p-1}(n)$$

$$=\frac{1}{\Gamma(b)(\mathbf{f})_\mathbf{m}}\sum_{n=0}^{\infty}\frac{\Gamma(b+n-a+1)x^n}{(b+p)_n}T^*_{p-1}(n)=\frac{\Gamma(b-a+1)}{\Gamma(b)(\mathbf{f})_\mathbf{m}}\sum_{n=0}^{\infty}\frac{(b-a+1)_n x^n}{(b+p)_n}T^*_{p-1}(n), \tag{36}$$

where

$$T^*_{p-1}(z)=\sum_{q=1}^{p}\frac{(-1)^{q-1}\Gamma(b+q-1)(\mathbf{f}-b-q+1)_\mathbf{m}}{\Gamma(b+q-a)(q-1)!(p-q)!}(b+z+q)_{p-q}(b+z+1-a)_{q-1}$$

is a polynomial of degree $p-1$. Setting $\boldsymbol{\lambda}^*=(\lambda_1^*,\ldots,\lambda_{p-1}^*)$ to be the roots of this polynomial, we have

$$T^*_{p-1}(n)=(-\boldsymbol{\lambda}^*)_1\frac{(-\boldsymbol{\lambda}^*+1)_n}{(-\boldsymbol{\lambda}^*)_n}\sum_{q=1}^{p}(-1)^{q-1}\frac{\Gamma(b+q-1)(\mathbf{f}-b-q+1)_\mathbf{m}}{\Gamma(b+q-a)(q-1)!(p-q)!}.$$

Substituting this expression into (36), we get (31). □

The remark made after Theorem 3 implies that $p=2$ case of (29) is the same (modulo some rearrangement) as [12, Corollary 5]. Setting $p=2$ in (31) we obtain the following

Corollary 3 *Suppose* $(b+1)(\mathbf{f}-b-1+\mathbf{m})\neq b(\mathbf{f}-b-1)$ *and* $(b-a+1)(\mathbf{f}-b-1+\mathbf{m})\neq b(\mathbf{f}-b-1)$. *Then the following identity holds:*

$$(1-x)^{a-1}{}_{r+2}F_{r+1}\left(\begin{matrix}a, b, \mathbf{f}+\mathbf{m}\\ b+2, \mathbf{f}\end{matrix}\middle| x\right)=\frac{b\,[(\mathbf{f}-b)_{\mathbf{m}}-(\mathbf{f}-b-1)_{\mathbf{m}}]}{(\mathbf{f})_{\mathbf{m}}}{}_3F_2\left(\begin{matrix}1, b+1-a, \lambda^*+1\\ b+2, \lambda^*\end{matrix}\middle| x\right)$$
$$+\sum_{l=0}^{m-1}[(b+1)Y_l(b, \mathbf{f}, \mathbf{m})-bY_l(b+1, \mathbf{f}, \mathbf{m})]\frac{(a)_l x^l}{(1-x)^{l+1}},$$

where

$$\lambda^*=(b-a+1)\frac{(b+1)(\mathbf{f}-b-1+\mathbf{m})-b(\mathbf{f}-b-1)}{(b-a+1)(\mathbf{f}-b-1+\mathbf{m})-b(\mathbf{f}-b-1)}$$

and Y_l is defined in (25).

The following theorem extends Theorem 3 in a different direction: we add two free parameters to ${}_{r+2}F_{r+1}$ on the left hand side.

Theorem 5 *Suppose $(e-d-m+1)_{m-1}\neq 0$. Then following identity holds:*

$$ {}_{r+3}F_{r+2}\left(\begin{matrix}a, d, b, \mathbf{f}+\mathbf{m}\\ e, b+1, \mathbf{f}\end{matrix}\middle| x\right)=\frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}{}_3F_2\left(\begin{matrix}a, d, b\\ e, b+1\end{matrix}\middle| x\right)$$
$$+\frac{(\mathbf{f})_{\mathbf{m}}-(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}(1-x)^{-a}{}_{m+1}F_m\left(\begin{matrix}a, e-d-m+1, \boldsymbol{\lambda}+1\\ e, \boldsymbol{\lambda}\end{matrix}\middle| \frac{x}{x-1}\right),$$

where $\boldsymbol{\lambda}$ is the vector of zeros of the polynomial

$$L_{m-1}(t)=L_{m-1}(e, d, b, c, \mathbf{f}, \mathbf{m}; t)=\sum_{k=0}^{m-1}(d)_k Y_k(b, \mathbf{f}, \mathbf{m})(t)_k(e-d-m+1-t)_{m-1-k}, \tag{37}$$

and $Y_k(b, \mathbf{f}, \mathbf{m})$ is given in (25). *If, in addition $(e-a-m+1)_{m-1}\neq 0$ and $(1+a+d-e)_{m-1}\neq 0$, then*

$$ {}_{r+3}F_{r+2}\left(\begin{matrix}a, d, b, \mathbf{f}+\mathbf{m}\\ e, b+1, \mathbf{f}\end{matrix}\middle| x\right)=\frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}{}_3F_2\left(\begin{matrix}a, d, b\\ e, b+1\end{matrix}\middle| x\right)$$
$$+\frac{(\mathbf{f})_{\mathbf{m}}-(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}(1-x)^{e-a-d-m+1}{}_{m+1}F_m\left(\begin{matrix}e-a-m+1, e-d-m+1, \boldsymbol{\lambda}^*+1\\ e, \boldsymbol{\lambda}^*\end{matrix}\middle| x\right),$$

where $\boldsymbol{\lambda}^*$ is the vector of zeros of the polynomial

$$\hat{L}_{m-1}(t)=\sum_{k=0}^{m-1}\frac{(-1)^k Y_k(b, \mathbf{f}, \mathbf{m})(a)_k(d)_k(t)_k}{(e-a-m+1)_k(e-d-m+1)_k}{}_3F_2\left(\begin{matrix}-m+1+k, t+k, e-a-d-m+1\\ e-a-m+1+k, e-d-m+1+k\end{matrix}\right). \tag{38}$$

Proof Let $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{m-1})$ be the roots of the polynomial $W_{m-1}(x) = W_{m-1}(b, \mathbf{f}, \mathbf{m}; x)$ defined in (22). Its definition implies that the leading coefficient of $W_{m-1}(x)$ equals $b/(\mathbf{f})_m$, while the free term is given by $W_{m-1}(0) = ((\mathbf{f})_{\mathbf{m}} - (\mathbf{f} - b)_{\mathbf{m}})/(\mathbf{f})_{\mathbf{m}}$. Hence,

$$W_{m-1}(n) = \frac{b(-\boldsymbol{\gamma})_1(n-\boldsymbol{\gamma})_1}{(\mathbf{f})_m(-\boldsymbol{\gamma})_1} = \frac{b(-\boldsymbol{\gamma})_1}{(\mathbf{f})_m}\frac{(-\boldsymbol{\gamma}+1)_n}{(-\boldsymbol{\gamma})_n}$$
$$= W_{m-1}(0)\frac{(-\boldsymbol{\gamma}+1)_n}{(-\boldsymbol{\gamma})_n} = \frac{((\mathbf{f})_{\mathbf{m}} - (\mathbf{f}-b)_{\mathbf{m}})(-\boldsymbol{\gamma}+1)_n}{(\mathbf{f})_{\mathbf{m}}(-\boldsymbol{\gamma})_n}. \tag{39}$$

By definition of the generalized hypergeometric function this leads to

$$ {}_{r+3}F_{r+2}\left(\begin{matrix}a, d, b, \mathbf{f}+\mathbf{m}\\ e, b+1, \mathbf{f}\end{matrix}\middle| x\right) = \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}\sum_{n=0}^{\infty}\frac{(a)_n(d)_n(b)_n x^n}{n!(e)_n(b+1)_n} + \sum_{n=0}^{\infty}\frac{(a)_n(d)_n x^n W_{m-1}(n)}{n!(e)_n}$$
$$= \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}{}_3F_2\left(\begin{matrix}a, d, b\\ e, b+1\end{matrix}\middle| x\right) + \sum_{n=0}^{\infty}\frac{(a)_n(d)_n x^n W_{m-1}(n)}{n!(e)_n}$$
$$= \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}{}_3F_2\left(\begin{matrix}a, d, b\\ e, b+1\end{matrix}\middle| x\right) + \frac{(\mathbf{f})_m - (\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}{}_{m+1}F_m\left(\begin{matrix}a, d, -\boldsymbol{\gamma}+1\\ e, -\boldsymbol{\gamma}\end{matrix}\middle| x\right).$$

It remains to apply the Miller-Paris transformations (3) and (4) (or (13) and (14)) to the function

$$ {}_{m+1}F_m\left(\begin{matrix}a, d, -\boldsymbol{\gamma}+1\\ e, -\boldsymbol{\gamma}\end{matrix}\middle| x\right).$$

Note the change of notation $b \to d$, $c \to e$, $m \to m-1$ as compared to (3), (4). To give explicit formulas for the characteristic polynomials (5) and (8) use (22) and (39) to get

$$ {}_{r+3}F_{r+2}\left(\begin{matrix}-k, b, \mathbf{f}+\mathbf{m}\\ b+1, \mathbf{f}\end{matrix}\right) = \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}{}_2F_1\left(\begin{matrix}-k, b\\ b+1\end{matrix}\right) + \frac{(\mathbf{f})_{\mathbf{m}} - (\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}{}_{m+1}F_m\left(\begin{matrix}-k, -\boldsymbol{\gamma}+1\\ -\boldsymbol{\gamma}\end{matrix}\right)$$
$$= \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}\frac{k!}{(b+1)_k} + \frac{(\mathbf{f})_{\mathbf{m}} - (\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}{}_{m+1}F_m\left(\begin{matrix}-k, -\boldsymbol{\gamma}+1\\ \boldsymbol{\gamma}\end{matrix}\right),$$

where the Chu-Vandermonde identity was applied in the second equality. Comparing this formula with (6) and (25) we immediately see that

$$C_{k,r}(-\boldsymbol{\gamma}, \mathbf{1}) = \frac{(-1)^k}{k!}{}_{m+1}F_m\left(\begin{matrix}-k, -\boldsymbol{\gamma}+1\\ -\boldsymbol{\gamma}\end{matrix}\middle| x\right) = \frac{(\mathbf{f})_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}} - (\mathbf{f}-b)_{\mathbf{m}}}Y_k(b, \mathbf{f}, \mathbf{m}).$$

Substituting this expression into (5) and (8) and canceling constant factors we arrive at (37) and (38), respectively. □

Taking $r = 1$, $m = 2$ in Theorem 5 after some elementary computations we arrive at

Corollary 4 *Suppose $e - d - 1 \neq 0$. Then following identities hold:*

$$f(f+1)_4F_3\left(\begin{matrix}a, d, b, f+2\\ e, b+1, f\end{matrix}\middle| x\right) - (f-b)(f-b+1)_3F_2\left(\begin{matrix}a, d, b\\ e, b+1\end{matrix}\middle| x\right)$$

$$= b(2f-b+1)(1-x)^{-a}{}_3F_2\left(\begin{matrix}a, e-d-1, \lambda+1\\ e, \lambda\end{matrix}\middle| \frac{x}{x-1}\right)$$

$$= b(2f-b+1)(1-x)^{e-a-d-1}{}_3F_2\left(\begin{matrix}e-a-1, e-d-1, \lambda^*+1\\ e, \lambda^*\end{matrix}\middle| x\right),$$

where

$$\lambda = \frac{(2f-b+1)(e-d-1)}{2f-b-d+1}, \quad \lambda^* = \frac{(2f-b+1)(e-a-1)(e-d-1)}{ad+(2f-b+1)(e-a-d-1)}.$$

For the second equality the additional restrictions $e-a-1 \neq 0$ and $1+a+d-e \neq 0$ must be imposed.

Similarly, taking $r = 2$, $m_1 = m_2 = 1$ in Theorem 5 we get

Corollary 5 *Suppose $e - d - 1 \neq 0$. Then following identities hold:*

$$(f_1f_2)_5F_4\left(\begin{matrix}a, d, b, f_1+1, f_2+1\\ e, b+1, f_1, f_2\end{matrix}\middle| x\right) - (f_1-b)(f_2-b)_3F_2\left(\begin{matrix}a, d, b\\ e, b+1\end{matrix}\middle| x\right)$$

$$= b(f_1+f_2-b)(1-x)^{-a}{}_3F_2\left(\begin{matrix}a, e-d-1, \lambda+1\\ e, \lambda\end{matrix}\middle| \frac{x}{x-1}\right)$$

$$= b(f_1+f_2-b)(1-x)^{e-a-d-1}{}_3F_2\left(\begin{matrix}e-a-1, e-d-1, \lambda^*+1\\ e, \lambda^*\end{matrix}\middle| x\right),$$

where

$$\lambda = \frac{(f_1+f_2-b)(e-d-1)}{f_1+f_2-b-d}, \quad \lambda^* = \frac{(f_1+f_2-b)(e-a-1)(e-d-1)}{ad+(f_1+f_2-b)(e-a-d-1)}.$$

For the second equality the additional restrictions $e-a-1 \neq 0$ and $1+a+d-e \neq 0$ must be imposed.

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Transmutation Operators For Ordinary Dunkl–Darboux Operators

S. P. Khekalo, V. V. Meshcheryakov, and K. O. Politov

Abstract The study is developed of transmutation operators for differential-difference operators, analogous to Dunkl operator. The basis for the study of operators' properties is the intertwining operator and Darboux transformations theories.

Keywords Dunkl operators · Dunkl–Darboux operators · Transmutation operators · Darboux transmutation

1 Introduction

Paper [1] introduced differential-difference operators, currently known as Dunkl operators. These operators have important applications for the theory of differential operators in partial differential equations (e.g., see [2, 3] and works cited therein). Equally curious applications of these operators were also discovered in the one-dimensional case (e.g., see [4] and works cited therein). In [5] for ordinary Dunkl operators and their analogs, i.e. Dunkl–Darboux operators, we can see the results, which have applications in the Sturm–Liouville operators theory.

Based on the transmutation operators (intertwining operators), we analyze certain aspects of building analogues of ordinary Dunkl operators.

This submission was made within the framework of the RFBR project No. 20-51-15008.

S. P. Khekalo (✉) · V. V. Meshcheryakov · K. O. Politov
State Educational Institution of Higher Education, State University of Humanities and Social Studies, Kolomna, Russia
e-mail: khekalo@mail.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_7

2 Dunkl–Darboux Operators

Let

$$\Omega = \{x \in \mathbf{R} \mid x \in \Omega \Rightarrow -x \in \Omega\}$$

be symmetrical with the respect to the point $x = 0$ open domain in $\mathbf{R}$;

$$\mathcal{F}(\Omega) = \{f \in \mathrm{C}^{\infty}(\Omega)\}$$

is a set of real functions which are infinitely differentiable on Ω.

On $\mathcal{F}(\Omega)$ we see the well defined differentiation operators $d/dx : \mathcal{F}(\Omega) \to \mathcal{F}(\Omega)$, multiplication by the function $k/x : \mathcal{F}(\Omega) \to \mathcal{F}(\Omega\backslash\{0\}),\ \ k \in \mathbf{Z}_+$, and inversion $s : \mathcal{F}(\Omega) \to \mathcal{F}(\Omega)$, functioning according to the rule

$$\forall f \in \mathcal{F}(\Omega) : s[f](x) = f(-x).$$

Thus, on $\mathcal{F}(\Omega)$ naturally works [1] Dunkl classical rational operator

$$\nabla_k = \frac{d}{dx} - \frac{k}{x} s, \quad k \in \mathbf{Z}_+.$$

It is obvious that

$$\nabla_k : \mathcal{F}_{\pm}(\Omega) \to \mathcal{F}_{\mp}(\Omega\backslash\{0\}), \tag{2.1}$$

where $\mathcal{F}_{\pm}$—subsets in $\mathcal{F}$ of respectively even and odd functions. It is also obvious that

$$\mathcal{L}_k := \nabla_k^2\Big|_{\mathcal{F}_+(\Omega)} = \frac{d^2}{dx^2} - \frac{k(k-1)}{x^2}; \quad \nabla_k^2\Big|_{\mathcal{F}_-(\Omega)} = \nabla_{k+1}^2\Big|_{\mathcal{F}_+(\Omega)} = \mathcal{L}_{k+1}. \tag{2.2}$$

Operators in (2.2) are Sturm–Liouville operators with special Stellmacher potentials, well known in mathematical physics (e.g., see [2, 6] and works cited therein).

Due to the design in (2.2) it is the case of Darboux transmutations [6]

$$\mathcal{L}_k = \left(\frac{d}{dx} + \frac{k}{x}\right)\left(\frac{d}{dx} - \frac{k}{x}\right) = \left(\frac{d}{dx} - \frac{k-1}{x}\right)\left(\frac{d}{dx} + \frac{k-1}{x}\right)$$

and the intertwining relations corresponding to it

$$
\begin{aligned}
\left(\frac{d}{dx}-\frac{k}{x}\right)\mathcal{L}_k &= \mathcal{L}_{k+1}\left(\frac{d}{dx}-\frac{k}{x}\right),\\
\left(\frac{d}{dx}+\frac{k}{x}\right)\mathcal{L}_{k+1} &= \mathcal{L}_k\left(\frac{d}{dx}+\frac{k}{x}\right).
\end{aligned}
\tag{2.3}
$$

On $\mathcal{F}(\Omega)$ we shall introduce Dunkl–Darboux operators using the following formula

$$
\nabla_\omega = \frac{d}{dx} - (\log|\omega(x)|)'\, s, \tag{2.4}
$$

where $\omega(x)$ is some even or odd in $\mathcal{F}(\Omega)$ function, analytical in its domain. It is obvious that

$$
\nabla_{|x|^k} = \nabla_k.
$$

Due to the equality of the function $|\omega(x)|$ for ∇_ω the analogue of the formula (2.1) is executed

$$
\nabla_\omega : \mathcal{F}_\pm(\Omega) \to \mathcal{F}_\mp(\Omega),
$$

and analogues of operators (2.2) may be written as

$$
\begin{aligned}
\mathcal{L}_\omega^+ &:= \nabla_\omega^2\Big|_{\mathcal{F}_+(\Omega)} = \frac{d^2}{dx^2} - \left(\left[(\log|\omega(x)|)'\right]^2 + (\log|\omega(x)|)''\right),\\
\mathcal{L}_\omega^- &:= \nabla_\omega^2\Big|_{\mathcal{F}_-(\Omega)} = \frac{d^2}{dx^2} - \left(\left[(\log|\omega(x)|)'\right]^2 - (\log|\omega(x)|)''\right).
\end{aligned}
\tag{2.5}
$$

It is apparent that for operators (2.5)

$$
\mathcal{L}^+_{|x|^k} = \mathcal{L}_k, \qquad \mathcal{L}^-_{|x|^k} = \mathcal{L}_{k+1}
$$

and the intertwining relations, analogous to (2.3) are met:

$$
\begin{aligned}
\left(\frac{d}{dx} - (\log|\omega(x)|)'\right)\mathcal{L}_\omega^+ &= \mathcal{L}_\omega^-\left(\frac{d}{dx} - (\log|\omega(x)|)'\right),\\
\left(\frac{d}{dx} + (\log|\omega(x)|)'\right)\mathcal{L}_\omega^- &= \mathcal{L}_\omega^+\left(\frac{d}{dx} + (\log|\omega(x)|)'\right).
\end{aligned}
\tag{2.6}
$$

Proposition 2.1 (For Darboux Transmutations, See e.g. [5]) *Let* $\omega(x) = \omega_k(x)$, *depends on the parameter* $k \in \mathbf{Z}_+, \quad \omega_0(x) = 1$. *Then the equation is satisfied*

$$\mathcal{L}^{+}_{\omega_{k+1}} = \mathcal{L}^{-}_{\omega_k}$$

with an accuracy to multiplicative constant factor and transformations of the type $x \to x + \text{const}$ *if and only if there is a equality*

$$\omega_k(x) = \frac{\mathcal{P}_k(|x|)}{\mathcal{P}_{k-1}(|x|)},$$

where $\mathcal{P}_k(x)$ *are Burchnall-Chaundi polynomials [2], defined by a recurrence equation*

$$\mathcal{P}_0(x) = 1; \quad \mathcal{P}_1(x) = x; \quad \mathcal{P}'_{k+1}(x)\mathcal{P}_{k-1}(x) - \mathcal{P}_{k+1}(x)\mathcal{P}'_{k-1}(x) = (2k+1)\,\mathcal{P}^2_k(x).$$

For the proof of necessity we need to establish that a partial Riccati equation solution

$$z' + z^2 = -\left(\log \mathcal{P}^2_k(|x|)\right)'',$$

has the following appearance

$$z = \left(\log \left|\frac{\mathcal{P}_{k+1}(|x|)}{\mathcal{P}_k(|x|)}\right|\right)'.$$

For the proof of sufficiency we need to establish that for $\omega_k(x) = \mathcal{P}_k(|x|)/\mathcal{P}_{k-1}(|x|)$ there is a equality

$$\left[(\log|\omega_{k+1}(x)|)'\right]^2 + (\log|\omega_{k+1}(x)|)'' = \left[(\log|\omega_k(x)|)'\right]^2 - (\log|\omega_k(x)|)''.$$

3 Darboux Transmutations for High Order Differential Operators

For the space $\mathbf{R}^{n+1}$ let us consider the deformation of wave operator by a time-dependent potential

$$\mathcal{L} = \Delta_n - \left(\frac{\partial^2}{\partial t^2} + c(t)\right).$$

Here Δ_n is the Laplace operator on $\mathbf{R}^n$ and $c(t)$ is an almost everywhere differentiable function.

If the function $\mu = \mu(t)$ is a solution to the equation

$$\mu''_{tt} + c(t)\mu = 0,$$

then there is an obvious transformation

$$\frac{\partial^2}{\partial t^2} + c(t) = \left(\frac{\partial}{\partial t} + \frac{\mu'}{\mu}\right)\left(\frac{\partial}{\partial t} - \frac{\mu'}{\mu}\right).$$

As a result, based on Darboux transmutations, the operator $\mathcal{L}$ can be written as [6] in the form

$$\mathcal{L} = \Delta_n + l^* l,$$

where

$$l = \frac{\partial}{\partial t} - \frac{\mu'}{\mu} \quad \text{and} \quad l^* = -\frac{\partial}{\partial t} - \frac{\mu'}{\mu}.$$

At the same time for operator, associated with $\mathcal{L}$

$$\tilde{\mathcal{L}} = \Delta_n + l l^*,$$

there is the following intertwining property

$$l\,\mathcal{L} = \tilde{\mathcal{L}}\,l. \tag{3.1}$$

Based on the intertwining property the Darboux transmutation technique (3.1) allows to construct new operators, the properties of which are described via the properties of the initial operator.

Example 3.1 Euler–Poisson–Darboux equation

$$\left(\Box_{n+1} - \frac{k(k+1)}{t^2}\right) u(t;x) = 0, \quad k = 0, 1, 2, \ldots,$$

results from the method mentioned. Here $\Box_{n+1}$ is the wave operator. Elementary solution of the Euler-Poisson-Darboux operator can be achieved based on the transmutation of the wave operator via step-by-step "destruction" of the parameter $k: \ k \to k-1 \to \ldots \to 1 \to 0$ based on the formula (3.1):

$$\tilde{\mathcal{L}} = \Box_{n+1} - \frac{k(k+1)}{t^2} \ \to \ \mathcal{L} = \Box_{n+1} - \frac{(k-1)k}{t^2} \to \ldots.$$

Example is complete.

Example 3.2 Wave operator deformations hierarchy by Lagnese-Stellmacher potentials [6]

$$\Box_{n+1} + u_k(t), \quad u_k(t) = 2\frac{\partial^2}{\partial t^2}\log|P_k(t)|,$$

are based on the classical Darboux transmutation.

Darboux transmutations are tightly linked to the concept of the so called gauge relation of differential operators, which, in our case is set by the equality [2]

$$\mathrm{ad}^{k(k+1)/2+1}_{\frac{d^2}{dt^2}-u_k(t),\frac{d^2}{dt^2}} \; P_k(t) = 0, \quad k = 1, 2, \ldots .$$

Here, ad-operator

$$\mathrm{ad}^k_{A,B} C = \mathrm{ad}^{k-1}_{A,B}(AC - CB), \quad \mathrm{ad}^0_{A,B} C = C,$$

– is a slant adjoint action operator. The example is complete.

Definition 3.1 ([7]) Operators $\mathcal{L}_k, \; k = 0, 1, \ldots, s,$ of some non-negative integer $s,$ meet the condition of step-by-step gauge equivalence by means of smooth nonzero function $f,$ if there is a equality

$$\mathrm{ad}^{1+1}_{\mathcal{L}_k,\mathcal{L}_{k-1}} \; f = 0$$

for all $k = 1, \ldots, s.$

Proposition 3.1 ([7]) *If operators* $\mathcal{L}_k, \; k = 0, 1, \ldots, s,$ *meet the condition of step-by-step gauge equivalence using the function* $f,$ *then the operator* $\mathcal{L}_s$ *is* s*-gauge related to the operator* $\mathcal{L}_0$ *using the function* f^s *:*

$$\mathrm{ad}^{1+1}_{\mathcal{L}_k,\mathcal{L}_{k-1}} \; f = 0, \quad k = 1, \ldots, s, \; \Rightarrow \mathrm{ad}^{s+1}_{\mathcal{L}_s,\mathcal{L}_0} \; f^s = 0.$$

Proposition 3.2 ([7]) *Let*

$$\mathcal{L}_0 = \sum_{j=0}^{K} a_j \frac{d^j}{dx^j}$$

– ordinary differential operator of an order K *on* $\mathbf{R}$ *with constant coefficients. Then, the differential operator*

$$\mathcal{L}_k = \mathcal{L}_0 + \sum_{j=0}^{K-2} \left(\sum_{p=0}^{K-j} (-1)^{p-1}(p-1)\binom{p+j}{j} \frac{(k,p)}{x^p} a_{p+j} \right) \frac{d^j}{dx^j}$$

of the order K meets the equality

$$\mathrm{ad}^{2}_{\mathcal{L}_k,\,\mathcal{L}_{k-1}}\, x = 0, \quad k = 1, 2, \ldots .$$

Here $\binom{p+j}{j}$ is the binomial coefficient, and (k, p) is the Pochhammer symbol.

Corollary 3.1 *Darboux transmutation analogue has the appearance*

$$\mathcal{L}_k\, l_k = l_k\, \mathcal{L}_{k-1}, \quad k = 1, 2, \ldots, \tag{3.2}$$

where

$$l_k = \sum_{j=0}^{K-1} \left(\sum_{p=1}^{K-j} (-1)^{p-1}\, p \binom{p+j}{j} \frac{(k, p-1)}{x^{p-1}} a_{p+j} \right) \frac{d^j}{dx^j}.$$

Formula (3.2) shall naturally be called Dunkl–Darboux transmutation. (E.g., see [5] and works cited therein).

Example 3.3 In the case $K = 2$ for

$$\mathcal{L}_k = \frac{d^2}{dx^2} - \frac{k(k+1)}{x^2}$$

Dunkl–Darboux transmutation (3.2) turns into a classical Darboux transmutation (2.3)

$$\left(\frac{d^2}{dx^2} - \frac{k(k+1)}{x^2} \right) \left(\mathrm{ad}_{\mathcal{L}_k,\mathcal{L}_{k-1}}\, x \right) = \left(\mathrm{ad}_{\mathcal{L}_k,\mathcal{L}_{k-1}}\, x \right) \left(\frac{d^2}{dx^2} - \frac{k(k-1)}{x^2} \right),$$

where $k = 1, 2, \ldots,$ and

$$l_k := \mathrm{ad}_{\mathcal{L}_k,\mathcal{L}_{k-1}}\, x = 2 \left(\frac{d}{dx} - \frac{k}{x} \right).$$

The example is complete.

4 Integral Dunkl–Darboux Transmutations

Let us designate via $\tilde{\mathcal{F}}$ the space of functions which are invariant under the action of operators ∇_ω and $\frac{d}{dx}$. In [4] describes a series of functions which are a formal solution to the equation $\nabla_\omega y = y$.

Specification of this series is based on the transmutation operator. Let linear operator $V : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$, satisfy the intertwining property

$$\tilde{\nabla}_\omega V = V \frac{d}{dx}.$$

If the intertwining operator V has the explicit form, then the function $f(x) = const \cdot V(e^x)$ is a solution to the equation $\nabla_\omega y = y$. We have

$$V(e^x) = \sum_{n=0}^{+\infty} \frac{V(x^n)}{n!}.$$

Let us define via $y_n = V(x^n)$, then

$$\nabla_\omega y_0 = 0, \quad \nabla_\omega y_n = n y_{n-1}, \; n \in \mathbf{N}.$$

If we limit ourselves to analytical functions, then the first equation shall have the unique solution (with an accuracy to the constant factor): $y_0 = \omega$. Induction by the parameter n establishes, that at even n the function y_n is even, at odd n it is odd. Based on this fact, the solution of the second equation (for natural n) can be written down in the form of Dunkl–Darboux integral transmutation [8]

$$y_n(x) = n(\omega(x))^{(-1)^n} \int (\omega(x))^{(-1)^{n-1}} y_{n-1}(x) dx, \tag{4.1}$$

where the integration constant is chosen so that the function y_n has the required type of equality.

Example 4.1 Immediately, we have

$$y_1 = \frac{1}{\omega} \int \omega^2 \, dx, \quad y_2 = 2\omega \int \left(\frac{1}{\omega^2} \int \omega^2 \, dx \right) dx, \quad \ldots .$$

Example is complete.

Using these integral transmutations (4.1) the solution f of the equation $\nabla_\omega y = y$ can be presented (within a constant factor) in the form of formal series

$$f(x) = \sum_{n=0}^{+\infty} \frac{y_n(x)}{n!}. \tag{4.2}$$

For some functions ω the sum of the last series (if we choose integration constants in a special way) it is possible to calculate immediately.

Example 4.2 If $\omega(x) = |x|^k$, then

$$f(x) = x^k \left(\mathcal{J}_{k-1/2}(x) + \frac{x}{2k+1} \mathcal{J}_{k+1/2}(x) \right) \qquad \text{at even } k\ ,$$

$$f(x) = x^{-k} \left(\mathcal{J}_{-k-1/2}(x) + \frac{x}{-2k+1} \mathcal{J}_{-k+1/2}(x) \right) \qquad \text{at odd } k\ ,$$

where

$$\mathcal{J}_\gamma(x) = \sum_{n=0}^{+\infty} \frac{\Gamma(\gamma+1)}{n!\Gamma(n+\gamma+1)} \left(\frac{x}{2}\right)^{2n}$$

is a modified Bessel function. Example is complete.

Example 4.3 If $\omega(x) = |\mathrm{tg}\frac{x}{2}|$, then $f(x) = -2e^x + e^x \mathrm{ctg} x + \dfrac{e^{-x}}{\sin x}$. Example is complete.

Example 4.4 If $\omega(x) = |\mathrm{th}\, x|$, then $f(x) = -\dfrac{2}{\mathrm{sh} 2x} e^{-x}$. Example is complete.

Let us show that for a function ω, with a specified choice of integration constants, the series of functions (4.2) converges at each set $[-b; -a] \cup [a; b]$ $(0 \leqslant a < b)$, where ω is positive and ω' is bounded. Thus, we shall determine the sequence of the functions f_n $(n \in \mathbf{Z}_+)$, set on $[-b; -a] \cup [a; b]$ by the equalities

$$f_0(x) = \omega(x), \quad f_n(x) = n(\omega(x))^{(-1)^n} \int_a^x (\omega(t))^{(-1)^{n-1}} f_{n-1}(t)dt, \quad n \in \mathbf{N}.$$

Outside the union $[-b; -a] \cup [a; b]$ we denote $f_n(x) = 0$ for all $n \in \mathbf{Z}_+$.

The following properties of functions f_n can be proved by an induction by n.

1. For all $n \in \mathbf{Z}_+$ we have $f_n(-x) = (-1)^n f_n(x)$;
2. For all $n \in \mathbf{Z}_+$ we have $|f_n(x)| \leqslant \dfrac{M^{n+1}}{m^n} \Big| |x| - a \Big|^n$;
3. For all $n \in \mathbf{Z}_+$ the function f_n is differentiable, however there are the following assessments

$$|f'_{2n+1}(x)| \leqslant (2n+1) \frac{M^{2n+1}}{m^{2n}} \Big| |x| - a \Big|^{2n} + \frac{M^{2n+2}}{m^{2n+2}} \Big| |x| - a \Big|^{2n+1} |\omega'(x)|,$$

$$|f'_{2n}(x)| \leqslant 2n \frac{M^{2n}}{m^{2n-1}} \Big| |x| - a \Big|^{2n-1} + \frac{M^{2n}}{m^{2n}} \Big| |x| - a \Big|^{2n} |\omega'(x)|.$$

Proposition 4.1 *Let ω be a positive and even or odd function, the derivative of which is limited on the interval $[a; b]$. Then, the series of functions $\sum\limits_{n=0}^{+\infty} \frac{f_n(x)}{n!}$ converges and its sum f is a solution to the equation $\nabla_\omega y = y$. In this connection*

$$f(a) = \omega(a) \ \text{and} \ |f(x)| \leqslant M e^{\frac{M}{m}\left||x|-a\right|},$$

where M and m are the greatest and the smallest values of the function ω on the interval $[a; b]$ respectively.

Proof Direct substitution (using property 1) shows that a series of functions $\sum\limits_{n=0}^{+\infty} \frac{f_n(x)}{n!}$ is the formal solution of the equation$\nabla_\omega y = y$. From property 2 it follows that in the set $[-b; -a] \cup [a; b]$, the series $\sum\limits_{n=0}^{+\infty} \frac{f_n(x)}{n!}$ converges absolutely and uniformly, whereas for the sum of f there is the following assessment $|f(x)| \leqslant M e^{\frac{M}{m}\left||x|-a\right|}$. Similarly, from property 3 follows that on the set$[-b; -a] \cup [a; b]$, the series $\sum\limits_{n=0}^{+\infty} \frac{f_n'(x)}{n!}$ converges absolutely and uniformly. Therefore, series $\sum\limits_{n=0}^{+\infty} \frac{f_n(x)}{n!}$ may be differentiated term-by-term, and therefore its sum f satisfies the equality $\nabla_\omega y = y$ in a substantial sense. Proof is complete.

5 Transmutation Operators for Dunkl–Darboux Operators in Cherednik Algebra

The question of ∇_ω operator integrability is closely linked with Cherednik algebra,

$$\mathcal{A} = \left\langle 1, \ x, \ \frac{d}{dx}, \ s \right\rangle,$$

the generators of which meet the following commutation ratios

$$[1, x] = \left[1, \frac{d}{dx}\right] = [1, s] = 0,$$

$$\left[\frac{d}{dx}, x\right] = 1, \ \ [x, s] = 2xs, \ \ \left[s, \frac{d}{dx}\right] = 2s\,\frac{d}{dx}.$$

Proposition 5.1 *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be linear differential operators. In algebra $\mathcal{A}$ there is an equivalence*

$$\mathcal{D}_1 + \mathcal{D}_2\, s = 0 \Leftrightarrow \mathcal{D}_1 = \mathcal{D}_2 = 0.$$

Proof Sufficiency is obvious. Let us check the necessity in several steps. Let

$$\mathcal{D}_1 = \sum_{i=0}^{N} d_{i1}(x)\,\frac{d^i}{dx^i}, \qquad \mathcal{D}_2 = \sum_{j=0}^{M} d_{j1}(x)\,\frac{d^j}{dx^j},$$

then

$$\begin{cases} (\mathcal{D}_1 + \mathcal{D}_2\, s)\,[1] = d_{01}(x) + d_{02}(x) = 0, \\ (\mathcal{D}_1 + \mathcal{D}_2\, s)\,[\mathrm{sgn}(x)] = (d_{01}(x) - d_{02}(x))\,\mathrm{sgn}(x) = 0, \end{cases}$$

and $d_{01}(x) = d_{02}(x) = 0$;

$$\begin{cases} (\mathcal{D}_1 + \mathcal{D}_2\, s)\,[x] = d_{11}(x) - d_{12}(x) = 0, \\ (\mathcal{D}_1 + \mathcal{D}_2\, s)\,[x\,\mathrm{sgn}(x)] = (d_{11}(x) + d_{12}(x))\,\mathrm{sgn}(x) = 0, \end{cases}$$

and $d_{11}(x) = d_{12}(x) = 0; \ldots;$

$$\begin{cases} (\mathcal{D}_1 + \mathcal{D}_2\, s)\,[x^n] = n!(d_{n1}(x) + (-1)^n d_{n2}(x)) = 0, \\ (\mathcal{D}_1 + \mathcal{D}_2\, s)\,[x^n \mathrm{sgn}(x)] = n!(d_{n1}(x) - (-1)^n d_{n2}(x))\,\mathrm{sgn}(x) = 0, \end{cases}$$

and $d_{n1}(x) = d_{n2}(x) = 0$. The proof is complete.

In algebra $\mathcal{A}$ we will consider the equation

$$\nabla_\omega\, V = V\,\frac{d}{dx} \tag{5.1}$$

for the operator $V \in \mathcal{A}$:

$$\begin{cases} V = \sum_{i=0}^{N} p_i(x)\,\frac{d^i}{dx^i} + \sum_{j=0}^{M} q_j(x)\,\frac{d^j}{dx^j}\, s, \\ p_N(x) \neq 0, \quad q_M(x) \neq 0. \end{cases} \tag{5.2}$$

Let us designate via

$$\varkappa = (\log|\omega(x)|)'\,.$$

Proposition 5.2 *If the operator V of a type* (5.2) *satisfies the intertwining property* (2.2)*, then* $N = M + 1$.

Proof Immediately we have

$$\nabla_\omega V = \sum_{i=0}^{N} p_i'(x)\,\frac{d^i}{dx^i} + \sum_{i=0}^{N} p_i(x)\,\frac{d^{i+1}}{dx^{i+1}},$$

$$+ \sum_{j=0}^{M} q_j'(x)\,\frac{d^j}{dx^j}\,s + \sum_{j=0}^{M} q_j(x)\,\frac{d^{j+1}}{dx^{j+1}}\,s$$

$$- \varkappa \sum_{i=0}^{N} (-1)^i\,p_i(-x)\,\frac{d^i}{dx^i}\,s - \varkappa \sum_{j=0}^{M} (-1)^j\,q_j(-x)\,\frac{d^j}{dx^j};$$

$$V\,\frac{d}{dx} = \sum_{i=0}^{N} p_i(x)\,\frac{d^{i+1}}{dx^{i+1}} - \sum_{j=0}^{M} q_j(x)\,\frac{d^{j+1}}{dx^{j+1}}\,s.$$

Thus, owing to Proposition 5.2, operator equality (5.1) in algebra $\mathcal{A}$ is equivalent to the system of the operator equations

$$\begin{cases} \displaystyle\sum_{i=0}^{N} p_i'(x)\,\frac{d^i}{dx^i} - \varkappa \sum_{j=0}^{M} (-1)^j\,q_j(-x)\,\frac{d^j}{dx^j} = 0, \\ \displaystyle\sum_{j=0}^{M} q_j'(x)\,\frac{d^j}{dx^j} + 2\sum_{j=0}^{M} q_j(x)\,\frac{d^{j+1}}{dx^{j+1}} - \varkappa \sum_{i=0}^{N} (-1)^i\,p_i(-x)\,\frac{d^i}{dx^i} = 0. \end{cases} \tag{5.3}$$

From the second equation of the system (4.1) we get that inequalities $p_N(x) \neq 0$ and $q_M(x) \neq 0$ result in $M + 1 = N$. The proof is complete.

Further, owing to proposition 4.3, we will write down the operator (5.2) in the form

$$V = \sum_{i=0}^{N} f_i(x)\,\frac{d^i}{dx^i} + \sum_{i=N+1}^{2N+1} f_i(x)\,\frac{d^{i-N-1}}{dx^{i-N-1}}\,s, \tag{5.4}$$

where, for further convenience,

$$f_{2N+1}(x) \equiv 0.$$

Proposition 5.3 *Equation* (5.1) *for operator* (5.4) *is equivalent to the system of* $2N+2$ *differential-difference equations*

$$\begin{cases} f_i'(x) - \varkappa\,(-1)^i\, f_{i+N+1}(-x) = 0, \quad i = 0, 1, \dots, N, \\ f_{i+N+1}'(x) + 2 f_{i+N}(x) - \varkappa\,(-1)^i\, f_i(-x) = 0, \quad i = 1, 2, \dots, N, \\ f_{N+1}'(x) - \varkappa\, f_0(-x) = 0. \end{cases} \tag{5.5}$$

at $2N+2$ *functions:* $\omega(x),\ f_0(x),\ f_1(x),\ \dots,\ f_{2N}(x)$ *with parameter* N.

Proof As per formula (5.3), taking into account the designations (5.4) and (5.2), we have:

$$\begin{cases} \displaystyle\sum_{i=0}^{N} f_i'(x)\,\frac{d^i}{dx^i} - \varkappa \sum_{i=0}^{N-1} (-1)^i\, f_{i+N+1}(-x)\,\frac{d^i}{dx^i} = 0, \\ \displaystyle\sum_{i=0}^{N-1} f_{i+N+1}'(x)\,\frac{d^i}{dx^i} + 2\sum_{i=0}^{N-1} f_{i+N+1}(x)\,\frac{d^{i+1}}{dx^{i+1}} - \varkappa \sum_{i=0}^{N} (-1)^i\, f_i(-x)\,\frac{d^i}{dx^i} = 0; \end{cases}$$

$$\begin{cases} \displaystyle\sum_{i=0}^{N-1} \Big(f_i'(x) - \varkappa\,(-1)^i\, f_{i+N+1}(-x)\Big)\,\frac{d^i}{dx^i} + f_N'(x)\frac{d^N}{dx^N} = 0, \\ \displaystyle\sum_{i=0}^{N-1} f_{i+N+1}'(x)\,\frac{d^i}{dx^i} + 2\sum_{i=1}^{N} f_{i+N}(x)\,\frac{d^i}{dx^i} - \varkappa \sum_{i=0}^{N} (-1)^i\, f_i(-x)\,\frac{d^i}{dx^i} = 0; \end{cases}$$

$$\begin{cases} f_N'(x) = 0, \\ f_i'(x) - \varkappa\,(-1)^i\, f_{i+N+1}(-x) = 0, \quad i = 0, 1, \dots, N-1, \\ f_{N+1}'(x) - \varkappa\, f_0(-x) = 0, \\ f_{i+N+1}'(x) + 2\, f_{i+N}(x) - \varkappa\,(-1)^i\, f_i(-x) = 0, \quad i = 1, 2, \dots, N-1, \\ 2\, f_{2N}(x) - \varkappa\,(-1)^N\, f_N(-x) = 0. \end{cases}$$

Taking into account equality $f_{2N+1}(x) = 0$, we get

$$\begin{cases} f_i'(x) - \varkappa\,(-1)^i\, f_{i+N+1}(-x) = 0, \quad i = 0, 1, \ldots, N, \\ f_{i+N+1}'(x) + 2\, f_{i+N}(x) - \varkappa\,(-1)^i\, f_i(-x) = 0, \quad i = 1, 2, \ldots, N, \\ f_{N+1}'(x) - \varkappa\, f_0(-x) = 0. \end{cases}$$

The proof is complete.

The following example shows that the system (5.5) has nonempty set of solutions.

Example 5.1 In the case $\omega(x) = |x|^k$, $k \in \mathbf{Z}_+$, and $N = k$ the solution of a system (5.5) determines coefficients of the intertwining operator (5.4) by the following equality

$$f_i(x) = \begin{cases} \dfrac{(k, k-i)\,(-k, k-i)}{2^{k-i}\,(k-i)!\,x^{k-i}}, \quad i = 0, 1, \ldots, k, \\ (-1)^k\, \dfrac{(k, 2k-i+1)\,(-k+1, 2k-i)}{2^{2k-i+1}(2k-i)!\,x^{2k-i+1}}, \quad i = k+1, \ldots, 2k, \end{cases}$$

where

$$(k, 0) = 1, \quad (k, n) = k(k+1)\ldots(k+n-1), \; n \in \mathbf{N},$$

is a Pochhammer symbol. Example is complete.

Comment 5.1 The system (5.5) can be rewritten as matrix equation

$$\frac{df}{dx}(x) = \Omega_\varkappa\, f(x), \tag{5.6}$$

where

$$f(x) = \begin{pmatrix} f_0(x) \\ \vdots \\ f_{2N}(x) \\ 0 \end{pmatrix}, \; \frac{df}{dx}(x) = \begin{pmatrix} \frac{df_0}{dx}(x) \\ \vdots \\ \frac{df_{2N}}{dx}(x) \\ 0 \end{pmatrix}, \quad \Omega_\varkappa = \begin{pmatrix} 0 & \varkappa E_{N+1}\, s \\ \varkappa E_{N+1}\, s & -2 I_{N+1} \end{pmatrix}$$

with matrices

$$E_{N+1} = ((-1)^{i+1}\delta_{ij}), \quad I_{N+1} = (\delta_{i,j+1})$$

of the order $N + 1$. Here δ_{ij} is the Kronecker symbol.

Example 5.2 In the case $N = 1$ the system (5.6) has the appearance

$$\begin{pmatrix} f_0'(x) \\ f_1'(x) \\ f_2'(x) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \varkappa s & 0 \\ 0 & 0 & 0 & -\varkappa s \\ \varkappa s & 0 & 0 & 0 \\ 0 & -\varkappa s & -2 & 0 \end{pmatrix} \begin{pmatrix} f_0(x) \\ f_1(x) \\ f_2(x) \\ 0 \end{pmatrix} \tag{5.7}$$

or

$$\begin{cases} f_0'(x) = \varkappa\, f_2(-x), \\ f_1'(x) = 0, \\ f_2'(x) = \varkappa\, f_0(-x), \\ \varkappa\, f_1(-x) + 2 f_2(x) = 0. \end{cases}$$

Without restriction to generality it is possible to consider that $f_1(x) = 1$. At the same time $f_2(x)$ is an odd function. Therefore, the last system can be rewritten as follows:

$$\begin{cases} f_0'(x) = -\varkappa\, f_2(x), \\ f_1(x) = 1, \\ f_2'(x) = -\varkappa\, f_0(x), \\ f_2(x) = -\dfrac{1}{2}\varkappa; \end{cases} \qquad \begin{cases} f_0(x) = \dfrac{1}{2}\dfrac{\varkappa'}{\varkappa}, \\ f_1(x) = 1, \\ f_2(x) = -\dfrac{1}{2}\varkappa, \\ (\log|\varkappa|)'' = \varkappa^2. \end{cases}$$

Let us multiply both members of the equation $(\log|\varkappa|)'' = \varkappa^2$ on $(\log|\varkappa|)'$ and then integrate:

$$\int (\log|\varkappa|)'\,(\log|\varkappa|)''\,dx = \int (\log|\varkappa|)'\,\varkappa^2\,dx,$$

$$\int (\log|\varkappa|)'\,d(\log|\varkappa|)' = \int \varkappa\,d\varkappa,$$

$$\left[(\log|\varkappa|)'\right]^2 = \varkappa^2 + \text{const}.$$

This equation is integrable in the elementary functions:

$$\varkappa = \pm\frac{1}{x}, \quad \varkappa = \pm\frac{1}{\sinh x}, \quad \varkappa = \pm\frac{1}{\sin x}.$$

Thus, the system of (5.7), to within elementary transformations, has the solutions determined by the function $\omega(x)$:

$$\omega = x;\ \ \omega = \frac{1}{x} - \text{ rational case;}$$

$$\omega = \tan\frac{x}{2};\ \ \omega = \cot\frac{x}{2} - \text{ trigonometrical case;}$$

$$\omega = \tanh\frac{x}{2};\ \ \omega = \coth\frac{x}{2} - \text{ hyperbolic case.}$$

Example is complete.

6 Recurrence Equations

Example 5.2 shows that the system (5.5) can be solved analytically: it is constructed and solved (if the later is possible) as an equation on $\varkappa$, and then on the basis of iterations the coefficients of the operator V, determined by formula (5.4) are found:

$$\varkappa \to f_{2N}(x) \to \cdots \to f_{N+1}(x) \to 1,$$

$$\varkappa \to f_{N-1}(x) \to \cdots \to f_0(x) \to \varkappa.$$

Indeed, from equalities (5.5) for $i = 1, 2, \ldots, N,$ follows

$$\begin{cases} f_{2N+1}(x) = 0, \\ f_{i+N}(x) = -\dfrac{1}{2}\, f'_{i+N+1}(x) + \dfrac{1}{2}\,\varkappa \displaystyle\int \varkappa\, f_{i+N+1}(x)dx, \\ f_N(x) = 1, \\ f_{i-1}(x) = -\dfrac{1}{2}\, f'_i(x) + \dfrac{1}{2}\, f_i(x)\,(\ln|\varkappa|)' + \dfrac{1}{2}\displaystyle\int \left(\varkappa^2 - (\ln|\varkappa|)''\right)\, f_i(x)dx, \\ \varkappa\, f''_0(x) - \varkappa'\, f'_0(x) - \varkappa^3\, f_0(x) = 0. \end{cases}$$

7 Transmutation Operators for Dunkl–Darboux Operators in Cherednik Pseudoalgebra

Let $\frac{d^{-1}}{dx^{-1}}$ be the pseudodifferential operator, inverse to $\frac{d}{dx}$:

$$\frac{d}{dx}\,\frac{d^{-1}}{dx^{-1}} = \frac{d^{-1}}{dx^{-1}}\,\frac{d}{dx} = 1.$$

In [9] the problem of finding the transmutation operator(5.1) is generalized for the case of Cherednik pseudoalgebra

$$A^* = \left\langle A, \frac{d^{-1}}{dx^{-1}} \right\rangle$$

with additional relations on the generators:

$$\left[1, \frac{d^{-1}}{dx^{-1}}\right] = \left[\frac{d}{dx}, \frac{d^{-1}}{dx^{-1}}\right] = 0, \quad \left[s, \frac{d^{-1}}{dx^{-1}}\right] = 2s\,\frac{d^{-1}}{dx^{-1}}, \quad \left[x, \frac{d^{-1}}{dx^{-1}}\right] = \frac{d^{-2}}{dx^{-2}}.$$

Equation (5.1) comes down to the solution of a special system of difference-differential equations for operator V coefficients and the function ω. In particular, it is sufficient that the function ω satisfies one of the following nonlinear differential equations

$$\left[\begin{array}{l} (\log|\varkappa|)'' - \varkappa^2 = 0, \\ \omega'''\,\omega'\,\omega - (\omega'')^2\,\omega - \omega''\,(\omega')^2 + 4\omega''\,\omega = 0, \\ \omega'''\,\omega'\,\omega - (\omega'')^2\,\omega - \omega''\,(\omega')^2 + 2\omega''\,\omega - 2\omega''\,\omega^3 + 4(\omega')^2\,\omega^2 = 0, \\ \omega'''\,\omega'\,\omega - (\omega'')^2\,\omega - \omega''\,(\omega')^2 + 2\omega''\,\omega + 2\omega''\,\omega^3 - 4(\omega')^2\,\omega^2 = 0. \end{array}\right. \tag{7.1}$$

The set (7.1) actually classifies the operators of the type (2.4) as rational, hyperbolic, trigonometrical and their combinations. For example,

$$\omega = x; \;\; \omega = \tan\frac{x}{2}; \;\; \omega = \tanh\frac{x}{2}; \; \omega = \frac{2x}{\tanh x} - 2.$$

Acknowledgements We would like to thank Savelev Sergey, Associate Professor, PhD for his assistance in drafting the English version of this article. The authors would also like to thank the participants of the Seminar on Analytic Theory of Differential Equations at Steklov Mathematical Institute of Russian Academy of Sciences. This paper was drafted within the framework of the research grant No.16-51-150005 of Russian Foundation for Basic Research.

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Theorems on Restriction of Fourier–Bessel and Multidimensional Bessel Transforms to Spherical Surfaces

A. A. Larin

Abstract The paper deals with problems of L_q-summability with a weight over spherical surface of Fourier–Bessel and n-dimensional Bessel transforms for functions from some weighted spaces. The results have applications to PDE theory. Results of this paper may be applied in transmutation theory, for example for estimating solutions of singular B-elliptic PDEs.

Keywords Fourier transform · Bessel transform · Sobolev spaces · Restriction to surfaces · Bessel functions · Generalized translation

MSC: 35A22

1 Introduction

In this paper we study problems of L_q-summability with a weight over spherical surface of Fourier–Bessel and n-dimensional Bessel transforms [1] for functions from some weighted spaces. Similar problems for Fourier transform were studied by Stein and Tomas [2] and Strichartz in [3], cf. also the monograph of Stein [4], ch. 8, 9, in which this topic was further investigated. Note that results in this direction are naturally applied to proofs of uniform Sobolev estimates and uniqueness theorems for PDEs [5]. Also results of this paper may be applied in transmutation theory [11, 12], for example for estimating solutions of singular B-elliptic PDE.

Let introduce necessary notions and notations.

Let E^+_{n+1} being a half-space of $(n+1)$ dimensional Euclidean space E_{n+1}, consisting of points $(x, y) = (x_1, \ldots, x_n, y)$ such that $y > 0$ $(n \geq 1)$. By $L_{p,\nu}\left(E^+_{n+1}\right)$

A. A. Larin (✉)
Military Educational-Research Centre of Air Force, "Air Force Academy named after Professors N.E. Zhukovsky and Y.A. Gagarin", Voronezh, Russia
e-mail: vaiu@mil.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_8

we denote a space of all measurable on E^+_{n+1} functions $f(x, y)$ with finite norm

$$\|f\|_{p,\,\nu} = \left(\int\limits_{E^+_{n+1}} |f(x, y)|^p y^{2\nu+1} dy dx \right)^{1/p},$$

$1 \le p < \infty$, ν is real and greater than $-1/2$. Mixed Fourier–Bessel transform of order ν on functions f from $L_{1,\,\nu}\left(E^+_{n+1}\right)$ is defined by

$$F_\nu f(x, y) = \int\limits_{E^+_{n+1}} e^{-i(x,\xi)} j_\nu(y\tau) f(\xi, \tau)\, \tau^{2\nu+1} d\tau d\xi,$$

with $j_\nu(t)$-normalized Bessel function defined by $j_\nu(t) = 2^\nu \Gamma(\nu + 1) J_\nu(t)/t^\nu$, $(x, \xi) = x_1\xi_1 + \cdots + x_n\xi_n$, $\Gamma(s)$ is the gamma-function. Parseval equation for functions f from $L_{2,\,\nu}\left(E^+_{n+1}\right)$ is [1]

$$\|F_\nu f\|_{2,\,\nu} = (2\pi)^{\frac{n}{2}}\, 2^\nu\, \Gamma(\nu + 1) \|f\|_{2,\,\nu}.$$

Now define n dimensional Bessel transform. Denote by E'_n a subset of E_n, $n \ge 2$, consisting of points $x = (x_1, \ldots, x_n)$ with positive coordinates. Let $\nu_1, \ldots, \nu_n$ being a set of n real numbers greater than $-1/2$ each. By $L_{p,\,\overline{\nu}}\left(E'_n\right)$ we denote a space of all measurable on E'_n functions with finite norm

$$\|f\|_{p,\,\overline{\nu}} = \left(\int\limits_{E'_n} |f(x)|^p \prod_{k=1}^{n} x_k^{2\nu_k+1} dx \right)^{1/p},$$

$$1 \le p < \infty.$$

N-dimensional Bessel transform for $f \in L_{1,\,\overline{\nu}}\left(E'_n\right)$ is defined by

$$F_{\overline{\nu}} f(x) = \int\limits_{E'_n} f(\xi) \prod_{k=1}^{n} j_{\nu_k}(x_k\xi_k)\, \xi_k^{2\nu_k+1} d\xi.$$

Corresponding Parseval equation for $f \in L_{2,\,\overline{\nu}}\left(E'_n\right)$ has a form

$$\|F_{\overline{\nu}} f\|_{2,\,\overline{\nu}} = \prod_{k=1}^{n} 2^{\nu_k} \Gamma(\nu_k + 1) \|f\|_{2,\,\overline{\nu}}.$$

Let $S^+_{n,R}$ being a part of sphere with radius R centered at 0 in E_{n+1} for points obeying $y > 0$. By $S'_{n-1,R}$ let denote a part of sphere with radius R centered at origin in E_n and belonging to E'_n. By $L_{p,\nu}\left(S^+_{n,1}\right)$ we denote a space of functions defined and measurable on unit half-sphere $S^+_{n,1}$ with finite norm

$$\|f\|_{L_{p,\nu}\left(S^+_{n,1}\right)} = \left(\int\limits_{S^+_{n,1}} |f(x,y)|^p\, y^{2\nu+1} dS\right)^{1/p}, \ 1 \le p < \infty.$$

(By dS here and further we denote a surface measure on a sphere of proper radius).

In the same way we define spaces $L_{p,\overline{\nu}}\left(S'_{n-1,1}\right)$, $1 \le p < \infty$. In case $p = \infty$ all spaces $L_{p,\nu}$ и$L_{p,\overline{\nu}}$ are defined analogously. We will denote them by L_∞.

2 Mixed Fourier–Bessel Transform

First we study properties of the transform F_ν. The main results of this section are the next statements.

Theorem 1 *Let* $1 \le p \le 2(n+2\nu+3)/(n+2\nu+5)$ *and* $f \in L_{p,\nu}\left(E^+_{n+1}\right)$. *Then* $F_\nu f \in L_{2,\nu}\left(S^+_{n,1}\right)$ *and with constant C not depending on function* f *the next inequality is valid:*

$$\|F_\nu f\|_{L_{2,\nu}\left(S^+_{n,1}\right)} \le C\, \|f\|_{p,\nu}\,. \tag{1}$$

Corollary *Let* $f \in L_{p,\nu}\left(E^+_{n+1}\right)$ *and* $1 \le p \le 2(n+2\nu+3)/(n+2\nu+5)$, *and let* $p' = p/(p-1)$, $q = p'(n+2\nu+1)/(n+2\nu+3)$. *Then* $F_\nu f \in L_{q,\nu}\left(S^+_{n,1}\right)$ *and with constant C not depending on function* f *the next inequality is valid:*

$$\|F_\nu f\|_{L_{q,\nu}\left(S^+_{n,1}\right)} \le C\, \|f\|_{p,\nu}\,.$$

The corollary is a consequence of M. Riesz interpolation theorem [6], an estimate (1) and obvious inequality

$$\|F_\nu f\|_{L_\infty\left(S^+_{n,1}\right)} \le \|f\|_{1,\nu}\,,$$

which is valid for any $f \in L_{1,\nu}\left(E^+_{n+1}\right)$.

In this section for points in E_{n+1} we use symbols $\widetilde{x} = (x, y)$, $\widetilde{\xi} = (\xi, \tau)$, $\widetilde{u} = (u, v)$.

To prove theorem 1 we need some auxiliary facts.

Lemma 1 *For all* $n \geq 1$ *it is valid that*

$$\int\limits_{S^{+}_{n,R}} e^{-i(x,\xi)} j_\nu(y\tau)\,\tau^{2\nu+1} dS = c(n,\nu)\,R^{n/2+\nu+1} J_{n/2+\nu}\,(R|\widetilde{x}|)\,/|\widetilde{x}|^{n/2+\nu}, \tag{2}$$

где $c(n,\nu) = 2^\nu (2\pi)^{n/2}\,\Gamma(\nu+1)$, $|\widetilde{x}|^2 = |x|^2 + y^2 = x_1^2 + \cdots + x_n^2 + y^2$.

Proof First consider the case $n \geq 2$. Using parametric representation of the half-sphere $S^{+}_{n,R}$

$$\tau = R\cos\Theta_1,\ \xi_1 = R\sin\Theta_1\cos\Theta_2,\ \xi_2 = R\sin\Theta_1\sin\Theta_2\cos\Theta_3, \ldots,\ \xi_{n-1} =$$

$$= R\sin\Theta_1\sin\Theta_2\ldots\sin\Theta_{n-1}\cos\Theta_n,\ \xi_n = R\sin\Theta_1\sin\Theta_2\ldots\sin\Theta_{n-1}\sin\Theta_n,$$

$$0 \leq \Theta_1 < \pi/2,\ 0 \leq \Theta_i \leq \pi,\ i = 2,\ldots,n-1,\ 0 \leq \Theta_n \leq 2\pi,$$

evaluate an integral from (2), which we denote as $I(\widetilde{x})$, in the form

$$I(\widetilde{x}) = \int\limits_0^{\frac{\pi}{2}}\int\limits_0^{\pi}\ldots\int\limits_0^{\pi}\int\limits_0^{2\pi} \exp(-i(x_1 R\sin\Theta_1\cos\Theta_2 + \cdots + x_n R\sin\Theta_1 \times$$

$$\times \sin\Theta_2\ldots\sin\Theta_n))\,j_\nu\,(yR\cos\Theta_1)\,R^{2\nu+1}\,(\cos\Theta_1)^{2\nu+1}\,R^n\,(\sin\Theta_1)^{n-1} \times$$

$$\times (\sin\Theta_2)^{n-2}\ldots\sin\Theta_{n-1} d\Theta_1\ldots d\Theta_n = R^{2\nu+2}\int\limits_0^{\frac{\pi}{2}} j_\nu\,(yR\cos\Theta_1)\,(\cos\Theta_1)^{2\nu+1} \times$$

$$\times (\int\limits_0^{\pi}\ldots\int\limits_0^{\pi}\int\limits_0^{2\pi} \exp(-i(x_1 R\sin\Theta_1\cos\Theta_2 + \cdots + x_n R\sin\Theta_1\sin\Theta_2\ldots\sin\Theta_n))$$

$$\times (R\sin\Theta_1)^{n-1}(\sin\Theta_2)^{n-2}(\sin\Theta_3)^{n-3}\ldots\sin\Theta_{n-1} d\Theta_2\ldots d\Theta_n)\,d\Theta_1. \tag{3}$$

Inner integral in (3) is over sphere in E_n centered at 0 with radius $R\sin\Theta_1$ от функции $e^{-i(x,\xi)}$. Using a known formula (cf. [6], p. 176)

$$\int\limits_{|\xi|=\rho} e^{-i(x,\xi)} dS = (2\pi)^{n/2}\rho^{n/2}|x|^{1-n/2} J_{n/2-1}\,(\rho|x|)\,,\ x \in E_n,$$

which is valid for $n \geq 2$ and connection of functions $j_\nu(t)$ и $J_\nu(t)$ we derive that

$$I(\widetilde{x}) = 2^\nu \Gamma(\nu+1)(2\pi)^{n/2}|x|^{1-n/2} y^{-\nu} R^{n/2+\nu+2} \times$$

$$\times \int_0^{\pi/2} J_\nu\left(yR\cos\Theta_1\right) J_{n/2-1}\left(|x|R\sin\Theta_1\right)\left(\cos\Theta_1\right)^{\nu+1}\left(\sin\Theta_1\right)^{n/2} d\Theta_1.$$

Now use the Sonine integral [7]

$$\int_0^{\pi/2} J_\mu(aq\cos\varphi)\, J_\lambda(az\sin\varphi)(\cos\varphi)^{\mu+1}(\sin\varphi)^{\lambda+1} d\varphi =$$

$$= q^\mu z^\lambda a^{-1} J_{\mu+\lambda+1}\left(a\sqrt{q^2+z^2}\right) / \left(\sqrt{q^2+z^2}\right)^{\mu+\lambda+1}, \tag{4}$$

$$\operatorname{Re}\mu > -1,\ \operatorname{Re}\lambda > -1,$$

taking $\mu = \nu$, $\lambda = n/2 - 1$, $a = R$, $q = y$, $z = |x|$ and derive the lemma's result for $n \geq 2$. In case $n = 1$ the evaluation of curvilinear integral (2) over half-circumference $\xi^2 + \tau^2 = R^2$, $\tau \geq 0$ using equality $\cos t = \sqrt{\pi t/2}\, J_{-1/2}(t)$ leads to

$$\int\limits_{\substack{\xi^2+\tau^2=R^2\\ \tau\geq 0}} e^{-ix\xi} j_\nu\left(y\tau\right)\tau^{2\nu+1} dS =$$

$$= 2^\nu \sqrt{2\pi}\,\Gamma(\nu+1)\, R^{3/2+\nu} J_{1/2+\nu}\left(R\sqrt{x^2+y^2}\right) / \left(\sqrt{x^2+y^2}\right)^{1/2+\nu},$$

and it coincides with (2) for $n = 1$.

The lemma is proved.

Lemma 2 *Let $z \in \mathbb{C}$, $Re\, z > 0$ and the function $F_z(\widetilde{x})$ is defined by the formula*

$$F_z(\widetilde{x}) = c(n,\nu)\, J_{n/2+\nu+z}(|\widetilde{x}|)/|\widetilde{x}|^{n/2+\nu+z}.$$

Then

$$F_\nu\left[F_z(\cdot)\right](\widetilde{\xi}) = (c(n,\nu))^2 \frac{2^{1-z}}{\Gamma(z)}\left(1-|\widetilde{\xi}|^2\right)_+^{z-1}, \tag{5}$$

with

$$\left(1-|\widetilde{\xi}|^2\right)_+ = \begin{cases} 1-|\widetilde{\xi}|^2, & \widetilde{\xi}\in E^+_{n+1},\ |\widetilde{\xi}|<1, \\ 0, & \widetilde{\xi}\in E^+_{n+1},\ |\widetilde{\xi}|>1. \end{cases}$$

Proof Evaluating an integral which define a function $F_\nu\,[F_z(\cdot)]$ let first integrate over the half-sphere $S^+_{n,r}$ with radius r centered at origin, and then at r in limits $(0,\infty)$. Taking into account (2) let derive

$$F_\nu\,[F_z(\cdot)]\left(\widetilde{\xi}\right) =$$

$$= c(n,\nu)\int\limits_0^\infty J_{n/2+\nu+z}(r)\, r^{-n/2-\nu-z}\left\{\int\limits_{S^+_{n,r}} e^{-i(x,\xi)}\, j_\nu(y\tau)\, y^{2\nu+1} dS\right\} dr =$$

$$= (c(n,\nu))^2|\widetilde{\xi}|^{-n/2-\nu}\int\limits_0^\infty J_{n/2+\nu+z}(r)\, J_{n/2+\nu}(r|\widetilde{\xi}|)\, r^{1-z} dr.$$

The last integral is taking by a formula

$$\int\limits_0^\infty J_\mu(ax)\, J_\lambda(yx)\, x^{\lambda-\mu+1}dx = \begin{cases} \frac{2^{\lambda-\mu+1}y^\lambda}{\Gamma(\mu-\lambda)a^\mu}(a^2-y^2)^{\mu-\lambda-1}, & 0<y<a, \\ 0, & a<y<\infty, \end{cases}$$

$$a>0,\ -1<\operatorname{Re}\lambda<\operatorname{Re}\mu$$

from [8], p. 49, in which we take $\lambda = n/2+\nu$, $\mu = n/2+\nu+z$, $a=1$, $y=|\widetilde{\xi}|$. From this we derive (5).

The Lemma 2 is proved.

Now the proof of Theorem 1 is based on the Stein interpolation theorem [6] and to exploit it we need an estimate for modulus of the function $J_\mu(t)/t^\mu$ with complex μ, $\operatorname{Re}\mu \geq -1/2$, which is uniform in $t\in(0,\infty)$.

The next fact is known.

Proposition (Watson [7], p. 217) *Let x и y being reals and $x > -1/2$. Then an inequality is valid*

$$\left|J_{x+iy}(\rho)\right| \leq A_x\, e^{2\pi|y|}/\sqrt{\rho},\ \ \rho\geq 1, \tag{6}$$

in which a constant A_x is not depending on y and ρ.

Note that estimate (6) may be derived from estimates for Hankel functions $H^{(1)}_\nu(z)$, $H^{(2)}_\nu(z)$ if use integral representations for them [9], p. 181, and the formula

$J_\nu(z) = (H_\nu^{(1)}(z) + H_\nu^{(2)}(z))/2$. Also note that if x is varying on some segment than values A_x are uniformly bounded on it. Using (6), Poisson integral representation and recurrences for functions $J_\nu(t)$ it is easy to derive the next proposition.

Lemma 3 *Let H being any real number greater than $-1/2$. Then a constant exists $C = C(H) > 0$ such that for any complex number $x + iy$ from the stripe $-1/2 \leq x \leq H$ and any real $t > 0$ the next inequality is valid*

$$\left| J_{x+iy}(t)/t^{x+iy} \right| \leq C\sqrt{1+y^2}\, e^{2\pi|y|}. \tag{7}$$

Proof of Theorem 1 Note that this theorem will be proved if we establish that inequality (1) is true for functions dense in $L_{p,\,\nu}\left(E_{n+1}^+\right)$. Let $f(\xi,\tau)$ being a simple finite function. Then $F_\nu f(x,y)$ is continuous function in $E_{n+1}^+ \bigcup \{y = 0\}$ and surface integral which define this function's norm $L_{2,\,\nu}\left(S_{n,1}^+\right)$ exists. Represent a value $\|F_\nu f\|^2_{L_{2,\,\nu}\left(S_{n,1}^+\right)}$ in the next form

$$\|F_\nu f\|^2_{L_{2,\,\nu}\left(S_{n,1}^+\right)} = \int\limits_{S_{n,1}^+} F_\nu f(x,y)\overline{F_\nu f}(x,y)\, y^{2\nu+1} dS = \int\limits_{S_{n,1}^+}\int\limits_{E_{n+1}^+}\int\limits_{E_{n+1}^+} e^{-i(x,\xi)} \times$$

$$\times e^{i(x,u)} j_\nu(y\tau) j_\nu(yv) f(\xi,\tau)\overline{f}(u,v)\, \tau^{2\nu+1} v^{2\nu+1} y^{2\nu+1} d\widetilde{\xi}\, d\widetilde{u}\, dS = \int\limits_{E_{n+1}^+}\int\limits_{E_{n+1}^+}$$

$$\left\{ \int\limits_{S_{n,1}^+} e^{-i(\xi-u,x)} j_\nu(y\tau) j_\nu(yv) y^{2\nu+1} dS \right\} f(\xi,\tau)\overline{f}(u,v)\, \tau^{2\nu+1} v^{2\nu+1} d\widetilde{\xi}\, d\widetilde{u}. \tag{8}$$

(we use standard notation for complex conjugation). In view that

$$j_\nu(\lambda y) j_\nu(\lambda x) = T_x^y j_\nu(\lambda x),$$

где T_x^y is a generalized translation operator defined by the formula

$$T_x^y f(x) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+1/2)} \int\limits_0^\pi f(\sqrt{x^2+y^2-2xy\cos\Theta})(\sin\Theta)^{2\nu} d\Theta,$$

and takin into account (2) we derive that inner integral in (8) reduced to $T_\tau^v K(\xi-u,\tau)$, with

$$K(\xi-u,\tau) = c(n,\nu)\, J_{n/2+\nu}(\sqrt{|\xi-u|^2+\tau^2})/(\sqrt{|\xi-u|^2+\tau^2})^{n/2+\nu}.$$

So

$$\|F_\nu f\|^2_{L_{2,\nu}\left(S^+_{n,1}\right)} = \int\limits_{E^+_{n+1}} (K * f)\overline{f}(u, v)\, v^{2\nu+1} d\widetilde{u}, \tag{9}$$

where $K * f$ is a generalized convolution of the kernel $K(\xi, \tau)$ with a function $f(\xi, \tau)$, defined by [10]

$$(K * f)(u, v) = \int\limits_{E^+_{n+1}} T^v_\tau K(\xi - u, \tau) f(\xi, \tau)\, \tau^{2\nu+1} d\widetilde{\xi}.$$

Now apply to the integral in (9) the Hölder inequality

$$\|F_\nu f\|^2_{L_{2,\nu}\left(S^+_{n,1}\right)} \le \|K * f\|_{p',\nu} \|f\|_{p,\nu},$$

$$1/p + 1/p' = 1,$$

and to prove inequality (1) it is enough to prove that for any fixed p, $1 \le p \le 2(n+2\nu+3)/(n+2\nu+5)$ with some constant $C > 0$ not depending on a function $f \in L_{p,\nu}\left(E^+_{n+1}\right)$ the next inequality is valid

$$\|K * f\|_{p',\nu} \le C \|f\|_{p,\nu}. \tag{10}$$

Let

$$p^* = 2(n+2\nu+3)/(n+2\nu+5),\ q^* = p^*/(p^*-1) = 2(n+2\nu+3)/(n+2\nu+1).$$

From M. Riesz interpolation theorem to prove inequality (10) we need to justify estimates

$$\|K * f\|_\infty \le C_1 \|f\|_{1,\nu}, \tag{11}$$

$$f \in L_{1,\nu}\left(E^+_{n+1}\right),$$

$$\|K * f\|_{q^*,\nu} \le C_2 \|f\|_{p^*,\nu}, \tag{12}$$

$$f \in L_{p^*,\nu}\left(E^+_{n+1}\right),$$

in which constants C_1, C_2 are not depending on functions from corresponding spaces.

Inequality (11) is obvious from

$$\left|T_\tau^\nu K(\xi-u,\tau)\right| \le T_\tau^\nu \left|K(\xi-u,\tau)\right| \le \sup_{\tau\ge 0}\left|K(\xi-u,\tau)\right| \le \sup_{\tau\ge 0,\ \xi\in E_n}\left|K(\xi,\tau)\right|$$

and boundedness of the function $J_{\frac{n}{2}+\nu}(t)/t^{\frac{n}{2}+\nu}$.

Now let us prove (12). For that we will apply complex interpolation. Let

$$\tau(z) = (n+2\nu+3)z/2 - (n+2\nu+1)/2,\ z = x+iy \in \mathbb{C}.$$

Define a kernel $K_z(\widetilde{\xi})$ by

$$K_z(\widetilde{\xi}) = c(n,\nu) J_{n/2+\nu+\tau(z)}(|\widetilde{\xi}|)/|\widetilde{\xi}|^{n/2+\nu+\tau(z)},\ 0 \le \operatorname{Re} z \le 1,$$

and introduce analytic family of convolution operators

$$T_z f = K_z * f,\ 0 \le \operatorname{Re} z \le 1.$$

From inequality (7) it follows that for any simple functions f and g from $L_{1,\nu}\left(E_{n+1}^+\right)$ the function

$$F(z) = \int\limits_{E_{n+1}^+} (T_z f)\cdot g \cdot y^{2\nu+1} d\widetilde{x}$$

has permissable growth. Let use the theorem of Stein [6] with $q_0 = \infty$, $p_0 = 1$, $q_1 = 2$, $p_1 = 2$ and establish that operator norms T_z at $z = iy$ and $z = 1+iy$, $-\infty < y < \infty$ satisfy to the needed growth conditions. Really if $z = iy$ then

$$K_{iy}(\widetilde{\xi}) = c(n,\nu) J_{-1/2+i\sigma(y)}(|\widetilde{\xi}|)/|\widetilde{\xi}|^{-1/2+i\sigma(y)},$$

with $\sigma(y) = (n+2\nu+3)y/2$, and from the inequality (7) it follows

$$\left\|T_{iy} f\right\|_\infty = \left\|K_{iy} * f\right\|_\infty \le C\sqrt{1+(\sigma(y))^2}\, e^{2\pi|\sigma(y)|}\, \|f\|_{1,\nu}.$$

If $z = 1+iy$, then

$$K_{1+iy}(\widetilde{\xi}) = c(n,\nu)\, J_{n/2+\nu+1+i\sigma(y)}(|\widetilde{\xi}|)/|\widetilde{\xi}|^{n/2+\nu+1+i\sigma(y)} \in L_{2,\nu}\left(E_{n+1}^+\right),$$

and so

$$\left\|T_{1+iy} f\right\|_{2,\nu} = \left\|K_{1+iy} * f\right\|_{2,\nu} =$$

$$= (2\pi)^{-n/2} 2^{-\nu} \left(\Gamma(\nu+1)\right)^{-1} \left\|F_\nu\left(K_{1+iy}\right)\cdot F_\nu f\right\|_{2,\nu} \le \sup_{E_{n+1}^+}\left|F_\nu\left(K_{1+iy}\right)\right| \cdot \|f\|_{2,\nu}.$$

From Lemma 2 and inequality $|\Gamma(1+iy)|^{-1} \le Ce^{\pi|y|/2}$, $-\infty < y < \infty$, in which the constant $C > 0$ does not depend on y it follows that

$$\left\| T_{1+iy} f \right\|_{2,\,\nu} \le Ce^{\pi|\sigma(y)|/2} \, \|f\|_{2,\,\nu} \, .$$

So operators norms in corresponding pairs of spaces have not more than exponential growth. Due to it putting in the above mentioned theorem $z = t_\nu = (n + 2\nu + 1)/(n + 2\nu + 3)$ and taking into account that $K_{t_\nu}(\widetilde{\xi}) = K(\widetilde{\xi})$ we derive an estimate (12), because

$$\frac{1}{p_{t_\nu}} = 1 - \frac{t_\nu}{2} = 1 - \frac{n+2\nu+1}{2(n+2\nu+3)} = \frac{n+2\nu+5}{2(n+2\nu+3)} = \frac{1}{p^*},$$

and also inequality (10) is valid. As functions f of considered form are dense in $L_{p,\,\nu}\left(E^+_{n+1}\right)$, the theorem is proved.

Remark An inequality (10) may be also written in the form

$$\left\| \int\limits_{S^+_{n,1}} F_\nu f(\xi, \tau) e^{i(x,\xi)} j_\nu(y\tau)\, \tau^{2\nu+1} dS \right\|_{p',\,\nu} \le C \, \|f\|_{p,\,\nu} \, ,$$

with $p \in [1; 2(n+2\nu+3)/(n+2\nu+5)]$.

3 *N*-Dimensional Bessel Transform

Now consider in E'_n the N-dimensional Bessel transform, $n \ge 2$. Let $\nu(n) = \sum\limits_{k=1}^{n} \nu_k$. The next result is valid

Theorem 2 *Let* $1 \le p \le 2(n+\nu(n)+1/2)/(n+\nu(n)+3/2)$ *and* $f \in L_{p,\,\overline{\nu}}\left(E'_n\right)$. *Then* $F_{\overline{\nu}} f \in L_{2,\,\overline{\nu}}\left(S'_{n-1,1}\right)$ *and with constant* C *not depending on function* f*, the next inequality is valid:*

$$\|F_{\overline{\nu}} f\|_{L_{2,\overline{\nu}}\left(S'_{n-1,1}\right)} \le C \, \|f\|_{p,\,\overline{\nu}} \, . \tag{13}$$

Corollary *Let* $f \in L_{p,\,\overline{\nu}}\left(E'_n\right)$ *and* $1 \le p \le 2(n+\nu(n)+1/2)/(n+\nu(n)+3/2)$, $p' = p/(p-1)$, $q = p'(n+\nu(n)-1/2)/(n+\nu(n)+1/2)$. *Then* $F_{\overline{\nu}} f \in L_{q,\,\overline{\nu}}\left(S'_{n-1,1}\right)$ *and with constant* C *not depending on a function* f*, the next* *inequality is valid:*

$$\| F_{\overline{\nu}} f \|_{L_{q,\overline{\nu}}\left(S'_{n-1,1}\right)} \le C \, \| f \|_{p,\overline{\nu}} \, .$$

Proof of the Theorem 2 is the same as that of the Theorem 1. Instead of Lemma 1 from section 2 in this case we use

Lemma 4 *For any $n \ge 2$ an equality is valid*

$$\int\limits_{S'_{n-1,R}} \prod_{k=1}^{n} j_{\nu_k}(x_k \xi_k) \xi_k^{2\nu_k+1} dS =$$

$$= \prod_{k=1}^{n} c(\nu_k) R^{n+\nu(n)} J_{n+\nu(n)-1} \left(R|x|\right) / |x|^{n+\nu(n)-1}, \tag{14}$$

with $c(\nu_k) = 2^{\nu_k} \Gamma(\nu_k + 1)$, $k = 1, \ldots, n$, $|x|^2 = x_1^2 + \cdots + x_n^2$.

Proof We prove equality (14) by induction. An equality (14) is true for $n = 2$, for it we evaluate an integral over one-fourth of the circumference. Suppose that the formula (14) is true for $n = m$. Let derive that then it is also true for $n = m + 1$. Denote the integral in this formula by $I(x)$. We will evaluate it by integrating first over a part of the parallel $\xi_{m+1} = R \cos \Theta$, that is over a part of the sphere E'_m centered at origin with radius $R \sin \Theta$, and then integrate by Θ from 0 to $\pi/2$.On every such a parallel a factor $j_{\nu_{m+1}}(x_{m+1}\xi_{m+1})\xi_{m+1}^{2\nu_{m+1}+1}$ is constant and using (14) with $n = m$ we derive

$$I(x) = \int\limits_{S'_{m,\,R}} \prod_{k=1}^{m} j_{\nu_k}(x_k \xi_k)\, \xi_k^{2\nu_k+1} \cdot j_{\nu_{m+1}}(x_{m+1}\xi_{m+1})\, \xi_{m+1}^{2\nu_{m+1}+1} dS =$$

$$= \prod_{k=1}^{m+1} c(\nu_k) R^{m+\nu(m+1)+2} x_{m+1}^{-\nu_{m+1}} |x'|^{-m-\nu(m)+1} \int\limits_{0}^{\pi/2} J_{\nu_{m+1}}(x_{m+1} R \cos \Theta) \times$$

$$\times J_{m+\nu(m)-1}(|x'| R \sin \Theta)(\cos \Theta)^{\nu_{m+1}+1} (\sin \Theta)^{m+\nu(m)} d\Theta,$$

with $|x'|^2 = x_1^2 + \cdots + x_m^2$. Evaluate the last integral using (4), it leads to

$$I(x) = \prod_{k=1}^{m+1} c(\nu_k) R^{m+\nu(m+1)+1} J_{m+\nu(m+1)} \left(R|x|\right) / |x|^{m+\nu(m+1)},$$

$$|x|^2 = |x'|^2 + x_{m+1}^2,$$

which coincides with (14) taking $n = m + 1$.

The lemma is proved.

Further, the formula (5) is also valid for the transform $F_{\overline{\nu}}$, acting on a function

$$F_z(x) = \prod_{k=1}^{n} c(\nu_k) J_{n+\nu(n)-1+z}\left(|x|\right)/|x|^{n+\nu(n)-1+z}.$$

The kernel of the convolution operator we interpolate, which is needed to prove an estimate of the type (12), has a form $K_z(x) = F_{\tau(z)}(x)$, with $\tau(z) = (n + \nu(n) + 1/2)z - n - \nu(n) + 1/2$, $0 \leq \operatorname{Re} z \leq 1$. Let note that the generalized convolution in the considered case is defined by iterative application of generalized translations to the kernel. After that applying Stein's interpolation theorem we derive inequality (12), from which the Theorem 2 essentially follows.

To conclude we want to point out that results of this paper may be applied in transmutation theory [11, 12], for example for estimating solutions of singular B-elliptic PDE.

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Necessary Condition for the Existence of an Intertwining Operator and Classification of Transmutations on Its Basis

Sergei M. Sitnik and Viktor I. Makovetsky

Abstract The authors study second-order ordinary differential operators with functional coefficients for all derivatives and the Volterra integral operator with a definite kernel. Results of the paper establish a hyperbolic equation and additional conditions that allow one to construct a kernel according to the ODE. The statements of the paper show the possibility of splitting the ODE into classes according to the type of the kernel of the Volterra operator. Examples are considered related to ODE with Pöschl-Teller type potentials, Bessel functions with complex arguments and Euler's relation for hypergeometric functions.

1 Introduction

The transmutation operator (the intertwining operator) [1–3] is a Volterra integral operator associated with other mathematical structures, which imposes a restriction on its construction. The article proves a theorem on conditions that interlaced ordinary differential operators of second order with variable coefficients for all derivatives impose on the form of the kernel of the Volterra operator. The inverse statement is also presented that for a given kernel, interlaced structures cannot be arbitrary, but are divided into classes of feasible functions, largely determined by the structure of the core, and the ODE coefficients of the highest derivative.

S. M. Sitnik (✉)
Belgorod State National Research University (BelGU), Belgorod, Russia
e-mail: mathsms@yandex.ru; sitnik@bsu.edu.ru

V. I. Makovetsky
South Sahalin Institute for Economics and Informatics, Yuzhno-Sahalinsk, Russia
e-mail: Makov-Vikror-Sakh@yandex.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_9

2 Problem Definition

Historically, the first intertwining operators, rebounding from generalized translation operators [4, 5], appeared in the form of a Volterra type II integral operator [6, ch I, лемма 1.1.1], [7, ch. I, (1.4)]. However, according to the traditional approach to integral equations, it is more natural to take the Volterra type I integral operator in a one-dimensional space ($T : L^2(I) \to L^2(I)$) defined by the formula

$$f_1(x) = Tf_0(x) = \int\limits_0^x K(x,t) f_0(t)dt \tag{1}$$

which reduced initial class of functions $f_0 \in E_0$ into reduced class $f_1 \in E_1$, при $I = [0, b]$, $K \in L^2(I \times I)$. Transition function $K(x, t)$ is called the kernel of the transmutation operator.

If in (1) kernel $K(x, x) = \gamma \neq 0$, then by differentiation (1) it traditionally turns into

$$f_1'(x) = \gamma f_0(x) + \int\limits_0^x \frac{dK(x,t)}{dx} f_0(t)dt$$

Due to this fact, only transformations of the first kind will be investigated in the future.

Comment The features of the kernel and the coefficients of the subsequent differential equations involved in the construction of $K(x, t)$, require a more correct record of the proposed definition. Exactly

$$Tf(x) = f_1(x) = \int\limits_\varepsilon^{x-\delta} K(x,t) f_0(t)dt$$

with the subsequent passage to the limit $\varepsilon \to 0$ and $\delta \to 0$. These clarifications will be clearly spelled out when installing the conditions imposed on the kernal of the Volterra operator.

The Transmutation Operator (Intertwining operator) (in the work [1]—the transformation operator for entities A and B) is a triplet of $\{A, B, T\}$ objects that satisfy the condition

$$TA = BT \tag{2}$$

where A, B are ordinary differential operators, traditionally defined by differential expressions and initial conditions

$$A = \begin{cases} \dfrac{d}{dt}\left(a_0(t)\dfrac{df_0(t)}{dt}\right) + \dfrac{d}{dt}\left(b_0(t)f_0(t)\right) + c_0(t)f_0(t) = 0 \\ \left.\dfrac{df_0(t)}{dt}\right|_{t=0} - h_0 * f_0(t)|_{t=0} = 0 \quad \text{или} \\ f_0(0) = 0; \quad \left.\dfrac{df_0(t)}{dt}\right|_{t=0} = H_0; \end{cases} \tag{3}$$

$$B = \begin{cases} a_1(x)\dfrac{d^2 f_1(x)}{dx^2} + b_1(x)\dfrac{df_1(x)}{dx} + c_1(x)f_1(x) = 0 \\ f_1(0) = 0; \ \left.\dfrac{df_1(x)}{dx}\right|_{x=0} = H_1; \end{cases} \tag{4}$$

and T is an integral operator, represented in (1). Note that the Sturm-Liouville operator (3) is written in the generally accepted divergent form (see Sturm-Liouville theory, Wikipedia) for favorable integration in parts, which is necessary in proving the following theorem. The transition from the divergent form to the usual one is not difficult and is, for example, registered in [8, Ch. 9]

The initial conditions for determining the entity (4) are associated with the tendency of the Volterra operator of the first kind to zero for $x \to 0$. Very often, when specifying the initial ratio (3), one of the standard constructions is used [9, ch. 8]

$$f_0(0) = 1; \quad f_0'(0) = 0; \quad or$$
$$f_0(0) = 0; \quad f_0'(0) = 1;$$

which contributes to the selection of the even or odd part of the solution $f_0(t)$. Then, after applying differentiation to transform (1) into a Volterra mapping of the second kind, the introduced transmutation operator corresponds to the transformation operators $\mathbf{K}_h$ and $\mathbf{K}_\infty$ used in Sturm-Liouville spectral theory [10, 11].

The proposed definition of the intertwining operator admits a generalization by modifying the operators A and B (for example, increasing the order of differential equations), as well as changing the form of the integral transform (1), but this extension is not intended.

In papers [1, 10, 11]) for $\{a_0(t) = a_1(x) = 1;\ b_0(t) = b_1(x) = 0,\ c_0(t) = q_0(t),\ -c_1(x) = q_1(x)\}$, [12]—for Bessel operators, [13]—in general, a relationship is established between the coefficients of differential operators and the type of the $K(x,t)$ transformation operator.

Theorem 1 *A necessary and sufficient conditions that the Volterra integral operator (1) be the transmutation operator for ordinary differential (3, 4) operators is:*

(a) The kernel of the transformation operator (1) must be a solution to the hyperbolic equation

$$\begin{cases} L\left[K(x,t)\right] = \left[\frac{\partial}{\partial t}\left(a_0(t)\frac{\partial K(x,t)}{\partial t}\right) - b_0(t)\frac{\partial K(x,t)}{\partial t} + c_0(t)K(x,t)\right] - \\ \qquad - \left[a_1(x)\frac{\partial^2 K(x,t)}{\partial x^2} + b_1(x)\frac{\partial K(x,t)}{\partial x} + c_1(x)K(x,t)\right] = 0 \end{cases} \tag{5a}$$

(b) On the characteristic t = x, the kernel K (x, t) and its first derivative with respect to t exist; at $t \to x - \delta$ и $\delta \to 0$

$$b1)\quad a_0(x) = a_1(x) = a(x)$$

$$b2)\quad 2a(x)\frac{dK(x,x-\delta)}{dx} + (b_1(x) - b_0(x))K(x,x-\delta) = 0 \tag{5b}$$

(c) With initial condition $t = \varepsilon \to 0$

$$\left\{a(\varepsilon)\left[\frac{dK(x,t)}{dt}\right]_{t=\varepsilon} - b_0(\varepsilon)K(x,\varepsilon) - h_0 a(\varepsilon)K(x,\varepsilon)\right\} f_0(\varepsilon) \to 0 \tag{5c}$$

(d) Condition at the edge. Under $\delta < \varepsilon, \quad \delta \to 0, \quad \varepsilon \to 0$

$$K(\varepsilon, \varepsilon - \delta) * f_0(\varepsilon) \to 0; \tag{5d}$$

Note that in (b) and (c) it is not necessary to know the explicit form of the function $f_0(x)$. What is important is the tempo of striving $f_0(\varepsilon)$ to zero with $\varepsilon \to 0$ to compensate for the singularity of the coefficients a, b, c at the origin point.

The proof of the theorem is based on the definition of the transmutation operator (2), which with respect to ordinary differential operators looks like

$$T(Af_0)(x) = B(Tf_0)(x); \quad \forall x;$$

The integral in the left component of the equality is taken in parts, and in the right component differentiation takes place according to its variable upper limit.

Proof of Theorem 1 Let us prove the assertions of the theorem, generalizing the method [1, 9–12]. For convenience and brevity of the record, we introduce the notation

$$K_0(x) = K(x, x-\delta); \quad f(x) = f_0(x);$$

$$\partial_t = \frac{d}{dt}; \quad \partial_{tt} = \frac{d^2}{dt^2}; \quad \partial_x = \frac{d}{dx}; \quad \partial_{xx} = \frac{d^2}{dx^2};$$

The first operation will be TA.

$$TA(f(x)) = \int_{\varepsilon}^{x-\delta} K(x,t)\{\partial_x(a_0(x)\partial_x f(x)) + \partial_x(b_0(x)f(x)) + c_0(x)f(x)\}dt$$

The integral with the first term is taken two times in parts. It is precisely at this moment that the record of the operator (3) in a divergent form is highly desirable. Similarly, in parts, the second addend will be transformed only once. This leads to the following result

$$\begin{aligned}
&\mathrm{TA}(f(x)) = a_0(x)K_0(x)\partial_x f(x) + \\
&+ \left[K_0(x)b_0(x) - a_0(x)\ \{\partial_t K(x,t)\}|_{t=x-\delta}\right] f(x) - a_0(\varepsilon)K(x,\varepsilon)\ \{\partial_t f(t)\}|_{t=\varepsilon} + \\
&+ a_0(\varepsilon)\ \{\partial_t K(x,t)\}|_{t=\varepsilon}\ f(\varepsilon) - b_0(\varepsilon)K(x,\varepsilon)f(\varepsilon) + \\
&+ \int_{\varepsilon}^{x-\delta} \{\partial_t[a_0(t)\partial_t K(x,t)] - b_0(t)\partial_t K(x.t) + c_0(t)K(x,t)\}\ f(t)dt
\end{aligned}$$

Further action is the study of the relationship BT

$$B(Tf(x)) = \left\{a_1(x)\partial_{x,x}(\circ) + b_1(x)\partial_x(\circ) + c_1(x)(\circ)\right\}\left\{\int_{\varepsilon}^{x-\delta} K(x,t)f(t)dt\right\}$$

The calculation of the derivative of the integral over a variable upper limit generates the equality

$$\begin{aligned}
&B(Tf(x)) = \{a_1(x)\partial_x K_0(x) + a_1(x)\ [\partial_x K(x,t)\rfloor|_{t=x-\delta} + \\
&b_1(x)K_0(x)\}f(x) + a_1(x)K_0(x)\partial_x f(x) + \\
&+ \int_{\varepsilon}^{x-\delta} K(x,t)\{a_1(x)\partial_{xx}K(x,t) + b_1(x)\partial_x K(x,t) + c_1(x)K(x,t)\}f(x)
\end{aligned}$$

Comparison of integrands implies (5a). Due to the arbitrariness of $f(x)$, the coefficients in front of the function and its first derivative should be separately equal to zero. Comparing the elements before the first derivative gives (5b.1). If we take into account this fact in the coefficient adjacent to f (x), as well as for $\delta \to 0$, use equality

$$\frac{dK(x, x-\delta)}{dx} = \left.\frac{\partial K(x,t)}{\partial x}\right|_{t=x} + \left.\frac{\partial K(x,t)}{\partial t}\right|_{t=x}$$

then the grouping of elements before $f(x)$ establishes a correspondence (5b.2). It remains to group the initial conditions string when $t = \varepsilon \to 0$. All its elements are entirely in the TA operator. There will be an expression

$$-a_2(\varepsilon)K(x,\varepsilon)\ \{\partial_t f(t)\}|_{t=\varepsilon} + a(\varepsilon)\ \{\partial_t K(x,t)\}|_{t=\varepsilon}\ f(\varepsilon) - b_2(\varepsilon)K(x,\varepsilon)f(\varepsilon)$$

The final result is fixed in the condition (5c). To formulate the condition at the vertex we take the derivative of (1)

$$\partial_x Tf(x) = K(x, x-\delta)f(x-\delta) + \int\limits_{\varepsilon}^{x-\delta} \partial_x K(x,t)f(t)dt$$

At the point $x = \delta + \varepsilon$

$$\partial\ Tf(x)|_{x=\delta+\varepsilon} = K(\delta+\varepsilon, \varepsilon) * f(\varepsilon)$$

In the end we take into account the initial conditions

$$\partial\ f_1(x)|_{x=\varepsilon} - h_1\ f_1(x)|_{x=\varepsilon} = 0; \quad \partial\ f_0(x)|_{x=\varepsilon} - h_0\ f_0(x)|_{x=\varepsilon} = 0;$$

what gives (5d).

The presented conditions refer to an arbitrary form of the kernel, but even they impose substantial restrictions on it and on the articulated operators A and B. First, the coefficients of the highest derivative in (3) and (4) must coincide with the accuracy of the free variable, that is, $a_0(t) = a_1(x)$ with $t = x$. The type of ordinary differential equation is largely determined by these coefficients, so often transmutation occurs between A and B with similar properties. Secondly, the absence of singularity of the kernel $K(x,t)$ and its derivative with respect to the argument t leaves outside the scope of this consideration intertwining transformations with special points, for example, the integral Mohler-Fock representation for Legendre functions and their generalizations [14]

$$P_{-\frac{1}{2}+\imath\nu}(\cosh x) = \frac{2}{\pi}\int\limits_0^x \frac{\cos(xt)}{\sqrt{2\,(\cosh x - \cosh t)}}\,dt$$

It is possible to overcome this difficulties with the help of integrals in the sense of the Hadamard finite part [15], but it requires a more detailed consideration of the presented structures.

An Example of the Theorem 1 Let us show that the Volterra operator performing the transformation for Gegenbauer polynomials

$$\int_0^x \left(x^2-t^2\right)^{\beta-1} C_{2n}^{2\nu}(t)dt = \frac{2\sqrt{\pi}\Gamma(\beta)}{\Gamma\left(\beta+\frac{1}{2}\right)} x^{2\beta-1} C_n^{\beta}\left(2x^2-1\right); \quad Re(\beta) > 2; \tag{6}$$

is a transmutation operator. The presented identity follows from [16, Vol II, 16.3, (19)] after replacing the variable and modifying the indices. It is easy to check that the function

$$f_0(t) = C_{2n}^{2\beta}(t)$$

turns out to be a solution of a differential operator (3) with coefficients

$$a_0(t) = (1-t^2); \quad b_0(t) = n(1-4\beta)t; \quad c_0(t) = 4n(n+2\beta)+(4\beta-1);$$

For even lower symbols (2n), the derivative of the Gegenbauer polynomials vanishes when t = 0, so the middle row is used as the initial condition in the operator (3) for $h_0 = 0$. Right part

$$f_1(x) = \frac{2\sqrt{\pi}\Gamma(\beta)}{\Gamma\left(\beta+\frac{1}{2}\right)} x^{2\beta-1} C_n^{\beta}\left(2x^2-1\right);$$

satisfies the operator (4) with coefficients

$$a_1(x) = (1-x^2); \quad b_1(x) = \frac{2(1-\beta)-3x^2}{x}; \quad c_1(x) = 4(n+\beta)^2-1;$$

If we substitute the kernel

$$K(x,t) = \left(x^2-t^2\right)^{\beta-1} \tag{7}$$

into a hyperbolic equation (5a) with the above groups of coefficients a, b, c, then it will turn it into a true equality. The core exponent ensures that the condition on the characteristic is met.

The left part of the initial condition (5c) is expanded in a series with the first member

$$h_0 \frac{4^n \sqrt{\pi}\Gamma(m+2\beta)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma(2n+1)\Gamma(2\beta)} x^2 + O(\varepsilon)$$

However, it was previously noted that $h_0 = 0$, and, therefore, is realized (5c). The condition at the vertex (5d) is an identity due to the type of kernel. As a result of the fulfillment of all conditions, the Volterra operator of the first kind becomes a transformation operator for ordinary differential operators (3) and (4).

3 Formulation and Specification of Reverse Statement

It can be seen from Theorem 1 that the kernal construction of the transmutation operator can be determined on the basis of the coefficients of intertwined ordinary differential operators (3–4). In this article, we make following inverse statement the cornerstone—'*The kernals of the $K(x,t)$ transmutation operator split the intertwined operators A and B into classes, causing the appearance of their coefficients.*'

This position is related to the conditions on the characteristic of the hyperbolic operator (5a). The work [3] noted that "the content of the Copson lemma is that the initial data on the characteristics cannot be specified arbitrarily, they must be connected by Bushman-Erdeyi operators of the first kind. The main point of the proposed current article is the opposite and extended statement".

Statement 1 Conditions on the characteristic of a hyperbolic equation (5a) together with (5b) are necessary to classify the linked operators A and B by classes of kernels $K(x,t)$.

Example for Statement 1 Let us find the classes of intertwined operators A and B for an already familiar kernel (7), but with a different coefficient in the main part. Exactly,

$$K(x,t) = \left(x^2 - t^2\right)^{\beta-1}; \quad a_0(t) = 1; \quad a_1(x) = 1;$$

Substituting the specified kernel into the hyperbolic equation (5a) leads to the relation

$$L[K(x,t)] = -\left(x^2 - t^2\right)^{\beta-2}$$
$$\left(-4(\beta-1)^2 + 2(\beta-1)tb_0(t) - 2(\beta-1)xb_1(x) + (x^2-t^2)(c_0(t) - c_1(x))\right)$$

in this embodiment, the result can be obtained directly, without using the ratio on the characteristic. It is easy to see that the right-hand side vanishes at constant and equal values of the free members of the 'c' and coefficients of the 'b', inversely proportional to their arguments

$$b_0(t) = \frac{b_0}{t}; \quad b_1(x) = \frac{b_1}{x};$$

In this case, the next identity must be satisfied

$$b_1 = b_0 + 2 - 2\beta$$

with arbitrary b_0. A change in b_0 leads to an extensive one-parameter class of possible representations of the operators A and B, but the most attractive results are obtained for $b_0 = -(2\,nu + 1)$. Then

$$a_0(t) = 1; \quad a_1(x) = 1;$$
$$b_0(t) = -\frac{2\nu + 1}{t}; \quad b_1(x) = -\frac{2(\beta + \nu) - 1}{x}; \quad c_0(t) = \omega^2; \quad c_1(x) = \omega^2;$$

The solutions of ordinary differential operators (3) and (4) with $h_0 = 0$ are Bessel functions, which makes it possible to write the transmutation operator [17, Vol II, No 2.12.4 (6)]

$$\int_0^x \left(x^2 - t^2\right)^{\beta-1} t^{\nu+1} J_\nu(\omega t)dt = \frac{2^{\beta-1}x^{\beta+\nu}}{\omega^\beta}\Gamma(\beta)J_{\beta+\nu}(\omega x); \tag{8}$$

Thus, we arrive at the following conclusion: *intertwined operators with a known form of the kernel $K(x, t)$ are not constructed in an arbitrary way and are largely determined by the type of this kernel*. Most often, the main factor in partitioning differential operators (3) and (4) into classes that are consistent with the kernel $K(x, t)$, is the main part of these operators $a(x) = a_0(x) = a_1(x)$. Recall the generality of the principal parts, up to a free variable, written in (5a).

With respect to the ad hoc kernels $K(x, t)$, statement 1 is strictly impossible to prove strictly, but it is well formalized for specific categories of $K(x, t)$. We introduce auxiliary expressions

$$\Upsilon_1(x) = 4a_0(x)\phi'(x)\Omega'(x) + \Omega(x)\left(\phi'(x)a_0'(x) + 2a_0(x)\phi''(x)\right)$$
$$\Upsilon_2(x) = -\Omega(x)\phi'(x)\left(b_1(x) - b_0(x)\right)$$

Lemma 1 *For operator class*

$$K(x, t) = K\left(\Omega(x)\sqrt{\phi(x) - \phi(t)}\right) \tag{9}$$

with kernel satisfying the requirements ((5a)–(5d)), the conditions on the characteristic impose the following restrictions on the coefficients of the intertwined operators A and B

$$\Psi_1(x) = K'(0)\,(2\Upsilon_2(x) - \Upsilon_1(x)) = 0 \tag{10a}$$

$$\Psi_2(x) = -2K(0)\,(c_1(x) - c_0(x)) + K''(0)\Omega(x)\,(\Upsilon_2(x) - \Upsilon_1(x)) \tag{10b}$$

For even functions, the first equality is automatically fulfilled, for odd functions—the second one. The proof of the lemma is carried out by substituting (9) into a hyperbolic operator (5a). As a result, when $t \to x$, an expression appears that contains singular and regular parts

$$\frac{\Psi_1(x)}{4\sqrt{\phi'(x)(x-t)}} + \Psi_2(x) + O(x-t)$$

In fact, a parametrix is constructed modulo smoothing operators used recently in hyperbolic equations [18], although the study of relations on characteristics has a rich history [19, Ch. 4]

Example 2 to Lemma 1 Consider the class of kernals of the form

$$K(x,t) = J_0\left(\Omega(x)\sqrt{\cosh(\mu x) - \cosh(\mu t)}\right) \tag{11}$$

under $a_0(t) = 1; a_1(x) = 1$. The condition (5b) immediately leads to the equality $b_1(x) = b_0(x)$, moreover, due to the parity of the Bessel function of zero index $J_0(\,xi)$, the first line in the condition (10a) is performed automatically. The second generates identity

$$\frac{1}{2}\mu\Omega(x)\left(\mu\Omega(x)\cosh(\mu x) + 2\sinh(\mu x)\Omega'(x)\right) = 0$$

Selection of $\Omega(x)$ in the form of an exponent makes possible the following kind of coefficients

$$b_0(t) = \frac{b}{\sinh(\mu t)}; \quad b_1(x) = \frac{b}{\sinh(\mu x)};$$
$$\Omega(x) = \exp\left(-\frac{\mu}{2}x\right);$$
$$c_1(x) = c_0(x) + \frac{1}{2}\mu^2\beta^2\exp(-2\mu x);$$

If these values are substituted into (5a), then we get an expression that includes two linearly independent terms, one of which contains the factor ‘b’, the second—the factor $c_0(t) - c_0(x)$. Equating $b = 0;\ c_0(\,xi) = \omega^2$, we arrive at the transmutation operator

$$f_1(x) = Tf_0(x) = \int_0^x J_0\left(\exp\left(-\frac{\mu}{2}x\right)\sqrt{\cosh(\mu x) - \cosh(\mu t)}\right) f_0(t)dt \tag{12}$$

intertwining ordinary differential operators

$$\frac{d^2}{dt^2} f_0(t) + \omega^2 f_0(t) = 0 \tag{13a}$$

$$\frac{d^2}{dx^2} f_1(x) + \left(\omega^2 + \frac{1}{2}\mu^2\beta^2 \exp(-2\mu x)\right) f_1(x) = 0 \tag{13b}$$

Earlier Sergey M. Sitnik obtained the kernel (11) by the method of fixed-point iteration, solving the integral equation given in the work of Marchenko [11, Ch. I].

If the initial condition is written in (3) in the traditional form with $h_0 = 0$, and a solution that satisfies the zero initial condition is selected in (4), then the transmutation operator taking into account [20, No.2.37 b] will give the following result

$$\int_0^x J_0\left(e^{-\frac{\mu}{2}x}\sqrt{\cosh(\mu x) - \cosh(\mu t)}\right) \cos(\omega t)dt =$$
$$= -\frac{\imath\pi}{2\mu}\frac{1}{\sinh\left(\frac{\pi\omega}{\mu}\right)}\left[J_{\frac{\imath\omega}{\mu}}\left(\frac{\beta}{\sqrt{2}}\right)J_{-\frac{\imath\omega}{\mu}}\left(\frac{\beta e^{-\mu x}}{\sqrt{2}}\right) - J_{-\frac{\imath\omega}{\mu}}\left(\frac{\beta}{\sqrt{2}}\right)J_{\frac{\imath\omega}{\mu}}\left(\frac{\beta e^{-\mu x}}{\sqrt{2}}\right)\right] \tag{14}$$

For $\mu \to 0$, the relation presented is reduced to the Vekua transformation operator [21, Ch. I, Par.12], created at the time to solve elliptic equations of mathematical physics. Its feature is the shift in spectral parameter

$$\int_0^x J_0\left(\beta\sqrt{x^2 - t^2}\right)\cos(\omega t)dt = \frac{\sin\left(\sqrt{\omega^2 + \beta^2}\,x\right)}{\sqrt{\omega^2 + \beta^2}} \tag{15}$$

Equalities (10a) lead to another class of transmutation operators.

An isolated class with respect to intertwined second-order operators are the Bushman-Erdei transformations, which include the Legendre functions [3]. It suffices to look at the tables [17, vol II, No 2.17-2.18] to see in most of the options the record of the transformed component of $f_1(x)$ by means of the generalized hypergeometric series ${}_pF_q$ with $p + q > 3$. Thus, a very significant set of second-order differential operators ‘B’ do not fit into the construction (4). But in rare exceptions, the method of studying a hyperbolic operator on the characteristic admits cases of finding new versions of the Bushman-Erdeia OP.

Let’s start with the traditional core of the Bushman-Erdeyi operator

$$K(x,t) = P_\nu\left(\frac{t}{x}\right) \tag{16}$$

where the Legendre function $P_\nu(z)$ is a solution of a differential equation [16, vol I, Ch. III]. The singularity in calculating $L[K(x,t)]$ with $t \to x$ is

$$\frac{1}{2x^2}\Big[-2\nu(\nu+1) + \nu(\nu+1)(b_1(x) - b_0(x))x - 2x^2(c_1(x) - c_0(x))\Big] + O(t-x)$$

For its elimination it is enough to put

$$a_0(t) = a_1(x) = 1; \quad b_0(t) = b_1(x) = 0; \quad c_0 = \omega^2; \quad c_(x) = \omega^2 - \frac{\nu(\nu+1)}{x^2}; \tag{17}$$

With such coefficients, the relations (3) and (4) taking into account the initial conditions in (3) and the finiteness of the solution at the origin for (4) are given for integer values the index ν [17, Vol II, No 2.17.7 (1)]. Exactly,

$$\int_0^x P_{2n+1}\left(\frac{t}{x}\right)\sin(\omega t)dt = (-1)^n\sqrt{\frac{\pi x}{2\omega}}J_{2n+\frac{3}{2}}(\omega x) \tag{18a}$$

and

$$\int_0^x P_{2n}\left(\frac{t}{x}\right)\cos(\omega t)dt = (-1)^n\sqrt{\frac{\pi x}{2\omega}}J_{2n+\frac{1}{2}}(\omega x) \tag{18b}$$

We show that the kernel (18) can be extended to wider classes of functions. In this case, we obtain the original solutions of hyperbolic equations and representations for some hypergeometric functions, including composite arguments. Consider the Bushman-Erdeia transmutation operators with kernels

$$K(x,t) = P_\nu\left(\frac{\sinh(\mu t)}{\sinh(\mu x)}\right) \tag{19}$$

The study of the relation on the characteristic $L[K(x,t)]$ with $t \to x$ leads to an estimate

$$L[K(x,t)] = -\nu(\nu+1)\mu^2 Csch^2(\mu x) + \frac{1}{2}\nu(\nu+1)(b_1(x) - b_0(x)) \\ - (c_1(x) - c_0(x) + O(t-x) \tag{20}$$

The selection of coefficients in the equations is not complicated. Initially, they are located so as to nullify the final term in (20), and then the final sorting takes place to turn (5a) into an identity. Finally, it found that the kernel (19) satisfies the hyperbolic equation

$$\frac{\partial^2 K(x,t)}{\partial t^2} + \omega^2 K(x,t) = \frac{\partial^2 K(x,t)}{\partial x^2} + \left(\omega^2 - \mu^2 \frac{\nu(\nu+1)}{\sinh^2(\mu x)}\right) K(x,t) \tag{21}$$

The steps involved in transforming an ordinary differential equation for the operator 'B' are well known from books on quantum mechanics [22, Problem 39]. Replace variables and the function sought are sequentially performed

$$y = -\sinh^2(\mu x); \quad f_1(y) = y^{\frac{\nu+1}{2}} v(y)$$

and also, parameter designation is introduced

$$a = \frac{\nu}{2} + \imath\frac{\omega}{2\mu}; \quad b = -\frac{\nu}{2} + \imath\frac{\omega}{2\mu};$$

The solution consists of a linear combination of the regular part tending to zero for $x \to 0$

$${}_2F_1\left(-a, b, \frac{1}{2} - \nu, -\sinh^2(\mu x)\right)(-\sinh^2(\mu x))^{\frac{\nu+1}{2}}$$

and singular part

$${}_2F_1\left(\frac{1}{2} - b, \frac{1}{2} + a, \frac{3}{2} + \nu, -\sinh^2(\mu x)\right)(-\sinh^2(\mu x))^{-\frac{\nu}{2}}$$

Clearly, the transmutation operator

$$\int_0^x P_\nu\left(\frac{\sinh(\mu t)}{\sinh(\mu x)}\right)\cos(\omega t)dt \tag{22}$$

correlates only with the regular component, however, due to the complexity of the parameters of the hypergeometric function, it is very difficult to trace the exact match. Nevertheless, the finite number of components in the Legendre polynomials for integer $\nu = 2n$ allows us to express and investigate the result in a much simpler form.

When $nu = 2$, the integral (22) takes the value

$$f_1(x) = -\frac{\sin(\omega x)}{2\omega} + \\ + \frac{3}{4}\frac{1}{\sinh^2(\mu x)}\left[-\frac{\sin\omega x}{\omega} + \frac{\omega\cosh(2\mu x)\sin(\omega x) + 2\mu\cos(\omega x)\sinh(2\mu x)}{\omega^2 + (2\mu)^2}\right]$$

When $\mu \to 0$, this representation completely coincides with the right-hand side of Eq. (18b) for $n = 1$. The presented examples with different integer indices describe a certain set of Bargman potentials [23, Ch. VI.I] and can be used for their construction and study. Calculations with kernels of the type (22) are carried out similarly. We present their results in the following lemma.

Lemma 2 *Bushman-Erdei transmutation operators with kernels*

$$K(x,t) = P_\nu\left(\frac{\sinh(\mu t)}{\sinh(\mu x)}\right); \qquad K(x,t) = P_\nu\left(\frac{\cosh(\mu x)}{\cosh(\mu t)}\right);$$
$$K(x,t) = P_\nu\left(\frac{\sin(\mu t)}{\sin(\mu x)}\right); \qquad K(x,t) = P_\nu\left(\frac{\cos(\mu x)}{\cos(\mu t)}\right); \tag{23}$$

connect the solution to the equation

$$\frac{d^2 f_0(t)}{dt^2} + \omega^2 f_0(t) = 0$$

with solutions of equations

$$\frac{d^2 f_1(x)}{dx^2} + \left(\omega^2 + V(x)\right) f_1(x) = 0$$

for potentials

$$V(x) = \mu^2\nu(\nu+1)U(x) \tag{24}$$

where respectively

$$U(x) = \frac{1}{\sinh^2(\mu x)}; \qquad U(x) = \frac{1}{\cosh^2(\mu x)};$$
$$U(x) = \frac{1}{\sin^2(\mu x)}; \qquad U(x) = \frac{1}{\cos^2(\mu x)}; \tag{25}$$

In quantum mechanics, the potentials presented are called Peschl-Teller potentials (modified and ordinary) [22, Problems No 38, 39]. Their use for integer $\nu = n$

is important when considering eigenvalues and eigenfunctions that are consistent with boundary or other quantization conditions [24]. The group of transmutation operators presented in Lemma 3 is an essential addition to the set of Bushman-Erdei operators given in the work [3].

4 Some Convolutions as Transmutation Operators and Their Modifications

Convolution type transformation operators have been extensively studied in the literature (see, for example, [25]), so we will only touch on those that are important from a transmutation point of view.

Lemma 3 *By definition, each transmutation operator is a Volterra operator of the first or second kind, the converse is false.*

Let us give an example of the last statement—the Kapteyn trigonometric integral [26, 12.21]

$$\int_0^x \cos(x-t) J_0(t) dt = x J_0(x)$$

Here the kernel is $K(x,t) = cos(x-t)$, and the coefficients in the ordinary differential operators (3) and (4) are

$$\begin{array}{l} a_0(t) = 1; \quad b_0(t) = \frac{1}{t}; \quad c_0(t) = 1 + \frac{1}{t^2}; \\ a_1(x) = 1; \; b_1(x) = -\frac{1}{x}; \; c_1(x) = 1 + \frac{1}{x^2}; \end{array}$$

It is easy to check the impracticability of the hyperbolic equation (5a) with a similar combination of elements necessary for the transmutation operator.

At the same time, extensive combinations of $K(x,t)$; $f_0(t)$; $f_1(x)$ associated with hypergeometric functions for which there is a possibility of linking. Imagine an initially simple illustration. It is easy to check that the coefficients

$$\begin{array}{lll} a_0(t) = t; & b_0(t) = (1-\beta) - t; & c_0(t) = \beta - \alpha; \\ a_1(x) = x; & b_1(x) = (2-\beta-\gamma) - x; & c_1(x) = \beta + \gamma - \alpha - 1; \end{array}$$

substituted into Eqs. (3) and (4) lead to Kummer intertwined functions. In this case, the kernal

$$K(x,t) = (x-t)^{\gamma-1}$$

Replacing the variables $t \to xt$ gives the well-known integral relation [27, Vol II, No 20.3 (2)]

$$\int_0^x (x-t)^{\gamma-1} t^{\beta-1} \, {}_1F_1(\alpha,\beta,t)dt = \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} x^{\beta+\gamma-1} \, {}_1F_1(\alpha,\beta+\gamma,x); \qquad (26)$$

A degenerate hypergeometric function with an integral nonpositive first argument is a generalized Laguerre polynomial

$${}_1F_1(-n,\beta,z) = L_n^{\beta}(x)$$

Together with (26), this leads to the transmutation operator [27, Vol II, No 16.6 (5)]

$$\int_0^1 (1-t)^{\beta-\alpha-1} t^{\alpha} L_n^{\alpha}(xt)dt = \frac{\Gamma(\alpha+n+1)\Gamma(\beta-\alpha)}{\Gamma(\beta+n+1)} L_n^{\beta}(x); \qquad (27)$$

5 Euler Transformation for Hypergeometric Functions as a Transmutation Operator

For the basis of further intertwining operators, we take the Euler transformation [28, Ch. 4]

$${}_{p+1}F_{q+1}\left(\begin{matrix} a_1 \dots a_p \ c \\ b_1 \dots b_q \ d \end{matrix}; \ z \right) =$$

$$= \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_0^1 \xi^{c-1}(1-\xi)^{d-c-1} \, {}_pF_q\left(\begin{matrix} a_1 \dots a_p \ c \\ b_1 \dots b_q \ d \end{matrix}; \ z\xi \right) d\xi$$

There are two directions in which it can develop. In the first case, this is a transition to the standard transformation operator, by replacing $t = z\xi$.

$$z^{d+1} \, {}_{p+1}F_{q+1}\left(\begin{matrix} a_1 \dots a_p \ c \\ b_1 \dots b_q \ d \end{matrix}; \ z \right) =$$

$$= \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_0^z (z-t)^{d-c-1} t^{c-1} \, {}_pF_q\left(\begin{matrix} a_1 \dots a_p \ c \\ b_1 \dots b_q \ d \end{matrix}; \ t \right) d\xi \qquad (28)$$

The second option is more interesting. The Euler transformation initially relies on $z = \kappa x^2$, where $\kappa = \pm 1$. Then the integral follows the replacement $\xi = \eta^2$ with the following substitution $t = z\eta$. The final ratio is as follows.

$$z^{2(d-1)}\,{}_{p+1}F_{q+1}\left(\begin{matrix} a_1 \dots a_p \ c \\ b_1 \dots b_q \ d \end{matrix}; \kappa z^2\right) =$$

$$= \frac{2\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int\limits_0^z (z^2 - t^2)^{d-c-1} t^{2c-1}\,{}_pF_q\left(\begin{matrix} a_1 \dots a_p \ c \\ b_1 \dots b_q \ d \end{matrix}; \kappa t^2\right) d\xi \qquad (29)$$

We emphasize that the integral relations (28) and (29) in this article are only postulated as Euler transformation operators and their modifications. The proof that they turn out to be intertwining operators in the general version is difficult, if only by replacing the standard hyperbolic equation (5a) with its generalized analogue. One of the works, highlighting the path of development in this direction [29].

Since the preimage ${}_0F_1$ satisfies the operator with the second, and, accordingly, the image of the Euler transformation ${}_1F_2$ to the operator with the third derivative, in the framework of second-order differential equations, only two types of hypergeometric functions [30, 31]:

$${}_0F_0(t) = F(;;t) = e^t; \quad {}_1F_0(t) = F(a;;t) = (1-t)^{-a};$$

The Euler transformation for ${}_0F_0$ leads to an integral representation of the Kummer function [16, раздел 6.5], [2, 32, 33]

$$\Theta(c,d,x) = x^{d-1}\,{}_1F_1(c;d;x) = \frac{\Gamma(d)}{\Gamma(d-c)\Gamma(c)} \int\limits_0^x (x-t)^{d-c-1} t^{c-1}\,{}_0F_0(t)dt \qquad (30)$$

provided that x is a real variable and $Re(d) > Re(c) > 0$. We note an important fact: the resulting transformation operator covers a smaller set of parameters than the series

$${}_1F_1(c;d;x) = \sum_{k=0}^{\infty} \frac{(c)_k}{(d)_k k!} x^k$$

where $(q)_k$ is a Pohgammer symbol, since the inequalities $Re(d) > Re(c) > 0$ impose significant restrictions on the domains of parameter changes.

We prove that the transformation operator (30) is a transmutation operator. The function ${}_0F_0(t) = e^t$, which is present under the integral sign, is a solution of a first order differential equation, but the conjugate form (3) allows you to artificially add another differentiation digit. Exactly if

$$f_0(t) = t^{c-1}\,{}_0F_0(t) = t^{c-1}e^t$$

$$a_0(t) = t; \quad b_0(t) = 1 - c - t; \quad c_0(t) = 0;$$

then the equality (3) takes the form

$$\frac{d}{dt}\left[a_0(t)\frac{df_0(t)}{dt}+b_0(t)f_0(t)\right]=0$$

For the transformed function $f_1(x)=x^{d-1}\,{}_1F_1(c,d,x)$, the identity (4) with the coefficients

$$a_1(x)=x;\quad b_1(x)=2-d-x;\quad c_1(x)=d-c-1;$$

It is easy to verify that with the coefficients indicated above, and $K(x,t)=(x-t)^{d-c-1}$, the hyperbolic equation (5a) holds.

Thus, the formula (30) is a two-parameter family of intertwining operators. According to the definition [34, Ch I, Def 2.1], it simultaneously belongs to the class of fractional integrals. The enumeration of the permissible values of the parameters [33, Ch 3] leads to many interesting results illustrating the significance of transmutation operators. For example, ratio

$$\Theta(1,2,x)=e^x-1;\quad \text{при}\quad x>0$$

with the help of the OP it turns out much easier to expand the Kummer function in a series. On the other hand, much less elementary results are possible. With real $x>0$

$$\Theta(\frac{3}{4},\frac{3}{2},x)=\sqrt{2}\sqrt[4]{x}e^{\frac{x}{2}}\Gamma\left(\frac{5}{4}\right)I_{\frac{1}{4}}\left(\frac{x}{2}\right)$$

with a modified Bessel function, a fractional argument—and this is not the highest bar of complexity.

At one time, the identity (28) was used by Leonard Euler to determine the traditional hypergeometric function. Because of the literal following (28), the definition will take on a different look.

$$x^{d-1}\frac{\pi}{\sin(\pi c)\Gamma(c)\Gamma(1-c)}\,{}_2F_1(a,c,d,x)=$$
$$=\frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int_0^x(x-t)^{d-c-1}t^{c-1}\,{}_1F_0(a,t)dt \tag{31}$$

or

$$x^{d-1}\frac{\pi}{\sin(\pi c)\Gamma(c)\Gamma(1-c)}\,{}_2F_1(a,c,d,x)=\frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int_0^x\frac{(x-t)^{d-c-1}t^{c-1}}{(1-t)^a}dt \tag{32}$$

The fact that the presented formula has a three-century history does not save it from checking for the agreement of all the equations for the transition from the variety of transformations to the class of intertwining operators. We have

$$a_0(t) = t(1-t); \quad b_0(t) = 2 - c - (3 + a - c) - \frac{da_0(t)}{dt};$$
$$a_1(x) = x(1-x); \quad b_1(x) = 2 - d - (3 + a + c - 2d)x;$$
$$c_0(t) = c - a - 1 - \frac{db_0(t)}{dt}; \quad c_1(x) = -(1 + a - d)(1 + c - d).$$

If, as before, $K(x,t) = (x-t)^{d-c-1}$, then the hyperbolic equation (5a) turns into an identity. Note again that the transformation is valid only for real $0 < x < 1$ and $Re(d) > Re(c) > 0$. In addition, the parameter c should not be an integer. The number of representatives of this transmutation operator with different variants of the coefficients is almost innumerable [33, Ch. 2, Section 2.4]

Lemma 4 *The Euler transformation (32) is the intertwining operator for the hypergeometric functions when selecting the coefficients in (3) and (4) mentioned above.*

We will not check the relations (29), but instead show how knowing the values on the characteristic of a hyperbolic equation helps to find a rather complicated integral that is close in some parameters, for example, 2 paragraph 1 of [35].

$$\int_0^x (x^2 - t^2)^\beta \cos(\omega t) dt \tag{33}$$

Here, the coefficients for the input function $f_0(t) = \cos(\omega t)$ are obvious

$$a_0(t) = 1; \quad b_0(t) = 0; \quad c_0(t) = \omega^2;$$

We substitute them in (5a), taking into account simultaneously (5b). For $t \to x$, a relation arises on the characteristic of a hyperbolic operator for the kernel $K(x,t) = (x^2 - t^2)^\beta$

$$L\,[K(x,t)]_{t\to x} = 2^\beta \beta(2\beta + b_1(x))(x(x-t))^{\beta-1} + (x(x-t))^\beta O(x-t);$$

The remaining coefficient $c_1(x)$ is easily chosen. As a result

$$a_1(x) = 1; \quad b_1(x) = -\frac{2\beta}{x}; \quad c_1(x) = \omega^2;$$

The solution of an ordinary differential equation (4) is a linear combination

$$x^{\frac{2\beta+1}{2}}\left[C_1 J_{\frac{2\beta+1}{2}}(\omega x) + C_2 Y_{\frac{2\beta+1}{2}}(\omega x)\right]$$

with Bessel and Neumann functions as components. The singularity at zero eliminates the coefficient C_2. For different interpretations of the result, it is convenient to use the relationship between the Bessel function and the hypergeometric function. The integral (33) takes the form

$$\int_0^x (x^2 - t^2)^\beta \cos(\omega t)dt = \frac{\sqrt{\pi}}{2}\Gamma(\beta + 1)x^{\frac{2\beta+1}{2}} J_{\frac{2\beta+1}{2}}(\omega x) =$$

$$= \frac{\sqrt{\pi}}{2}\Gamma(\beta + 1)_0F_1\left(; \beta + \frac{3}{2}; -\left(\frac{\omega x}{2}\right)^2\right)$$

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Polynomial Quantization on Line Bundles

V. F. Molchanov

Abstract We expand polynomial quantization on G/H to the case when a representation of the group G on functions on G/H is induced by a character of the subgroup H.

Keywords Polynomial quantization · Representations · Berezin Transform

MSC Primary 22E46; Secondary 47L15

In [1] we constructed quantization in the spirit of Berezin on para-Hermitian symmetric spaces G/H, see also [3]. In [2] we showed that this quantization, anyway polynomial quantization—the most algebraic variant of quantization, can be considered as a part of the representation theory. In present paper we offer to expand polynomial quantization to case when a representation of the group G on functions on G/H is induced by a character of the subgroup H. Here we restrict ourselves to a hyperboloid of one sheet in $\mathbb{R}^3$. As it is well known, the main content of the representation theory is based on intertwining operators—intertwining transforms, transmutations. In this paper we focus to the Berezin transform. It connects symbols of different types.

We use the following notation:

$$a^{[s]} = a\,(a+1)\ldots(a+s-1), \quad a^{(s)} = a\,(a-1)\ldots(a-s+1),$$

Supported by grants of Minobrnauki: 3.8515.2017.

V. F. Molchanov (✉)
Derzhavin Tambov State University, Tambov, Russia
e-mail: imfit@tsutmb.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_10

here a is a number or an operator, $s \in \mathbb{N} = \{0, 1, 2, \ldots\}$.

$$t^{\lambda,\nu} = |t|^{\lambda} \mathrm{sgn}^{\nu} t\,, \quad t \in \mathbb{R}^* = \mathbb{R} \setminus \{0\},\ \lambda \in \mathbb{C},\ \nu = 0, 1,$$

1 The Group SL(2, $\mathbb{R}$) and Its Representations

The group $G = \mathrm{SL}(2, \mathbb{R})$ consists of real matrices of the second order with unit determinant:

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1. \tag{1}$$

Changing in (1) $\alpha \leftrightarrow \delta$ and $\beta \leftrightarrow \gamma$, we obtain an involution $g \mapsto \widehat{g}$ in G given by

$$\widehat{g} = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}$$

The Lie algebra $\mathfrak{g}$ of the group G consists of real matrices of the second order with zero trace. A basis in $\mathfrak{g}$ consists of matrices:

$$L_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad L_+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{2}$$

The commutation relations are:

$$[L_+, L_-] = -2L_1\,, \quad [L_+, L_1] = -L_+\,, \quad [L_1, L_-] = -L_-\,.$$

Denote by $\mathrm{Env}\,(\mathfrak{g})$ the universal enveloping algebra of the Lie algebra $\mathfrak{g}$.

The center of $\mathrm{Env}\,(\mathfrak{g})$ is generated by the element (the Casimir element, up to a factor)

$$\Omega_{\mathfrak{g}} = L_1^2 + \frac{1}{2}\,(L_+L_- + L_-L_+).$$

Recall some material on representations of G. For $\sigma \in \mathbb{C}$, $\nu = 0, 1$, let us denote by $\mathcal{D}_{\sigma,\nu}(\mathbb{R})$ the space of functions f in $C^\infty(\mathbb{R})$ such that the function $\widehat{f}(t) = t^{2\sigma,\nu} f(1/t)$ belongs to $C^\infty(\mathbb{R})$ too. The representation $\pi_{\sigma,\nu}$ of the group G acts on $\mathcal{D}_{\sigma,\nu}(\mathbb{R})$ by (we consider that G acts from the right):

$$\big(\pi_{\sigma,\nu}(g) f\big)\,(t) = f\,(t \cdot g)\,\,(\beta t + \delta)^{2\sigma,\nu}, \quad t \cdot g = \frac{\alpha t + \gamma}{\beta t + \delta}\,.$$

The *contragredient* representation $\widehat{\pi}_{\sigma,\nu}$ is defined by the involution $g \mapsto \widehat{g}$, so that

$$\left(\widehat{\pi}_{\sigma,\nu}(g) f\right)(t) = f\,(t \cdot \widehat{g})\ (\gamma t + \alpha)^{2\sigma,\nu}.$$

Representations $\pi_{\sigma,\nu}$ and $\widehat{\pi}_{\sigma,\nu}$ are equivalent by means of the operator $f \mapsto \widehat{f}$.

Any irreducible finite-dimensional representation T_k of the group G is labelled by the number k (the *highest weight*) such that $2k \in \mathbb{N} = \{0, 1, 2, \ldots\}$. It acts on the space V_k of polynomials $\varphi(t)$ in t of degree $\leqslant 2k$ (so that $\dim V_k = 2k + 1$) by

$$(T_k(g)\,\varphi)\,(t) = \varphi\,(t \cdot g)\ (\beta t + \delta)^{2k}\,.$$

Operators corresponding to elements of $\mathfrak{g}$ and Env $(\mathfrak{g})$ in representations $\pi_{\sigma,\nu}$ do not depend on ν, so we do not write ν in indexes. For basis elements (2) we have

$$\pi_\sigma(L_-) = \frac{d}{dt},\ \ \pi_\sigma(L_1) = t\,\frac{d}{dt} - \sigma\,,\ \ \pi_\sigma(L_+) = t^2\,\frac{d}{dt} - 2\sigma t. \tag{3}$$

and $\widehat{\pi}_\sigma(L_\pm) = -\pi_\sigma(L_\mp)$, $\widehat{\pi}_\sigma(L_1) = -\pi_\sigma(L_1)$. Replacing in (3) σ by k, we obtain formulae for T_k.

On monomials t^m the representation π_σ of $\mathfrak{g}$ is:

$$\begin{aligned} \pi_\sigma(L_-)\,t^m &= m\,t^{m-1}. \\ \pi_\sigma(L_1)\,t^m &= (m - \sigma)\,t^m, \\ \pi_\sigma(L_+)\,t^m &= (2\sigma - m)\,t^{m+1}, \end{aligned}$$

A bilinear form

$$\langle F, f\rangle = \int_{-\infty}^{\infty} F(t)\,g(t)\,dt \tag{4}$$

is invariant with respect to pairs $(\pi_{\sigma,\varepsilon},\ \pi_{-\sigma-1,\varepsilon})$ and $(\widehat{\pi}_{\sigma,\varepsilon},\ \widehat{\pi}_{-\sigma-1,\varepsilon})$:

$$\langle \pi_{\sigma,\varepsilon}(g) f,\ h\rangle = \langle f,\ \pi_{-\sigma-1,\varepsilon}(g^{-1}) h\rangle \tag{5}$$

and similarly for $\widehat{\pi}$.

An operator $A_{\sigma,\nu}$ defined by

$$(A_{\sigma,\nu} f)(t) = \int_{-\infty}^{\infty} (1 - ts)^{-2\sigma-2,\,\nu}\ f(s)\,ds$$

intertwines $\pi_{\sigma,\nu}$ and $\widehat{\pi}_{-\sigma-1,\nu}$:

$$\widehat{\pi}_{-\sigma-1,\nu}(g) A_{\sigma,\nu} = A_{\sigma,\nu}\pi_{\sigma,\nu}(g)\,,$$

and also $\widehat{\pi}_{\sigma,\nu}$ and $\pi_{-\sigma-1,\nu}$. The composition $A_{\sigma,\nu}$ and $A_{-\sigma-1,\nu}$ is a scalar operator:

$$A_{-\sigma-1,\nu}\, A_{\sigma,\nu} = \frac{1}{c(\sigma,\nu)} \cdot E, \tag{6}$$

where

$$c(\sigma,\nu) = \frac{2\sigma+1}{2\pi} \cdot \frac{(-1)^\nu + \cos 2\sigma\pi}{\sin 2\sigma\pi} .$$

Let Z be the space of distributions on $\mathbb{R}$ concentrated at the point $t = 0$. It consists of linear combinations of the Dirac delta function $\delta(t)$ and its derivatives $\delta^{(p)}(t)$. Formulae (3) with $\sigma = k$ define a representation of the Lie algebra $\mathfrak{g}$ on Z, denote it by T_k again. Moreover, such a representation is defined not only for $k \in (1/2)\mathbb{N}$ but for arbitrary $k \in \mathbb{R}$. We need $k \in (1/2)\mathbb{Z}$. In particular, the representation T_{-k-1}, $k \in (1/2)\mathbb{N}$, acts on $\delta^{(p)}(t)$ as follows:

$$\begin{aligned}
&T_{-k-1}\,(L_-)\,\delta^{(p)}(t) = \delta^{(p+1)}(t)\\
&T_{-k-1}\,(L_1)\,\delta^{(p)}(t) = (k-p)\,\delta^{(p)}(t)\,,\\
&T_{-k-1}\,(L_+)\,\delta^{(p)}(t) = p\,(2k+1-p)\,\delta^{(p-1)}(t)\,.
\end{aligned}$$

It has an invariant subspace spanned by $\delta^{(p)}(t)$, $p \geqslant 2k+1$. The corresponding factor space will be denoted Z_{-k-1}, it is spanned by $\delta^{(p)}(t)$, $p \leqslant 2k$. The factor representation T_{-k-1} on Z_{-k-1} is equivalent to T_k, hence it arises to the group G. It is more convenient to take another basis in Z_{-k-1}, namely, distributions

$$b_p(t) = p!\,\delta^{(2k-p)}(t), \quad p = 0, 1, \ldots, 2k.$$

We have

$$\begin{aligned}
&T_{-k-1}\,(L_-)\,b_p = p b_{p-1}\\
&T_{-k-1}\,(L_1)\,b_p = (p-k)\,b_p\,,\\
&T_{-k-1}\,(L_+)\,b_p = (p-2k)\,b_{p+1}\,.
\end{aligned}$$

We see that the action of the Lie algebra $\mathfrak{g}$ in the basis b_p precisely coincides with the action in the basis t^p.

The bilinear form (4) can be extended to the product $Z \times V_k$:

$$\langle \zeta\,,\,\varphi \rangle = \int_{-\infty}^{\infty} \zeta(t)\,\varphi(t)\,dt\,, \quad \zeta \in Z,\ \varphi \in V_k\,. \tag{7}$$

It is invariant with respect to the pair (T_{-k-1}, T_k), i.e.

$$\langle T_{-k-1}\,(g^{-1})\,\zeta\;,\;\varphi\rangle = \langle \zeta\;,\;T_k\,(g)\,\varphi\rangle\,, \quad g \in G. \tag{8}$$

2 Tensor Products

The tensor product $V_l \otimes V_m$ consists of polynomials $f(x, y)$ of degree $\leqslant 2l$ in x and degree $\leqslant 2m$ in y. The tensor product $T_l \otimes \widehat{T}_m$ of representations T_l and $\widehat{T}_m$ of G acts on $V_l \otimes V_m$ by

$$\big((T_l \otimes \widehat{T}_m)(g) f\big)\,(x, y) = f(x \cdot g\,,\; y \cdot \widehat{g}\,) \cdot (\beta x + \delta)^{2l} \cdot (\gamma y + \alpha)^{2m}.$$

Denote

$$r = m - l.$$

It is well known that $T_l \otimes \widehat{T}_m$ decomposes into the direct multiplicity free sum

$$T_l \otimes \widehat{T}_m = \sum_k T_k\,,$$

where k ranges over the set

$$|r|,\; |r| + 1,\; \ldots, l + m - 1,\; l + m\,. \tag{9}$$

Respectively, $V_l \otimes V_m$ decomposes into the direct sum

$$V_l \otimes V_m = \sum_k W_k\,. \tag{10}$$

Subspaces W_k are invariant and irreducible with respect to $T_l \otimes \widehat{T}_m$. To simplify writing, we do not show dependence on l, m—both on the right side of (10) and later on. The restriction of $T_l \otimes \widehat{T}_m$ to W_k is equivalent to T_k.

3 Hyperboloid of One Sheet

Let us "seat" $T_l \otimes \widehat{T}_m$ and $V_l \otimes V_m$ on the hyperboloid of one sheet $\mathcal{X}$ in $\mathbb{R}^3$ defined by equation $-x_1^2 + x_2^2 + x_3^2 = 1$. Realize $\mathcal{X}$ as the set of matrices

$$x = \frac{1}{2}\begin{pmatrix} 1 - x_3 & x_2 - x_1 \\ x_2 + x_1 & 1 + x_3 \end{pmatrix} \tag{11}$$

with determinant equal to 0. The group G acts on it: $x \mapsto g^{-1}xg$, transitively. The stabilizer of the point $x^0 = (0, 0, 1)$ is the subgroup H of diagonal matrices

$$h = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

Let us introduce on $\mathcal{X}$ horospherical coordinates ξ, η:

$$x = \frac{1}{N} \begin{pmatrix} -\eta\xi & -\eta \\ \xi & 1 \end{pmatrix}, \quad N = 1 - \xi\eta. \tag{12}$$

The action of G in these coordinates splits: if a point x has coordinates ξ, η, then the point $g^{-1}xg$ has coordinates $\widetilde{\xi} = \xi \cdot g, \widehat{\eta} = \eta \cdot \widehat{g}$. The initial point x^0 has coordinates $\xi = 0, \eta = 0$. The element

$$g_x = \begin{pmatrix} 1/N & \eta/N \\ \xi & 1 \end{pmatrix}. \tag{13}$$

moves x^0 to the point x with coordinates ξ, η. The G-invariant measure on $\mathcal{X}$ is

$$dx(\xi, \eta) = \frac{d\xi \, d\eta}{(1 - \xi\eta)^2}.$$

Recall $r = m - l$. Further we consider that

$$r \geqslant 0.$$

The map

$$D : f \mapsto F = N^{-2m} f, \quad f \in V_l \otimes V_m,$$

transfers $V_l \otimes V_m$ to a space $\mathcal{M}^{lm}(\mathcal{X})$ of some rational functions F of ξ, η. In virtue of (11) and (12) these rational functions are *polynomials* in x_1, x_2, x_3 on $\mathcal{X}$. For example, the polynomial $f = 1$ in $V_l \otimes V_m$ goes to the polynomial $((x_3 + 1)/2)^{2m}$. Denote

$$W_k^{(r)}(\mathcal{X}) = D(W_k^{lm}).$$

These subspaces depend on r and k (not on l and m).

The map D intertwines $T_l \otimes \widehat{T}_m$ and the representation U_r of G induced by the character $h \mapsto \lambda^{-2r}$ of the subgroup H. The representation U_r acts on polynomials on $\mathcal{X}$ and in horospherical coordinates is:

$$(U_r(g)F)(\xi, \eta) = F\left(\widetilde{\xi}, \widehat{\eta}\right)(\beta\xi + \delta)^{-2r}.$$

Under U_r the Casimir element $\Omega_{\mathfrak{g}}$ goes to the operator $\Delta_r = U_r(\Omega_{\mathfrak{g}})$. Let us call it the Casimir operator. Here is its expression:

$$\Delta_r = N^2 \frac{\partial^2}{\partial\xi\,\partial\eta} - 2Nr\,\eta\frac{\partial}{\partial\eta} + r^2 - r.$$

For $r = 0$ it is the Laplace–Beltrami operator.

The restriction of the representation U_r to the space $\mathcal{M}^{lm}(\mathcal{X})$ decomposes (as $T_l \otimes \widehat{T}_m$ does) into the direct multiplicity free sum $\sum T_k$ where k ranges over the set (9). Respectively, $\mathcal{M}^{lm}(\mathcal{X})$ decomposes into the direct sum

$$\mathcal{M}^{lm}(\mathcal{X}) = \sum_k W_k^{(r)}(\mathcal{X})\,.$$

Subspaces $W_k^{(r)}(\mathcal{X})$ are invariant and irreducible with respect to U_r.

4 Poisson Transform

Let k belong to the set (9). We consider operators $\mathcal{P}_{r,k} : V_k \to V_l \otimes V_m$ intertwining representations $T_l \otimes \widehat{T}_m$ and T_k, i.e.,

$$\mathcal{P}_{r,k}\, T_k(g) = \left(T_l \otimes \widehat{T}_m\right)(g)\mathcal{P}_{r,k}\,,$$

where $g \in G$. We call the operator $\mathcal{P}_{r,k}$ the *Poisson transform*.

First we define the Poisson kernel $P_{r,k}(x;t)$, $k \in \mathbb{N}$, where $x \in \mathcal{X}$, $t \in \mathbb{R}$. Recall $k - r \in \mathbb{N}$. The monomial t^{k-r} is an eigenvector for the subgroup H with the eigenvalue λ^{2r}. We obtain the Poisson kernel $P_{r,k}(x;t)$ applying the operator $T_k(g_x^{-1})$ to this monomial, namely, we set

$$P_{r,k}(x;t) = T_k(g_x^{-1})t^{k-r}$$

and find

$$P_{r,k}(x;t) = \frac{(t-\xi)^{k-r}(1-t\eta)^{k+r}}{N^{k+r}}\,, \quad N = 1 - \xi\eta.$$

where ξ, η are horospherical coordinates of the point x.

An important property of this kernel is that it is a fixed vector in the tensor product $U_r \otimes T_k$:

$$\left(U_r(g) \otimes T_k(g)P_{r,k}\right)(x;t) = P_k(x;t), \quad g \in G.$$

Therefore, the kernel $P_{r,k}(x;t)$ is a generating function for polynomials in $W_k^{(r)}$. Namely, a coefficient at t^a in this kernel is a polynomial in $W_k^{(r)}(\mathcal{X})$ that is an eigenvector for the subgroup H with the eigenvalue $\lambda^{-2k+2r+2a}$.

Now define the Poisson transform $\mathcal{P}_{r,k} : V_k \to V_l \otimes V_m$. A distribution corresponding to the monomial t^{k-r} is $b_{k-r}(t) = (k-r)!\,\delta^{(k+r)}(t)$. It is convenient to change the factor in front of delta, namely, instead of b_{k-r} we take the distribution $\beta_{k-r}(t) = (-1)^{k+r}\,\delta^{(k+r)}(t)$.

This distribution is an eigenfunction for the subgroup H with the eigenvalue λ^{2r}. Therefore, the map $\mathcal{P}_{r,k} : V_k \to M^{lm}$ defined by

$$\left(\mathcal{P}_{r,k}\varphi\right)(x) = \langle T_{-k-1}\,(g_x^{-1})\,\beta_{k-r}\,,\ \varphi\rangle, \tag{14}$$

intertwines T_k with U_r. Therefore, the image of this map is just $M_k^{(r)}$. Because of (8), (7), (13) and (1) we get:

$$\left(T_{-k-1}(g_x^{-1})\beta_{k-r}\right)(t) = (-1)^{k+r}N^{k+1-r}\delta^{(k+r)}\left(\frac{t-\xi}{1-t\eta}\right)(1-t\eta)^{-2k-2}\,. \tag{15}$$

By (14) and (15) we have

$$\left(\mathcal{P}_{r,k}\,\varphi\right)(x) = (-1)^{k+r}N^{k+1-r}\int \delta^{(k+r)}\left(\frac{t-\xi}{1-t\eta}\right)(1-t\eta)^{-2k-2}\varphi(t)\,dt.$$

By means of the change $s = \dfrac{t-\xi}{1-t\eta}$ we compute

$$\left(\mathcal{P}_{r,k}\,\varphi\right)(x) = N^{-k-r}\left(\frac{d}{ds}\right)^{k+r}\bigg|_{s=0}\varphi\left(\frac{s+\xi}{s\eta+1}\right)(s\eta+1)^{2k}\,. \tag{16}$$

Theorem 1 *The Poisson transform $\mathcal{P}_{r,k} : V_k \to V_l \otimes V_m$ can be written in one of the following two forms. Let $\varphi \in V_k$. Then*

$$\left(\mathcal{P}_{r,k}\,\varphi\right)(x) = \sum_{s=0}^{k+r}\binom{k+r}{s}(k-r+1)^{[s]}\left(\frac{\eta}{N}\right)^s\varphi^{(k+r-s)}(\xi)\,, \tag{17}$$

$$= N^{k-r+1}\left(\frac{d}{d\xi}\right)^{k+r}N^{-(k-r+1)}\,\varphi(\xi)\,, \tag{18}$$

where $N = 1-\xi\eta$, $r = m-l$. a point x has horospherical coordinates ξ, η.

Proof Formula (17) follows from formula (16), for example, by induction on $k+r$. Formula (18) can be checked immediately, for example, by induction. □

The Poisson transform $\mathcal{P}_{r,k}$ moves the minimal element $t^0 = 1$ in V_k to the minimal element

$$f_k^{(r)} = \left(\frac{\eta}{N}\right)^{k+r}$$

in $W_k^{(r)}(\mathcal{X})$—with a factor $(k-r+1)^{[k+r]}$.

5 Polynomial Quantization

Recall some material on *polynomial quantization*, see, for example [3]. For initial algebra of operators we take the algebra of operators $\pi_{\sigma,\nu}(X)$, $X \in \mathrm{Env}\,(\mathfrak{g})$, with the complex parameter σ, acting on functions $\varphi(\xi)$, $\xi \in \mathbb{R}$. As a supercomplete system we take the kernel of the intertwining operator $A_{-\sigma-1,\nu}$, namely,

$$\Phi_{\sigma,\nu}(\xi,\eta) = N(\xi,\eta)^{2\sigma,\nu}\,.$$

This function has an invariance property

$$\Big[\pi_\sigma(g) \otimes \widehat{\pi}_\sigma(g)\Big]\Phi_{\sigma,\nu}(\xi,\eta) = \Phi_{\sigma,\nu}(\xi,\eta).$$

This formula can be rewritten as

$$\Big(\pi_\sigma(g^{-1}) \otimes 1\Big)\,\Phi_{\sigma,\nu}(\xi,\eta) = \Big(1 \otimes \widehat{\pi}_\sigma(g)\Big)\,\Phi_{\sigma,\nu}(\xi,\eta). \tag{19}$$

For elements L of the Lie algebra $\mathfrak{g}$, formula (5) gives:

$$-\Big(\pi_\sigma(L) \otimes 1\Big)\,\Phi_{\sigma,\nu}(\xi,\eta) = \Big(1 \otimes \widehat{\pi}_\sigma(L)\Big)\,\Phi_{\sigma,\nu}(\xi,\eta).$$

Let us define symbols (covariant and contravariant) of operators $D = \pi_\sigma(X)$, $X \in \mathrm{Env}\,(\mathfrak{g})$. Recall that operators corresponding to elements of $\mathrm{Env}\,(\mathfrak{g})$ in representations $\pi_{\sigma,\nu}$ do not depend on ν, so we do not write ν in indexes.

Formula (6) can be rewritten as a reproducing formula:

$$\varphi(\xi) = c(\sigma,\nu)\int \Phi_{\sigma,\nu}(\xi,v)\,\Phi_{-\sigma-1,\nu}(u,v)\varphi(u)\,du\,dv, \tag{20}$$

where $c = c(\sigma,\nu)$. Let us apply to both sides of this formula an operator $D = \pi_\sigma(X)$, acting on functions of ξ. We obtain

$$D\varphi(\xi) = c\int (D\otimes 1)\Phi_{\sigma,\nu}(\xi,v)\,\Phi_{-\sigma-1,\nu}(u,v)\varphi(u)\,du\,dv \tag{21}$$

Let us introduce a function

$$F(\xi,\eta)=\frac{1}{\Phi_{\sigma,\nu}(\xi,\eta)}\Big(D\otimes 1)\Phi_{\sigma,\nu}(\xi,\eta\Big), \tag{22}$$

then (21) becomes

$$D\varphi(\xi)=c\int F(\xi,v)\,\Phi_{\sigma,\nu}(\xi,v)\,\Phi_{-\sigma-1,\nu}(u,v)\varphi(u)\,du\,dv \tag{23}$$

$$=c\int F(\xi,v)\,\frac{\Phi_{\sigma,\nu}(\xi,v)}{\Phi_{\sigma,\nu}(u,v)}\varphi(u)\,dx(u,v) \tag{24}$$

The function $F(\xi,\eta)$ given by (22) is called by the *covariant symbol* of the operator $D=\pi_\sigma(X)$, $X\in\mathrm{Env}\,(\mathfrak{g})$. Formulae (23) and (24) recover the operator by its symbol.

Let us define contravariant symbols. In formula (20) instead of φ we take $D\varphi$. We get

$$D\varphi(\xi)=c\int \Phi_{\sigma,\nu}(\xi,v)\,\Phi_{-\sigma-1,\nu}(u,v)D\varphi(u)\,du\,dv,$$

Let us transfer the operator D in the interior integral to Φ by means of (4). By (5), the operator conjugate to $D=\pi_\sigma(X)$ with respect to form (4) is $D^*=\pi_{-\sigma-1}(X^\vee)$, where the map $X\to X^\vee$ is the principal anti-involution of $\mathrm{Env}\,(\mathfrak{g})$ corresponding to the map $g\to g^{-1}$ in G. We obtain

$$D\varphi(\xi)=c\int \Phi_{\sigma,\nu}(\xi,v)\,\left(D^*\otimes 1\right)\Phi_{-\sigma-1,\nu}(u,v)\,\varphi(u)\,du\,dv, \tag{25}$$

Let us introduce a function

$$F^\natural(\xi,\eta)=\frac{1}{\Phi_{-\sigma-1,\nu}(\xi,\eta)}\left(\pi_{-\sigma-1}(X^\vee)\otimes 1\right)\Phi_{-\sigma-1,\nu}(\xi,\eta) \tag{26}$$

Then formula (25) becomes

$$D\varphi(\xi)=c\int F^\natural(u,v)\Phi_{\sigma,\nu}(\xi,v)\,\Phi_{-\sigma-1,\nu}(u,v)\,\varphi(u)\,du\,dv, \tag{27}$$

$$=c\int F^\natural(u,v)\frac{\Phi_{\sigma,\nu}(\xi,v)}{\Phi_{\sigma,\nu}(u,v)}\,\varphi(u)\,dx(u,v). \tag{28}$$

Using (19) we can rewritten (26) as follows

$$F^\natural(\xi,\eta)=\frac{1}{\Phi_{-\sigma-1,\nu}(\xi,\eta)}\left(1\otimes\widehat{\pi}_{-\sigma-1}(X)\right)\Phi_{-\sigma-1,\nu}(\xi,\eta) \tag{29}$$

The function $F^{\natural}(\xi, \eta)$ given by (26) or (29) is called by the *contravariant symbol* of the operator $D = \pi_\sigma(X)$, $X \in \mathrm{Env}\,(\mathfrak{g})$. Formulae (27) and (28) recover the operator by its symbol.

Covariant and contravariant symbols turn out to be *polynomials* on $\mathcal{X}$. It is why this theory is called the polynomial quantization.

In particular, covariant symbols for basis elements (2) are multiplied by $(-\sigma)$ polynomials $x_1 - x_2, x_3, x_1 + x_2$, respectively.

The multiplication of operators gives rise to the multiplication of covariant and contravariant symbols. But in this paper we do not concern this theme.

The passage from F to $F^{\natural}$ is called by the Berezin transform. We consider this transform in the next section in more general setting.

6 Berezin Transform for Induced Representation

Let us take *two* complex numbers σ and τ and construct for them covariant symbols $F_\sigma(\xi, \eta)$ and contravariant symbols $F^{\natural}_\tau(\xi, \eta)$. We consider that they are connected by:

$$\sigma - \tau = r.$$

Let us write an operator $B_{\sigma,\tau}$ which moves the covariant symbol $F_\sigma(\xi, \eta)$ to the contravariant symbols $F^{\natural}_\tau(\xi, \eta)$ of operators corresponding to the same element $X \in \mathrm{Env}\,(\mathfrak{g})$. Let us call this operator the Berezin transform. Using formulae from Sect. 5 we find

$$F_\sigma(\xi, \eta) = c(\sigma, \nu) \int B_{\sigma,\tau}(\xi, \eta; u, v)\, F^{\natural}_\tau(u, v)\, dx(u, v),$$

where

$$B_{\sigma,\tau}(\xi, \eta; u, v) = c(\sigma, \nu) \frac{\Phi_{\tau,\nu}(\xi, v)\Phi_{\sigma,\nu}(u, \eta)}{\Phi_{\tau,\nu}(u, v)\Phi_{\sigma,\nu}(\xi, \eta)}.$$

This transform commutes with the representation U_r of the group G:

$$U_r(g) B_{\sigma,\tau} = B_{\sigma,\tau} U_r(g), \quad g \in G. \tag{30}$$

It follows from the formula

$$B_{\sigma,\tau}(\widetilde{\xi}, \widehat{\eta}; \widetilde{u}, \widehat{v}) = B_{\sigma,\tau}(\xi, \eta; u, v) \left(\frac{\beta\xi + \delta}{\beta u + \delta}\right)^{2r}$$

Theorem 2 *For generic σ and τ (not entering in $(1/2)\mathbb{Z}$) the Berezin transform $B_{\sigma,\tau}$ is defined on the space of all polynomials on $\mathcal{X}$ and preserves subspaces $W_k^{(r)}(\mathcal{X})$. On every this subspace it is a scalar operator, the multiplication by a number. This number is*

$$b_k(\sigma,\tau) = \frac{\Gamma(-2\sigma+k)\,\Gamma(-2\tau-k-1)}{\Gamma(-2\sigma)\,\Gamma(-2\tau-1)}\,. \tag{31}$$

Proof The Berezin transform $B_{\sigma,\tau}$ acts on symbols, that are polynomials. In virtue of permutability of this transform with U_r, see (30), and irreducibility of subspaces $W_k^{(r)}(\mathcal{X})$ with respect to U_r, the Berezin transform preserves these subspaces. Therefore on every this subspace it is the multiplication by a number $b_k(\sigma,\tau)$.

Let us compute this number. For the element L_-^k, the corresponding operator D is $(\partial/\partial\xi)^k$ for both representations π_σ and π_τ. Covariant and contravariant symbols for it are respectively

$$F = (-1)^k(2\sigma)^{(k)}\left(\frac{\eta}{N}\right)^k, \quad F^\natural = (-2\tau-2)^{(k)}\left(\frac{\eta}{N}\right)^k$$

Therefore,

$$b_k(\sigma,\tau) = (-1)^k\frac{(2\sigma)^{(k)}}{(-2\tau-2)^{(k)}}$$

which is just (31). □

Theorem 3 *The Berezin transform $B_{\sigma,\tau}$ is expressed in terms of the Casimir operator Δ_r as follows:*

$$B_{\sigma,\tau} = \left.\frac{\Gamma(-2\sigma+r+A)\,\Gamma(-2\tau-r-A-1)}{\Gamma(-2\sigma)\,\Gamma(-2\tau-1)}\right|_{A(A+1)=\Delta_r}. \tag{32}$$

Proof Let us apply the Casimir operator Δ_r to the minimal vector $(\eta/N)^k$ in $W_k^{(r)}(\mathcal{X})$. We get

$$\Delta_r\left(\frac{\eta}{N}\right)^k = (k-r)(k-r+1)\left(\frac{\eta}{N}\right)^k.$$

So that $\Delta_r = (k-r)(k-r+1)$ on $W_k^{(r)}(\mathcal{X})$. The expression (31) does not change when we replace k by $-k+2r-1$, or $k-r$ by $-(k-r)-1$. Then we select in the nominator of (31) $A = k-r$ and obtain (32). □

Theorem 4 *There is the following decomposition of the Berezin transform $B_{\sigma,\tau}$:*

$$B_{\sigma,\tau} = \sum_{s=0}^{\infty}\frac{1}{s!(2\tau+2)^{[s]}}\prod_{p=0}^{s-1}\left[\Delta_r-(p-r)(p-r+1)\right]. \tag{33}$$

Proof Using properties of the Gamma function, we can rewrite (31) as follows:

$$b_k(\sigma,\tau) = \frac{\Gamma(2\sigma+1)\,\Gamma(2\tau+2)}{\Gamma(2\sigma+1-k)\,\Gamma(2\tau+2+k)}\,.$$

It is the value of the Gauss hypergeometric function at 1:

$$b_k(\sigma,\tau) = F(-k,\ k+1-2r;\ 2\tau+2;\ 1),$$

so that

$$b_k(\sigma,\tau) = \sum_{s=0}^{\infty} \frac{k^{(s)}\,(k+1-2r)^{[s]}}{(-2\tau-2)^{(s)}\,s!}\,. \tag{34}$$

Denote $\lambda = (k-r)(k-r+1)$. It is the eigenvalue of Δ_r on $W_k^{(r)}(\mathcal{X})$. The numerator in (34) can be written as

$$\Big(\lambda - r(r-1)\Big)\Big(\lambda - (r-1)(r-2)\Big)\ldots\Big(\lambda - (r-s+1)(r-s)\Big).$$

But it just the eigenvalue on $W_k^{(r)}(\mathcal{X})$ of the product of operators in (33). □

We see from (33) that on the space $\mathcal{M}^{lm}(\mathcal{X})$ the Berezin transform is a differential operator (some polynomial in Δ_r).

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Fourier–Bessel Transforms of Measures and Qualitative Properties of Solutions of Singular Differential Equations

A. B. Muravnik

Abstract In this paper, we review a number of results about the Fourier–Bessel transformation of nonnegative functions. For the specified case, weighted L_∞-norms of the spherical mean of $|\hat{f}|^2$ are estimated by its weighted L_1-norms; note that such a phenomenon does not take place in the general case, i.e., without the requirement of the nonnegativity of f. Moreover, unlike the classical case of the Fourier transform, this phenomenon takes place for one-variable functions as well: weighted L_∞-norms of the Fourier–Bessel transform are estimated by its weighted L_2-norms. Those results are applied to the investigation of singular differential equations containing Bessel operators acting with respect to selected spatial variables (the so-called *special variables*); equations of such kind arise in models of mathematical physics with degenerative heterogeneities and in axially symmetric problems. The proposed approach provides a priori estimates for weighted L_∞-norms of the solutions (for ordinary differential equations) and for weighted spherical means of the squared solutions (for partial differential equations).

This work was financially supported by the Ministry of Education and Science of the Russian Federation on the program to improve the competitiveness of Peoples' Friendship University (RUDN University) among the world's leading research and education centers in the 2016–2020, by the Russian Foundation for Basic Research (grant No. 17-01-00401), and by the President Grant for the Government Support of the Leading Scientific Schools of the Russian Federation, No. 4479.2014.1.

A. B. Muravnik (✉)
JSC "Concern "Sozvezdie" & Peoples Friendship University of Russia (RUDN University), Voronezh, Russia
e-mail: amuravnik@gmail.com

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_11

1 Introduction

In [6], it is proved that if $f \geq 0$, then, for each α from $\left(0, \frac{n-1}{2}\right]$, we have the estimate

$$\|r^{\alpha}\sigma(f)\|_{\infty} \leq C\|r^{\alpha-1}\sigma(f)\|_{1}, \tag{1}$$

where $\sigma(f)(r)$ is the mean value of $|\hat{f}|^2$ over the sphere of radius r, centered at the origin, and C depends only on the dimension of the space.

Note that, generally, (1) does not hold because one can construct a sequence $\{f_m\}_{m=1}^{\infty}$ such that $\|r^{\alpha-1}\sigma(f_m)\|_1$ does not depend on m, but $\sigma(f_m)(1)$ tends to infinity as $m \to \infty$. Thus, imposing the nonnegativity requirement on f, we prohibit the above type of the behavior. Actually, that requirement implies a certain restriction on the shape of the graph of $\hat{f}$.

One can expect that, in the one-dimensional case, inequality (1) provides a similar estimate for the function itself instead of its mean. However, in the said case, the integral at the right-hand part of (1) diverges for any nonnegative f: since $\hat{f}(0)$ is equal to the integral of f over the whole real line, it follows that there is a nonintegrable singularity at the origin. It turns out that, unlike the classical regular case, the mentioned one-dimensional phenomenon takes place for the Fourier–Bessel transformation applied in the theory of differential equations containing singular Bessel operators: a weighted L_∞-norm of the transform is estimated from above by its weighted L_2-norm. The mentioned equations arise in models of mathematical physics with degenerative space heterogeneities.

In the present paper, we provide a review of results about estimates of that kind (both for the one-dimensional and multi-dimensional cases) as well as their applications to singular (ordinary and partial) differential equations. Using the fact that the Bessel operator acts as a multiplier in Fourier–Bessel images, we find estimates of the kinds

$$\|u\|_{\infty} \leq u(0) \tag{2}$$

and

$$\|r^{\frac{\alpha}{2}}u\|_{\infty} \leq C\|r^{\frac{\alpha-1}{2}}u\|_{2} \tag{3}$$

for norms of solutions of ordinary differential equations and estimates of the kind

$$\|r^{\gamma}S_{p,q}u\|_{\infty} \leq C\|r^{\gamma-1}S_{\alpha,\beta}u\|_{1} \tag{4}$$

for norms of solutions of partial differential equations, where $S_{p,q}$ denotes the weighted hemispherical mean value of $|\cdot|^2$ with weight $|x|^p \prod\limits_{l=1}^{m} y_l^{q_l}$, while the constant C and the allowed values of the parameters p, $q = (q_1, \dots, q_m)$, α, $\beta = (\beta_1, \dots, \beta_m)$, and γ are determined by the dimension of the space and by the parameters at the singularities of the Bessel operators contained in the equation and acting with respect to the special variables $y_1, \dots, y_m$ (or, which is the same, the indices of the Bessel functions from the kernel of the corresponding integral transformation).

To obtain estimates (2)–(4), we require the nonnegativity of the Fourier–Bessel transform of the right-hand part of the equation, divided by the symbol of the operator at its left-hand part.

The presented results are mainly obtained in [7–10].

2 Notation and Definitions

Let $k \stackrel{\text{def}}{=} 2\nu + 1$ and μ be positive parameters. We introduce

$$\mathbb{R}^{n+1}_{+} \stackrel{\text{def}}{=} \left\{(x, y) \,\middle|\, x \in \mathbb{R}^n,\ y > 0\right\},$$

$$L_{p,\mu}(\mathbb{R}^{n+1}_{+}) \stackrel{\text{def}}{=} \left\{f \,\middle|\, \|f\| = \left(\int\limits_{\mathbb{R}^{n+1}_{+}} y^{\mu} |f(x, y)|^p \, dxdy\right)^{\frac{1}{p}} < \infty\right\} \text{ for finite } p,$$

and

$$L_{\infty,\mu}(\mathbb{R}^{n+1}_{+}) \stackrel{\text{def}}{=} \left\{f \,\middle|\, \|f\| = \operatorname{vrai\,sup} y^{\mu} |f(x, y)| < \infty\right\}.$$

The set of infinitely smooth compactly supported functions defined on $\mathbb{R}^{n+1}$ is denoted by $C_0^{\infty}(\mathbb{R}^{n+1})$. We consider the subset of $C_0^{\infty}(\mathbb{R}^{n+1})$ consisting of even functions with respect to y. Then we denote by $C^{\infty}_{0,\text{even}}(\mathbb{R}^{n+1}_{+})$ the set of restrictions of elements of that subset to $\mathbb{R}^{n+1}_{+}$. The set $C^{\infty}_{0,\text{even}}(\mathbb{R}^{n+1}_{+})$ plays the role of the *space of test functions*.

Generalized functions (distributions) on $C^{\infty}_{0,\text{even}}(\mathbb{R}^{n+1}_{+})$ are introduced with respect to the degenerative measure $y^k dxdy$: if a linear continuous functional on $C^{\infty}_{0,\text{even}}(\mathbb{R}^{n+1}_{+})$ can be identified with a function $f \in L_{1,k,loc}(\mathbb{R}^{n+1}_{+})$ (an *ordinary* function) according to the rule

$$\langle f, \varphi\rangle \stackrel{\text{def}}{=} \int\limits_{\mathbb{R}^{n+1}_{+}} y^k f(x, y)\varphi(x, y)\, dxdy \quad \text{for any } \varphi \in C^{\infty}_{0,\text{even}}(\mathbb{R}^{n+1}_{+}), \tag{5}$$

then that functional f is called *regular*. Other linear continuous functionals on $C^{\infty}_{0,\text{even}}(\mathbb{R}^{n+1}_{+})$ are called *singular*.

The Fourier–Bessel transformation is defined as follows:

$$\hat{f}(\xi_1,\dots,\xi_n,\eta) \overset{\text{def}}{=} \mathcal{F}_{\text{b}}(f)(\xi,\eta) \overset{\text{def}}{=} \int\limits_{\mathbb{R}^{n+1}_{+}} y^k j_\nu(\eta y) e^{-i\xi\cdot x} f(x,y)\,dxdy,$$

where $j_\nu(z) = \dfrac{2^\nu \Gamma(\nu+1)}{y^\nu} J_\nu(y)$ is the normalized Bessel function.

Note that $f(x,y) = C \int\limits_{\mathbb{R}^{n+1}_{+}} \eta^k j_\nu(\eta y) e^{i\xi\cdot x} \hat{f}(\xi,\eta)\,d\xi d\eta$.

In the one-dimensional case, we define the *pure* Fourier–Bessel transformation

$$\tilde{f}(\eta) \overset{\text{def}}{=} \mathcal{F}_{\text{b}}(f)(\eta) \overset{\text{def}}{=} \int\limits_0^\infty y^k j_\nu(\eta y) f(y)\,dy$$

and note that $f(y) = C\int\limits_0^\infty \eta^k j_\nu(\eta y)\tilde{f}(\eta)\,d\eta$.

The Fourier–Bessel transformation maps $L_{2,k}(\mathbb{R}^{n+1}_{+})$ onto itself.

The generalized translation operator corresponding to the considered degenerative measure is defined (for the one-dimensional case) as follows:

$$T^h_y f(y) \overset{\text{def}}{=} C \int\limits_0^\pi f\left(\sqrt{y^2+h^2-2yh\cos\theta}\right) \sin^{k-1}\theta d\theta. \tag{6}$$

Then $\int\limits_0^\infty \eta^k g(\eta) T^\eta_y f(y) d\eta = \int\limits_0^\infty \eta^k f(\eta) T^\eta_y g(y) d\eta$. Therefore, one can introduce the generalized convolution $(f*g)(y) \overset{\text{def}}{=} \int\limits_0^\infty \eta^k f(\eta) T^\eta_y g(y) d\eta$ such that $\widetilde{f*g} = \tilde{f}\tilde{g}$.

In the general case of the mixed Fourier–Bessel transformation, the generalized translation operator is constructed as a superposition of operator (6) acting with respect to the *special* variable y and classical translation operators acting with respect to the remaining *nonspecial* variables.

3 Estimates for the One-Dimensional Case

In Sects. 3 and 4, the dimension n is assigned to be equal to zero, i. e., we consider the case where there are no nonspecial variables and the only independent variable is singular.

The following assertion is valid.

Theorem 3.1 *There exists $C = C(k)$ such that the inequality*

$$\sup_{(0,+\infty)} r^\alpha \tilde{f}^2(r) \le C \int_0^\infty y^{\alpha-1} \tilde{f}^2(y) dy \tag{7}$$

holds for any nonnegative f from $L_{1,k}(0,+\infty) \cap L_{2,k}(0,+\infty)$ and any α from $\left(0, \frac{k}{2}\right]$.

Proof The first stage of the proof is to prove the validity of (7) for the greatest claimed value of α.

Under our assumptions, $\tilde{f}$ is an ordinary function from $L_{2,k}(0,+\infty)$. Then the integral $\int_0^\infty y^k f(y) j_\nu(ry) dy$ absolutely converges for a. e. positive r. Therefore,

$$\tilde{f}^2(r) = \int_0^\infty \int_0^\infty y^k \eta^k f(y) f(\eta) j_\nu(ry) j_\nu(r\eta) dy d\eta.$$

Taking into account that $j_\nu(ry) j_\nu(r\eta) = T_y^\eta j_\nu(ry)$, we see that the last integral is equal to

$$\int_0^\infty \int_0^\infty y^k \eta^k f(y) f(\eta) T_y^\eta j_\nu(ry) dy d\eta = \int_0^\infty \eta^k f(\eta) \int_0^\infty y^k j_\nu(ry) T_y^\eta f(y) dy d\eta.$$

We note that $f \ge 0$ and the generalized translation operator preserves the sign; on the other hand, $|j_\nu(ry)| = \left| \frac{J_\nu(ry)}{(ry)^\nu} \right| \le \frac{C}{(ry)^{\nu+\frac{1}{2}}} = \frac{C}{(ry)^{\frac{k}{2}}}$. Hence,

$$\tilde{f}^2(r) \le C \int_0^\infty \eta^k f(\eta) \int_0^\infty y^k \frac{1}{(ry)^{\frac{k}{2}}} T_y^\eta f(y) dy d\eta \tag{8}$$

under the assumption that the integral at the right-hand part converges. To prove its convergence, we represent it as

$$\frac{1}{r^{\frac{k}{2}}}\int\limits_0^\infty \eta^k f(\eta)\int\limits_0^\infty y^k f(y)T_y^\eta y^{-\frac{k}{2}}dyd\eta = \frac{1}{r^{\frac{k}{2}}}\int\limits_0^\infty \eta^k f(\eta)\Big(f * y^{-\frac{k}{2}}\Big)(\eta)d\eta$$

$$= r^{-\frac{k}{2}}\langle f, f{*}y^{-\frac{k}{2}}\rangle = r^{-\frac{k}{2}}\langle \tilde f, \mathcal{F}_{\mathrm{b}}(f{*}y^{-\frac{k}{2}})\rangle = r^{-\frac{k}{2}}\langle \tilde f\,\widetilde{y^{-\frac{k}{2}}}, \tilde f\rangle = r^{-\frac{k}{2}}\langle \widetilde{y^{-\frac{k}{2}}}, \tilde f^2\rangle.$$

Using the well-known Fourier transforms of the Riesz kernel and of radial functions, we see that $\widetilde{y^{-s}} = C_s y^{s-k-1}$ for $s \in (0, k+1)$. Therefore,

$$r^{-\frac{k}{2}}\langle \widetilde{y^{-\frac{k}{2}}}, \tilde f^2\rangle = r^{-\frac{k}{2}}\langle y^{\frac{k}{2}-k-1}, \tilde f^2\rangle = \frac{1}{r^{\frac{k}{2}}}\int\limits_0^\infty y^{\frac{k}{2}-1}\tilde f^2(y)dy. \tag{9}$$

The last integral converges by virtue of the following reasons:

(i) the singularity at the origin is integrable since $\frac{k}{2}-1 > -1$ and $\tilde f(0)$ is equal to the converging integral $\int\limits_0^\infty y^k f(y)dy$ (the said convergence holds because $f \in L_{1,k}(0,+\infty)$);

(ii) the rate of the decay at infinity is sufficient because $\frac{k}{2}-1 < k$ and $\tilde f \in L_{2,k}(0,+\infty)$ (since $f \in L_{2,k}(0,+\infty)$).

The proved convergence means that all the operations leading from (8) to (9) are really legible, i.e.,

$$\tilde f^2(r) \le \frac{C}{r^{\frac{k}{2}}}\int\limits_0^\infty y^{\frac{k}{2}-1}\tilde f^2(y)dy \text{ on } (0,+\infty). \tag{10}$$

The last inequality is actually the claimed estimate (7) for $\alpha = \frac{k}{2}$. To extend it to the whole claimed interval, we apply estimate (10) to the function $f_\gamma \overset{\text{def}}{=} f * y^{\gamma-k-1}$, where $\gamma \overset{\text{def}}{=} \frac{1}{2}\Big(\frac{k}{2}-\alpha\Big) > 0$. However, to do that, we need f_γ to satisfy the assumptions of the theorem. To show that it belongs to $L_{2,k}(0,+\infty)$, we note that

$$\int\limits_0^\infty y^k \tilde f_\gamma^2(y)dy = C_\gamma\int\limits_0^\infty y^{k-2\gamma}\tilde f^2(y)dy = C_\gamma\int\limits_0^\infty y^{\frac{k}{2}+\alpha}\tilde f^2(y)dy < \infty$$

because there is no singularity at the origin since $\tilde{f}(0) = \int\limits_0^\infty y^k f(y)dy < \infty$ and the decay rate at infinity is sufficient because the integral converges even for the "greater" weight y^k. Now, using the nonnegativity of f_γ (note that it is the convolution of nonnegative functions), we can follow the same line of the proof as for f beyond the point where its weight summability is used; for the case of f_γ, we directly prove the convergence of the majorizing integral $\int\limits_0^\infty y^{\alpha-1}\tilde{f}_\gamma^2(y)dy$ instead. □

4 Estimates for One-Variable Compactly Supported Functions

In [13], it is proved that the additional (besides the nonnegativity) assumptions of the compactness of the support and of the radiality improve the result of [6] as follows: the weights at the right-hand part and the left-hand part of the estimate are bound less strictly than in [6] (actually, they are bound by an *inequality*) and the weight powers belong to a wider interval.

In this section, the method of [13] is applied to the Fourier–Bessel transformation to improve the results of Sect. 3 under the additional assumption of the compactness of the support.

In this section, all the absolute constants generally depend on ν and α.

4.1 The Case Where the Weight Power Does Not Exceed the Parameter at the Singularity

Let $k > 0$ and $\alpha \le k$. Let f be a nonnegative function such that it belongs to $L_{1,k}(0,+\infty) \cap L_{2,k}(0,+\infty)$ and $\operatorname{supp} f \subset [0,1]$.

Then $\tilde{f}(r) = \int\limits_0^1 s^k j_\nu(rs)f(s)ds$.

Let $\beta < \alpha$. Then $\nu + \frac{1}{2} = \frac{k}{2} \ge \frac{\alpha}{2} > \frac{\beta}{2}$ and, therefore, $|j_\nu(t)| \le \frac{C}{t^{\nu+\frac{1}{2}}} \le \frac{C}{t^{\frac{\beta}{2}}}$ on $[1,+\infty)$.

On the other hand, $j_\nu(0) = 1$; hence, there exists C such that $j_\nu(0) \le \frac{C}{t^{\frac{\beta}{2}}}$ on $(0,1]$.

Thus, $j_\nu(t) \le Ct^{-\frac{\beta}{2}}$ on $(0,+\infty)$.

This implies that

$$|\tilde{f}(r)| \leq C \left| \int_0^1 \frac{1}{(rs)^{\frac{\beta}{2}}} s^k f(s) ds \right| = C r^{-\frac{\beta}{2}} \int_0^\infty s^k s^{-\frac{\beta}{2}} f(s) ds$$

$$= C r^{-\frac{\beta}{2}} \int_0^\infty s^k \widetilde{s^{-\frac{\beta}{2}}} \tilde{f}(s) ds = C_\beta r^{-\frac{\beta}{2}} \int_0^\infty s^k s^{\frac{\beta}{2}-k-1} \tilde{f}(s) ds$$

$$= C_\beta r^{-\frac{\beta}{2}} \int_0^1 s^{\frac{\beta}{2}-1} \tilde{f}(s) ds + C_\beta r^{-\frac{\beta}{2}} \int_1^\infty s^{\frac{\beta}{2}-1} \tilde{f}(s) ds.$$

Note that $\left| \int_0^1 s^{\frac{\beta}{2}-1} \tilde{f}(s) ds \right| \leq \|\tilde{f}\|_\infty \int_0^1 s^{\frac{\beta}{2}-1} ds = C_\beta \int_0^\infty y^k f(y) dy$ because f is a nonnegative function. Thus, the last integral is equal to

$$\left(\int_0^\infty y^k f(y) dy \right)^{\frac{1}{2}} \left(\int_0^\infty x^k f(x) dx \right)^{\frac{1}{2}} = \left(\int_0^1 \int_0^1 x^k y^k f(x) f(y) dx dy \right)^{\frac{1}{2}}.$$

Let $y \in [0, 1]$ and $h \in [0, 1]$; then

$$T_y^h y^{-\alpha} \geq \frac{C}{(y+h)^\alpha} \int_0^\pi \sin^{k-1} \theta d\theta \stackrel{\text{def}}{=} \frac{C}{(y+h)^\alpha} \geq \frac{C}{2^\alpha} \stackrel{\text{def}}{=} \frac{1}{C}.$$

This means that

$$\int_0^1 \int_0^1 x^k y^k f(x) f(y) dx dy \leq C \int_0^1 x^k f(x) \int_0^1 y^k f(y) T_y^x y^{-\alpha} dy dx$$

$$= C \langle \tilde{f}, \widetilde{f * y^{-\alpha}} \rangle = C \int_0^\infty y^{\alpha-1} \tilde{f}^2(y) dy \stackrel{\text{def}}{=} C I_\alpha(f).$$

Thus,

$$|\tilde{f}(r)| \leq C_\beta r^{-\frac{\beta}{2}} \Big(I_\alpha(f) \Big)^{-\frac{1}{2}} + C r^{-\frac{\beta}{2}} \int_1^\infty s^{\frac{\beta}{2}-1} \tilde{f}(s) ds.$$

Hence,

$$\tilde{f}^2(r) \le C_\beta r^{-\beta} I_\alpha(f) + C_\beta r^{-\beta}\Big(\int_1^\infty s^{\frac{\beta-\alpha-1}{2}} s^{\frac{\alpha-1}{2}} \tilde{f}(s)ds\Big)^2$$

$$\le C_\beta r^{-\beta} I_\alpha(f) + C_\beta r^{-\beta}\int_1^\infty s^{\beta-\alpha-1}ds \int_1^\infty s^{\alpha-1}\tilde{f}^2(s)ds = C_\beta r^{-\beta} I_\alpha(f).$$

Therefore, the following assertion is proved.

Theorem 4.1 *Let* $k > 0$ *and* $\alpha \in (0, k]$. *Then for any* β *from* $(0, \alpha)$ *there exists* C_β *such that if* $f \in L_{1,k}(0, +\infty) \cap L_{2,k}(0, +\infty)$, $\operatorname{supp} f \subset [0, 1]$, *and* $f \ge 0$, *then*

$$\tilde{f}^2(r) \le C_\beta r^{-\beta}\int_0^\infty y^{\alpha-1}\tilde{f}^2(y)dy \quad for\ any\ positive\ r. \tag{11}$$

Now, let $\operatorname{supp} f \subset [0, R]$, $R > 1$. Define $f_R(y) \overset{\text{def}}{=} f(Ry)$; then $\operatorname{supp} f_R \subset [0, 1]$. Hence, Theorem 4.1 is valid for $f_R(y)$, i. e., for each β from $(0, \alpha)$ there exists C_β such that for any positive r the following inequality holds:

$$\widetilde{f_R}^2(r) \le C_\beta r^{-\beta}\int_0^\infty y^{\alpha-1}\widetilde{f_R}^2(y)dy.$$

On the other hand,

$$\widetilde{f_R}(y) = \int_0^\infty s^k j_\nu(ys) f_R(s)ds = \frac{1}{R^{k+1}}\int_0^\infty s^k j_\nu\Big(\frac{y}{R}s\Big)f(s)ds = \frac{1}{R^{k+1}}\tilde{f}\Big(\frac{1}{R}y\Big).$$

Therefore,

$$\tilde{f}^2\Big(\frac{r}{R}\Big) \le C_\beta r^{-\beta}\int_0^\infty y^{\alpha-1}\tilde{f}^2(\frac{1}{R}y)dy = C_\beta R^\alpha r^{-\beta}\int_0^\infty y^{\alpha-1}\tilde{f}^2(y)dy$$

provided that $R > 1$ and $r > 0$.

Then $\tilde{f}^2(r) \le C_\beta R^{\alpha-\beta} r^{-\beta}\int_0^\infty y^{\alpha-1}\tilde{f}^2(y)dy$ for any positive r.

Thus, the following assertion is proved.

Corollary 4.2 *Let $k > 0$ and $\alpha \in (0, k]$. Then for any β from $(0, \alpha)$ there exists C_β such that for each nonnegative compactly supported f from $L_{1,k}(0, +\infty) \cap L_{2,k}(0, +\infty)$ the following estimate holds:*

$$\tilde{f}^2(r) \leq C_\beta R^{\alpha-\beta} r^{-\beta} \int_0^\infty y^{\alpha-1} \tilde{f}^2(y)dy \quad for\ any\ positive\ r, \tag{12}$$

i.e.,

$$\|r^{\frac{\beta}{2}} \tilde{f}\|_\infty \leq C_\beta R^{\frac{\alpha-\beta}{2}} \|r^{\frac{\alpha-1}{2}} \tilde{f}\|_2, \tag{13}$$

where R is the right-hand boundary of supp f.

The next assertion shows the unimprovability of the obtained results.

Corollary 4.3 *Estimate* (11) *(and, therefore, estimates* (12)–(13)*) is not valid if $\beta > \alpha$.*

Proof Suppose, to the contrary, that there exists β exceeding α such that (11) is valid. Let f from $L_{1,k}(0, +\infty) \cap L_{2,k}(0, +\infty)$ be nonnegative and supp $f \subset [0, 1]$. Define $f_R(y)$ as $f(Ry)$ for R exceeding 1; then supp $f_R \subset \left[0, \frac{1}{R}\right] \subset [0, 1]$. Hence, (11) is valid for $f_R(y)$. Then, as in Corollary 4.2, $\tilde{f}^2(r) \leq C_\beta R^{\alpha-\beta} r^{-\beta} I_\alpha(f)$ for any positive r. Hence, $\tilde{f}^2(1) \leq C_\beta R^{\alpha-\beta} I_\alpha(f)$ for each R exceeding 1. Since $\alpha < \beta$, it follows that $\tilde{f}^2(1) = 0$.

We obtain a contradiction. □

4.2 The Case Where the Weight Power Exceeds the Parameter at the Singularity

Theorem 4.4 *Let $k > 0$ and $\alpha \in (k, k + 1)$. Then for any β from $(0, k]$ there exists C_β such that if a nonnegative f belongs to $L_{1,k}(0, +\infty) \cap L_{2,k}(0, +\infty)$ and* supp $f \subset [0, 1]$, *then*

$$\tilde{f}^2(r) \leq C_\beta r^{-\beta} I_\alpha(f) \quad for\ any\ positive\ r.$$

Proof Since $\nu = \frac{k}{2} - \frac{1}{2}$, it follows that $\nu + \frac{1}{2} \geq \frac{\beta}{2}$. Hence, $|j_\nu(t)| \leq \frac{C}{t^{\frac{\beta}{2}}}$ on $[1, +\infty)$.

On the other hand, $j_\nu(0) = 1$. Hence, there exists C such that $|j_\nu(t)| \le \dfrac{C}{t^{\frac{\beta}{2}}}$ on $(0, +\infty)$.

Then we can repeat the proof of Theorem 4.1 completely because the integral $\int\limits_1^\infty s^{\beta-\alpha-1} ds$ converges. □

As above, we derive the following corollary.

Corollary 4.5 *Let $k > 0$ and $\alpha \in (k, k+1)$. Then for any β from $(0, k]$ there exists C_β such that for any nonnegative compactly supported f from $L_{1,k}(0, +\infty) \cap L_{2,k}(0, +\infty)$ the following estimate holds:*

$$\tilde{f}^2(r) \le C_\beta R^{\alpha-\beta} r^{-\beta} \int\limits_0^\infty y^{\alpha-1} \tilde{f}^2(y) dy \quad for\ any\ positive\ r,$$

i.e.,

$$\|r^{\frac{\beta}{2}} \tilde{f}\|_\infty \le C_\beta R^{\frac{\alpha-\beta}{2}} \|r^{\frac{\alpha-1}{2}} \tilde{f}\|_2 \Big),$$

where R is the right-hand boundary of supp f.

The unimprovability of the results of this section is established by the following assertion.

Theorem 4.6 *Theorem 4.4 is not valid if $\beta > k$.*

Proof Let $f(s) = f_R(s) = e^{-iRs}\varphi(s)$, $\varphi \in C^\infty(\mathbf{R})$, $\operatorname{supp}\varphi \subset (0, 1)$, $\varphi \ge 0$, and $\varphi \equiv 1$ on $\left(\dfrac{5}{8}, \dfrac{7}{8}\right)$.

Then

$$\tilde{f}(R) = \int\limits_0^\infty s^k j_\nu(Rs) f(s) ds = \int\limits_0^\infty s^k j_\nu(Rs) e^{-iRs} \varphi(s) ds.$$

Taking into account that

$$\begin{aligned} j_\nu(t) = \frac{C}{t^{\nu+\frac{1}{2}}} \left(\sum_{l=0}^{m-1} \frac{C_l}{t^l} \cos\left[t - \frac{\pi}{2}\Big(\nu - l + \frac{1}{2}\Big)\right] + \mathcal{O}\Big(\frac{1}{t^m}\Big) \right) \\ = \frac{C}{t^{\nu+\frac{1}{2}}} \cos\Big(t - \frac{\pi}{4}k\Big) + \mathcal{O}\left(\frac{1}{t^{\frac{k}{2}+1}}\right) \\ = \frac{C}{t^{\frac{k}{2}}} \Big(e^{i(t-\frac{\pi}{4}k)} + e^{-i(t-\frac{\pi}{4}k)}\Big) + \mathcal{O}\left(\frac{1}{t^{\frac{k}{2}+1}}\right) \text{ as } t \to \infty, \end{aligned}$$

we conclude that

$$\tilde{f}(R) = C_1 R^{-\frac{k}{2}} \int\limits_0^1 s^{\frac{k}{2}} \varphi(s)ds + C_2 R^{-\frac{k}{2}} \int\limits_0^1 s^{\frac{k}{2}} e^{-i2Rs} \varphi(s)ds + \mathcal{O}\Big(R^{-\frac{k}{2}-1}\Big) \text{ as } R \to \infty,$$

where $C_1 \stackrel{\text{def}}{=} Ce^{-i\frac{\pi}{4}k}$ and $C_2 \stackrel{\text{def}}{=} Ce^{i\frac{\pi}{4}k}$.

Denote the coefficient at $R^{-\frac{k}{2}}$ by A and prove that $\lim\limits_{R\to\infty} A \neq 0$. We have

$$\frac{A}{C} = \cos\frac{\pi}{4}k \int\limits_0^1 s^{\frac{k}{2}} \varphi(s)ds + \cos\frac{\pi}{4}k \int\limits_0^1 s^{\frac{k}{2}} \varphi(s)\cos 2Rsds + \sin\frac{\pi}{4}k \int\limits_0^1 s^{\frac{k}{2}} \varphi(s)\sin 2Rsds + i\Big(- \sin\frac{\pi}{4}k \int\limits_0^1 s^{\frac{k}{2}} \varphi(s)ds + \sin\frac{\pi}{4}k \int\limits_0^1 s^{\frac{k}{2}} \varphi(s)\cos 2Rsds + \cos\frac{\pi}{4}k \int\limits_0^1 s^{\frac{k}{2}} \varphi(s)\sin 2Rsds\Big).$$

Suppose, to the contrary, that $\lim\limits_{R\to\infty} \mathrm{Re}A = \lim\limits_{R\to\infty} \mathrm{Im}A = 0$.

Then $\lim\limits_{R\to\infty} \Big(\mathrm{Re}A \cos\frac{\pi}{4}k - \mathrm{Im}A \sin\frac{\pi}{4}k\Big)$ vanishes too. On the other hand, the last limit is equal to

$$\int\limits_0^1 s^{\frac{k}{2}} \varphi(s)ds + \cos\frac{\pi}{2}k \int\limits_0^1 s^{\frac{k}{2}} \varphi(s)\cos 2Rsds + \sin\frac{\pi}{2}k \int\limits_0^1 s^{\frac{k}{2}} \varphi(s)\sin 2Rsds.$$

Taking into account that the integrated function is nonnegative, we conclude that the last expression is greater than or equal to

$$\int\limits_{\frac{5}{8}}^{\frac{7}{8}} s^{\frac{k}{2}} \varphi(s)\left[1 + \cos\Big(\frac{\pi}{2}k - 2Rs\Big)\right] ds = \int\limits_{\frac{5}{8}}^{\frac{7}{8}} s^{\frac{k}{2}} \left[1 + \cos\Big(\frac{\pi}{2}k - 2Rs\Big)\right] ds$$

bounded from below by

$$\left(\frac{5}{8}\right)^{\frac{k}{2}} \int_{\frac{5}{8}}^{\frac{7}{8}} \left[1 + \cos\left(\frac{\pi}{2}k - 2Rs\right)\right] ds = \left(\frac{5}{8}\right)^{\frac{k}{2}} \left(\frac{1}{4} - \frac{\sin\left(\frac{\pi}{2}k - 2Rs\right)}{2R}\Bigg|_{\frac{5}{8}}^{\frac{7}{8}}\right).$$

This tends to the positive number $\frac{1}{4}\left(\frac{5}{8}\right)^{\frac{k}{2}}$ as $R \to \infty$, which yields a contradiction.

Thus, A does not tend to zero as $R \to \infty$.

Moreover there exists a positive C $\left(\text{e. g., one can take } \frac{1}{8}\left(\frac{5}{8}\right)^{\frac{k}{2}}\right)$ such that if R is sufficiently large, then

$$\mathrm{Re}A \cos\frac{\pi}{4}k - \mathrm{Im}A \sin\frac{\pi}{4}k \geq C.$$

Therefore, $C \leq |\mathrm{Re}A| + |\mathrm{Im}A|$. Hence, $C^2 \leq (|\mathrm{Re}A| + |\mathrm{Im}A|)^2 \leq 2|A|^2$, i. e., $|A| \geq C_0 \stackrel{\text{def}}{=} \sqrt{\frac{C}{2}}$ for sufficiently large values of R. Hence, $|\widetilde{f_R}(R)| \geq \dfrac{C_0}{2} R^{-\frac{k}{2}}$ provided that R is sufficiently large.

On the other hand, the following assertion is true.

Lemma 4.7 *The inequality* $|\widetilde{f_R}(R)|^2 \leq C_\beta R^{-\beta}$ *holds for any positive* R.

Proof Introduce $f_l(s)$, $l = \overline{1,4}$, as follows:

$$f_1(x) \stackrel{\text{def}}{=} \begin{cases} \mathrm{Re} f(x) \text{ if } \mathrm{Re} f(x) > 0 \\ 0 \text{ otherwise} \end{cases}, \quad f_2(x) \stackrel{\text{def}}{=} \begin{cases} 0 \text{ if } \mathrm{Re} f(x) > 0 \\ -\mathrm{Re} f(x) \text{ otherwise} \end{cases},$$

$$f_3(x) \stackrel{\text{def}}{=} \begin{cases} \mathrm{Im} f(x) \text{ if } \mathrm{Im} f(x) > 0 \\ 0 \text{ otherwise} \end{cases}, \quad \text{and} \quad f_4(x) \stackrel{\text{def}}{=} \begin{cases} 0 \text{ if } \mathrm{Im} f(x) > 0 \\ -\mathrm{Im} f(x) \text{ otherwise} \end{cases}.$$

Then $0 \leq f_l(x) \leq |f(x)|$, $l = \overline{1,4}$, and $f(s) = f_1(s) - f_2(s) + i[f_3(s) - f_4(s)]$. Hence, $|\tilde{f}(s)|^2 \leq C \sum_{l=1}^{4} \widetilde{f_l}^2(s) \leq C_\beta s^{-\beta} \sum_{l=1}^{4} I_\alpha(f_l)$ due to Theorem 4.4.

Now, observe that

$$I_\alpha(|f|) = \langle |f|, |f| y^{\alpha-k-1} \rangle = \langle \widetilde{|f|}, \widetilde{|f|} \widetilde{y^{-\alpha}} \rangle = C \langle |f|, |f| * y^{-\alpha} \rangle$$

$$= C \int_0^\infty \int_0^\infty x^k y^k |f(x)||f(y)| T_y^x y^{-\alpha} dx dy \geq C \int_0^\infty \int_0^\infty x^k y^k f_l(x) f_l(y) T_y^x y^{-\alpha} dx dy$$

$$= C I_\alpha(f_l), \; l = \overline{1,4}.$$

Thus, $|\widetilde{f_R}(r)|^2 \leq C_\beta r^{-\beta} I_\alpha(|f_R|)$. However, $|f_R|$ does not depend on R. □

Thus, there exists C_β such that

$$R^{-k} \leq C_\beta R^{-\beta}$$

provided that R is sufficiently large. This means that $k \geq \beta$. □

Remark 4.8 If $k < \alpha < k+1$, then C does not depend on β.

Really, the said dependence arises (see the prove of Theorem 4.1) at the two following points:

$$C_{\frac{\beta}{2}} \int_0^1 s^{\frac{\beta}{2}-1} ds \text{ where } \widetilde{s^{-\frac{\beta}{2}}} = C_{\frac{\beta}{2}} s^{\frac{\beta}{2}-k-1} \quad \text{and} \quad \int_1^\infty s^{\beta-\alpha-1} ds.$$

Since $C_{\frac{\beta}{2}} = \dfrac{\Gamma\left(\frac{k-\frac{\beta}{2}-1}{2}\right)}{\Gamma(\frac{\beta}{4})} 2^{k-\frac{\beta}{2}-1} C_k$ (see, e. g., [12, pp. 157–158]), it follows that $C_{\frac{\beta}{2}} \int_0^1 s^{\frac{\beta}{2}-1} ds = \frac{2}{\beta} C_{\frac{\beta}{2}}$ is bounded by a constant depending only on k (because $\Gamma(\frac{\beta}{4})$ has a simple pole at the origin).

On the other hand, $\int_1^\infty s^{\beta-\alpha-1} ds = \dfrac{1}{\alpha-\beta} \leq \dfrac{1}{\alpha-k}$, which does not depend on β.

Note that the dependence of C on α cannot be removed (at least, on that way) because C becomes infinite at both ends of $(k, k+1)$; really, $C_\alpha = \dfrac{\Gamma(\frac{k-\alpha-1}{2})}{\Gamma(\frac{\alpha}{2})}$.

Also, note that if $0 < \alpha \leq k$, then the dependence of C on β cannot be removed either because $\int_1^\infty s^{\beta-\alpha-1} ds = \dfrac{1}{\alpha-\beta} \longrightarrow \infty$ as $\beta \to \alpha$.

5 Multi-Dimensional Estimates: The Prototype Case

In this section, we consider the case where the special variable y is unique. This is called the *prototype case* because it is traditionally assumed that main properties of singular problems and their core differences from regular ones can be explained and observed on problems with a single special variable.

Denoting the upper hemisphere in $\mathbb{R}^{n+1}_+$ of radius r, centered at the origin, by $S_+(r)$, we introduce $\sigma^{p,q}$ as follows:

$$\sigma^{p,q}(f) \stackrel{\text{def}}{=} \int\limits_{S_+(1)} |x|^p y^q |\hat{f}(rx, ry)|^2 dS_{x,y}, \tag{14}$$

where p and q are real parameters and dS denotes the surface (spherical) measure with respect to the corresponding variables.

The following assertion is valid.

Theorem 5.1 *Suppose that* $f \in L_{1,k}(\mathbb{R}^{n+1}_+) \cap L_{2,k}(\mathbb{R}^{n+1}_+)$ *and* $f \geq 0$.

If $n > 1$, *then for any* p *exceeding* $-n$ *and any* q *exceeding* -1 *there exists* C *such that the estimate*

$$\|r^{\alpha+\beta}\sigma^{p+\alpha,q+\beta}(f)\|_\infty \leq C\|r^{\alpha+\beta-1}\sigma^{\alpha-n,\beta-1}(f)\|_1 \tag{15}$$

holds provided that $\alpha \in \left(0, \frac{n-1}{2}\right)$ *and* $\beta \in \left(0, \frac{k}{2}\right)$ *or* $(\alpha, \beta) = \left(\frac{n-1}{2}, \frac{k}{2}\right)$. *If* $n = 1$, *then for any* p *exceeding* -1 *and any* q *exceeding* $\frac{k}{2} - 1$ *there exists* C *such that*

$$\|r^{\frac{k}{2}}\sigma^{p,q}(f)\|_\infty \leq C\|\eta^{\frac{k}{2}-1}|\hat{f}\Big|_{\xi=0}|^2\|_1. \tag{16}$$

Proof The main idea of the proof is the same as in [6]: the generalized function of the weighted spherical averaging (in the sequel, it is denoted by σ_r) is singular, but its Fourier–Bessel transform is regular and could be computed explicitly.

We have

$$\begin{aligned}
\langle \widehat{\sigma_r}, \varphi \rangle = C\langle \sigma_r, \check{\varphi} \rangle &= C \int\limits_{S_+(1)} |\xi|^p \eta^q \check{\varphi}(r\xi, r\eta)\, dS_{\xi,\eta} \\
&= \int\limits_{S_+(1)} |\xi|^p \eta^q \int\limits_{\mathbb{R}^{n+1}_+} y^k \varphi(x, y) e^{irx\cdot\xi} j_\nu(ry\eta)\, dx dy dS_{\xi,\eta} \\
&= \int\limits_{\mathbb{R}^{n+1}_+} y^k \varphi(x, y) \int\limits_{S_+(1)} |\xi|^p \eta^q e^{irx\cdot\xi} j_\nu(ry\eta)\, dS_{\xi,\eta} dx dy,
\end{aligned}$$

where φ is an arbitrary test function. The inner integral in the last relation is equal to

$$Cr^{(2-n)/2-\nu}|x|^{(2-n)/2}\,y^{-\nu}$$
$$\times\int_0^1 \eta^{q-\nu}(1-\eta^2)^{(n+2p-2)/4}J_{(n-2)/2}\left(r\sqrt{1-\eta^2}|x|\right)J_\nu(ry\eta)\,d\eta$$

(see, e. g., [14, p. 155]). This implies the estimate

$$|\widehat{\sigma_r}(x,y)|\le Cr^{-(n+k-1)/2}|x|^{-(n-1)/2}y^{-k/2} \tag{17}$$

(since now, C depends on p and q as well).

Next, we observe that $\sigma(f)(r)\overset{\text{def}}{=}\langle\sigma_r,|\hat f|^2\rangle=\langle\sigma_r,\hat f\,\overline{\hat f}\,\rangle=\langle\hat f\sigma_r,\hat f\rangle$.

On the other hand, $\langle f*\widehat{\sigma_r},f\rangle = C\langle\widehat{f*\widehat{\sigma_r}},\hat f\rangle = C\langle\hat f\widehat{\widehat{\sigma_r}},\hat f\rangle = C\langle\hat f\sigma_r(-x,y),\hat f\rangle$. However, we have just proved that $\widehat{\sigma_r}$ is an ordinary function even with respect to each nonspecial variable. Thus, $\sigma(f)(r)=\langle f*\widehat{\sigma_r},f\rangle$.

Using the known properties of the generalized translation operator and generalized convolution (see [3, 4]), we find that

$$\sigma(f)(r)=\int_{S_+(1)}|\xi|^p\eta^q|\hat f(r\xi,r\eta)|^2\,dS_{\xi,\eta}$$
$$\le Cr^{(1-n-k)/2}\big\langle\mathcal{F}_{\mathrm{b}}\big(|x|^{-(n-1)/2}y^{-k/2}\big),|\hat f|^2\big\rangle$$

in the case where $n>1$. Also, $\mathcal{F}_{\mathrm{b}}\big(|x|^{-(n-1)/2}y^{-k/2}\big)$ may be computed explicitly (if $n>1$ and $k>0$):

$$\mathcal{F}_{\mathrm{b}}\big(|x|^{(1-n)/2}y^{-k/2}\big)=\int_0^\infty y^k j_\nu(y\eta)y^{-k/2}\,dy\int_{\mathbb{R}^n}e^{-ix\cdot\xi}|x|^{(1-n)/2}\,dx$$
$$=C\eta^{-k/2-1}|\xi|^{-(n+1)/2}.$$

This leads to (15) with $\alpha=\alpha_0\overset{\text{def}}{=}(n-1)/2$ and $\beta=\beta_0\overset{\text{def}}{=}k/2$. To extend (15) to the whole claimed intervals, we, apply (as it is done in [6]) the already proved inequality to the new function

$$f_{\gamma,\delta}\overset{\text{def}}{=}f*(|x|^{\gamma-n}y^{\delta-k-1}),$$

where

$$\gamma\overset{\text{def}}{=}\frac{\alpha_0-\alpha}{2}>0\quad\text{and}\quad\delta\overset{\text{def}}{=}\frac{\beta_0-\beta}{2}>0.$$

In fact, the mixed Fourier–Bessel transform of $|x|^{\gamma-n}y^{\delta-k-1}$ is decomposed $\big($in the same way as the mixed Fourier–Bessel transform of $|x|^{-(n-1)/2}y^{-k/2}$ is decomposed above$\big)$ into the product of the Fourier transform of the Riesz kernel $|x|^{\gamma-n}$ and the pure Fourier–Bessel transform of the power function $y^{\delta-k-1}$. From this, we conclude that $\widehat{f_{\gamma,\delta}} = C_{\gamma,\delta}\hat{f}|x|^{-\gamma}y^{-\delta}$. Now, it remains to go back from γ, δ to α, β and obtain the powers in (15). We note that $C_{\gamma,\delta}$ appears at both parts of the inequality. Therefore, C in (15) does not really depend on α and β.

In the critical case (i. e, for $n = 1$), the variable x vanishes at the right-hand part of inequality (17), and the above arguing leads to (16). □

Remark 5.2 Inequalities (15)–(16) hold for each nonnegative f such that the right-hand part of the inequality converges (in particular, for each nonnegative f from $L_{1,k}(\mathbb{R}^{n+1}_+) \cap L_{2,k}(\mathbb{R}^{n+1}_+)$). However, the inequality remains to be valid formally even if its right-hand part diverges.

6 Estimates for the Case of Several Special Variables

6.1 Preliminaries

In this section, we extend the necessary notation and definitions of Sect. 2 to the considered case. Also, we recall the necessary properties of the Fourier–Bessel transformation.

Let $k \overset{\text{def}}{=} (k_1,\dots,k_m) \overset{\text{def}}{=} (2\nu_1+1,\dots,2\nu_m+1)$ denote a positive multi-index such that $k_l > 0$ for each $l = \overline{1,m}$. Let $|k|$ denote the length $k_1+\dots+k_m$ of k.

Introduce $\mathbb{R}^m_{(+)} \overset{\text{def}}{=} \Big\{y = (y_1,\dots,y_m)\,\Big|\, y_l > 0 \text{ for each } l = \overline{1,m}\Big\}$ and $\mathbb{R}^{n+m}_+ \overset{\text{def}}{=} \Big\{(x,y)\,\Big|\, x\in\mathbb{R}^n,\ y\in\mathbb{R}^m_{(+)}\Big\}$.

In the sequel, all the absolute constants generally depend on k, m, and n.

Let $S(r)$ denote the sphere $\Big\{x \in \mathbb{R}^n \,\Big|\, |x| = r\Big\}$, $S^l_+(r)$ denote the spherical segment $\Big\{(x,y) \in \mathbb{R}^{n+l}_+ \,\Big|\, |x|^2+|y|^2 = r^2\Big\}$, where $l = \overline{1,m}$, $S^+(r)$ denote the spherical segment $\Big\{y \in \mathbb{R}^m_{(+)} \,\Big|\, |y| = r\Big\}$, and $B^+(r)$ denote the following segment of the ball: $\Big\{y \in \mathbb{R}^m_{(+)} \,\Big|\, |y| \le r\Big\}$.

Now, we can introduce

$$L_{p,k}(\mathbb{R}^{n+m}_+) \overset{\text{def}}{=} \Big\{f \,\Big|\, \|f\| = \Big(\int\limits_{\mathbb{R}^{n+m}_+}\prod_{l=1}^m y_l^{k_l}|f(x,y)|^p dxdy\Big)^{\frac{1}{p}} < \infty\Big\},\ p<\infty,$$

and

$$L_{\infty,k}(\mathbb{R}^{n+m}_{+}) \stackrel{\text{def}}{=} \left\{ f \,\middle|\, \|f\| = \sup \prod_{l=1}^{m} y_l^{k_l} |f(x,y)| < \infty \right\}.$$

The set of infinitely smooth functions with compact supports, defined on $\mathbb{R}^{n+m}$, is denoted by $C_0^\infty(\mathbb{R}^{n+m})$.

The subset of $C_0^\infty(\mathbb{R}^{n+m})$ formed by functions that are even with respect to each y_l, $l = \overline{1,m}$, is observed; the set of restrictions of elements of that subset to $\mathbb{R}^{n+m}_{+}$ is denoted by $C^\infty_{0,even}(\mathbb{R}^{n+m}_{+})$ and is treated as the *space of test functions*.

Generalized functions (distributions) on $C^\infty_{0,even}(\mathbb{R}^{n+m}_{+})$ are introduced (following, e. g., [1]) with respect to the degenerative measure $\prod_{l=1}^{m} y_l^{k_l} dxdy$:

$$\langle f, \varphi\rangle \stackrel{\text{def}}{=} \int\limits_{\mathbb{R}^{n+m}_{+}} \prod_{l=1}^{m} y_l^{k_l} f(x,y)\varphi(x,y)dxdy \ \text{ for each } \varphi \text{ from } \ C^\infty_{0,even}(\mathbb{R}^{n+m}_{+}). \tag{18}$$

Thus, all linear continuous functionals on $C^\infty_{0,even}(\mathbb{R}^{n+m}_{+})$ that could be given by (18) (with f from $L_{1,k,loc}(\mathbb{R}^{n+m}_{+})$) are called *regular* (and the corresponding function f is called *ordinary*).

The Fourier–Bessel transformation is introduced according to [2, 5]:

$$\hat{f}(\xi,\eta) \stackrel{\text{def}}{=} \mathcal{F}_{\text{b}} f \stackrel{\text{def}}{=} \int\limits_{\mathbb{R}^m_{(+)}} \int\limits_{\mathbb{R}^n} \prod_{l=1}^{m} y_l^{k_l} j_{\nu_l}(\eta_l y_l) e^{-ix\cdot\xi} f(x,y)dxdy.$$

Note (see [2]) that

$$f(x,y) = C \int\limits_{\mathbb{R}^m_{(+)}} \int\limits_{\mathbb{R}^n} \prod_{l=1}^{m} \eta_l^{k_l} j_{\nu_l}(\eta_l y_l) e^{ix\cdot\xi} \hat{f}(\xi,\eta)d\xi d\eta.$$

The generalized convolution is introduced according to [4, 5]:

$$\hat{f}(\xi,\eta) \stackrel{\text{def}}{=} (f * g)(\xi,\eta) \stackrel{\text{def}}{=} \int\limits_{\mathbb{R}^m_{(+)}} \int\limits_{\mathbb{R}^n} \prod_{l=1}^{m} y_l^{k_l} T_y^\eta f(x_1-\xi_1,\dots,x_m-\xi_m,y_1,\dots,y_m)dxdy.$$

It satisfies the relation $\widehat{f * g} = \hat{f}\hat{g}$ (see also [3]).

Here, $T_y^h f(x,y) \stackrel{\text{def}}{=} T_{y_1,\dots,y_m}^{h_1,\dots,h_m} f(x,y) \stackrel{\text{def}}{=} T_{y_1}^{h_1} T_{y_2}^{h_2} \cdots T_{y_m}^{h_m} f(x,y)$, where $T_{y_l}^{h_l}$ denotes (for $l = \overline{1,m}$) the generalized translation operator with respect to the corresponding special variable, defined by relation (6) above:

$$T_{y_l}^{h_l} f(y) = C \int_0^\pi f\left(x, y_1, \dots, y_{l-1}, \sqrt{y_l^2 + h_l^2 - 2y_l h_l \cos\theta}, y_{l+1}, \dots, y_m\right) \times \sin^{k_l - 1}\theta d\theta.$$

Note (see [4, 5]) that

$$\int_{\mathbb{R}_{(+)}^m} \prod_{l=1}^m \eta_l^{k_l} g(\eta) T_y^\eta f(y) d\eta = \int_{\mathbb{R}_{(+)}^m} \prod_{l=1}^m \eta_l^{k_l} f(\eta) T_y^\eta g(y) d\eta.$$

6.2 Estimates for the General Case

We start our investigation from the case of several *nonspecial* variables, i. e., the case where $n > 1$; the critical cases of $n = 1$ and $n = 0$ are considered in Sects. 6.3 and 6.4 respectively, while the case of a single special variable, i. e., the case where $m = 1$, is investigated in Sect. 5; hence, in the sequel, we assume that $m > 1$.

Thus, let $m \geq 2$ and $n \geq 2$.

Now, as in Sect. 5, we have to define the weighted spherical mean and the corresponding generalized function of the weighted spherical averaging:

$$\sigma^{p,q}(f)(r) \stackrel{\text{def}}{=} \sigma(f)(r) \stackrel{\text{def}}{=} \int_{S_+^m(1)} |x|^p y_1^{q_1} \cdots y_m^{q_m} |\hat{f}(rx, ry)|^2 dS_{x,y} \stackrel{\text{def}}{=} \langle \sigma_r, |\hat{f}|^2 \rangle \stackrel{\text{def}}{=} \langle \sigma_r^{p,q}, |\hat{f}|^2 \rangle,$$

where $p > -n$ and $q_l > -1$, $l = \overline{1,m}$.

The following assertion is valid.

Theorem 6.1 *If $m \geq 2$, $n \geq 2$, k is a positive multi-index, $p > -n$, and $q_l > -1$ ($l = \overline{1,m}$), then there exists C such that the inequality*

$$\|r^{\alpha+|\beta|} \sigma^{p+\alpha, q+\beta}(f)\|_\infty \leq C \|r^{\alpha+|\beta|-1} \sigma^{\alpha-n, \beta-1}(f)\|_1 \tag{19}$$

holds for each nonnegative f from $L_{1,k}(\mathbb{R}_+^{n+m}) \cap L_{2,k}(\mathbb{R}_+^{n+m})$ provided that $\alpha \in \left(0, \frac{n-1}{2}\right)$ and $\beta_l \in \left(0, \frac{k_l}{2}\right)$ ($l = \overline{1,m}$) or $(\alpha, \beta) = \left(\frac{n-1}{2}, \frac{k_1}{2}, \dots, \frac{k_m}{2}\right)$.

Proof Estimate

$$g_r(x,y) \stackrel{\text{def}}{=} \widehat{\sigma_r} = \int\limits_{S_+^m(1)} |\xi|^p e^{irx\cdot\xi} \prod_{l=1}^{m} \eta_l^{q_l} j_{\nu_l}(ry_l\eta_l) dS_{\xi,\eta}$$

$$= \int\limits_0^1 \eta_m^{q_m} j_{\nu_m}(ry_m\eta_m) \int\limits_{S_+^{m-1}(\sqrt{1-\eta_m^2})} |\xi|^p e^{irx\cdot\xi}$$

$$\times \prod_{l=1}^{m-1} \eta_l^{q_l} j_{\nu_l}(ry_l\eta_l) dS_{\xi,\eta_1,\dots,\eta_{m-1}} \frac{d\eta_m}{\sqrt{1-\eta_m^2}}$$

$$= \int\limits_0^1 \eta_m^{q_m} j_{\nu_m}(ry_m\eta_m) \int\limits_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}} j_{\nu_{m-1}}(ry_{m-1}\eta_{m-1})$$

$$\times \int\limits_{S_+^{m-2}(\sqrt{1-\eta_m^2-\eta_{m-1}^2})} |\xi|^p e^{irx\cdot\xi}$$

$$\times \prod_{l=1}^{m-2} \eta_l^{q_l} j_{\nu_l}(ry_l\eta_l) dS_{\xi,\eta_1,\dots,\eta_{m-2}} \frac{\sqrt{1-\eta_m^2}\,d\eta_{m-1}}{\sqrt{1-\eta_{m-1}^2-\eta_m^2}} \frac{d\eta_m}{\sqrt{1-\eta_m^2}}$$

$$\dots$$

$$= \int\limits_0^1 \eta_m^{q_m} j_{\nu_m}(ry_m\eta_m) \int\limits_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}} j_{\nu_{m-1}}(ry_{m-1}\eta_{m-1}) \dots$$

$$\dots \int\limits_0^{\sqrt{1-\eta_m^2-\dots-\eta_2^2}} \eta_1^{q_1} j_{\nu_1}(ry_1\eta_1)(1-|\eta|^2)^{\frac{p}{2}} \int\limits_{S\left(\sqrt{1-|\eta|^2}\right)} e^{irx\cdot\xi} dS_\xi$$

$$\times \frac{1}{\sqrt{1-|\eta|^2}} d\eta_1 \dots d\eta_m$$

(see, e. g., [14, p. 155])

$$= \int_0^1 \eta_m^{q_m} j_{\nu_m}(r y_m \eta_m) \int_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}} j_{\nu_{m-1}}(r y_{m-1} \eta_{m-1}) \dots$$

$$\dots \int_0^{\sqrt{1-\eta_m^2-\dots-\eta_2^2}} \eta_1^{q_1} j_{\nu_1}(r y_1 \eta_1) \frac{(1-|\eta|^2)^{\frac{n+p-1}{2}}}{\sqrt{1-|\eta|^2}}$$

$$\times (r\sqrt{1-|\eta|^2}|x|)^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(r\sqrt{1-|\eta|^2}|x|) d\eta$$

$$= r^{-|\nu|}|x|^{\frac{2-n}{2}} \prod_{l=1}^{m} y_l^{-\nu_l} \int_0^1 \eta_m^{q_m-\nu_m} J_{\nu_m}(r y_m \eta_m)$$

$$\times \int_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}-\nu_{m-1}} J_{\nu_{m-1}}(r y_{m-1} \eta_{m-1}) \dots \int_0^{\sqrt{1-\eta_m^2-\dots-\eta_2^2}} \eta_1^{q_1-\nu_1} J_{\nu_1}(r y_1 \eta_1)$$

$$\times \frac{(1-|\eta|^2)^{\frac{n+p-1}{2}+\frac{2-n}{4}}}{\sqrt{1-|\eta|^2}} r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(r\sqrt{1-|\eta|^2}|x|) d\eta$$

$$= r^{\frac{2-n}{2}-|\nu|}|x|^{\frac{2-n}{2}} \prod_{l=1}^{m} y_l^{-\nu_l} \int_0^1 \eta_m^{q_m-\nu_m} J_{\nu_m}(r y_m \eta_m)$$

$$\times \int_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}-\nu_{m-1}} J_{\nu_{m-1}}(r y_{m-1} \eta_{m-1}) \dots$$

$$\dots \int_0^{\sqrt{1-\eta_m^2-\dots-\eta_2^2}} \eta_1^{q_1-\nu_1} J_{\nu_1}(r y_1 \eta_1)(1-|\eta|^2)^{\frac{n+2p-2}{4}} J_{\frac{n-2}{2}}(r\sqrt{1-|\eta|^2}|x|) d\eta.$$

Since $|J_\mu(t)| \le \dfrac{C}{\sqrt{t}}$ provided that $t > 0$, it follows that

$$|g_r(x,y)| \le C r^{\frac{m+2-n-|k|}{2}} |x|^{\frac{1-n}{2}} r^{-\frac{m+1}{2}} \prod_{l=1}^{m} y_l^{-\nu_l - \frac{1}{2}}$$
$$\times \int\limits_{B^+(1)} (1-|\eta|^2)^{\frac{n+2p-3}{2}} \prod_{l=1}^{m} \eta_l^{q_l-\nu_l-\frac{1}{2}} d\eta = C r^{\frac{1-n-|k|}{2}} |x|^{\frac{1-n}{2}} \prod_{l=1}^{m} y_l^{-\frac{k_l}{2}}$$

provided that $p > -\dfrac{n+1}{2}$ and $q_l > \dfrac{k_l}{2} - 1,\ l = \overline{1,m}$.

This means that the Fourier–Bessel transform of σ_r is a regular generalized function (though σ_r itself is a singular generalized function).

Further, similarly to Sect. 5, we conclude that $\sigma(f)(r) = \langle f * g_r, f \rangle$. Then, due to the nonnegativity of f, the last expression is less than or equal to

$$\int\limits_{\mathbb{R}^{n+m}_+} \prod_{l=1}^{m} \eta_l^{k_l} f(\xi,\eta) \int\limits_{\mathbb{R}^{n+m}_+} \prod_{l=1}^{m} y_l^{k_l} |g_r(x,y)|\, T_y^\eta f(x-\xi, y)\, dx dy d\xi d\eta$$

since the generalized translation operator preserves the sign.

Hence,

$$\sigma(f)(r) \le C r^{\frac{1-n-|k|}{2}} \Big\langle f, f * \Big(|x|^{-\frac{n-1}{2}} \prod_{l=1}^{m} y_l^{-\frac{k_l}{2}} \Big) \Big\rangle$$
$$= C r^{\frac{1-n-|k|}{2}} \Big\langle \mathcal{F}_{\mathrm{b}} \Big(|x|^{-\frac{n-1}{2}} \prod_{l=1}^{m} y_l^{-\frac{k_l}{2}} \Big), |\hat{f}|^2 \Big\rangle.$$

On the other hand,

$$\mathcal{F}_{\mathrm{b}} \Big(|x|^{-\frac{n-1}{2}} \prod_{l=1}^{m} y_l^{-\frac{k_l}{2}} \Big) = C |\xi|^{-\frac{n+1}{2}} \prod_{l=1}^{m} \eta_l^{-\frac{k_l}{2}-1}$$

(see, e. g., [14, p. 155] and Sect. 3).

Therefore, for each nonnegative f from $L_{1,k}(\mathbb{R}^{n+m}_+) \cap L_{2,k}(\mathbb{R}^{n+m}_+)$, the inequality

$$\sigma(f)(r) \le C r^{\frac{1-n-|k|}{2}} \int\limits_{\mathbb{R}^m_{(+)}} \int\limits_{\mathbb{R}^n} |x|^{-\frac{n+1}{2}} \prod_{l=1}^{m} y_l^{\frac{k_l}{2}-1} |\hat{f}(x,y)|^2 dx dy$$

holds on $(0, +\infty)$.

Actually, this is the claimed estimate (19) for $(\alpha, \beta) = \left(\frac{n-1}{2}, \nu + \frac{1}{2}\right)$. In order to extend it to each α from $\left(0, \frac{n-1}{2}\right)$ and each $(\beta_1, \dots, \beta_m)$ from $\prod_{l=1}^{m}\left(0, \nu_l + \frac{1}{2}\right)$, we have to introduce $f_{\gamma,\delta} \overset{\text{def}}{=} f * \left(|x|^{\gamma-n} \prod_{l=1}^{m} y_l^{\delta_l - k_l - 1}\right)$, where $\gamma = \frac{n-2\alpha-1}{4} > 0$ and $\delta_l = \frac{2\nu_l - 2\beta_l + 1}{4} > 0$, $l = \overline{1, m}$ (cf. Sects. 3 and 5). Then we apply the last inequality to this new function $f_{\gamma,\delta}$.

This yields the inequality

$$\sup_{\mathbb{R}_+} r^{\alpha+|\beta|} \sigma^{p-\frac{n-1}{2}+\alpha, q-\frac{k}{2}+\beta}(f)(r) \leq C \int_{\mathbb{R}^m_{(+)}} \int_{\mathbb{R}^n} |x|^{\alpha-n} \prod_{l=1}^{m} y_l^{\beta_l - 1} |\hat{f}(x, y)|^2 dx dy \tag{20}$$

(valid under the same assumptions about f).

The right-hand part of (20) is equal to $C \int_0^\infty r^{\alpha+|\beta|-1} \sigma^{\alpha-n, \beta-1} f(r) dr$. □

6.3 The Case of a Single Nonspecial Variable

Let $n = 1$. Then, similarly to Sect. 6.2, we have

$$g_r(x, y) = \int_{S^m_+(1)} |\xi|^p e^{irx\xi} \prod_{l=1}^{m} \eta_l^{q_l} j_{\nu_l}(r y_l \eta_l) dS_{\xi,\eta}$$

$$= \int_0^1 \eta_m^{q_m} j_{\nu_m}(r y_m \eta_m) \int_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}} j_{\nu_{m-1}}(r y_{m-1} \eta_{m-1}) \dots$$

$$\dots \int_0^{\sqrt{1-|\eta|^2+\eta_1^2}} (1-|\eta|^2)^{\frac{p}{2}} \eta_1^{q_1} j_{\nu_1}(r y_1 \eta_1) \left(e^{ix\sqrt{1-|\eta|^2} r} + e^{-ix\sqrt{1-|\eta|^2} r}\right) \frac{d\eta}{\sqrt{1-|\eta|^2}}.$$

This is equal to

$$= 2\int\limits_0^1 \eta_m^{q_m} j_{\nu_m}(r y_m \eta_m) \int\limits_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}} j_{\nu_{m-1}}(r y_{m-1}\eta_{m-1}) \dots$$

$$\dots \int\limits_0^{\sqrt{1-|\eta|^2+\eta_1^2}} \eta_1^{q_1} j_{\nu_1}(r y_1 \eta_1)(1-|\eta|^2)^{\frac{p-1}{2}} \cos rx\sqrt{1-|\eta|^2} d\eta.$$

Since $\left|\cos rx\sqrt{1-|\eta|^2}\right| \leq 1$, it follows that the inequality

$$|\widehat{\sigma_r}| = |g_r(x, y)| \leq \frac{C}{r^{\frac{|k|}{2}} \prod\limits_{l=1}^{m} y_l^{\frac{k_l}{2}}}$$

holds provided that $p > -1$ and $q_l > \dfrac{k_l}{2} - 1$, $l = \overline{1, m}$.

Therefore, for each positive r, we have the relation

$$\sigma^{p,q}(f)(r) \leq C r^{-\frac{|k|}{2}} \left\langle \mathcal{F}_{\mathrm{b}}\Big(\prod_{l=1}^{m} y_l^{\frac{k_l}{2}}\Big), |\hat{f}|^2 \right\rangle$$

$$= C r^{-\frac{|k|}{2}} \int\limits_{\mathbb{R}^m_{(+)}} \prod_{l=1}^{m} y_l^{\frac{k_l}{2}-1} |\hat{f}(0, y)|^2 dy \qquad (21)$$

(cf. [7, (1.3)]).

6.4 The Case of Absence of Nonspecial Variables

Let $n = 0$. Then

$$\tilde{f}(\eta) \stackrel{\text{def}}{=} \int\limits_{\mathbb{R}^m_{(+)}} y_1^{k_1} \dots y_m^{k_m} f(y_1, \dots, y_m) j_{\nu_1}(y_1\eta_1) \dots j_{\nu_m}(y_m \eta_m) dy.$$

Therefore,

$$g_r(x,y) \stackrel{\text{def}}{=} \widetilde{\sigma_r} = \int\limits_{S^+(1)} \prod_{l=1}^{m} \eta_l^{q_l} j_{\nu_l}(ry_l\eta_l)dS_\eta$$

$$= r^{-|\nu|}\prod_{l=1}^{m} y_l^{-\nu_l}\int\limits_0^1 \eta_m^{q_m-\nu_m} J_{\nu_m}(ry_m\eta_m)\int\limits_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}-\nu_{m-1}} J_{\nu_{m-1}}(ry_{m-1}\eta_{m-1})\dots$$

$$\dots\int\limits_0^{\sqrt{1-\eta_m^2-\dots-\eta_2^2}} \eta_1^{q_1-\nu_1} J_{\nu_1}(ry_1\eta_1)d\eta_1\dots d\eta_m.$$

Hence, if $q_l > \dfrac{k_l}{2} - 1,\ l = \overline{1,m}$, then

$$|\widetilde{\sigma_r}| \le Cr^{-|\nu|}\prod_{l=1}^{m} y_l^{-\nu_l-\frac{1}{2}} r^{-\frac{m}{2}} \int\limits_{S^+(1)} \prod_{l=1}^{m} \eta_l^{q_l-\frac{k_l}{2}} dS_\eta \stackrel{\text{def}}{=} Cr^{-\frac{|k|}{2}} \prod_{l=1}^{m} y_l^{-\frac{k_l}{2}}.$$

Therefore, for each nonnegative f from $L_{1,k}(\mathbb{R}^m_{(+)}) \cap L_{2,k}(\mathbb{R}^m_{(+)})$, the inequality

$$\sigma^{0,q}(f)(r) \le Cr^{-\frac{|k|}{2}} \int\limits_{\mathbb{R}^m_{(+)}} \prod_{l=1}^{m} y_l^{\frac{k_l}{2}-1} \tilde{f}^2(y)dy$$

holds on $(0,+\infty)$ and (similarly to Sect. 6.2) the inequality

$$\sup_{\mathbb{R}_+} r^{|\beta|}\sigma^{0,q+\beta}(f)(r) \le C\int\limits_{\mathbb{R}^m_{(+)}} \prod_{l=1}^{m} y_l^{\beta_l-1}\tilde{f}^2(y)dy \tag{22}$$

holds for each β_l from $\left(0, \dfrac{k_l}{2}\right)$, $l = \overline{1,m}$.

The right-hand side of the inequality (22) is equal to

$$C\int\limits_0^\infty \int\limits_{S^+(r)} \prod_{l=1}^{m} y_l^{\beta_l-1}\tilde{f}^2(y)dS_y dr$$

$$= C\int\limits_0^\infty \int\limits_{S^+(1)} r^{|\beta|-m}\prod_{l=1}^{m} \eta_l^{\beta_l-1}\tilde{f}^2(r\eta)r^{m-1}dS_\eta dr = C\|r^{|\beta|-1}\sigma^{0,\beta-1}(f)\|_1.$$

Thus the following statement is true:

Theorem 6.2 *Let $m \geq 2$, $n = 0$, and $q_l > -1$, $l = \overline{1, m}$. Then there exists C such that for any nonnegative f from $L_{1,k}(\mathbb{R}^m_{(+)}) \cap L_{2,k}(\mathbb{R}^m_{(+)})$, the inequality*

$$\|r^{|\beta|}\sigma^{0,q+\beta}(f)\|_\infty \leq C\|r^{|\beta|-1}\sigma^{0,\beta-1}(f)\|_1$$

holds provided that $\beta = \left(\frac{k_1}{2}, \dots, \frac{k_m}{2}\right)$ *or* $\beta \in \prod_{l=1}^{m}\left(0, \frac{k_l}{2}\right)$.

7 Applications to Singular Equations

7.1 Estimates of Solutions of Singular Ordinary Differential Equations

In this section, we apply the above one-dimensional results to estimate norms of solutions of the singular ordinary differential equation

$$P(-B)u = f(y), \tag{23}$$

where $Bu \stackrel{\text{def}}{=} B_k u \stackrel{\text{def}}{=} \frac{1}{y^k}\frac{d}{dy}\left(y^k\frac{du}{dy}\right)$ is the Bessel operator and P is a polynomial with real coefficients.

Let u from $L_{2,k}(0, +\infty)$ satisfy Eq. (23) in the sense of generalized functions. Then $\tilde{u}$ belongs to $L_{2,k}(0, +\infty)$ as well (see [2]) and

$$P(\eta^2)\tilde{u}(\eta) = \tilde{f}(\eta). \tag{24}$$

Since $P(\eta^2) \in L_{2,k,loc}(0, +\infty)$ and $\tilde{u}(\eta) \in L_{2,k,loc}(0, +\infty)$, it follows that $\tilde{f}(\eta) \in L_{1,k,loc}(0, +\infty)$, i. e., $\tilde{f}(\eta)$ is an ordinary function.

Thus, (24) bounds ordinary functions and, therefore the following division is legible:

$$\tilde{u}(\eta) = \frac{\tilde{f}(\eta)}{P(\eta^2)} \in L_{2,k}(0, +\infty).$$

Now, we denote $\frac{\tilde{f}(\eta)}{P(\eta^2)}$ by $g(\eta)$ and assume that g is nonnegative and belongs to $L_{1,k}(0, +\infty)$; also, we assume that $\operatorname{supp}\tilde{f} \subset [0, R]$. Then g satisfies the assumptions of Theorems 4.1–4.4 and $u = \tilde{g}$.

This implies the validity of the following assertion.

Theorem 7.1 *Let* $\alpha \in (0, k+1)$, $0 \le \dfrac{\tilde{f}(\eta)}{P(\eta^2)} \in L_{1,k}(0, +\infty)$, $\operatorname{supp} \tilde{f} \subset [0, R]$, *and* u *from* $L_{2,k}(0, +\infty)$ *satisfy Eq.* (23) *in the sense of generalized functions. Then, if* $\beta \min(\alpha, k)$, *then there exists* $C = C(\alpha, \beta, k)$ *such that the estimate*

$$\|r^{\frac{\beta}{2}} u\|_\infty \le C R^{\frac{\alpha-\beta}{2}} \|r^{\frac{\alpha-1}{2}} u\|_2 \tag{25}$$

holds. If the above is satisfied and $\alpha > k$, *then* C *does not depend on* β.

Remark 7.2 In the same way, Corollary 4.3 and Theorem 4.6 imply that Theorem 7.1 is not valid if $\beta > \min(\alpha, k)$.

Remark 7.3 If $\alpha = \beta \le \dfrac{k}{2}$, then the constant C depends only on k and the compactness condition for $\operatorname{supp} \tilde{f}$ is taken off (see Sect. 3).

7.2 Estimates of Solutions of Singular Partial Differential Equations

In this section we apply the above results to estimate norms of solutions of

$$P(-\Delta_{\mathcal{B}})u = f(x, y), \tag{26}$$

where $\Delta_{\mathcal{B}} u \overset{\text{def}}{=} \sum\limits_{j=1}^{n} \dfrac{\partial^2 u}{\partial x_j^2} + \sum\limits_{l=1}^{m} \dfrac{1}{y_l^{k_l}} \dfrac{\partial}{\partial y_l}\Big(y_l^{k_l} \dfrac{\partial u}{\partial y_l}\Big)$, $l = \overline{1, m}$, and P is a polynomial with real coefficients.

Consider the general case of several special variables.

Let u from $L_{2,k}(\mathbb{R}_+^{n+m})$ satisfy (26) in the sense of generalized functions. Then $\hat{u}$ belongs to $L_{2,k}(\mathbb{R}_+^{n+m})$ as well (see [2]) and

$$P(|\xi|^2 + |\eta|^2)\hat{u}(\xi, \eta) = \hat{f}(\xi, \eta). \tag{27}$$

Since $P(|\xi|^2 + |\eta|^2) \in L_{2,k,loc}(\mathbb{R}_+^{n+m})$ and $\hat{u}(\xi, \eta) \in L_{2,k,loc}(\mathbb{R}_+^{n+m})$, it follows that $\hat{f}(\xi, \eta) \in L_{1,k,loc}(\mathbb{R}_+^{n+m})$, i. e., $\hat{f}(\xi, \eta)$ is an ordinary function.

Thus, (27) bounds ordinary functions and, therefore, the following division is legible:

$$\hat{u}(\xi, \eta) = \frac{\hat{f}(\xi, \eta)}{P(|\xi|^2 + |\eta|^2)} \in L_{2,k}(\mathbb{R}_+^{n+m}).$$

Now, we denote $\dfrac{\hat{f}(\xi,\eta)}{P(|\xi|^2+|\eta|^2)}$ by $g(\xi,\eta)$ and assume that g is nonnegative and belongs to $L_{1,k}(\mathbb{R}_+^{n+m})$. Then g satisfies the conditions of Theorem 6.1 and $u=\hat{g}$.

This yields the following assertion.

Theorem 7.4 *Let* $\dfrac{\hat{f}(\xi,\eta)}{P(|\xi|^2+|\eta|^2)}$ *be a nonnegative function from* $L_{1,k}(\mathbb{R}_+^{n+m})$ *and* u *from* $L_{2,k}(\mathbb{R}_+^{n+m})$ *satisfy Eq.* (26) *at least in the sense of generalized functions. Let* $p>-n$ *and* $q_l>-1$, $l=\overline{1,m}$. *Then there exists* C *such that*

$$\|r^{\alpha+|\beta|-p-|q|}\sigma^{\alpha,\beta}u\|_\infty \le C\|r^{\alpha+|\beta|-p-|q|-1}\sigma^{\alpha-n-p,\beta-q-1}u\|_1$$

for $(\alpha,\beta)=(\alpha,\beta_1,\dots,\beta_m)=\left(p+\dfrac{n-1}{2},q_1+\dfrac{k_1}{2},\dots,q_m+\dfrac{k_m}{2}\right)$ *and for each* (α,β) *such that* $\alpha\in\left(p,p+\dfrac{n-1}{2}\right)$ *and* $\beta_l\in\left(q_l,q_l+\dfrac{k_l}{2}\right)$, $l=\overline{1,m}$.

In the same way, (21) yields the following assertion.

Theorem 7.5 *Let* $\dfrac{\hat{f}(\xi,\eta)}{P(\xi^2+|\eta|^2)}$ *be a nonnegative function from* $L_{1,k}(\mathbb{R}_+^{1+m})$ *and* u *from* $L_{2,k}(\mathbb{R}_+^{1+m})$ *satisfy Eq.* (26) *at least in the sense of generalized functions. Let* $p>-1$ *and* $q_l>\dfrac{k_l}{2}-1$, $l=\overline{1,m}$. *Then there exists* C *such that*

$$\|r^{\frac{|k|}{2}}\sigma^{p,q}u\|_\infty \le C\|\prod_{l=1}^m y_l^{\frac{k_l}{2}-1}u^2(0,y)\|_1.$$

Finally, Theorem 6.2 yields the following assertion.

Theorem 7.6 *Let* $\dfrac{\tilde{f}(\eta)}{P(|\eta|^2)}\in L_{1,k}(\mathbb{R}_{(+)}^m)$, $\dfrac{\tilde{f}(\eta)}{P(|\eta|^2)}\ge 0$, *and a function* u *from* $L_{2,k}(\mathbb{R}_{(+)}^m)$ *satisfy Eq.* (26) *at least in the sense of generalized functions. Let* $q_l>-1$, $l=\overline{1,m}$. *Then*

$$\|r^{|\beta|-|q|}\sigma^{0,\beta}u\|_\infty \le C\|r^{|\beta|-|q|-1}\sigma^{0,\beta-q-1}u\|_1$$

for $\beta=(\beta_1,\dots,\beta_m)=\left(q_1+\dfrac{k_1}{2},\dots,q_m+\dfrac{k_m}{2}\right)$ *and for each* β *such that* $\beta_l\in\left(q_l,q_l+\dfrac{k_l}{2}\right)$, $l=\overline{1,m}$.

Remark 7.7 Under the assumptions of Theorems 7.4–7.6, the right-hand parts of the corresponding inequalities converge and the constant C depends only on m, n, k, p, and q.

Remark 7.8 In general, to ensure the right-hand part of an inequality of kind (2) to be well defined, we need a greater smoothness than the one assumed by the above theorems. However, since we assume the nonnegativity of the corresponding integral transform of the solution, we can well define the said value as the norm of the said integral transform in the space $L_{1,k}(\mathbb{R}^1_+)$. Indeed, such a definition is entirely coordinated with the case of smooth functions: if u is smooth, then $u(0) = \int\limits_0^{+\infty} \eta^k j_\nu(\eta y)\hat{u}(\eta)\, d\eta\Big|_{y=0} = \int\limits_0^{+\infty} \eta^k \hat{u}(\eta)\, d\eta$, which is equal to the said norm because $\hat{u}$ is assumed to be nonnegative.

All the above results with low-dimensional traces (e. g., estimate (16)) are treated in the same sense.

Remark 7.9 In [11], results of this section are extended to the case of pseudodifferential equations.

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Inversion of Hyperbolic B-Potentials

E. L. Shishkina

Abstract The paper is devoted to the study of the fractional integral operator which is a negative real power of the singular wave operator generated by Bessel operator and its inverse using weighted generalized functions. Such operators are called **hyperbolic B-potentials**. Boundedness, Green formula, inversion were proved for hyperbolic B-potentials here.

Keywords Hyperbolic Riesz B-potential · Fractional power of singular hyperbolic operator · Lorentz distance · Singular Bessel differential operator · Generalized translation · Multidimensional Hankel transform · Green formula

1 Introduction

1.1 Transmutation Operators

Method of transmutation operators is important and powerful approach to study problems connected with singular operators such, for example, the Bessel operator. Non-zero operator T is called transmutation operator for two operators A and B if $TA = BT$. In the framework of this method special classes of transmutation operators such as Sonine, Poisson, Buschman–Erdélyi and others are used (see [1–14]).

In order to construct transmutation operators *integral transform composition method* is used (see [5, 6, 15, 16]) The method is based on the representation of transmutation operators as compositions of basic integral transforms. The formal algorithm of integral transform composition method is the next. Let us take as input a pair of arbitrary operators A, B, and also connecting with them integral transforms F_A, F_B, which are invertible and act by the formulas

$$F_A A = g(t) F_A, \quad F_B B = g(t) F_B, \tag{1}$$

E. L. Shishkina (✉)
University of Information Technology and Management in Rzeszow, Rzeszow, Poland
e-mail: shishkina@amm.vsu.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_12

where t is a dual variable, g is an arbitrary function with suitable properties. Then we can obtain formally a pair of transmutation operators P and S by the next formulas:

$$S = F_B^{-1}\frac{1}{w(t)}F_A, \qquad P = F_A^{-1}w(t)F_B \tag{2}$$

with arbitrary function $w(t)$. When P and S are transmutation operators intertwining A and B:

$$SA = BS, \qquad PB = AP. \tag{3}$$

A formal checking of (3) can be obtained by direct substitution. The main difficulty is the calculation of compositions (2) in an explicit integral form, as well as the choice of domains of operators P and S. Here we use integral transform composition method for constructing hyperbolic B-potential.

1.2 A Brief History of the Potentials Operators

In recent years, the interest to the Fractional Calculus has been increasing due to its applications in many fields. As for multidimensional case the most developed type of fractional integrals are Riesz potentials which are generalized both Newton potential to the fractional case and Riemann-Liouville fractional integral to the multidimensional case.

Let us start from the classical mechanic Newton potential. If f is integrable function with compact support then the Newton potential of f is the convolution product (see [17])

$$V_N f(x) = \int\limits_{\mathbb{R}^n} v(x-y)f(y)dy,$$

where

$$v(x) = \begin{cases} \frac{1}{2\pi}\log|x|, & n=2; \\ \frac{1}{n(2-n)\omega_n}|x|^{2-n}, & n\neq 2, \end{cases} \qquad \omega_n \quad \text{is a volume of unit ball } \mathbb{R}^n.$$

Newton potential V_N of f is the solution to the Poisson equation

$$\Delta V_N = f, \qquad \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}, \qquad x=(x_1,\ldots,x_n)\in\mathbb{R}^n,$$

therefore, it can be considered as a negative degree of the Laplace operator:

$$V_N f = \Delta^{-1} f.$$

Along with the Newtonian potential, the wave potential of the function f has found wide applications (see [17])

$$V_W f(x) = \int\limits_{\mathbb{R}^n} \varepsilon(x-y) f(y) dy,$$

where ε is a fundamental solution of the wave operator. For the wave potential V_W the next equality is true

$$\Box V_W = f,$$

therefore, it can be considered as a negative degree of the D'Alembert operator: $V_W f = \Box^{-1} f$.

Marsel Riesz was a Hungarian mathematician who first established the fractional powers of the Laplace and D'Alembert operators (see [18] and [19]). Such potentials are called the **Riesz potentials** now and have the forms

$$I_\Delta^\alpha f(P) = \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} f(Q) r^{\alpha-n} dQ$$

and

$$I_\Box^\alpha f(P) = \frac{1}{H_n(\alpha)} \int\limits_{D} f(Q) r_{PQ}^{\alpha-n} dQ,$$

where $P = (x_1, \ldots, x_n)$, $Q = (\xi_1, \ldots, \xi_n)$, $\gamma_n(\alpha)$, $H_n(\alpha)$ is normalizing constant,

$$r = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + \ldots + (x_n-\xi_n)^2}$$

is the Euclidean distance,

$$r_{PQ} = \sqrt{(x_1-\xi_1)^2 - (x_2-\xi_2)^2 - \ldots - (x_n-\xi_n)^2}$$

is the Lorentz distance, $D = \{x : x_1^2 \geq x_2^2 + \ldots + x_n^2\}$ is the positive cone.

In [19] was shown that

$$\Delta I_\Delta^{\alpha+2} f(P) = -I_\Delta^\alpha f(P)$$

and

$$\Box I_{\Box}^{\alpha+2} f(P) = I_{\Box}^{\alpha} f(P).$$

For further properties such as conditions of existence, semigroup property and inversion see [19–23]. The theory of hyperbolic potentials introduced in [22] was developed in the articles [24, 25].

More attention was paid to Riesz potentials with Euclidean distance (see [26–30]). In [31] and [32] kernels of fractional powers which are the set of all positive powers of the operator generated by the Green function for the Laplace equation were studied. In [33–39] optimal embedding of spaces of Bessel and Riesz types potentials are obtained.

As for classical Riesz potentials with Lorentz distance we refer to [23, 40].

The theory of fractional powers of elliptic operators with Bessel operator

$$B_\nu = D^2 + \frac{\nu}{x} D, \qquad D = \frac{d}{dx}$$

acting instead of all or some second derivatives in Δ is well developed (see [41–58]).

Fractional powers of hyperbolic operators, with Bessel operators instead of all or some second derivatives, are much less studied. Such operators have wide areas of application such as singular differential equations, differential geometry and random walks.

In this article we study real powers of

$$\Box_\gamma = B_{\gamma_1} - B_{\gamma_2} - \ldots - B_{\gamma_n}, \qquad B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \qquad i = 1, \ldots, n.$$

Composition method (see [5, 8, 11, 16]) was used for construction of $(\Box_\gamma)^{-\frac{\alpha}{2}}, \alpha > 0$.

1.3 Basic Definitions

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space,

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad x_1 > 0, \ldots, x_n > 0\},$$

$\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multi-index consisting of positive fixed real numbers γ_i, $i = 1, \ldots, n$, and $|\gamma| = \gamma_1 + \ldots + \gamma_n$.

Let Ω be finite or infinite open set in $\mathbb{R}^n$ symmetric with respect to each hyperplane $x_i = 0$, $i = 1, \ldots, n$, $\Omega_+ = \Omega \cap \mathbb{R}^n_+$ and $\overline{\Omega}_+ = \Omega \cap \overline{\mathbb{R}}^n_+$ where $\overline{\mathbb{R}}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_1 \geq 0, \ldots, x_n \geq 0\}$. We deal with the class $C^m(\Omega_+)$ consisting of m times differentiable on Ω_+ functions and denote by $C^m(\overline{\Omega}_+)$ the subset of functions from $C^m(\Omega_+)$ such that all derivatives of these functions with respect to x_i for any $i = 1, \dots, n$ are continuous up to x_i=0. Class $C^m_{ev}(\overline{\Omega}_+)$ consists of all functions from $C^m(\overline{\Omega}_+)$ such that $\left.\frac{\partial^{2k+1} f}{\partial x_i^{2k+1}}\right|_{x=0} = 0$ for all non-negative integer $k \le \frac{m-1}{2}$ (see [59] and [60], p. 21). In the following we will denote $C^m_{ev}(\overline{\mathbb{R}}^n_+)$ by C^m_{ev}. We set

$$C^\infty_{ev}(\overline{\Omega}_+) = \bigcap C^m_{ev}(\overline{\Omega}_+)$$

with intersection taken for all finite m and $C^\infty_{ev}(\overline{\mathbb{R}}_+) = C^\infty_{ev}$.

As the space of basic functions we will use the subspace of the space of rapidly decreasing functions:

$$S_{ev} = \left\{ f \in C^\infty_{ev} : \sup_{x \in \mathbb{R}^n_+} \left| x^\alpha D^\beta f(x) \right| < \infty \quad \forall \alpha, \beta \in \mathbb{Z}^n_+ \right\},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ are integer non-negative numbers, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, $D^\beta = D^{\beta_1}_{x_1} \dots D^{\beta_n}_{x_n}$, $D_{x_j} = \frac{\partial}{\partial x_j}$.

Let $L^\gamma_p(\mathbb{R}^n_+) = L^\gamma_p$, $1 \le p < \infty$, be the space of all measurable in $\mathbb{R}^n_+$ functions even with respect to each variable x_i, $i = 1, \dots, n$ such that

$$\int\limits_{\mathbb{R}^n_+} |f(x)|^p x^\gamma dx < \infty,$$

where and further

$$x^\gamma = \prod_{i=1}^n x_i^{\gamma_i}.$$

For a real number $p \ge 1$, the L^γ_p-norm of f is defined by

$$||f||_{L^\gamma_p(\mathbb{R}^n_+)} = ||f||_{p,\gamma} = \left(\int\limits_{\mathbb{R}^n_+} |f(x)|^p x^\gamma dx \right)^{1/p}.$$

Weighted measure of Ω_+ is denoted by $\text{mes}_\gamma(\Omega_+)$ and is defined by formula

$$\text{mes}_\gamma(\Omega_+) = \int\limits_{\Omega_+} x^\gamma dx.$$

For every measurable function $f(x)$ defined on $\mathbb{R}^n_+$ we consider

$$\mu_\gamma(f,t)=\text{mes}_\gamma\{x\in\mathbb{R}^n_+:\,|f(x)|>t\}=\int\limits_{\{x:\,|f(x)|>t\}^+} x^\gamma dx$$

where $\{x{:}|f(x)|{>}t\}^+{=}\{x{\in}\mathbb{R}^n_+{:}|f(x)|{>}t\}$. We will call the function $\mu_\gamma=\mu_\gamma(f,t)$ a *weighted distribution function* $|f(x)|$.

Space $L^\gamma_\infty(\mathbb{R}^n_+){=}L^\gamma_\infty$ is the space of all measurable in $\mathbb{R}^n_+$ functions even with respect to each variable x_i, $i=1,\dots,n$ for which the norm

$$||f||_{L^\gamma_\infty(\mathbb{R}^n_+)}=||f||_{\infty,\gamma}=\underset{x\in\mathbb{R}^n_+}{\text{ess}\,\text{sup}_\gamma}|f(x)|=\inf_{a\in\mathbb{R}}\{\mu_\gamma(f,a)=0\}$$

is finite.

Statement 1 *Norms of spaces L^γ_p and L^γ_∞ related by equality*

$$||f||_{\infty,\gamma}=\lim_{p\to\infty}||f||_{p,\gamma},\qquad f\in L^\gamma_\infty. \tag{4}$$

For $1\leq p\leq\infty$ the $L^\gamma_{p,loc}(\mathbb{R}^n_+)=L^\gamma_{p,loc}$ is the set of functions $u(x)$ defined almost everywhere in $\mathbb{R}^n_+$ such that $uf\in L^\gamma_p$ for any $f\in S_{ev}$.

Definition 1 The **space of weighted distributions** $S'_{ev}(\mathbb{R}^n_+)=S'_{ev}$ is a class of continuous linear functionals that map a set of test functions $f\in S_{ev}$ into the set of real numbers. Each function $u(x)\in L^\gamma_{1,loc}$ will be identified with the functional $u\in S'_{ev}(\mathbb{R}^n_+)=S'_{ev}$ acting according to the formula

$$(u,f)_\gamma=\int\limits_{\mathbb{R}^n_+}u(x)\,f(x)\,x^\gamma\,dx,\qquad f\in S_{ev}. \tag{5}$$

Functionals $u\in S'_{ev}$ acting by the formula (5) will be called **regular weighted functionals**. All other continuous linear functionals $u\in S'_{ev}$ will be called **singular weighted functionals**.

We consider regular generalized functions

$$(\mathcal{P}^\lambda_\gamma,\varphi)_\gamma=\int\limits_{\mathbb{R}^n_+}\mathcal{P}^\lambda(x)\varphi(x)x^\gamma dx,\qquad x^\gamma=\prod_{i=1}^n x_i^{\gamma_i}, \tag{6}$$

where $\mathcal{P}(x)=\alpha_1x_1^2+\ldots+\alpha_nx_n^2$ is quadratic form with complex coefficients, φ is appropriate basic function. Let $P{=}x_1^2-x_2^2-\ldots-x_n^2$, and $P'=\varepsilon(x_1^2+\ldots+$

$x_n^2)$, $\varepsilon > 0$. Weighted generalized functions $(P \pm i0)_\gamma^\lambda$ are defined by

$$(P \pm i0)_\gamma^\lambda = \lim_{\varepsilon \to 0} (P \pm i P')_\gamma^\lambda$$

in which we passing to the limit under the integral sign in (6).

Generalized function δ_γ is defined by the equality (by analogy with [61])

$$(\delta_\gamma, \varphi)_\gamma = \varphi(0), \quad \varphi(x) \in S_{ev}.$$

Definition 2 The **multidimensional generalized translation** is defined by the equality

$$({}^\gamma \mathbf{T}_x^y f)(x) = {}^\gamma \mathbf{T}_x^y f(x) = ({}^{\gamma_1} T_{x_1}^{y_1} \ldots {}^{\gamma_n} T_{x_n}^{y_n} f)(x), \tag{7}$$

where each of one-dimensional generalized translation ${}^{\gamma_i} T_{x_i}^{y_i}$ acts for $i=1, \ldots, n$ according to

$$({}^{\gamma_i} T_{x_i}^{y_i} f)(x) = \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma_i}{2}\right)} \times$$

$$\times \int_0^\pi f(x_1, \ldots, x_{i-1}, \sqrt{x_i^2 + \tau_i^2 - 2x_i y_i \cos\varphi_i}, x_{i+1}, \ldots, x_n)\ \sin^{\gamma_i - 1} \varphi_i \, d\varphi_i.$$

We will use the generalized convolution product defined by the formula

$$(f * g)_\gamma(x) = \int_{\mathbb{R}_+^n} f(y)({}^\gamma \mathbf{T}_x^y g)(x) y^\gamma dy, \qquad f, g \in S_{ev}$$

where ${}^\gamma \mathbf{T}_x^y$ is multidimensional generalized translation (7).

The generalized convolution $(u * f)_\gamma$ of a weighted distribution $u \in S'_{ev}$ and a function $f \in S_{ev}$ is defined by

$$(u * f)_\gamma(x) = (u, {}^\gamma \mathbf{T}_x^{\cdot} f)_\gamma$$

where the right-hand side denotes u acting on ${}^\gamma \mathbf{T}_x^y f$ as a function of y.

Based on the multidimensional generalized translation ${}^\gamma \mathbf{T}_x^y$ the weighted spherical mean $M_t^\gamma[f(x)]$ of a suitable function is defined by the formula (see [62–64])

$$M_t^\gamma[f(x)] = \frac{1}{|S_1^+(n)|_\gamma} \int_{S_1^+(n)} {}^\gamma \mathbf{T}_x^{t\theta} f(x) \theta^\gamma dS, \tag{8}$$

where $\theta^\gamma = \prod\limits_{i=1}^{n} \theta_i^{\gamma_i}$, $S_1^+(n) = \{\theta : |\theta| = 1, \theta \in \mathbb{R}_+^n\}$ and

$$|S_1^+(n)|_\gamma = \frac{\prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}. \tag{9}$$

It is easy to see that

$$M_0^\gamma[f(x)] = f(x), \qquad \frac{\partial}{\partial t} M_t^\gamma[f(x)]\bigg|_{t=0} = 0. \tag{10}$$

We will deal with the **singular Bessel differential operator** B_ν (see, for example, [60], p. 5):

$$(B_\nu)_t = \frac{\partial^2}{\partial t^2} + \frac{\nu}{t}\frac{\partial}{\partial t} = \frac{1}{t^\nu}\frac{\partial}{\partial t} t^\nu \frac{\partial}{\partial t}, \qquad t > 0$$

and the elliptical singular operator or the Laplace-Bessel operator $\triangle_\gamma$:

$$\triangle_\gamma = (\triangle_\gamma)_x = \sum_{i=1}^{n} (B_{\gamma_i})_{x_i} = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x}\right) = \sum_{i=1}^{n} \frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i} x_i^{\gamma_i}\frac{\partial}{\partial x_i}. \tag{11}$$

The operator (11) belongs to the class of B-elliptic operators by I. A. Kipriyanovs' classification (see [60]).

Statement 2 ([14]) *The weighted spherical mean* $M_t^\gamma[f(x)]$ *is the transmutation operator intertwining* $(\Delta_\gamma)_x$ *and* $(B_{n+|\gamma|-1})_t$ *for the* $f \in C_{ev}^2$*:*

$$(B_{n+|\gamma|-1})_t M_t^\gamma[f(x)] = M_t^\gamma[(\Delta_\gamma)_x f(x)]. \tag{12}$$

The natural method for the investigation of operators associated with the Bessel differential operator is using the multidimensional Hankel transform instead of Fourier transform.

Definition 3 The **Hankel transform** of a function $f \in L_1^\gamma(\mathbb{R}_+^n)$ is expressed as

$$\mathbf{F}_\gamma[f](\xi) = \mathbf{F}_\gamma[f(x)](\xi) = \widehat{f}(\xi) = \int\limits_{\mathbb{R}_+^n} f(x)\,\mathbf{j}_\gamma(x;\xi) x^\gamma dx,$$

where

$$\mathbf{j}_\gamma(x;\xi)=\prod_{i=1}^{n} j_{\frac{\gamma_i-1}{2}}(x_i\xi_i),\qquad \gamma_1>0,\dots,\gamma_n>0,$$

the symbol j_ν is used for the normalized Bessel function:

$$j_\nu(r)=\frac{2^\nu\Gamma(\nu+1)}{r^\nu}J_\nu(r) \tag{13}$$

and $J_\nu(r)$ is the Bessel function of the first kind of order ν (see [65]).

For $f\in S_{ev}$ inverse Hankel transform is defined by

$$\mathbf{F}_\gamma^{-1}[\widehat{f}(\xi)](x)=f(x)=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}_+^n}\mathbf{j}_\gamma(x,\xi)\widehat{f}(\xi)\xi^\gamma\,d\xi.$$

If $g\in S'_{ev}$ then equality

$$(\mathbf{F}_\gamma g,\varphi)_\gamma=(g,\mathbf{F}_\gamma\varphi)_\gamma,\qquad \varphi\in S_{ev} \tag{14}$$

defines Hankel transform of functional $g\in S'_{ev}$.

In [66] the space Ψ_V consisting of functions vanished on a given closed set V of measure zero was considered. The Lizorkin–Samko space Φ_V is dual to Φ_V in the sense of Fourier transforms. We introduce the space Ψ_V^γ of functions S_{ev} vanished with all their derivatives on a given closed set V:

$$\Psi_V^\gamma=\{\psi\in S_{ev}(\mathbb{R}_+^n):(D^k\psi)(x)=0,x\in V,|k|=0,1,2,\dots\}.$$

Space Ψ_V^γ is dual to Φ_V^γ in the sense of Hankel transforms:

$$\Phi_V^\gamma=\{\varphi:\mathbf{F}_\gamma\varphi\in\Psi_V^\gamma\}. \tag{15}$$

Statement 3 ([14]) *Integral* $\int\limits_{S_1^+(n)}\mathbf{j}_\gamma(r\theta,\xi)\theta^\gamma\,dS$ *is calculated by the formula*

$$\int\limits_{S_1^+(n)}\mathbf{j}_\gamma(r\theta,\xi)\theta^\gamma\,dS=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}j_{\frac{n+|\gamma|}{2}-1}(r|\xi|). \tag{16}$$

A linear operator A is of *strong type* $(p,q)_\gamma$, $1\le p\le\infty$, $1\le q\le\infty$ if it is defined from L_p^γ to L_q^γ and the following inequality is valid:

$$||Af||_{q,\gamma}\le h||f||_{p,\gamma},\quad \forall f\in L_p^\gamma, \tag{17}$$

where constant h does not depend on f.

We say that a linear operator A is an operator of *weak type* $(p,q)_\gamma$ $(1\le p\le\infty, 1\le q<\infty)$ if

$$\mu_\gamma(Af,\lambda)\le\left(\frac{h||f||_{p,\gamma}}{\lambda}\right)^q,\quad \forall f\in L_p^\gamma, \tag{18}$$

where h does not depend on f and λ, $\lambda>0$.

If $q=\infty$ then a linear operator A is an operator of weak type $(p,q)_\gamma$ when is has strong type $(p,q)_\gamma$.

Let $p,q,r\in[1,\infty]$ and

$$\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}. \tag{19}$$

If $f\in L_p^\gamma$, $g\in L_q^\gamma$, $1\le p,q,r\le\infty$, $\frac{1}{q}=\frac{1}{p}+\frac{1}{r}-1$ then a generalized convolution$(f*g)_\gamma$ is bounded almost everywhere and Hausdorff-Young inequality is valid

$$||(f*g)_\gamma||_{r,\gamma}\le||f||_{p,\gamma}||g||_{q,\gamma}. \tag{20}$$

Inequality

$$||(f*g)_\gamma||_{\infty,\gamma}\le||f||_{p,\gamma}|||g||_{q,\gamma} \tag{21}$$

is obtained from (20) by tending to the limit with $r\to\infty$ using (4) (p and q should be such that $1/p+1/q=1$).

We present here the Marcinkiewicz interpolation theorem in the following form (see [67]).

Theorem 1 *Let* $1\le p_i\le q_i<\infty$, *(*$i=1,2$*)*, $q_1\ne q_2$, $0<\tau<1$, $\frac{1}{p}=\frac{1-\tau}{p_1}+\frac{\tau}{p_2}$, $\frac{1}{q}=\frac{1-\tau}{q_1}+\frac{\tau}{q_2}$. *If a linear operator* A *has simultaneously weak types* $(p_1,q_1)_\gamma$ *and* $(p_2,q_2)_\gamma$ *then an operator* A *has a strong type* $(p,q)_\gamma$ *and*

$$||Af||_{q,\gamma}\le M||f||_{p,\gamma}, \tag{22}$$

where a constant $M=M(\gamma,\tau,\kappa,p_1,p_2,q_1,q_2)$ *and does not depend on* f *and* A.

Appell hypergeometric function $F_4(a, b, c_1, c_2; x, y)$ (see [68], p. 658) for $|x|^{1/2} + |y|^{1/2} < 1$ has the form

$$F_4(a, b, c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m (c_2)_n \, m! \, n!} \, x^m y^n. \tag{23}$$

For $|x|^{1/2} + |y|^{1/2} \geq 1$ function $F_4(a, b; c_1, c_2; x, y)$ is understood as an analytical continuation, which is determined by the formulas from [69].

2 Hyperbolic B-Potentials and Their Properties

2.1 Definitions of the Hyperbolic B-Potentials

We consider fractional powers of the hyperbolic expression with Bessel operators

$$\Box_\gamma = B_{\gamma_1} - B_{\gamma_2} - \ldots - B_{\gamma_n}, \qquad B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \qquad i = 1, \ldots, n$$

in S_{ev} and L_p^γ. Negative real powers of $\Box_\gamma$ we will call **hyperbolic B-potentials**.

Definition 4 Hyperbolic B-potentials $I^\alpha_{P\pm i0,\gamma}$ for $\alpha > n + |\gamma| - 2$ are defined by formulas

$$(I^\alpha_{P\pm i0,\gamma} f)(x) = \frac{e^{\pm\frac{n-1+|\gamma'|}{2} i\pi}}{H_{n,\gamma}(\alpha)} \int\limits_{\mathbb{R}^n_+} (P \pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}} ({}^\gamma \mathbf{T}^y_x f)(x) y^\gamma dy, \qquad y^\gamma = \prod_{i=1}^n y_i^{\gamma_i}, \tag{24}$$

where $\gamma' = (\gamma_2, \ldots, \gamma_n)$, $|\gamma'| = \gamma_2 + \ldots + \gamma_n$,

$$H_{n,\gamma}(\alpha) = \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)}{2^{n-\alpha} \Gamma\left(\frac{n+|\gamma|-\alpha}{2}\right)}.$$

For $0 \leq \alpha \leq n + |\gamma| - 2$ hyperbolic B-potentials $I^\alpha_{P\pm i0,\gamma}$ are defined as

$$\begin{aligned}(I^\alpha_{P\pm i0,\gamma} f)(x) &= (\Box_\gamma)^k (I^{\alpha+2k}_{P\pm i0,\gamma} f)(x) \\ &= \frac{e^{\pm\frac{n-1+|\gamma'|}{2} i\pi}}{H_{n,\gamma}(\alpha+2k)} (\Box_\gamma)^k \int\limits_{\mathbb{R}^n_+} (P \pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} ({}^\gamma \mathbf{T}^y_x f)(x) y^\gamma dy,\end{aligned} \tag{25}$$

where $k=\left[\frac{n+|\gamma|-\alpha}{2}\right]$.

It is well known (see for example [60]) that generalized convolution of weighted generalized functions and regular function is a regular function.

Using property of weighted generalized functions $(P\pm i0)^{\lambda}_{\gamma}$ see [70] we can rewrite formulas (24) as

$$(I^{\alpha}_{P\pm i0,\gamma}f)(x)=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha)}\left[\int\limits_{K^+}r^{\alpha-n-|\gamma|}(y)({}^{\gamma}\mathbf{T}^y_x f)(x)y^{\gamma}dy+\right.$$
$$\left.+e^{\pm\frac{\alpha-n-|\gamma|}{2}\pi i}\int\limits_{K^-}|r(y)|^{\alpha-n-|\gamma|}({}^{\gamma}\mathbf{T}^y_x f)(x)y^{\gamma}dy\right],\tag{26}$$

where

$$K^+=\{x:x\in\mathbb{R}^n_+:P(x)\geq 0\},\qquad K^-=\{x:x\in\mathbb{R}^n_+:P(x)\leq 0\},$$
$$r(y)=\sqrt{P(y)}=\sqrt{y_1^2-y_2^2-\ldots-y_n^2}.$$

Function $r(y)$ is a Lorentz distance and K^+ is a part of light cone.

Introducing the notations

$$(I^{\alpha}_{P_+,\gamma}f)(x)=\int\limits_{K^+}r^{\alpha-n-|\gamma|}(y)({}^{\gamma}\mathbf{T}^y_x f)(x)y^{\gamma}dy,\tag{27}$$

$$(I^{\alpha}_{P_-,\gamma}f)(x)=\int\limits_{K^-}|r(y)|^{\alpha-n-|\gamma|}({}^{\gamma}\mathbf{T}^y_x f)(x)y^{\gamma}dy,\tag{28}$$

we can write formulas (24) as

$$(I^{\alpha}_{P\pm i0,\gamma}f)(x)=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha)}\left[(I^{\alpha}_{P_+,\gamma}f)(x)+e^{\pm\frac{\alpha-n-|\gamma|}{2}\pi i}(I^{\alpha}_{P_-,\gamma}f)(x)\right].\tag{29}$$

Remark 1 Let $y'=(y_2,\ldots,y_n)$, $|y'|=\sqrt{y_2^2+\ldots+y_n^2}$, $(y')^{\gamma'}=y_2^{\gamma_2}\ldots y_n^{\gamma_n}$, $\alpha>n+|\gamma|-2$.

For $n \geq 3$ we have

$$(I^{\alpha}_{P_+,\gamma} f)(x) = \int\limits_0^{\infty} y_1^{\gamma_1} dy_1 \int\limits_{\{|y'|<y_1\}^+} (y_1^2 - |y'|^2)^{\frac{\alpha-n-|\gamma|}{2}} ({}^{\gamma}\mathbf{T}^y_x f)(x)(y')^{\gamma'} dy', \tag{30}$$

$$(I^{\alpha}_{P_-,\gamma} f)(x) = \int\limits_0^{\infty} y_1^{\gamma_1} dy_1 \int\limits_{\{|y'|>y_1\}^+} (|y'|^2 - y_1^2)^{\frac{\alpha-n-|\gamma|}{2}} ({}^{\gamma}\mathbf{T}^y_x f)(x)(y')^{\gamma'} dy', \tag{31}$$

where $\{|y'| < y_1\}^+ = \{y \in \mathbb{R}^n_+ : |y'| < y_1\}$, $\{|y'| > y_1\}^+ = \{y \in \mathbb{R}^n_+ : |y'| > y_1\}$.
For $n = 2$ we have

$$(I^{\alpha}_{P_+,\gamma} f)(x) = \int\limits_0^{\infty} y_1^{\gamma_1} dy_1 \int\limits_0^{y_1} (y_1^2 - y_2^2)^{\frac{\alpha-2-|\gamma|}{2}} ({}^{\gamma}\mathbf{T}^y_x f)(x) y_2^{\gamma_2} dy_2,$$

$$(I^{\alpha}_{P_-,\gamma} f)(x) = \int\limits_0^{\infty} y_1^{\gamma_1} dy_1 \int\limits_{y_1}^{\infty} (y_2^2 - y_1^2)^{\frac{\alpha-2-|\gamma|}{2}} ({}^{\gamma}\mathbf{T}^y_x f)(x) y_2^{\gamma_2} dy_2.$$

Passing to the spherical coordinates $y' = \rho\sigma$ in (30) and in (31) we obtain

$$(I^{\alpha}_{P_+,\gamma} f)(x) = |S_1^+(n-1)|_{\gamma} \times$$

$$\times \int\limits_0^{\infty} y_1^{\gamma_1} dy_1 \int\limits_0^{y_1} (y_1^2 - \rho^2)^{\frac{\alpha-n-|\gamma|}{2}} \rho^{n+|\gamma'|-2} ({}^{\gamma_1}T^{y_1}_{x_1})(M^{\gamma'}_{\rho})_{x'}[f(x_1, x')] d\rho, \tag{32}$$

$$(I^{\alpha}_{P_-,\gamma} f)(x) = |S_1^+(n-1)|_{\gamma} \times$$

$$\int\limits_0^{\infty} y_1^{\gamma_1} dy_1 \int\limits_{y_1}^{\infty} (\rho^2 - y_1^2)^{\frac{\alpha-n-|\gamma|}{2}} \rho^{n+|\gamma'|-2} ({}^{\gamma_1}T^{y_1}_{x_1})(M^{\gamma'}_{\rho})_{x'}[f(x_1, x')] d\rho, \tag{33}$$

where

$$(M^{\gamma'}_{\rho})_{x'}[f(x_1, x')] = \frac{1}{|S_1^+(n-1)|_{\gamma}} \int\limits_{S_1^+(n-1)} {}^{\gamma'}\mathbf{T}^{\rho\sigma}_{x'} f(x_1, x') \sigma^{\gamma'} dS$$

is weighted spherical mean (8).

If $f(x)=\varphi(x_1)G(x')$ then (32) and (33) have forms

$$(I^{\alpha}_{P_+,\gamma}f)(x)=|S^+_1(n-1)|_{\gamma}\times$$

$$\int\limits_0^{\infty}({}^{\gamma_1}T^{y_1}_{x_1})[\varphi(x_1)]y_1^{\gamma_1}dy_1\int\limits_0^{y_1}(M^{\gamma'}_{\rho})_{x'}[G(x')](y_1^2-\rho^2)^{\frac{\alpha-n-|\gamma|}{2}}\rho^{n+|\gamma'|-2}d\rho, \tag{34}$$

$$(I^{\alpha}_{P_-,\gamma}f)(x)=|S^+_1(n-1)|_{\gamma}\times$$

$$\times\int\limits_0^{\infty}({}^{\gamma_1}T^{y_1}_{x_1})[\varphi(x_1)]y_1^{\gamma_1}dy_1\int\limits_{y_1}^{\infty}(M^{\gamma'}_{\rho})_{x'}[G(x')](\rho^2-y_1^2)^{\frac{\alpha-n-|\gamma|}{2}}\rho^{n+|\gamma'|-2}d\rho. \tag{35}$$

2.2 Absolute Convergence and Boundedness

Theorem 2 *Let $f\in S_{ev}$ and $\alpha>n+|\gamma|-2$. Then integrals $(I^{\alpha}_{P\pm i0,\gamma}f)(x)$ converge absolutely for $x\in\mathbb{R}^n_+$.*

Proof Let prove absolute convergence of each term in (26). Passing in (26) to spherical coordinates $y{=}\rho\sigma$, $\rho{=}|y|$, $\sigma'{=}(\sigma_2,\ldots,\sigma_n)$ we obtain

$$\int\limits_{K^+}r^{\alpha-n-|\gamma|}(y)({}^{\gamma}\mathbf{T}^y_xf)(x)y^{\gamma}dy=$$

$$=\int\limits_0^{\infty}\rho^{\alpha-1}d\rho\int\limits_{\{S^+_1(n),|\sigma'|<\sigma_1\}}(\sigma_1^2-|\sigma'|^2)^{\frac{\alpha-n-|\gamma|}{2}}({}^{\gamma}\mathbf{T}^{\rho\sigma}f)(x)\sigma^{\gamma}dS,$$

where

$$\{S^+_1(n),|\sigma'|<\sigma_1\}=\{\sigma'\in\mathbb{R}^{n-1}_+:\sigma_1^2+|\sigma'|^2=1,\ |\sigma'|<\sigma_1\}.$$

Using formula ${}^{\gamma}\mathbf{T}^y_xf(x){=}{}^{\gamma}\mathbf{T}^x_yf(y)$, inequality $|{}^{\gamma}\mathbf{T}^y_xf(x)|{\leq}\sup\limits_{\mathbb{R}^n_+}|f(x)|$ (see [71, p. 124]) and considering that $f\in S_{ev}$ we get

$$\left|\int\limits_{K^+}r^{\alpha-n-|\gamma|}(y)({}^{\gamma}\mathbf{T}^y_xf)(x)y^{\gamma}dy\right|\leq$$

$$\leq C\int\limits_0^{\infty}\frac{\rho^{\alpha-1}}{(1+\rho^2)^{\frac{\alpha+1}{2}}}d\rho\int\limits_{S^+_1(n),|\sigma'|<\sigma_1}(\sigma_1^2-|\sigma'|^2)^{\frac{\alpha-n-|\gamma|}{2}}\sigma^{\gamma}dS<\infty,$$

for $\alpha > n + |\gamma| - 2$. Similarly, we get that (28) converges absolutely for $\alpha > n+|\gamma|-2$. So for $\alpha > n+|\gamma|-2$ integrals $(I^{\alpha}_{P\pm i0,\gamma} f)(x)$ converge absolutely. □

Theorem 3 *Let* $n + |\gamma| - 2 < \alpha < n + |\gamma|$, $1 \leq p < \frac{n+|\gamma|}{\alpha}$. *For the next estimate*

$$||I^{\alpha}_{P\pm i0,\gamma} f||_{q,\gamma} \leq C_{n,\gamma,p}||f||_{p,\gamma}, \qquad f(x) \in S_{ev} \tag{36}$$

to be valid it is necessary and sufficient that $q = \frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}$. *Constant* $C_{n,\gamma,p}$ *does not depend on* f.

Proof *(Necessity)* Let $n + |\gamma| - 2 < \alpha < n + |\gamma|$, $1 < p < \frac{n+|\gamma|}{\alpha}$ and for some q an inequality

$$||I^{\alpha}_{P\pm i0,\gamma} f||_{q,\gamma} \leq C_{n,\gamma,p}||f||_{p,\gamma}, \qquad f(x) \in S_{ev} \tag{37}$$

is hold.

We show that the inequality (37) is valid only for $q=\frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}$. Let obtain the required inequality for each term in the representation (29).

Let consider the extension operator $\tau_\delta : (\tau_\delta f)(x) = f(\delta x)$, $\delta > 0$. We have

$$||\tau_\delta f||_{p,\gamma} = \left(\int\limits_{\mathbb{R}^n_+} f^p(\delta x) x^\gamma dx\right)^{\frac{1}{p}} = \left(\delta^{-n-|\gamma|}\int\limits_{\mathbb{R}^n_+} f^p(y) y^\gamma dy\right)^{\frac{1}{p}}.$$

Therefore

$$||\tau_\delta f||_{p,\gamma} = \delta^{-\frac{n+|\gamma|}{p}}||f||_{p,\gamma}. \tag{38}$$

For $(I^{\alpha}_{P_+,\gamma} f)(x)$ we obtain

$$(I^{\alpha}_{P_+,\gamma} f)(x) = \int\limits_{K^+} [y_1^2 - y_2^2 - \ldots - y_n^2]^{\frac{\alpha-n-|\gamma|}{2}} ({}^\gamma \mathbf{T}^y_x \tau_\delta f)(y) y^\gamma dy =$$

$$= 2^{2n-|\gamma|}C(\gamma)\int\limits_{K^+} \frac{[y_1^2 - y_2^2 - \ldots - y_n^2]^{\frac{\alpha-n-|\gamma|}{2}} y^\gamma dy}{(xy)^{\gamma-1}} \times$$

$$\times \int\limits_{|x_1-y_1|}^{x_1+y_1} \ldots \int\limits_{|x_n-y_n|}^{x_n+y_n} f(\delta z) \prod_{i=1}^n z_i [(z_i^2 - (x_i - y_i)^2)((x_i + y_i)^2 - z_i^2)]^{\frac{\gamma_i}{2}-1} dz =$$

$$= \{\delta z = s\} =$$

$$= 2^{2n-|\gamma|}C(\gamma)\int\limits_{K^+} \frac{[y_1^2-y_2^2-\ldots-y_n^2]^{\frac{\alpha-n-|\gamma|}{2}} y^\gamma dy}{(xy)^{\gamma-1}} \int\limits_{\delta|x_1-y_1|}^{\delta(x_1+y_1)} \ldots \int\limits_{\delta|x_n-y_n|}^{\delta(x_n+y_n)} f(s)\delta^{-n}\times$$

$$\times\prod_{i=1}^n \frac{s_i}{\delta}\left[\left(\frac{s_i^2}{\delta^2}-(x_i-y_i)^2\right)\left((x_i+y_i)^2-\frac{s_i^2}{\delta^2}\right)\right]^{\frac{\gamma_i}{2}-1} ds =$$

$$= \delta^{2n-2|\gamma|}2^{2n-|\gamma|}C(\gamma)\int\limits_{K^+} \frac{[y_1^2-y_2^2-\ldots-y_n^2]^{\frac{\alpha-n-|\gamma|}{2}} y^\gamma dy}{(xy)^{\gamma-1}}\times$$

$$\times\int\limits_{\delta|x_1-y_1|}^{\delta(x_1+y_1)} \ldots \int\limits_{\delta|x_n-y_n|}^{\delta(x_n+y_n)} f(s)\prod_{i=1}^n s_i[(s_i^2-\delta^2(x_i-y_i)^2)(\delta^2(x_i+y_i)^2-s_i^2)]^{\frac{\gamma_i}{2}-1}ds =$$

$$= \{\delta y = t\} =$$

$$= \delta^{2n-2|\gamma|}2^{2n-|\gamma|}C(\gamma)\int\limits_{K^+} \frac{\delta^{n+|\gamma|-\alpha}[t_1^2-t_2^2-\ldots-t_n^2]^{\frac{\alpha-n-|\gamma|}{2}}\delta^{-n-|\gamma|} t^\gamma dt}{\delta^{n-|\gamma|}(xt)^{\gamma-1}}\times$$

$$\times\int\limits_{|\delta x_1-t_1|}^{\delta x_1+t_1} \ldots \int\limits_{|\delta x_n-t_n|}^{\delta x_n+t_n} f(s)\prod_{i=1}^n s_i[(s_i^2-(\delta x_i-t_i)^2)((\delta x_i+t_i)^2-s_i^2)]^{\frac{\gamma_i}{2}-1}ds =$$

$$= \delta^{-\alpha}2^{2n-|\gamma|}C(\gamma)\int\limits_{K^+} \frac{[t_1^2-t_2^2-\ldots-t_n^2]^{\frac{\alpha-n-|\gamma|}{2}} t^\gamma dt}{\delta^{|\gamma|-n}(xt)^{\gamma-1}} \int\limits_{|\delta x_1-t_1|}^{\delta x_1+t_1} \ldots \int\limits_{|\delta x_n-t_n|}^{\delta x_n+t_n} f(s)\times$$

$$\times\prod_{i=1}^n s_i[(s_i^2-(\delta x_i-t_i)^2)((\delta x_i+t_i)^2-s_i^2)]^{\frac{\gamma_i}{2}-1}ds =$$

$$= \delta^{-\alpha}\int\limits_{K^+} ({}^\gamma\mathbf{T}_t^{\delta x} f(t))[t_1^2-t_2^2-\ldots-t_n^2]^{\frac{\alpha-n-|\gamma|}{2}} t^\gamma dt = \delta^{-\alpha}\tau_\delta(I^\alpha_{P_+,\gamma}f)(x).$$

Then

$$(I^\alpha_{P_+,\gamma}f)(x) = \delta^\alpha \tau_\delta^{-1}(I^\alpha_{P_+,\gamma}\tau_\delta f)(x). \tag{39}$$

Next we have

$$||\tau_\delta^{-1} I^\alpha_{P_+,\gamma} f||_q^\gamma = \left(\int\limits_{\mathbb{R}^n_+} (\tau_\delta^{-1}(I^\alpha_{P_+,\gamma} f)(x))^q x^\gamma dx \right)^{\frac{1}{q}} =$$

$$= \left(\int\limits_{\mathbb{R}^n_+} \left(\int\limits_{K^+} [y_1^2 - y_2^2 - \ldots - y_n^2]^{\frac{\alpha-n-|\gamma|}{2}} ({}^\gamma \mathbf{T}_y^{\frac{x}{\delta}} f)(y) y^\gamma dy \right)^q x^\gamma dx \right)^{\frac{1}{q}} = \left(\frac{x}{\delta} = t \right) =$$

$$= \delta^{\frac{n+|\gamma|}{q}} ||I^\alpha_{P_+,\gamma} f||_q^\gamma$$

hence

$$||\tau_\delta^{-1} I^\alpha_{P_+,\gamma} f||_q^\gamma = \delta^{\frac{n+|\gamma|}{q}} ||I^\alpha_{P_+,\gamma} f||_q^\gamma. \tag{40}$$

Using (38)–(40) we get

$$||I^\alpha_{P_+,\gamma} f||_{q,\gamma} = \delta^\alpha ||\tau_\delta^{-1} I^\alpha_{P_+,\gamma} \tau_\delta f||_{q,\gamma} =$$

$$= \delta^{\frac{n+|\gamma|}{q}+\alpha} ||I^\alpha_{P_+,\gamma} \tau_\delta f||_{q,\gamma} \leq C_{n,\gamma,p} \delta^{\frac{n+|\gamma|}{q}+\alpha} ||\tau_\delta f||_{p,\gamma} =$$

$$= C_{n,\gamma,p} \delta^{\frac{n+|\gamma|}{q} - \frac{n+|\gamma|}{p} + \alpha} ||f||_{p,\gamma}$$

or

$$||I^\alpha_{P_+,\gamma} f(x)||_{q,\gamma} \leq C_{n,\gamma,p}\ \delta^{\frac{n+|\gamma|}{q} - \frac{n+|\gamma|}{p} + \alpha} ||f(x)||_{p,\gamma}. \tag{41}$$

If $\frac{n+|\gamma|}{q} - \frac{n+|\gamma|}{p} + \alpha > 0$ or $\frac{n+|\gamma|}{q} - \frac{n+|\gamma|}{p} + \alpha < 0$ then passing to the limit at $\delta \to 0$ or at $\delta \to \infty$ in (41) accordingly we obtain that for all functions $f \in L_p^\gamma$ equality

$$||I^\alpha_{P_+,\gamma} f||_{q,\gamma} = 0,$$

is hold what is wrong. That means that inequality (41) is possible only if $\frac{n+|\gamma|}{q} - \frac{n+|\gamma|}{p} + \alpha = 0$, i.e. for $q = \frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}$. Necessity is proved.

Sufficiency Let $x' = (x_2, \ldots, x_n)$, $|x'| = \sqrt{x_2^2 + \ldots + x_n^2}$, $(x')^{\gamma'} = x_2^{\gamma_2} \ldots x_n^{\gamma_n}$. Without loss of generality, we will assume that $f(x) \geq 0$ and $||f||_{p,\gamma} = 1$.

Let $0 < \delta < 1$. Consider the operators

$$(I^{\alpha}_{P_+,\gamma,\delta} f)(x) = \int\limits_{\delta y_1^2 \geq |y'|^2} r^{\alpha-n-|\gamma|}(y) ({}^{\gamma}\mathbf{T}^y_x f)(y) y^{\gamma} dy$$

and

$$(I^{\alpha}_{P_-,\gamma,\delta} f)(x) = \int\limits_{y_1^2 \leq \delta |y'|^2} r^{\alpha-n-|\gamma|}(y) ({}^{\gamma}\mathbf{T}^y_x f)(y) y^{\gamma} dy.$$

Let μ is some fixed real number. We introduce the notations

$$G^0_{\delta,\mu} = \{y \in \mathbb{R}^n_+ : \delta y_1^2 \geq |y'|^2,\ 0 \leq y_1 \leq \mu\},$$

$$G^{\infty}_{\delta,\mu} = \{y \in \mathbb{R}^n_+ : \delta y_1^2 \geq |y'|^2,\ \mu < y_1\},$$

$$K^+_{0,\delta}(y) = \begin{cases} r^{\alpha-n-|\gamma|}(y), & y \in G^0_{\delta,\mu}; \\ 0, & y \in \mathbb{R}^n_+ \setminus G^0_{\delta,\mu}, \end{cases}$$

$$K^+_{\infty,\delta}(y) = \begin{cases} r^{\alpha-n-|\gamma|}(y), & y \in G^{\infty}_{\delta,\mu}; \\ 0, & y \in \mathbb{R}^n_+ \setminus G^{\infty}_{\delta,\mu}, \end{cases}$$

$$H^0_{\delta,\mu} = \{y \in \mathbb{R}^n_+ : y_1^2 \leq \delta |y'|^2,\ |y'| \leq \mu\},$$

$$H^{\infty}_{\delta,\mu} = \{y \in \mathbb{R}^n_+ : y_1^2 \leq \delta |y'|^2,\ \mu < |y'|\},$$

$$M^+_{0,\delta}(y) = \begin{cases} r^{\alpha-n-|\gamma|}(y), & y \in H^0_{\delta,\mu}; \\ 0, & y \in \mathbb{R}^n_+ \setminus H^0_{\delta,\mu}, \end{cases}$$

$$M^+_{\infty,\delta}(y) = \begin{cases} r^{\alpha-n-|\gamma|}(y), & y \in H^{\infty}_{\delta,\mu}; \\ 0, & y \in \mathbb{R}^n_+ \setminus H^{\infty}_{\delta,\mu}. \end{cases}$$

In these notations we have

$$(I^{\alpha}_{P_+,\gamma,\delta} f)(x) = (K^+_{0,\delta} * f)_{\gamma} + (K^+_{\infty,\delta} * f)_{\gamma}, \tag{42}$$

$$(I^{\alpha}_{P_-,\gamma,\delta} f)(x) = (M^+_{0,\delta} * f)_{\gamma} + (M^+_{\infty,\delta} * f)_{\gamma}. \tag{43}$$

To apply Marcinkiewicz's theorem, we should prove that the operators $I^{\alpha}_{P_\pm,\gamma,\delta}$ have a weak type $(p_1,q_1)_\gamma$ and $(p_2,q_2)_\gamma$, where p_1, q_1, p_2, q_2 such that $\frac{1}{p}=\frac{1-\tau}{p_1}+\frac{\tau}{p_2}$, $\frac{1}{q}=\frac{1-\tau}{q_1}+\frac{\tau}{q_2}$, $0<\tau<1$. In order to do it we will be interested in the estimate of the

$$\sup_{0<\lambda<\infty}\lambda(\mu_\gamma(I^{\alpha}_{P_\pm,\gamma,\delta}f,\lambda))^{1/p}=$$

$$=\sup_{0<\lambda<\infty}\lambda\left(\mathrm{mes}_\gamma\{x\in\mathbb{R}^n_+:\,|(I^{\alpha}_{P_\pm,\gamma,\delta}f)(x)|>\lambda\}\right).$$

Considering (42) and (43) it is enough to estimate

$$\mathrm{mes}_\gamma\{x\in\mathbb{R}^n_+:\,|(K^+_{0,\delta}*f)_\gamma|>\lambda\},$$

$$\mathrm{mes}_\gamma\{x\in\mathbb{R}^n_+:\,|(K^+_{\infty,\delta}*f)_\gamma|>\lambda\},$$

$$\mathrm{mes}_\gamma\{x\in\mathbb{R}^n_+:\,|(M^+_{0,\delta}*f)_\gamma|>\lambda\},$$

$$\mathrm{mes}_\gamma\{x\in\mathbb{R}^n_+:\,|(M^+_{\infty,\delta}*f)_\gamma|>\lambda\}$$

and then to apply inequality

$$\mathrm{mes}_\gamma\{x\in\mathbb{R}^n_+:\,|A+B|>\lambda\}\le\mathrm{mes}_\gamma\{x\in\mathbb{R}^n_+:\,|A|>\lambda\}+\mathrm{mes}_\gamma\{x\in\mathbb{R}^n_+:\,|B|>\lambda\}.$$

To estimate the generalized convolution, we will use Young's inequality (20).

We have

$$||K^+_{0,\delta}||_{1,\gamma}=\int\limits_{\mathbb{R}^n_+}K^+_{0,\delta}(y)y^\gamma dy=\int\limits_{G^0_{\delta,\mu}}(y_1^2-y_2^2-\ldots-y_n^2)^{\frac{\alpha-n-|\gamma|}{2}}y^\gamma dy=$$

$$=\int\limits_0^\mu y_1^{\gamma_1}dy_1\int\limits_{|y'|^2\le\delta y_1^2}(y_1^2-|y'|^2)^{\frac{\alpha-n-|\gamma|}{2}}(y')^{\gamma'}dy'=\{y'=y_1z',\,z'\in\mathbb{R}^{n-1}_+\}=$$

$$=\int\limits_0^\mu y_1^{\alpha-1}dy_1\int\limits_{|z'|^2\le\delta}(1-|z'|^2)^{\frac{\alpha-n-|\gamma|}{2}}(z')^{\gamma'}dz'\le$$

$$\leq \int\limits_0^{\mu} y_1^{\alpha-1}dy_1 \int\limits_{|z'|^2\leq 1} (1-|z'|^2)^{\frac{\alpha-n-|\gamma|}{2}}(z')^{\gamma'}dz' =$$

$$= \frac{\mu^\alpha}{\alpha} \int\limits_{|z'|\leq 1} (1-|z'|^2)^{\frac{\alpha-n-|\gamma|}{2}}(z')^{\gamma'}dz' = C^1_{\alpha,n,\gamma}\,\mu^\alpha,$$

where $C^1_{\alpha,n,\gamma} = 2^{1-n}\frac{\Gamma\left(\frac{\alpha-n-|\gamma|+2}{2}\right)\prod\limits_{i=2}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\alpha\Gamma\left(\frac{\alpha-\gamma_1+1}{2}\right)}$ does not depend on δ. Therefore

$$||K^+_{0,\delta}||_{1,\gamma} \leq C^1_{\alpha,n,\gamma}\,\mu^\alpha, \tag{44}$$

and $K^+_{0,\delta}\in L^\gamma_1$.

Now let's consider $M^+_{0,\delta}$:

$$||M^+_{0,\delta}||_{1,\gamma} = \int\limits_{\mathbb{R}^n_+} M^+_{0,\delta}(y)y^\gamma dy = \int\limits_{H^0_{\delta,\mu}} (y_1^2-y_2^2-\ldots-y_n^2)^{\frac{\alpha-n-|\gamma|}{2}}y^\gamma dy =$$

$$= \int\limits_{|y'|\leq\mu} (y')^{\gamma'}dy' \int\limits_{y_1^2\leq\delta|y'|^2} (|y'|^2-y_1^2)^{\frac{\alpha-n-|\gamma|}{2}}y_1^{\gamma_1}dy_1 = \{y_1=|y'|z_1,\ z_1\in\mathbb{R}^1_+\} =$$

$$= \int\limits_{|y'|\leq\mu} |y'|^{\alpha-n-|\gamma|+\gamma_1+1}(y')^{\gamma'}dy' \int\limits_{z_1^2\leq\delta} (1-z_1^2)^{\frac{\alpha-n-|\gamma|}{2}}z_1^{\gamma_1}dz_1 \leq$$

$$\leq D^1_{\alpha,n,\gamma} \int\limits_{|y'|\leq\mu} |y'|^{\alpha-n-|\gamma|+\gamma_1+1}(y')^{\gamma'}dy',$$

where $D^1_{\alpha,n,\gamma} = \int\limits_{z_1^2\leq 1}(1-z_1^2)^{\frac{\alpha-n-|\gamma|}{2}}z_1^{\gamma_1}dz_1$ does not depend on δ. When going over to spherical coordinates $y'=\rho\sigma$ we obtain

$$||M^+_{0,\delta}||_{1,\gamma} \leq D^2_{\alpha,n,\gamma}\int\limits_0^{\mu}\rho^{\alpha-1}d\rho = D^3_{\alpha,n,\gamma}\mu^\alpha,$$

where $D^3_{\alpha,n,\gamma} = \frac{1}{\alpha}\int\limits_{S_1^+(n-1)}\sigma^{\gamma'}dS$.

Now we estimate the norm $K^+_{\infty,\delta}$. Let's take p' such that $\frac{1}{p}+\frac{1}{p'}=1$. First consider $||K^+_{\infty,\delta}||_{p',\gamma}$. Let $p\neq 1$ (i.e. $p'\neq\infty$) then

$$||K^+_{\infty,\delta}||_{p',\gamma}=\left(\int\limits_{\mathbb{R}^n_+}|K^+_{0,\delta}(y)|^{p'}y^\gamma dy\right)^{1/p'}=\left(\int\limits_{G^\infty_{\delta,\mu}}(y_1^2-|y'|^2)^{\frac{\alpha-n-|\gamma|}{2}\,p'}y^\gamma dy\right)^{1/p'}=$$

$$=\left(\int\limits_\mu^\infty y_1^{\gamma_1}dy_1\int\limits_{|y'|^2\leq\delta y_1^2}(y_1^2-|y'|^2)^{\frac{\alpha-n-|\gamma|}{2}\,p'}(y')^{\gamma'}dy'\right)^{1/p'}=\{y'=y_1z',\,z'\in\mathbb{R}^{n-1}_+\}=$$

$$=\left(\int\limits_\mu^\infty y_1^{(\alpha-n-|\gamma|)p'+n+|\gamma|-1}dy_1\int\limits_{|z'|^2\leq\delta}(1-|z'|^2)^{\frac{\alpha-n-|\gamma|}{2}\,p'}(z')^{\gamma'}dz'\right)^{1/p'}\leq$$

$$\leq\frac{\prod\limits_{i=2}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma'|+1}{2}\right)}(1-\delta)^{\frac{\alpha-n-|\gamma|}{2}}\left(\int\limits_\mu^\infty y_1^{(\alpha-n-|\gamma|)p'+n+|\gamma|-1}dy_1\right)^{1/p'}=$$

$$=C^2_{\alpha,n,\gamma}(1-\delta)^{\frac{\alpha-n-|\gamma|}{2}}\mu^{-\frac{n+|\gamma|}{q}},$$

$$C^2_{\alpha,n,\gamma,p}=\frac{2^{-n}\prod\limits_{i=2}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{((n+|\gamma|-\alpha)p'-n-|\gamma|)^{1/p'}\Gamma\left(\frac{n+|\gamma'|+1}{2}\right)}.$$

Here we take into account that $\alpha-n-|\gamma|<0$, $p'=\frac{p}{p-1}$, $p<\frac{n+|\gamma|}{\alpha}$ and $q=\frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}$. Then

$$||K^+_{\infty,\delta}||_{p',\gamma}\leq C^2_{\alpha,n,\gamma,p}(1-\delta)^{\frac{\alpha-n-|\gamma|}{2}}\mu^{-\frac{n+|\gamma|}{q}},\qquad\frac{1}{p}+\frac{1}{p'}=1,\tag{45}$$

hence $K^+_{\infty,\delta}\in L^\gamma_{p'}$, $p'<\infty$.

Passing to the limit as $p'\to\infty$ in (45) we obtain

$$||K^+_{\infty,\delta}||_{\infty,\gamma}\leq C^2_{\alpha,n,\gamma,1}(1-\delta)^{\frac{\alpha-n-|\gamma|}{2}}\mu^{-\frac{n+|\gamma|}{q}},\qquad C^2_{\alpha,n,\gamma,1}=\frac{e^{\frac{n+|\gamma'|}{n+|\gamma|-\alpha}}\prod\limits_{i=2}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma'|+1}{2}\right)}.\tag{46}$$

Let estimate the norm $||M^+_{\infty,\delta}||_{p',\gamma}$. Let p' such that $\frac{1}{p}+\frac{1}{p'}=1$ and $p\neq 1$ (i.e. $p'\neq\infty$), then

$$||M^+_{\infty,\delta}||_{p',\gamma}=\left(\int\limits_{\mathbb{R}^n_+}|M^+_{\infty,\delta}(y)|^{p'}y^\gamma dy\right)^{1/p'}=\left(\int\limits_{H^\infty_{\delta,\mu}}(|y'|^2-y_1^2)^{\frac{\alpha-n-|\gamma|}{2}p'}y^\gamma dy\right)^{1/p'}=$$

$$=\left(\int\limits_{\mu\leq|y'|}(y')^{\gamma'}dy'\int\limits_{y_1^2\leq\delta|y'|^2}(|y'|^2-y_1^2)^{\frac{\alpha-n-|\gamma|}{2}p'}y_1^{\gamma_1}dy_1\right)^{1/p'}=\{y_1=|y'|z_1,\,z_1\in\mathbb{R}^1_+\}=$$

$$=\left(\int\limits_{\mu\leq|y'|}|y'|^{(\alpha-n-|\gamma|)p'+\gamma_1+1}(y')^{\gamma'}dy'\int\limits_{z_1^2\leq\delta}(1-z_1^2)^{\frac{\alpha-n-|\gamma|}{2}p'}z_1^{\gamma_1}dz_1\right)^{1/p'}\leq$$

$$\leq D^4_{\alpha,n,\gamma}(1-\delta^2)^{\frac{\alpha-n-|\gamma|}{2}}\left(\int\limits_{\mu\leq|y'|}|y'|^{(\alpha-n-|\gamma|)p'+\gamma_1+1}(y')^{\gamma'}dy'\right)^{1/p'}.$$

Going over to spherical coordinates $y'=\rho\sigma$ we get

$$||M^+_{\infty,\delta}||_{1,\gamma}\leq D^5_{\alpha,n,\gamma}(1-\delta^2)^{\frac{\alpha-n-|\gamma|}{2}}\left(\int\limits_{\mu}^{\infty}\rho^{(\alpha-n-|\gamma|)p'+n+|\gamma|-1}d\rho\right)^{1/p'}=$$

$$=D^5_{\alpha,n,\gamma}(1-\delta^2)^{\frac{\alpha-n-|\gamma|}{2}}\mu^{-\frac{n+|\gamma|}{q}}.$$

Here we take into account that $\alpha-n-|\gamma|<0$, $p'=\frac{p}{p-1}$, $p<\frac{n+|\gamma|}{\alpha}$ and $q=\frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}$. Then

$$||M^+_{\infty,\delta}||_{p',\gamma}\leq D^5_{\alpha,n,\gamma,p}(1-\delta)^{\frac{\alpha-n-|\gamma|}{2}}\mu^{-\frac{n+|\gamma|}{q}},\qquad \frac{1}{p}+\frac{1}{p'}=1,\tag{47}$$

hence $M^+_{\infty,\delta}\in L^\gamma_{p'}$, $p'<\infty$.

Passing to the limit as $p'\to\infty$ in (47) we obtain

$$||M^+_{\infty,\delta}||_{\infty,\gamma}\leq D^6_{\alpha,n,\gamma,1}(1-\delta)^{\frac{\alpha-n-|\gamma|}{2}}\mu^{-\frac{n+|\gamma|}{q}}.\tag{48}$$

So we have

$$(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||K^{+}_{\infty,\delta}||_{p',\gamma} \leq C\mu^{-\frac{n+|\gamma|}{q}}, \qquad 1 \leq p' \leq \infty$$

and

$$(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||M^{+}_{\infty,\delta}||_{p',\gamma} \leq C\mu^{-\frac{n+|\gamma|}{q}}, \qquad 1 \leq p' \leq \infty.$$

Then applying (21) we can write

$$(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||(K^{+}_{\infty,\delta} * f)_{\gamma}||_{\infty,\gamma} \leq (1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||K^{+}_{\infty,\delta}||_{p',\gamma} \leq C\mu^{-\frac{n+|\gamma|}{q}}$$

and

$$(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||(M^{+}_{\infty,\delta} * f)_{\gamma}||_{\infty,\gamma} \leq (1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||M^{+}_{\infty,\delta}||_{p',\gamma} \leq C\mu^{-\frac{n+|\gamma|}{q}}.$$

If we choose μ such that $C\mu^{-\frac{n+|\gamma|}{q}} = \lambda$ then

$$\text{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : (1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(K^{+}_{\infty,\delta} * f)_{\gamma}| > \lambda\} = 0$$

and

$$\text{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : (1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(M^{+}_{\infty,\delta} * f)_{\gamma}| > \lambda\} = 0.$$

Considering (42) and (43) and applying the Young's inequality (20) we obtain

$$\text{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : (1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(I^{\alpha}_{P_+,\delta} f)(x)| > 2\lambda\} \leq$$

$$\leq \text{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : (1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(K^{+}_{0,\delta} * f)_{\gamma}| > \lambda\} +$$

$$+\text{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : (1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(K^{+}_{\infty,\delta} * f)_{\gamma}| > \lambda\} =$$

$$= \text{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : (1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(K^{+}_{0,\delta} * f)_{\gamma}| > \lambda\} \leq (1-\delta)^{\frac{n+|\gamma|-\alpha}{2}} \frac{||(K^{+}_{0,\delta} * f)_{\gamma}||^{p}_{p,\gamma}}{\lambda^{p}} \leq$$

$$\leq \frac{(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}p}||K^{+}_{0,\delta}||^{p}_{1,\gamma}||f||^{p}_{p,\gamma}}{\lambda^{p}} \leq$$

$$\leq \frac{(C^1_{\alpha,n,\gamma})^p(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}\,p}\,\mu^{p\alpha}}{\lambda^p} =$$

$$= C^7(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}\frac{1}{\lambda^q}.$$

Similarly,

$$\mathrm{mes}_\gamma\{x\in\mathbb{R}^n_+ : (1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(I^\alpha_{P_-,\delta}f)(x)|>2\lambda\}\leq C^7(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}\frac{1}{\lambda^q}.$$

It was shown that the operators $I^\alpha_{P_\pm,\gamma,\delta}$ have a week type $(p,q)_\gamma$, where p and q related by equality $q=\frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}$. Let $0<\tau<1$, $p_1{=}\frac{p(1-\tau)}{1-\tau p}$, $p_1\in\left[1,\frac{n+|\gamma|}{\alpha}\right)$. The operators $I^\alpha_{P_\pm,\gamma,\delta}$ have a week type $\left(1,\frac{n+|\gamma|}{n+|\gamma|-\alpha}\right)_\gamma$ and a week type $\left(p_1,\frac{(n+|\gamma|)p_1}{n+|\gamma|-\alpha p_1}\right)_\gamma$. Then by Marcinkiewicz's Theorem 1 the operators $I^\alpha_{P_\pm,\gamma,\delta}$ have a strong type $\left(p,\frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}\right)_\gamma$ and the next inequality

$$||(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}(I^\alpha_{P_\pm,\gamma,\delta}f)(x)||_{q,\gamma}\leq M(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||f||_{p,\gamma}$$

is true. So

$$||(I^\alpha_{P_\pm,\gamma,\delta}f)(x)||_{q,\gamma}\leq M||f||_{p,\gamma},\qquad 1{\leq}p{<}\frac{n+|\gamma|}{\alpha},\qquad n+|\gamma|-2{<}\alpha{<}n+|\gamma|. \tag{49}$$

Since $f(x)\geq 0$ then for $0<\delta_1\leq\delta_2\leq\ldots\leq\delta_m\leq\ldots<1$ we have

$$(I^\alpha_{P_\pm,\gamma,\delta_1}f)(x)\leq(I^\alpha_{P_\pm,\gamma,\delta_2}f)(x)\leq\ldots\leq(I^\alpha_{P_\pm,\gamma,\delta_m}f)(x)\leq\ldots$$

Due to the fact that

$$\lim_{\delta\to 1}(I^\alpha_{P_\pm,\gamma,\delta}f)(x)=(I^\alpha_{P_\pm,\gamma}f)(x)$$

Passing to the limit as $\delta\to 1$ in (49) we obtain

$$||(I^\alpha_{P_\pm,\gamma}f)(x)||_{q,\gamma}\leq M||f||_{p,\gamma},\qquad 1{\leq}p{<}\frac{n+|\gamma|}{\alpha},\qquad n{+}|\gamma|{-}2{<}\alpha{<}n{+}|\gamma|.$$

The theorem is proved. □

Further operators $I^\alpha_{P_\pm,\gamma}$ on functions we L^γ_p we will define as continuations of operators (24) with preservation of boundedness. If integral (24) converges absolutely for $f \in L_p^\gamma$ then these continuations are representable as

$$(I^\alpha_{P\pm i0,\gamma} f)(x) = \frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{\gamma_{n,\gamma}(\alpha)} \int\limits_{\mathbb{R}^n_+} (P \pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}} ({}^\gamma\mathbf{T}^y_x f)(x) y^\gamma dy, \quad y^\gamma = \prod_{i=1}^n y_i^{\gamma_i}.$$

Next we show how hyperbolic B-potentials are connected with the operator $(\square_\gamma)^k$, $k \in \mathbb{N}$.

Theorem 4 *If $f \in S_{ev}$, $n + |\gamma| - 2 < \alpha$ and $k \in \mathbb{N}$ then*

$$(\square_\gamma)^k I^{\alpha+2k}_{P\pm i0,\gamma} f = I^\alpha_{P\pm i0,\gamma} f, \tag{50}$$

where $\square_\gamma = B_{\gamma_1} - \sum\limits_{i=2}^n B_{\gamma_i}$.

Proof Using representation (24) and the property ${}^{\gamma_i}T^{y_i}_{x_i}(B_{\gamma_i})_{x_i} = (B_{\gamma_i})^{\gamma_i}_{x_i} T^{y_i}_{x_i}$ (see formula 1.8.3 from [60]) we obtain

$$(\square_\gamma)^k (I^{\alpha+2k}_{P\pm i0,\gamma} f)(x) =$$

$$= \frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha+2k)} (\square_\gamma)^k \int\limits_{\mathbb{R}^n_+} (P \pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} ({}^\gamma\mathbf{T}^y_x f)(x) y^\gamma dy =$$

$$= \frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha+2k)} \int\limits_{\mathbb{R}^n_+} \left({}^\gamma\mathbf{T}^y_x (\square_\gamma)^k (P \pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} \right) f(y) y^\gamma dy.$$

For function $(P \pm i0)^\lambda_\gamma$ the next equality is true (see [70])

$$(\square_\gamma)^k (P \pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} = 2^{2k} \frac{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+k+1\right)}{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+1\right)} \frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(\frac{\alpha}{2}\right)} (P \pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}}. \tag{51}$$

Since

$$2^{2k} \frac{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+k+1\right)}{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+1\right)} \frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \cdot \frac{1}{H_{n,\gamma}(\alpha+2k)} = \frac{1}{H_{n,\gamma}(\alpha)},$$

then using (51) we get

$$(\Box_\gamma)^k (I^{\alpha+k}_{P\pm i0,\gamma} f)(x) =$$

$$= \frac{e^{\pm \frac{n-1+|\gamma'|}{2} i\pi}}{H_{n,\gamma}(\alpha)} \int\limits_{\mathbb{R}^n_+} \left({}^\gamma \mathbf{T}^y_x (P \pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}} \right) f(y) y^\gamma dy = (I^\alpha_{P\pm i0,\gamma} f)(x).$$

and the proof is complete. □

Theorem 5 *If* $f \in S_{ev}$, $n + |\gamma| - 2 < \alpha$ *and* $k \in \mathbb{N}$ *then*

$$I^{\alpha+2k}_{P\pm i0,\gamma} (\Box_\gamma)^k f = I^\alpha_{P\pm i0,\gamma} f, \tag{52}$$

where $\Box_\gamma = B_{\gamma_1} - \sum\limits_{i=2}^n B_{\gamma_i}$ *and* $x_i^{\gamma_i} \frac{\partial}{\partial x_i} f \Big|_{x_i=0} = 0$, $i = 1, \dots, n$.

Proof Using the formula 1.8.3 from [60] of the form ${}^{\gamma_i} T^{y_i}_{x_i} (B_{\gamma_i})_{x_i} = (B_{\gamma_i})^{\gamma_i}_{x_i} T^{y_i}_{x_i}$ we get

$$(I^{\alpha+2k}_{P\pm i0,\gamma} \Box^k_\gamma f)(x) = \frac{e^{\pm \frac{n-1+|\gamma'|}{2} i\pi}}{\gamma_{n,\gamma}(\alpha+2k)} \int\limits_{\mathbb{R}^n_+} (P \pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} ({}^\gamma \mathbf{T}^y_x (\Box_\gamma)^k_x f)(x) y^\gamma dy =$$

$$= \frac{e^{\pm \frac{n-1+|\gamma'|}{2} i\pi}}{\gamma_{n,\gamma}(\alpha+2)} \int\limits_{\mathbb{R}^n_+} (P \pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} \left[(\Box_\gamma)_y \, {}^\gamma \mathbf{T}^y_x (\Box_\gamma)^{k-1}_x f)(x) \right] y^\gamma dy =$$

$$= \frac{e^{\pm \frac{n-1+|\gamma'|}{2} i\pi}}{\gamma_{n,\gamma}(\alpha+2)} \int\limits_{\mathbb{R}^n_+} (P \pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} \left[\left((B_{\gamma_1})_{y_1} - \sum_{i=2}^n (B_{\gamma_i})_{y_i} \right) {}^\gamma \mathbf{T}^y_x (\Box_\gamma)^{k-1}_x f(x) \right] y^\gamma dy =$$

$$= \frac{e^{\pm \frac{n-1+|\gamma'|}{2} i\pi}}{\gamma_{n,\gamma}(\alpha+2)} \int\limits_{\mathbb{R}^n_+} (P \pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} \left[(B_{\gamma_1})_{y_1} \, {}^\gamma \mathbf{T}^y_x (\Box_\gamma)^{k-1}_x f(x) \right] y^\gamma dy -$$

$$- \frac{e^{\pm \frac{n-1+|\gamma'|}{2} i\pi}}{\gamma_{n,\gamma}(\alpha+2)} \sum_{i=2}^n \int\limits_{\mathbb{R}^n_+} (P \pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} \left[(B_{\gamma_i})_{y_i} \, {}^\gamma \mathbf{T}^y_x (\Box_\gamma)^{k-1}_x f(x) \right] y^\gamma dy. \tag{53}$$

Integrating by part at $j=1,\dots,n$ we get

$$\int\limits_0^\infty (P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}(B_{\gamma_j})_{y_j}\left[{}^\gamma\mathbf{T}_x^y(\square_\gamma)_x^{k-1}f(x)\right]y_j^{\gamma_j}dy_j=$$

$$=\int\limits_0^\infty (P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\left[\frac{\partial}{\partial y_j}y_j^{\gamma_j}\frac{\partial}{\partial y_j}{}^\gamma\mathbf{T}_x^y(\square_\gamma)_x^{k-1}f(x)\right]dy_j=$$

$$=\left\{u=(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}},\ dv=\frac{\partial}{\partial y_j}y_j^{\gamma_j}\frac{\partial}{\partial y_j}{}^\gamma\mathbf{T}_x^y(\square_\gamma)_x^{k-1}f(x)dy_j\right\}=$$

$$=(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}y_j^{\gamma_j}\frac{\partial}{\partial y_j}{}^\gamma\mathbf{T}_x^y(\square_\gamma)_x^{k-1}f(x)\Big|_{y_j=0}^{\infty}-$$

$$-\int\limits_0^\infty y_j^{\gamma_j}\frac{\partial}{\partial y_j}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\left[\frac{\partial}{\partial y_j}{}^\gamma\mathbf{T}_x^y(\square_\gamma)_x^{k-1}(\square_\gamma)_x^{k-1}(\square_\gamma)_x^{k-1}f(x)\right]dy_j=$$

$$=-\int\limits_0^\infty y_j^{\gamma_j}\frac{\partial}{\partial y_j}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\left[\frac{\partial}{\partial y_j}{}^\gamma\mathbf{T}_x^y(\square_\gamma)_x^{k-1}f(x)\right]dy_j=$$

$$=\left\{u=y_j^{\gamma_j}\frac{\partial}{\partial y_j}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}},\ dv=\frac{\partial}{\partial y_j}{}^\gamma\mathbf{T}_x^y(\square_\gamma)_x^{k-1}f(x)dy_j\right\}=$$

$$=-y_j^{\gamma_j}\left[\frac{\partial}{\partial y_j}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\right]{}^\gamma\mathbf{T}_x^y(\square_\gamma)_x^{k-1}f(x)\Big|_{y_j=0}^{\infty}+$$

$$+\int\limits_0^\infty\left[\frac{\partial}{\partial y_j}y_j^{\gamma_j}\frac{\partial}{\partial y_j}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\right]{}^\gamma\mathbf{T}_x^y f(x)dy_j=$$

$$=\int\limits_0^\infty\left[(B_{\gamma_j})_{y_j}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\right]{}^\gamma\mathbf{T}_x^y(\square_\gamma)_x^{k-1}f(x)y_j^{\gamma_j}dy_j.$$

Returning to (53) we obtain

$$(I^{\alpha+2k}_{P\pm i0,\gamma}\Box^k_\gamma f)(x)=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{\gamma_{n,\gamma}(\alpha+2k)}\int\limits_{\mathbb{R}^n_+}\left[(\Box_\gamma)_y(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\right]({}^\gamma\mathbf{T}^y_x(\Box_\gamma)^{k-1}_x f)(x)y^\gamma dy.$$

Consistently applying these actions k times we get

$$(I^{\alpha+2k}_{P\pm i0,\gamma}\Box^k_\gamma f)(x)=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{\gamma_{n,\gamma}(\alpha+2k)}\int\limits_{\mathbb{R}^n_+}\left[(\Box_\gamma)^k_y(P\pm i0)_\gamma^{\frac{\alpha+2-n-|\gamma|}{2}}\right]({}^\gamma\mathbf{T}^y_x f)(x)y^\gamma dy.$$

Now applying (51) we obtain the required statement

$$(I^{\alpha+2k}_{P\pm i0,\gamma}\Box^k_\gamma f)(x)=(I^{\alpha}_{P\pm i0,\gamma} f)(x).$$

□

By virtue of the density S_{ev} in L^γ_p equalities (50) and (52) spread on function from L^γ_p for $1<p<\frac{n+|\gamma|}{\alpha}$ when integrals $I^\alpha_{P\pm i0,\gamma}f$ converge absolutely for $f\in L^\gamma_p$.

Example 1 Let $n{=}3$, $\gamma_1{=}2$, $\alpha{=}4$, $|\gamma'|{=}\gamma_2{+}\gamma_3{<}1$, $f(x){=}x_1^2e^{-x_1}\mathbf{j}_{\gamma'}(x',b)$, $|b|{=}1$, $x{\in}\mathbb{R}^3_+$. We have

$$I^4_{P_+,\gamma}x_1^2e^{-x_1}\mathbf{j}_{\gamma'}(x',b)=$$

$$=2^{-\frac{3}{2}}\prod_{i=2}^3\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{1-|\gamma'|}{2}\right)\mathbf{j}_{\gamma'}(x';b)\int\limits_0^\infty e^{-y_1}\left({}^2T^{y_1}_{x_1}J_{\frac{1}{2}}(x_1)x_1^{\frac{1}{2}}\right)y_1^4dy_1=$$

$$=2^{-\frac{3}{2}}\prod_{i=2}^3\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{1-|\gamma'|}{2}\right)\frac{C(2)\sqrt{2}}{\sqrt{\pi}x_1}\mathbf{j}_{\gamma'}(x';b)\times$$

$$\times\left(\int\limits_0^\infty e^{-y_1}\left(\sin(x_1+y_1)-(x_1+y_1)\cos(x_1+y_1)\right)y_1^3dy_1-\right.$$

$$\left.-\int\limits_0^\infty e^{-y_1}\left(\sin(|x_1-y_1|)-|x_1-y_1|\cos(|x_1-y_1|)\right)y_1^3dy_1\right)=$$

$$=\prod_{i=2}^3\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{1-|\gamma'|}{2}\right)\frac{1}{4\sqrt{\pi}}\mathbf{j}_{\gamma'}(x';b)\left(\frac{6(\cos x_1-e^{x_1})}{x_1}-12-6x_1-x_1^2\right)$$

and

$$I^4_{P_-,\gamma}x_1^2e^{-x_1}\mathbf{j}_{\gamma'}(x',b)=2^{-\frac{3}{2}}\prod_{i=2}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{1-|\gamma'|}{2}\right)\mathbf{j}_{\gamma'}(x';b)\int\limits_0^\infty e^{-y_1}\times$$

$$\times\left(\left({}^2T^{y_1}_{x_1}J_{\frac{1}{2}}(x_1)x_1^{\frac{1}{2}}\right)\cos\frac{3+|\gamma'|}{2}\pi+\left({}^2T^{y_1}_{x_1}J_{-\frac{1}{2}}(x_1)x_1^{\frac{1}{2}}\right)\sin\frac{1+|\gamma'|}{2}\pi\right)y_1^4dy_1=$$

$$=2^{-\frac{3}{2}}\prod_{i=2}^{3}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{1-|\gamma'|}{2}\right)\frac{C(2)\sqrt{2}}{\sqrt{\pi}x_1}\mathbf{j}_{\gamma'}(x';b)\Bigg(\cos\left(\frac{3+|\gamma'|}{2}\right)\pi\times$$

$$\times\int\limits_0^\infty e^{-y_1}y_1^3\left(\sin(x_1+y_1)-(x_1+y_1)\cos(x_1+y_1)-\sin(|x_1-y_1|)+\right.$$

$$\left.+|x_1-y_1|\cos(|x_1-y_1|)\right)dy_1+$$

$$+\sin\left(\frac{1+|\gamma'|}{2}\right)\pi\int\limits_0^\infty e^{-y_1}y_1^3\left(\cos(x_1+y_1)+(x_1+y_1)\sin(x_1+y_1)-\right.$$

$$\left.-\cos(|x_1-y_1|)-|x_1-y_1|\sin(|x_1-y_1|)dy_1\right)\Bigg)=$$

$$=\prod_{i=2}^{3}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{1-|\gamma'|}{2}\right)\frac{1}{4\sqrt{\pi}}\mathbf{j}_{\gamma'}(x';b)\times$$

$$\times\Bigg(\cos\left(\frac{3+|\gamma'|}{2}\right)\pi\left(\frac{6(\cos x_1-e^{-x_1})}{x_1}-12-6x_1-x_1^2\right)-$$

$$-6\sin\left(\frac{1+|\gamma'|}{2}\right)\pi\,\frac{\sin x_1}{x_1}\Bigg).$$

Considering that

$$H_{3,\gamma}(4)=\frac{\sqrt{\pi}\prod\limits_{i=2}^{3}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{|\gamma'|+1}{2}\right)},$$

we obtain

$$I^4_{P\pm i0,\gamma} x_1^2 e^{-x_1} \mathbf{j}_{\gamma'}(x',b)=$$

$$=\frac{e^{\pm\frac{2+|\gamma'|}{2}i\pi}}{H_{3,\gamma}(4)}\left[I^4_{P_+,\gamma} x_1^2 e^{-x_1} \mathbf{j}_{\gamma'}(x',b)+e^{\mp\frac{|\gamma'|+1}{2}\pi i}I^4_{P_-,\gamma} x_1^2 e^{-x_1} \mathbf{j}_{\gamma'}(x',b)\right]=$$

$$=\frac{e^{\pm\frac{2+|\gamma'|}{2}i\pi}}{4\sin\left(\frac{1+|\gamma'|}{2}\right)\pi}\mathbf{j}_{\gamma'}(x';b)\Bigg(\frac{6(\cos x_1-e^{x_1})}{x_1}-12-6x_1-x_1^2+$$

$$+e^{\mp\frac{|\gamma'|+1}{2}\pi i}\left(\cos\left(\frac{3+|\gamma'|}{2}\right)\pi\left(\frac{6(\cos x_1-e^{-x_1})}{x_1}-12-6x_1-x_1^2\right)-\right.$$

$$\left.-6\sin\left(\frac{1+|\gamma'|}{2}\right)\pi\cdot\frac{\sin x_1}{x_1}\right)=$$

$$=\mathbf{j}_{\gamma'}(x';b)\,\frac{e^{-x_1}(6-6e^{(1+i)x_1}+x_1(12+x_1(6+x_1)))}{4x_1}.$$

3 Green's Second Identity for the Hyperbolic B-Potentials

3.1 Divergence Theorem for Weighted Nabla Operator

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\vec{e}=(e_1,\ldots,e_n)$ is orthonormal basis in $\mathbb{R}^n$,

$$\overline{\mathbb{R}}{}^n_+=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n,\ \ x_1\geq 0,\ldots,x_n\geq 0\},$$

$$\nabla'_\gamma=\left(\frac{1}{x_1^{\gamma_1}}\frac{\partial}{\partial x_1},\ldots,\frac{1}{x_n^{\gamma_n}}\frac{\partial}{\partial x_n}\right)$$

is the first weighted nabla operator,

$$\vec{F}=\vec{F}(x)=(F_1(x),\ldots,F_n(x))$$

is a vector field,

$$(\nabla'_\gamma\cdot\vec{F})=\frac{1}{x_1^{\gamma_1}}\frac{\partial F_1}{\partial x_1}+\ldots+\frac{1}{x_i^{\gamma_n}}\frac{\partial F_n}{\partial x_n}$$

is the weighted divergence. Let

$$\lozenge_\gamma = \left(x_1^{\gamma_1} \frac{\partial}{\partial x_1}, -x_2^{\gamma_{p+1}} \frac{\partial}{\partial x_2}, \ldots, -x_n^{\gamma_n} \frac{\partial}{\partial x_n} \right),$$

then

$$(\nabla'_\gamma \cdot \lozenge_\gamma) = \Box_\gamma.$$

In $\overline{\mathbb{R}}^n_+$ let consider a domain G^+ bounded by a piecewise smooth surface $S^+ \in \overline{\mathbb{R}}^n_+$. Thus, a surface can be represented as a union $S^+ = \bigcup\limits_{k=1}^{q} S_k^+$ of a finite number of its parts S_k^+ without common internal points. Let for each interior point there is a neighborhood within which the surface S_k^+ is represented by parametric equations of the form

$$x_i = \chi_i(y_1, \ldots, y_{n-1}), \qquad i = 1, \ldots, n,$$

where $\chi_i(y)$, $y = (y_1, \ldots, y_{n-1})$ has continuous first derivatives and the rank of the Jacobi matrix $\left\| \frac{\partial(\chi_1,\ldots,\chi_n)}{\partial(y_1,\ldots,y_{n-1})} \right\|$ is equal to $n-1$. Vector

$$\vec{N} = \begin{Vmatrix} e_1 & \ldots & e_n \\ \frac{\partial \chi_1(y)}{\partial y_1} & \ldots & \frac{\partial \chi_n(y)}{\partial y_1} \\ \ldots & \ldots & \ldots \\ \frac{\partial \chi_1(y)}{\partial y_{n-1}} & \ldots & \frac{\partial \chi_n(y)}{\partial y_{n-1}} \end{Vmatrix}$$

is normal in each point $y \in S^+$ to the surface S^+ with the exception of the junction points of surfaces S_k^+, $k = 1, .., q$, where it is not defined unambiguously and will not be considered. Vector

$$\vec{\nu} = \frac{\vec{N}}{|\vec{N}|}$$

is determined to within sign. Of the two possible directions $\vec{\nu}$, we choose the external with respect to the domain G^+. Such a vector will be called the unit normal vector to the surface S^+ at the point y. Let denote η_i the angle which forms a vector $\vec{\nu}$ with an axis x_j, then

$$\vec{\nu} = e_1 \cos \eta_1 + \ldots + e_n \cos \eta_n.$$

Theorem 6 *Let G^+ is the domain in $\overline{\mathbb{R}}^n_+$ such that each line perpendicular to the plane $x_i = 0$, $i = 1, \ldots, n$ either does not cross G^+ or has one common segment* *with G^+ (maybe degenerating to a point) of the form*

$$\alpha_i(x') \le x_i \le \beta_i(x'), \qquad x'=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n), \qquad i=1,\ldots,n.$$

If $\vec{F} = (F_1(x),\ldots,F_n(x))$, $F_1(x) = x_1^{\gamma_1} g_1(x),\ldots,F_n(x) = x_n^{\gamma_n} g_n(x)$, $\vec{g} = (g_1(x),\ldots,g_n(x))$ is a continuously differentiable in G^+ vector field, then the next formula is valid

$$\int\limits_{G^+} (\nabla'_\gamma \cdot \vec{F})\, x^\gamma dx = \int\limits_{S^+} (\vec{g}\cdot\vec{\nu})\; x^\gamma\, dS\,, \tag{54}$$

where $\vec{\nu}$ is the external unit normal vector S^+.

Proof Let $i=1,\ldots,n$ is fixed. The part of the surface S^+ defined by the equation $x_i=\beta_i(x')$ is denoted by S^+_u and the part of the surface S^+ defined by the equation $x_i=\alpha_i(x')$ is denoted by S^+_d, then

$$(\vec{\nu},e_i) = \begin{cases} -\dfrac{1}{\sqrt{1+\left(\frac{\partial\alpha_i}{\partial x_1}\right)^2+\ldots+(\frac{\partial\alpha_i}{\partial x_{i-1}})^2+\left(\frac{\partial\alpha_i}{\partial x_{i+1}}\right)^2+\ldots+\left(\frac{\partial\alpha_i}{\partial x_n}\right)^2}}, & x\in S^+_d \\ \dfrac{1}{\sqrt{1+\left(\frac{\partial\beta_i}{\partial x_1}\right)^2+\ldots+\left(\frac{\partial\beta_i}{\partial x_{i-1}}\right)^2+\left(\frac{\partial\beta_i}{\partial x_{i+1}}\right)^2+\ldots+\left(\frac{\partial\beta_i}{\partial x_n}\right)^2}}, & x\in S^+_u \end{cases}$$

We have

$$\int\limits_{G^+} (\nabla'_\gamma \cdot \vec{F})\, x^\gamma dx = \sum_{i=1}^n \int\limits_{G^+} \frac{1}{x_i^{\gamma_i}}\frac{\partial F_i}{\partial x_i}\, x^\gamma dx.$$

Let consider

$$\int\limits_{G^+} \frac{1}{x_i^{\gamma_i}}\frac{\partial F_i}{\partial x_i}\, x^\gamma dx = \int\limits_{Q} x_1^{\gamma_1}\ldots x_{i-1}^{\gamma_{i-1}} x_{i+1}^{\gamma_{i+1}}\ldots x_n^{\gamma_n} dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_n \int\limits_{\alpha_i(x')}^{\beta_i(x')} \frac{\partial F_i}{\partial x_i}dx_i,$$

where Q is a projection of the G^+ to $x_i=0$. Integrating by x_i we obtain

$$\int\limits_{G^+} \frac{1}{x_i^{\gamma_i}}\frac{\partial F_i}{\partial x_i}\, x^\gamma dx = \int\limits_{Q} F_i(x)|^{x_i=\beta_i(x')}_{x_i=\alpha_i(x')} x_1^{\gamma_1}\ldots x_{i-1}^{\gamma_{i-1}} x_{i+1}^{\gamma_{i+1}}\ldots x_n^{\gamma_n} dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_n.$$

Let $(x')^{\gamma'} = x_1^{\gamma_1}\ldots x_{i-1}^{\gamma_{i-1}} x_{i+1}^{\gamma_{i+1}}\ldots x_n^{\gamma_n}$, $dx' = dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_n$, then

$$\int\limits_{G^+} \frac{1}{x_i^{\gamma_i}}\frac{\partial F_i}{\partial x_i}\, x^\gamma dx =$$

$$= \int\limits_{Q} F_i(x_1, \ldots, x_{i-1}, \beta_i(x'), x_{i+1}, \ldots, x_n)(x')^{\gamma'} dx'$$

$$- \int\limits_{Q} F_i(x_1, \ldots, x_{i-1}, \alpha_i(x'), x_{i+1}, \ldots, x_n)(x')^{\gamma'} dx' =$$

$$= \int\limits_{Q} F_i(x_1, \ldots, x_{i-1}, \beta_i(x'), x_{i+1}, \ldots, x_n)(\vec{\nu}, e_i)\times$$

$$\times\sqrt{1 + \left(\frac{\partial\beta_i}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial\beta_i}{\partial x_{i-1}}\right)^2 + \left(\frac{\partial\beta_i}{\partial x_{i+1}}\right)^2 + \ldots + \left(\frac{\partial\beta_i}{\partial x_n}\right)^2}(x')^{\gamma'} dx' +$$

$$+ \int\limits_{Q} F_i(x_1, \ldots, x_{i-1}, \alpha_i(x'), x_{i+1}, \ldots, x_n)(\vec{\nu}, e_i)\times$$

$$\times\sqrt{1 + \left(\frac{\partial\alpha_i}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial\alpha_i}{\partial x_{i-1}}\right)^2 + \left(\frac{\partial\alpha_i}{\partial x_{i+1}}\right)^2 + \ldots + \left(\frac{\partial\alpha_i}{\partial x_n}\right)^2}(x')^{\gamma'} dx' =$$

$$= \int\limits_{S_u^+} F_i(x)(\vec{\nu}, e_i)(x')^{\gamma'} dS_u + \int\limits_{S_d^+} F_i(x)(\vec{\nu}, e_i)(x')^{\gamma'} dS_d$$

$$= \int\limits_{S_u^+} g_i(x)(\vec{\nu}, e_i)x^{\gamma} dS_u + \int\limits_{S_d^+} g_i(x)(\vec{\nu}, e_i)x^{\gamma} dS_d =$$

$$= \int\limits_{S^+} g_i(x)\cos\eta_i\, x^{\gamma} dS.$$

Then

$$\int\limits_{G^+} (\nabla'_{\gamma} \cdot \vec{F})\, x^{\gamma} dx = \sum_{i=1}^{n} \int\limits_{S^+} g_i(x)\cos\eta_i\, x^{\gamma} dS = \int\limits_{S^+} (\vec{g} \cdot \vec{\nu})\; x^{\gamma}\, dS.$$

□

Remark 1 Suppose that a domain $G^+ \in \overline{\mathbb{R}}^n_+$ is a union of domains $G_1^+, \ldots, G_m^+$ without common internal points. Let each G_j^+ is the domain in $\overline{\mathbb{R}}^n_+$ such that each line perpendicular to the plane $x_i = 0, i = 1, \ldots, n$ either does not cross G_j^+ or has one common segment with G_j^+ (maybe degenerating to a point) of the form

$$\alpha_i^j(x') \leq x_i \leq \beta_i^j(x'), \qquad x'=(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \qquad i = 1, \ldots, n$$

and $\vec{F} = (F_1(x), \ldots, F_n(x))$, $F_1(x) = x_1^{\gamma_1} g_1(x), \ldots, F_n(x) = x_n^{\gamma_n} g_n(x)$, $\vec{g} = (g_1(x), \ldots, g_n(x))$ is a continuously differentiable in G^+ vector field, then the next formula is valid

$$\int\limits_{G^+} (\nabla'_\gamma \cdot \vec{F})\, x^\gamma dx = \int\limits_{S^+} (\vec{g} \cdot \vec{\nu})\ x^\gamma\ dS\,, \tag{55}$$

where $S^+ \in \overline{\mathbb{R}}^n_+$ a piecewise smooth surface boundary, $\vec{\nu}$ is the external unit normal vector S^+.

3.2 Green's Second Identities for the $\Box_\gamma$ and for the Hyperbolic B-Potentials

Theorem 7 *Let G^+ satisfies to conditions in Remark 1. If φ, ψ are twice continuously differentiable functions defined on G^+, such that*

$$\left.\frac{\partial\varphi}{\partial x_i}\right|_{x_i=0} = 0, \qquad \left.\frac{\partial\psi}{\partial x_i}\right|_{x_i=0} = 0, \qquad i = 1, \ldots, n$$

then the Green's second identity for B-ultra-hyperbolic operator has the form

$$\int\limits_{G^+} (\psi \Box_\gamma \varphi - \varphi \Box_\gamma \psi)\, x^\gamma dx = \int\limits_{S^+} \left(\psi \frac{\partial\varphi}{\partial\vec{\tau}} - \varphi \frac{\partial\psi}{\partial\vec{\tau}}\right) x^\gamma\, dS, \tag{56}$$

where $\vec{\tau} = (\cos\eta_1, -\cos\eta_2, \ldots, -\cos\eta_n)$.

Proof Let

$$\vec{F} = \psi \Diamond_\gamma \varphi - \varphi \Diamond_\gamma \psi = \left(x_1^{\gamma_1}\left(\psi\frac{\partial\varphi}{\partial x_1} - \varphi\frac{\partial\psi}{\partial x_1}\right),\right.$$
$$\left. -x_2^{\gamma_2}\left(\psi\frac{\partial\varphi}{\partial x_2} - \varphi\frac{\partial\psi}{\partial x_2}\right), \ldots, -x_n^{\gamma_n}\left(\psi\frac{\partial\varphi}{\partial x_n} - \varphi\frac{\partial\psi}{\partial x_n}\right)\right),$$

so

$$\vec{g} = \left(\psi\frac{\partial\varphi}{\partial x_1} - \varphi\frac{\partial\psi}{\partial x_1}, -\left(\psi\frac{\partial\varphi}{\partial x_2} - \varphi\frac{\partial\psi}{\partial x_2}\right), \ldots, -\left(\psi\frac{\partial\varphi}{\partial x_n} - \varphi\frac{\partial\psi}{\partial x_n}\right)\right),$$

$$(\nabla'_\gamma \cdot \vec{F}) = \psi \Box_\gamma \varphi - \varphi \Box_\gamma \psi,$$

$$(\vec{g} \cdot \vec{\nu}) = \left(\psi\frac{\partial\varphi}{\partial x_1}\cos\eta_1 - \varphi\frac{\partial\psi}{\partial x_1}\cos\eta_1\right) - \sum_{i=2}^n \left(\psi\frac{\partial\varphi}{\partial x_i}\cos\eta_i - \varphi\frac{\partial\psi}{\partial x_i}\cos\eta_i\right) = \psi\frac{\partial\varphi}{\partial\vec{\tau}} - \varphi\frac{\partial\psi}{\partial\vec{\tau}},$$

where $\vec{\tau} = (\cos\eta_1, -\cos\eta_2, \ldots, -\cos\eta_n)$. Then applying Theorem 6 we obtain (56). □

Let g such that $({}^\gamma\mathbf{T}_x^y g)(x)$ vanishes outside the domain G^+, where satisfies to conditions in Remark 1, then

$$(I^\alpha_{P\pm i0,\gamma}g)(x)=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha)}\int\limits_{G^+}(P\pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}}({}^\gamma\mathbf{T}_x^y g)(x)y^\gamma dy.$$

Applying (56) to the hyperbolic B-potentials $I^\alpha_{P\pm i0,\gamma}g$ we get

$$\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha+2)}\int\limits_{G^+}((P\pm i0)_\gamma^{\frac{\alpha+2-n-|\gamma|}{2}}\Box_\gamma({}^\gamma\mathbf{T}_x^y g)(x)-({}^\gamma\mathbf{T}_x^y g)(x)\Box_\gamma(P\pm i0)_\gamma^{\frac{\alpha+2-n-|\gamma|}{2}})x^\gamma dx=$$

$$=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha+2)}\int\limits_{S^+}\left((P\pm i0)_\gamma^{\frac{\alpha+2-n-|\gamma|}{2}}\frac{\partial({}^\gamma\mathbf{T}_x^y g)(x)}{\partial\vec{\tau}}-({}^\gamma\mathbf{T}_x^y g)(x)\frac{\partial(P\pm i0)_\gamma^{\frac{\alpha+2-n-|\gamma|}{2}}}{\partial\vec{\tau}}\right)x^\gamma\, dS.$$

Using the equality $\frac{\Box_\gamma(P\pm i0)_\gamma^{\frac{\alpha+2-n-|\gamma|}{2}}}{H_{n,\gamma}(\alpha+2)}=\frac{(P\pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}}}{H_{n,\gamma}(\alpha)}$ and the fact that $\Box_\gamma({}^\gamma\mathbf{T}_x^y g)(x)$ $=({}^\gamma\mathbf{T}_x^y\Box_\gamma g)(x)$ we obtain Green's second identities for the hyperbolic B-potentials $I^\alpha_{P\pm i0,\gamma}g$:

$$(I^\alpha_{P\pm i0,\gamma}g)(x)=(I^{\alpha+2}_{P\pm i0,\gamma}\Box_\gamma g)(x)-$$

$$-\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha+2)}\int\limits_{S^+}\left((P\pm i0)_\gamma^{\frac{\alpha+2-n-|\gamma|}{2}}\frac{\partial({}^\gamma\mathbf{T}_x^y g)(x)}{\partial\vec{\tau}}-({}^\gamma\mathbf{T}_x^y g)(x)\frac{\partial(P\pm i0)_\gamma^{\frac{\alpha+2-n-|\gamma|}{2}}}{\partial\vec{\tau}}\right)x^\gamma\, dS.$$

4 Inversion of the Hyperbolic B-Potentials

4.1 Method of Approximative Inverse Operators

Here we describe one approach for inverting potential type operators, based on the idea of approximative inverse operators developed in [40, 72].

The problem to invert this or that convolution operator $Af = a*f$ reduces to multiplication of the some convenient integral transform of a function f by the reciprocal $\frac{1}{\hat{a}}$ of chosen integral transform of the kernel:

$$Af=a*f,\qquad \widehat{Af}=\hat{a}\cdot\hat{f},\qquad \widehat{A^{-1}f}=\frac{1}{\hat{a}}\cdot\hat{f}.$$

Indeed we have

$$g=Af,\qquad \widehat{A^{-1}g}=\frac{1}{\hat{a}}\cdot\hat{a}\cdot\hat{f}=\hat{f}.$$

However, in the case of potentials, the multiplier $\frac{1}{\hat{a}}$, is unbounded at infinity and, maybe, on some sets. In this case we use the multiplier m_ε, which is dependent on ε such that $\frac{m_\varepsilon}{\hat{a}}$ vanishes at those sets on which it is necessary and $\lim\limits_{\varepsilon\to 0} m_\varepsilon = 1$. So we can construct $\widehat{A_\varepsilon^{-1} f} = \frac{m_\varepsilon}{\hat{a}} \cdot \hat{f}$. Applying the inverse integral transform and passing to the limit $\varepsilon \to 0$ we obtain A^{-1}. Next it is necessary to prove that the resulting operator will be inverse to the operator to A in some appropriate space. Therefore, the factor m_ε should be chosen so that inverse integral transform of $\frac{m_\varepsilon}{\hat{a}} \cdot \hat{f}$ provides a fairly good class of functions.

In our case, we take the Hankel transform. Considering that

$$\mathbf{F}_\gamma I^\alpha_{P\pm i0,\gamma} f = (P \mp i0)_\gamma^{-\frac{\alpha}{2}} \mathbf{F}_\gamma f,$$

where $f\in\Phi_V^\gamma$, $V=\{x\in\mathbb{R}^n_+ : P(x)=0\}$ we take

$$M_{\varepsilon,\delta} = \frac{(P \mp i0)^m e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m}.$$

So we should prove that left inverse operators to $I^\alpha_{P\pm i0,\gamma}$ are

$$(I^\alpha_{P\pm i0,\gamma})^{-1} f) = \overset{L_p^\gamma}{\lim_{\delta\to 0}} \overset{L_2^\gamma}{\lim_{\varepsilon\to 0}} \left(\left(\mathbf{F}_\gamma^{-1} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m}\right)(x) * f(x)\right)_\gamma .$$

We denote

$$(I^\alpha_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta} f = \left(\left(\mathbf{F}_\gamma^{-1} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m}\right)(x) * f(x)\right)_\gamma =$$

$$= \int\limits_{\mathbb{R}^n_+} {}^{\mp}g^\alpha_{\varepsilon,\delta}(y)({}^\gamma\mathbf{T}^y_x f(x)) y^\gamma dy,$$

where

$${}^{\mp}g^\alpha_{\varepsilon,\delta}(x) = \left(\mathbf{F}_\gamma^{-1} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m}\right)(x) =$$

$$= \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int\limits_{\mathbb{R}^n_+} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m} \mathbf{j}_\gamma(x,\xi)\xi^\gamma\, d\xi,$$

$m \geq n + |\gamma| - \frac{\alpha}{2}$, $n + |\gamma| - 2 < \alpha < n + |\gamma|$.

4.2 General Poisson Kernel

In this section, we consider a certain function used for solving the problem of inverting a hyperbolic B-potential. Based on the type and properties of this function, we will call it the general Poisson kernel.

We first prove an auxiliary lemma.

Lemma 1 *Hankel transform of the* $e^{-\delta|x|}$ *is*

$$\mathbf{F}_\gamma[e^{-\delta|x|}](\xi)=\frac{2^{|\gamma|}\delta\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}}. \tag{57}$$

Proof We have

$$\mathbf{F}_\gamma[e^{-\delta|x|}](\xi)=\int\limits_{\mathbb{R}^n_+}e^{-\delta|x|}\mathbf{j}_\gamma(x;\xi)x^\gamma dx=\{x=\rho\sigma\}=$$

$$=\int\limits_0^\infty e^{-\delta\rho}\rho^{n+|\gamma|-1}d\rho\int\limits_{S_1^+(n)}\mathbf{j}_\gamma(\rho\sigma;\xi)\sigma^\gamma dS.$$

Applying the formula (13) we obtain

$$\mathbf{F}_\gamma[e^{-\delta|x|}](\xi)=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty e^{-\delta\rho}j_{\frac{n+|\gamma|}{2}-1}(\rho|\xi|)\rho^{n+|\gamma|-1}d\rho=$$

$$=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{\frac{n-|\gamma|}{2}}|\xi|^{\frac{n+|\gamma|}{2}-1}}\int\limits_0^\infty e^{-\delta\rho}J_{\frac{n+|\gamma|}{2}-1}(\rho|\xi|)\rho^{\frac{n+|\gamma|}{2}}d\rho.$$

Applying the formula 2.12.8.4 from [68] p. 164 of the form

$$\int\limits_0^\infty x^{\nu+2}e^{-px}J_\nu(cx)dx=\frac{2p(2c)^\nu\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}(p^2+c^2)^{\nu+\frac{3}{2}}},\qquad \mathrm{Re}\,\nu>-1$$

we get

$$\int_0^\infty e^{-\delta\rho} J_{\frac{n+|\gamma|}{2}-1}(\rho|\xi|)\rho^{\frac{n+|\gamma|}{2}} d\rho = \frac{2\delta(2|\xi|)^{\frac{n+|\gamma|}{2}-1}\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}}$$

and therefore

$$\mathbf{F}_\gamma[e^{-\delta|x|}](\xi) = \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{\frac{n-|\gamma|}{2}}|\xi|^{\frac{n+|\gamma|}{2}-1}} \frac{2\delta(2|\xi|)^{\frac{n+|\gamma|}{2}-1}\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}} =$$

$$= \frac{2^{|\gamma|}\delta \prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}}.$$

□

We give the formula from [73] that will be used further

$$\int\limits_{S_1^+(n)} \mathcal{P}_\xi^\gamma f(\langle\xi,x\rangle)x^\gamma d\omega_x = \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}2^{n-1}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)} \int_{-1}^1 f(|\xi|p)(1-p^2)^{\frac{n+|\gamma|-3}{2}} dp, \tag{58}$$

where $f(t)(1-t^2)^{\frac{n+|\gamma|-3}{2}} \in L_1(-1,1)$.

Definition 5 Function

$$P_\gamma(x,\delta) = \frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^n \Gamma\left(\frac{\gamma_j+1}{2}\right)} \delta\,(\delta^2+|x|^2)^{-\frac{n+|\gamma|+1}{2}}, \qquad \delta>0 \tag{59}$$

is called the **general Poisson kernel**.

Lemma 2 *For $P_\gamma(x,\delta)$ next properties are valid*

1. $\mathbf{F}_\gamma[P_\gamma(x,\delta)](\xi) = e^{-\delta|\xi|}$,
2. $\int\limits_{\mathbb{R}_+^n} P_\gamma(x,\delta)x^\gamma dx = \int\limits_{\mathbb{R}_+^n} P_\gamma(x,1)x^\gamma dx = 1$,
3. $P_\gamma(x,\delta) \in L_p^\gamma$, $1 \le p \le \infty$.

Proof

1. From Lemma 1 we get

$$\mathbf{F}_\gamma^{-1}[e^{-\delta|x|}](\xi)=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\mathbf{F}_\gamma[e^{-\delta|x|}](\xi)=$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\frac{2^{|\gamma|}\delta\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}}=$$

$$=\frac{2^n\delta\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}\frac{1}{(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}}=P_\gamma(x,\delta).$$

And when we obtain $\mathbf{F}_\gamma[P_\gamma(x,\delta)](\xi)=e^{-\delta|\xi|}$.

2. Consider the integral $\int\limits_{\mathbb{R}^n_+}P_\gamma(x,\delta)x^\gamma dx$. We have

$$\int\limits_{\mathbb{R}^n_+}P_\gamma(x,\delta)x^\gamma dx=\frac{2^n\delta\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\frac{x^\gamma\,dx}{(\delta^2+|x|^2)^{\frac{n+|\gamma|+1}{2}}}=\{x=\delta y\}=$$

$$=\frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\frac{y^\gamma\,dy}{(1+|y|^2)^{\frac{n+|\gamma|+1}{2}}}=\int\limits_{\mathbb{R}^n_+}P_\gamma(x,1)x^\gamma dx.$$

Let us show now that $\int\limits_{\mathbb{R}^n_+}P_\gamma(x,1)x^\gamma dx=1$. Going over to spherical coordinates and using (9) we obtain

$$\int\limits_{\mathbb{R}^n_+}\frac{y^\gamma\,dy}{(1+|y|^2)^{\frac{n+|\gamma|+1}{2}}}=\{y=\rho\sigma\}=\int\limits_0^\infty\frac{\rho^{n+|\gamma|-1}\,d\rho}{(1+\rho^2)^{\frac{n+|\gamma|+1}{2}}}\int\limits_{S_1^+(n)}\sigma^\gamma\,dS=$$

$$=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty\frac{\rho^{n+|\gamma|-1}\,d\rho}{(1+\rho^2)^{\frac{n+|\gamma|+1}{2}}}=\{\rho^2=r\}=$$

$$=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty\frac{r^{\frac{n+|\gamma|}{2}-1}}{(1+r)^{\frac{n+|\gamma|+1}{2}}}dr.$$

Using the formula 2.2.5.24 from [74], p. 239 of the form

$$\int\limits_0^\infty \frac{x^{\alpha-1}}{(x+z)^\beta}dx = z^{\alpha-\beta}B(\alpha,\beta-\alpha), \qquad 0<\mathrm{Re}\,\alpha<\mathrm{Re}\,\beta,$$

we obtain

$$2\int\limits_0^\infty \frac{\rho^{n+|\gamma|-1}\,d\rho}{(1+\rho^2)^{\frac{n+|\gamma|+1}{2}}} = \int\limits_0^\infty \frac{r^{\frac{n+|\gamma|}{2}-1}}{(1+r)^{\frac{n+|\gamma|+1}{2}}}dr = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\Gamma\left(\frac{n+|\gamma|+1}{2}\right)} \tag{60}$$

and

$$\int\limits_{\mathbb{R}^n_+} \frac{y^\gamma\,dy}{(1+|y|^2)^{\frac{n+|\gamma|+1}{2}}} = \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|}{2}\right)}\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\Gamma\left(\frac{n+|\gamma|+1}{2}\right)} = \frac{\sqrt{\pi}\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}.$$

Finally,

$$\int\limits_{\mathbb{R}^n_+} P_\gamma(x,1)x^\gamma dx = \frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^n \Gamma\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+} \frac{y^\gamma\,dy}{(1+|y|^2)^{\frac{n+|\gamma|+1}{2}}} =$$

$$= \frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^n \Gamma\left(\frac{\gamma_j+1}{2}\right)}\frac{\sqrt{\pi}\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)} = 1.$$

3. We finally prove that $P_\gamma(x,\delta)\in L_p^\gamma$, $1\le p\le\infty$. We have

$$\int\limits_{\mathbb{R}^n_+} \frac{x^\gamma dx}{(\delta^2+|x|^2)^{p\,\frac{n+|\gamma|+1}{2}}} = \delta^{(n+|\gamma|)(1-p)-p}\int\limits_{\mathbb{R}^n_+} \frac{x^\gamma dx}{(|x|^2+1)^{p\,\frac{n+|\gamma|+1}{2}}} =$$

$$= \{x=\rho\sigma, |x|=\rho\} = \delta^{(n+|\gamma|)(1-p)-p}\int\limits_0^\infty \frac{\rho^{n+|\gamma|-1}d\rho}{(\rho^2+1)^{p\,\frac{n+|\gamma|+1}{2}}}\int\limits_{S_1^+(n)} \sigma^\gamma dS.$$

Applying (9) and (60) for $1 \le p < \infty$ we get

$$||P_\gamma(x,\delta)||_{p,\gamma} = \left(\delta^{(n+|\gamma|)(1-p)-p} \frac{\sqrt{\pi}\Gamma\left(\frac{n+|\gamma|}{2}\right)}{2\Gamma\left(\frac{n+|\gamma|+1}{2}\right)} \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)} \right)^{\frac{1}{p}} =$$

$$= \left(\delta^{(n+|\gamma|)(1-p)-p} \frac{\sqrt{\pi}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)} \right)^{\frac{1}{p}} < \infty.$$

For $p=\infty$ we get inequality $||P_\gamma(x,\delta)||_{\infty,\gamma} < \infty$ using (4).

□

Following [75] (see Theorem 1.18, p. 17) we prove that a generalized convolution of a function with the Poisson kernel tends to a function in L_p^γ.

Let

$$(\mathbf{P}_{\gamma,\delta} f)(x) = (f(x) * P_\gamma(x,\delta))_\gamma. \tag{61}$$

Lemma 3 *If $f \in L_p^\gamma$, $1 \le p \le \infty$ or $f \in C_0 \subset L_\infty^\gamma$ then*

$$||(\mathbf{P}_{\gamma,\delta} f)(x) - f(x)||_{p,\gamma} \to 0 \qquad \textit{with} \qquad \delta \to 0.$$

Proof Considering the property 2 from Lemma 2 we can write

$$(f(x) * P_\gamma(x,\delta))_\gamma - f(x) = \int\limits_{\mathbb{R}_+^n} [\,{}^\gamma\mathbf{T}_x^y f(x) - f(y)] P_\gamma(y,\delta) y^\gamma dy.$$

Hence, applying the generalized Minkowski inequality, we obtain

$$||(f(x) * P_\gamma(x,\delta))_\gamma - f(x)||_{p,\gamma} \le$$

$$\le \int\limits_{\mathbb{R}_+^n} \left(\int\limits_{\mathbb{R}_+^n} [\,{}^\gamma\mathbf{T}_x^y f(x) - f(x)]^p x^\gamma dx \right)^{\frac{1}{p}} |P_\gamma(y,\delta)| y^\gamma dy = \{y = \delta t\} =$$

$$= \int\limits_{\mathbb{R}_+^n} \left(\int\limits_{\mathbb{R}_+^n} [\,{}^\gamma\mathbf{T}_x^{\delta t} f(x) - f(x)]^p x^\gamma dx \right)^{\frac{1}{p}} |P_\gamma(t,1)| t^\gamma dt. \tag{62}$$

From [76] (see the Lemma 3.6, p. 166) it follows that for $f \in L_p^\gamma$

$$||\,{}^\gamma\mathbf{T}_x^{\delta t} f(x) - f(x)||_{p,\gamma} \leq c||f(x)||_{p,\gamma},$$

and from [77] (see the preposition 4.1, p. 182) and [78] p. 50 follows that

$$\lim_{\delta\to 0}\left(\int\limits_{\mathbb{R}_+^n}[\,{}^\gamma\mathbf{T}_x^{\delta t} f(x) - f(x)]^p x^\gamma dx\right)^{\frac{1}{p}} = 0.$$

Then, by the Lebesgue theorem on dominated convergence, the integral (62) tends to zero when $\delta \to 0$, since the integrand is majorized by the integrable function $c||f||_{p,\gamma}|P_\gamma(t,1)|t^\gamma$.

□

4.3 Representation of the Kernel $^\mp g^\alpha_{\varepsilon,\delta}$

In this section we get the integral kernel representation $^\mp g^\alpha_{\varepsilon,\delta}$.

Theorem 8 *Function $^\mp g^\alpha_{\varepsilon,\delta}$ can be presented in the form*

$$^\mp g^\alpha_{\varepsilon,\delta}(x) = \frac{2^{2-|\gamma|}}{\delta^{n+|\gamma|+\alpha}\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}\frac{\Gamma(n+|\gamma|+\alpha)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times$$

$$\times\int\limits_0^\infty r^{n+|\gamma'|-2}\frac{(1-r^2\mp i0)^{m+\frac{\alpha}{2}}}{(1+r^2)^{\frac{n+|\gamma|+\alpha}{2}}(1-r^2+i\varepsilon(1+r^2))^m}\times$$

$$\times F_4\left(\frac{\beta}{2},\frac{\beta+1}{2};\frac{\gamma_1+1}{2},\frac{n+|\gamma'|-1}{2};-\frac{x_1^2}{\delta^2(1+r^2)},-\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right)dr.$$

where $\beta = n+|\gamma|+\alpha$ $F_4(a,b,c_1,c_2;x,y)$ is the Appell hypergeometric function (23).

Proof We represent the function $^\mp g^\alpha_{\varepsilon,\delta}(t)$ as the sum

$$^\mp g^\alpha_{\varepsilon,\delta}(x) = \mathbf{F}_\gamma^{-1}\frac{(P\mp i0)^{m+\frac{\alpha}{2}}e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} =$$

$$= \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}_+^n}\frac{(P\mp i0)^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi =$$

$$= \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n} \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \left[\int\limits_{\{P(\xi)>0\}^+} \frac{P^{m+\frac{\alpha}{2}}(\xi)e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m} \mathbf{j}_\gamma(x,\xi)\xi^\gamma \, d\xi + \right.$$

$$\left. + e^{\mp(m+\frac{\alpha}{2})\pi i} \int\limits_{\{P(\xi)<0\}^+} \frac{|P(\xi)|^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m} \mathbf{j}_\gamma(x,\xi)\xi^\gamma \, d\xi \right].$$

Let

$$J_1 = \int\limits_{\{P(\xi)>0\}^+} \frac{P^{m+\frac{\alpha}{2}}(\xi)e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m} \mathbf{j}_\gamma(x,\xi)\xi^\gamma \, d\xi,$$

$$J_2 = \int\limits_{\{P(\xi)<0\}^+} \frac{|P(\xi)|^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m} \mathbf{j}_\gamma(x,\xi)\xi^\gamma \, d\xi.$$

Going in J_1 over to spherical coordinates $\xi' = \rho\sigma$, $\sigma \in \mathbb{R}_+^{n-1}$, $\rho = |\xi'|$ we obtain

$$J_1 = \int\limits_0^\infty j_{\frac{\gamma_1-1}{2}}(x_1\xi_1)\xi_1^{\gamma_1} \, d\xi_1 \times$$

$$\times \int\limits_{|\xi'|^2<\xi_1^2} \frac{(\xi_1^2-|\xi'|^2)^{m+\frac{\alpha}{2}} e^{-\delta\sqrt{\xi_1^2+|\xi'|^2}}}{(\xi_1^2-|\xi'|^2+i\varepsilon(\xi_1^2+|\xi'|^2))^m} \mathbf{j}_\gamma(x',\xi')(\xi')^{\gamma'} \, d\xi' =$$

$$= \int\limits_0^\infty j_{\frac{\gamma_1-1}{2}}(x_1\xi_1)\xi_1^{\gamma_1} \, d\xi_1 \int\limits_0^{\xi_1} \rho^{n+|\gamma'|-2} \frac{(\xi_1^2-\rho^2)^{m+\frac{\alpha}{2}} e^{-\delta\sqrt{\xi_1^2+\rho^2}}}{(\xi_1^2-\rho^2+i\varepsilon(\xi_1^2+\rho^2))^m} d\rho \times$$

$$\times \int\limits_{S_1^+(n-1)} \mathbf{j}_\gamma(x',\rho\sigma)(\sigma)^{\gamma'} \, dS.$$

The next formula

$$\int\limits_{S_1^+(n-1)} \mathbf{j}_\gamma(x',\rho\sigma)(\sigma)^{\gamma'} \, dS = \frac{\prod\limits_{i=2}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)} \, j_{\frac{n-1+|\gamma'|}{2}-1}(\rho|x'|),$$

is valid (see (16)), therefore

$$J_1=\frac{\prod\limits_{i=2}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)}\int\limits_0^\infty j_{\frac{\gamma_1-1}{2}}(x_1\xi_1)\xi_1^{\gamma_1}\,d\xi_1\times$$

$$\times\int\limits_0^{\xi_1}\rho^{n+|\gamma'|-2}j_{\frac{n-1+|\gamma'|}{2}-1}(\rho|x'|)\frac{(\xi_1^2-\rho^2)^{m+\frac{\alpha}{2}}e^{-\delta\sqrt{\xi_1^2+\rho^2}}}{(\xi_1^2-\rho^2+i\varepsilon(\xi_1^2+\rho^2))^m}d\rho=\{\rho=\xi_1 r\}=$$

$$=\frac{\prod\limits_{i=2}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)}\int\limits_0^\infty j_{\frac{\gamma_1-1}{2}}(x_1\xi_1)\xi_1^{n+|\gamma|-1+\alpha}\,d\xi_1\times$$

$$\times\int\limits_0^1 r^{n+|\gamma'|-2}j_{\frac{n-1+|\gamma'|}{2}-1}(r\xi_1|x'|)\frac{(1-r^2)^{m+\frac{\alpha}{2}}e^{-\delta\xi_1\sqrt{1+r^2}}}{(1-r^2+i\varepsilon(1+r^2))^m}dr=$$

$$=\frac{\prod\limits_{i=2}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)}\frac{2^{\frac{\gamma_1-1}{2}}\Gamma\left(\frac{\gamma_1+1}{2}\right)}{x_1^{\frac{\gamma_1-1}{2}}}\frac{2^{\frac{n-1+|\gamma'|}{2}-1}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)}{|x'|^{\frac{n-1+|\gamma'|}{2}-1}}\times$$

$$\times\int\limits_0^1 r^{\frac{n+|\gamma'|-1}{2}}\frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1-r^2+i\varepsilon(1+r^2))^m}dr\times$$

$$\times\int\limits_0^\infty \xi_1^{\frac{n+|\gamma|}{2}+\alpha+1}e^{-\delta\xi_1\sqrt{1+r^2}}J_{\frac{\gamma_1-1}{2}}(x_1\xi_1)J_{\frac{n-1+|\gamma'|}{2}-1}(r\xi_1|x'|)\,d\xi_1=$$

$$=\frac{2^{\frac{|\gamma|-n}{2}}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{x_1^{\frac{\gamma_1-1}{2}}|x'|^{\frac{n-1+|\gamma'|}{2}-1}}\int\limits_0^1 r^{\frac{n+|\gamma'|-1}{2}}\frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1-r^2+i\varepsilon(1+r^2))^m}dr\times$$

$$\times\int\limits_0^\infty \xi_1^{\frac{n+|\gamma|}{2}+\alpha+1}e^{-\delta\xi_1\sqrt{1+r^2}}J_{\frac{\gamma_1-1}{2}}(x_1\xi_1)J_{\frac{n-1+|\gamma'|}{2}-1}(r\xi_1|x'|)\,d\xi_1.$$

To calculate the internal integral, apply the formula 2.12.38.2 from [68], p. 194 of the form

$$\int\limits_0^\infty x^{a-1}\,e^{-px}\,J_\mu(bx)\,J_\nu(cx)dx=\frac{b^\mu c^\nu}{2^{\mu+\nu}p^{a+\mu+\nu}}\frac{\Gamma(a+\mu+\nu)}{\Gamma(\mu+1)\Gamma(\nu+1)}\times$$

$$\times F_4\left(\frac{a+\mu+\nu}{2},\frac{a+\mu+\nu+1}{2};\mu+1,\nu+1;-\frac{b^2}{p^2},-\frac{c^2}{p^2}\right),$$

$$\mathrm{R}e\,(a+\mu+\nu)>0;\ \mathrm{R}e\,p>0.$$

We have

$$a=\frac{n+|\gamma|}{2}+\alpha+2,\ \ p=\delta\sqrt{1+r^2},\ \ \mu=\frac{\gamma_1-1}{2},$$

$$\nu=\frac{n-1+|\gamma'|}{2}-1,\ \ b=x_1,\ \ c=r|x'|$$

and

$$\int\limits_0^\infty \xi_1^{\frac{n+|\gamma|}{2}+\alpha+1}e^{-\delta\xi_1\sqrt{1+r^2}}J_{\frac{\gamma_1-1}{2}}(x_1\xi_1)J_{\frac{n-1+|\gamma'|}{2}-1}(r\xi_1|x'|)\,d\xi_1=$$

$$=\frac{x_1^{\frac{\gamma_1-1}{2}}(r|x'|)^{\frac{n+|\gamma'|-3}{2}}}{2^{\frac{n+|\gamma|}{2}-2}(\delta\sqrt{1+r^2})^{n+|\gamma|+\alpha}}\frac{\Gamma(n+|\gamma|+\alpha)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times$$

$$\times F_4\left(\frac{\beta}{2},\frac{\beta+1}{2};\frac{\gamma_1+1}{2},\frac{n+|\gamma'|-1}{2};-\frac{x_1^2}{\delta^2(1+r^2)},-\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right),$$

where $\beta=n+|\gamma|+\alpha$. Then

$$J_1=\frac{2^{\frac{|\gamma|-n}{2}}\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{x_1^{\frac{\gamma_1-1}{2}}|x'|^{\frac{n-1+|\gamma'|}{2}-1}}\int\limits_0^1 r^{\frac{n+|\gamma'|-1}{2}}\frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1-r^2+i\varepsilon(1+r^2))^m}dr\times$$

$$\times\frac{x_1^{\frac{\gamma_1-1}{2}}(r|x'|)^{\frac{n+|\gamma'|-3}{2}}}{2^{\frac{n+|\gamma|}{2}-2}(\delta\sqrt{1+r^2})^{n+|\gamma|+\alpha}}\frac{\Gamma(n+|\gamma|+\alpha)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times$$

$$\times F_4\left(\frac{\beta}{2},\frac{\beta+1}{2};\frac{\gamma_1+1}{2},\frac{n+|\gamma'|-1}{2};-\frac{x_1^2}{\delta^2(1+r^2)},-\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right).$$

$$=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\delta^{\beta}}\frac{\Gamma(n+|\gamma|+\alpha)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times$$

$$\times\int\limits_0^1 r^{n+|\gamma'|-2}\frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1+r^2)^{\frac{n+|\gamma|+\alpha}{2}}(1-r^2+i\varepsilon(1+r^2))^m}\times$$

$$\times F_4\left(\frac{\beta}{2},\frac{\beta+1}{2};\frac{\gamma_1+1}{2},\frac{n+|\gamma'|-1}{2};-\frac{x_1^2}{\delta^2(1+r^2)},-\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right)dr.$$

Similarly, we find

$$J_2=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\delta^{\beta}}\frac{\Gamma(\beta)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times$$

$$\times\int\limits_1^{\infty} r^{n+|\gamma'|-2}\frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1+r^2)^{\frac{n+|\gamma|+\alpha}{2}}(1-r^2+i\varepsilon(1+r^2))^m}\times$$

$$\times F_4\left(\frac{\beta}{2},\frac{\beta+1}{2};\frac{\gamma_1+1}{2},\frac{n+|\gamma'|-1}{2};-\frac{x_1^2}{\delta^2(1+r^2)},-\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right)dr.$$

Multiplying by the corresponding constants, adding $J_1(x)$ with $J_2(x)$ and taking into account that

$$(1-r^2\mp i0)^{m+\frac{\alpha}{2}}=(1-r^2)_+^{m+\frac{\alpha}{2}}+e^{\mp(m+\frac{\alpha}{2})\pi i}(1-r^2)_-^{m+\frac{\alpha}{2}}$$

we obtain the statement of the Theorem. □

4.4 Belonging of the $(I^{\alpha}_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta}$ to the Class L^{γ}_{P}

Consider a convolution operator

$$Af=(T*f)_{\gamma},\qquad f\in S_{ev}. \tag{63}$$

In the images of Hankel transform we can write

$$\mathbf{F}_\gamma[Af] = \mathbf{F}_\gamma[T] \cdot \mathbf{F}_\gamma[f].$$

Definition 6 Let $M \in S'_{ev}$. The weighted generalized function is called **B-multiplier in** L_p^γ, if for all $f \in S_{ev}$ the generalized convolution $(\mathbf{F}_\gamma^{-1} M * f)_\gamma$ belongs to L_p^γ and the supremum

$$\sup_{||f||_{p,\gamma}=1} ||(\mathbf{F}_\gamma^{-1} M * f)_\gamma||_{p,\gamma} \tag{64}$$

is finite. Linear space of all such M is denoting by the $M_{p,\gamma} = M_{p,\gamma}(\mathbb{R}^n_+)$. Norm in $M_{p,\gamma}$ is the supremum (64).

Consider a singular differential operator

$$(D_B)^{\beta_i}_{x_i} = \begin{cases} B_{\gamma_i}^{\frac{\beta_i}{2}} & , \ \beta = 0, 2, 4, \ldots, \\ D_{x_i} B_{\gamma_i}^{\frac{\beta_i - 1}{2}} & , \ \beta = 1, 3, 5, \ldots, \end{cases}$$

where $B_{\gamma_i} = \frac{\partial^2}{\partial x_i} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}$.

In the article [79] proved the following criterion of B-multiplier of the type of Mikhlin criterion.

Theorem 9 *Let $M(\xi) \in C^k_{ev}(\mathbb{R}^n_+) \backslash \{0\}$, where k is even number grate then $\frac{n+|\gamma|}{2}$ and there is a constant A which does not depend on $\beta = (\beta_1, \ldots, \beta_m)$, $|\beta| < k$, such that for $\xi \neq 0$, $\xi \in \overline{\mathbb{R}}^n_+$ the condition*

$$\left| \xi^\beta \, (D_B)^\beta_\xi \, M(\xi) \right| \leq A$$

is valid Then $M(\xi)$ is B-multiplier for $1 < p < \infty$.

Lemma 4 *Let $\varepsilon, \delta > 0$ are fixed numbers and $m \geq n + |\gamma| - \frac{\alpha}{2}$. Function*

$$M^{\mp}_{\alpha,\varepsilon,\delta}(\xi) = \begin{cases} \frac{(P \mp i0)^{m + \frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m}, & P(\xi) \neq 0; \\ 0, & P(\xi) = 0 \end{cases}$$

is B-multiplier for $1 < p < \infty$.

Proof We prove the estimate

$$\left| \xi_1^{\beta_1} \ldots \xi_n^{\beta_n} (D_B)^{\beta_1}_{\xi_1} \ldots (D_B)^{\beta_n}_{\xi_n} M^{\mp}_{\alpha,\varepsilon,\delta}(\xi) \right| \leq C(\varepsilon, \delta). \tag{65}$$

For $\xi \notin V = \{\xi \in \mathbb{R}^n_+ : P(\xi) = 0\}$ we have

$$|(D_B)^j_\xi (P \mp i0)^{m+\frac{\alpha}{2}}| \le C_1|\xi^j| \cdot |P(\xi)|^{m+\frac{\alpha}{2}-|j|},$$

$$|(D_B)^k_\xi (P(\xi) + i\varepsilon|\xi|^2)^{-m}| \le C_2|\xi^k| \cdot |P^2(\xi) + \varepsilon^2|\xi|^4|^{-\frac{m+|k|}{2}},$$

$$|(D_B)^r_\xi e^{-\delta|\xi|}| \le C_3|\xi^r| \cdot \frac{e^{-\delta|\xi|}}{|\xi|^{2r-1}}.$$

Using these estimates and the formula of the type of Leibniz formula for B-differentiation of the following form (see [80]):

$$B^l_i(u\,v) = \sum_{k=0}^{2l} C^k_{2l}\left(D^{2l-k}_{B_i}u\right)\left(D^k_{B_i}v\right) + \sum_{m=1}^{2l-2}\frac{1}{x_i^m}\mathbf{P}_{2l-m}\left(D_{B_i}v;\, D_{B_i}u\right),$$

where

$$\mathbf{P}_{2l-m}\left(D_{B_i}v;\, D_{B_i}u\right) = \sum_{j=1}^{2l-v-1} a_{2l-m-j,j}(\gamma_j)\left(D^{2l-m-j}_{B_i}u\right)\left(D^j_{B_i}v\right),$$

we get the required estimate (65).

If $\xi \in V$ then the estimate (65) follows from the continuity of the function $M^{\mp}_{\alpha,\varepsilon,\delta}(\xi)$ and its derivatives on V. □

Lemma 5 *Function* ${}^{\mp}g^\alpha_{\varepsilon,\delta}(x)$ *belongs to space* L^γ_p, $1 < p < \infty$.

Proof Since the function

$${}^{\mp}g^\alpha_{\varepsilon,\delta}(t) = \mathbf{F}^{-1}_\gamma \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|x|}}{(P(x) + i\varepsilon|x|^2)^m}$$

is representable by an operator generated by the B-multiplier $M^{\mp}_{\alpha,\varepsilon,\delta}(\xi)$ in L^γ_p then ${}^{\mp}g^\alpha_{\varepsilon,\delta} \in L^\gamma_p$. □

Lemma 6 *Let* $f \in S_{ev}$. *The operator*

$$(I^\alpha_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta} f(x) = \int\limits_{\mathbb{R}^n_+} {}^{\mp}g^\alpha_{\varepsilon,\delta}(t)({}^\gamma\mathbf{T}^t_x f(x)) t^\gamma dt$$

is bounded in L^γ_p, $1 < p < \infty$.

Proof By definition of the operator

$$(I^{\alpha}_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta}f=\left(\left(\mathbf{F}^{-1}_{\gamma}\frac{(P\mp i0)^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\right)(x)*f(x)\right)_{\gamma}$$

it is a generalized convolution $(\mathbf{F}^{-1}_{\gamma}M^{\mp}_{\alpha,\varepsilon,\delta}*f)_{\gamma}$ with the B-multiplier $M^{\mp}_{\alpha,\varepsilon,\delta}(\xi)$ therefore belongs to L^{γ}_{p}. □

4.5 Theorems About the Inversion of the Hyperbolic B-Potential

Lemma 7 *Let* $f\in\Phi^{\gamma}_{V}$, $V=\{\xi\in\mathbb{R}^n_+:P(\xi)=0\}$ *then*

$$((I^{\alpha}_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta}I^{\alpha}_{P\pm i0,\gamma}f)(x)=(\mathbf{P}_{\gamma,\delta}f)(x)+\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\sum_{k=0}^{m}C^k_m(-i\varepsilon)^k(\mathbf{A}^{\gamma,\delta,\varepsilon}_k f)(x),$$

where $(\mathbf{P}_{\gamma,\delta}f)(x)$ *is a generalized convolution with the Poisson kernel* (61)

$$(\mathbf{A}^{\gamma,\delta,\varepsilon}_k f)(x)=(A^{\gamma,\delta,\varepsilon}_k(x)*f(x))_{\gamma},\qquad A^{\gamma,\delta,\varepsilon}_k(x)=\int\limits_{\mathbb{R}^n_+}\frac{|\xi|^{2k}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k}\mathbf{j}_{\gamma}(x,\xi)\xi^{\gamma}d\xi.$$

Proof Let $I^{\alpha}_{P\pm i0,\gamma}f=g$. We have

$$\mathbf{F}_{\gamma}((I^{\alpha}_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta}g)(x)=\mathbf{F}_{\gamma}\left(\left(\mathbf{F}^{-1}_{\gamma}\frac{(P\mp i0)^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\right)(x)*g(x)\right)_{\gamma}=$$

$$=\frac{(P\mp i0)^{m+\frac{\alpha}{2}}_{\gamma}e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}\cdot\mathbf{F}_{\gamma}g=\frac{(P\mp i0)^{m+\frac{\alpha}{2}}_{\gamma}e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}\cdot(P\mp i0)^{-\frac{\alpha}{2}}_{\gamma}\mathbf{F}_{\gamma}f=$$

$$=\frac{(P\mp i0)^{m}_{\gamma}e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}\cdot\mathbf{F}_{\gamma}f.$$

Then

$$((I^{\alpha}_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta}I^{\alpha}_{P\pm i0,\gamma}f)(x)=\mathbf{F}^{-1}_{\gamma}\left(\frac{(P\mp i0)^{m}_{\gamma}e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}\cdot\mathbf{F}_{\gamma}f\right)=$$

$$=\left(\mathbf{F}^{-1}_{\gamma}\frac{(P\mp i0)^{m}_{\gamma}e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}*f\right)_{\gamma}.\qquad(66)$$

Applying to $\mathbf{F}_\gamma^{-1}\frac{(P\mp i0)^m_\gamma e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}$ the Newton's binomial formula we obtain

$$\mathbf{F}_\gamma^{-1}\frac{(P\mp i0)^m_\gamma e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\left[\int\limits_{\{\xi_1>|\xi'|\}^+}\frac{(\xi_1^2-|\xi'|^2)^m e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi+\right.$$

$$\left.+e^{\mp m\pi i}\int\limits_{\{\xi_1<|\xi'|\}^+}\frac{(|\xi'|^2-\xi_1^2)^m e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi\right]=$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\left[\int\limits_{\{\xi_1>|\xi'|\}^+}\left((1-\frac{i\varepsilon|\xi|^2}{P(\xi)+i\varepsilon|\xi|^2}\right)^m e^{-\delta|\xi|}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi+\right.$$

$$\left.+e^{\mp m\pi i}(-1)^m\int\limits_{\{\xi_1<|\xi'|\}^+}\left((1-\frac{i\varepsilon|\xi|^2}{P(\xi)+i\varepsilon|\xi|^2}\right)^m e^{-\delta|\xi|}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi\right]=$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\sum_{k=0}^m C_m^k(-i\varepsilon)^k\left[\int\limits_{\{\xi_1>|\xi'|\}^+}\frac{|\xi|^{2k}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi+\right.$$

$$\left.+\int\limits_{\{\xi_1<|\xi'|\}^+}\frac{|\xi|^{2k}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi\right]=$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\sum_{k=0}^m C_m^k(-i\varepsilon)^k\int\limits_{\mathbb{R}^n_+}\frac{|\xi|^{2k}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi.$$

For $m=0$ applying of (57) gives

$$(\mathbf{F}_\gamma^{-1}e^{-\delta|\xi|})(x)=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\frac{2^{|\gamma|}\delta\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|x|^2)^{\frac{n+|\gamma|+1}{2}}}=$$

$$=\frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^n\Gamma\left(\frac{\gamma_j+1}{2}\right)}\delta\,(\delta^2+|x|^2)^{-\frac{n+|\gamma|+1}{2}}=P_\gamma(x,\delta). \tag{67}$$

Here $P_\gamma(x,\delta)$ is general Poisson kernel (59). By the Lemma 2 $P_\gamma(x,\delta)\in L_p^\gamma$.
Introducing the notation

$$A_k^{\gamma,\delta,\varepsilon}(x)=\int\limits_{\mathbb{R}_+^n}\frac{|\xi|^{2k}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi=\mathbf{F}_\gamma\frac{|x|^{2k}e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^k}$$

for $m>0$ we get

$$\mathbf{F}_\gamma^{-1}\frac{(P\mp i0)_\gamma^m e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\sum_{k=0}^m C_m^k(-i\varepsilon)^k A_k^{\gamma,\delta,\varepsilon}(x). \tag{68}$$

Substituting (67) and (68) in (66) we obtain the statement of the theorem for $f\in\Phi_V^\gamma$. □

Theorem 10 *Let* $f\in\Phi_V^\gamma$, $V=\{\xi\in\mathbb{R}_+^n:P(\xi)=0\}$, $1<p<\frac{n+|\gamma|}{\alpha}$, $p\le 2$, $n+|\gamma|-2<\alpha<n+|\gamma|$, *then*

$$((I_{P\pm i0,\gamma}^\alpha)^{-1}I_{P\pm i0,\gamma}^\alpha f)(x)=f(x),$$

where

$$(I_{P\pm i0,\gamma}^\alpha)^{-1}f)=\overset{L_p^\gamma}{\lim_{\delta\to0}}\overset{L_2^\gamma}{\lim_{\varepsilon\to0}}\left(\left(\mathbf{F}_\gamma^{-1}\frac{(P\mp i0)^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\right)(x)*f(x)\right)_\gamma$$

here the limit by ε *is understood by norm in* L_2^γ *and the limit by* δ *is understood by norm in* L_p^γ.

Proof From Lemma 7 follows that it is enough to show

$$\overset{L_p^\gamma}{\lim_{\delta\to0}}\overset{L_2^\gamma}{\lim_{\varepsilon\to0}}\left[(\mathbf{P}_{\gamma,\delta}f)(x)+\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\sum_{k=0}^m C_m^k(-i\varepsilon)^k(\mathbf{A}_k^{\gamma,\delta,\varepsilon}f)(x)\right]=f(x).$$

Find the limit for ε in L_2^γ. We have

$$(\mathbf{A}_k^{\gamma,\delta,\varepsilon}f)(x)=(A_k^{\gamma,\delta,\varepsilon}(x)*f(x))_\gamma=$$

$$=\int\limits_{\mathbb{R}_+^n}\mathbf{F}_\gamma\left[\frac{|x|^{2k}e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^k}\right](y)({}^\gamma\mathbf{T}_x^y f)(x)y^\gamma\,dy=$$

$$=\int\limits_{\mathbb{R}^n_+}\mathbf{F}_\gamma\left[\frac{|x|^{2k}e^{-\frac{\delta}{2}|x|}}{(P(x)+i\varepsilon|x|^2)^k}\,e^{-\frac{\delta}{2}|x|}\right](y)({}^\gamma\mathbf{T}^y_x f)(x)y^\gamma\,dy=$$

$$=\int\limits_{\mathbb{R}^n_+}\mathbf{F}_\gamma\left[\frac{|x|^{2k}e^{-\frac{\delta}{2}|x|}}{(P(x)+i\varepsilon|x|^2)^k}\mathbf{F}_\gamma\left[P_\gamma\left(z,\frac{\delta}{2}\right)\right](x)\right](y)({}^\gamma\mathbf{T}^y_x f)(x)y^\gamma\,dy.$$

Using Parseval Equation to Hankel transform (see [60], p. 20) we obtain

$$||(-i\varepsilon)^k(\mathbf{A}_k^{\gamma,\delta,\varepsilon}f)(x)||^2_{2,\gamma}=||(A_k^{\gamma,\delta,\varepsilon}(x)*f(x))_\gamma||^2_{2,\gamma}=||\mathbf{F}_\gamma A_k^{\gamma,\delta,\varepsilon}(x)\cdot\mathbf{F}_\gamma f(x))_\gamma||^2_{2,\gamma}=$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\left|\frac{(-i\varepsilon)^k|x|^{2k}e^{-\frac{\delta}{2}|x|}}{(P(x)+i\varepsilon|x|^2)^k}\mathbf{F}_\gamma\left[P_\gamma\left(x,\frac{\delta}{2}\right)\right]\mathbf{F}_\gamma f(x)\right|^2x^\gamma dx=$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\left|\frac{(-i\varepsilon)^k|x|^{2k}e^{-\frac{\delta}{2}|x|}}{(P(x)+i\varepsilon|x|^2)^k}\mathbf{F}_\gamma\left[(\mathbf{P}_{\gamma,\delta}f)(x)\right]\right|^2x^\gamma dx.$$

Considering that

$$\left|\frac{(-i\varepsilon)^k|x|^{2k}e^{-\frac{\delta}{2}|x|}}{(P(x)+i\varepsilon|x|^2)^k}\mathbf{F}_\gamma\left[(\mathbf{P}_{\gamma,\delta}f)(x)\right]\right|^2\leq e^{-\delta|x|}\left|\mathbf{F}_\gamma\left[(\mathbf{P}_{\gamma,\delta}f)(x)\right]\right|^2$$

and $e^{-\delta|x|}\left|\mathbf{F}_\gamma\left[P_\gamma\left(x,\frac{\delta}{2}\right)\right]\right|^2\in L_1^\gamma$ on the basis of the Lebesgue dominated convergence theorem, we obtain that

$$(-i\varepsilon)^k(\mathbf{A}_k^{\gamma,\delta,\varepsilon}f)(x)\to 0\qquad\text{for}\qquad\varepsilon\to 0\qquad\text{in}\qquad L_2^\gamma.$$

The fact that

$$||(\mathbf{P}_{\gamma,\delta}f)(x)-f(x)||_{p,\gamma}\to 0\qquad\text{for}\qquad\delta\to 0$$

was proved in the Lemma 3. Thus, the theorem is proved. □

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One-Dimensional and Multi-Dimensional Integral Transforms of Buschman–Erdélyi Type with Legendre Functions in Kernels

Sergei M. Sitnik and Oksana V. Skoromnik

Abstract This paper consists of two parts. In the first part we give a brief survey of results on Buschman–Erdélyi operators, which are transmutations for the Bessel singular operator. Main properties and applications of Buschman–Erdélyi operators are outlined. In the second part of the paper we consider multi-dimensional integral transforms of Buschman–Erdélyi type with Legendre functions in kernels. Complete proofs are given in this part, main tools are based on Mellin transform properties and usage of Fox H-functions.

Keywords Buschman–Erdélyi operators · Multidimensional Buschman–Erdélyi operators · Transmutations · Mellin transform · Fox H-function

MSC 35A22

1 Buschman–Erdélyi Operators

For a given pair of operators (A, B) an operator T is called transmutation (or intertwining) operator if on elements of some functional spaces the following property is valid

$$T\,A = B\,T. \tag{1}$$

And how the transmutations usually works? Suppose we study properties for a rather complicated operator A. But suppose also that we know the corresponding properties for a model more simple operator B and transmutation (1) readily exists.

S. M. Sitnik
Belgorod State National Research University (BelGU), Belgorod, Russia
e-mail: sitnik@bsu.edu.ru

O. V. Skoromnik (✉)
Polotsk State University "PSU", Novopolotsk, Belarus
e-mail: post@psu.by

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_13

Then we usually may copy results for the model operator B to corresponding ones for the more complicated operator A. This is shortly the main idea of transmutations.

Let us consider for example an equation $Au = f$, then applying to it a transmutation with property (1) we consider a new equation $Bv = g$, with $v = Tu$, $g = Tf$. So if we can solve the simpler equation $Bv = g$, then the initial one is also solved and has solution $u = T^{-1}v$. Of course, it is supposed that the inverse operator exists and its explicit form is known. This is a simple application of the transmutation technique for finding and proving formulas for solutions of ordinary and partial differential equations.

The monographs [2, 6–8, 17, 23, 57, 59] are completely devoted to the transmutation theory and its applications, note also author's survey [50]. Moreover, essential parts of monographs [9, 12, 24, 30–32, 34–39, 45, 60], include material on transmutations, the complete list of books which investigate some transmutational problems is now near of 100 items.

The term "Buschman–Erdélyi transmutations" was introduced by the author and is now accepted. Integral equations with these operators were studied in mid-1950th. The author was first to prove the transmutational nature of these operators. The classical Sonine and Poisson operators are special cases of the Buschman–Erdélyi transmutations and Sonine–Dimovski and Poisson–Dimovski transmutations are their generalizations for the hyper-Bessel equations and functions.

The Buschman–Erdélyi transmutations have many modifications. The author introduced convenient classification of them. Due to this classification we introduce Buschman–Erdélyi transmutations of the first kind, their kernels are expressed in terms of Legendre functions of the first kind. In the limiting case we define Buschman–Erdélyi transmutations of zero order smoothness being important in applications. The kernels of Buschman–Erdélyi transmutations of the second kind are expressed in terms of Legendre functions of the second kind. Some combination of operators of the first kind and the second kind leads to operators of the third kind. For the special choice of parameters they are unitary operators in the standard Lebesgue space. The author proposed the terms "Sonine–Katrakhov" and "Poisson–Katrakhov" transmutations in honor of V. Katrakhov who introduced and studied these operators.

The study of integral equations and invertibility for the Buschman–Erdélyi operators was started in 1960-th by P. Buschman and A. Erdélyi, [4, 5, 14, 15]. These operators also were investigated by Higgins, Ta Li, Love, Habibullah, K. N. Srivastava, Ding Hoang An, Smirnov, Virchenko, Fedotova, Kilbas, Skoromnik and others. During this period, for this class of operators were considered only problems of solving integral equations, factorization and invertibility, cf. [44].

The most detailed study of the Buschman–Erdélyi transmutations was taken by the author in 1980–1990th [20, 46, 47] and continued in [19–22, 46–49, 51–56] and some other papers. Interesting and important results were proved by N. Virchenko and A. Kilbas and their disciples [26, 27, 61].

Let us first consider the most well-known transmutations for the Bessel operator and the second derivative:

$$T\left(B_\nu\right) f=\left(D^2\right) T f, \; B_\nu=D^2+\frac{2\nu+1}{x} D, \; D^2=\frac{d^2}{d x^2}, \quad \nu \in \mathbb{C}. \tag{2}$$

Definition 1 The Poisson transmutation is defined by

$$P_\nu f=\frac{1}{\Gamma(\nu+1) 2^\nu x^{2\nu}} \int_0^x\left(x^2-t^2\right)^{\nu-\frac{1}{2}} f(t)\, d t, \;\; \Re \nu>-\frac{1}{2}. \tag{3}$$

Respectively, the Sonine transmutation is defined by

$$S_\nu f=\frac{2^{\nu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-\nu)} \frac{d}{d x} \int_0^x\left(x^2-t^2\right)^{-\nu-\frac{1}{2}} t^{2\nu+1} f(t)\, d t, \;\; \Re \nu<\frac{1}{2}. \tag{4}$$

The operators (3)–(4) intertwine by the formulas

$$S_\nu B_\nu=D^2 S_\nu, \quad P_\nu D^2=B_\nu P_\nu. \tag{5}$$

The definition may be extended to $\nu \in \mathbb{C}$. We will use more historically exact term as the Sonine–Poisson–Delsarte transmutations [50].

An important generalization for the Sonine–Poisson–Delsarte are the transmutations for the hyper-Bessel operators and functions. Such functions were first considered by Kummer and Delerue. The detailed study on these operators and hyper-Bessel functions was done by Dimovski and further, by Kiryakova. The corresponding transmutations have been called by Kiryakova [31] as the Sonine–Dimovski and Poisson–Dimovski transmutations. In hyper-Bessel operators theory the leading role is for the Obrechkoff integral transform [10, 11, 13, 31]. It is a transform with Meijer's G-function kernel which generalizes the Laplace, Meijer and many other integral transforms introduced by different authors. Various results on the hyper-Bessel functions, connected equations and transmutations were many times reopened. The same is true for the Obrechkoff integral transform. It my opinion, the Obrechkoff transform together with the Laplace, Fourier, Mellin, Stankovic transforms are essential basic elements from which many other transforms are constructed with corresponding applications.

Let us define and study some main properties of the Buschman–Erdélyi transmutations of the first kind. This class of transmutations for some choice of parameters generalize the Sonine–Poisson–Delsart transmutations, Riemann–Liouville and Erdélyi–Kober fractional integrals, Mehler–Fock transform.

Definition 2 Define the Buschman–Erdélyi operators of the first kind by

$$B_{0+}^{\nu,\mu} f = \int_0^x \left(x^2 - t^2\right)^{-\frac{\mu}{2}} P_\nu^\mu \left(\frac{x}{t}\right) f(t) d\,t, \tag{6}$$

$$E_{0+}^{\nu,\mu} f = \int_0^x \left(x^2 - t^2\right)^{-\frac{\mu}{2}} \mathbb{P}_\nu^\mu \left(\frac{t}{x}\right) f(t) d\,t, \tag{7}$$

$$B_{-}^{\nu,\mu} f = \int_x^\infty \left(t^2 - x^2\right)^{-\frac{\mu}{2}} P_\nu^\mu \left(\frac{t}{x}\right) f(t) d\,t, \tag{8}$$

$$E_{-}^{\nu,\mu} f = \int_x^\infty \left(t^2 - x^2\right)^{-\frac{\mu}{2}} \mathbb{P}_\nu^\mu \left(\frac{x}{t}\right) f(t) d\,t. \tag{9}$$

Here $P_\nu^\mu(z)$ is the Legendre function of the first kind, $\mathbb{P}_\nu^\mu(z)$ is this function on the cut $-1 \leq t \leq 1$ ([1]), $f(x)$ is a locally summable function with some growth conditions at $x \to 0,\ x \to \infty$. The parameters are $\mu, \nu \in \mathbb{C}$, $\Re\mu < 1$, $\Re\nu \geq -1/2$.

Now consider some main properties for this class of transmutations, following essentially [46, 47], and also [48, 50]. All functions further are defined on positive semiaxis. So we use notations L_2 for the functional space $L_2(0,\infty)$ and $L_{2,k}$ for power weighted space $L_{2,k}(0,\infty)$ equipped with norm

$$\int_0^\infty |f(x)|^2 x^{2k+1}\, dx, \tag{10}$$

$\mathbb{N}$ denotes the set of naturals, $\mathbb{N}_0$-positive integer, $\mathbb{Z}$-integer and $\mathbb{R}$-real numbers.

First, add to Definition 2 a case of parameter $\mu = 1$. It defines a very important class of operators.

Definition 3 Define for $\mu = 1$ the Buschman–Erdélyi operators of zero order smoothness by

$$B_{0+}^{\nu,1} f = {}_1S_{0+}^{\nu} f = \frac{d}{dx} \int_0^x P_\nu \left(\frac{x}{t}\right) f(t)\, dt, \tag{11}$$

$$E_{0+}^{\nu,1} f = {}_1P_{-}^{\nu} f = \int_0^x P_\nu \left(\frac{t}{x}\right) \frac{df(t)}{dt}\, dt, \tag{12}$$

$$B_{-}^{\nu,1} f = {}_1S_{-}^{\nu} f = \int_x^\infty P_\nu \left(\frac{t}{x}\right) (-\frac{df(t)}{dt})\, dt, \tag{13}$$

$$E_{-}^{\nu,1} f = {}_1P_{0+}^{\nu} f = (-\frac{d}{dx}) \int_x^\infty P_\nu \left(\frac{x}{t}\right) f(t)\, dt, \tag{14}$$

where $P_\nu(z) = P_\nu^0(z)$ is the Legendre function.

Theorem 1 *The next formulas hold true for factorizations of Buschman–Erdélyi transmutations for suitable functions via Riemann–Liouville fractional integrals and*

Buschman–Erdélyi operators of zero order smoothness:

$$B_{0+}^{\nu,\mu} f = I_{0+}^{1-\mu}\, {}_1S_{0+}^{\nu} f,\ \ B_{-}^{\nu,\mu} f = {}_1P_{-}^{\nu}\, I_{-}^{1-\mu} f, \tag{15}$$

$$E_{0+}^{\nu,\mu} f = {}_1P_{0+}^{\nu}\, I_{0+}^{1-\mu} f,\ \ E_{-}^{\nu,\mu} f = I_{-}^{1-\mu}\, {}_1S_{-}^{\nu} f. \tag{16}$$

These formulas allow to separate parameters ν and μ. We will prove soon that operators (11)–(14) are isomorphisms of $L_2(0,\infty)$ except for some special parameters. So, operators (6)–(9) roughly speaking are of the same smoothness in L_2 as integrodifferentiations $I^{1-\mu}$ and they coincide with them for $\nu = 0$. It is also possible to define Buschman–Erdélyi operators for all $\mu \in \mathbb{C}$.

Definition 4 Define the number $\rho = 1 - Re\,\mu$ as smoothness order for Buschman–Erdélyi operators (6)–(9).

So for $\rho > 0$ (otherwise for $Re\,\mu > 1$) the Buschman–Erdélyi operators are smoothing and for $\rho < 0$ (otherwise for $Re\,\mu < 1$) they decrease smoothness in L_2 spaces. Operators (11)–(14) for which $\rho = 0$ due to Definition 4 are of zero smoothness order in accordance with their definition.

For some special parameters $\nu,\ \mu$ the Buschman–Erdélyi operators of the first kind are reduced to other known operators. So for $\mu = -\nu$ or $\mu = \nu + 2$ they reduce to Erdélyi–Kober operators, for $\nu = 0$ they reduce to fractional integrodifferentiation $I_{0+}^{1-\mu}$ or $I_{-}^{1-\mu}$, for $\nu = -\frac{1}{2}$, $\mu = 0$ or $\mu = 1$ kernels reduce to elliptic integrals, for $\mu = 0$, $x = 1$, $\nu = it - \frac{1}{2}$ the operator $B_{-}^{\nu,0}$ differs only by a constant from Mehler–Fock transform.

As a pair for the Bessel operator consider a connected one

$$L_\nu = D^2 - \frac{\nu(\nu+1)}{x^2} = \left(\frac{d}{dx} - \frac{\nu}{x}\right)\left(\frac{d}{dx} + \frac{\nu}{x}\right), \tag{17}$$

which for $\nu \in \mathbb{N}$ is an angular momentum operator from quantum physics. Their transmutational relations are established in the next theorem.

Theorem 2 *For a given pair of transmutations X_ν, Y_ν*

$$X_\nu L_\nu = D^2 X_\nu,\ Y_\nu D^2 = L_\nu Y_\nu \tag{18}$$

define the new pair of transmutations by formulas

$$S_\nu = X_{\nu-1/2}\, x^{\nu+1/2},\ P_\nu = x^{-(\nu+1/2)}\, Y_{\nu-1/2}. \tag{19}$$

Then for the new pair S_ν, P_ν the next formulas are valid:

$$S_\nu B_\nu = D^2 S_\nu,\ P_\nu D^2 = B_\nu P_\nu. \tag{20}$$

Theorem 3 *Let Re* $\mu \le 1$. *Then an operator* $B_{0+}^{\nu,\,\mu}$ *on proper functions is a Sonine type transmutation and* (18) *is valid.*

The same result holds true for other Buschman–Erdélyi operators, $E_{-}^{\nu,\,\mu}$ is Sonine type and $E_{0+}^{\nu,\,\mu}$, $B_{-}^{\nu,\,\mu}$ are Poisson type transmutations.

From these transmutation connections, we conclude that the Buschman–Erdélyi operators link the corresponding eigenfunctions for the two operators. They lead to formulas for the Bessel functions via exponents and trigonometric functions, and vice versa which generalize the classical Sonine and Poisson formulas.

Now consider factorizations of the Buschman–Erdélyi operators. First let us list the main forms of fractional integrodifferentiations: Riemann–Liouville, Erdélyi–Kober, fractional integral by function $g(x)$, cf. [44],

$$I_{0+,x}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \tag{21}$$

$$I_{-,x}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt,$$

$$I_{0+,2,\eta}^{\alpha} f = \frac{2x^{-2(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^x \left(x^2-t^2\right)^{\alpha-1} t^{2\eta+1} f(t)dt, \tag{22}$$

$$I_{-,2,\eta}^{\alpha} f = \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^\infty \left(t^2-x^2\right)^{\alpha-1} t^{1-2(\alpha+\eta)} f(t)dt,$$

$$I_{0+,g}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^x (g(x)-g(t))^{\alpha-1} g'(t) f(t)dt, \tag{23}$$

$$I_{-,g}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_x^\infty (g(t)-g(x))^{\alpha-1} g'(t) f(t)dt.$$

In all cases $\Re\alpha > 0$ and the operators may be further defined for all α, see [44]. In the case of $g(x) = x$ (23) reduces to the Riemann–Liouville integral, in the case of $g(x) = x^2$ (23) reduces to the Erdélyi–Kober operator, and in the case of $g(x) = \ln x$—to the Hadamard fractional integrals.

Theorem 4 *The following factorization formulas are valid for the Buschman–Erdélyi operators of the first kind via the Riemann–Liouville and Erdélyi–Kober fractional integrals:*

$$B_{0+}^{\nu,\,\mu} = I_{0+}^{\nu+1-\mu} I_{0+;\,2,\,\nu+\frac{1}{2}}^{-(\nu+1)} \left(\frac{2}{x}\right)^{\nu+1}, \tag{24}$$

$$E_{0+}^{\nu,\,\mu} = \left(\frac{x}{2}\right)^{\nu+1} I_{0+;\,2,\,-\frac{1}{2}}^{\nu+1} I_{0+}^{-(\nu+\mu)}, \tag{25}$$

$$B_{-}^{\nu,\,\mu}=\left(\frac{2}{x}\right)^{\nu+1} I_{-;\,2,\,\nu+1}^{-(\nu+1)} I_{-}^{\nu-\mu+2}, \tag{26}$$

$$E_{-}^{\nu,\,\mu}=I_{-}^{-(\nu+\mu)} I_{-;\,2,\,0}^{\nu+1}\left(\frac{x}{2}\right)^{\nu+1}. \tag{27}$$

The Sonine–Poisson–Delsarte transmutations also are special cases for this class of operators.

Now let us study the properties of the Buschman–Erdélyi operators of zero order smoothness, defined by (11)–(14). A similar operator was introduced by Katrakhov by multiplying the Sonine operator with a fractional integral, his aim was to work with transmutation obeying good estimates in $L_2(0,\infty)$.

We use the Mellin transform defined by [40]

$$g(s)=Mf(s)=\int_0^\infty x^{s-1} f(x)\,dx. \tag{28}$$

The Mellin convolution is defined by

$$(f_1 * f_2)(x)=\int_0^\infty f_1\left(\frac{x}{y}\right) f_2(y)\,\frac{dy}{y}, \tag{29}$$

so the convolution operator with kernel K acts under the Mellin transform as a multiplication on multiplicator

$$M[Af](s)=M\left[\int_0^\infty K\left(\frac{x}{y}\right) f(y)\,\frac{dy}{y}\right](s)=M[K*f](s)=m_A(s)Mf(s), \tag{30}$$

$$m_A(s)=M[K](s).$$

We observe that the Mellin transform is a generalized Fourier transform on semiaxis with Haar measure $\frac{dy}{y}$, [18]. It plays important role for the theory of special functions, for example the gamma function is a Mellin transform of the exponential. With the Mellin transform the important breakthrough in evaluating integrals was done in 1970th when mainly by O. Marichev, the famous Slater's theorem was adapted for calculations. The Slater's theorem taking the Mellin transform as input gives the function itself as output via hypergeometric functions, see [40]. This theorem occurred to be the milestone of powerful computer method for calculating integrals for many problems in differential and integral equations. The package *Mathematica* of Wolfram Research is based on this theorem in calculating integrals.

Theorem 5 *The Buschman–Erdélyi operator of zero order smoothness* ${}_1S^{\nu}_{0+}$ *defined by (11) acts under the Mellin transform as convolution (30) with multiplicator*

$$m(s) = \frac{\Gamma(-s/2 + \frac{\nu}{2} + 1)\Gamma(-s/2 - \frac{\nu}{2} + 1/2)}{\Gamma(1/2 - \frac{s}{2})\Gamma(1 - \frac{s}{2})} \tag{31}$$

for $\Re s < \min(2 + \Re\nu, 1 - \Re\nu)$. *Its norm is a periodic in* ν *and equals*

$$\|B^{\nu,1}_{0+}\|_{L_2} = \frac{1}{\min(1, \sqrt{1 - \sin \pi\nu})}. \tag{32}$$

This operator is bounded in $L_2(0,\infty)$ *if* $\nu \neq 2k + 1/2, k \in \mathbb{Z}$ *and unbounded if* $\nu = 2k + 1/2, k \in \mathbb{Z}$.

Corollary 1 *The norms of operators* (11)–(14) *are periodic in* ν *with period 2* $\|X^{\nu}\| = \|X^{\nu+2}\|$, X^{ν} *is any of operators* (11)–(14).

Corollary 2 *The norms of the operators* ${}_1S^{\nu}_{0+}$, ${}_1P^{\nu}_{-}$ *are not bounded in general, every norm is greater or equals to 1. The norms are equal to 1 if* $\sin \pi\nu \leq 0$. *The operators* ${}_1S^{\nu}_{0+}$, ${}_1P^{\nu}_{-}$ *are unbounded in* L_2 *if and only if* $\sin \pi\nu = 1$ *(or* $\nu = (2k) + 1/2,\ k \in \mathbb{Z}$*).*

Corollary 3 *The norms of the operators* ${}_1P^{\nu}_{0+}$, ${}_1S^{\nu}_{-}$ *are all bounded in* ν, *every norm is not greater then* $\sqrt{2}$. *The norms are equal to 1 if* $\sin \pi\nu \geq 0$. *The operators* ${}_1P^{\nu}_{0+}$, ${}_1S^{\nu}_{-}$ *are bounded in* L_2 *for all* ν. *The maximum of norm equals* $\sqrt{2}$ *is achieved if and only if* $\sin \pi\nu = -1$ *(или* $\nu = -1/2 + (2k),\ k \in \mathbb{Z}$*).*

The most important property of the Buschman–Erdélyi operators of zero order smoothness is the unitarity for integer ν. It is just the case if we interpret for these parameters the operator L_ν as angular momentum operator in quantum mechanics.

Theorem 6 *The operators* (11)–(14) *are unitary in* L_2 *if and only if the parameter* ν *is an integer. In this case the pairs of operators* $({}_1S^{\nu}_{0+}, {}_1P^{\nu}_{-})$ *and* $({}_1S^{\nu}_{-}, {}_1P^{\nu}_{0+})$ *are mutually inverse.*

To formulate an interesting special case, let us suppose that operators (11)–(14) act on functions permitting outer or inner differentiation in integrals, it is enough to suppose that $xf(x) \to 0$ for $x \to 0$. Then for $\nu = 1$

$${}_1P^{1}_{0+} f = (I - H_1) f,\ {}_1S^{1}_{-} f = (I - H_2) f, \tag{33}$$

and H_1, H_2 are the famous Hardy operators,

$$H_1 f = \frac{1}{x}\int\limits_0^x f(y)dy, \; H_2 f = \int\limits_x^\infty \frac{f(y)}{y}dy, \tag{34}$$

I is the identic operator.

Corollary 4 *The operators (33) are unitary in L_2 and mutually inverse. They are transmutations for the pair of differential operators d^2/dx^2 and $d^2/dx^2 - 2/x^2$.*

The unitarity of the shifted Hardy operators (33) in L_2 is a known fact [33]. Below in application section, we introduce a new class of generalizations for the classical Hardy operators.

Now we list some properties of the operators acting as convolutions by the formula (30) and with some multiplicator under the Mellin transform and being transmutations for the second derivative and angular momentum operator in quantum mechanics.

Theorem 7 *Let an operator S_ν act by formulas (30) and (18). Then:*

(a) its multiplicator satisfies a functional equation

$$m(s) = m(s-2)\frac{(s-1)(s-2)}{(s-1)(s-2)-\nu(\nu+1)}; \tag{35}$$

(b) if any function $p(s)$ is periodic with period 2 ($p(s) = p(s-2)$), then a function $p(s)m(s)$ is a multiplicator for a new transmutation operator S_2^ν also acting by the rule (18).

This theorem confirms the importance of studying transmutations in terms of the Mellin transform and multiplicator functions.

Define the Stieltjes transform by (cf. [44])

$$(Sf)(x) = \int\limits_0^\infty \frac{f(t)}{x+t}dt.$$

This operator also acts by the formula (30) with multiplicator $p(s) = \pi/\sin(\pi s)$, it is bounded in L_2. Obviously $p(s) = p(s-2)$. So from Theorem 7 it follows a convolution of the Stieltjes transform with bounded transmutations (11)–(14), also transmutations of the same class bounded in L_2.

In this way many new classes of transmutations were introduced with special functions as kernels.

Now we construct transmutations which are unitary for all ν. They are defined by formulas

$$S_U^\nu f = -\sin\frac{\pi\nu}{2}\, {}_2S^\nu f + \cos\frac{\pi\nu}{2}\, {}_1S_-^\nu f, \tag{36}$$

$$P_U^\nu f = -\sin\frac{\pi\nu}{2}\, {}_2P^\nu f + \cos\frac{\pi\nu}{2}\, {}_1P_-^\nu f. \tag{37}$$

For all values $\nu \in \mathbb{R}$ they are linear combinations of Buschman–Erdélyi transmutations of the first and second kinds of zero order smoothness. Also they are in the defined below class of Buschman–Erdélyi transmutations of the third kind. The following integral representations are valid:

$$S_U^\nu f = \cos\frac{\pi\nu}{2}\left(-\frac{d}{dx}\right)\int_x^\infty P_\nu\left(\frac{x}{y}\right) f(y)\,dy \tag{38}$$

$$+\frac{2}{\pi}\sin\frac{\pi\nu}{2}\left(\int_0^x (x^2-y^2)^{-\frac{1}{2}} Q_\nu^1\left(\frac{x}{y}\right) f(y)\,dy - \int_x^\infty (y^2-x^2)^{-\frac{1}{2}} \mathbb{Q}_\nu^1\left(\frac{x}{y}\right) f(y)\,dy\right),$$

$$P_U^\nu f = \cos\frac{\pi\nu}{2}\int_0^x P_\nu\left(\frac{y}{x}\right)\left(\frac{d}{dy}\right) f(y)\,dy \tag{39}$$

$$-\frac{2}{\pi}\sin\frac{\pi\nu}{2}\left(-\int_0^x (x^2-y^2)^{-\frac{1}{2}} \mathbb{Q}_\nu^1\left(\frac{y}{x}\right) f(y)\,dy - \int_x^\infty (y^2-x^2)^{-\frac{1}{2}} Q_\nu^1\left(\frac{y}{x}\right) f(y)\,dy\right).$$

Theorem 8 *The operators* (36)–(37)*,* (38)–(39) *for all $\nu \in \mathbb{R}$ are unitary, mutually inverse and conjugate in L_2. They are transmutations acting by* (17)*. S_U^ν is a Sonine type transmutation and P_U^ν is a Poisson type one.*

Transmutations like (38)–(39) but with kernels in more complicated form with hypergeometric functions were first introduced by Katrakhov in 1980. Due to this, the author proposed terms for this class of operators as Sonine–Katrakhov and Poisson–Katrakhov. In author's papers these operators were reduced to more simple form of Buschman–Erdélyi ones. It made possible to include this class of operators in general composition (or factorization) method [20, 21, 49].

2 Multi-Dimensional Integral Transforms of Buschman–Erdélyi Type with Legendre Functions in Kernels

In this part we consider generalisations of Buschman–Erdélyi operators for multi-dimensional case.

First introduce integral transforms:

$$\left(\mathrm{H}^{1}_{\sigma,\kappa} f\right)(\mathbf{x}) = \mathbf{x}^{\sigma} \int_{\mathbf{0}}^{\mathbf{x}} \mathrm{H}^{\mathbf{m},\,\mathbf{n}}_{\mathbf{p},\,\mathbf{q}}\left[\frac{\mathbf{x}}{\mathbf{t}}\middle| \begin{matrix} (\mathbf{a}_i, \alpha_i)_{1,p} \\ (\mathbf{b}_j, \beta_j)_{1,q} \end{matrix}\right] \mathbf{t}^{\kappa} f(\mathbf{t}) \frac{\mathbf{dt}}{\mathbf{t}} (\mathbf{x} > 0); \tag{40}$$

$$\left(\mathrm{P}^{\gamma}_{\delta,1} f\right)(\mathbf{x}) = \int_{\mathbf{0}}^{\mathbf{x}} \left(\mathbf{x}^2 - \mathbf{t}^2\right)^{-\gamma/2} \mathrm{P}^{\gamma}_{\delta}\left(\frac{\mathbf{x}}{\mathbf{t}}\right) f(\mathbf{t})\mathbf{dt} = g(\mathbf{x}) \ \ (\mathbf{x} > 0); \tag{41}$$

$$\left(\mathrm{P}^{\gamma}_{\delta,2} f\right)(\mathbf{x}) = \int_{\mathbf{0}}^{\mathbf{x}} \left(\mathbf{x}^2 - \mathbf{t}^2\right)^{-\gamma/2} \mathrm{P}^{\gamma}_{\delta}\left(\frac{\mathbf{t}}{\mathbf{x}}\right) f(\mathbf{t})\mathbf{dt} = g(\mathbf{x}) \ \ (\mathbf{x} > 0); \tag{42}$$

here (see [[43], Section 28.4]) $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathrm{R}^n$; $\mathbf{t} = (t_1, t_2, \dots, t_n) \in \mathrm{R}^n$, R^n Euclidean n-space; $\mathbf{x} \cdot \mathbf{t} = \sum\limits_{n=1}^{n} x_n t_n$ denotes their scalar product; in particular, $\mathbf{x} \cdot \mathbf{1} = \sum\limits_{n=1}^{n} x_n$ for $\mathbf{1}$= (1,...,1). The expression $\mathbf{x} > \mathbf{t}$ means that $x_1 > t_1, \dots, x_n > t_n$, the nonstrict inequality $\geq$ has similar meaning; $\int\limits_{\mathbf{0}}^{\mathbf{x}} = \int\limits_{\mathbf{0}}^{x_1} \int\limits_{\mathbf{0}}^{x_2} \cdots \int\limits_{\mathbf{0}}^{x_n}$; by $\mathrm{N} = \{1, 2, \dots\}$ we denote the set of positive integers, $\mathrm{N}_0 = \mathrm{N} \bigcup \{0\}$,$\mathrm{N}_0^n = \mathrm{N}_0 \times \mathrm{N}_0 \times \dots \times \mathrm{N}_0$, $\mathrm{R}^n_+ = \{\mathbf{x} \in \mathrm{R}^n,\ \mathbf{x} > 0\}$;

$\mathbf{m} = (m_1, m_2, \dots, m_n) \in \mathrm{N}_0^n$ and $m_1 = m_2 = \dots = m_n$; $\mathbf{n} = (\overline{n}_1, \overline{n}_2, \dots, \overline{n}_n) \in \mathrm{N}_0^n$ and $\overline{n}_1 = \overline{n}_2 = \dots = \overline{n}_n$; $\mathbf{p} = (p_1, p_2, \dots, p_n) \in \mathrm{N}_0$ and $p_1 = p_2 = \dots = p_n$; $\mathbf{q} = (q_1, q_2, \dots, q_n) \in \mathrm{N}_0$ and $q_1 = q_2 = \dots = q_n$) $(0 \leq \mathbf{m} \leq \mathbf{q}, 0 \leq \mathbf{n} \leq \mathbf{p})$;

$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in \mathrm{C}^n$; $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n) \in \mathrm{C}^n$;

$\delta = (\delta_1, \delta_2, \dots, \delta_n) \in \mathrm{R}^n$; $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n) \in \mathrm{R}^n$; $0 < \gamma < 1$;

$\mathbf{a}_i = (a_{i_1}, a_{i_2}, \dots, a_{i_n})$, $1 \leq i \leq p$, $a_{i_1}, a_{i_2}, \dots, a_{i_n} \in \mathrm{C}$ $(1 \leq i_1 \leq p_1, \dots, 1 \leq i_n \leq p_n)$;

$\mathbf{b}_j = (b_{j_1}, b_{j_2}, \dots, b_{j_n})$, $1 \leq j \leq q$, $b_{j_1}, b_{j_2}, \dots, b_{j_n} \in \mathrm{C}$ $(1 \leq j_1 \leq q_1, \dots, 1 \leq j_n \leq q_n)$;

$\alpha_i = (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_n})$, $1 \leq i \leq p$, $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_n} \in \mathrm{R}_1^+$ $(1 \leq i_1 \leq p_1, \dots, 1 \leq i_n \leq p_n)$;

$\beta_j = (\beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_n})$, $1 \leq j \leq q$, $\beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_n} \in \mathrm{R}_1^+$ $(1 \leq j_1 \leq q_1, \dots, 1 \leq j_n \leq q_n)$;

$\mathbf{k} = (k_1, k_2, \ldots, k_n) \in \mathrm{N}_0^n = \mathrm{N}_0 \times \ldots \times \mathrm{N}_0$ $(k_i \in \mathrm{N}_0,\ i = 1, 2, \ldots, n)$ is a multi-index with $\mathbf{k}! = k_1! \cdots k_n!$ and $|\mathbf{k}| = k_1 + k_2 + \ldots + k_n$; for $l = (l_1, l_2, \ldots, l_n) \in \mathrm{R}_+^n$
$\mathbf{D}^l = \frac{\partial^{|l|}}{(\partial x_1)^{l_1} \cdots (\partial x_n)^{l_n}}$, $\mathbf{dt} = dt_1 \cdot dt_2 \cdots dt_n$; $\mathbf{t}^l = t^{l_1} \cdots t^{l_n}$;
$\mathbf{x}^2 - \mathbf{t}^2 = (x_1^2 - t_1^2) \cdots (x_n^2 - t_n^2)$; $f(\mathbf{t}) = f(t_1, t_2, \ldots, t_n)$; we introduce the function

$$\mathrm{H}_{\mathbf{p},\,\mathbf{q}}^{\mathbf{m},\,\mathbf{n}}\left[\frac{\mathbf{x}}{\mathbf{t}}\middle| \begin{matrix} (\mathbf{a}_i, \alpha_i)_{1,p} \\ (\mathbf{b}_j, \beta_j)_{1,q} \end{matrix}\right] = \prod_{k=1}^{n} \mathrm{H}_{p_k,\,q_k}^{m_k,\,\overline{n}_k}\left[\frac{x_k}{t_k}\middle| \begin{matrix} (a_{i_k}, \alpha_{i_k})_{1,p_k} \\ (b_{j_k}, \beta_{j_k})_{1,q_k} \end{matrix}\right], \tag{43}$$

which is the product of the H-functions $\mathrm{H}_{p,\,q}^{m,\,n}[z]$. Such a function is defined by

$$\mathrm{H}_{p,\,q}^{m,n}[z] \equiv \mathrm{H}_{p,q}^{m,n}\left[z\middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right] = \frac{1}{2\pi i} \int\limits_L \mathcal{H}_{p,q}^{m,n}(s) z^{-s} ds, \ \ z \neq 0, \tag{44}$$

where

$$\mathcal{H}_{p,\,q}^{m,n}(s) \equiv \mathcal{H}_{p,\,q}^{m,n}\left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\middle| s\right] = \frac{\prod\limits_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod\limits_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod\limits_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod\limits_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)}. \tag{45}$$

Here L—is a specially chosen infinite contour and empty product, if it occurs, being taken to be one. Note that most of the elementary and special functions are special cases of the H-function (44), and one may find its properties in the books by Mathai and Saxena [41, Chapter 2], Srivastava et al. [58, Chapter 1], Prudnikov et al. [42, Section 8.3] and Kilbas and Saigo [25, Chapters 1 and 2].

We introduce the function

$$\mathrm{P}_{\delta}^{\gamma}[\mathbf{z}] = \prod_{k=1}^{n} \mathrm{P}_{\delta_k}^{\gamma_k}[z_k], \tag{46}$$

which is the product of the Legendre functions $\mathrm{P}_{\overline{\delta}}^{\overline{\gamma}}(z)$ of the first kind. For complex $\overline{\gamma}$, $Re(\overline{\gamma}) < 1$, and $\overline{\delta}, z \in \mathrm{C}$ this function is defined by

$$\mathrm{P}_{\overline{\delta}}^{\overline{\gamma}}(z) = \frac{1}{\Gamma(1 - \overline{\gamma})} \left(\frac{z+1}{z-1}\right)^{\frac{\overline{\gamma}}{2}} {}_2\mathrm{F}_1\left(-\overline{\delta},\ 1 + \overline{\delta};\ 1 - \overline{\gamma};\ \frac{1-z}{2}\right),\ |arg(z-1)| < \pi, \tag{47}$$

$$\mathrm{P}_{\overline{\delta}}^{\overline{\gamma}}(x) = \frac{1}{\overline{\Gamma}(1 - \overline{\gamma})} \left(\frac{1+x}{1-x}\right)^{\frac{\overline{\gamma}}{2}} {}_2\mathrm{F}_1\left(-\overline{\delta},\ 1 + \overline{\delta};\ 1 - \overline{\gamma};\ \frac{1-x}{2}\right), \quad -1 < x < 1, \tag{48}$$

see ([[16], Formulas 3.2(3) and 3.4(6)], [[42], Section 11.18]), where ${}_2\mathrm{F}_1(-\overline{\delta},\ 1+\overline{\delta};\ 1-\overline{\gamma};\ z)$—is the Gauss hypergeometric function [[16], Section 2.1].

Our paper is devoted to the study of transforms $\mathrm{P}^{\gamma}_{\delta,k} f$ $(k = 1, 2)$ in the weighted spaces $\mathfrak{L}_{\overline{v},\overline{2}}$ summable functions $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ on $\mathrm{R}^{\mathbf{n}}_{+}$, such that:

$$\|f\|_{\overline{v},\overline{2}} = \{\int_{R^1_+} x_n^{v_n\cdot 2-1}\{\cdot\cdot\cdot\{\int_{R^1_+} x_2^{v_2\cdot 2-1}\times$$
$$\times [\int_{R^1_+} x_1^{v_1\cdot 2-1}|f(x_1,\ldots,x_n)|^2 dx_1]dx_2\}\cdot\cdot\cdot\}dx_n\}^{1/2} < \infty \tag{49}$$

$(\overline{2} = (2, \ldots, 2), \overline{v} = (v_1, \ldots, v_n) \in \mathrm{R}^n, v_1 = v_2 = \ldots = v_n)$.

Our investigations are based on representations of Eqs. (41) and (42) via the modified H-transform of the form (40). Mapping properties such as the boundedness the range, the representation and the inversion of the considered transforms are established.

Preliminaries

Denote by $[X, Y]$ a set of bounded linear operators acting from a Banach space X into a Banach space Y.

The n-dimensional Mellin transform $(\mathfrak{M}f)(\mathbf{x})$ of a function $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathrm{R}^n_+$, is defined by

$$(\mathfrak{M}f)(\mathbf{s}) = \int_0^\infty f(\mathbf{t})\mathbf{t}^{\mathbf{s}-1}\mathbf{dt},\ \ Re(\mathbf{s}) = \overline{v}, \tag{50}$$

$\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathrm{C}^n$; while the inverse Mellin transform is given for $\mathbf{x} \in \mathrm{R}^n_+$ by the formula

$$(\mathfrak{M}^{-1}g)(\mathbf{x}) = \mathfrak{M}^{-1}[g(\mathbf{p})](\mathbf{x}) = \frac{1}{(2\pi i)^n}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\cdots\int_{\gamma_n-i\infty}^{\gamma_n+i\infty}\mathbf{x}^{-\mathbf{s}}g(\mathbf{s})\mathbf{ds}, \tag{51}$$

with $\gamma_j = Re(s_j)$ $(j = 1, \cdots, n)$. The theory for these multidimensional Mellin transforms appears in the book by Brychkov [3], see also [29, Chapter 1].

Let $\mathbf{M}_\zeta$, $\mathbf{R}$ be elementary operators (see [29, Chapter 1]):

$$(\mathbf{M}_\zeta f)(\mathbf{x}) = \mathbf{x}^\zeta f(\mathbf{x})\ (\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathrm{C}^n),\ \ (\mathbf{R}f)(\mathbf{x}) = \frac{1}{\mathbf{x}}f\left(\frac{1}{\mathbf{x}}\right). \tag{52}$$

There holds the following assertion, which follows from [29] formulas (1.4.44), (1.4.45), (1.4.46)] [25, Lemma 3.2].

Lemma 1 *Let* $\overline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in R^n$ $(\nu_1 = \nu_2 = \ldots = \nu_n)$ *and* $1 \le \overline{r} < \infty$.

(a) $\mathbf{M}_\zeta$ *is isometric isomorphism of* $\mathfrak{L}_{\overline{\nu},\overline{r}}$ *onto* $\mathfrak{L}_{\overline{\nu}-\mathrm{Re}(\zeta),\overline{r}}$ *and if* $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$ $(1 \le \overline{r} \le 2)$*, then*

$$(\mathfrak{M}\mathbf{M}_\zeta f)(\mathbf{s}) = (\mathfrak{M} f)(\mathbf{s}+\zeta) \ \ (\mathrm{Re}(\mathbf{s}) = \overline{\nu} - \mathrm{Re}(\zeta)). \tag{53}$$

(b) $\mathbf{R}$ *is an isometric isomorphism of* $\mathfrak{L}_{\overline{\nu},\overline{r}}$ *onto* $\mathfrak{L}_{1-\overline{\nu},\overline{r}}$ *and if* $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$ $(1 \le \overline{r} \le 2)$, *then*

$$(\mathfrak{M}\mathbf{R} f)(\mathbf{s}) = (\mathfrak{M} f)(1-\mathbf{s}) \ \ (\mathrm{Re}(\mathbf{s}) = \overline{\nu}). \tag{54}$$

Let $\mathrm{I}^{\alpha}_{0+;\,\sigma,\eta}$ and $\mathrm{I}^{\alpha}_{-;\,\sigma,\eta}$ be the Erdelyi-Kober operators of fractional integration, defined for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathrm{C}^n$ $(\mathrm{Re}(\alpha) > 0)$, $\sigma > 0$, $\eta \in C^n$ by:

$$\left(\mathrm{I}^{\alpha}_{0+;\,\sigma,\,\eta} f\right)(\mathbf{x}) = \frac{\sigma \mathbf{x}^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^{\mathbf{x}} \left(\mathbf{x}^\sigma - \mathbf{t}^\sigma\right)^{\alpha-1} \mathbf{t}^{\sigma\eta+\sigma-1} f(\mathbf{t})\mathbf{dt} \ (\mathbf{x} > 0), \tag{55}$$

$$\left(\mathrm{I}^{\alpha}_{-;\,\sigma,\,\eta} f\right)(\mathbf{x}) = \frac{\sigma \mathbf{x}^{\sigma\eta}}{\Gamma(\alpha)} \int_{\mathbf{x}}^{\infty} \left(\mathbf{t}^\sigma - \mathbf{x}^\sigma\right)^{\alpha-1} \mathbf{t}^{\sigma(1-\alpha-\eta)-1} f(\mathbf{t})\mathbf{dt} \ (\mathbf{x} > 0). \tag{56}$$

2.1 $\mathfrak{L}_{\overline{\nu},\overline{2}}$–*Theory and the Inversion Formulas for the Modified* **H**-*Transform*

To formulate the results presented $\mathfrak{L}_{\overline{\nu},\overline{2}}$-theory and the inversion formulas for the modified H-transform (40) we need the following constants, analogical for one-dimensional case defined via the parameters of the H-function (44) [[25], (3.4.1), (3.4.2), (1.1.7), (1.1.8), (1.1.10)]:

$$\alpha_1 = \begin{cases} -\min\limits_{1\le j_1\le m_1}\left[\frac{\mathrm{Re}(b_{j_1})}{\beta_{j_1}}\right], & m_1 > 0, \\ 0, & m_1 = 0; \end{cases} \qquad \beta_1 = \begin{cases} \min\limits_{1\le i_1\le \overline{n}_1}\left[\frac{1-\mathrm{Re}(a_{i_1})}{\alpha_{i_1}}\right], & \overline{n}_1 > 0, \\ 0, & \overline{n}_1 = 0; \end{cases}$$

$$\alpha_2 = \begin{cases} -\min\limits_{1\le j_2\le m_2}\left[\frac{\mathrm{Re}(b_{j_2})}{\beta_{j_2}}\right], & m_2 > 0, \\ 0, & m_2 = 0; \end{cases} \qquad \beta_2 = \begin{cases} \min\limits_{1\le i_2\le \overline{n}_2}\left[\frac{1-\mathrm{Re}(a_{i_2})}{\alpha_{i_2}}\right], & \overline{n}_2 > 0, \\ 0, & \overline{n}_2 = 0; \end{cases}$$

and so on

$$\alpha_n = \begin{cases} -\min\limits_{1\le j_n\le m_n}\left[\frac{\mathrm{Re}(b_{j_n})}{\beta_{j_n}}\right], & m_n > 0, \\ 0, & m_2 = 0; \end{cases} \qquad \beta_n = \begin{cases} \min\limits_{1\le i_n\le \overline{n}_n}\left[\frac{1-\mathrm{Re}(a_{i_n})}{\alpha_{i_n}}\right], & \overline{n}_n > 0, \\ 0, & \overline{n}_n = 0; \end{cases} \tag{57}$$

$$a_1^* = \sum_{i=1}^{\overline{n}_1} \alpha_{i_1} - \sum_{i=\overline{n}_1+1}^{p_1} \alpha_{i_1} + \sum_{j=1}^{m_1} \beta_{j_1} - \sum_{j=m_1+1}^{q_1} \beta_{j_1}, \quad \Delta_1 = \sum_{j=1}^{q_1} \beta_{j_1} - \sum_{i=1}^{p_1} \alpha_{i_1},$$

$$a_2^* = \sum_{i=1}^{\overline{n}_2} \alpha_{i_2} - \sum_{i=\overline{n}_2+1}^{p_2} \alpha_{i_2} + \sum_{j=1}^{m_2} \beta_{j_2} - \sum_{j=m_1+1}^{q_2} \beta_{j_2}, \quad \Delta_2 = \sum_{j=1}^{q_2} \beta_{j_2} - \sum_{i=1}^{p_2} \alpha_{i_2},$$

and so on

$$a_n^* = \sum_{i=1}^{\overline{n}_n} \alpha_{i_n} - \sum_{i=\overline{n}_n+1}^{p_n} \alpha_{i_n} + \sum_{j=1}^{m_n} \beta_{j_n} - \sum_{j=m_n+1}^{q_n} \beta_{j_n}, \quad \Delta_n = \sum_{j=1}^{q_n} \beta_{j_n} - \sum_{i=1}^{p_n} \alpha_{i_n}; \tag{58}$$

$$\mu_1 = \sum_{j=1}^{q_1} b_{j_1} - \sum_{i=1}^{p_1} a_{i_1} + \frac{p_1 - q_1}{2}, \mu_2 = \sum_{j=1}^{q_2} b_{j_2} - \sum_{i=1}^{p_2} a_{i_2} + \frac{p_2 - q_2}{2}, \ldots,$$

$$\mu_n = \sum_{j=1}^{q_n} b_{j_n} - \sum_{i=1}^{p_n} a_{i_n} + \frac{p_n - q_n}{2}; \tag{59}$$

$$\alpha_0^1 = \begin{cases} 1 + \max\limits_{m_1+1 \le j_1 \le q_1} \left[\frac{\mathrm{Re}(b_{j_1})-1}{\beta_{j_1}}\right], & q_1 > m_1, \\ \infty, & q_1 = m_1, \end{cases} \quad \beta_0^1 = \begin{cases} 1 + \min\limits_{\overline{n}_1+1 \le i_1 \le p_1} \left[\frac{\mathrm{Re}(a_{i_1})}{\alpha_{i_1}}\right], & p_1 > \overline{n}_1, \\ \infty, & p_1 = \overline{n}_1; \end{cases}$$

$$\alpha_0^2 = \begin{cases} 1 + \max\limits_{m_2+1 \le j_2 \le q_2} \left[\frac{\mathrm{Re}(b_{j_2})-1}{\beta_{j_2}}\right], & q_2 > m_2, \\ \infty, & q_2 = m_2, \end{cases} \quad \beta_0^2 = \begin{cases} 1 + \min\limits_{\overline{n}_2+1 \le i_2 \le p_2} \left[\frac{\mathrm{Re}(a_{i_2})}{\alpha_{i_2}}\right], & p_2 > \overline{n}_2, \\ \infty, & p_2 = \overline{n}_2; \end{cases} \ldots$$

$$\alpha_0^n = \begin{cases} 1 + \max\limits_{m_n+1 \le j_n \le q_n} \left[\frac{\mathrm{Re}(b_{j_n})-1}{\beta_{j_n}}\right], & q_n > m_n, \\ \infty, & q_n = m_n, \end{cases} \quad \beta_0^n = \begin{cases} 1 + \min\limits_{\overline{n}_n+1 \le i_n \le p_2} \left[\frac{\mathrm{Re}(a_{i_n})}{\alpha_{i_n}}\right], & p_n > \overline{n}_n, \\ \infty, & p_n = \overline{n}_n. \end{cases} \tag{60}$$

The exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of a function $\overline{\mathcal{H}}_{\mathbf{p},\mathbf{q}}^{\mathbf{m},\mathbf{n}}(\mathbf{s})$:

$$\overline{\mathcal{H}}_{\mathbf{p},\,\mathbf{q}}^{\mathbf{m},\mathbf{n}}(\mathbf{s}) \equiv \overline{\mathcal{H}}_{\mathbf{p},\,\mathbf{q}}^{\mathbf{m},\mathbf{n}}\left[\begin{matrix} (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}} \end{matrix} \middle| \mathbf{s}\right] = \prod_{k=1}^{n} \mathcal{H}_{p_k,\,q_k}^{m_k,\overline{n}_k}\left[\begin{matrix} (a_{i_k}, \alpha_{i_k})_{1,\,p_k} \\ (b_{j_k}, \beta_{j_k})_{1,q_k} \end{matrix} \middle| s\right], \tag{61}$$

is called a set of vectors $\overline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in R^n$ $(\nu_1 = \nu_2 = \ldots = \nu_n)$, such that $\alpha_1 < 1 - \nu_1 < \beta_1$, $\alpha_2 < 1 - \nu_2 < \beta_2, \ldots, \alpha_n < 1 - \nu_n < \beta_n$, and functions $\mathcal{H}_{p_1,\,q_1}^{m_1,\overline{n}_1}(s_1)$, $\mathcal{H}_{p_2,\,q_2}^{m_2,\overline{n}_2}(s_2), \ldots, \mathcal{H}_{p_n,\,q_n}^{m_n,\overline{n}_n}(s_n)$, have zeros on lines $Re(s_1) < 1 - \nu_1$, $Re(s_2) < 1 - \nu_2, \ldots, Re(s_n) < 1 - \nu_n$, respectively.

Applying multidimensional Mellin transform (50) to (40), taking into account the results for the one-dimensional case [25, Formulae (5.1.14)], we obtain:

$$(\mathfrak{M}\mathrm{H}^{1}_{\sigma,\kappa} f)(\mathbf{s}) = \overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}\left[\begin{array}{c|c} (\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}} & \\ (\mathbf{b}_j,\beta_j)_{1,\mathbf{q}} & \mathbf{s}+\sigma \end{array}\right](\mathfrak{M}f)(\mathbf{s}+\sigma+\kappa). \tag{62}$$

The following assertion presents the $\mathfrak{L}_{\overline{\nu},\overline{2}}$-theory of the modified H-transform (40). One dimensional case see in [25, Theorem 5.37].

Theorem 9 *Let*

$$\alpha_1 < \nu_1 - \mathrm{Re}(\kappa_1) < \beta_1, \alpha_2 < \nu_2 - \mathrm{Re}(\kappa_2) < \beta_1, \ldots, \alpha_n$$
$$< \nu_n - \mathrm{Re}(\kappa_1) < \beta_n, \nu_1 = \nu_2 = \ldots = \nu_n;$$

$$a_1^* = 0, a_2^* = 0, \ldots, \; a_n^* = 0; \; \Delta_1[\nu_1 - \mathrm{Re}(\kappa_1)] + \mathrm{Re}(\mu_1) \le 0,$$

$$\Delta_2[\nu_2 - \mathrm{Re}(\kappa_2)] + \mathrm{Re}(\mu_2) \le 0, \ldots, \Delta_n[\nu_n - \mathrm{Re}(\kappa_n)] + \mathrm{Re}(\mu_n) \le 0. \tag{63}$$

There hold the following assertions:

(a) *There exists a one-to-one map* $\mathrm{H}^1_{\sigma,\kappa} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{\overline{\nu}-\mathrm{Re}(\kappa+\sigma),\overline{2}}]$ *such the relation (62) holds for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $\mathrm{Re}(\mathbf{s}) = \overline{\nu} - \mathrm{Re}(\kappa+\sigma)$.
If $a_1^* = 0, a_2^* = 0, \ldots, a_n^* = 0; \Delta_1[\nu_1 - \mathrm{Re}(\kappa_1)] + \mathrm{Re}(\mu_1) = 0, \Delta_2[\nu_2 - \mathrm{Re}(\kappa_2)] + \mathrm{Re}(\mu_2) = 0, \ldots, \Delta_n[\nu_n - \mathrm{Re}(\kappa_n)] + \mathrm{Re}(\mu_n) = 0$ *and* $1 - \overline{\nu} + \mathrm{Re}(\kappa) \notin \mathcal{E}_{\overline{\mathcal{H}}}$, *then* $\mathrm{H}^1_{\sigma,\kappa}$ *maps* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *onto* $\mathfrak{L}_{\overline{\nu}-\mathrm{Re}(\kappa+\sigma),\overline{2}}$.
(b) *The transform* $\mathrm{H}^1_{\sigma,\kappa}$ *does not depend on* $\overline{\nu}$ *in the sense if* $\overline{\nu}$ *and* $\widetilde{\overline{\nu}}$ *satisfy Eq. (63) and if the transforms* $\mathrm{H}^1_{\sigma,\kappa}$ *and* $\widetilde{\mathrm{H}}^1_{\sigma,\kappa}$ *are defined in respective spaces* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ и $\mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}}$ *by Eq. (62), then* $\mathrm{H}^1_{\sigma,\kappa} f = \widetilde{\mathrm{H}}^1_{\sigma,\kappa} f$ *for* $f \in \mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}} \bigcap \mathfrak{L}_{\overline{\nu},\overline{2}}$.
(c) *If* $a_1^* = 0, a_2^* = 0, \ldots, a_n^* = 0; \Delta_1[\nu_1 - \mathrm{Re}(\kappa_1)] + \mathrm{Re}(\mu_1) < 0, \Delta_2[\nu_2 - \mathrm{Re}(\kappa_2)] + \mathrm{Re}(\mu_2) < 0, \ldots, \Delta_n[\nu_n - \mathrm{Re}(\kappa_n)] + \mathrm{Re}(\mu_n) < 0$*; then for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ $\mathrm{H}^1_{\sigma,\kappa} f$ *is given by Eq. (40).*
(d) *Let* $\overline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathrm{C}^n$, $\overline{h} = (h_1, \ldots, h_n) > 0$, *and* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$. *If* $\mathrm{Re}(\overline{\lambda}) > (\overline{\nu} - \mathrm{Re}(\kappa))\overline{h} - 1$, *then* $\mathrm{H}^1_{\sigma,\kappa} f$ *is represented in the form*

$$\left(\mathrm{H}^1_{\sigma,\kappa} f\right)(\mathbf{x}) = \overline{h}\mathbf{x}^{\sigma+1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathbf{d}}{\mathbf{dx}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \times$$
$$\times \int_0^\infty \mathrm{H}^{\mathbf{m},\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1}\left[\frac{\mathbf{x}}{\mathbf{t}} \left| \begin{array}{c} (-\overline{\lambda},\overline{h}), (\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j,\beta_j)_{1,\mathbf{q}}, \; (-\overline{\lambda}-1,\overline{h}) \end{array}\right.\right] \mathbf{t}^{\kappa-1} f(\mathbf{t})\mathbf{dt}. \tag{64}$$

while for $\mathrm{Re}(\overline{\lambda}) < (\overline{\nu} - \mathrm{Re}(k))\overline{h} - 1$ *is given by*

$$\left(\mathrm{H}^1_{\sigma,\kappa} f\right)(\mathbf{x}) = -\overline{h}\mathbf{x}^{\sigma+1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathbf{d}}{\mathbf{dx}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \times \\ \times \int_0^\infty \mathrm{H}^{\mathbf{m}+1,\mathbf{n}}_{\mathbf{p}+1,\mathbf{q}+1}\left[\frac{\mathbf{x}}{\mathbf{t}}\middle| \begin{matrix} (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}}, (-\overline{\lambda}, \overline{h}) \\ (-\overline{\lambda}-1, \overline{h}), (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}} \end{matrix}\right] \mathbf{t}^{\kappa-1} f(\mathbf{t})\mathbf{dx}. \tag{65}$$

(e) If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $g \in \mathfrak{L}_{1-\overline{\nu}+\mathrm{Re}(\kappa+\sigma),\overline{2}}$*, then there holds the relation:*

$$\int_0^\infty f(\mathbf{x})\left(\mathrm{H}^1_{\sigma,\kappa} g\right)(\mathbf{x})\mathbf{dx} = \int_0^\infty \left(\mathrm{H}^2_{\sigma,\kappa} f\right)(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}, \tag{66}$$

where

$$\left(\mathrm{H}^2_{\sigma,\kappa} f\right)(\mathbf{x}) = \mathbf{x}^\sigma \int_0^\infty \mathrm{H}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}\left[\frac{\mathbf{t}}{\mathbf{x}}\middle| \begin{matrix} (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}} \end{matrix}\right] \mathbf{t}^\kappa f(\mathbf{t}) \frac{\mathbf{dt}}{\mathbf{x}}. \tag{67}$$

Inversion formulas for the transform $\mathrm{H}^1_{\sigma,\kappa}$ *are given by the following equalities (one-dimensional case see in [[25], (5.5.23) and (5.5.24)]):*

$$f(\mathbf{x}) = -\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}-\kappa} \frac{\mathbf{d}}{\mathbf{dx}} \mathbf{x}^{-(\overline{\lambda}+1)/\overline{h}} \times \\ \times \int_0^\infty \mathrm{H}^{\mathbf{q}-\mathbf{m},\mathbf{p}-\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1}\left[\frac{\mathbf{t}}{\mathbf{x}}\middle| \begin{matrix} (-\overline{\lambda}, \overline{h}), (1-\mathbf{a}_i-\alpha_i, \alpha_i)_{\mathbf{n}+1,\mathbf{p}}, (1-\mathbf{a}_i-\alpha_i, \alpha_i)_{1,\mathbf{n}} \\ (1-\mathbf{b}_j-\beta_j, \beta_j)_{\mathbf{m}+1,\mathbf{q}}, (1-\mathbf{b}_j-\beta_j, \beta_j)_{1,\mathbf{m}} (-\overline{\lambda}-1, \overline{h}) \end{matrix}\right] \\ \times \mathbf{t}^{-\sigma}(\mathrm{H}^1_{\sigma,\kappa} f)(\mathbf{t})\mathbf{dt} \tag{68}$$

or

$$f(\mathbf{x}) = \overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}-1} \frac{\mathbf{d}}{\mathbf{dx}} \mathbf{x}^{-(\overline{\lambda}+1)/\overline{h}} \times \\ \times \int_0^\infty \mathrm{H}^{\mathbf{q}-\mathbf{m}+1,\mathbf{p}-\mathbf{n}}_{\mathbf{p}+1,\mathbf{q}+1}\left[\frac{\mathbf{t}}{\mathbf{x}}\middle| \begin{matrix} (1-\mathbf{a}_i-\alpha_i, \alpha_i)_{\mathbf{n}+1,\mathbf{p}}, (1-\mathbf{a}_i-\alpha_i, \alpha_i)_{1,\mathbf{n}}, (-\overline{\lambda}, \overline{h}) \\ (-\overline{\lambda}-1, \overline{h}), (1-\mathbf{b}_j-\beta_j, \beta_j)_{\mathbf{m}+1,\mathbf{q}}, (1-\mathbf{b}_j-\beta_j, \beta_j)_{1,\mathbf{m}} \end{matrix}\right] \\ \times \mathbf{t}^{-\sigma}(\mathrm{H}^1_{\sigma,\kappa} f)(\mathbf{t})\mathbf{dt}. \tag{69}$$

Condition for the validity of these formulas are given by the following assertion (one-dimensional case see in [25, Theorem 5.47]).

Theorem 10 *Let* $a_1^* = 0, a_2^* = 0, \dots, a_n^* = 0;$ $\alpha_1 < \nu_1 - \mathrm{Re}(\kappa_1) < \beta_1,$ $\alpha_2 < \nu_2 - \mathrm{Re}(\kappa_2) < \beta_2, \dots,$ $\alpha_n < \nu_n - \mathrm{Re}(\kappa_n) < \beta_n;$ $\alpha_0^1 < 1 - \nu_1 + \mathrm{Re}(\kappa_1) < \beta_0^1,$ $\alpha_0^2 < 1-\nu_2+\mathrm{Re}(\kappa_2) < \beta_0^2, \dots, \alpha_0^n < 1-\nu_n+\mathrm{Re}(\kappa_n) < \beta_0^n;$ *and let* $\overline{\lambda} \in C^n,$ $\overline{h} > 0.$

If $\Delta_1[\nu_1 - \mathrm{Re}(\kappa_1)] + \mathrm{Re}(\mu_1) = 0,$ $\Delta_2[\nu_2 - \mathrm{Re}(\kappa_2)] + \mathrm{Re}(\mu_2) = 0, \dots,$ $\Delta_n[\nu_n - \mathrm{Re}(\kappa_n)] + \mathrm{Re}(\mu_n) = 0,$ *and* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}(\nu_1, \nu_2, \dots, \nu_n)$*, then the inversion formulas (68) and (69) are valid for* $\mathrm{Re}(\overline{\lambda}) > (1 - \overline{\nu} + \mathrm{Re}(\kappa))\overline{h} - 1$ *and* $\mathrm{Re}(\overline{\lambda}) < (1 - \overline{\nu} + \mathrm{Re}(\kappa))\overline{h} - 1,$ *respectively.*

2.2 Representations in the Form of Modified H-Transform

Introduce so-called one-sided functions

$$\mathrm{K}_1(\mathbf{x}) = (\mathbf{x}^2 - 1)_+^{-\gamma/2}\ \mathrm{P}_\gamma^\delta(\mathbf{x}) = \begin{cases} (\mathbf{x}^2 - 1)^{-\gamma/2}\mathrm{P}_\gamma^\delta(\mathbf{x}), & \mathbf{x} > 1, \\ 0, & 0 < \mathbf{x} < 1; \end{cases} \tag{70}$$

$$\mathrm{K}_2(\mathbf{x}) = (1 - \mathbf{x}^2)_+^{-\gamma/2}\ \mathrm{P}_\gamma^\delta(\mathbf{x}) = \begin{cases} (1 - \mathbf{x}^2)^{-\gamma/2}\mathrm{P}_\gamma^\delta(\mathbf{x}), & 0 < \mathbf{x} < 1, \\ 0, & \mathbf{x} > 1. \end{cases} \tag{71}$$

Using notations in (52) and (70), (71), present transforms (41) and (42) in respective forms

$$\left(\mathrm{P}_{\delta,1}^\gamma f\right)(\mathbf{x}) = \int_0^\infty \mathrm{K}_1\left(\frac{\mathbf{x}}{\mathbf{t}}\right)\left(\mathrm{M}_{-\gamma} f\right)(\mathbf{t})d\mathbf{t}; \tag{72}$$

$$\left(\mathrm{P}_{\delta,2}^\gamma f\right)(\mathbf{x}) = \mathbf{x}^{1-\gamma}\int_0^\infty \left(\mathrm{R}\mathrm{K}_2\right)\left(\frac{\mathbf{x}}{\mathbf{t}}\right)\left(\mathrm{M}_{-1} f\right)(\mathbf{t})d\mathbf{t}. \tag{73}$$

The following assertion yields the Mellin transform formulas (50) of $\mathrm{K}_1(\mathbf{x})$ and $\mathrm{K}_2(\mathbf{x})$ in (70) and (71).

Lemma 2 *Let* $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n),$ $\delta = (\delta_1, \delta_2, \dots, \delta_n),$ $\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathrm{C}^n.$

(a) If $\mathrm{Re}(\gamma) < 1,$ $\mathrm{Re}(\mathbf{s}) < 1 + \mathrm{Re}(\gamma + \delta),$ $\mathrm{Re}(\mathbf{s}) < \mathrm{Re}(\gamma - \delta)$*, then*

$$\left(\mathfrak{M}\mathrm{K}_1\right)(\mathbf{s}) = 2^{\gamma-1}\frac{\Gamma\left(\frac{1+\gamma+\delta-\mathbf{s}}{2}\right)\Gamma\left(\frac{\gamma-\delta-\mathbf{s}}{2}\right)}{\Gamma\left(1 - \frac{\mathbf{s}}{2}\right)\Gamma\left(\frac{1-\mathbf{s}}{2}\right)}. \tag{74}$$

(b) If $\mathrm{Re}(\gamma) < 1,\ \mathrm{Re}(\mathbf{s}) > 0,$ *then*

$$\big(\mathfrak{M}\mathrm{K}_2\big)(\mathbf{s}) = 2^{\gamma-1}\frac{\Gamma\big(\frac{\mathbf{s}}{2}\big)\Gamma\big(\frac{\mathbf{s}+1}{2}\big)}{\Gamma\big(\frac{1-\gamma-\delta+\mathbf{s}}{2}\big)\Gamma\big(1+\frac{\delta-\gamma+\mathbf{s}}{2}\big)}. \tag{75}$$

Proof By [42, 2.172.9], under conditions in (a) there holds the formula

$$\big(\mathfrak{M}\mathrm{K}_1\big)(\mathbf{s}) = \frac{2^{\gamma_1-s_1-1}}{\sqrt{\pi}}\frac{\Gamma\big(\frac{1+\gamma_1+\delta_1-s_1}{2}\big)\Gamma\big(\frac{\gamma_1-\delta_1-s_1}{2}\big)}{\Gamma(1-s_1)}\frac{2^{\gamma_2-s_2-1}}{\sqrt{\pi}}\frac{\Gamma\big(\frac{1+\gamma_2+\delta_2-s_2}{2}\big)\Gamma\big(\frac{\gamma_2-\delta_2-s_2}{2}\big)}{\Gamma(1-s_2)}\cdots$$

$$\frac{2^{\gamma_1-s_n-1}}{\sqrt{\pi}}\frac{\Gamma\big(\frac{1+\gamma_n+\delta_1-s_n}{2}\big)\Gamma\big(\frac{\gamma_n-\delta_n-s_n}{2}\big)}{\Gamma(1-s_n)} = \frac{2^{\gamma-\mathbf{s}-1}}{\sqrt{\pi}}\frac{\Gamma\big(\frac{1+\gamma+\delta-\mathbf{s}}{2}\big)\Gamma\big(\frac{\gamma-\delta-\mathbf{s}}{2}\big)}{\Gamma(1-\mathbf{s})}. \tag{76}$$

Using the duplication formula for the gamma function

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\Big(z+\frac{1}{2}\Big) \tag{77}$$

with $z = \frac{1-s}{2}$, from Eq. (76) we deduce Eq. (74).

If conditions in (b) are satisfied, then according to [42, 2.172].

$$\big(\mathfrak{M}\mathrm{K}_2\big)(s) = 2^{\gamma-s}\sqrt{\pi}\frac{\Gamma(s)}{\Gamma\big(\frac{1-\gamma-\delta+s}{2}\big)\Gamma\big(1+\frac{\delta-\gamma+s}{2}\big)}. \tag{78}$$

Applying Eq. (77) with $z = \frac{s}{2}$, from Eq. (78) we deduce Eq. (75). Lemma is proved. □

Applying the convolution Mellin formula [29, (1.4.56)]

$$\Big(\mathfrak{M}\int_0^\infty \mathrm{K}\Big(\frac{\mathbf{x}}{\mathbf{t}}\Big)y(\mathbf{t})\frac{d\mathbf{t}}{\mathbf{t}}\Big)(\mathbf{s}) = \big(\mathfrak{M}\mathrm{K}\big)(\mathbf{s})\big(\mathfrak{M}f\big)(\mathbf{s}),\ (\mathbf{x}\in\mathrm{R}_+^n), \tag{79}$$

being valid for suitable $\mathrm{K}\big(\frac{\mathbf{x}}{\mathbf{t}}\big) = \mathrm{K}\big(\frac{x_1}{t_1}, \frac{x_2}{t_2}, \ldots, \frac{x_n}{t_n}\big)$ and $y(\mathbf{x})$, and formulas (53) and (54) for Mellin transform of $\mathrm{M}_\zeta f$, $\mathrm{R}f$, we find the Mellin transform of Eqs. (72) and (73) for suitable f.

Applying (74), we have for $\big(\mathrm{P}_{\delta,1}^{\gamma}f\big)(\mathbf{x})$:

$$(\mathfrak{M}\mathrm{P}_{\delta,1}^{\gamma}f)(\mathbf{s}) = \Big(\mathfrak{M}\int_0^\infty \mathrm{K}_1\Big(\frac{\mathbf{x}}{\mathbf{t}}\Big)\Big(\mathrm{M}_{1-\gamma}f\Big)(\mathbf{t})\frac{d\mathbf{t}}{\mathbf{t}}\Big)(\mathbf{s}) = \Big(\mathfrak{M}\mathrm{K}_1\Big)(\mathbf{s})\Big(\mathfrak{M}\mathrm{M}_{1-\gamma}f\Big)(\mathbf{s}) =$$

$$= 2^{\gamma-1}\frac{\Gamma\Big((1+\gamma+\delta-\mathbf{s})/2\Big)\Gamma\Big((\gamma-\delta-\mathbf{s})/2\Big)}{\Gamma\Big(1-\mathbf{s}/2\Big)\Gamma\Big((1-\mathbf{s})/2\Big)}\Big(\mathfrak{M}f\Big)(1-\gamma+\mathbf{s}). \tag{80}$$

In accordance with (61), relation (80) takes the form

$$\left(\mathfrak{M}\mathrm{P}^{\gamma}_{\delta,1} f\right)(\mathbf{s}) = 2^{\gamma-1}\frac{\Gamma\Big((1+\gamma+\delta-\mathbf{s})/2\Big)\Gamma\Big((\gamma-\delta-\mathbf{s})/2\Big)}{\Gamma\Big(1-\mathbf{s}/2\Big)\Gamma\Big((1-\mathbf{s})/2\Big)}\Big(\mathfrak{M}f\Big)(1-\gamma+\mathbf{s}) =$$

$$2^{\gamma-1}\overline{\mathcal{H}}^{0,2}_{2,2}\left[\begin{matrix}\left(\frac{1-\gamma-\delta}{2},\frac{1}{2}\right), & \left(1+\frac{\delta-\gamma}{2},\frac{1}{2}\right)\\ \left(0,\frac{1}{2}\right), & \left(\frac{1}{2},\frac{1}{2}\right)\end{matrix}\middle|\mathbf{s}\right]\Big(\mathfrak{M}f\Big)(\mathbf{s}+1-\gamma). \tag{81}$$

Therefore, by (62), the initial integral transform (41) is modified H-transform (40) with $\sigma = 0,\ \kappa = 1-\gamma$:

$$\left(\mathrm{P}^{\gamma}_{\delta,1} f\right)(\mathbf{s}) = 2^{\gamma-1}\int\limits_0^{\infty}\mathrm{H}^{0,2}_{2,2}\left[\frac{\mathbf{x}}{\mathbf{t}}\middle|\begin{matrix}\left(\frac{1-\gamma-\delta}{2},\frac{1}{2}\right) & \left(1+\frac{\delta-\gamma}{2},\frac{1}{2}\right)\\ \left(0,\frac{1}{2}\right) & \left(\frac{1}{2},\frac{1}{2}\right)\end{matrix}\right]\mathbf{t}^{-\gamma} f(\mathbf{t})d\mathbf{t}. \tag{82}$$

Similarly to the above, using Eq. (75) we have for $\left(\mathrm{P}^{\gamma}_{\delta,2} f\right)(\mathbf{x})$:

$$\left(\mathfrak{M}\mathrm{P}^{\gamma}_{\delta,2} f\right)(\mathbf{s}) = \left(\mathfrak{M}\left(\mathbf{x}^{1-\gamma}\int\limits_0^{\infty}\left(\mathrm{RK}_2\right)\left(\frac{\mathbf{x}}{\mathbf{t}}\right)\left(\mathrm{M}_{-1} f\right)(\mathbf{t})d\mathbf{t}\right)\right)(\mathbf{s})$$

$$= \left(\mathfrak{M}\int\limits_0^{\infty}\left(\mathrm{RK}_2\right)\left(\frac{\mathbf{x}}{\mathbf{t}}\right) f(\mathbf{t})\frac{d\mathbf{t}}{\mathbf{t}}\right)(\mathbf{s}+1-\gamma) =$$

$$= \left(\mathfrak{M}\left(\mathrm{RK}_2\right)\right)(\mathbf{s}+1-\gamma)\left(\mathfrak{M}f\right)(\mathbf{s}+1-\gamma) = \left(\mathfrak{M}\mathrm{K}_2\right)(\gamma-\mathbf{s})\left(\mathfrak{M}f\right)(\mathbf{s}+1-\gamma) =$$

$$= 2^{\gamma-1}\frac{\Gamma\big((\gamma-\mathbf{s})/2\big)\Gamma\big((\gamma-\mathbf{s}+1)/2\big)}{\Gamma\big((1-\delta-\mathbf{s})/2\big)\Gamma\big(1+(\delta-\mathbf{s})/2\big)}\left(\mathfrak{M}f\right)(1-\gamma+\mathbf{s}). \tag{83}$$

According to Eq. (61), relation (83) takes the form:

$$\left(\mathfrak{M}\mathrm{P}^{\gamma}_{\delta,2} f\right)(\mathbf{s}) = 2^{\gamma-1}\frac{\Gamma\big((\gamma-\mathbf{s})/2\big)\Gamma\big((\gamma-\mathbf{s}+1)/2\big)}{\Gamma\big((1-\delta-\mathbf{s})/2\big)\Gamma\big(1+(\delta-\mathbf{s})/2\big)}\left(\mathfrak{M}f\right)(1-\gamma+\mathbf{s})$$

$$= 2^{\gamma-1}\overline{\mathcal{H}}^{0,2}_{2,2}\left[\begin{matrix}\left(\frac{1-\gamma}{2},\frac{1}{2}\right), & \left(1-\frac{\gamma}{2},\frac{1}{2}\right)\\ \left(\frac{1+\delta}{2},\frac{1}{2}\right), & \left(-\frac{\delta}{2},\frac{1}{2}\right)\end{matrix}\middle|\mathbf{s}\right]\Big(\mathfrak{M}f\Big)(\mathbf{s}+1-\gamma), \tag{84}$$

and hence, in accordance with Eq. (62), the initial transform $\left(\mathrm{P}^{\gamma}_{\delta,2}f\right)(\mathbf{x})$ is also modified H-transform (40), with $\sigma = 0,\ \kappa = 1-\gamma$:

$$\left(\mathrm{P}^{\gamma}_{\delta,2}f\right)(\mathbf{s}) = 2^{\gamma-1}\int\limits_0^\infty \mathrm{H}^{0,2}_{2,2}\left[\frac{\mathbf{x}}{\mathbf{t}}\middle|\begin{matrix}\left(1-\frac{\gamma}{2},\frac{1}{2}\right), & \left(\frac{1-\gamma}{2},\frac{1}{2}\right)\\ \left(\frac{1+\delta}{2},\frac{1}{2}\right), & \left(-\frac{\delta}{2},\frac{1}{2}\right)\end{matrix}\right]\mathbf{t}^{-\gamma}f(\mathbf{t})d\mathbf{t}. \tag{85}$$

$\mathfrak{L}_{\overline{\nu},\overline{2}}$-Theory of Transforms $\mathbf{P}^{\gamma}_{\delta,k}f$ $(k=1,2)$

$\mathfrak{L}_{\nu,\overline{2}}$-theory of transforms (41)–(42) follows from Eqs. (82) and (85) with using Theorem 9 for the $\mathrm{H}^1_{\sigma,\kappa}$-transform.

By Eqs. (82), (85), and (40), $a_1^* = a_2^* = \ldots = a_n^* = 0$; $\Delta_1 = \Delta_2 = \ldots = \Delta_n = 0$; $\mathbf{p} = (p_1, p_2, \ldots, p_n) = (2, 2, \ldots, 2)$; $\mathbf{q} = (q_1, q_2, \ldots, q_n) = (2, 2, \ldots, 2)$, $\alpha_i = (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}) = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, $\beta_j = (\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_n}) = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ $(i = 1, \ldots, p;\ j = 1, \ldots, q)$; $\mu = \gamma - 1$.

As for $\mathbf{m}$, $\mathbf{n}$ and other parameters in Eqs. (57) and (59), we have:

$$\mathbf{m} = 0,\ \mathbf{n} = 2,\ \alpha = -\infty,\ \beta = \min[\mathrm{Re}(1+\gamma+\delta),\ \mathrm{Re}(\gamma-\delta)]; \tag{86}$$

$$\mathbf{m} = 0,\ \mathbf{n} = 2,\ \alpha = -\infty,\ \beta = \mathrm{Re}(\gamma); \tag{87}$$

respectively for the operators (41) and (42).

According to (80), $1-\overline{\nu}$ does not belong to exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of the $\overline{\mathcal{H}}^{0,2}_{2,2}$-function in the right-hand side of (81), if:

$$\mathbf{s} \neq 2m+1,\ \mathbf{s} \neq 2l+2\ (l = (l_1, l_2, \ldots, l_n);\ m = (\overline{m}_1, \overline{m}_2, \ldots, \overline{m}_n) \in N_0^n), \tag{88}$$

for $\mathrm{R}e(\mathbf{s}) = 1-\overline{\nu}$.

According to (83), $1-\overline{\nu}$ does not belong to exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of the $\overline{\mathcal{H}}^{2,0}_{2,2}$-function in the right-hand side of (84), if:

$$\mathbf{s} \neq -\delta+2m+1,\ \mathbf{s} \neq \delta+2l+2\ (l = (l_1, l_2, \ldots, l_n);\ m = (\overline{m}_1, \overline{m}_2, \ldots, \overline{m}_n) \in N_0^n), \tag{89}$$

for $\mathrm{R}e(\mathbf{s}) = 1-\overline{\nu}$.

By Eqs. (82), (85) and (86), (87), from Theorem 9 we deduce $\mathfrak{L}_{\overline{\nu},\overline{2}}$-theory of the transforms $\mathrm{P}^{\gamma}_{\delta,k}f$ $(k = 1, 2)$.

Theorem 11 *Let*

$$-\infty < \nu_1 - \mathrm{Re}(1-\gamma_1) < \min[\mathrm{Re}(1+\gamma_1+\delta_1),\ \mathrm{Re}(\gamma_1-\delta_1)],\ \mathrm{Re}(\gamma_1-1) \leq 0;$$

$$-\infty < \nu_2 - \mathrm{Re}(1-\gamma_2) < \min[\mathrm{Re}(1+\gamma_2+\delta_2),\ \mathrm{Re}(\gamma_2-\delta_2)],\ \mathrm{Re}(\gamma_2-1) \leq 0; \ldots;$$

$$-\infty < \nu_n - \mathrm{Re}(1-\gamma_n) < \min[\mathrm{Re}(1+\gamma_n+\delta_n),\ \mathrm{Re}(\gamma_n-\delta_n)],\ \mathrm{Re}(\gamma_n-1) \leq 0. \tag{90}$$

There hold the following assertions:

(a) *There exists a one-to-one map* $P^{\gamma}_{\delta,1} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{\overline{\nu}-\mathrm{Re}(1-\gamma),\overline{2}}]$ *such that the relation (81) holds for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $\mathrm{Re}(\mathbf{s}) = \overline{\nu} - \mathrm{Re}(1-\gamma)$. *If* $\mathrm{Re}(\gamma - 1) = 0$ *and Eq. (88) holds, then* $P^{\gamma}_{\delta,1}$ *is one-to-one on* $\mathfrak{L}_{\overline{\nu},\overline{2}}$.

(b) *The transform* $P^{\gamma}_{\delta,1} f$ *does not depend on* $\overline{\nu}$ *in the sense if* $\overline{\nu}_1$ *and* $\overline{\nu}_2$ *satisfy Eq. (90) and if the transforms* $P^{\gamma}_{\delta,1} f$ *and* $\widetilde{P}^{\gamma}_{\delta,1} f$ *are defined in respective spaces* $\mathfrak{L}_{\overline{\nu}_1,\overline{2}}$ *and* $\mathfrak{L}_{\overline{\nu}_2,\overline{2}}$ *by Eq. (81), then* $P^{\gamma}_{\delta,1} f = \widetilde{P}^{\gamma}_{\delta,1} f$ *for* $f \in \mathfrak{L}_{\overline{\nu}_1,\overline{2}} \bigcap \mathfrak{L}_{\overline{\nu}_2,\overline{2}}$.

(c) *If* $\mathrm{Re}(\gamma - 1) < 0$, *then for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ $P^{\gamma}_{\delta,1} f$ *is given by Eqs. (41) and (82).*

(d) *Let* $\overline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathrm{C}^n$, $\overline{h} = (h_1, \ldots, h_n) > 0$, *and* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$. *If* $\mathrm{Re}(\overline{\lambda}) > (\overline{\nu} - \mathrm{Re}(1-\gamma))\overline{h} - 1$, *then* $P^{\gamma}_{\delta,1} f$ *is represented in the form*

$$\left(P^{\gamma}_{\delta,1} f\right)(\mathbf{x}) = 2^{\gamma-1}\overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathbf{d}}{\mathbf{dx}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \times$$

$$\times \int_0^\infty \mathrm{H}^{0,3}_{3,3}\left[\frac{\mathbf{x}}{\mathbf{t}} \middle| \begin{matrix} (-\overline{\lambda}, h),\ \left(\frac{1-\gamma-\delta}{2}, \frac{1}{2}\right),\ \ \left(1 + \frac{\delta-\gamma}{2}, \frac{1}{2}\right) \\ \left(0, \frac{1}{2}\right),\ \ \left(\frac{1}{2}, \frac{1}{2}\right),\ (-\overline{\lambda} - 1, \overline{h}) \end{matrix}\right] \mathbf{t}^{-\gamma} f(\mathbf{t})\mathbf{dt}, \tag{91}$$

while for $\mathrm{Re}(\overline{\lambda}) < (\overline{\nu} - \mathrm{Re}(1-\gamma))\overline{h} - 1$ *is given by*

$$\left(P^{\gamma}_{\delta,1} f\right)(\mathbf{x}) = -2^{\gamma-1}\overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathbf{d}}{\mathbf{dx}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \times$$

$$\times \int_0^\infty \mathrm{H}^{1,2}_{3,3}\left[\frac{\mathbf{x}}{\mathbf{t}} \middle| \begin{matrix} \left(\frac{1-\gamma-\delta}{2}, \frac{1}{2}\right),\ \ \left(1 + \frac{\delta-\gamma}{2}, \frac{1}{2}\right),\ (-\overline{\lambda}, \overline{h}) \\ (-\overline{\lambda} - 1, \overline{h}),\ \left(0, \frac{1}{2}\right),\ \ \left(\frac{1}{2}, \frac{1}{2}\right) \end{matrix}\right] \mathbf{t}^{-\gamma} f(\mathbf{t})\mathbf{dt}. \tag{92}$$

(e) *If* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $g \in \mathfrak{L}_{1-\overline{\nu}+\mathrm{Re}(1-\gamma),\overline{2}}$, *then there holds the relation:*

$$\int_0^\infty f(\mathbf{x})\left(P^{\gamma}_{\delta,1} g\right)(\mathbf{x})\mathbf{dx} = \int_0^\infty 2^{\gamma-1}\left(P^{*\gamma}_{\delta,2} f\right)(\mathbf{x}) g(\mathbf{x})\mathbf{dx}, \tag{93}$$

where $\left(P^{*\gamma}_{\delta,2} f\right)(\mathbf{x})$ *is the transform*

$$\left(P^{*\gamma}_{\delta,2} f\right)(\mathbf{x}) = \int_{\mathbf{x}}^\infty \left(\mathbf{t}^2 - \mathbf{x}^2\right)^{-\gamma/2} P^{\gamma}_{\delta}\left(\frac{\mathbf{t}}{\mathbf{x}}\right) f(\mathbf{t})\mathbf{dt} = g(\mathbf{x}) \ \ (\mathbf{x} > 0). \tag{94}$$

Theorem 12 *Let*

$$-\infty < \nu_1 - \mathrm{Re}(1-\gamma_1) < \mathrm{Re}(\gamma_1),\ \mathrm{Re}(\gamma_1 - 1) \le 0;$$

$$-\infty < \nu_2 - \mathrm{Re}(1-\gamma_2) < \mathrm{Re}(\gamma_2),\ \mathrm{Re}(\gamma_2 - 1) \le 0; \ldots;$$

$$-\infty < \nu_n - \mathrm{Re}(1-\gamma_n) < \mathrm{Re}(\gamma_n),\ \mathrm{Re}(\gamma_n - 1) \le 0. \tag{95}$$

There hold the following assertions:

(a) *There exists a one-to-one map* $\mathrm{P}^{\gamma}_{\delta,2} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{\overline{\nu}-\mathrm{Re}(1-\gamma),2}]$ *such that the relation (84) holds for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $\mathrm{Re}(\mathbf{s}) = \overline{\nu} - \mathrm{Re}(1-\gamma)$. *If* $\mathrm{Re}(\gamma-1) = 0$ *and Eq. (89) holds, then* $\mathrm{P}^{\gamma}_{\delta,2}$ *is one-to-one on* $\mathfrak{L}_{\overline{\nu},\overline{2}}$.

(b) *The transform* $\mathrm{P}^{\gamma}_{\delta,2}f$ *does not depend on* $\overline{\nu}$ *in the sense if* $\overline{\nu}_1$ *and* $\overline{\nu}_2$ *satisfy Eq. (95) and if the transforms* $\mathrm{P}^{\gamma}_{\delta,2}f$ *and* $\widetilde{\mathrm{P}}^{\gamma}_{\delta,2}f$ *are defined in respective spaces* $\mathfrak{L}_{\overline{\nu}_1,\overline{2}}$ *and* $\mathfrak{L}_{\overline{\nu}_2,\overline{2}}$ *by Eq. (84), then* $\mathrm{P}^{\gamma}_{\delta,2}f = \widetilde{\mathrm{P}}^{\gamma}_{\delta,2}f$ *for* $f \in \mathfrak{L}_{\overline{\nu}_1,\overline{2}} \bigcap \mathfrak{L}_{\overline{\nu}_2,\overline{2}}$.

(c) *If* $\mathrm{Re}(\gamma-1) < 0$, *then for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ $\mathrm{P}^{\gamma}_{\delta,2}f$ *is given by Eqs. (42) and (85).*

(d) *Let* $\overline{\lambda} \in \mathrm{C}^n$, $\overline{h} > 0$, *and* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$. *If* $\mathrm{Re}(\overline{\lambda}) > (\overline{\nu} - \mathrm{Re}(1-\gamma))\overline{h} - 1$, *then* $\mathrm{P}^{\gamma}_{\delta,2}f$ *is represented in the form*

$$\left(\mathrm{P}^{\gamma}_{\delta,2}f\right)(\mathbf{x}) = 2^{\gamma-1}\overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}}\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}}\times$$
$$\times\int_0^\infty \mathrm{H}^{0,3}_{3,3}\left[\frac{\mathbf{x}}{\mathbf{t}}\left|\begin{matrix}(-\overline{\lambda},\overline{h}),\ \left(1-\frac{\gamma}{2},\frac{1}{2}\right),\ \left(\frac{1-\gamma}{2},\frac{1}{2}\right)\\ \left(\frac{1+\delta}{2},\frac{1}{2}\right),\ \left(-\frac{\delta}{2},\frac{1}{2}\right),\ (-\overline{\lambda}-1,\overline{h})\end{matrix}\right.\right]\mathbf{t}^{-\gamma}f(\mathbf{t})\mathbf{dt}, \tag{96}$$

while for $\mathrm{Re}(\overline{\lambda}) < (\overline{\nu} - \mathrm{Re}(1-\gamma))\overline{h} - 1$ *is given by*

$$\left(\mathrm{P}^{\gamma}_{\delta,2}f\right)(\mathbf{x}) = -2^{\gamma-1}\overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}}\frac{\mathbf{x}}{\mathbf{dx}}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}}\times$$
$$\times\int_0^\infty \mathrm{H}^{1,2}_{3,3}\left[\frac{\mathbf{x}}{\mathbf{t}}\left|\begin{matrix}\left(1-\frac{\gamma}{2},\frac{1}{2}\right),\ \left(\frac{1-\gamma}{2},\frac{1}{2}\right),\ (-\overline{\lambda},\overline{h})\\ (-\overline{\lambda}-1,\overline{h}),\ \left(\frac{1+\delta}{2},\frac{1}{2}\right),\ \left(-\frac{\delta}{2},\frac{1}{2}\right)\end{matrix}\right.\right]\mathbf{t}^{-\gamma}f(\mathbf{t})\mathbf{dt}. \tag{97}$$

(e) *If* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $g \in \mathfrak{L}_{1-\overline{\nu}+\mathrm{Re}(1-\gamma),\overline{2}}$, *then there holds the relation:*

$$\int_0^\infty f(\mathbf{x})\left(\mathrm{P}^{\gamma}_{\delta,2}g\right)(\mathbf{x})\mathbf{dx} = \int_0^\infty 2^{\gamma-1}\left(\mathrm{P}^{*\gamma}_{\delta,2}f\right)(\mathbf{x})g(\mathbf{x})\mathbf{dx}, \tag{98}$$

where $\left(\mathrm{P}^{*\gamma}_{\delta,2}f\right)$ *is given by*

$$\left(\mathrm{P}^{*\gamma}_{\delta,2}f\right)(\mathbf{x}) = \int_{\mathbf{x}}^\infty \left(\mathbf{t}^2-\mathbf{x}^2\right)^{-\gamma/2}\mathrm{P}^{\gamma}_{\delta}\left(\frac{\mathbf{x}}{t}\right)f(\mathbf{t})\mathbf{dt} = g(\mathbf{x})\ \ (\mathbf{x} > 0). \tag{99}$$

Inversion Formulas of Transforms $\mathbf{P}^{\gamma}_{\delta,k}f$ ($k = 1, 2$)

By substitution Eqs. (82), (85), and (40) parameters in Eq. (60) leads to

$$\alpha_0 = 0,\ \beta_0 = \infty; \tag{100}$$

$$\alpha_0 = 1 + \max[\mathrm{Re}(\delta-1),\ \mathrm{Re}(-\delta-2)],\ \beta_0 = \infty; \tag{101}$$

respectively for the operators (41), (42).

According to Eq. (82) the relation formulas (68) and (69) for $\mathrm{P}^{\gamma}_{\delta,1} f$ take the forms:

$$f(\mathbf{x}) = -2^{1-\gamma}\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}-1+\gamma}\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{-(\overline{\lambda}+1)/\overline{h}}\times$$

$$\times\int\limits_0^\infty \mathrm{H}^{2,1}_{3,3}\left[\frac{\mathbf{t}}{\mathbf{x}}\left|\begin{matrix} -(\overline{\lambda},\overline{h}), & \left(\frac{\gamma+\delta}{2},\frac{1}{2}\right), & \left(\frac{\gamma-\delta-1}{2},\frac{1}{2}\right) \\ \left(\frac{1}{2},\frac{1}{2}\right), & \left(0,\frac{1}{2}\right), & (-\overline{\lambda}-1,\overline{h}) \end{matrix}\right.\right]\left(\mathrm{P}^{\gamma}_{\delta,1} f\right)(\mathbf{t})\mathbf{dt}, \tag{102}$$

or

$$f(\mathbf{x}) = 2^{1-\gamma}\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}-1}\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{-(\overline{\lambda}+1)/\overline{h}}\times$$

$$\times\int\limits_0^\infty \mathrm{H}^{3,0}_{3,3}\left[\frac{\mathbf{t}}{\mathbf{x}}\left|\begin{matrix} \left(\frac{\gamma+\delta}{2},\frac{1}{2}\right), & \left(\frac{\gamma-\delta-1}{2},\frac{1}{2}\right), & (-\overline{\lambda},\overline{h}) \\ (-\overline{\lambda}-1,\overline{h}), & \left(\frac{1}{2},\frac{1}{2}\right), & \left(0,\frac{1}{2}\right) \end{matrix}\right.\right]\left(\mathrm{P}^{\gamma}_{\delta,1} f\right)(\mathbf{t})\mathbf{dt}. \tag{103}$$

According to Eq. (85) the relation formulas (68) and (69) for $\mathrm{P}^{\gamma}_{\delta,4} f$ take the forms:

$$f(\mathbf{x}) = -2^{1-\gamma}\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}-1+\gamma}\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{-(\overline{\lambda}+1)/\overline{h}}\times$$

$$\times\int\limits_0^\infty \mathrm{H}^{2,1}_{3,3}\left[\frac{\mathbf{t}}{\mathbf{x}}\left|\begin{matrix} (-\overline{\lambda},\overline{h}), & \left(\frac{\gamma-1}{2},\frac{1}{2}\right), & \left(\frac{\gamma}{2},\frac{1}{2}\right) \\ \left(-\frac{\delta}{2},\frac{1}{2}\right), & \left(\frac{\delta+1}{2},\frac{1}{2}\right), & (-\overline{\lambda}-1,\overline{h}) \end{matrix}\right.\right]\left(\mathrm{P}^{\gamma}_{\delta,4} f\right)(\mathbf{t})\mathbf{dt}, \tag{104}$$

or

$$f(\mathbf{x}) = 2^{1-\gamma}\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}-1}\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{-(\overline{\lambda}+1)/\overline{h}}\times$$

$$\times\int\limits_0^\infty \mathrm{H}^{3,0}_{3,3}\left[\frac{\mathbf{t}}{\mathbf{x}}\left|\begin{matrix} \left(\frac{\gamma-1}{2},\frac{1}{2}\right), & \left(\frac{\gamma}{2},\frac{1}{2}\right), & (-\overline{\lambda},\overline{h}) \\ (-\overline{\lambda}-1,\overline{h}), & \left(-\frac{\delta}{2},\frac{1}{2}\right), & \left(\frac{\delta+1}{2},\frac{1}{2}\right) \end{matrix}\right.\right]\left(\mathrm{P}^{\gamma}_{\delta,4} f\right)(\mathbf{t})\mathbf{dt}. \tag{105}$$

Theorem 13 *Let* $\mathrm{Re}(\gamma) = 1$, $-\infty < \overline{\nu} < \min[1, \mathrm{Re}(2+\delta), \mathrm{Re}(1-\delta)]$ *and let* $\overline{\lambda} \in \mathrm{C}^n$, $\overline{h} > 0$.

If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, *then the inversion formulas (102) and (103) are valid for* $\mathrm{Re}(\overline{\lambda}) > (1-\overline{\nu})\overline{h} - 1$ *and* $\mathrm{Re}(\overline{\lambda}) < (1-\overline{\nu})\overline{h} - 1$, *respectively.*

Theorem 14 *Let* $\mathrm{Re}(\gamma) = 1$, $-\infty < \overline{\nu} < \min[1, \mathrm{Re}(1-\delta), \mathrm{Re}(2+\delta)]$ *and let* $\overline{\lambda} \in \mathrm{C}^n$, $\overline{h} > 0$.

If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, *then the inversion formulas (104) and (105) are valid for* $\mathrm{Re}(\overline{\lambda}) > (1-\overline{\nu})\overline{h} - 1$ *and* $\mathrm{Re}(\overline{\lambda}) < (1-\overline{\nu})\overline{h} - 1$, *respectively.*

In the second part of the paper we summarize the corresponding results for the one-dimensional case, obtained in [28].

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Distributions, Non-smooth Manifolds, Transmutations and Boundary Value Problems

Vladimir B. Vasilyev

Abstract One discusses the problem of constructing the theory of pseudo differential equations on manifolds with a non-smooth boundary. Using special factorization principle and transmutation operators we consider some general boundary value problems for elliptic pseudo-differential equations in canonical non-smooth manifolds.

Keywords Non-smooth manifold · Pseudo-differential operator · Elliptic symbol · Boundary value problem

2010 Mathematics Subject Classification Primary: 35S15; Secondary: 58J05

1 Introduction

We study Fredholm properties of elliptic pseudo-differential operators (or equations) in Sobolev–Slobodetskii spaces on manifolds with a boundary but in our case the boundary may be non-smooth.

Basic principles for studying such equations are the following:

- a local principle or freezing coefficients principle;
- factorizability principle for an elliptic symbol at boundary point;
- a pluralism principle for singular boundary points which implies distinct types of local operators.

Local principle and factorizability was first introduced in papers I.B. Simonenko [16] (for multidimensional singular integral operators in Lebesgue L_p-spaces) and M.I. Vishik–G.I. Eskin [2] (for pseudo-differential operators in Sobolev–Slobodetskii H^s-spaces). For manifolds with a smooth boundary one uses an idea

V. B. Vasilyev (✉)
Chair of Differential Equations, Belgorod National Research University, Belgorod, Russia
e-mail: vbv57@inbox.ru; vladimir.b.vasilyev@gmail.com

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*,
Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_14

of “rectification of a boundary”, and the problems reduces to a half-space case, for which a factorizability principle holds immediately because under localization at a boundary point and applying the Fourier transform we obtain well known one-dimensional classical Riemann boundary value problem for upper and lower complex half-planes with a multidimensional parameter. This approach does not work if a boundary has at least one singular point like a conical point. One needs here other considerations and approaches.

The wave factorization principle was introduced by the author in 90th [18, 19] to extend the Vishik–Eskin theory to manifolds with a singular boundary. Such approach requires a special factorization for an elliptic symbol, and it leads to multidimensional variant of classical Riemann boundary value problem and multidimensional analogues of the Cauchy type integrals. It was shown [20, 28] these multidimensional analogues transform to the Cauchy type integral with a parameter for limit cases.

The third principle asserts that there are a lot of singularities at a boundary. Every singularity requires a separate studying to obtain solvability conditions for corresponding model equation. Common part of such studying is requiring the wave factorization for an elliptic symbol with respect to corresponding cone. If we have such factorization then we can describe needed solvability conditions (see, for example, [22–27]).

2 Domains and Operators

We consider a certain integro-differential operator A on m-dimensional compact manifold M with a boundary. This operators is defined by the function $A(x,\xi), (x,\xi) \in \mathbb{R}^{2m}$. There are some smooth compact sub-manifolds M_k of dimension $0 \le k \le m-1$ on the boundary ∂M of manifold M which are singularities of a boundary. These singularities are described by a local representative of operator A in a point $x_0 \in M$ on the map $U \ni x_0$ in the following way

$$(A_{x_0}u)(x) = \int\limits_{D_{x_0}} \int\limits_{\mathbb{R}^m} e^{i\xi\cdot(x-y)} A(\varphi(x_0),\xi)u(y)d\xi dy, \quad x \in D_{x_0}, \tag{1}$$

where $\varphi : U \to D_{x_0}$ is a diffeomorphism, and the canonical domain D_{x_0} has a distinct form depending on a placement of the point x_0 on manifold M. We consider the following canonical domains D_{x_0}: $\mathbb{R}^m, \mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x', x_m), x_m > 0\}$, $W^k = \mathbb{R}^k \times C^{m-k}$, where C^{m-k} is a convex cone in $\mathbb{R}^{m-k}$ non-including a whole line.

Such an operator A will be considered in Sobolev–Slobodetskii spaces $H^s(M)$, and local variants of such spaces will be spaces $H^s(D_{x_0})$. Local principle asserts that for a Fredholm property of the operator A it is necessary and sufficient an invertibility for all “local operators” $A_{x_0}, x_0 \in M$. So, we need to describe the conditions for unique solvability all model equations of the following type

$$(A_{x_0}u)(x) = v(x), \quad x \in D_{x_0}, \tag{2}$$

in corresponding local Sobolev–Slobodetskii spaces $H^s(D_{x_0})$.

2.1 Paired Equations

Such equations appear together with Eq. (2). Paired equation is called the following equation

$$(AP_+ + BP_-)U(x) = V(x), \quad x \in \mathbb{R}^m,$$

where A, B are model elliptic pseudo-differential operators, P_+ is restriction operator on canonical domain D, P_- is restriction operator on $\mathbb{R}^m \setminus D$. It is easily to show that solving the Eq. (2) is equivalent to solving the paired equation with $A = A_{x_0}$ and $B = I$ (identity). For solving such paired equations they apply the factorization technique and complex variables [2].

2.2 Singularities and Distributions

Author's point of view is the following. Each boundary point of manifold M is served by a special distribution. Such a distribution is the Fourier transform of an indicator of canonical domain. Using these distributions we reduce the Eq. (2) to a certain variant of the Riemann boundary value problem in the function theory of complex variables (one or many) [1, 2, 4, 7, 9, 19–21].

2.3 Complex Variables and Wave Factorization

To obtain the conditions for unique solvability for the Eq. (2) (or equivalently invertibility conditions for the operator (1)) we introduce the following concept. Let us denote [32]

$$\overset{*}{C}{}^{m-k} = \{x \in \mathbb{R}^{m-k} : x \cdot y > 0, y \in C^{m-k}\}$$

Taking into account local principle we will consider only symbols non-depending on spatial variables x and satisfying the condition

$$c_1(1+|\xi|)^\alpha \le |A(\xi)| \le c_2(1+|\xi|)^\alpha. \tag{3}$$

Definition 1 k-Wave factorization of elliptic symbol $A(\xi)$ with respect to the C^{m-k} is called its representation in the form

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi),$$

where the factors $A_{\neq}(\xi)$, $A_{=}(\xi)$ must satisfy the following conditions:

(1) $A_{\neq}(\xi)$, $A_{=}(\xi)$ are defined forall $\xi \in \mathbb{R}^m$ without may be the points $\mathbb{R}^k \times \partial\left(\overset{*}{C}{}^{m-k} \cup (-\overset{*}{C}{}^{m-k})\right)$;

(2) $A_{\neq}(\xi)$, $A_{=}(\xi)$ admit analytic continuation into radial tube domains $T(\overset{*}{C}{}^{m-k})$, $T(-\overset{*}{C}{}^{m-k})$ for almost all $\xi'' \in \mathbb{R}^k$ respectively with estimates

$$|A_{\neq}^{\pm 1}(\xi'', \xi' + i\tau)| \le c_1(1+|\xi|+|\tau|)^{\pm æ_k},$$

$$|A_{=}^{\pm 1}(\xi'', \xi' - i\tau)| \le c_2(1+|\xi|+|\tau|)^{\pm(\alpha - æ_k)}, \ \forall \tau \in \overset{*}{C}{}^{m-k}.$$

The number $æ_k \in \mathbb{R}$ is called index of k-wave factorization.

Existence of such factorization permits to describe solvability picture for model pseudo-differential equation (2) for $m-k=2$ [19, 20], but in a general case we need to know the general form of a distribution supported on a conical surface (we can't find such form in [5]). We try to reduce the problem to a half-space case using transmutation operators.

3 Transmutations, Distributions and the Fourier Transform

Below we consider the case $k=0$ because all conclusions will be the same, only k-dimensional parameter can be appear. Let C be a convex cone in the space $\mathbb{R}^m$, and this cone does not include any whole straight line, it is important because we use the theory of analytic functions of several complex variables [1, 31, 32]. Moreover we suppose that a surface of this cone is given by the equation $x_m = \varphi(x')$, $x' = (x_1, \cdots, x_{m-1})$, where $\varphi : \mathbb{R}^{m-1} \to \mathbb{R}$ is a smooth function in $\mathbb{R}^{m-1} \setminus \{0\}$, and $\varphi(0) = 0$.

Let us introduce the following change of variables [14, 29, 30]

$$\begin{cases} t_1 = x_1 \\ t_2 = x_2 \\ \cdots \\ t_{m-1} = x_{m-1} \\ t_m = x_m - \varphi(x') \end{cases}$$

and we denote this operator by $T_\varphi : \mathbb{R}^m \to \mathbb{R}^m$.

Obviously, this is a smooth transformation excluding an origin. Let f be a local integrable function which generates a distribution defined by the formula

$$(f, \psi) = \int_{\mathbb{R}^m} f(x)\psi(x)dx.$$

We define a functional $T_\varphi f$ by the formula

$$(T_\varphi f, \psi) = (f, T_\varphi^{-1}\psi).$$

According to the Schwartz theorem on one-dimensional distribution from $S'(\mathbb{R})$ supported at the origin 0 [5, 32] we can conclude that if a distribution $f \in S'(\mathbb{R}^m)$ supported in the hyper-plane $x_m = 0$ then it has the following form

$$f(x) = \sum_{k=0}^{n} c_k(x') \otimes \delta^{(k)}(x_m), \quad x = (x', x_m),$$

where $c_k \in S'(\mathbb{R}^{m-1})$, $k = 0, 1, \cdots, n$, are arbitrary distributions.

Therefore we can assert that if a distribution $f \in S'(\mathbb{R}^m)$ is supported on ∂C then $T_\varphi f$ is supported on $\mathbb{R}^{m-1}$.

An arbitrary distribution $f \in S'(\mathbb{R}^m)$ supported on conical surface ∂C can written in the form

$$f(x) = T_\varphi^{-1}\left(\sum_{k=0}^{n} c_k(y') \otimes \delta^{(k)}(y_m)\right), \tag{4}$$

where $c_k \in S'(\mathbb{R}^{m-1})$, $k = 0, 1, \cdots, n$, are arbitrary distributions.

Further, for functions $u(x)$ from $S(\mathbb{R}^m)$ their Fourier transform is defined by the formula

$$(Fu)(\xi) \equiv \tilde{u}(\xi) = \int_{\mathbb{R}^m} e^{ix\cdot\xi}u(x)dx.$$

The Fourier transform for distributions is defined as follows

$$(Ff, \psi) = (f, F\psi),$$

therefore

$$(FT_\varphi f, \psi) = (f, T_\varphi^{-1} F\psi).$$

Let $f \in S'(\mathbb{R}^m)$ be a distribution supported on ∂C. According to the above conclusions it has the special form (4). Using properties of T_φ and F we will find

$$Ff = V_\varphi \left(\sum_{k=0}^{n} \tilde{c}_k(\xi') \xi_m^k \right),$$

where

$$FT_\varphi^{-1} F^{-1} \equiv V_\varphi.$$

For a distribution $f \in S'(\mathbb{R}^m)$ the transform V_φ is given by the formula

$$(V_\varphi \tilde{f}, \psi) \equiv (\tilde{f}, V_{-\varphi}\psi), \quad \forall \psi \in S(\mathbb{R}^m).$$

If $\hat{u}(x', \xi_m)$ denotes the Fourier transform of the function $u(x', x_m)$ with respect to a variable x_m then one can make the following conclusion. Let us denote

$$F_{x' \to \xi'}(e^{-i\xi_m \varphi(x')}) \equiv K_\varphi(\xi', \xi_m),$$

and after this we obtain an integral representation for the operator V_φ:

$$(FT_\varphi^{-1} u)(\xi) = \int\limits_{\mathbb{R}^m} K_\varphi(\xi' - \eta', \xi_m) \tilde{u}(\eta', \xi_m) d\eta'.$$

3.1 Examples

3.1.1 Plane Sector

The case $m = 2$ is a very good, there is only one mentioned cone. We write it as follows

$$C_+^a = \{x \in \mathbb{R}^2 : x = (x_1, x_2), x_2 > a|x_1|, a > 0\},$$

and further evaluate:

$$(FT_\varphi^{-1}u)(\xi) = \frac{\tilde{u}(\xi_1 + a\xi_2, \xi_2) + \tilde{u}(\xi_1 - a\xi_2, \xi_2)}{2} +$$

$$v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{u}(\eta, \xi_2)d\eta}{\xi_1 + a\xi_2 - \eta} - v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{u}(\eta, \xi_2)d\eta}{\xi_1 - a\xi_2 - \eta} \equiv (V_\varphi\tilde{u})(\xi).$$

We denote by $S_1\tilde{u}$ the operator

$$(S_1\tilde{u})(\xi_1, \xi_2) = v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{u}(\eta, \xi_2)d\eta}{\xi_1 - \eta}$$

and analogously S_2 for the second variable.

3.1.2 Standard Cone

As it was shown the kernel K_φ is computable for concrete function $\varphi(x')$. Let $\varphi(x') = a|x'|, a > 0,$. If we will look at the formulas from [31] (see also [16] in which a real analogue of these formulas is given as the Poisson kernel) we will find

$$K_\varphi(\xi', \xi_m) = \frac{a2^{m-1}\pi^{\frac{m-2}{2}}\Gamma(m/2)}{\left(|\xi'|^2 - a^2\xi_m^2\right)^{m/2}}.$$

Therefore for such multidimensional cone the operator V_φ looks as follows

$$(V_\varphi\tilde{u})(\xi) = \int_{\mathbb{R}^{m-1}}\frac{a2^{m-1}\pi^{\frac{m-2}{2}}\Gamma(m/2)\tilde{u}(\eta', \xi_m)d\eta'}{\left(|\xi' - \eta'|^2 - a^2\xi_m^2\right)^{m/2}}.$$

In our opinion we could call it *a conical potential*.

Of course this formula should be treated in a distribution sense. Below we give such definition for the operator V_φ in the space $S'(\mathbb{R}^m)$.

3.1.3 Three-Wedged Pyramid

This cone looks as follows

$$C_+^{\mathbf{a}} = \{x \in \mathbb{R}^3 : x_3 > a_1|x_1| + a_2|x_2|, a_1, a_2 > 0\}$$

For this case the operator V_φ is constructed exactly using two operators S_1, S_2 (see below)

4 Potentials Generated by Transmutations

4.1 General Situation

Now we see that the main problem is to study

Let C be a convex cone non-including a whole straight line. Let us introduce the Bochner kernel [1, 31, 32]

$$B_m(z) = \int_C e^{ix\cdot z}dx, \quad z = \xi + i\tau,$$

and related integral operator

$$(B_m u)(x) = \lim_{\tau\to 0+} \int_{\mathbb{R}^m} B_m(x - y + i\tau)u(y)dy, \quad x \in \mathbb{R}^m.$$

Theorem 1 *If the symbol $A(\xi)$ admits the wave factorization with the index* æ, $æ - s = n+\delta, n \in \mathbb{N}, |\delta| < 1/2$, *then a general solution of the Eq.* (2) *in Fourier images is given by the formula*

$$\tilde{u}_+(\xi) = A_{\neq}^{-1}(\xi)Q_n(\xi)B_m Q_n^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{lf}(\xi)+$$

$$+A_{\neq}^{-1}(\xi)V_\varphi^{-1}F\left(\sum_{k=1}^{n} c_k(x')\delta^{(k-1)}(x_m)\right),$$

where $c_k(x') \in H^{s_k}(\mathbb{R}^{m-1})$ are arbitrary functions, $s_k = s - æ + k - 1/2,\ k = 1, 2, \ldots, n$, lf is an arbitrary continuation of f onto $H^{s-\alpha}(\mathbb{R}^m)$, Q_n is an arbitrary polynomial satisfying the condition (3) *for $\alpha = n$.*

Using these results one needs to add some additional conditions to determine uniquely unknown functions c_k. We will consider certain particular case in the next section.

Some special cases are very interesting, for example if $C = C_+^a = \{x \in \mathbb{R}^m : x = (x', x_m), x_m > a|x'|, a > 0\}$. Using evaluations from [17] we can obtain the following result.

Corollary 1 *If* $f \equiv 0, n = 1$, *then we have the following form for a general solution in the space* $H^s(C^a_+)$

$$\tilde{u}_+(\xi) = A^{-1}_{\neq}(\xi) \int\limits_{\mathbb{R}^{m-1}} \frac{a2^{m-1}\pi^{\frac{m-2}{2}}\Gamma(m/2)\tilde{c}(\eta')d\eta'}{\left(|\xi'-\eta'|^2 - a^2\xi_m^2\right)^{m/2}},$$

where $c(x') \in H^{s-æ+1/2}(\mathbb{R}^{m-1})$ *is an arbitrary function.*

5 Boundary Value Problems

According to Theorem 1 we can consider different types of boundary value problems with boundary conditions or with co-boundary operators.

Let us consider a simple boundary value problem for the equation

$$(Au)(x) = 0, \quad x \in C^{\mathbf{a}}_+ \tag{5}$$

for the case $æ - s = 1 + \delta, |\delta| < 1/2$, where A is an elliptic pseudo-differential operator with the symbol $A(\xi)$ satisfying the condition (3) and admitting the wave factorization with respect to the cone $C^{\mathbf{a}}_+$.

According to Theorem 1 we have the formula for a general solution, for our case it can be written as

$$\tilde{u}(\xi) = A^{-1}_{\neq}(\xi)(V_{-\mathbf{a}}Fc_0)(\xi), \tag{6}$$

where $c_0(x')$ is an arbitrary function from $H^{s_0}(\mathbb{R}^2)$.

Now we will write an expression for $V_{-\mathbf{a}}Fc_0$ and then we will see what kind of conditions for a solution u is more preferable. Direct calculations led to the following expression

$$\begin{aligned} A_{\neq}(\xi)\tilde{u}(\xi) = \tilde{C}_1(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3) + \tilde{C}_2(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3) + \\ \tilde{C}_3(\xi_1 + a_1\xi_3, \xi_2 - a_2\xi_3) + \tilde{C}_1(\xi_1 + a_1\xi_3, \xi_2 + a_2\xi_3), \end{aligned} \tag{7}$$

where

$$\tilde{C}_1(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3) = \frac{1}{4}\tilde{c}_0(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3) - \frac{1}{2}(S_1\tilde{c}_0)(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3) -$$

$$-\frac{1}{2}(S_2\tilde{c}_0)(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3) + (S_1S_2\tilde{c}_0)(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3);$$

$$\tilde{C}_2(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3) = \frac{1}{4}\tilde{c}_0(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3) - \frac{1}{2}(S_1\tilde{c}_0)(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3) +$$

$$+\frac{1}{2}(S_2\tilde{c}_0)(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3) - (S_1S_2\tilde{c}_0)(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3);$$

$$\tilde{C}_3(\xi_1+a_1\xi_3, \xi_2-a_2\xi_3) = \frac{1}{4}\tilde{c}_0(\xi_1+a_1\xi_3, \xi_2-a_2\xi_3)+\frac{1}{2}(S_1\tilde{c}_0)(\xi_1+a_1\xi_3, \xi_2-a_2\xi_3)-$$

$$-\frac{1}{2}(S_2\tilde{c}_0)(\xi_1 + a_1\xi_3, \xi_2 - a_2\xi_3) - (S_1S_2\tilde{c}_0)(\xi_1 + a_1\xi_3, \xi_2 - a_2\xi_3);$$

$$\tilde{C}_4(\xi_1+a_1\xi_3, \xi_2+a_2\xi_3) = \frac{1}{4}\tilde{c}_0(\xi_1+a_1\xi_3, \xi_2+a_2\xi_3)+\frac{1}{2}(S_1\tilde{c}_0)(\xi_1+a_1\xi_3, \xi_2+a_2\xi_3)+$$

$$+\frac{1}{2}(S_2\tilde{c}_0)(\xi_1 + a_1\xi_3, \xi_2 + a_2\xi_3) + (S_1S_2\tilde{c}_0)(\xi_1 + a_1\xi_3, \xi_2 + a_2\xi_3).$$

It seems the problem of finding the unknown function $c_0(\xi_1, \xi_2)$ is very hard, but we suppose that we know the following function $\tilde{u}(\xi_1, \xi_2, 0)$. It means that we know the following integral

$$\int_{-\infty}^{+\infty} u(x_1, x_2, x_3)dx_3 \equiv g(x_1, x_2), \tag{8}$$

thus

$$\tilde{u}(\xi_1, \xi_2, 0) = \tilde{g}(\xi_1, \xi_2). \tag{9}$$

The formula (7) includes a representation for $V_{-\mathbf{a}}\tilde{c}_0$, where $\tilde{c}_0(\xi')$ is a function of two variables. Thus, if $\tilde{c}_0(\xi_1, \xi_2)$ depends on two variables ξ_1, ξ_2 then $V_{-\mathbf{a}}\tilde{c}_0$ depends on all three variables ξ_1, ξ_2, ξ_3.

Substituting (9) into (7) and collecting similar summands we obtain the following equation for the unknown $\tilde{c}_0(\xi')$

$$A_{\neq}^{-1}(\xi', 0)(\tilde{c}_0(\xi')) = \tilde{g}(\xi'),$$

or if we designate $A_{\neq}(\xi', 0)\tilde{g}(\xi') \equiv f(\xi')$

$$\tilde{c}_0(\xi') = \tilde{f}(\xi')$$

Now if we have found $\tilde{c}_0(\xi')$ we have the solution of the problem (5) and (8).

Also we can give a priori estimates for the solution.

Theorem 2 *Let $A(\xi)$ admits the wave factorization with respect to the $C_+^{\mathbf{a}}$. Then the boundary value problem (5) and (8) has a unique solution for an arbitrary $g \in H^{s+1/2}(\mathbb{R}^2)$ in the space $H^s(C_+^{\mathbf{a}})$. This solution can be constructed explicitly by the*

Fourier transform and the one-dimensional singular integral operator. The a priori estimate

$$||u||_s \leq c[g]_{s+1/2}$$

holds for $-1/2 < \delta < 0$.

6 Thin Cones

As we see all local operators includes some parameters (sizes of cones) which can be small or large. These situations correspond to so called thin cones or a half-space case (see, for example, [20] were some calculations were given). Singularities at a boundary can be of distinct dimensions and it is possible such singularities of a low dimension can be obtained from analogous singularities of full dimension. It means we need to find distributions for limit cases when some of parameters of singularities tend to zero. This approach was partially realized in author's papers [22, 23], and the latest paper [27] is devoted to multi-dimensional constructions. The further author's idea is the following. If we know the limit operator for a thin singularity then possible it is zero approximation for a such thin singularity. It is desirable to obtain an asymptotic expansion with a small parameter for the distribution corresponding to a such singularity. We will consider here a two-dimensional case.

To describe a solvability picture for a model elliptic pseudo differential equation with an operator A

$$(Au)(x) = v(x), \tag{10}$$

in two-dimensional cone $C_+^a = \{x \in \mathbb{R}^2 : x_2 > a|x_1|, a > 0\}$ the author earlier considered a special singular integral operator [18, 19]

$$(K_a u)(x) = \frac{a}{2\pi^2} \lim_{\tau\to 0+} \int\limits_{\mathbb{R}^2} \frac{u(y)dy}{(x_1-y_1)^2 - a^2(x_2-y_2+i\tau)^2}.$$

This operator served a conical singularity in the general theory of boundary value problems for elliptic pseudo differential equations on manifolds with a non-smooth boundary. This operator is a convolution operator, and the parameter a is a size of an angle, $x_2 > a|x_1|$, $a = \cot\alpha$.

We will consider two spaces of basic functions for distributions. If $D(\mathbb{R}^2)$ denotes a space of infinitely differentiable functions with a compact support then $D'(\mathbb{R}^2)$ is the corresponding space of distributions over the space $D(\mathbb{R}^2)$, analogously if $S(\mathbb{R}^2)$ is the Schwartz space of infinitely differentiable rapidly decreasing at infinity functions then $S'(\mathbb{R}^2)$ is a corresponding space of distributions over $S(\mathbb{R}^2)$.

When $a \to +\infty$ one obtains [20] the following limit distribution

$$\lim_{a\to\infty} \frac{a}{2\pi^2} \frac{1}{\xi_1^2 - a^2\xi_2^2} = \frac{i}{2\pi} \mathcal{P}\frac{1}{\xi_1} \otimes \delta\left(\xi_2\right),$$

where the notation for distribution $\mathcal{P}$ is taken from V.S. Vladimirov's books [31, 32], and $\otimes$ denotes the direct product of distributions. Here δ denotes one-dimensional Dirac mass-function which acts on $\varphi \in D(\mathbb{R})$ by the following way

$$(\delta, \varphi) = \varphi(0),$$

and the distribution $\mathcal{P}\frac{1}{x}$ is defined by the formula

$$(\mathcal{P}\frac{1}{x}, \varphi) = v.p. \int\limits_{-\infty}^{+\infty} \frac{\varphi(x)dx}{x} \equiv \lim_{\varepsilon\to 0+} \left(\int\limits_{-\infty}^{-\varepsilon} + \int\limits_{\varepsilon}^{+\infty} \right) \frac{\varphi(x)dx}{x}.$$

We would like to obtain an asymptotical expansion for the two-dimensional distribution

$$K_a(\xi_1, \xi_2) \equiv \frac{a}{2\pi^2} \frac{1}{\xi_1^2 - a^2\xi_2^2}$$

with respect to small a^{-1}. It is defined by the corresponding formula $\forall \varphi \in D(\mathbb{R}^2)$

$$(K_a, \varphi) = \frac{a}{2\pi^2} \int\limits_{\mathbb{R}^2} \frac{\varphi(\xi_1, \xi_2)d\xi}{\xi_1^2 - a^2\xi_2^2}.$$

For $K_a \in D'(\mathbb{R}^2)$ we can suggest the following decomposition [28]

$$K_a(\xi_1, \xi_2) = \frac{i}{2\pi} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!a^n} \mathcal{P}\frac{1}{\xi_1} \otimes \delta^{(n)}(\xi_2).$$

But for $K_a \in S'(\mathbb{R}^2)$ we have more explicit result [28].

Theorem 3 *The following formula*

$$K_a(\xi_1, \xi_2) = \frac{i}{2\pi} \mathcal{P}\frac{1}{\xi_1} \otimes \delta(\xi_2) + \sum_{m,n} c_{m,n}(a) \widetilde{\delta^{(m)}}(\xi_1) \otimes \delta^{(n)}(\xi_2),$$

where $c_{m,n}(a) \to 0$, $a \to +\infty$, *holds in a distribution sense.*

Let us return to the Eq. (10). For $|æ - s| < 1/2$ one has the existence and uniqueness theorem [18]

$$\widetilde{u}(\xi) = A_{\neq}^{-1}(\xi)(K_a\widetilde{lv})(\xi),$$

where lv is an arbitrary continuation of v on the whole $H^s(\mathbb{R}^2)$.

Below we denote $lv \equiv V$.

Theorem 4 *If the symbol $A(\xi)$ admits a wave factorization with respect to the cone C_+^a and $|æ - s| < 1/2$ then Eq. (1) has a unique solution in the space $H^s(C_+^a)$, and for a large a it can be represented in the form*

$$\widetilde{u}(\xi) = \frac{i}{2\pi}A_{\neq}^{-1}(\xi)v.p.\int_{-\infty}^{+\infty}\frac{(A_{=}^{-1}\widetilde{V})(\eta_1,\xi_2)d\eta_1}{\xi_1-\eta_1}+$$

$$A_{\neq}^{-1}(\xi)\sum_{m,n}c_{m,n}(a)\int_{-\infty}^{+\infty}(\xi_1-\eta_1)^m(A_{=}^{-1}\widetilde{V})_{\xi_2}^{(n)}(\eta_1,\xi_2)d\eta_1$$

assuming $\widetilde{V} \in S(\mathbb{R}^2)$, $A_{=}^{-1}\widetilde{V}$ means the function $A_{=}^{-1}(\xi)\widetilde{V}(\xi)$.

7 Conclusion

This paper is a brief description of latest author's studies on elliptic pseudo-differential equations and boundary value problems on manifolds with non-smooth boundaries. Other approaches, similar problems, interesting statements can be found in books and monographs [3, 6–8, 10–13, 15].

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Part II
Transmutations in ODEs, Direct and Inverse Problems

On a Transformation Operator Approach in the Inverse Spectral Theory of Integral and Integro-Differential Operators

Sergey Buterin

Abstract A brief survey is given on using transformation operators in the inverse spectral theory of integral and integro-differential operators possessing a convolutional term to be recovered. The central place of this approach is occupied by reducing the inverse problem to solving some nonlinear equation, which can be solved globally. We illustrate this scheme on several examples, among which there are: one-dimensional perturbation of the convolution operator, Sturm–Liouville-type integro-differential operators and an integro-differential Dirac system.

Keywords Integral operator · Convolution · Integro-differential operator · Nonlocal operator · Transformation operator · Inverse spectral problem · Nonlinear integral equation

AMS Classification (2010) 34A55, 34B09, 45J05, 45P05, 45G05, 45G15

1 Introduction

Inverse spectral problems consist in recovering operators from their spectral characteristics. The greatest success in the inverse spectral theory has been achieved for the Sturm–Liouville and Dirac differential operators (see, e.g., [1–6] and references therein) and afterwards for higher-order differential operators and differential systems with an arbitrary location of roots of characteristic polynomial [5–9]. The classical methods of inverse spectral theory that allow to obtain global solutions of

S. Buterin (✉)
Department of Mathematics, Saratov State University, Saratov, Russia
e-mail: buterinsa@info.sgu.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_15

inverse problems for differential operators (the transformation operator method (i.e. the so-called Gel'fand–Levitan method) [2–5] and the method of spectral mappings [4–8]), do not work for integral, integro-differential and other classes of nonlocal operators (see [10–46] and references therein). At the same time, the transformation operator itself is a common tool (see also monograph [47]) and it is widely used in the spectral theory. In the present paper a brief survey is given on one way of using transformation operators in the inverse spectral theory of integral and integro-differential operators possessing a convolution term to be recovered.

In [16] for a one-dimensional perturbation of the Volterra convolution operator a special approach was suggested, which was based on a structure of the transformation operator kernel. Within this approach, the inverse problem was reduced to some nonlinear integral equation, which was solved globally. This allowed to obtain a global solution of the inverse problem of recovering the convolution term from the spectrum, provided that the perturbation term was known a priori (for more details see Sect. 2). Moreover, this approach appeared to be successful for studying inverse problems for convolution integro-differential operators [19, 25, 34] (see Sect. 3).

Later on, a further development of this approach was given in [21] and then in [35, 39, 46], where the global solution was obtained for the inverse problem of recovering a convolutional perturbation of the Sturm–Liouville operator. The main difficulty there was connected with a general implicit structure of the transformation operator kernel. However, a detailed analysis of its dependence on the convolution kernel allowed to prove the global solvability of the main equation (see Sect. 4).

Another important promotion relates to integro-differential Dirac systems [28, 30, 38, 40]. Unlike the scalar case, here the main equation of the inverse problem is a vectorial nonlinear integral equation of a special form, whose global solvability has been also established (see Sect. 5). Among other essential directions of applying this approach one should mention inverse problems for fractional order integro-differential operators [34, 35], integro-differential operators with discontinuity conditions [26, 31, 39], integro-differential operators on geometrical graphs [32], integro-differential pencils [43–45]. Moreover, it appeared to be successful also for solving the so-called half inverse problems [27, 33, 40].

As was mentioned above, in each case the main equation of the inverse problem may take a special form peculiar namely to the considered class of operators, which usually causes the necessity to carry out the proof of solvability of the main equation in each new case. For this reason, in [48] a general approach has been developed for solving nonlinear equations of this type by introducing some abstract equation and proving its global solvability. Moreover, in [48] uniform stability of such nonlinear equations was established, which has not been studied before even in simple cases.

2 One-Dimensional Perturbation of a Convolution Operator

2.1 Historical Notes

Consider the integral operator

$$Af = Mf + g(x)\int_0^\pi f(t)v(t)\,dt, \quad Mf = \int_0^x M(x,t)f(t)\,dt, \quad 0 \le x \le \pi, \tag{1}$$

which is a one-dimensional perturbation of the Volterra integral operator M. It is known that the inverse operator to the Sturm–Liouville one is an operator of the form (1). Moreover, the inverse operators to differential and Volterra integro-differential ones of an arbitrary order on the segment $[0, \pi]$ with separated boundary conditions, from which only one is imposed at the point π, have the form (1) too. Boundary conditions of any possible form (including integral ones) correspond to a finite-dimensional perturbation. We note that *direct* spectral problems for finite-dimensional perturbations of Volterra operators have been investigated fairly completely (see [49] and the references therein). Regarding *inverse* spectral problems for the operator (1), in [12, 18, 36, 41] inverse problems were studied of recovering the functions $g(x)$, $v(x)$ from spectral data, provided that the function $M(x,t)$ was known a priori. Also a connection with the inverse Sturm–Liouville problem was established (see [12]).

In [15–17] another inverse problem for A was studied, namely, the problem of recovering the operator M from the spectrum, provided that the functions $g(x)$, $v(x)$ are known a priori. Since a solution of this inverse problem is not unique, the special case was considered, when the kernel $M(x,t)$ depends only on the difference of its arguments, i.e. M is a convolution operator. Nonlinear inclusion of M into the representation of the characteristic function of A essentially complicates studying this inverse problem. However, a special form of the transformation operator kernel, connected with M allowed to reduce the inverse problem to some nonlinear integral equation with singularity, which was solved globally. This allowed to prove the uniqueness theorem and to obtain a constructive procedure for solving the inverse problem along with necessary and sufficient conditions of its solvability. In the next subsections we illustrate key points of this approach.

2.2 Statement of the Inverse Problem

Consider the operator $A = A(M, g, v)$ of the form (1) with

$$Mf = \int_0^x M(x-t)f(t)\,dt, \quad 0 \le x \le \pi,$$

where $M(x)$ is a complex-valued function, $M(x) \in W_2^2[0,T]$ for all $T \in (0,\pi)$, $(\pi - x)M''(x) \in L_2(0,\pi)$ and $M(0) = -i,\ M'(0) = 0$. Under these assumptions the operator M^{-1} has the form

$$M^{-1}y = \ell_1 y := iy'(x) + \int_0^x H(x-t)y(t)\,dt, \quad y(0) = 0, \tag{2}$$

where the function $H(x)$, $(\pi - x)H(x) \in L_2(0,\pi)$, is connected with $M(x)$ by the relation

$$M''(x) = H(x) + i\int_0^x M''(t)\,dt \int_0^{x-t} H(\tau)\,d\tau$$

We also assume that $g(x),\ v(x) \in W_2^1[0,\pi]$ and $a_1a_2 \neq 0$, where

$$a_1 = 1 + ig(0)v(0) + \int_0^\pi v(x)\ell_1 g(x)\,dx, \quad a_2 = ig(0)v(\pi). \tag{3}$$

Under all above assumptions we say that the operator A belongs to the class $\mathcal{A}$. The operator A has infinitely many characteristic numbers $\lambda_k,\ k \in \mathbb{Z}$, of the form (see [16])

$$\lambda_k = 2k + \alpha + \varkappa_k, \quad \lambda_k \neq 0, \quad \alpha \in \mathbb{C}, \quad \{\varkappa_k\} \in l_2, \tag{4}$$

which, in turn, coincide with zeros of the characteristic function

$$\mathcal{L}(\lambda) = 1 - \lambda\int_0^\pi v(x)g(x,\lambda)\,dx, \tag{5}$$

where

$$g(x,\lambda) = (I - \lambda M)^{-1}g = g(x) + \lambda\int_0^x M(x-t,\lambda)g(t)\,dt. \tag{6}$$

Here I is the identity operator, $M(x-t,\lambda)$ is the kernel of the integral operator $R_\lambda(M) = (E - \lambda M)^{-1}M$. Consider the following inverse problem.

Inverse Problem 1 Given the spectrum $\{\lambda_k\}_{k\in\mathbb{Z}}$; find the function $M(x)$, provided that the functions $g(x)$ and $v(x)$ are known a priori.

2.3 Transformation Operator

In this subsection we obtain a representation for the function $M(x,\lambda)$.

Lemma 1 *The relation $M(x,\lambda) = -iy(x)$ holds with $y(x)$ being a solution of the Cauchy problem*

$$\ell_1 y(x) = \lambda y(x), \quad 0 < x < \pi, \quad y(0) = 1, \tag{7}$$

where ℓ_1 is determined in (2).

Proof Since $R_\lambda(M) = M + \lambda M R_\lambda(M)$, the functions $M(x)$ and $M(x,\lambda)$ are connected by the relation

$$M(x,\lambda) = M(x) + \lambda \int_0^x M(x-t)M(t,\lambda)\,dt.$$

Hence, they have the same smoothness with respect to x, and $M(0,\lambda) = -i$. Applying the operator $(R_\lambda(M))^{-1} = M^{-1} - \lambda I$ to the function $z = R_\lambda(M)f$ with $f \in L_2(0,\pi)$, we arrive at the relation

$$i\int_0^x \frac{\partial}{\partial x}M(x-t,\lambda)f(t)\,dt + \int_0^x f(t)\,dt \int_t^x H(x-\tau)M(\tau-t,\lambda)\,d\tau = \lambda\int_0^x M(x-t,\lambda)f(t)\,dt,$$

which, by virtue of arbitrariness of f, finishes the proof. □

Denote

$$f*g(x) = \int_0^x f(x-t)g(t)\,dt, \quad f^{*1}(x) = f(x), \quad f^{*(\nu+1)}(x) = f*f^{*\nu}(x), \ \nu \ge 1.$$

Lemma 2 *The solution of the Cauchy problem (7) has the form*

$$y(x) = \exp(-i\lambda x) + \int_0^x P(x,t)\exp(-i\lambda(x-t))\,dt, \tag{8}$$

where

$$P(x,t) = \sum_{\nu=1}^{\infty} i^\nu \frac{(x-t)^\nu}{\nu!} H^{*\nu}(t). \tag{9}$$

Proof Substituting (8) into the equation in (7), we get the relation

$$P(x,x) + \int_0^x \frac{\partial}{\partial x}P(x,t)\exp(-i\lambda(x-t))\,dt = i\int_0^x H(t)\exp(-i\lambda(x-t))\,dt$$

$$+i\int_0^x \exp(-i\lambda(x-t))\,dt \int_0^t H(t-\tau)P(x-t+\tau,\tau)\,d\tau, \quad 0 < x < \pi.$$

Thus, representation (8) holds, if the function $P(x,t)$ is a solution of the Cauchy problem

$$\frac{\partial}{\partial x}P(x,t) = iH(t) + i\int_0^t H(t-\tau)P(x-t+\tau,\tau)\,d\tau, \quad P(x,x)=0, \quad 0<t<x<\pi,$$

which, in turn, is equivalent to the integral equation

$$P(x,t) = i(x-t)H(t) + i\int_0^{x-t} ds \int_0^t H(t-\tau)P(s+\tau,\tau)\,d\tau, \quad 0\le t\le x\le \pi. \tag{10}$$

For solving it by the method of successive approximations we put

$$P_1(x,t) = i(x-t)H(t), \quad P_{\nu+1}(x,t) = i\int_0^{x-t} ds\int_0^t H(t-\tau)P_\nu(s+\tau,\tau)\,d\tau.$$

Then by induction we get

$$P_\nu(x,t) = i^\nu \frac{(x-t)^\nu}{\nu!}H^{*\nu}(t).$$

The series in the right-hand side of (9) converges uniformly for $0\le t\le x\le\pi$ and gives the solution of (10). □

Formula (8) determines the *transformation operator* $I+P$, where $Pf = \int_0^x P(x,x-t)f(t)\,dt$, which connects the solution $y_0(x) = \exp(-i\lambda x)$ of the unperturbed Cauchy problem (7) possessing $H(x)=0$ with the solution $y(x)$ of the problem (7) with arbitrary $H(x)$, i.e. $y(x) = (I+P)y_0(x)$.

Transformation operators are closely related to the notion of similarity (or linear equivalence) of linear operators. For example, one can show that the following relation holds:

$$M(I+P) = (I+P)M_0, \quad M_0 f = -i\int_0^x f(t)\,dt. \tag{11}$$

2.4 Main Nonlinear Integral Equation

Denote

$$\mu_0(x) = \int_x^\pi v(t)g(t-x)\,dt, \quad \mu(x) = \mu_0(x) + \int_x^\pi P(t,t-x)\mu_0(t)\,dt, \tag{12}$$

then, in particular, $\mu(x)\in W_2^2[0,\pi]$.

Lemma 3 *The characteristic function of the operator A has the form*

$$\mathcal{L}(\lambda) = a_1 - a_2 \exp(-i\lambda\pi) + \int_0^\pi w(x)\exp(-i\lambda x)\,dx, \quad w(x) \in L_2(0,\pi). \tag{13}$$

Here the numbers a_1, a_2 are determined by (3) and

$$w(x) = -i\mu''(x). \tag{14}$$

Proof Substituting (6) into (5) and changing the order of integration, we obtain

$$\mathcal{L}(\lambda) = 1 - \mu_0(0)\lambda - \lambda^2 \int_0^\pi \mu_0(x) M(x,\lambda)\,dx.$$

Substituting here $M(x,\lambda) = -iy(x)$, where $y(x)$ is determined by (8), and using (12) we get

$$\mathcal{L}(\lambda) = 1 - \mu_0(0)\lambda + i\lambda^2 \int_0^\pi \mu(x)\exp(-i\lambda x)\,dx.$$

Integrating by parts twice and taking into account that $\mu(0) = \mu_0(0)$, $\mu(\pi) = 0$, $\mu'(0) = i(a_1 - 1)$, $\mu'(\pi) = ia_2$ we arrive at (13). □

Relation (14) can be considered as a nonlinear equation with respect to $H(x)$. Indeed, differentiating (12) twice and using (14) we get the equation

$$(\pi - x)H(x) = \varphi(x) + \sum_{\nu=1}^{\infty}\Big(b_\nu(x)H^{*\nu}(x) + \int_0^x B_\nu(x,t)H^{*\nu}(t)\,dt\Big), \quad 0 < x < \pi, \tag{15}$$

where

$$\varphi(x) = \frac{iw(\pi - x) - \check{\mu}_0''(x)}{a_2}, \quad \check{\mu}_0(x) = \mu_0(\pi - x), \quad b_1(x) \equiv 0, \quad b_\nu(x) = i^{\nu+1}\frac{(\pi - x)^\nu}{\nu!}, \quad \nu \geq 2,$$

$$B_\nu(x,t) = -\frac{i^\nu}{a_2}\frac{(\pi - x)^{\nu-2}}{\nu!}\Big(\nu(\nu-1)\check{\mu}_0(x-t) - 2\nu(\pi - x)\check{\mu}_0'(x-t) + (\pi - x)^2\check{\mu}_0''(x-t)\Big). \tag{16}$$

Equation (15) is called *main nonlinear integral equation* of Inverse Problem 1. Its solution is complicated both by its nonlinearity and also by the singularity connected with presence of the multiplier $(\pi - x)$ in the left-hand side. The following theorem holds (see Theorem 2.1 in [16]).

Theorem 1 *For any function $\varphi(x) \in L_2(0,\pi)$, satisfying the condition*

$$\int_0^\pi (\pi - x)\varphi(x)\,dx = 0, \tag{17}$$

Eq. (15) has a unique solution $H(x)$, $(\pi - x)H(x) \in L_2(0,\pi)$.

2.5 Solution of a Nonlinear Equation Without Singularity

Consider the equation

$$y(x) = \xi(x) + \sum_{\nu=1}^{\infty} \Big(\psi_\nu(x) y^{*\nu}(x) + \int_0^x \Psi_\nu(x,t) y^{*\nu}(t)\,dt\Big), \quad 0 < x < T, \tag{18}$$

where $\psi_1(x) = 0$. Let the functions $\psi_\nu(x)$, $\Psi_\nu(x,t)$ be square-integrable and let there exist square-integrable functions $u(x)$, $U(x,t)$ such that $|\psi_\nu(x)| \le u(x)$, $|\Psi_\nu(x,t)| \le U(x,t)$, $0 < t < x < T$, for all ν. The following theorem holds (see Theorem 2.2 in [16]).

Theorem 2 *For any function $\xi \in L_2(0,T)$ Eq. (18) has a unique solution $y \in L_2(0,T)$.*

Proof Let us show that for sufficiently small $\delta > 0$ Eq. (18) has a unique solution $y(x)$, $0 < x < \delta$, in the domain $B_\delta = \{y : \|y\|_\delta \le 1/2\}$, where $\|\cdot\|_\delta$ is the norm in $L_2(0,\delta)$. Denote

$$\psi_\nu y = \psi_\nu(x) y^{*\nu}(x) + \int_0^x \Psi_\nu(x,t) y^{*\nu}(t)\,dt, \quad \Psi y = \xi + \sum_{\nu=1}^{\infty} \psi_\nu y. \tag{19}$$

Let $y, \tilde y \in L_2(0,\delta)$. The Cauchy–Bunyakovsky–Schwarz inequality yields $|y * \tilde y(x)| \le \|y\|_\delta \|\tilde y\|_\delta$ for all $x \in [0,\delta]$. For convenience it is assumed that $\delta \le 1$. Then by induction we get the estimate $|y^{*\nu}(x)| \le \|y\|_\delta^\nu$, $\nu \ge 2$, and consequently

$$\|\psi_\nu y\|_\delta \le C_\delta \|y\|_\delta^\nu, \quad \text{where} \quad C_\delta = \|u\|_\delta + \Big(\int_0^\delta \int_0^x U^2(x,t)\,dt\,dx\Big)^{\frac12}. \tag{20}$$

Moreover, since

$$y^{*\nu} - \tilde y^{*\nu} = (y - \tilde y) * (y^{*(\nu-1)} + y^{*(\nu-2)} * \tilde y^{*1} + \ldots + \tilde y^{*(\nu-1)}), \quad \nu \ge 2,$$

and $\|y^{*\nu}\|_\delta \le \|y\|_\delta^\nu$, then we arrive at the estimate

$$\|\psi_\nu y - \psi_\nu \tilde y\|_\delta \le C_\delta \nu (\max\{\|y\|_\delta, \|\tilde y\|_\delta\})^{\nu-1} \|y - \tilde y\|_\delta. \tag{21}$$

Let us choose δ, so that $C_\delta < 1/4$, $\|\xi\|_\delta \le 1/4$. Then it follows from (20), (21) that the operator Ψ maps B_δ into B_δ and it is a contraction in B_δ. Indeed let $y,\ \tilde{y} \in B_\delta$, then

$$\|\Psi y\|_\delta \le \|\xi\|_\delta + \sum_{\nu=1}^{\infty} \|\psi_\nu y\|_\delta \le \|\xi\|_\delta + C_\delta \sum_{\nu=1}^{\infty} \|y\|_\delta^\nu \le \|\xi\|_\delta + C_\delta < \frac{1}{2},$$

$$\|\Psi y - \Psi\tilde{y}\|_\delta \le \sum_{\nu=1}^{\infty} \|\psi_\nu y - \psi_\nu \tilde{y}\|_\delta \le C_\delta \sum_{\nu=1}^{\infty} \nu(\max\{\|y\|_\delta, \|\tilde{y}\|_\delta\})^{\nu-1} \|y - \tilde{y}\|_\delta \le \alpha \|y - \tilde{y}\|_\delta,$$

where

$$\alpha = C_\delta \sum_{\nu=1}^{\infty} \frac{\nu}{2^{\nu-1}} = 4C_\delta < 1.$$

The contracting mapping principle yields that Eq. (18) has a unique solution in B_δ.

Suppose that $y = y_1(x)$ is a solution of Eq. (18) for $0 < x < \delta$, $\delta \in (0, \pi)$. Let us show that (18) has a solution $y(x)$ in $L_2(0, 2\delta)$, which coincides with $y_1(x)$ on $(0, \delta)$. We seek $y(x)$ in the form $y(x) = y_1(x) + y_2(x)$, where $y_1(x) = 0$ for $\delta < x < 2\delta$ and $y_2(x) = 0$ for $0 < x < \delta$. By induction one can prove the following representation

$$y^{*\nu}(x) = (y_1 + y_2)^{*\nu}(x) = y_1^{*\nu}(x) + \sum_{k=1}^{\nu-1} \binom{\nu}{k} (y_1^{*(\nu-k)} * y_2^{*k})(x) + y_2^{*\nu}(x), \quad \nu \ge 2, \tag{22}$$

where $\binom{\nu}{k} = \nu!/(k!(\nu-k)!)$. Since $y_2(x) = 0$ for $(0, \delta)$, then $y_2^{*2}(x) \equiv 0$ on $[0, \delta]$ and

$$y_2^{*2}(x) = \int_\delta^x y_2(t) y_2(x-t)\, dt = \int_0^{x-\delta} y_2(x-t) y_2(t)\, dt = 0, \quad \delta \le x \le 2\delta.$$

Consequently, for $\nu \ge 2$ we have $y_2^{*\nu}(x) \equiv 0$ on $[0, 2\delta]$ and, according to (22), we get the expression

$$y^{*\nu}(x) = y_1^{*\nu}(x) + \nu(y_1^{*(\nu-1)} * y_2)(x), \quad 0 \le x \le 2\delta, \quad \nu \ge 2. \tag{23}$$

Substituting (23) into (18) we arrive at the linear Volterra equation of second order with respect to the function $y_2(x)$:

$$y_2(x) = \zeta(x) + \int_\delta^x \Psi(x, t) y_2(t)\, dt, \quad \delta < x < 2\delta, \tag{24}$$

where the functions

$$\zeta(x) = \xi(x) + \sum_{\nu=1}^{\infty} \Big(\psi_\nu(x) y_1^{*\nu}(x) + \int_0^x \Psi_\nu(x,t) y_1^{*\nu}(t)\, dt \Big),$$

$$\Psi(x,t) = \Psi_1(x,t) + \sum_{\nu=2}^{\infty} \nu \Big(\psi_\nu(x) y_1^{*(\nu-1)}(x-t) + \int_0^{x-t} \Psi_\nu(x,t+\tau) y_1^{*(\nu-1)}(\tau)\, d\tau \Big)$$

are square-integrable. Equation (24) has a unique solution and, therefore, the function $y(x) = y_1(x) + y_2(x)$ is a solution of Eq. (18) in $L_2(0, 2\delta)$ that coincides with $y_1(x)$ on $(0, \delta)$. Continuing this process after a finite number of steps we obtain a solution of (18) on the entire interval $(0, T)$, which coincides with $y_1(x)$ on $(0, \delta)$. Note that a solution with this property is unique. Thus, the existence of the solution is proved. Let $\tilde{y} \in L_2(0, T)$ be another solution of (18). For sufficiently small $\delta > 0$ both the functions $y(x)$ and $\tilde{y}(x)$ belong to the domain B_δ. According to the first part of the proof, they are equal a.e. on $(0, \delta)$. Hence, $y(x) = \tilde{y}(x)$ a.e. on $(0, T)$. □

Remark 1 The operator Ψ determined in (19) belongs to the class $\mathcal{E}_T$ (see [39]) as well as to the class $\mathcal{E}_{T,1}$ (see [48]). Thus, Theorem 2 can be obtained as a corollary from Theorem 4.2 in [39] or from Theorem 1 in [48].

2.6 Proof of Theorem 1

Note that, according to Theorem 2, the main Eq. (15) has a unique locally square-integrable solution $H(x)$, i.e. such that $H(x) \in L_2(0, T)$ for all $T \in (0, \pi)$. However, so far it is impossible to say anything about integrability of $H(x)$ in the vicinity of the point π. In the present subsection we show how to prove that $(\pi - x)H(x) \in L_2(0, \pi)$ using condition (17) as well as the special structure of the functions $b_\nu(x),\ B_\nu(x,t),\ \nu \ge 1$.

By virtue of Theorem 2, there exists a unique square-integrable solution $H(x) = H_1(x)$ of Eq. (15) on the interval $(0, \pi/2)$. As in its proof we seek the solution on $(0, \pi)$ in the form $H(x) = H_1(x) + H_2(x)$, where $H_1(x) = 0$ on $(\pi/2, \pi)$ and $H_2(x) = 0$ on $(0, \pi/2)$, and arrive at the following equation with respect to $H_2(x)$:

$$(\pi - x) H_2(x) = \zeta(x) + \int_{\frac{\pi}{2}}^{x} B(x,t) H_2(t)\, dt, \quad \frac{\pi}{2} < x < \pi, \tag{25}$$

where

$$\zeta(x) = \varphi(x) + \sum_{\nu=1}^{\infty} \Big(b_\nu(x) H_1^{*\nu}(x) + \int_0^x B_\nu(x,t) H_1^{*\nu}(t)\, dt \Big),$$

$$B(x,t) = B_1(x,t) + \sum_{\nu=2}^{\infty} \nu \Big(b_\nu(x) H_1^{*(\nu-1)}(x-t) + \int_0^{x-t} B_\nu(x, t+\tau) H_1^{*(\nu-1)}(\tau)\, d\tau \Big).$$

Denote $h_2(x) = (\pi - x) H_2(x)$. According to (16), the following equation is equivalent to (25):

$$h_2(x) = \zeta(x) + 2 \int_{\frac{\pi}{2}}^{x} \frac{h_2(t)\, dt}{\pi - t} + \int_{\frac{\pi}{2}}^{x} G(x,t) h_2(t)\, dt, \quad \frac{\pi}{2} < x < \pi, \tag{26}$$

where the function

$$G(x,t) = \frac{i}{a_2(\pi - t)} \Big\{ 2 \int_t^x \check{\mu}_0''(x-\tau)\, d\tau - (\pi - x) \check{\mu}_0''(x-t) \Big\}$$

$$+ \frac{1}{\pi - t} \sum_{\nu=2}^{\infty} \nu \Big(b_\nu(x) H_1^{*(\nu-1)}(x-t) + \int_0^{x-t} B_\nu(x, t+\tau) H_1^{*(\nu-1)}(\tau)\, d\tau \Big)$$

is square-integrable on the triangle $\pi/2 < t < x < \pi$. It remains to show that $h_2(x) \in L_2(\pi/2, \pi)$. For this purpose we need the following two lemmas.

Lemma 4 *Let $\eta,\ \theta$ be real numbers, $\theta \ge 0$. The solution $y(x)$ of the equation*

$$y(x) = f(x) + \eta \int_a^x \frac{y(t)\, dt}{b - t}, \quad a < x < b,$$

satisfies the condition $(b-x)^\theta y(x) \in L_2(a,b)$ if and only if one of the following conditions (depending on the difference $\eta - \theta$) holds:

(1) $(b-x)^\theta f(x) \in L_2(a,b)$ *for* $\eta - \theta < 1/2$;
(2) $(b-x)^\theta f(x) \in L_2(a,b)$,

$$\int_a^b (b-x)^{\eta - 1} f(x)\, dx = 0$$

for $\eta - \theta > 1/2$.

Lemma 5 *Fix $\eta \geq 0$ and let $(b-x)^\eta f(x) \in L_2(a,b)$. Then the equation*

$$y(x) = f(x) + \eta \int_a^x \frac{y(t)\,dt}{b-t} + \int_a^x G(x,t)y(t)\,dt, \quad a < x < b,$$

where

$$\int_a^b \int_a^x |G(x,t)|^2\,dt\,dx < \infty,$$

has a unique solution $y(x)$, $(b-x)^\eta y(x) \in L_2(a,b)$.

In this subsection we will use Lemmas 4 and 5 for $\eta = 2$. For this case they were proved in [16]. For arbitrary η the proof can be found in [39].

Applying Lemma 5 to Eq. (26), we get $(\pi - x)^2 h_2(x) \in L_2(\pi/2, \pi)$, i.e. $(\pi - x)^3 H(x) \in L_2(0,\pi)$. It remains to show that (17) implies $(\pi - x)H(x) \in L_2(0,\pi)$. According to (15) and (16), we have

$$h(x) = \varphi(x) + \alpha_1(x) + \alpha_2(x) + 2\int_0^x \frac{h(t)\,dt}{\pi - t}, \tag{27}$$

where $h(x) = (\pi - x)H(x)$ and

$$\alpha_1(x) = i\int_0^x \frac{h(t)}{a_2(\pi - t)}\Big\{2\int_0^{x-t} \check{\mu}_0''(\tau)\,d\tau - (\pi - x)\check{\mu}_0''(x-t)\Big\}\,dt,$$

$$\alpha_2(x) = \sum_{\nu=2}^{\infty}\Big(b_\nu(x)H^{*\nu}(x) + \int_0^x B_\nu(x,t)H^{*\nu}(t)\,dt\Big).$$

The following two lemmas hold (see [16]).

Lemma 6 *Let $(\pi - x)^\theta h(x) \in L_2(0,\pi)$ for some $\theta \in [1/2, 2]$. Then the functions $(\pi - x)^{\theta - 1/2}\alpha_1(x)$ and $(\pi - x)^{\theta - 1}\alpha_2(x)$ also belong to $L_2(0,\pi)$.*

Lemma 7 *If $(\pi - x)^\theta h(x) \in L_2(0,\pi)$ for some $\theta < 2$, then*

$$\int_0^\pi (\pi - x)\alpha_k(x)\,dx = 0, \quad k = 1, 2. \tag{28}$$

Consider relation (27). Since $(\pi - x)^2 h(x) \in L_2(0,\pi)$, by Lemma 6 we have $(\pi - x)^{3/2}\alpha_k(x) \in L_2(0,\pi)$, $k = 1, 2$. By virtue of Lemma 4, we arrive at $(\pi - x)^{3/2+\varepsilon} h(x) \in L_2(0,\pi)$ for all $\varepsilon > 0$. Then Lemma 7 gives (28). Taking also (17) into account and applying Lemma 6 along with Lemma 4 four more times, we finally arrive at $h(x) \in L_2(0,\pi)$, which finishes the proof of Theorem 1.

2.7 Solution of Inverse Problem 1

Based on the solution of the main Eq. (15), i.e. on Theorem 1, one can obtain a global solution of the inverse problem (for more details see [16]). In particular, the following uniqueness theorem holds.

Theorem 3 *Specification of the spectrum* $\{\lambda_k\}_{k\in\mathbb{Z}}$ *uniquely determines the function* $M(x)$, *provided that the functions* $g(x)$ *and* $v(x)$ *are known a priori.*

The next theorem gives necessary and sufficient conditions for solvability of the inverse problem.

Theorem 4 *Let arbitrary complex-valued functions* $g(x),\ v(x) \in W_2^1[0,\pi]$, $g(0)v(\pi) \neq 0$, *be given. For an arbitrary complex sequence* $\{\lambda_k\}_{k\in\mathbb{Z}}$ *to be the spectrum of a certain operator* $A = A(M, g, v) \in \mathcal{A}$ *it is necessary and sufficient to have the form (4) and to satisfy the so-called concordance conditions*

$$p = -\int_0^\pi g(x)v(x)\,dx, \qquad \gamma\exp(i\alpha\pi) = ig(0)v(\pi),$$

where

$$p = \begin{cases} \dfrac{i\pi}{\exp(-i\alpha\pi)-1} + \displaystyle\sum_{k=-\infty}^{\infty}\left(\frac{1}{2k+\alpha} - \frac{1}{\lambda_k}\right), & \exp(i\alpha\pi) \neq 1, \\ \dfrac{\pi}{2i} - \dfrac{1}{\lambda_{-\frac{\alpha}{2}}} + \displaystyle\sum_{k=-\infty,\, k\neq -\frac{\alpha}{2}}^{\infty}\left(\frac{1}{2k+\alpha} - \frac{1}{\lambda_k}\right), & \exp(i\alpha\pi) = 1, \end{cases}$$

$$\gamma = \begin{cases} (1-\exp(i\alpha\pi))^{-1}\displaystyle\prod_{k=-\infty}^{\infty}\frac{2k+\alpha}{\lambda_k}, & \exp(i\alpha\pi) \neq 1, \\ \dfrac{i}{\pi\lambda_{-\frac{\alpha}{2}}}\displaystyle\prod_{k=-\infty,\, k\neq -\frac{\alpha}{2}}^{\infty}\frac{2k+\alpha}{\lambda_k}, & \exp(i\alpha\pi) = 1. \end{cases}$$

The proof is constructive and gives an algorithm for solving the inverse problem (see [17]). We also note that Theorem 4, in particular, implies that in Theorem 3 it is sufficient to specify characteristic numbers with exception of any two.

3 Convolution Integro-Differential Operator

3.1 Statement of the Inverse Problem and Main Results

In this section we illustrate the above approach to studying an inverse problem for the so-called convolution integro-differential operator of the second order [19].

Let $\{\lambda_k\}_{k\ge 1}$ be the spectrum of the boundary value problem $L = L(M)$ of the form

$$\ell y := -y'' + \int_0^x M(x-t)y'(t)\,dt = \lambda y, \quad 0 < x < \pi, \quad y(0) = y(\pi) = 0, \tag{29}$$

where $M(x)$ is a complex-valued function and $(\pi - x)M(x) \in L_2(0,\pi)$. By the standard method (see, e.g., [3]) one can obtain the asymptotics of $\{\lambda_k\}_{k\ge 1}$. Namely, the following theorem holds.

Theorem 5 *Eigenvalues λ_k, $k \ge 1$, of the problem L have the form*

$$\lambda_k = (k + \varkappa_k)^2, \quad \{\varkappa_k\} \in l_2. \tag{30}$$

Consider the following inverse problem.

Inverse Problem 2 Given $\{\lambda_k\}_{k\ge 1}$; find $M(x)$.

The following theorem gives uniqueness of solution of Inverse Problem 2 along with its global solvability.

Theorem 6

(i) For arbitrary complex numbers λ_k, $k \ge 1$, of the form (30) *there exists a unique (up to values on a set of measure zero) function $M(x)$, $(\pi - x)M(x) \in L_2(0,\pi)$, such that $\{\lambda_k\}_{k\ge 1}$ is the spectrum of the corresponding eigenvalue problem $L(M)$. In other words, asymptotics (30) is a necessary and sufficient condition for solvability of Inverse Problem 2.*

(ii) The function $M(x)$ satisfies the additional smoothness condition: $M(x) \in W_2^1[0,T]$ for each $T \in (0,\pi)$, $(\pi - x)M'(x) \in L_2(0,\pi)$ if and only if

$$\lambda_k = \Big(k + \frac{\omega}{k} + \frac{\varkappa_{k,1}}{k}\Big)^2, \quad \{\varkappa_{k,1}\} \in l_2, \quad \omega - const. \tag{31}$$

Moreover, $M(0) = 2\omega$.

The proof of Theorem 6 is constructive and gives an algorithm for solving the inverse problem (see [19] for details). This proof is based on a special form of the kernel of a transformation operator for (29) (see the next subsection). This form allows one to reduce Inverse Problem 2 to solving some nonlinear integral equation, whose global solvability is proved in Sect. 3.3. For the Robin boundary conditions $y'(0) - hy(0) = y'(\pi) + Hy(\pi) = 0$ analogous results were obtained in [25].

3.2 Transformation Operator

Let $y = S(x, \lambda)$ be a solution of the equation in (29) satisfying the initial conditions

$$y(0) = 0, \quad y'(0) = 1. \tag{32}$$

Eigenvalues of L coincide with zeros of its characteristic function $\Delta(\lambda) := S(\pi, \lambda)$. In order to obtain an appropriate representation for $S(x, \lambda)$, we consider the function $H(x)$ that satisfies the equation

$$M(x) = 2iH(x) + \int_0^x dt \int_0^t H(t-\tau)H(\tau)\,d\tau, \quad 0 < x < \pi. \tag{33}$$

Note that unique solvability of Eq. (33) follows, e.g., from Theorem 2. Moreover, it is easy to show that $(\pi - x)H(x) \in L_2(0, \pi)$. As will be seen below it is convenient to recover first $H(x)$ and then one can construct $M(x)$ via (33).

Lemma 8 *Let $\rho^2 = \lambda$. Then the following representation holds:*

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x P(x, t)\frac{\sin \rho(x-t)}{\rho}\,dt, \tag{34}$$

where the function $P(x, t)$ is determined in (9).

Proof Consider the integro-differential operator ℓ_1 determined in (2). Let $y(x) \in W_2^2[0, T]$ for each $T \in (0, \pi)$, then we can calculate

$$\ell_1(\ell_1 y) = -y'' + \int_0^x \Big(2iH(t) + \int_0^t H^{*2}(\tau)\,d\tau\Big)y'(x-t)\,dt + y(0)\Big(iH(x) + \int_0^x H^{*2}(t)\,dt\Big).$$

Thus, by virtue of (33), we have $\ell u = \ell_1(\ell_1 u)$ for any sufficiently smooth function $u(x)$, $u(0) = 0$. Let $y = e(x, \lambda)$ be a solution of the Cauchy problem (7). Recall that, by virtue of Lemma 2, it has the form (8). Hence, taking into account that $e(0, \rho) = 1$ and $e'(0, \rho) = -i\rho$, we get the identity

$$S(x, \lambda) = \frac{e(x, -\rho) - e(x, \rho)}{2i\rho},$$

which along with (8) give (34). □

Remark 2 The proof of Lemma 8 is actually based on extracting the square root from a convolution operator (more precisely from its inverse). For more details on extraction of roots from convolution operators see also [50, 51]. Formula (34) means that the transformation operator $I + P$ introduced in Sect. 2.3 for a first-order integro-differential equation applies also for the second-order one. Namely, it connects the solution $y_0(x) = \rho^{-1} \sin \rho x$ of the Cauchy problem for the equation in (29) possessing $M(x) = 0$ under the initial conditions (32) with the solution $y(x)$ of the corresponding Cauchy problem with arbitrary $M(x)$.

The same transformation operator $I + P$ can be used also for convolution integro-differential equations of arbitrary order (see [37]). Namely, the solution $S_n(x, \lambda)$ of the Cauchy problem

$$i^n y^{(n)} + \int_0^x M(x-t) y^{(n-1)}(t)\, dt = \lambda y, \quad 0 < x < \pi, \qquad y^{(j)}(0) = \delta_{j,n-1}, \quad j = \overline{0, n-1}, \tag{35}$$

has the form

$$S_n(x, \lambda) = S_{n,0}(x, \lambda) + \int_0^x P(x,t) S_{n,0}(x-t, \lambda)\, dt, \tag{36}$$

where the function

$$S_{n,0}(x,\lambda) = \frac{1}{n(-i\rho)^{n-1}} \sum_{j=1}^{n} \omega_j \exp(-i\omega_j \rho x), \quad \rho^n = \lambda, \quad \omega_j = \exp\Big(\frac{2\pi i (j-1)}{n}\Big), \quad j = \overline{1, n},$$

is the solution of the unperturbed Cauchy problem (35) with $M(x) = 0$, while $P(x,t)$ is determined by formula (9) with $H(x)$ being a solution of the nonlinear integral equation

$$M(x) = n i^{n-1} H(x) + \sum_{j=2}^{n} \binom{n}{j} i^{n-j} \int_0^x \frac{(x-t)^{j-2}}{(j-2)!} H^{*j}(t)\, dt, \quad 0 < x < \pi.$$

More briefly this effect can be demonstrated by using (11). Namely, it is easy to see that relation (11) implies $M^n(I+P) = (I+P)M_0^n$ for all $n \in \mathbb{N}$ with one and the same P.

Eventually, according to Lemma 8, we have

$$\Delta(\lambda) = \frac{\sin \rho\pi}{\rho} + \int_0^\pi w(x) \frac{\sin \rho x}{\rho}\, dx, \quad \rho^2 = \lambda, \quad w(x) \in L_2(0,\pi),$$

where

$$w(\pi - x) = \sum_{\nu=1}^{\infty} i^\nu \frac{(\pi - x)^\nu}{\nu!} H^{*\nu}(x), \quad 0 < x < \pi. \tag{37}$$

3.3 The Main Equation

Relation (37) can be considered as a nonlinear integral equation with respect to $H(x)$, which is called *main equation* of Inverse Problem 2. For briefness denote

$$W_l := \{f(x) : \ f(x) \in W_2^l[0, T] \ \text{ for each } \ T \in (0, \pi), \ (\pi - x) f^{(l)}(x) \in L_2(0, \pi)\}.$$

In particular, $W_0 = \{f(x) : \ (\pi - x) f(x) \in L_2(0, \pi)\}$.

Theorem 7

(i) For any $w(x) \in L_2(0, \pi)$ Eq. (37) has a unique solution $H(x) \in W_0$.
(ii) The function $H(x)$ belongs to W_1 if and only if $w(x) \in W_2^1[0, \pi]$ and $w(0) = 0$. Moreover, in this case $w(\pi) = i\pi H(0)$.

Proof (i) After division by $\pi - x$ Eq. (37) takes the form (18) on intervals $(0, T)$, $T \in (0, \pi)$. Then, by virtue of Theorem 2, it has a unique locally square-integrable solution $H(x) \in L_2(0, T)$, $T \in (0, \pi)$. Representing it in the form $H(x) = H_1(x) + H_2(x)$ where $H_1(x) \in L_2(0, \pi)$ and $H_2(x) = 0$ on $(0, \pi/2)$ we have

$$H^{*\nu}(x) = H_1^{*\nu}(x) + \nu H_1^{*(\nu-1)} * H_2(x), \quad \nu \geq 2. \tag{38}$$

Substituting this into (37) we arrive at

$$w(\pi - x) - \mu_1(x) - i(\pi - x) H_1(x) = i(\pi - x) H_2(x) + \int_0^x Q(x, t)\, i(\pi - t) H_2(t)\, dt,$$

where

$$\mu_1(x) = \sum_{\nu=2}^{\infty} i^\nu \frac{(\pi - x)^\nu}{\nu!} H_1^{*\nu}(x), \quad Q(x, t) = \frac{\pi - x}{\pi - t} \sum_{\nu=1}^{\infty} i^\nu \frac{(\pi - x)^\nu}{\nu!} H_1^{*\nu}(x - t)$$

are square-integrable functions. Hence $H(x) \in W_0$.

(ii) Necessity is obvious. Let us prove sufficiency. Denote

$$\mu(x) = \sum_{\nu=2}^{\infty} i^\nu \frac{(\pi - x)^\nu}{\nu!} H^{*\nu}(x).$$

It is sufficient to show that there exists a function $N(x) \in W_0$ such that

$$H(x) = \frac{w(\pi)}{i\pi} + \int_0^x N(t)\, dt.$$

Substituting this into (37) and differentiating we arrive at the following nonlinear equation with respect to $N(x)$:

$$i(\pi - x)N(x) = -w'(\pi - x) + \frac{w(\pi)}{\pi} + i\int_0^x N(t)\,dt - \mu'(x), \quad 0 < x < \pi, \tag{39}$$

which, after division by $i(\pi - x)$, takes the form (18) on intervals $(0, T)$, $T \in (0, \pi)$. Then, by virtue of Theorem 2, it has a unique locally square-integrable solution $N(x) \in L_2(0, T)$, $T \in (0, \pi)$. Representing the function $h(x) := i(\pi - x)N(x)$ in the form $h(x) = h_1(x) + h_2(x)$ where $h_1(x) = 0$ on $(\pi/2, \pi)$ and $h_2(x) = 0$ on $(0, \pi/2)$ and putting

$$H_1(x) = \frac{w(\pi)}{i\pi} - i\int_0^x \frac{h_1(t)}{\pi - t}\,dt, \quad H_2(x) = -i\int_0^x \frac{h_2(t)}{\pi - t}\,dt$$

we have (38). Then the following relation is equivalent to (39):

$$h(x) = -w'(\pi - x) + \frac{w(\pi)}{\pi} + \int_0^x \frac{h(t)\,dt}{\pi - t} + \alpha(x), \quad 0 < x < \pi, \tag{40}$$

where

$$\alpha(x) = -\mu'(x) = -\mu_1'(x) - \int_0^x G(x,t)h_2(t)\,dt, \tag{41}$$

$$G(x,t) = \frac{1}{\pi - t}\sum_{\nu=1}^{\infty} i^\nu \frac{(\pi - x)^\nu}{\nu!}\Big((\pi - x)H_1^{*\nu}(x - t) - (\nu + 1)\int_0^{x-t} H_1^{*\nu}(\tau)\,d\tau\Big).$$

It remains to prove that $h(x) \in L_2(0, \pi)$. Applying Lemma 5 (for $\eta = 1$) to (40), (41) we deduce that $(\pi - x)h_2(x) \in L_2(0, \pi)$. According to (41), this yields $\alpha(x) \in L_2(0, \pi)$. Further, besides

$$\int_0^\pi \alpha(x)\,dx = -\mu(\pi) = -\Big(\mu_1(x) + \sum_{\nu=2}^{\infty} i^\nu \frac{(\pi - x)^\nu}{(\nu - 1)!}\int_0^x H_1^{*(\nu-1)}(x - t)H_2(t)\,dt\Big)\Big|_{x\to\pi} = 0,$$

the assumption on the function $w(x)$ gives

$$\int_0^\pi \Big[-w'(\pi - x) + \frac{w(\pi)}{\pi}\Big]dx = w(0) = 0.$$

Thus, applying Lemma 4 (for $\eta = 1$, $\theta = 0$) to (40), we arrive at $h(x) \in L_2(0, \pi)$. □

4 Convolutional Perturbation of the Sturm–Liouville Operator

4.1 Historical Notes and the Main Result

In the present section a more general situation is considered, when the transformation operator kernel cannot be represented as a series of convolutional powers of the unknown function. However, a detailed study of dependence of the kernel on the unknown function allows one to prove the global solvability of the corresponding nonlinear equation (see Sect. 4.3).

Let $\{\lambda_n\}_{n\geq 1}$ be the spectrum of the boundary value problem $L = L(q, M)$ of the form

$$-y'' + q(x)y + \int_0^x M(x-t)y(t)\,dt = \lambda y, \quad 0 < x < \pi, \quad y(0) = y(\pi) = 0, \tag{42}$$

where $q(x)$ and $M(x)$ are complex-valued functions such that $q(x) \in L_2(0, \pi)$ and

$$(\pi - x)M(x) \in L_2(0, \pi). \tag{43}$$

Then the following asymptotics holds (see [14]):

$$\lambda_n = \left(n + \frac{\omega}{n} + \frac{\varkappa_n}{n}\right)^2, \quad \omega = \frac{1}{2\pi}\int_0^\pi q(x)\,dx, \quad \{\varkappa_n\} \in l_2. \tag{44}$$

Consider the following inverse problem.

Inverse Problem 3 Given the spectrum $\{\lambda_n\}_{n\geq 1}$; find the function $M(x)$, provided that the potential $q(x)$ is known a priori.

The first detailed study of this inverse problem was undertaken in [14]. In particular, the uniqueness theorem was proved and a local solvability of the inverse problem was established. Specifically, it was proved that a complex sequence $\{\tilde{\lambda}_n\}_{n\geq 1}$ is the spectrum of a certain problem $L(q, \tilde{M})$, if it is sufficiently close in the l_2-metric to the spectrum $\{\lambda_n\}_{n\geq 1}$ of some model problem $L(q, M)$. Moreover, the stability of the solution was established. The proof was based on developing Borg's idea [1] for the classical Sturm–Liouville operator (see also [4]).

By the development of the approach illustrated in the previous sections, in [21] the global solution of this inverse problem was obtained. Namely, the following theorem holds.

Theorem 8 *Let a complex-valued function $q(x) \in L_2(0, \pi)$ be given. Then for any sequence of complex numbers $\{\lambda_n\}_{n\geq 1}$ of the form (44) there exists a unique (up to values on a set of measure zero) function $M(x)$, satisfying condition (43), such that $\{\lambda_n\}_{n\geq 1}$ is the spectrum of the corresponding boundary value problem $L(q, M)$.*

Thus, asymptotics (44) is a necessary and sufficient condition for solvability of Inverse Problem 3.

A generalization of this result to the case of Robin boundary conditions was obtained in [46].

Remark 3 In [14] the following conditions on the function $M(x)$ were imposed:

$$(\pi - x)M(x), \quad \int_0^x M(t)\,dt \in L(0,\pi), \quad (\pi - x)M(x) - \int_0^x M(t)\,dt \in L_2(0,\pi). \tag{45}$$

However, from Theorem 8 along with the uniqueness theorem in [14] it follows that (45) is equivalent to (43). This can also be proved directly using Lemma 4 for $\eta = 1$.

In the next subsection the transformation operators related to (42) is studied.

4.2 Transformation Operator

Consider the linear integral equation

$$\begin{aligned} F(x,t,\tau) = F_0(x,t,\tau) + \frac{1}{2}\Big(\int_t^x q(s)\,ds \int_\tau^t F(s,\xi,\tau)\,d\xi + \int_{\frac{t+\tau}{2}}^t q(s)\,ds \int_\tau^{2s-t} F(s,\xi,\tau)\,d\xi \\ - \int_{\frac{\tau-t}{2}+x}^x q(s)\,ds \int_\tau^{2(s-x)+t} F(s,\xi,\tau)\,d\xi + \int_0^{t-\tau} M(s)\,ds \int_t^x d\xi \int_\tau^{t-s} F(\xi - s,\eta,\tau)\,d\eta \\ + \int_0^{t-\tau} M(s)\,ds \int_{\frac{t+\tau+s}{2}}^t d\xi \int_\tau^{2\xi-t-s} F(\xi - s,\eta,\tau)\,d\eta \\ - \int_0^{t-\tau} M(s)\,ds \int_{\frac{s+\tau-t}{2}+x}^x d\xi \int_\tau^{2(\xi-x)+t-s} F(\xi - s,\eta,\tau)\,d\eta \Big), \quad 0 \le \tau \le t \le x \le \pi, \end{aligned} \tag{46}$$

where the free term $F_0(x,t,\tau)$ is a continuous function. By the method of successive approximations one can prove that Eq. (46) has a unique solution $F(x,t,\tau) = F(x,t,\tau;M)$, which is also a continuous function (see Lemma 2.1 in [39]).

Note that $\tau \in [0,\pi)$ in (46) is actually a parameter, i.e. it can be fixed. Another important property of Eq. (46) is that for any fixed $\delta \in (0,\pi]$ it can be narrowed down to the set

$$\mathcal{D}_\delta := \Big\{ (x,t,\tau) : 0 \le x \le \pi, \ 0 \le \tau \le t \le \min\{\delta, x\} \Big\}.$$

In other words, for $(x,t,\tau) \in \mathcal{D}_\delta$ the right-hand side of (46) depends on values of $F(x,t,\tau)$ only on the set $\mathcal{D}_\delta$. Moreover, on $\mathcal{D}_\delta$ the solution of the "narrowed" equation coincides with the solution of the initial one. Hence, the function $F(x,t,\tau;M)$ on $\mathcal{D}_\delta$ depends on values of the function $M(s)$ only on $(0,\delta)$. This property allows solving the main equation in Sect. 4.3 by steps.

Let $y = S(x, \lambda)$ be a solution of the equation in (42) obeying the initial conditions $S(0, \lambda) = 0$, $S'(0, \lambda) = 1$. Eigenvalues of the boundary value problem L coincide with zeros of its characteristic function $\Delta(\lambda) = S(\pi, \lambda)$. The following representation holds (see Lemma 2.2 in [39]):

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x P(x, t) \frac{\sin \rho(x - t)}{\rho} \, dt, \quad \rho^2 = \lambda, \tag{47}$$

which gives the *transformation operator* associated with the equation in (42), where the kernel

$$P(x, t) = P(x, t; M) = F(x, t, 0; M) \tag{48}$$

is a solution of Eq. (46) for $\tau = 0$ and with the free term

$$F_0(x, t, 0) = \frac{1}{2}\left(\int_{t/2}^{x-t/2} q(s)\, ds + \int_0^t (x - t) M(s)\, ds \right). \tag{49}$$

Denote $R(x, t; M) := \dfrac{\partial}{\partial t} P(x, t; M)$. According to (46)–(49), the characteristic function has the form

$$\Delta(\lambda) = \frac{\sin \rho\pi}{\rho} - \omega\pi \frac{\cos \rho\pi}{\rho^2} + \int_0^\pi v(x) \frac{\cos \rho x}{\rho^2} \, dx, \quad v(x) \in L_2(0, \pi).$$

Moreover, we have

$$-v(\pi - x) = R(\pi, x; M), \quad 0 < x < \pi, \tag{50}$$

$$\int_0^\pi v(x)\, dx = \omega\pi, \tag{51}$$

where the value ω is determined in (44).

4.3 The Main Equation

Relation (50) can be considered as a nonlinear equation with respect to the function $M(x)$, which is called the *main equation* of Inverse Problem 3. Thus, having initially assumed that the function $M(x)$ obeys condition (43), we arrived at $v(x) \in L_2(0, \pi)$ as well as at (51). The following inverse assertion holds.

Theorem 9 *For any function $v(x) \in L_2(0, \pi)$, satisfying condition (51), the main Eq. (50) has a unique solution $M(x)$, $(\pi - x)M(x) \in L_2(0, \pi)$.*

Proof Fix $\delta \in (0, \pi/2]$ and put

$$M_1(x) = \begin{cases} M(x), & x \in (0, \delta), \\ 0, & x \in (\delta, 2\delta), \end{cases} \qquad M_2(x) = \begin{cases} 0, & x \in (0, \delta), \\ M(x), & x \in (\delta, 2\delta). \end{cases}$$

Solving equation (50) is based on the following representation (see Lemma 3.2 in [39]):

$$P(x, t; M) = P(x, t; M_1) + \int_0^t F(x, t, \tau; M_1) M_2(\tau)\, d\tau, \quad 0 \le t \le \min\{2\delta, x\}, \quad x \le \pi, \tag{52}$$

where the function $F(x, t, \tau; M_1)$ is a solution of Eq. (46) for $0 \le \tau \le t \le \min\{2\delta, x\}$, $x \le \pi$ with $M_1(x)$ instead of $M(x)$ and with the free term

$$F_0(x, t, \tau) = \frac{1}{2}\Big(x - t + \int_t^x ds \int_0^{t-\tau} P(s-\tau, \xi; M_1)\, d\xi +$$

$$+ \int_{\frac{t+\tau}{2}}^t ds \int_0^{2s-t-\tau} P(s-\tau, \xi; M_1)\, d\xi - \int_{\frac{\tau-t}{2}+x}^x ds \int_0^{2(s-x)+t-\tau} P(s-\tau, \xi; M_1)\, d\xi \Big).$$

By the contracting mappings principle, one can prove that for sufficiently small $\delta > 0$ in the ball $B_\delta := \{f : \int_0^\delta |f(x)|^2\, dx \le 1\}$ Eq. (50) has a unique solution $M(x) = M_1(x)$, $0 < x < \delta$. Continuing the function $M_1(x)$ by zero on $(\delta, 2\delta)$, we look for the solution of (50) on $(0, 2\delta)$ in the form $M(x) = M_1(x) + M_2(x)$, where $M_2(x) = 0$ on $(0, \delta)$. Differentiating representation (52) with respect to t and substituting $x = \pi$, we arrive at the following linear equation with respect to $M_2(t)$:

$$g(t) = \frac{\pi - t}{2} M_2(t) + \int_\delta^t \Phi(\pi, t, \tau; M_1) M_2(\tau)\, dt, \quad \delta < t < 2\delta, \tag{53}$$

where the functions

$$g(t) = -v(\pi - t) - R(\pi, t; M_1), \quad \Phi(\pi, t, \tau; M_1) = \frac{\partial}{\partial t} F(\pi, t, \tau; M_1)$$

are square-integrable in their domains of definition. Equation (53) has a unique solution $M_2(x)$, which belongs to $L_2(\delta, 2\delta)$ as soon as $2\delta < \pi$. Obviously, the obtained function $M(x) = M_1(x) + M_2(x)$ is a unique solution of Eq. (50) on $(0, 2\delta)$ that coincides with $M_1(x)$ a.e. on $(0, \delta)$. Continuing this process, we obtain the solution $M(x)$ on the entire interval $(0, \pi)$ such that $M(x) \in L_2(0, T)$ for any $T \in (0, \pi)$. It is easy to see that this solution is unique. Indeed, let $\tilde{M}(x)$ be another solution, then for sufficiently small $\delta > 0$ the both solutions belong to the ball B_δ and hence, they coincide a.e. on $(0, \delta)$. By virtue of uniqueness of the continuation of the solution, they coincide a.e. on the entire interval $(0, \pi)$.

Further, using properties of the solution of Eq. (46) along with Lemmas 4 and 5 for $\eta = 1$ as in the second part of the proof of Theorem 7 one can prove that condition (51) implies (43) (for more details see [39]). □

Remark 4 For any $T \in (0, \pi)$ the operator $2R(\pi, t; M) - (\pi - t)M(t)$ belongs to the class $\mathcal{E}_T$ (see [39]) as well as to the class $\mathcal{E}_{T,1}$ (see [48]). Thus, existence of a unique locally square-integrable solution of the main Eq. (50) is a corollary from Theorem 4.2 in [39] or from Theorem 1 in [48].

5 Integro-Differential Dirac Systems

5.1 Statement of the Inverse Problem and Main Results

Consider the integro-differential Dirac system of the form

$$By' + \int_0^x M(x-t)y(t)\,dt = \lambda y, \quad 0 < x < \pi, \tag{54}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \quad M(x) = \begin{pmatrix} M_1(x) & M_2(x) \\ M_3(x) & M_4(x) \end{pmatrix},$$

the functions $M_k(x)$ are complex-valued and $(\pi - x)M_k(x) \in L_2(0, \pi)$, $k = \overline{1, 4}$.

For $j = 1, 2$ let $\{\lambda_{n,j}\}_{n\in\mathbb{Z}}$ be the spectrum of the boundary value problem $D_j := D_j(M)$ for Eq. (54) under the boundary conditions

$$y_1(0) = y_j(\pi) = 0.$$

In this section we illustrate the generalization of the above approach for solving the following inverse problem [38].

Inverse Problem 4 Given the spectra $\{\lambda_{n,j}\}_{n\in\mathbb{Z}}$, $j = 1, 2$; find the matrix-function $M(x)$.

In [28] the inverse problem was studied of recovering the matrix-function $M(x)$ in the particular case, when $M_1(x) = M_4(x)$ and $M_2(x) = -M_3(x)$, from given one spectrum $\{\lambda_{n,1}\}_{n\in\mathbb{Z}}$. Specifically, the uniqueness theorem was proved and a constructive procedure was obtained for solving the inverse problem along with necessary and sufficient conditions for its solvability in terms of asymptotics of the spectrum. In [30] analogous results were obtained for the situation, when $M_1(x) = -M_4(x)$ and $M_2(x) = -M_3(x)$. In [33, 40] for the particular case from [28] the half inverse problem was studied, when $M(x)$ was to be found on subintervals $(a,\pi) \subset (0,\pi)$ from appropriate subspectra of D_1, provided that on $(0,a)$ the matrix-function $M(x)$ was known a priori. We note that analogous half inverse problems for scalar integro-differential operators were studied in [23] and [27].

The case of independent components of the matrix-function $M(x)$ is much more difficult. The following uniqueness theorem holds (see [38]).

Theorem 10 *Specification of the spectra* $\{\lambda_{n,1}\}_{n\in\mathbb{Z}}$ *and* $\{\lambda_{n,2}\}_{n\in\mathbb{Z}}$ *uniquely determines the matrix-function* $M(x)$.

More deep results are connected with obtaining necessary and sufficient conditions for solvability of Inverse Problem 4. For this purpose a special subclass $\mathfrak{M}$ of kernels $M(x)$ was chosen in which such conditions could take a sufficiently concise form. Namely, we say that $M(x) \in \mathfrak{M}$, if the following two requirements are fulfilled:

(1) $M_k(x) \in L_2(0,\pi)$, $k=\overline{1,4}$;
(2) $(\pi - x)(M_1 + M_4)(x)$, $(\pi - x)(M_2 - M_3)(x) \in W_2^1[0,\pi]$.

The next theorem gives necessary and sufficient conditions for solvability of Inverse Problem 4 in the class $\mathfrak{M}$ (see [38]).

Theorem 11 *For two arbitrary sequences of complex numbers* $\{\lambda_{n,1}\}_{n\in\mathbb{Z}}$ *and* $\{\lambda_{n,1}\}_{n\in\mathbb{Z}}$ *to be the spectra of the boundary value problems* $D_1(M)$ *and* $D_2(M)$, *respectively, with a common kernel-function* $M(x) \in \mathfrak{M}$ *it is necessary and sufficient to have the asymptotics*

$$\lambda_{n,j} = n + \frac{\delta_{2,j}}{2} + \frac{\omega}{\pi n} + \frac{\varkappa_{n,j}}{n}, \quad \{\varkappa_{n,j}\} \in l_2, \quad j=1,2,$$

with a common complex coefficient ω, *where* $\delta_{2,j}$ *is Kronecker's delta; and to satisfy the condition*

$$n\Big(\Delta_1\Big(n+\frac{1}{2}\Big) + \Delta_2(n)\Big) = o(1), \quad n \to \infty,$$

where $\Delta_1(\lambda)$ *and* $\Delta_2(\lambda)$ *are entire functions constructed by the formulae*

$$\Delta_1(\lambda) = \pi(\lambda - \lambda_{0,1}) \prod_{k\neq 0} \frac{\lambda_{k,1}-\lambda}{k} \exp\Big(\frac{\lambda}{k}\Big), \quad \Delta_2(\lambda) = -\prod_{k\in\mathbb{Z}} \frac{\lambda_{k,2}-\lambda}{k+\frac{1}{2}} \exp\Big(\frac{\lambda}{k+\frac{1}{2}}\Big).$$

The central place in the proof of Theorems 10 and 11 is occupied by the main equation of Inverse Problem 4, which is a nonlinear vectorial integral equation (see Sect. 5.4). In the next subsection we construct a transformation operator connected with system (54).

5.2 Transformation Operator

Let $y = S(x,\lambda) = (S_1(x,\lambda), S_2(x,\lambda))^T$ be a solution of system (54), satisfying the initial conditions

$$S_1(0,\lambda) = 0, \quad S_2(0,\lambda) = -1, \tag{55}$$

where T is the transposition sign. In the following lemma we introduce the transformation operator that connects $S(x,\lambda)$ with the solution of the Cauchy problem (54), (55) having the trivial kernel $M(x) = 0$, i.e. with the vector-function $S_0(x,\lambda) = (\sin\lambda x, -\cos\lambda x)^T$.

Lemma 9 *The following representation holds:*

$$S(x,\lambda) = S_0(x,\lambda) + \int_0^x K(x,t)S_0(t,\lambda)\,dt, \quad K(x,t) = \begin{pmatrix} K_{11}(x,t) & K_{12}(x,t) \\ K_{21}(x,t) & K_{22}(x,t) \end{pmatrix},$$

where $K_{lm}(x,t)$, $l,m = 1,2$, *are square-integrable functions.*

Moreover, for each $x \in (0,\pi]$ *the functions* $K_{lm}(x,t)$ *are determined for almost all* $t \in (0,x)$ *and* $|K_{lm}(x,t)| \le f(x-t)$, $l,m = 1,2$, *for some function* $f(x) \in L_2(0,\pi)$.

The proof of Lemma 9 can be found in [38], Moreover, in [38] further representations for the functions $K_{lm}(x,t)$ were obtained. In order to provide them, we introduce the following notations:

$$g_n(x) := \frac{x^n}{n!}, \quad n \ge 0, \quad f * g_{-1} = g_{-1} * f := f, \quad f^{*0} * u = u * f^{*0} := u$$

for any functions f and u. By C we will denote different positive constants in estimates.

Proposition 1 *For* $l,m = 1,2$ *the following representation holds:*

$$K_{lm}(x,t) = \sum_{n=1}^{\infty} K_{lm,n}(x,t), \quad K_{lm,n}(x,t) = \sum_{j=1}^{m_n}\sum_{k=0}^{n} a_{njk}^{lm} g_k(t)\Big(Q_{nj}[M] * g_{n-1-k}\Big)(x-t), \tag{56}$$

where $Q_{nj}[M]$, $j = \overline{1, m_n}$, *are all possible convolutional monomials of the form*

$$Q_{nj}[M] = M_{i_1} * M_{i_2} * \cdots * M_{i_n}, \quad 1 \le i_1 \le i_2 \le \cdots \le i_n \le 4,$$

and a_{njk}^{lm} are some constant coefficients satisfying the estimate

$$S_n^{lm} := \sum_{j=1}^{m_n} \sum_{k=0}^{n} |a_{njk}^{lm}| \le C4^n, \quad n \in \mathbb{N}. \tag{57}$$

Proposition 2 *The functions $K_{lm,1}(x,t)$, $l, m = 1, 2$, in (56) have the form*

$$\left.\begin{aligned}
K_{11,1}(x,t) &= -\frac{t}{2}(M_2 - M_3)(x-t),\\
K_{12,1}(x,t) &= \frac{t}{2}(M_1 + M_4)(x-t) - \frac{1}{2}\int_0^{x-t} (M_1 - M_4)(\tau)\,d\tau,\\
K_{21,1}(x,t) &= -\frac{t}{2}(M_1 + M_4)(x-t),\\
K_{22,1}(x,t) &= -\frac{t}{2}(M_2 - M_3)(x-t) - \frac{1}{2}\int_0^{x-t} (M_2 + M_3)(\tau)\,d\tau.
\end{aligned}\right\} \tag{58}$$

Proposition 3 *In (56) the coefficients a_{nj0}^{l1} vanish for all $n \ge 1$, $j = \overline{1, m_n}$ and $l = 1, 2$.*

5.3 Characteristic Functions

Eigenvalues of the boundary value problems D_j, $j = 1, 2$, coincide with zeros of their characteristic functions $\Delta_j(\lambda) = S_j(\pi, \lambda)$, $j = 1, 2$, respectively, which, according to Lemma 9, have the form

$$\Delta_1(\lambda) = \sin\lambda\pi + \int_0^\pi \Big(v_{11}(x)\sin\lambda x + v_{12}(x)\cos\lambda x\Big)\,dx, \quad v_{1m}(x) \in L_2(0,\pi), \quad m = 1, 2,$$

$$\Delta_2(\lambda) = -\cos\lambda\pi + \int_0^\pi \Big(v_{21}(x)\sin\lambda x + v_{22}(x)\cos\lambda x\Big)\,dx, \quad v_{2m}(x) \in L_2(0,\pi), \quad m = 1, 2,$$

where

$$v_{lm}(x) = (-1)^{m+1} K_{lm}(\pi, x). \tag{59}$$

The following lemma reveals a connection between the characteristic functions $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$.

Lemma 10 *The following interrelations of the functions $v_{lm}(x)$ hold:*

$$x(v_{11} + v_{22})(x) \in W_2^1[0,\pi], \quad x(v_{12} - v_{21})(x) \in W_2^1[0,\pi]. \tag{60}$$

The proof of Lemma 10 is based on the following representations, which were obtained as a result of further analysis of the transformation operator kernels (see [38]):

$$(v_{11} + v_{22})(x) = \frac{1}{2} \int_0^{\pi - x} (M_2 + M_3)(t)\, dt + \sum_{n=2}^{\infty} \sum_{j=1}^{m_n} \sum_{k=0}^{n-1} A_{njk}^{(1)} g_k(x) \Big(Q_{nj}[M] * g_{n-1-k} \Big)(\pi - x), \tag{61}$$

$$(v_{12} - v_{21})(x) = \frac{1}{2} \int_0^{\pi - x} (M_1 - M_4)(t)\, dt + \sum_{n=2}^{\infty} \sum_{j=1}^{m_n} \sum_{k=0}^{n-1} A_{njk}^{(2)} g_k(x) \Big(Q_{nj}[M] * g_{n-1-k} \Big)(\pi - x), \tag{62}$$

where it is important that the summation index k does not exceed $n - 1$. Moreover, (57) implies

$$\sum_{j=1}^{m_n} \sum_{k=0}^{n-1} |A_{njk}^{(1)}| \le S_n^{11} + S_n^{22} \le C4^n, \quad \sum_{j=1}^{m_n} \sum_{k=0}^{n-1} |A_{njk}^{(2)}| \le S_n^{12} + S_n^{21} \le C4^n, \quad n \ge 2. \tag{63}$$

5.4 *The Main Equation*

By virtue of (56), (58) and (59), we get the representations

$$\left.\begin{aligned} v_{11}(x) &= -\frac{x}{2}(M_2 - M_3)(\pi - x) + u_{11}(x), \\ v_{12}(x) &= -\frac{x}{2}(M_1 + M_4)(\pi - x) + \frac{1}{2} \int_0^{\pi - x} (M_1 - M_4)(t)\, dt - u_{12}(x), \\ v_{21}(x) &= -\frac{x}{2}(M_1 + M_4)(\pi - x) + u_{21}(x), \\ v_{22}(x) &= \frac{x}{2}(M_2 - M_3)(\pi - x) + \frac{1}{2} \int_0^{\pi - x} (M_2 + M_3)(t)\, dt - u_{22}(x), \end{aligned}\right\} \tag{64}$$

where

$$u_{lm}(x) = \sum_{n=2}^{\infty} \sum_{j=1}^{m_n} \sum_{k=0}^{n} a_{njk}^{lm} g_k(x) \Big(Q_{nj}[M] * g_{n-1-k} \Big)(\pi - x), \quad l, m = 1, 2. \tag{65}$$

Denote

$$\left.\begin{aligned} w_1(x) &:= -v_{21}(\pi-x)-(\pi-x)(v_{12}-v_{21})'(\pi-x), \\ w_2(x) &:= -v_{11}(\pi-x)-(\pi-x)(v_{11}+v_{22})'(\pi-x), \\ w_3(x) &:= v_{11}(\pi-x)-(\pi-x)(v_{11}+v_{22})'(\pi-x), \\ w_4(x) &:= -v_{21}(\pi-x)+(\pi-x)(v_{12}-v_{21})'(\pi-x). \end{aligned}\right\} \tag{66}$$

We note that, according to Lemma 10, the derivatives in (66) exist and $w_\nu(x) \in L_2(0,\pi)$, $\nu=\overline{1,4}$. Moreover, by virtue of (61) and (62), we have

$$(v_{11}+v_{22})(\pi)=(v_{12}-v_{21})(\pi)=0, \tag{67}$$

which gives the bijectivity of (66). In other words, the following assertion holds.

Lemma 11 *For arbitrary functions $w_\nu(x) \in L_2(0,\pi)$, $\nu=\overline{1,4}$, the system (66) has a unique solution $v_{jk}(x) \in L_2(0,\pi)$, $j,k=1,2$, satisfying (60) and (67).*

By virtue of (61), (62), (64)–(66) and Proposition 3, we have

$$w_\nu(x)=(\pi-x)M_\nu(x)+\sum_{n=2}^{\infty}\sum_{j=1}^{m_n}\sum_{k=1}^{n} b_{njk}^{(\nu)} g_k(\pi-x)\Big(Q_{nj}[M]*g_{n-1-k}\Big)(x), \quad \nu=\overline{1,4}, \tag{68}$$

where $b_{nj}^{(\nu)}$ are constant coefficients, which, according to (57) and (63), satisfy the estimates

$$\sum_{j=1}^{m_n}\sum_{k=1}^{n}|b_{njk}^{(\nu)}| \le Cn4^n, \quad n\ge 2, \quad \nu=\overline{1,4}.$$

The relations in (68) can be considered as a system of nonlinear integral equations with respect to the functions $M_\nu(x)$, $\nu=\overline{1,4}$, which is called *main nonlinear vectorial integral equation* (or the *main equation*) of Inverse Problem 4. The following theorem gives global solvability of the main equation (see [38]).

Theorem 12 *For any functions $w_\nu(x) \in L_2(0,\pi)$, $\nu=\overline{1,4}$, the main Eq. (68) has a unique solution $(M_\nu(x))_{\nu=\overline{1,4}}^T$ obeying $(\pi-x)M_\nu(x) \in L_2(0,\pi)$, $\nu=\overline{1,4}$.*

Note that existence of a unique locally square-integrable solution of (68), i.e. the vector-function $(M_\nu(x))_{\nu=\overline{1,4}}^T$, which belongs to $(L_2(0,b))^4$ for all $b\in(0,\pi)$, can be obtained as a corollary from Theorem 3 in [48]. Afterwards, the belonging to the class $(\pi-x)M_\nu(x) \in L_2(0,\pi)$, $\nu=\overline{1,4}$, can be proved developing the trick used in part (i) of the proof of Theorem 7 above (see also [38]).

Theorem 12 plays a central role in the proof of Theorem 10 as well as in justification of the constructive procedure for solving the inverse problem.

However, for the proof of Theorem 11 one needs a more deep analysis of both the transformation operator kernels and the main equation (for more details see [38]).

Acknowledgement This work was supported by Grant 17-11-01193 of the Russian Science Foundation.

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Expansion in Terms of Appropriate Functions and Transmutations

Ahmed Fitouhi and Wafa Binous

Abstract This work presents and summarizes the main steps of the work of Fitouhi et al. on the expansions in series of appropriate functions, namely the Bessel functions of the first kind for second-order differential Bessel perturbed operators. By changing functions or variables we can reduce the operators associated with certain polynomials and special functions to the operators considered like the Jacobi polynomials and the Whittaker functions. Taking into account that the principal part of these operators is closely related to the function of Bessel and that these latter verify recursive relations, we show that their eigenfunctions can be developed in series of Bessel functions which induce two integral representations of Mehler and Sonine type.

1 Introduction

This work presents and summarizes the main steps of the work of Fitouhi et al. on the expansions in series of appropriate functions, namely the Bessel functions of the first kind for second-order differential Bessel perturbed operators. By changing functions or variables we can reduce the operators associated with certain polynomials and special functions to the operators considered like the Jacobi polynomials and the Whittaker functions. Taking into account that the principal part of these operators is closely related to the function of Bessel and that these latter verify recursive relations, we show that their eigenfunctions can be developed in series of Bessel functions which induce two integral representations of Mehler and Sonine type. These representations suggest to define transmutation operators with the second derivative operator for the first one and with the Bessel operator for the second. This new approach is different from that studied by Levitan, Marchenko, Sitnik, and many other authors. It allows in particular to give a series

A. Fitouhi (✉) · W. Binous
Faculty of Sciences of Tunis, University of Tunis El Manar, Tunis, Tunisia
e-mail: ahmed.fitouhi@fst.rnu.tn

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_16

development of the kernels of the transmutation operator and its inverse. In the same direction, further work on the expansion in polynomials of Laguerre and Gegenbauer concerning the perturbed operators with discrete spectrum operators has been the subject of other works but the study of related transmutations do not make up to day.

Finally we would like to thank M.M. Hamza and H. Chebli for their collaboration for establishing some result at the beginning.

2 Presentation of the Class of the Operators and Expansion

On the interval $(0, \infty)$ we consider the class of second-order singular differential equations given for λ complex and $q(x)$ a suitable function by

$$(A): \quad u''(x) - \Big(\frac{\alpha^2 - 1/4}{x^2} + \lambda^2 + q(x)\Big)u(x) = 0, \quad \alpha > -1/2.$$

This class contains many differential equations related to special polynomials and functions of Legendre, Gegenbauer, Jacobi, Whittaker, ...type. We note in particular that the radial parts of the Laplace Beltrami operator in the riemanann symmetric space of rank one is included in this class. More precisely, we find after change of function that the following operators

$$\frac{1}{x^{2\alpha+1}C(x)}\Big[x^{2\alpha+1}C(x)u'(x)\Big]' - (\lambda^2 + q(x))u(x) = 0.$$

The principal part of the operators (A) $(q(x) = 0)$ suggests that it is natural to seek a solution $V_\lambda(x)$ of the operator (A) satisfying

$$2^\alpha \Gamma(\alpha + 1)V_\lambda(x) \sim x^{\alpha+1/2}, \quad x \to 0^+$$

in a formal series, of the form:

$$(B): \quad V_\lambda(x) = \sum_{p=0}^{+\infty} A_p \sqrt{x} \frac{J_{\alpha+p}(\lambda x)}{\lambda^{\alpha+p}}$$

where $J_\alpha(x)$ denotes the Bessel function of first kind of index α. We recall that $\sqrt{x} J_\alpha(\lambda)$ is a solution of the equation:

$$u''(x) = \Big(\frac{\alpha^2 - 1/4}{x^2} - \lambda^2\Big)u(x)$$

and that the function $J_\alpha(x)$ satisfy the recurrence relations:

$$J_{\alpha+1}(x) + J_{\alpha-1}(x) = \frac{2\alpha}{x} J_\alpha(x),$$

$$J_{\alpha+1}(x) - J_{\alpha-1}(x) = 2J'_\alpha(x).$$

Putting L_α the operator defined by

$$L_\alpha u = u'' - \frac{\alpha^2 - 1/4}{x} u.$$

After computation and using the previous relations, we discover that the series (B) is to be a formal solutions this implies that the coefficients $A_p(x)$ must satisfy the following relations.

$$A'_0(x) = 0$$

$$A'_{p+1}(x) = A''_p(x) + \frac{1-(\alpha+p)}{x} A'_p(x) + \Big(\frac{p(p+2\alpha)}{x^2} + q(x)\Big) A_p(x).$$

If we put $A_p(x) = x^p B_p(x)$, we obtain that:

$$B_{p+1}(x) = -\frac{1}{2x^{p+1}} \int_0^x t^p \Big(B''_p(t) + \frac{1-2\alpha}{t} B'_p(t) + q(t) B_p(t)\Big) dt, \quad \text{if } x \neq 0.$$

and

$$B_{p+1}(0) = -\frac{1}{2(1-\alpha)} \{B''_p(0) + q(0) B_p(0)\}.$$

Hence the functions $B_p(x)$ are even and entire functions as the nature of $q(x)$.

For precision and detail of computation, we invite the interest reader to refer to [1] and [4].

Thus we have the formal solution

$$V_\lambda(x) = \sum_{p=0}^{+\infty} x^p B_p \frac{\sqrt{x} J_{\alpha+p}(x)}{\lambda^{\alpha+p}}$$

where $B_p(x)$ are defined above.

A crucial problem is to study the convergence of this serie.To show that we use the technic of complex variable. We prove that if $q(z)$ is assumed to be holomorphic in the disc $D(0, 2R) = \{z \in \mathbb{C}, \ |z| < 2R\}$ the series converges uniformly on every subinterval of $(0, (1 + |1 - 2\alpha|)^{-1/2} R/e)$. This is achieved owing the following lemma which derive immediately from Cauchy formula.

Lemma 1 *Let f be a holomorphic function on a bounded domain D_2 and D_1 a sub-domain in D_2 such that $d = \inf\{|z_1 - z|;\ z_1 \in D_1;\ z \notin D_2\}$ is positive, then*

$$|f''|_{D_1} \le \frac{2}{d}|f|_{D_2},$$

where $|u|_D = \sup\{|u(z)|;\ \ z \in D\}$.

Now we are in situation to state the main result.

Theorem 1 *Suppose that $q(z)$ is an even holomorphic function on $\mathbb{C}$.Then the series*

$$V_\lambda(x) = \sum_{k=0}^{+\infty} x^k B_k(x)\, \sqrt{x}\, \frac{J_{\alpha+k}(x)}{\lambda^{\alpha+k}}$$

converges on $(0, \infty)$, uniformly on every compact subinterval of $(0, \infty)$.

Proof We give here the principal step of proof. We begin by applying the previous Lemma in taking $f = B_{k+1}$, $D_2 = D(0, (1 + 1/k)R)$ and $D_1 = (0, R)$, then we establish an inequality involving B_k. We reapplying for $f = B_k$, $D_2 = D(0, (1 + 2/k)R)$ and $D_1 = (0, (1 + 1/k)R)$ and iterate the processus. Using the convexity of the function x^k, we find for some constant c that

$$|B_{k+1}|_{D(0,R)} \le \frac{1}{2(k+1)!}\Big[2^k c^k \frac{k^{2k}}{R^{2k}} + M^k\Big]|B_1|_{D(0,2R)}.$$

The estimation

$$\frac{J_{\alpha+k+1}(\lambda x)}{\lambda^{\alpha+k+1}} \le \frac{|x|^{\alpha+k} e^{|\lambda x|}}{2^{\alpha+k+1}\Gamma(\alpha+k+1)}$$

gives an entire series with big radius of convergence see [1] for more detail.

3 Integral Representations

Bessel functions have the integral representations respectively of Mehler and Sonine type:

$$\sqrt{x} J_\nu(\lambda x) = \frac{\lambda^\nu}{2^{\nu-1}\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^x (x^2 - u^2)^{\nu-1/2} \cos(\lambda \nu) du, \quad \nu > -1/2;$$

$$x^{\nu+k} J_{\nu+k}(\lambda x) = \frac{\lambda^k}{2^{k-1}\Gamma(k)} \int_0^x u^{\nu+1}(x^2 - u^2)^{k-1} J_\nu(\lambda u) du, \quad k \ge 1.$$

From these representations with the previous theorem, we have following result.

Theorem 2 *The solution $V_\lambda(x)$ has:*

1. *the Mehler type integral representation:*

$$V_\lambda(x) = \int_0^x M(x,u)\cos(\lambda u)du,$$

where

$$M(x,u) = (x^2-u^2)^{\alpha-1/2}\sum_0^{+\infty}\frac{x^k B_k(x)}{\sqrt{\pi}2^{\alpha+k-1}\Gamma(\alpha+k-1/2)}(x^2-u^2)^k.$$

2. *the Sonine integral representation:*

$$V_\lambda(x) = \sqrt{x}\frac{J_\alpha(\lambda x)}{\lambda^\alpha} + \int_0^x S(x,u)\sqrt{u}\frac{J_\alpha(\lambda u)}{\lambda^\alpha},$$

where

$$S(x,u) = \frac{u^{\alpha+1/2}}{x^{\alpha-1/2}}\sum_1^{+\infty}\frac{B_k(x)}{2^{k-1}\Gamma(k)}(x^2-u^2)^{k-1}.$$

the functions $B_k(x)$ being defined above and these representations hold in any bounded interval of $(0,\infty)$.

These representations link the eigenfunctions $V_\lambda(\lambda x)$ of the operator A to $\cos(\lambda x)$ and $J_\alpha(\lambda)$ eigenfunctions of the second derivative operator and the Bessel operator. So we can used it to built an operator which transmutes a perturbed Bessel operator into these operators. Here we focus our attention on the Sonine integral representation which is a good approach to transmute perturbed Bessel operators.

4 Transmutation

For $\alpha > -1/2$, $r(x)$ and $s(x)$ two real entire functions, we consider two perturbed differential Bessel operators L_r and L_s,

$$L_r(u)(x) = u''(x) + \Big(\frac{1/4-\alpha^2}{x^2} + r(x)\Big)u(x),$$

$$L_s(u)(x) = u''(x) + \Big(\frac{1/4-\alpha^2}{x^2} + s(x)\Big)u(x),$$

We look for an operator χ of the form:

$$(\chi f)(x) = f(x) + \int_0^x K(x,u) f(u) du,$$

such that

$$\chi L^r = L^s \chi$$

on suitable space of functions.
When you make computation as those treated by some authors in this field, it is seen that to built a such transmutation operator it suffices that the kernel satisfies:

$$\frac{d}{dx} K(x,x) = -\frac{1}{2}(s(x) - r(x)),$$

$$L_u^r K(x,u) = L_x^s K(x,u),$$

$$\lim_{u \to 0_+} u^{\alpha - 1/2} K(x,u) = 0.$$

Hence

$$K(x,x) = -\frac{1}{2} \int_0^x (s(t) - r(t)) dt.$$

The expression and the definition of the operator χ suggest that we seek the kernel $K(x,u)$ of the form:

$$K(x,u) = \sum_1^{\infty} \frac{g_{k(x,u)}}{2^{k-1}\Gamma(k)} (x^2 - u^2)^{k-1}.$$

The kernel satisfies the hyperbolic equation, simple computation shows that the functions $g_k(x,u)$ must satisfy:

$$(L^s - L^r) g_{k(x,u)} = -2\Big(x \frac{\partial g_{k+1}(x,u)}{\partial x} + u \frac{\partial g_{k+1}(x,u)}{\partial u} + (k+1) g_{k+1}(x,u) \Big)$$

we are in situation to state

Theorem 3

1. The operator χ_1 defined on the space of even entire functions by

$$\chi_1(f)(x) = f(x) + \int_0^x S(x,u) f(u) du$$

Where

$$S(x,u) = \frac{u^{\alpha+1/2}}{x^{\alpha-1/2}} \sum_{1}^{+\infty} \frac{B_k(x)}{2^{k-1}\Gamma(k)}(x^2-u^2)^{k-1}$$

verifies $L^q \chi_1 = L^0 \chi_1$

2. *The operator* χ_2 *defined on the space of even entire functions by*

$$\chi_2(f)(x) = f(x) + \int_0^x H(x,u)(u)du.$$

where

$$H(x,u) = -S(u,x)$$

verifies $L^0 \chi_2 = \chi_2 L^q$.

To show (1) we use the previous theorem in putting $g_k(x,u) = \frac{u^{\alpha+1}}{x^{\alpha-1/2}} c_k(x)$ we discover that the functions $c_k(x)$satisfy the same recursive relations than $B_k(x)$.
to show (2) we proceed with the same manner in putting $g_k(x,u) = (\frac{x^{\alpha+1/2}}{u^{\alpha-1/2}} d_k(u)$.

Now consider the operators

$$L_\alpha u = u'' + \frac{2\alpha+1}{x} u'$$

and

$$Lu = L_\alpha u + q(x)$$

the function $q(x)$ is given even entire. Let us denote by χ and $\tilde{\chi}$ the following operators

$$\chi(f)(x) = \frac{f(x)}{2^\alpha \Gamma(\alpha+1)} + \frac{1}{2^\alpha \Gamma(\alpha+1)} \int_0^x s(x,u) f(u) u^{2\alpha+1} du,$$

where

$$s(x,u) = x^{-2\alpha} \sum_{1}^{\infty} \frac{B_k(x)}{2^{k-1}\Gamma(k)}(x^2-u^2)^{k-1},$$

and

$$\tilde{\chi}(f)(x) = f(x) + \int_0^x h(x,u) f(x) u^{2\alpha+1} du,$$

where

$$h(x,u)=u^{-2\alpha}\sum_{1}^{\infty}\frac{(-1)^k B_k(u)}{2^{k-1}\Gamma(k)}(x^2-u^2)^{k-1}.$$

By change of function and the above theorem, we have

$$L\chi=\chi L_\alpha, \quad \text{and} \quad \tilde{\chi}L=L_\alpha\tilde{\chi}.$$

Now using the well known Riemann-Liouville integral transform which transmute L_α and the second derivative operator one can prove:

Proposition 1 *the operators* χ *and* $\tilde{\chi}$ *are isomorphism of the space of* C^∞ *functions and are inverse:*

$$\chi^{-1}=\tilde{\chi}.$$

Estimation on the Kernel $s(x,u)$
To look for an estimation of the kernel $s(x,u)$, we first refer to the work of M. Coz and V. Coudray [2] who study the link between some kernel operator and his perturbed, second with the aid the Riemann method via the Guelfand-Levitan domain, we find that

Proposition 2 *The kernel associated to the transmutation operator*χ *satisfies*

$$(xu)^{\alpha+1/2}s(x,u)\leq\frac{1}{2}\int_0^{\frac{x+u}{2}}|q(t)|dt\ \exp(\int_0^x t|q(t)|dt).$$

For deep study we invite the reader to consult the work of the authors [1, 3–6].

5 Some Applications

We return to the operator L and we impose following assumptions on the functions $A(x)=x^{2\alpha+1}C(x)$:
$A(x)$ is increasing, $A'/A(x)$ is decreasing et tends to 0 at infinity.

Many authors defined a generalized Fourier transform for f a C^∞ even function of compact support by

$$\mathbf{F}(f)(\lambda)=\int_0^\infty f(x)\phi(\lambda,x)x^{2\alpha+1}C(x)dx,\quad \lambda\in\mathbb{C}$$

where $\phi(\lambda,x)$ is the eigenfunctions of the operator L.

This integral transform is an isometry between $L^2((0,\infty), A(x)dx)$ and $L^2((0,\infty), \lambda^{2\alpha+1}|c(\lambda)|d\lambda)$ where $c(\lambda)$ is the Harich-Sandra function related to the operator L.The generalized Fourier inverse is given by

$$\mathbf{F}^{-1}(f)(\lambda) = \int_0^\infty \mathbf{F}(\lambda)\phi(\lambda, x)\lambda^{2\alpha+1}|c(\lambda|^2 d\lambda).$$

One can proof that

$$\chi(f)(x) = \mathbf{F}^{-1}(\lambda^{2\alpha+0}|c(\lambda)|^2(\mathbf{F_b})(\lambda))(x),$$

where $\mathbf{F_b}$ is the Bessel transform associated with the Bessel operator the case $A(x) = x^{2\alpha+1}$.

From the properties of the transformations $\mathbf{F}$ and $\mathbf{F_b}$ we deduce some properties related to the transmutation operator χ in particular

$$||\chi(f)||_{L^2_{(A(x)dx)}} \leqslant ||f||_{L^2(x^{2\alpha+1})}$$

N being constant.

Finally, on the space of even C^∞ functions of compact support, we define the transposed operator χ by

$$\chi(f)(x) = \sqrt{C(x)}f(x) + \int_0^\infty \sqrt{C(u)}s(u, x)f(u)u^{2\alpha+1}du.$$

one can show that

$$\mathbf{F} = F_b\chi.$$

We have used essentially the paper of the first author and all published in Journal of Mathematical Analysis and Applications Vol. 181. No. 3. 1994. The readers must consult the references therein.

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Transmutation Operators as a Solvability Concept of Abstract Singular Equations

A. V. Glushak

Abstract One of the methods of studying differential equations is the transmutation operators method. Detailed study of the theory of transmutation operators with applications may be found in the literature. Application of transmutation operators establishes many important results for different classes of differential equations including singular equations with Bessel operator. In this paper transmutation operators are used in more general case when in Euler–Poisson–Darboux equation as the space-variable Laplace operator is replaced by some abstract operator acting in Banach space. Also some other abstract singular equations are studied by this method.

1 Introduction

One of the method of studying to differential equations is transmutation operators method. Detailed study of the theory of transmutation operators with applications may be found in [1, 2]. Application of transmutation operators establishes many important results for different classes of differential equations including singular differential equations with Bessel operator

$$B_k = \frac{d^2}{dt^2} + \frac{k}{t}\frac{d}{dt}, \quad k \in \mathbb{R}.$$

For example, singular PDE named Euler–Poisson–Darboux equation (EPD) has the form

$$\frac{\partial^2 u(t,x)}{\partial t^2} + \frac{k}{t}\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x), \quad k > 0, \ \ x \in \mathbb{R}^n,$$

A. V. Glushak (✉)
Belgorod State National Research University (BelGU), Belgorod, Russia
e-mail: sitnik@bsu.edu.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_17

where Δ is the space-variable Laplace operator. In the paper [3] singular EPD was leading to a simpler wave equation (with $k = 0$) using the appropriate transmutation operator. In this case, the formulas for the solution are written using spherical means acting by spatial variables.

In this paper transmutation operators are used in more general case when in EPD equation the space-variable Laplace operator is replaced by some abstract operator acting in Banach space. Also some other abstract singular equations will be studied by this method.

In the future we will assume that A is a closed operator in a Banach space E with a dense in E domain $D(A)$.

2 Euler–Poisson–Darboux Equation: Bessel Operator Function

Consider the Euler–Poisson–Darboux equation expressed as follows:

$$u''(t) + \frac{k}{t}u'(t) = Au(t), \quad t > 0 \tag{1}$$

for $k > 0$ in Banach space E.

As we will see further the correct initial condition for the EPD equation (1) are

$$u(0) = u_0, \quad u'(0) = 0. \tag{2}$$

Wherein, if $k \geq 1$ then the initial condition $u'(0) = 0$ is not needed that is usual situation for some equations with a singularity in the coefficients at $t = 0$.

Correct choice of initial conditions depending on the parameter $k \in \mathbb{R}$ and solution to the problem (1)–(2) when A is space-variable Laplace operator is given in Chapter 1 in [3]. Next results on the theory of singular equations in partial derivatives can be found, for example, in the papers [4–10] and their bibliography.

The problem (1)–(2) for $k = 0$ studied in details in [11–13]. In these papers the fact that the problem (1)–(2) is uniformly correct only if the operator A is the generator of the cosine operator function (COF) $C(t)$ was established. For terminology, see [11–14]. In the same papers, necessary and sufficient conditions that the operator A is a generator COF are given. These conditions are formulated in terms of the estimation of the norm of the resolvent $R(\lambda) = (\lambda I - A)^{-1}$ and its derivatives of the operator A

As for abstract EPD equation (1), then is was studied in [15], in Chapter 1 in [16, 17] under various assumptions about the operator A.

The Cauchy problem (1)–(2) was studied in [18], in which the necessary and sufficient solvability conditions are formulated in terms of the estimation of the norm of the resolvent $R(\lambda)$ and its weighted derivatives. In the present paper, unlike in [18], we give the necessary and sufficient condition for operator A is formulated in terms of the fractional degree of the resolvent and its non-weight derivatives as in the case $k = 0$.

Denote by $C^n(I, E_0)$ a space of n times strongly continuously differentiable for $t \in I$ functions with values in $E_0 \subset E$. Let $L(E)$ is the space of linear bounded operators.

Definition 1 A solution of Eq. (1) is a function $u(t)$ that is twice strongly continuously differentiable for $t \geq 0$ which takes values belonging to $D(A)$ for $t > 0$. That means $u(t) \in C^2(\bar{R}_+, E) \cap C(R_+, D(A))$, and satisfies Eq. (1).

Definition 2 Problem (1)–(2) is called uniformly well posed if there exists a commuting on $D(A)$ with the A operator function $Y_k(\cdot) : [0, \infty) \to L(E)$ and numbers $M \geq 1$, $\omega \geq 0$, such that for all $u_0 \in D(A)$ function $Y_k(t)u_0$ is its unique solution and

$$\|Y_k(t)\| \leq M \exp(\omega t), \tag{3}$$

$$\left\|Y_k'(t)u_0\right\| \leq Mt\ \exp(\omega t)\,\|Au_0\|\,. \tag{4}$$

Function $Y_k(t)$ is the Bessel operator function (OFB) of the problem (1)–(2) and the set of operators for which the problem (1)–(2) is uniformly correct, denoted by G_k. Moreover, G_0 is the set of generators of the operator cosine function, and $Y_0(t) = C(t)$.

In Definition 2 and throughout the following, we use the notation $Y_k'(t)u_0 = (Y_k(t)u_0)'$.

Theorem 1 ([19]) *Let problem (1)–(2) be uniformly well posed for values of parameter $m \geq 0$ ($A \in G_m$). Then this problem is also uniformly well posed fork $> m \geq 0$ ($A \in G_k \supset G_m$). The corresponding Bessel operator function $Y_k(t)$ has the form*

$$Y_k(t) = \Pi_{k,m} Y_m(t) = \mu_{k,m} \int\limits_0^1 s^m (1-s^2)^{(k-m)/2-1} Y_m(ts)\, ds, \tag{5}$$

$$\mu_{k,m} = \frac{2\Gamma(k/2+1/2)}{\Gamma(m/2+1/2)\Gamma(k/2-m/2)},$$

where $\Gamma(\cdot)$ is the Euler gamma-function.

The equality (5) written on the initial element u_0 is called the translation formula by the parameter k for the solution of the Cauchy problem for Eq. (1).

The integral on the right side of Eq. (5) called the Poisson integral, and $\Pi_{k,m}$ is transmutation operator intertwining differential operators B_m and B_k (for terminology see [1]). Operator $\Pi_{k,m}$ is the particular case of Erdelyi–Kober operator (see. [20]) preserving the initial conditions (2).

Note that in this paper we get along with the concept of an integral of a continuous function, but if necessary, we can use the Bochner integral of a function with a value in a Banach space.

If operator $A \in G_0$ is a COF generator $C(t)$ then (see. [21]) uniformly by $t \in [0, t_0]$, $t_0 > 0$ for $u_0 \in E$. When $k \to 0$ operator $P_{k,0}$ strongly converges to a identity operator I and OFB $Y_k(t)$ strongly converges to a COF $C(t)$:

$$\lim_{k\to 0} Y_k(t)u_0 = C(t)u_0.$$

Let $\rho(A)$ is resolvent operator set of A, $K_\nu(\cdot)$ is Macdonald function or modified Bessel function of the third kind of order ν.

Theorem 2 ([19]) *If problem (1)–(2) is uniformly well posed and Re $\lambda > \omega$, then $\lambda^2 \in \rho(A)$ and the representation for resolvent of operator A*

$$\lambda^{(1-k)/2} R(\lambda^2)x = \frac{2^{(1-k)/2}}{\Gamma(k/2+1/2)} \int_0^\infty K_{(k-1)/2}(\lambda t) t^{(k+1)/2} Y_k(t)x\, dt.$$

holds for each $x \in E$.

Theorem 3 ([19]) *Let the problem (1)–(2) is uniformly well posed and $Y_k(t)$ is OFB of this problem. Then the operator A is the generator of a C_0—semigroup $T(t)$, and this semigroup admits the representation*

$$T(t)x = \frac{1}{2^k\, \Gamma(k/2+1/2)\, t^{k/2+1/2}} \int_0^\infty s^k \exp\left(-\frac{s^2}{4t}\right) Y_k(s)x\, ds, \quad x \in E. \tag{6}$$

The semigroup $T(t)$ defined by the equality (6) can be extended to an operator function that is analytic in some sector Ξ_φ and get the representation (see [22], p. 269)

$$T(z) = \frac{1}{2\pi i} \int_{\Gamma_1 \bigcup \Gamma_2} e^{\lambda z} R(\lambda)\, d\lambda,$$

where $\Gamma_1 \bigcup \Gamma_2$ is a contour consisting of rays $\lambda = \sigma + \rho \exp(-i\varphi)$, $0 \le \rho < \infty$ and $\lambda = \sigma + \rho \exp(i\varphi)$, $0 \le \rho < \infty$, $\sigma \ge \omega_0$, $\dfrac{\pi}{2} < \varphi < \dfrac{\pi}{2} + \arcsin \dfrac{1}{M_0(k)}$. Therefore, to find a criterion for the uniform well-posedness of problem (1)–(2) one can restrict considerations to the class of operators that are generators of analytic C_0–semigroups $T(t)$. We denote this class of operators by G. In [23] can be found that if $A \in G$ then for Re $\lambda > \omega$ and for $\alpha > 0$ there exists a fractional degree of the resolvent $R(\lambda)$ which has the form

$$R^\alpha(\lambda)x = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \exp(-\lambda t) T(t)x\, dt, \quad x \in E.$$

A necessary condition for the uniform well-posedness of problem (1)–(2) is obtained in the following assertion.

Theorem 4 ([19]) *If problem (1)–(2) is uniformly well posed and Re* $\lambda > \omega$, *then* λ^2 *belongs to the resolvent set* $\rho(A)$ *of the operator A, and the fractional power of the resolvent admits the representation*

$$R^{1+k/2}(\lambda^2) = \frac{1}{\Gamma(k+1)\,\lambda} \int\limits_0^\infty t^k \exp(-\lambda t) Y_k(t)\, dt$$

in addition,

$$\left\| \frac{d^n}{d\lambda^n} \left(\lambda R^{1+k/2}\left(\lambda^2\right)\right) \right\| \leq \frac{M\,\Gamma(k+n+1)}{(Re\,\lambda - \omega)^{k+n+1}}, \quad n = 0,\ 1,\ 2, \ldots \tag{7}$$

In fact the estimates (7) are sufficient for the uniform well-posedness of problem (1)–(2).

Theorem 5 (Criterion of the Uniform Well-Posedness [19]) *Let* $A \in G$ *is a generator of an analytic* C_0*–semigroup. For the problem (1)–(2) to be uniformly well posed it is necessary and sufficient that for some constants* $M \geq 1,\ \omega \geq 0$ *the number* λ^2 *with Re* $\lambda > \omega$ *belonged to the resolvent set of the operator A and for the fractional degree of the resolvent of the A operator estimates (7) were correct.*

Example 1 Let $m > 0$ and $E = L^2_{x^m}(0,\infty)$ is a Hilbert space of complex-valued functions $v(x)$, $x \in (0,\infty)$, squared integrable with the weight x^m and with the norm

$$\|v(x)\|^2 = \int_0^\infty x^m\ |v(x)|^2\, dx.$$

Consider presented into[24] set of the form

$$S = \left\{ v(x) \in C^\infty(-\infty,\infty),\ v(-x) = v(x),\ \left|v^{(n)}(x)\right| \leq \frac{M_n}{(1+x^2)^N} \right\},$$

where $n \geq 0,\ N \geq 0$ are arbitrary integers, M_n are constants independent of x, and operator A is Bessel operator

$$A = \frac{d^2}{dx^2} + \frac{m}{x}\frac{d}{dx}$$

on functions from the set S considering on $[0,\infty)$. Obviously, $\overline{D(A)} = L^2_{x^m}(0,\infty)$ and operator A is a symmetric upper semibounded operator, i.e. $(Av, v) \leq 0$. By the Friedrichs theorem, its closure $\overline{A}$ is a selfadjoint operator.

Following [24, 25], we define the Fourier–Bessel transform on functions in S by the formulas

$$\hat{v}(s) = \int_0^\infty x^{2p+1}\, j_p(sx)\, v(x)\, dx,$$

$$v(x) = \gamma_p \int_0^\infty s^{2p+1}\, j_p(sx)\, \hat{v}(s)\, ds,$$

$$m = 2p+1,\ \gamma_p = \frac{1}{2^{2p}\,\Gamma^2(p+1)},\ j_p(x) = \frac{2^p\,\Gamma(p+1)}{x^p}\, J_p(x),$$

where $J_p(x)$ is the Bessel function. The set S is invariant under the one-to-one Fourier–Bessel transform.

For Re $\lambda > 0$, the operator $\overline{A}$ has the resolvent $R(\lambda)$ defined by the formula

$$R(\lambda)v(x) = \gamma_p \int_0^\infty \frac{s^{2p+1}}{s^2+\lambda}\, j_p(sx)\, \hat{v}(s)\, ds, \qquad v(x) \in L^2_{x^{2p+1}}(0,\infty),$$

and, by virtue of the Parseval relation, the following estimate holds:

$$\|R(\lambda)v(x)\|^2 = \gamma_p^2 \int_0^\infty s^{2p+1}\frac{|\hat{v}(s)|^2}{|s^2+\lambda|^2}\, ds \le \frac{\gamma_p^2}{|\lambda|^2}\|\hat{v}(s))\|^2 = \frac{\gamma_p^2}{|\lambda|^2}\|v(x))\|^2, \ \ \text{Re}\,\lambda > 0.$$

Consequently, the operator $\overline{A} \in G$, i.e., it is the generator of an analytic semigroup, which admits the representation

$$T_{2p+1}(t)v(x) = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t} R(\lambda)v(x)\, d\lambda = \frac{\gamma_p}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\left(\int_0^\infty \frac{s^{2p+1}}{s^2+\lambda} j_p(sx)\hat{v}(s)\, ds\right) d\lambda =$$

$$= \gamma_p \int_0^\infty s^{2p+1} j_p(sx)\hat{v}(s)\left(\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{e^{\lambda t}}{s^2+\lambda}\, d\lambda\right) ds = \gamma_p \int_0^\infty \exp(-s^2 t)s^{2p+1} j_p(sx)\hat{v}(s)\, ds =$$

$$= \gamma_p^2 \int_0^\infty \exp(-s^2 t)s^{2p+1} j_p(sx)\left(\int_0^\infty \tau^{2p+1} j_p(s\tau)v(\tau)\, d\tau\right) ds =$$

$$= \frac{1}{x^p}\int_0^\infty \tau^{p+1}v(\tau)\left(\int_0^\infty s\ \exp(-s^2 t)J_p(sx)J_p(s\tau)\, ds\right) d\tau =$$

$$= \frac{1}{2t\, x^p}\int_0^\infty \tau^{p+1}\exp\left(-\frac{x^2+\tau^2}{4t}\right) I_p\left(\frac{x\tau}{2t}\right) v(\tau)\, d\tau, \tag{8}$$

here we have used the integral 2.12.39.3 [26], where $I_p(\cdot)$ is the modified Bessel function.

Let us show that the resolvent of the operator $\overline{A}$ satisfies the estimates (7). By using relation (8) we obtain

$$R^{1+k/2}(\lambda^2)v(x) = \frac{1}{\Gamma(k/2+1)}\int_0^\infty t^{k/2}\exp(-\lambda^2 t)T_{2p+1}(t)v(x)\,dt =$$

$$= \frac{\gamma_p}{\Gamma(k/2+1)}\int_0^\infty t^{k/2}\exp(-\lambda^2 t)\int_0^\infty \exp(-s^2 t)s^{2p+1}j_p(sx)\hat{v}(s)\,dsdt =$$

$$= \gamma_p\int_0^\infty \frac{s^{2p+1}}{(s^2+\lambda^2)^{1+k/2}}j_p(sx)\hat{v}(s)\,ds, \quad v(x)\in L^2_{x^{2p+1}}(0,\infty).$$

Next, by virtue of the Parseval relation, the representation

$$\left\|\frac{d^n}{d\lambda^n}\left(\lambda R^{1+k/2}\left(\lambda^2\right)\right)v(x)\right\|^2 = \gamma_p^2\int_0^\infty s^{2p+1}\left|\frac{d^n}{d\lambda^n}\left(\frac{\lambda}{(s^2+\lambda^2)^{1+k/2}}\right)\right|^2|\hat{v}(s)|^2\,ds \tag{9}$$

holds for $\operatorname{Re}\lambda > 0$.

By differentiating the relation (see [26] 2.12.8.4)

$$\frac{\lambda}{(s^2+\lambda^2)^{1+k/2}} = \frac{\sqrt{\pi}}{2(2s)^{(k-1)/2}\Gamma(k/2+1)}\int_0^\infty t^{(k+1)/2}e^{-\lambda t}J_{(k-1)/2}(ts)\,dt,$$

with respect to λ, we obtain

$$\frac{d^n}{d\lambda^n}\left(\frac{\lambda}{(s^2+\lambda^2)^{1+k/2}}\right) = \frac{(-1)^n\sqrt{\pi}}{2(2s)^{(k-1)/2}\Gamma(k/2+1)}\int_0^\infty t^{(k+1)/2+n}e^{-\lambda t}J_{(k-1)/2}(ts)\,dt. \tag{10}$$

By taking into account relation (10), from the representation (9) we obtain the estimate

$$\left\|\frac{d^n}{d\lambda^n}\left(\lambda R^{1+k/2}\left(\lambda^2\right)\right)v(x)\right\|^2 \le \frac{\pi\,\gamma_p^2}{2^{k+1}\,\Gamma^2(k/2+1)} \quad\times$$

$$\times\quad\int_0^\infty s^{2p-k+2}\left|\int_0^\infty t^{(k+1)/2+n}e^{-\lambda t}J_{(k-1)/2}(ts)\,dt\right|^2|\hat{v}(s)|^2\,ds =$$

$$= \frac{\pi\,\gamma_p^2}{2^{k+1}\,\Gamma^2(k/2+1)}\int_0^\infty s^{2p-2k-2n-1}\left|\int_0^\infty \tau^{k+n}e^{-\lambda t/s}\tau^{(1-k)/2}J_{(k-1)/2}(\tau)\,d\tau\right|^2|\hat{v}(s)|^2\,ds \le$$

$$\leq \frac{M_0\,\pi\,\gamma_p^2}{2^{k+1}\,\Gamma^2(k/2+1)}\int_0^\infty s^{2p-2k-2n-1}\left|\int_0^\infty \tau^{k+n}e^{-\lambda t/s}\,d\tau\right|^2|\hat{v}(s)|^2\,ds\leq$$

$$\leq \frac{M_1\,\Gamma^2(k+n+1)}{(\mathrm{Re}\,\lambda)^{2(k+n+1)}}\|v(x)\|^2,\quad n=0,\ 1,\ 2,\ldots$$

Therefore, the estimates (7) hold. Theorem 5 is true for the considered operator $\overline{A}$, and $\overline{A}\in G_k$ for each $k\geq 0$. In particular, $\overline{A}\in G_0$ and the corresponding cosine operator function has the form

$$C(t)v(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma-i\infty}e^{\lambda t}\,\lambda\,R(\lambda^2)v(x)\,d\lambda=$$

$$=\frac{\gamma_p}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\lambda e^{\lambda t}\int_0^\infty\frac{s^{2p+1}}{s^2+\lambda^2}j_p(sx)\hat{v}(s)\,dsd\lambda=\gamma_p\int_0^\infty s^{2p+1}j_p(sx)\cos st\;\hat{v}(s)\,ds.$$

for $\sigma>0$

It is convenient to use relation (5) to find the function $Y_k(t)$. For $k>0$, we have the representation

$$Y_k(t)v(x)=\frac{2}{B(1/2,k/2)}\int_0^1(1-\tau^2)^{k/2-1}C(t\tau)v(x)\,d\tau=$$

$$=\frac{2}{B(1/2,k/2)}\int_0^1(1-\tau^2)^{k/2-1}\gamma_p\int_0^\infty s^{2p+1}j_p(sx)\cos st\tau\;\hat{v}(s)\,dsd\tau=$$

$$=\gamma_p\int_0^\infty s^{2p+1}j_p(sx)\;j_{(k-1)/2}(st)\;\hat{v}(s)\,ds.$$

Example 2 Let $m>0$ and let $E=L^2_{x^m}\left(R_2^+\right)$ be the Hilbert space of complex-valued functions $v(x,y)$, $(x,y)\in R_2^+$ that are square integrable with weight x^m and with the norm

$$\|v(x,y)\|^2=\int_{-\infty}^\infty\int_0^\infty x^m\,|v(x,y)|^2\,dxdy.$$

Consider the set

$$S_2=\left\{v(x,y)\in C^\infty(R_2),\ v(-x,y)=v(x,y),\ \left|\frac{\partial^n}{\partial x^n}\frac{\partial^j}{\partial y^j}v(x,y)\right|\leq\frac{M_{n,j}}{(1+x^2+y^2)^N}\right\},$$

where $n,\ j\ \geq\ 0,\ N\ \geq\ 0$ are arbitrary integers and the $M_{n,j}$ are constants independent of $x,\ y$, and define the operator A by the differential expression

$$A=\frac{\partial^2}{\partial x^2}+\frac{m}{x}\frac{\partial}{dx}+\frac{\partial^2}{\partial y^2}$$

on functions in the set S_2 considered on R_2^+. Obviously, $\overline{D(A)}=L^2_{x^m}\left(R_2^+\right)$ and A is a symmetric upper semibounded operator; i.e., $(Av,v)\leq 0$. By the Friedrichs theorem, its closure $\overline{A}$ is a selfadjoint operator.

In addition to the Fourier–Bessel transform on functions in the set S_2, we define the Fourier transform (with respect to the variable y) by the formulas

$$\tilde{w}(x,\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\xi y}\ w(x,y)\,dy,\quad w(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\xi y}\ \tilde{w}(x,\xi)\,d\xi.$$

The Fourier–Bessel and Fourier transforms are one-to-one mappings of S_2 onto S_2. For Re $\lambda>0$, the operator $\overline{A}$ has the resolvent $R(\lambda)$ defined by the formula

$$R(\lambda)v(x,y)=\frac{\gamma_p}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^{\infty}e^{i\xi y}\ \frac{s^{2p+1}}{s^2+\xi^2+\lambda}\ j_p(xs)\tilde{\hat{v}}(s,\xi)\,dsd\xi,$$

in addition, by virtue of the Parseval relation, we have the estimate

$$\|R(\lambda)v(x,y)\|^2\leq\frac{\gamma_p^2}{|\lambda|^2}\|v(x,y)\|^2,\ \ \text{Re}\,\lambda>0.$$

Consequently, the operator $\overline{A}\ \in\ G$, i.e., it is the generator of the analytic semigroup

$$T_{2p+1}(t)v(x,y)=\frac{\gamma_p}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\exp(-s^2t-\xi^2t+i\xi y)\ s^{2p+1}\ j_p(sx)\ \tilde{\hat{v}}(s,\xi)\,dsd\xi=$$

$$=\frac{1}{4\pi\sqrt{2}\,t\sqrt{t}\,x^p}\int_0^{\infty}\tau^{p+1}\exp\left(-\frac{x^2+\tau^2}{4t}\right)I_p\left(\frac{x\tau}{2t}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{(\eta-y)^2}{4t}\right)v(\tau,\eta)\,d\eta d\tau.$$

By analogy with Example 1, one can prove the estimates

$$\left\|\frac{d^n}{d\lambda^n}\left(\lambda R^{1+k/2}\left(\lambda^2\right)\right)v(x)\right\|^2\leq\frac{M_1\ \Gamma^2(k+n+1)}{(\text{Re}\,\lambda)^{2(k+n+1)}}\|v(x)\|^2,\ \ n=0,\ 1,\ 2,\ldots,$$

and consequently, $\overline{A} \in G_k$ for any $k \geq 0$, in addition,

$$C(t)v(x, y) = \frac{\gamma_p}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_0^{\infty} e^{i\xi y} s^{2p+1} j_p(sx) \cos\left(t\sqrt{s^2 + \xi^2}\right) \tilde{\hat{v}}(s, \xi)\, ds d\xi,$$

$$Y_k(t)v(x, y) = \frac{\gamma_p}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_0^{\infty} e^{i\xi y} s^{2p+1} j_p(sx)\; j_{(k-1)/2}\left(t\sqrt{s^2 + \xi^2}\right) \tilde{\hat{v}}(s, \xi)\, ds d\xi.$$

Example 3 Let $E = L_\infty(0, \infty)$ is a space of measurable functions $v(x)$ of variable $x \in (0, \infty)$ with norm $\|v(x)\| = \operatorname{ess} \sup_{(0,\infty)} |v(x)|$.

Operator A is the Bessel differential expression for $m = 2$ on considered on the semiaxis $[0, \infty)$ even functions $v(x)$ from $L_\infty(-\infty, \infty)$ such that $v''(x) + 2/x\; v'(x) \in E$. Then A is closed operator with a dense domain of definition and the problem

$$\frac{\partial^2 u}{\partial t^2} + \frac{2}{t}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x}, \quad t,\ x > 0, \quad u(0, x) = v(x), \quad \frac{\partial u(0, x)}{\partial t} = 0$$

has the unique solution of the form

$$u(t, x) = T_x^t v(x) = \frac{1}{2} \int_0^{\pi} v\left(\sqrt{x^2 + t^2 - 2xt\cos\varphi}\right) \sin\varphi\, d\varphi.$$

Function $u(t, x)$ for each $t \geq 0$ belongs to E and estimates (3), (4) with $\omega = 0$ are valid. Therefore, $A \in G_2$. We show that the operator A is not a generator COF, i.e. $A \notin G_0$. Indeed, the unique solution to the problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x}, \quad t,\ x > 0, \quad u(0, x) = v(x) \in D(A), \quad \frac{\partial u(0, x)}{\partial t} = 0$$

is

$$u(t, x) = \frac{(x+t)v(x+t) + (x-t)v(x-t)}{2x} = \frac{v(x+t) + v(x-t)}{2} + \frac{t}{2x} \int_{t-x}^{t+x} v'(s)\, ds. \tag{11}$$

Obviously for defined by equality (11) function $u(t)$ evaluation (3) for $k = \omega = 0$ is not valid and, therefore, $A \notin G_0$. Based on this example, it can be argued that the statement is the opposite of Theorem 1 is, generally speaking, false, i.e. for $k > 0$ enclosure $G_0 \subset G_k$ is strict.

Here are some more properties of OFB $Y_k(t)$. Let $u_0 \in D(A)$ then for OFB $Y_k(t)$ the relations

$$Y_k'(t)u_0 = \frac{t}{k+1} Y_{k+2}(t) A u_0, \qquad \lim_{t \to 0} Y_k''(t) u_0 = \frac{1}{k+1} A u_0,$$

$$Y_k(t) Y_k(s) = T_s^t Y_k(s)$$

are valid. Here T_s^t is generalized translation corresponding to Eq. (1), defined by the equality (see [25])

$$T_s^t H(s) = \frac{1}{B(k/2, 1/2)} \int_0^{\pi} H\left(\sqrt{s^2 + t^2 - 2st\cos\varphi}\right) \sin^{k-1}\varphi \, d\varphi.$$

Along with Eq. (1) for $m > 0$ we consider the equation perturbed by the operator coefficient B:

$$u''(t) + \frac{m}{t} u'(t) + Bu(t) = Au(t), \quad t > 0. \tag{12}$$

In [27] investigated the question of belonging of the operator $A - B$ to the correctness class G_m when $A \in G_k$ and $B \in L(E)$ is bounded operator and it is established that $A - B \in G_m, \; m \ge k$.

Theorem 6 ([27]) *Let for some $k > 0$ $A \in G_k$, B is bounded operator, $Y_k(t; A)$ and B commute. Then $A - B \in G_m$ for any $m \ge k$ and*

$$Y_k(t; A - B) = Y_k(t; A) + \frac{(-1/2)^{N+1} \Gamma(k/2 + 1/2) t^2 B}{\Gamma(k/2+1)\Gamma(N + 1/2)} \times$$

$$\times \int_0^1 s^{2N} \left(\frac{1}{s}\frac{d}{ds}\right)^N (1 - s^2)^{k/2} \, {}_1F_2\left(1; k/2 + 1, 2; t^2(s^2 - 1)B/4\right) Y_{2N}(ts; A) \, ds,$$

where N is the smallest integer such that $2N \ge k$, ${}_1F_2(\alpha; \beta, \gamma; \cdot)$ is generalized hypergeometric function and $Y_m(t; A - B)$ for $m > k$ determined through $Y_k(t; A - B)$ by the formula (5), written for the operator $A - B$.

If $(-B) \in G_p$, then ib [28] it is established that the closure of the operator $A - B$ belongs to $G_m, \; m \ge k + p + 1$.

Theorem 7 ([28]) *Let for some $k \ge 0$ $A \in G_k$ and $(-B) \in G_{m-k-1}$ for $m \ge k + 1$, $Y_k(t; A)$. Let operators $Y_{m-k-1}(t; -B)$ commute on $D = D(A) \bigcap D(B)$, $\overline{D} = E$.*

Then the closure of the operator $A - B$ belongs to G_m and

$$Y_m(t; \overline{A - B}) = \frac{2\Gamma(m/2 + 1/2)}{\Gamma(k/2 + 1/2)\Gamma(m/2 - k/2)} \times$$

$$\times \int_0^1 s^k (1 - s^2)^{(m-k)/2-1} Y_{m-k-1}\left(t\sqrt{1 - s^2}; -B\right) Y_k(ts; A)\, ds.$$

In the general case of the sum of n operators, the following theorem is established.

Theorem 8 ([28]) *Let $A_j \in G_{k_j}$, $k_j \geq 0$, $j = 1, \dots, n$. If for $i \neq j$ A_i and A_j commute on $D = \bigcap_{j=1}^{n} D(A_j)$ and $\overline{D} = E$, then operator closure $A = \sum_{j=1}^{n} A_j$ belongs to G_k for $k = n - 1 + \sum_{j=1}^{n} k_j$ and*

$$Y_k(t; \overline{A}) = \frac{2^{n-1}\Gamma(k/2 + 1/2)}{\prod_{j=1}^{n} \Gamma(k_j/2 + 1/2)} \int_\Omega \prod_{j=1}^{n} y_j^{k_j}\, Y_{k_j}(ty_j; A_j)\, dy,$$

where $\Omega = \{|y| = 1,\ y_1, \dots, y_n \geq 0\}$.

Theorem 3 established that OFB $Y_k(t; B)$, $B \in G_k$ generates a semigroup $T(t; B)$, which allows us to solve the corresponding Dirichlet problem.

Theorem 9 ([29]) *Let $u_0 \in D(B)$, in Eq. (12) $A = 0$ and operator B is a generator of a uniformly bounded C_0-semigroup $T(t; B)$. Then for $m < 1$ the function*

$$u(t) = \frac{(t/2)^{1-m}}{\Gamma(1/2 - m/2)} \int_0^\infty s^{m/2-3/2} \exp\left(-\frac{t^2}{4s}\right) T(s; B) u_0\, ds$$

is the unique limited solution to Eq. (12) for $A = 0$, satisfying to condition $u(0) = u_0$.

Weakening requirements for resolving operators of the Cauchy problem for abstract differential equations of the first and second orders led (see [30–33]) to the concept of an integrated semigroup and an integrated cosine operator function (ICOF).

Lower bound of the resolvent $R(\lambda^2, A)$ of the operator A of the form

$$\left\| \frac{d^n}{d\lambda^n} \left(\lambda^{1-\alpha} R(\lambda^2, A)\right) \right\| \leq \frac{M\, n!}{(\lambda - \omega)^{n+1}}, \quad \lambda > \omega, \quad n = 0, 1, \dots$$

is the criterion for existence of the generator of ICOF $C_\alpha(t)$ (see, for example, theorem 2.2.5 from [33]).

Let $P_\nu(t)$ is the Legendre spherical function (see [25], p. 205). In papers [34], [35] formulas that associate ICOF with a resolving operator $Y_k(t)$ of (1), (2) are established and the following theorem is proved.

Theorem 10 *Let $k = 2\alpha > 0$ and operator A is a α-times generator ICOF $C_\alpha(t)$, $u_0 \in D(A)$. Then the problem (1), (2) uniformly correct, i.e., $A \in G_k$, and corresponding OFB has a form*

$$Y_k(t)u_0 = \frac{2^\alpha \Gamma(\alpha + 1/2)}{\sqrt{\pi}\, t^\alpha} \left(C_\alpha(t)u_0 - \int\limits_0^1 P'_{\alpha-1}(\tau) C_\alpha(t\tau)u_0\, d\tau \right).$$

In the end of this section we note that if $0 < k < 1$ then OFB $Y_k(t)$ can be used to solve the weighted Cauchy problem for the EPD equation (1) with conditions

$$u(0) = u_0, \quad \lim_{t\to 0} t^k u'(t) = u_1. \tag{13}$$

For $u_0,\ u_1 \in D(A)$ and $A \in G_k \subset G_{2-k}$ the unique solution to the Cauchy problem (1)–(13) is (see [36])

$$u(t) = Y_k(t)u_0 + \frac{1}{1-k} t^{1-k} Y_{2-k}(t)u_1.$$

3 Euler–Poisson–Darboux Equation: Bessel Operator Function with Negative Index

In this section for EPD equation (1) for $k < 0$ we consider the initial problem

$$u(0) = 0, \quad \lim_{t\to 0+} t^k u'(t) = u_1, \tag{14}$$

which, due to the presence of a factor in front of the derivative in the second initial condition, will be called the weighted Cauchy problem.

Correct setting of initial conditions depending on the parameter $k \in \mathbb{R}$ for the EPD equation (1) in the case when A is the Laplace operator with respect to spatial variables is given in Ch. 1 of [3] and the initial conditions for the abstract EPD equation are considered in [36]. We also note that for $k < 0$ Cauchy problem for EPD equation (1) with conditions

$$u(0) = 0, \quad u'(0) = u_1$$

is not correct due to loss of uniqueness (see [37]).

Definition 3 The problem (1), (14) is called uniformly correct if there exists a commuting on $D(A)$ with the A operator function $Z_k(\cdot) : [0, \infty) \to B(E)$ and numbers $M \geq 1, \omega \geq 0$ such that for any $u_1 \in D(A)$ function $Z_k(t)u_1$ is its unique solution and at the same time

$$\|Z_k(t)\| \leq M\, t^{1-k} \exp(\omega t),$$

$$\left\| Z_k'(t)u_1 \right\| \leq M\, t^{-k} \exp(\omega t) \left(\|u_1\| + t\|Au_1\| \right).$$

Operator function $Z_k(t)$ for $k < 0$ we will call the Bessel operator function with a negative index (OBFNI) of the problem (1), (14). Set of operators for which the problem (1), (14) is uniformly correct we will denote by H_k. In addition, we denote $H_0 = G_2$ and $Z_0(t) = tY_2(t)$.

Here we present the main statements about OFBNI from the article [38], which are analogues of the corresponding properties OFB.

Theorem 11 *Let the problem (1), (14) is uniformly correct, i.e., $A \in H_k$ and $u_1 \in D(A)$. Then this problem is uniformly correct and for $m < k \leq 0$, i.e., $A \in H_m$. The corresponding Bessel operator function with a negative index $Z_m(t)$ has the form*

$$Z_m(t)u_1 = \mu_{k,m}\, t^{k-m} \int_0^1 s(1-s^2)^{(k-m)/2-1}\, Z_k(ts)u_1\, ds,$$

$$\mu_{k,m} = \frac{2(1-k)}{(1-m)\; B(3/2-k/2, k/2-m/2)},$$

where $B(\cdot, \cdot)$ is Euler beta-function.

Theorem 12 *If the problem (1), (14) is uniformly correct and Re $\lambda > \omega$, then λ^2 belongs to the resolvent set $\rho(A)$ and for any $x \in E$ the representation*

$$\lambda^{(k-1)/2} R(\lambda^2)x = \frac{2^{(k-1)/2}(1-k)}{\Gamma(3/2-k/2)} \int_0^\infty K_\nu(\lambda t) t^{(k+1)/2} Z_k(t)x\, dt$$

is valid.

Theorem 13 *Let the problem (1), (14) is uniformly correct and let $Z_k(t)$ is the Bessel operator function with a negative index for this problem. Then operator A is generator of C_0-semigroups $T(t)$ and for this semigroup, the representation*

$$T(t)x = \frac{1-k}{2^{2-k}\; \Gamma(3/2-k/2)\; t^{3/2-k/2}} \int_0^\infty s \exp\left(-\frac{s^2}{4t}\right) Z_k(s)x\, ds, \quad x \in E$$

is valid.

Theorem 14 *If the problem (1), (14) is uniformly correct and Re* $\lambda > \omega$, *then* λ^2*belongs to the resolvent set* $\rho(A)$ *of the operator A and for the fractional degree of the resolvent the representation*

$$R^{2-k/2}(\lambda^2)x = \frac{1-k}{\Gamma(3-k)\,\lambda}\int\limits_0^\infty t\exp(-\lambda t)Z_k(t)x\,dt, \quad x \in E$$

is valid. Also inequalities

$$\left\| \frac{d^n}{d\lambda^n}\left(\lambda R^{2-k/2}\left(\lambda^2\right)\right)\right\| \le \frac{M\,\Gamma(n-k+3)}{(Re\,\lambda-\omega)^{n-k+3}}, \quad n = 0,\ 1,\ 2, \ldots \tag{15}$$

are true.

Theorem 15 (Criterion for Uniform Correctness of the Weighted Cauchy Problem) *Let operator A is a generator of the analytic* C_0*–semigroup. In order to the problem (1), (14) was uniformly correct, it is necessary and sufficient that for some constants* $M \ge 1,\ \omega \ge 0$ *the number* λ^2 *with Re* $\lambda > \omega$ *belonged to the resolvent set of the operator A and for the fractional degree of the resolvent of the operator A the estimates (15) were valid.*

Theorem 16 *Suppose that the conditions of Theorem 16 are satisfied, then for* $k \le 0$ *the equality* $H_k = G_{2-k}$ *holds true and, moreover,* $Z_k(t) = \dfrac{1}{1-k}t^{1-k}Y_{2-k}(t)$.

Note that examples of operators belonging to G_{2-k}*, and, therefore, and* H_k*, are given in Sect. 2.*

Theorem 17 *Let* $\alpha < 0$ *and the operator A a generator of* $1-\alpha$*–timesintegrated COF* $C_{1-\alpha}(t)$*. Then* $A \in H_{2\alpha}$*, wherein the corresponding Bessel operator function with a negative index* $Z_{2\alpha}(t)$ *has the form*

$$Z_{2\alpha}(t) = \frac{2^{1-\alpha}\Gamma(3/2-\alpha)}{\sqrt{\pi}(1-2\alpha)t^\alpha}\left(C_{1-\alpha}(t) - \int\limits_0^1 P'_{-\alpha}(\tau)C_{1-\alpha}(t\tau)\,d\tau\right).$$

If the operator A is a generator of $(-\alpha)$*–times integrated COF* $C_{-\alpha}(t)$*, then*

$$Z_{2\alpha}(t) = \frac{2^{-\alpha}\Gamma(1/2-\alpha)t^{1-\alpha}}{\sqrt{\pi}}\int\limits_0^1 P_{-\alpha}(\tau)C_{-\alpha}(t\tau)\,d\tau.$$

4 The Bessel-Struve Equation: Operator Function Struve

In this section, for $k > 0$, we consider the equation

$$u''(t) + \frac{k}{t}\big(u'(t) - u'(0)\big) = Au(t), \quad t > 0, \tag{16}$$

which, unlike Eq. (1), contains the value of the derivative of an unknown function at the point $t = 0$.

Scalar equation of the form (16) is called the Bessel-Struve equation and it was previously met in [39–42]. Equation (16), following to [43, 44], can also be called a lightly loaded EPD equation. The growing interest in studying loaded differential equations is explained by the expanding scope of their applications and the fact that loaded equations constitute a special class of functional differential equations with their own specific tasks. A review of publications on loaded differential equations can be found in monographs [43, 44].

It is important to note that the presence in Eq. (16) given at $t = 0$ load changes the formulation of the initial problem. In contrast to the weighted problem (1), (13) for $k > 0$ we establish the well-posedness of the Cauchy problem

$$u(0) = u_0, \quad u'(0) = u_1 \tag{17}$$

for the Bessel-Struve equation (16) and we indicate the explicit form of the resolving operator.

First, we make a remark about the point $t = \tau, \ \tau \geq 0$, at which load value, i.e. the value of an unknown function or its derivative entering the equation.

Let consider the equation

$$u''(t) + \frac{k}{t}u'(t) = Au(t) + B_0 u(\tau), \quad t > 0 \tag{18}$$

with bounded operator B_0 and $A \in G_k$.

For $0 < k < 1$ solution to the problem (18), (2) satisfies equality (see [36])

$$u(t) = Y_k(t)u_0 + \frac{1}{1-k}\left(t^{1-k}Y_{2-k}(t)\int_0^t s^k Y_k(s)B_0 u(\tau)\,ds - Y_k(t)\int_0^t sY_{2-k}(s)B_0 u(\tau)\,ds\right). \tag{19}$$

Putting in (19) $t = \tau$ in order to find $u(\tau)$ we get the equation

$$(I - \Theta(\tau))u(\tau) = Y_k(\tau)u_0, \tag{20}$$

where

$$\Theta(\tau) = \frac{B_0}{1-k}\left(\tau^{1-k} Y_{2-k}(\tau)\int\limits_0^\tau s^k Y_k(s)\,ds - Y_k(\tau)\int\limits_0^\tau sY_{2-k}(s)\,ds\right).$$

In particular, if the inverse operator A^{-1} exists then $\Theta(\tau) = B_0(Y_k(\tau) - 1)A^{-1}$ (see [45]).

For sufficiently small τ, the norm of the bounded operator $\Theta(\tau)$ satisfies the inequality $\|\Theta(\tau)\| < 1$ and, therefore, from Eq. (20) can be determined

$$u(\tau) = (I - \Theta(\tau))^{-1} Y_k(\tau) u_0,$$

after which the solution to the problem (18), (2) found by the formula (19).

A similar situation arises if, in the EPD equation, instead of the load $B_0 u(\tau)$, a load of the form $B_1 u'(\tau)$ or $B_2 u''(\tau)$ is introduced.

The operator $(I - \Theta(\tau))^{-1}$ in the formula (19) makes it difficult to find explicit representations for the resolving operator of initial problems. Finding such a representation is simplified if the equation contains a load at the point $\tau = 0$, then the problem with a given load is actually solved. Here are some examples.

Let consider two equations

$$u''(t) + \frac{k}{t}u'(t) = A(u(t) - b_0 u(0)), \quad t > 0 \;\; b_0 \neq 0, \tag{21}$$

$$u''(t) + b_2 u''(0) + \frac{k}{t}u'(t) = Au(t), \quad t > 0, \;\; b_2 \neq 0. \tag{22}$$

It is easy to verify that for $0 \le k < 1$, $A \in G_k$ (note that if $b_0 = 1$ or $b_2 = k+1$, then the condition on the operator A can be changed and require that $A \in G_{2-k} \supset G_k$) the unique solution to the Cauchy weighted problem (21), (13) is

$$u(t) = (1 - b_0) Y_k(t) u_0 + \frac{1}{1-k} t^{1-k} Y_{2-k}(t) u_1 + b_0 u_0$$

and the unique solution to the (22), (13) unloaded is

$$u(t) = \frac{k - b_2 + 1}{k+1} Y_k(t) u_0 + \frac{1}{1-k} t^{1-k} Y_{2-k}(t) u_1 + \frac{b_2}{k+1} u_0.$$

Also note that in the paper [21] an explicit formula for a solution to a Cauchy problem for a weekly stressed Malmsteen equation was found in the form

$$u''(t) + \frac{k}{t}u'(t) + \frac{l}{t^2}(u(t) - u(0)) = Au(t), \quad t > 0. \tag{23}$$

If $A \in G_m$ for some $m \geq 0$ and $k > m,\ l \leq (k-1)^2/4$ then a function

$$u(t) = \frac{2\,\Gamma(p+1)\Gamma(q+1)}{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{k-m}{2}\right)} \int_0^1 s^k \left(1-s^2\right)^{(k-m)/2-1} {}_2F_1\left(p, q; \frac{k-m}{2}; 1-s^2\right) Y_m(ts)u_0\, ds, \tag{24}$$

${}_2F_1\,(p, q; r; z)$—Gauss hypergeometric function, $p,\ q$—real roots of quadratic equation

$$x^2 + \frac{1-k}{2}x + \frac{l}{4} = 0, \quad l \leq \frac{(k-1)^2}{4},$$

is the unique solution of (23) satisfying conditions (2).

If $p = (k-m)/2,\ q = (m-1)/2,\ l = (k-m)(m-1) \leq (k-1)^2/4$ then (24) has a form

$$u(t) = (k-m) \int_0^1 s \left(1-s^2\right)^{(k-m)/2-1} Y_m(ts)u_0\, ds. \tag{25}$$

More interesting is a problem of finding explicit solution to the Cauchy problem (16), (17), which leads to a new notion of operator function—Struve operator function. Let go to its introduction.

Consider the Cauchy problem (16), (17) in case $u_0 = 0$.

Theorem 18 ([46]) *Let $u_0 = 0,\ u_1 \in D(A),\ k = 2\alpha > 0$ and operator A is a generator of the operator cosine function α times of $C_\alpha(t)$. Then a function $u(t) = L_k(t)u_1$, with*

$$L_k(t)u_1 = \frac{2^\alpha\ \Gamma(\alpha+1)}{t^{\alpha-1}} \int_0^1 P_{\alpha-1}(\tau)C_\alpha(t\tau)u_1\, d\tau,$$

is a solution to a problem (16), (17).

In formulations of theorems 10, 17 and 18 some integral operators are involved with spherical Legendre functions in kernel $P_\nu(t)$. These are Buschman–Erdélyi transmutations, they are extensively studied cf. [1, 2, 47–51].

Remark 1 If A is an operator of multiplication by a number then

$$Y_k(t) = \Gamma(k/2+1/2)\sum_{j=0}^{\infty} \frac{\left(t^2A/4\right)^j}{j!\ \Gamma(j+k/2+1/2)} = \Gamma(k/2+1/2)\left(t\sqrt{A}/2\right)^{1/2-k/2} I_{k/2-1/2}\left(t\sqrt{A}\right),$$

with $I_\nu(z)$ being a modified Bessel function,

$$L_k(t) = \frac{\sqrt{\pi}}{2}\,\Gamma\left(k/2+1\right)\sum_{j=0}^{\infty}\frac{t\ \left(t^2 A/4\right)^j}{\Gamma\left(j+3/2\right)\Gamma\left(j+k/2+1\right)} =$$

$$= \frac{2^{k/2-1/2}\sqrt{\pi}\ \Gamma(k/2+1)}{A^{k/4+1/4}\ t^{k/2-1/2}}\,\mathbf{L}_{k/2-1/2}\left(t\sqrt{A}\right),$$

with $\mathbf{L}_\nu(z)$ being a Struve function. Due to it we call $Y_k(t)$ as operator Bessel function (OBF) and $L_k(t)$ operator Struve function (OSF).

Remark 2 Let $u_0 = 0$, then a condition on operator A in the theorem 18 to existence of only OCF $L_k(t)$ may be weakened. If operator A is a generator $\alpha+1$ times COF $C_{\alpha+1}(t)$, then the next representation is valid

$$L_k(t)u_1 = \frac{2^\alpha\ \Gamma(\alpha+1)}{t^\alpha}\left(C_{\alpha+1}(t)u_1 - \int\limits_0^1 P'_{\alpha-1}(\tau)C_{\alpha+1}(t\tau)u_1\,d\tau\right).$$

Also it is interesting to find formulas representing COF via OFB and These formulas follows from theorem 17 [47] and have the next form

$$C_\alpha(t) = \frac{\sqrt{\pi}}{2^\alpha\Gamma(\alpha+1/2)}\left(t^\alpha Y_{2\alpha}(t) + \int\limits_0^t P'_{\alpha-1}(t/\xi)\xi^{\alpha-1}Y_{2\alpha}(\xi)\,d\xi\right),$$

$$C_{\alpha+1}(t) = \frac{1}{2^\alpha\Gamma(\alpha+1)}\left(t^\alpha L_{2\alpha}(t) + \int\limits_0^t P'_{\alpha-1}(t/\xi)\xi^{\alpha-1}L_{2\alpha}(\xi)\,d\xi\right).$$

For OSF $L_k(t)$ and also OBF $Y_k(t)$ (cf. theorem 1) the next shift parameter formula is valid.

Theorem 19 ([46]) *Let $k = 2\alpha$ and operator A is a generator $\alpha+1$ times OCF $C_{\alpha+1}(t)$ and $m > k \geq 0$. Then operator function*

$$L_m(t) = \frac{2}{B(k/2+1, m/2-k/2)}\int\limits_0^1 s^k\,(1-s^2)^{(m-k)/2-1}\,L_k(ts)\,ds$$

is an OSF for a problem (16), (17) for a parameter choice m.

OBFs $Y_k(t)$ and OCFs $L_k(t)$ give solving operator to a problem (16), (17).

Theorem 20 ([46]) *Let $u_0,\ u_1 \in D(A)$, $k = 2\alpha > 0$ and operator A is a generator α times OCF $C_\alpha(t)$. Then a function $u(t) = Y_k(t)u_0 + L_k(t)u_1$ with OSF $Y_k(t)$ and OCF $L_k(t)$, which are defined in 10 and 18, is a unique solution to the Cauchy problem (16), (17).*

Let $u_1 \in D(A)$ then for OCF $L_k(t)$ the next is valid

$$L'_k(t)u_1 = \frac{t}{k+2}L_{k+2}(t)Au_1 + u_1, \quad \lim_{t\to 0+} L''_k(t)u_1 = 0.$$

OBF and OSF give solutions to a Cauchy problem for the stressed Malmsteen equation (23) for $l = -k$ and $A \in G_{k+2}$. From properties of these function we find a solution to

$$u''(t) + \frac{k}{t}u'(t) - \frac{k}{t^2}\left(u(t) - u(0)\right) = Au(t), \quad t > 0, \tag{26}$$

satisfying (17), and it has a form

$$u(t) = tY_{k+2}(t)u_1 + \frac{t}{k+2}L_{k+2}(t)Au_0 + u_0,$$

and equality (25) for $m = 0$ has a form

$$u(t) = \frac{t}{k+2}L_{k+2}(t)Au_0 + u_0.$$

So it may be stated that a pass from abstract wave equation $u''(t) = Au(t)$ to Euler–Poisson–Darboux (EPD) equation (1) with coefficient $k > 0$ a set of admissible operators A for which an initial problem with a condition (2) is correct, is expanded from G_0 to G_k, $G_0 \subset G_k$, and a further pass from EPD equation (1) to Eq. (26) expand this set to G_{k+2}, $G_k \subset G_{k+2}$.

We also note the relations

$$L_k(t)x = \int_0^t \frac{\xi}{\sqrt{t^2-\xi^2}}Y_{k+1}(\xi)x\,d\xi, \quad A \in G_{k+1}, \quad x \in E,$$

$$L_k(t)x = \frac{\sqrt{\pi}\,\Gamma(k/2+1)}{\Gamma(k/2+1/2)}\int_0^t {}_2F_1\left(\frac{1}{2}, \frac{k}{2}; 1; 1-\frac{t^2}{\tau^2}\right)Y_k(\tau)x\,d\tau, \quad A \in G_k, \quad x \in E.$$

If the problem (1), (2) is uniformly correct, i.e., $A \in G_k$ and $Y_k(t)$ is OFB of this problem then operator A is a generator of a strongly continuous semigroup $T(t)$ and for this semigroup the representation through OFB is valid (see Theorem 3).

We also indicate a formula that allows us to express this semigroup in terms of OFS $L_k(t)$

$$T(t)x = \frac{1}{\sqrt{\pi}\, 2^k\, \Gamma(k/2+1)\, t^{k/2+1}} \int\limits_0^\infty s^k \exp\left(-\frac{s^2}{4t}\right) \Psi\left(-\frac{1}{2}, \frac{k+1}{2}; \frac{s^2}{4t}\right) L_k(s)x\, ds,$$

where $\Psi(a, b; \cdot)$ *is confluent hypergeometric Tricomi function (see. ([52], p. 365 or [25], p. 309).*

5 The Legendre Equation: Legendre Operator Function

The study of many physical processes is based on solving equations containing the Laplace operator. Using the separating of variables in curvilinear coordinate systems one can lead to differential equations containing a singularity. If there is a certain symmetry, these equations turn into the Euler–Poisson–Darboux and Legendre equations. The initial problem for the abstract EPD equation was considered in Sect. 2. In this section, we study the Cauchy problem for another abstract singular equation, namely for the Legendre equation.

For $k > 0$ we consider the Legendre equation

$$L_k u(t) \equiv u''(t) + k \coth t\, u'(t) + (k/2)^2 u(t) = Au(t), \quad t > 0. \tag{27}$$

Differential operator L_k in the left part of (27) occurs when solving the Laplace equation in coordinates of an elongated ellipsoid of revolution [53], p. 138. If A is scalar multiplication operator then for $k = 2$ spherical functions (considered in [54], p. 53) satisfy to Eq. (27).

Also note papers [5, 55–58], in which partial differential equations containing a singular operator of the type under consideration were studied.

As follows from the results of the paper [59], correct statement of the initial conditions for the abstract Legendre equation (27) consists in setting the initial conditions at the point $t = 0$

$$u(0) = u_0, \quad u'(0) = 0, \tag{28}$$

in this case, if $k \geq 1$ then initial condition $u'(0) = 0$ removed. The definition of uniform correctness of the problem (27), (28) formulated similarly to the Definition 2.

In [59] found that set of operators A with which the problem (27), (28) correct uniformly coincides with the set G_k introduced in Section. The resolving operator of this problem is denoted by $P_k(t)$ and called operator Legendre function (OLF).

OLF can also be used and for solving the weighted Cauchy problem for the Legendre equation. If $0 < k < 1$ then more general then in (28) initial conditions are correct. Let consider the initial conditions of the form

$$u(0) = u_0, \quad \lim_{t\to 0} \left(\frac{\sinh t}{t}\right)^k u'(t) = u_1. \tag{29}$$

For $u_0,\ u_1 \in D(A)$ and $A \in G_k \subset G_{2-k}$ the unique solution to the Cauchy problem (27), (29) has the form (see [59])

$$u(t) = P_k(t)u_0 + \frac{1}{1-k}\left(\frac{\sinh t}{t}\right)^{1-k} P_{2-k}(t)u_1.$$

Note that if $A \in G_k$ and $k \geq 1$ then the problem (27), (29) is not correct.

Theorem 21 ([59]) *Let the problem (27), (28) uniformly correct when parameter $m \geq 0$ ($A \in G_m$) then this problem uniformly correct and for $k > m \geq 0$ ($A \in G_k \supset G_m$). While corresponding OLF $P_k(t)$ is*

$$P_k(t) = \Upsilon_{k,m} P_m(t) = \frac{2^{(k-m)/2}\sinh^{1-k} t}{B(k/2 - m/2, m/2 + 1/2)} \int_0^t (\cosh t - \cosh s)^{(k-m)/2-1} \sinh^m y\ P_m(s)\, ds. \tag{30}$$

The equality (30) written on the initial element u_0 is called the formula of a shift by the parameter k of the solution of the Cauchy problem for Eq. (27) and $\Upsilon_{k,m}$ is transmutation operator transmuting differential operators L_m and L_k and preserving initial conditions (28).

In addition, the equality

$$P_k'(t)u_0 = \frac{\sinh t}{k+1}\ P_{k+2}(t)\left(A - \frac{k^2}{4}I\right)u_0$$

is valid. From this equality follows that the first and the second producing operators of OLF $P_k(t)$ are equal to zero and to $\dfrac{1}{k+1}\left(A - \dfrac{k^2}{4}I\right)$, respectively.

In the particular case when the operator $A = (\delta + 1/2)^2,\ \delta \in \mathbb{R}$ is the operator of multiplication by a number then OLF $P_k(t)$ is expressed through the associated Legendre function of the first kind $\mathbf{P}_\delta^\beta(\cdot)$ (see [52], p. 661)

$$P_k(t) = \Gamma(1-\beta)\left(\frac{1}{2}\sinh t\right)^\beta \mathbf{P}_\delta^\beta(\cosh t), \quad \beta = \frac{1-k}{2}.$$

As indicated in Theorem 3, the operator $A \in G_k$ is a generator of the semigroup $T(t)$ which in case of even k can be represented (see [59]) through the OLF $P_k(t)$ (see [59])

$$T(t) = \frac{1}{\Gamma(k/2+1/2)\sqrt{t}} \int_0^\infty \sinh^k s \left(-\frac{1}{2\sinh s}\frac{d}{ds}\right)^{k/2} \exp\left(-\frac{s^2}{4t}\right) P_k(s)\, ds.$$

In the case of integer $k/2$ semigroup $T(t)$ can be represented through OLF $P_k(t)$ using for

$$\left(-\frac{1}{2\sinh s}\frac{d}{ds}\right)^{k/2},$$

the definition of a fractional derivative.

In conclusion of this section, we note that the OFL $P_k(t)$ was used by the author in [60] to establish the criterion for stabilizing the solution of the Cauchy problem for an abstract differential equation of the first order.

6 The Loaded Legendre Equation

In this section, we consider the equation

$$u''(t)+k\coth t\left(u'(t) - \frac{\cosh^{2-k}(t/2)}{\cosh t}u'(0)\right)+\frac{k^2}{4}u(t) = Au(t), \quad t > 0, \tag{31}$$

which, unlike Eq. (27), contains the value of the derivative of the unknown function at the point $t = 0$ and which we will call the weakly loaded Legendre equation.

The presence in Eq. (31) given at $t = 0$ load changes the setting of the initial problem. Unlike the weighted problem (27)–(29) for $k > 0$ we will establish the correctness of the Cauchy problem

$$u(0) = u_0, \quad u'(0) = u_1 \tag{32}$$

for a lightly loaded equation (31) and indicate the explicit form of the resolving operator.

In this section, we will further assume $g(t) = \cosh t$ and

$$\mu_k = \frac{2^{k/2}\Gamma(k/2+1/2)}{\sqrt{\pi}\,\Gamma(k/2)}.$$

To prove the following statements, it is convenient to use the concept of a fractional integral of a function $f(t)$ by the function $g(t) = \cosh t$ (see [20], p. 248)

$$I_g^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\cosh t - \cosh s)^{\alpha-1} \sinh s \; f(s)\, ds.$$

Let an operator A is a generator COF $C(t)$, $u_0 \in D(A)$. Then from Theorem 21 the following representation follows for OFL $P_k(t)$

$$P_k(t)u_0 = \mu_k \sinh^{1-k} t \int_0^t (\cosh t - \cosh s)^{k/2-1} C(s)u_0 \, ds = \mu_k \; \Gamma(k/2) \sinh^{1-k} t \;\, I_g^{k/2} \left[\frac{C(t)}{\sinh t}\right] u_0 \tag{33}$$

is valid.

Next, we consider the Cauchy problem (31)–(32) in case when $u_0 = 0$. Let $\nu_k = k2^{k/2-1}$ and

$$S(t) = \int_0^t C(s)\, ds$$

is a sine operator function (SOF).

Theorem 22 ([61]) *If $u_0 = 0, \; u_1 \in D(A)$ and the operator A is a generator COF $C(t)$, then function $u(t) = Q_k(t)u_1$, where*

$$Q_k(t)u_1 = \nu_k \sinh^{1-k} t \int_0^t (\cosh t - \cosh \tau)^{k/2-1} S(\tau)u_1 \, d\tau = \nu_k \; \Gamma(k/2) \sinh^{1-k} t \;\, I_g^{k/2} \left[\frac{S(t)}{\sinh t}\right] u_1 \tag{34}$$

is the solution to the problem (31)–(32), and wherein

$$Q_k'(t)u_1 = \frac{\sinh t}{k+2}\, Q_{k+2}(t) \left(A - \frac{k^2}{4} I\right) u_1 + \frac{u_1}{\cosh^k (t/2)}.$$

Theorem 23 ([61]) *Let $u_0, \; u_1 \in D(A)$ and operator A is a generator COF $C(t)$. Then function $u(t) = P_k(t)u_0 + Q_k(t)u_1$, where operator functions $P_k(t)$ and $Q_k(t)$ are given by (33), (34), is the unique solution to the Cauchy problem (31)–(32).*

7 Nonlocal Problems

Opposite to Sect. 1 of this paper let find a solution $u(t) \in C^2([0,1],E) \cap C((0,1],D(A))$ to EPD equation (1), with nonlocal integral condition

$$\lim_{t\to 1} I_{\nu,\beta}\, u(t) = u_1 \tag{35}$$

and condition

$$u'(0) = 0, \tag{36}$$

with $\nu = (k-1)/2,\ \beta > 0,\ I_{\nu,\beta}$ being an Erdélyi–Kober operator defined by (cf. [20], p. 246)

$$I_{\nu,\beta}\, u(t) = \frac{2}{\Gamma(\beta)\, t^{2(\beta+\nu)}} \int\limits_0^t s^{2\nu+1}(t^2-s^2)^{\beta-1} u(s)\, ds.$$

The problem (1), (35), (36) with nonlocal condition (35) in general is not correct. Many ill-posed problems for differential–operator equations may be reduced to operator equations of the first kind $Bx = y,\ x, y \in E$ and the main problem is to prove its solvability. We formulate conditions for an operator A and element $u_1 \in E$ which are sufficient for unique solvability.

Let refer to papers on solvability of nonlocal problems with integral condition for abstract first order equation [62] and [63]. Necessary and sufficient condition for solution's uniqueness was found in [64].

As it follows from the results of the first section of this work correct initial problem for EPD equation (1) include given values at $t = 0$ and a condition (36), which is dropped for $k \geq 1$,

$$u(0) = u_0 \in D(A). \tag{37}$$

Further let fix a condition $A \in G_k$ as valid, it means uniform correctness of the problem (1), (37), (36), and below we consider a determination of initial element u_0 in condition (37) by nonlocal condition (35). This nonlocal problem is reduced to an operator equation of the first kind $Y_k(1)u_0 = y$ which we solve on a subset $D(A)$.

Let introduce an entire function

$$\cosh\, i_{k,\beta}(\lambda) = \frac{\Gamma((k+1)/2)}{\Gamma((k+1)/2+\beta)}\, {}_0F_1\left(\frac{k+1}{2}+\beta; \frac{\lambda}{4}\right),$$

which is called characteristic function for nonlocal condition (35).

Theorem 24 ([65]) *Let A being a bounded operator and $u_1 \in E$. For unique solvability of the problem (1), (35), (36) is necessary and sufficient for the next condition being valid on a spectrum $\sigma(A)$ of operator A*

$$\cosh\, i_{k,\beta}(\lambda) \neq 0, \quad \lambda \in \sigma(A). \tag{38}$$

From the Theorem 24 it follows that position of zeroes of the function $\cosh\, i_{k,\beta}(\lambda)$ *is responsible for the unique solvability of the problem (1), (35), (36) with a bounded operator A. For EPD equation with unbounded operator A the condition (38) will not be sufficient for the unique solvability, though position of zeroes is also important.*

Now let find necessary condition for the uniqueness of a solution for the inverse problem (1), (35), (36) with an unbounded operator A.

Theorem 25 ([65]) *Let A being a linear closed operator in E. Propose that nonlocal problem (1), (35), (36) has a solution $u(t)$. Then for this solution being unique it is necessary that all zeroes μ_j, $j = 1, 2, \dots$ of the entire function* $\cosh\, i_{k,\beta}(\lambda)$ *do not belong to the set of eigenvalues of operator A.*

In contrast to Theorem 24, the proof of a sufficient condition for unique solvability requires additional conditions.

Theorem 26 ([65]) *Let the operator $A \in G_k$ and each zero μ_j, $j = 1, 2, \dots$ of function* $\cosh\, i_{k,\beta}(\lambda)$ *belongs to the resolvent set $\rho(A)$. Let also exists such $d > 0$ that $\sup\limits_{j=1,2,\dots} \|R(\mu_j)\| \leq d$. If $u_1 \in D(A^{n+1})$, where $n \in \mathbb{N}$ chosen so that the inequality $n > \max\{(k + \beta + 1)/2, (k/2 + \beta + 2)/2\}$ is true then the problem (1), (35), (36) has a unique solution.*

A similar nonlocal problem for the abstract Malmsten equation, which is a generalization of the EPD equation, was considered in [66].

We also point out that the nonlocal problem for the Legendre equation (27) with conditions

$$\lim_{t\to 1} I_g^{\beta}\left(\sinh^{k-1} t\, u(t)\right) = u_1, \quad u'(0) = 0$$

and the boundary control problem for a lightly loaded Legendre equation (31) with conditions

$$u(1) = u_2, \quad u'(1) = u_3$$

were studied in [61]. Results on the solvability of a nonlocal problem for the Bessel-Struve equation (16) with two nonlocal conditions containing Erdeyi-Kober operators were announced in [67].

8 Dirichlet Problem for the Bessel-Struve Equation

Boundary problems for Eq. (16) for $A \in G_k$ (hyperbolic case), generally speaking, they are not correct, but the need to solve these incorrect problems is now generally recognized (see introduction [68–70] and their bibliography). The second chapter of the monograph [68] explores the correctness of general boundary value problems for a first-order differential-operator equation and for an abstract wave equation $u''(t) = Au(t)$.

We will look for a solution $u(t) \in C^2([0,1], E) \cap C((0,1], D(A))$ of Eq. (16) for $t \in [0,1]$, satisfying to the boundary conditions

$$u(0) = u_0, \quad u(1) = u_1. \tag{39}$$

Dirichlet Problem (16), (39) can be reformulated as the inverse problem of finding a function $u(t)$ and an element $p \in D(A)$ which is the second initial condition in (17). So $u(t)$ and p should be found from the equation

$$u''(t) + \frac{k}{t}u'(t) = Au(t) + \frac{k}{t}p \tag{40}$$

by initial and final conditions from equality (39). A detailed review of the work on various inverse problems can be found in [71].

Returning to the problem we are considering (40), (39), note that, taking into account the Theorem 20, we should define an element $p \in D(A)$ from the operator equation

$$L_k(1)p = u_2, \tag{41}$$

where $u_2 = u_1 - Y_k(1)u_0$.

To establish the solvability of Eq. (41) we impose an additional condition to the resolvent of the operator A. An important role will be played by the entire function

$$\cosh i_k(\lambda) = \frac{2^{k/2-1/2}\sqrt{\pi}\,\Gamma(k/2+1)}{\lambda^{k/4+1/4}}\,\mathbf{L}_{k/2-1/2}\left(\sqrt{\lambda}\right), \tag{42}$$

Condition 1 Each zero $\mu_j,\ j = 1, 2, \ldots$ defined by equality (42) of entire function $\cosh i_k(\lambda)$ belongs to the resolvent set $\rho(A)$ and there is such $d > 0$ then

$$\sup_{j=1,2,\ldots} \|R(\mu_j)\| \le d.$$

Note that in the general case for $k > 0$ distribution of zeros μ_j of function $\cosh i_k(\lambda)$ we do not know, but in special cases for $k = 0$ and $k = 2$ zeros μ_j are calculated explicitly. In these particular cases, respectively, we have:

$$\cosh\, i_0(\lambda) = \frac{\sinh\sqrt{\lambda}}{\sqrt{\lambda}}, \quad \mu_j = -\pi^2 j^2, \quad j \in \mathbb{N},$$

$$\cosh\, i_2(\lambda) = \frac{2\left(\cosh\sqrt{\lambda} - 1\right)}{\lambda}, \quad \mu_j = -4\pi^2 j^2, \quad j \in \mathbb{N}.$$

Let the condition 1 is valid. Since each zero μ_j, $j = 1, 2, \ldots$ of the function $\cosh\, i_k(\lambda)$ belongs to $\rho(A)$, then it belongs to $\rho(A)$ together with a circular neighborhoodΩ_j with the radius $\frac{1}{d}$, whose boundary is traversed along clockwise, we denote γ_j.

Condition 2 For some n, such that

$$n > \frac{1}{4}\left(k + 7 - \max\{3 - k, 1\}\right),$$

series

$$\sum_{j=1}^{\infty} \int_{\gamma_j} \frac{R(z)\; dz}{\cosh\, i_k(z)\; (z - \lambda_0)^n}, \quad \lambda_0 \in \rho(A), \quad \mathrm{Re}\lambda_0 > \sigma$$

absolutely converges.

We formulate a theorem on the solvability of the Dirichlet problem for the Bessel-Struve equation, which was announced in [72].

Theorem 27 *Let $A \in G_k$ and conditions 1, 2 are valid. If $u_0, u_1 \in D(A^{n+1})$ then the problem (16), (39) has a unique solution.*

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On the Bessel-Wright Operator and Transmutation with Applications

Ilyes Karoui, Wafa Binous, and Ahmed Fitouhi

Abstract In this paper we summarize and complete the study of the Bessel-Wright operator and the transmutation operator recently introduced in Fitouhi et al. (Anal Math 44:1–19, 2018). Special motivation is given for the translation operator and the wavelet transform and for the resolution of the associated wave and heat equation.

Keywords Bessel-Wright functions · Bessel-Wright transform · Heat kernel · Wave kernel · Translation operator · Wavelet transform

2000 AMS Mathematics Subject Classification Primary 33D15, 47A05

1 Introduction

In [5], Fitouhi et al. introduced a family of second-order differential operators with double indices α and β

$$\mathbf{\Delta}_{(\alpha,\beta)}u(x) = \frac{d^2u}{dx^2}(x) + \frac{2(\alpha+\beta)+1}{x}\frac{du}{dx}(x) + \frac{4\alpha\beta}{x^2}\left[u(x) - u(0)\right] \tag{1}$$

These operators are very important in pure Mathematics and especially in the special function and Harmonic analysis area [2, 11]. Throughout these operators, several known mathematical analytic structures related to the Bessel operator are generalized, as for instance, taken $\beta = 0$, we regain the Bessel differential operator

$$\mathbf{\Delta}_{(\alpha,0)}u(x) = \frac{d^2u}{dx^2}(x) + \frac{2\alpha+1}{x}\frac{du}{dx}(x) = -\lambda^2 u(x). \tag{2}$$

I. Karoui · A. Fitouhi (✉)
Faculté des Sciences de Tunis, Tunis, Tunisia
e-mail: Ahmed.Fitouhi@fst.rnu.tn

W. Binous
Institut préparatoire aux études d'ingénieurs de Tunis, Tunis, Tunisia

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_18

This difference differential operator admits as eigenfunction with $-\lambda^2$ as eigenvalue the Bessel-Wright function

$$j_{(\alpha,\beta)}(\lambda x)=\sum_{n=0}^{+\infty}(-1)^n\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}\left(\frac{\lambda x}{2}\right)^{2n},\ \lambda\in\mathbb{C}$$

which is even and symmetric in α and β and coincide when $\alpha=0$ or $\beta=0$ with the normalized Bessel function [10]:

$$j_\alpha(\lambda x)=\sum_{n=0}^{+\infty}(-1)^n\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+n)}\left(\frac{\lambda x}{2}\right)^{2n},\ \lambda\in\mathbb{C}.$$

To study an harmonic analysis associated with the Bessel-Wright operators, the authors have introduced the Riemann Liouville operator that links the Bessel-Wright function to the classical Bessel function. In fact, for x,α,β reals, the following integral representation holds

$$\begin{aligned} j_{(\alpha,\beta)}(x) &= 2\beta\int_0^1(1-t^2)^{\beta-1}j_\alpha(tx)t\,dt, \quad \text{if } \alpha>-1 \text{ and } \beta>0 \qquad (3)\\ &= 2\alpha\int_0^1(1-t^2)^{\alpha-1}j_\beta(tx)t\,dt, \quad \text{if } \alpha>0 \text{ and } \beta>-1, \end{aligned}$$

where $j_\alpha(.)$ is the normalized Bessel function.

These last integral representations use a simple integral unlike these concerning the Bessel functions of index vector with r components which involve $r-1$ multiple integrals [6, 8].

A second important consequence of this integral representation is the possibility to build an operator linking the operator $\mathbf{\Delta}_{(\alpha,\beta)}$ and $\mathbf{\Delta}_{(\alpha,0)}$.

In this paper, we return to work with the objective of studying the translation operators for the Bessel-Wright operator. We investigate mainly in the translation operator mapping properties, the Bessel-Wright function product formula and the dual of the translation operator.

As application, we define and study the Bessel-Wright wavelets and the continuous wavelet transform. We will prove, by using the Bessel-Wright wavelets, the Plancherel, the inversion formulas and we will end this paper by solving the Bessel-Wright heat and wave equation.

2 The Bessel-Wright Transmutation Operator

In the sequel, we assume that $\alpha>0$ and $\beta>-1$. We need the following functionals spaces:

By $\mathcal{C}_0$ we denote the space of real continuous functions on $]0,+\infty[$ having 0 as limit at infinity.

By $\mathcal{L}^p_\beta$, for $p \in [1,+\infty[$, the Banach space of real-valued functions f, measurable on $]0,+\infty[$ such that

$$\|f\|_{p,\beta} = \left[\int_0^\infty |f(x)|^p\, x^{2\beta+1} dx\right]^{1/p} < +\infty,\ \forall p \in [1,+\infty[$$

We equipped the space $\mathcal{L}^2_\beta$ by the inner product given by:

$$\langle f, g\rangle_\beta = \int_0^\infty f(t)g(t)t^{2\beta+1}dt. \tag{4}$$

Let $\mathcal{E}$ be the space of real $\mathcal{C}^\infty$-functions on $]0,+\infty[$, provided with the topology of uniform convergence on every compact of the functions and their derivatives. This topology is defined by the semi-norms

$$\forall n \in \mathbb{N}, a \geq 0 \quad p_{n,a}(f) = \sup_{0\leq k\leq n,\ x\in[-a,a]} \left|\frac{d^k}{dx^k} f(x)\right| < \infty.$$

Let $\mathcal{S}$ be the space of real $\mathcal{C}^\infty$-functions on $]0,+\infty[$ rapidly decreasing together with their derivatives equipped with the seminorms

$$p_{m,n}(f) = \sup_{x>0} \left(1+x^2\right)^m \left|\frac{d^n}{dx^n} f(x)\right|,\ n, m \in \mathbb{N}.$$

The seminorms $p_{m,n}$ define the topology of $\mathcal{S}$.

We denote by $\mathcal{D}_a$ the subspace of $\mathcal{S}$ of function with compact support of the form $[\tau, a], \quad \tau < a$.

The Paley-Winer space $\mathcal{PW}_a$ is the set of entire functions of exponential type and rapidly decreasing. The topology on $\mathcal{PW}_a$ is defined by the seminorms

$$P_m(f) = \sup_{\lambda\in\mathbb{C}} \left(1+|\lambda|^2\right)^m |f(\lambda)|\, e^{-a|\mathrm{Im}\lambda|},\ m \in \mathbb{N}.$$

Consider the Riemann Liouville operator

$$R_\alpha g(x) = 2\alpha \int_0^1 g(xt)(1-t^2)^{\alpha-1} t\,dt, \quad \alpha > 0, x > 0.$$

which can be written as follows:

$$R_\alpha g(x) = \frac{2\alpha}{x^{2\alpha}} \int_0^x g(u)\left[x^2-u^2\right]^{\alpha-1} u\,du, \quad x > 0.$$

It is easy to verify that

$$\mathbf{\Delta}_{(\alpha,\beta)} \circ R_\alpha = R_\alpha \circ \mathbf{\Delta}_{(0,\beta)}$$

and

$$\mathbf{\Delta}_{(\alpha,\beta)} \circ R_\beta = R_\beta \circ \mathbf{\Delta}_{(\alpha,0)}$$

$\mathbf{\Delta}_{(0,\beta)}$ and $\mathbf{\Delta}_{(\alpha,0)}$ being the Bessel operator.

In [5], we have proved the that:

Theorem 1 *If we assume that*

$$1 + \beta < p$$

then

$$R_\alpha : \mathcal{L}^p_\beta \to \mathcal{L}^p_\beta$$

is a bounded linear operator. In particular for all $-1 < \beta < 0$ *we obtain*

$$R_\alpha : \mathcal{L}^1_\beta \to \mathcal{L}^1_\beta,$$

and if $-1 < \beta < 1$ *then*

$$R_\alpha : \mathcal{L}^2_\beta \to \mathcal{L}^2_\beta.$$

Proposition 1 *The operator* R_α *is continuous from* $\mathcal{E}$ *into itself.*

Proof For all $n \in \mathbb{N}$, the function $x \mapsto R_\alpha(f)(x)$ is C^n on $[0, +\infty[$ and we have then the function

$$x \mapsto R_\alpha(f)(x) \in \mathcal{E}.$$

and the fact that

$$p_{n,a}(R_\alpha(f)) \le C p_{n,a}(f), \ C > 0,$$

leads to the result. ■

Proposition 2 *The dual of the Riemann-Liouville operator relative to the inner product (4) is given by*

$$R^t_{(\alpha,\beta)} g(u) = 2\alpha \int_1^\infty g(ut) \left[t^2 - 1\right]^{\alpha-1} t^{2(\beta-\alpha)+1} dt$$

which is valid for any function belongs to the space $\mathcal{S}$.

Proof In fact we have

$$\begin{aligned}\langle R_\alpha f, g\rangle &= \int_0^\infty R_\alpha f(x)g(x)x^{2\beta+1}dx \\ &= \int_0^\infty \left[2\alpha\int_0^x f(u)\left[x^2-u^2\right]^{\alpha-1} udu\right] g(x)x^{2\beta+1-2\alpha}dx \\ &= 2\alpha\int_0^\infty f(u)u\left[\int_u^\infty g(x)\left[x^2-u^2\right]^{\alpha-1} x^{2\beta+1-2\alpha}dx\right]du \\ &= \int_0^\infty f(u)R^t_{(\alpha,\beta)}g(u)u^{2\beta+1}du = \left\langle f, R^t_{(\alpha,\beta)}g\right\rangle.\end{aligned}$$

Then we obtain

$$\begin{aligned}R^t_{(\alpha,\beta)}g(u) &= 2\alpha u^{-2\beta}\int_u^\infty g(x)\left[x^2-u^2\right]^{\alpha-1} x^{2\beta+1-2\alpha}dx \\ &= 2\alpha u^{-2\beta}\int_1^\infty g(ut)\left[u^2t^2-u^2\right]^{\alpha-1} t^{2\beta+1-2\alpha}u^{2\beta+2-2\alpha}dt \\ &= 2\alpha\int_1^\infty g(ut)\left[t^2-1\right]^{\alpha-1} t^{2(\beta-\alpha)+1}dt,\end{aligned}$$

which leads to the result. ■

As important results concerning the operator $R^t_{(\alpha,\beta)}$, we have proved in [5] that:

Theorem 2 *If we assume that*

$$\beta - \frac{1}{p}(1+\beta) < 0$$

then

$$R^t_{(\alpha,\beta)} : \mathcal{L}^p_\beta \to \mathcal{L}^p_\beta$$

is a bounded linear operator. In particular for all $\beta > -1$ *we obtain*

$$R^t_{(\alpha,\beta)} : \mathcal{L}^1_\beta \to \mathcal{L}^1_\beta,$$

and if $-1 < \beta < 1$ *then*

$$R^t_{(\alpha,\beta)} : \mathcal{L}^2_\beta \to \mathcal{L}^2_\beta.$$

Proposition 3 *For k integer and* $0 < \alpha < 1$ *we have*

$$R_{k+\alpha}^{-1} g(x) = \frac{1}{\Gamma(k+1)(k+\alpha)(1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)} x \left(\frac{1}{2x}\frac{d}{dx}\right)^{k+1} x^{2(1-\alpha)} R_{1-\alpha} x^{2(k+\alpha)} g(x)$$

and

$$R_k^{-1} g(x) = \frac{1}{\Gamma(k+1)} \left(\frac{1}{2x}\frac{d}{dx}\right)^{k} x^{2k} g(x)$$

which is valid for any function belongs to the space $\mathcal{S}$.

Corollary 1 *For k integer and* $0 < \alpha < 1$ *we have*

$$\left(R^t_{(\alpha+k,\beta)}\right)^{-1} g(x) = (-1)^{k+1} \frac{1}{2^k \Gamma(k+1)(k+\alpha)\Gamma(\alpha)\Gamma(1-\alpha)} x^{2(k+1)} \times \int_1^\infty \left[\left(\frac{d}{dx}\frac{1}{x} + \frac{2\beta+1}{x^2}\right)^{k+1} xg\right](xt)\left(t^2-1\right)^{-\alpha} t^{2\beta+1} dt.$$

which is valid for any function belongs to the space $\mathcal{S}$.

Finally, the following properties are summarized as:

Proposition 4 *The operators* R_α *and* $R^t_{(\alpha,\beta)}$ *satisfied the following properties*

1. R_α *is a linear operator from* $\mathcal{D}_a$ *into itself.*
2. R_α *is a topological isomorphism from* $\mathcal{E}$ (*resp.* $\mathcal{S}$ *and* $\mathcal{D}$) *into itself.*
3. $R^t_{(\alpha,\beta)}$ *is a linear operator from* $\mathcal{D}_a$ *into itself.*
4. $R^t_{(\alpha,\beta)}$ *is a topological isomorphism from* $\mathcal{E}$ (*resp.* $\mathcal{S}$ *and* $\mathcal{D}$) *into itself.*

3 Applications

3.1 The Bessel-Wright Transform

Definition 1 We define the Bessel-Wright transform on $\mathcal{L}^1_\beta$ by

$$\forall \lambda \in \mathbb{R}_+, \qquad \mathcal{F}_{(\alpha,\beta)}(f)(\lambda) = c_\beta \int_0^\infty f(x)\, j_{(\alpha,\beta)}(\lambda x)\, x^{2\beta+1} dx.$$

where

$$c_\beta = \frac{1}{2^\beta \Gamma(\beta+1)}.$$

Proposition 5 *For f and g in $\mathcal{L}^1_\beta$, we have*

$$\int_0^\infty \mathcal{F}_{(\alpha,\beta)}(f)(x)\, g(x)\, x^{2\beta+1}dx = \int_0^\infty f(x)\mathcal{F}_{(\alpha,\beta)}(g)(x)x^{2\beta+1}dx$$

Since

$$j_{(\alpha,\beta)}(\lambda x) = R_\alpha \left(j_\beta\right)(\lambda x)$$

It is proved in [5] that:

Proposition 6 *The Bessel-Wright transform is related to the Bessel-Fourier transform via*

$$\mathcal{F}_{(\alpha,\beta)} = \mathcal{F}_\beta \circ R^t_{(\alpha,\beta)} \tag{5}$$

where $\mathcal{F}_\beta$ is the classical Bessel-Fourier transform defined by

$$\mathcal{F}_\beta(g)(\lambda) = c_\beta \int_0^{+\infty} g(x)\, j_\beta(\lambda x) x^{2\beta+1}dx.$$

We recall that the following results holds.

Theorem 3 *The Bessel-Wright transform $\mathcal{F}_{(\alpha,\beta)}$ satisfies the following mapping properties:*

(i) $\mathcal{F}_{(\alpha,\beta)}$ *is a bounded linear operator from $\mathcal{L}^1_\beta$ to $\mathcal{C}_0$.*
(ii) *Let $p \in]1,2]$ and $q = \frac{p}{p-1}$. If we assume that*

$$\beta - \frac{1}{p}(1+\beta) < 0$$

then the Bessel-Wright transform $\mathcal{F}_{(\alpha,\beta)}$ extends to a bounded linear operator from $\mathcal{L}^p_\beta$ to $\mathcal{L}^q_\beta$.
(iii) $\mathcal{F}_{(\alpha,\beta)}$ *a topological isomorphism from $\mathcal{S}$ (resp. $\mathcal{E}$)into itself.*
(iv) $\mathcal{F}_{(\alpha,\beta)}$ *is a linear operator from $\mathcal{D}_a$ into $\mathcal{PW}_a$.*

3.2 The Bessel-Wright Transform Inversion Formula

Like we already proved in [5], using the transmutation operator $R^t_{(\alpha,\beta)}$ the Bessel-Wright transform could be inverted in the Schwartz space. This result is larger than the formula of Cooke [1, 7, 9].

Theorem 4 *Let k integer and $0 < \alpha < 1$. The inversion of the Bessel-Wright transform is given by*

$$\mathcal{F}^{-1}_{(k+\alpha,\beta)}g(x) = (-1)^{k+1}\frac{1}{2^k\Gamma(k+1)(k+\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}x^{2(k+1)}$$
$$\times \int_1^{\infty}\left[\left(\frac{d}{dx}\frac{1}{x}+\frac{2\beta+1}{x^2}\right)^{k+1} x\mathcal{F}_\beta(g)\right](xt)\left(t^2-1\right)^{-\alpha}t^{2\beta+1}dt,$$

and

$$\mathcal{F}^{-1}_{(k,\beta)}g(x) = \frac{1}{2^k\Gamma(k+1)}x^{2k}\left(\frac{d}{dx}\frac{1}{x}+\frac{2\beta+1}{x^2}\right)^{k}\mathcal{F}_\beta g(x),$$

which is valid for any function belongs to the space $\mathcal{S}$.

Proof Using formula (5) we obtain

$$\mathcal{F}^{-1}_{(\alpha,\beta)} = \left(R^t_{(\alpha,\beta)}\right)^{-1}\mathcal{F}^{-1}_\beta$$
$$= \left(R^t_{(\alpha,\beta)}\right)^{-1}\mathcal{F}_\beta.$$

and Corollary 1 we get the result. ■

3.3 The Bessel-Wright Translation Operator and Its Dual

Let's recall some results about the harmonic analysis generated by the Bessel operator Δ_α. We stake out especially properties that will serve us in below section. For more details we refer the reader to [4].

Recall that the Fourier-Bessel transform of order $\mathcal{F}_\alpha$ is defined by

$$\mathcal{F}_\alpha(g)(\lambda) = c_\alpha\int_0^{+\infty} g(x)j_\alpha(\lambda x)x^{2\alpha+1}dx$$

where

$$c_\alpha = \frac{1}{2^\alpha\Gamma(\alpha+1)}.$$

where j_α is the normalized Bessel function defined by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\alpha+n+1)\,n!}\left(\frac{x}{2}\right)^{2n},\ \alpha \geq -\frac{1}{2}$$

The Bessel translation operator τ_α^x has been investigated in [3]. It has been proved that τ_α^x has the following integral representation

$$\tau_\alpha^x(f)(y) = \int_{|x-y|}^{x+y} f(z)\,W_\alpha(x,y,z)\,z^{2\alpha+1}dz,\ x,y>0 \tag{6}$$

where

$$W_\alpha(x,y,z) = \frac{2^{1-\alpha}[\Gamma(\alpha+1)]^2}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \times \frac{[(x+y)^2-z^2]^{\alpha-\frac{1}{2}}[z^2-(x-y)^2]^{\alpha-\frac{1}{2}}}{(xyz)^{2\alpha}}$$

with a change of variables τ_α^x can written in the following form

$$\tau_\alpha^x(f)(y) = k_\alpha\int_0^{\pi} f\left(\sqrt{x^2+y^2+2xy\cos\theta}\right)(\sin\theta)^{2\alpha}\,d\theta$$

where

$$k_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}$$

Hence, the Bessel convolution product of two functions f, g on $[0,\infty[$ could be defined by the relation

$$f *_\alpha g(x) = \int_0^{\infty}\tau_\alpha^x f(y)\,g(y)\,y^{2\alpha+1}dy,\ x\geq 0$$

The below theorems highlight the most important properties of the Bessel translation operator. For detailed proof we refer the reader to [3, 8]

Theorem 5 *The Bessel translation operator satisfies the following properties*

1. $\tau_\alpha^x f$ *is the solution of the hyperbolic equation*

$$\begin{cases} \Delta_\alpha^x u(x,y) = \Delta_\alpha^y u(x,y) \\ u(x,0) = f(x) \\ \frac{\partial}{\partial y}u(x,0) = 0 \end{cases}$$

2. *Let* $p \in [1, \infty]$ *and* $f \in \mathcal{L}^p_\alpha$*. Then for all* $x \geq 0$, $\tau^x_\alpha f \in \mathcal{L}^p_\alpha$ *and*

$$\left\| \tau^x_\alpha f \right\|_{p,\alpha} \leq \| f \|_{p,\alpha}$$

3. *For* $f \in \mathcal{L}^p_\alpha$, $p = 1$ *or* 2*, we have*

$$\mathcal{F}_\alpha \left(\tau^x_\alpha f \right) (\lambda) = j_\alpha (\lambda x) \mathcal{F}_\alpha (f) (\lambda)$$

4. *Let* $p, q \in [1, \infty]$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. If* $f \in \mathcal{L}^p_\alpha$ *and* $g \in \mathcal{L}^q_\alpha$*, then for every* $x \geq 0$ *we have*

$$\int_0^\infty \tau^x_\alpha f (y) g (y) y^{2\alpha+1} dy = \int_0^\infty f (y) \tau^x_\alpha g (y) y^{2\alpha+1} dy$$

5. *Let* $p, q, r \in [1, \infty]$ *such that* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$*. If* $f \in \mathcal{L}^p_\alpha$ *and* $g \in \mathcal{L}^q_\alpha$*, then* $f *_\alpha g \in \mathcal{L}^r_\alpha$ *and*

$$\| f *_\alpha g \|_{r,\alpha} \leq \| f \|_{p,\alpha} \| g \|_{q,\alpha}$$

6. *For* $f \in \mathcal{L}^1_\alpha$ *and* $g \in \mathcal{L}^p_\alpha$, $p = 1$ *or* 2*, we have*

$$\mathcal{F}_\alpha (f *_\alpha g) = \mathcal{F}_\alpha (f) \mathcal{F}_\alpha (g)$$

7.

$$\tau^x_\alpha : \mathcal{D}_a \to \mathcal{D}_{a+x} \tag{7}$$

3.3.1 The Bessel-Wright Translation Operator

Definition 2 For $f \in \mathcal{E}$, we define the Bessel-Wright translation operator as follow

$$\tau^x_{(\alpha,\beta)} f (y) = (R_\alpha)_x (R_\alpha)_y \tau^x_\beta \left(R^{-1}_\alpha (f) \right) (y) \tag{8}$$

where τ^y_β is the Bessel translation operator defined by formula (6).

Theorem 6 *The Bessel-Wright translation operator verifies the following properties*

1. *For all* $f \in \mathcal{E}$

$$\tau^0_{(\alpha,\beta)} f = f$$

2. *Given* $\lambda \in \mathbb{C}$, *and* $x, y \in \left]0, +\infty\right[$ *,we have the Bessel-Wright product formula*

$$\tau^{x}_{(\alpha,\beta)} \left(j_{(\alpha,\beta)} (\lambda .)\right) (y) = j_{(\alpha,\beta)} (\lambda x)\, j_{(\alpha,\beta)} (\lambda y)$$

3. *The Bessel-Wright translation operator is a linear operator from* $\mathcal{E}$ *into itself.*

$$\tau^{x}_{(\alpha,\beta)} : \mathcal{E} \to \mathcal{E}$$

4. *The Bessel-Wright translation operator is a linear operator from* $\mathcal{S}$ *into itself.*

$$\tau^{x}_{(\alpha,\beta)} : \mathcal{S} \to \mathcal{S}$$

5. *The Bessel-Wright translation operator is a linear operator from* $\mathcal{D}$ *into itself.*

$$\tau^{x}_{(\alpha,\beta)} : \mathcal{D} \to \mathcal{D}$$

Proof Recall that the Bessel translation verifies for all $f \in \mathcal{E}$

$$\tau^{0}_{\beta} (f) (y) = f (y)$$

So

$$\begin{aligned}
\tau^{0}_{(\alpha,\beta)} f (y) &= (R_\alpha)_0 (R_\alpha)_y \,\tau^{0}_{\beta} \left(R^{-1}_{\alpha} (f)\right) (y) \\
&= (R_\alpha)_0 (R_\alpha)_y \,\tau^{0}_{\beta} \left(R^{-1}_{\alpha} (f)\right) (y) \\
&= 2\alpha \int_0^1 f (y) (1 - t^2)^{\alpha - 1} t dt \\
&= f (y)
\end{aligned}$$

And Since $j_{(\alpha,\beta)} \in \mathcal{E}$, we get the product formula

$$\begin{aligned}
\tau^{x}_{(\alpha,\beta)} \left(j_{(\alpha,\beta)} (\lambda .)\right) (y) &= (R_\alpha)_x (R_\alpha)_y \,\tau^{x}_{\beta} \left(R^{-1}_{\alpha} \left(j_{(\alpha,\beta)} (\lambda .)\right)\right) (y) \\
&= (R_\alpha)_x (R_\alpha)_y \,\tau^{x}_{\beta} \left(R^{-1}_{\alpha} \left(j_{\beta} (\lambda .)\right)\right) (y) \\
&= (R_\alpha)_x (R_\alpha)_y \left[j_\beta (\lambda x)\, j_\beta (\lambda y)\right] \\
&= j_{(\alpha,\beta)} (\lambda x)\, j_{(\alpha,\beta)} (\lambda y)
\end{aligned}$$

Using the R_α properties in Proposition 4 and Theorem 3.3, we get the rest of the proof. ∎

3.3.2 The Dual of the Bessel-Wright Translation Operator

Definition 3 For each $x, y \in \mathbb{R}$, the dual of the translation operator ${}^t\tau^x_{(\alpha,\beta)}$ is defined on $\mathcal{D}$ (resp. $\mathcal{S}$) by

$$ {}^t\tau^x_{(\alpha,\beta)} f(y) = (R_\alpha)_x \left(R^{*-1}_{\alpha,\beta}\right)_y \tau^x_\beta \left(R^*_{\alpha,\beta}(f)\right)(y) $$

Remark 1 Note that for $\alpha = 0$, ${}^t\tau^x_{(\alpha,\beta)}$ reduces to the well know Bessel translation operator

$$ {}^t\tau^x_{(0,\beta)} f(y) = \tau^x_\beta f(y) $$

since

$$ R_0 = R^*_{0,\beta} = id. $$

Theorem 7 *The operator ${}^t\tau^x_{(\alpha,\beta)}$ verifies the following properties*

1. *${}^t\tau^x_{(\alpha,\beta)}$ is a linear operator from $\mathcal{E}$ into itself.*

$$ {}^t\tau^x_{(\alpha,\beta)} : \mathcal{E} \to \mathcal{E} $$

2. *${}^t\tau^x_{(\alpha,\beta)}$ is a linear operator from $\mathcal{S}$ into itself.*

$$ {}^t\tau^x_{(\alpha,\beta)} : \mathcal{S} \to \mathcal{S} $$

3. *${}^t\tau^x_{(\alpha,\beta)}$ is a linear operator from $\mathcal{D}$ into itself.*

$$ {}^t\tau^x_{(\alpha,\beta)} : \mathcal{D} \to \mathcal{D} $$

4. *For $f \in \mathcal{E}$, $g \in \mathcal{D}$ (resp. $\mathcal{S}$), we have*

$$ \int_0^\infty \tau^x_{(\alpha,\beta)}(f)(y)\, g(y)\, y^{2\beta+1} dy = \int_0^\infty f(y)^t\, \tau^x_{(\alpha,\beta)}(g)(y)\, y^{2\beta+1} dy $$

5. *For all $f \in \mathcal{S}$,*

$$ \mathcal{F}_{(\alpha,\beta)}\left({}^t\tau^x_{(\alpha,\beta)}(f)\right)(\lambda) = j_{(\alpha,\beta)}(\lambda x)\, \mathcal{F}_{(\alpha,\beta)}(f)(\lambda) $$

Proof Using mainly the $R^t_{\alpha,\beta}$ properties in Proposition 4 and Theorem 3.3, we deduce the mapping properties of the operator ${}^t\tau^x_{(\alpha,\beta)}$.

And the fact that

$$ {}^{t}\tau^{x}_{(\alpha,\beta)} : \mathcal{S} \to \mathcal{S} $$

will be the entry to justify that $\mathcal{F}_{(\alpha,\beta)}\left({}^{t}\tau^{x}_{(\alpha,\beta)}(f)\right)$ exists, and we have

$$ \begin{aligned} \mathcal{F}_{(\alpha,\beta)}\left({}^{t}\tau^{x}_{(\alpha,\beta)}(f)\right)(\lambda) &= c_\beta \int_0^\infty \left[{}^{t}\tau^{x}_{(\alpha,\beta)}\right] f(t)\, j_{(\alpha,\beta)}(\lambda t)\, t^{2\beta+1} dt \\ &= c_\beta \int_0^\infty f(t)\, \tau^{x}_{(\alpha,\beta)}\left(j_{(\alpha,\beta)}\right)(\lambda t)\, t^{2\beta+1} dt \\ &= j_{(\alpha,\beta)}(\lambda x)\, \mathcal{F}_{(\alpha,\beta)}(f)(\lambda). \end{aligned} $$

■

3.4 Generalized Wavelet Transform

In this subsection, we show that the transmutation operators are crucial to define and study the generalized wavelet transform.

3.4.1 Preliminaries

Definition 4 We define the Bessel wavelet as a measurable function satisfying the admissibility condition [4]

$$ 0 < C^{\alpha}_{g} = \int_0^\infty |\mathcal{F}_\alpha(g)(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty $$

Definition 5 Let $a, b > 0$, the Bessel continuous wavelet transform is defined for suitable functions f on $[0, \infty[$ by

$$ S^{\alpha}_{g}(f)(a,b) = \int_0^\infty f(x)\, \overline{g^{\alpha}_{a,b}(x)} x^{2\alpha+1} dx $$

Where

$$ g^{\alpha}_{a,b}(x) = \frac{1}{a^{2\alpha+2}} \tau^{b}_{\alpha}(g_a)(b) $$

and

$$g_a(x) = g_a\left(\frac{x}{a}\right)$$

Theorem 8 *Let $g \in \mathcal{L}_\alpha^2$ be a Bessel wavelet of order α. We have*

1. *For all $f \in \mathcal{L}_\alpha^2$ we have the Plancherel formula*

$$\int_0^\infty |f(x)|^2 x^{2\alpha+1} dx = \frac{1}{C_g^\alpha} \int_0^\infty \int_0^\infty \left|S_g^\alpha(f)(a,b)\right|^2 b^{2\alpha+1} db \frac{da}{a}$$

2. *Assume that $\|F_\alpha(g)\|_\infty < \infty$. For $f \in \mathcal{L}_\alpha^2$ and $0 < \varepsilon < \delta < \infty$, the function*

$$f^{\varepsilon,\delta} = \frac{1}{C_g} \int_\varepsilon^\delta \int_0^\infty S_g^\alpha(f)(a,b)\, g_{a,b}^\alpha(x)\, b^{2\alpha+1} db \frac{da}{a}$$

belongs to $\mathcal{L}_\alpha^2$ and satisfies

$$\lim_{\varepsilon \to 0,\, \delta \to \infty} \left\| f^{\varepsilon,\delta} - f \right\|_{2,\alpha} = 0$$

3. *For $f \in \mathcal{L}_\alpha^1$ such that $\mathcal{F}_\alpha(f) \in \mathcal{L}_\alpha^1$, we have*

$$f(x) = \frac{1}{C_g^\alpha} \int_0^\infty \left(\int_0^\infty S_g^\alpha(f)(a,b)\, g_{a,b}^\alpha(x)\, b^{2\alpha+1} db \right) \frac{da}{a}$$

for almost all $x \geq 0$.

3.4.2 The Bessel-Wright Wavelet

Definition 6 We define the Bessel-Wright wavelet as a measurable function satisfying the admissibility condition

$$0 < C_g^{\alpha,\beta} = \int_0^\infty \left|\mathcal{F}_{(\alpha,\beta)}(g)(\lambda)\right|^2 \frac{d\lambda}{\lambda} < \infty$$

Definition 7 Let $a, b > 0$, and let g be a Bessel-Wright wavelet. We consider the family $g_{a,b}^{\alpha,\beta}$ of Bessel-Wright wavelets on $]0, +\infty[$ in $\mathcal{L}_\alpha^2$, defined by

$$g_{a,b}^{\alpha,\beta}(x) = \frac{1}{a^{2\alpha+2}}\, {}^t\tau_{(\alpha,\beta)}^b(g_a)(x)$$

where

$$g_a(x) = g(x/a)$$

and $\tau^b_{(\alpha,\beta)}$ the Bessel-Wright translation operator 8.

Proposition 7 *For all* $a > 0,\ b > 0$, *we have*

$$g^{\alpha,\beta}_{a,b}(x) = (R_\alpha)_b \left(R^t_{\alpha,\beta}\right)^{-1}_x \left(R^t_{\alpha,\beta} g\right)^{\beta}_{a,b}(x) \tag{9}$$

where $g^{\alpha}_{a,b}$ *is the classical Bessel wavelet.*

Proof we have

$$\begin{aligned} g^{\alpha,\beta}_{a,b}(x) &= \frac{1}{a^{2\alpha+2}} {}^t\tau^b_{(\alpha,\beta)}(g_a)(x) \\ &= \frac{1}{a^{2\alpha+2}} (R_\alpha)_b \left(R^t_{(\alpha,\beta)}\right)^{-1}_x \tau^b_\beta \left(R^*_{\alpha,\beta}(g_a)\right)(x) \\ &= (R_\alpha)_b \left(R^t_{(\alpha,\beta)}\right)^{-1}_x \tau^b_\beta \left(R^*_{\alpha,\beta} g\right)_a (x) \\ &= (R_\alpha)_b \left(R^{*-1}_{\alpha,\beta}\right)_x \left(R^*_{\alpha,\beta} g\right)^{\beta}_{a,b}(x) \end{aligned}$$

which ends the proof. ■

Proposition 8 *A function g is a Bessel-Wright wavelet in* $\mathcal{D}$ (*resp.* $\mathcal{S}$) *if and only if the function* $R^t_{(\alpha,\beta)}g$ *is a Bessel wavelet, and we have*

$$C^{\alpha,\beta}_g = C^{\beta}_{\mathcal{R}^t_{(\alpha,\beta)}(g)}$$

Proof We deduce these results, since the following formula is valid in $\mathcal{D}$ (resp. $\mathcal{S}$),

$$\mathcal{F}_{(\alpha,\beta)} = \mathcal{F}_\beta \circ \mathcal{R}^t_{(\alpha,\beta)}$$

Which may end the proof. ■

Definition 8 Let $a, b > 0$, the Bessel-Wright continuous wavelet transform is defined for suitable functions f on $]0,\infty[$ by

$$S^{\alpha,\beta}_g(f)(a,b) = \int_0^\infty f(x)\, \overline{g^{\alpha,\beta}_{a,b}(x)} x^{2\beta+1} dx$$

Proposition 9 *For all $a, b > 0$, the Bessel-Wright continuous wavelet transform*

1. *Given a function $f \in \mathcal{S}$, the Bessel-Wright wavelet transform is linked to the classical Bessel wavelet transform via the following formula*

$$S_g^{\alpha,\beta}(f)(a,b) = (R_\alpha)_b \, S^\beta_{R^t_{(\alpha,\beta)}g}\left(R_\alpha^{-1} f\right)(a,b) \tag{10}$$

2. *$S_g^{\alpha,\beta}$ is a linear operator from $\mathcal{S}$ into itself.*

$$S_g^{\alpha,\beta} : \mathcal{S} \to \mathcal{S}$$

3. *$S_g^{\alpha,\beta}$ is a linear operator from $\mathcal{D}$ into itself.*

$$S_g^{\alpha,\beta} : \mathcal{D} \to \mathcal{D}$$

Proof From (9), we get

$$\begin{aligned} S_g^{\alpha,\beta}(f)(a,b) &= \int_0^\infty f(x)\, \overline{g_{a,b}^{\alpha,\beta}(x)} x^{2\beta+1} dx \\ &= \int_0^\infty f(x) (R_\alpha)_b \left(R^{*-1}_{\alpha,\beta}\right)_x \left(R^*_{\alpha,\beta} g\right)^\beta_{a,b}(x)\, x^{2\beta+1} dx \\ &= (R_\alpha)_b \int_0^\infty R_\alpha^{-1}(f)(x) \left(R^*_{\alpha,\beta} g\right)^\beta_{a,b}(x)\, x^{2\beta+1} dx \\ &= (R_\alpha)_b \, S^\beta_{R^*_{\alpha,\beta} g}\left(R_\alpha^{-1} f\right)(a,b). \end{aligned}$$

and so we deduce that

$$S_g^{\alpha,\beta} : \mathcal{S} \to \mathcal{S}$$

and.

$$S_g^{\alpha,\beta} : \mathcal{D} \to \mathcal{D}$$

■

Theorem 9 (Plancherel Formula) *Let $f \in \mathcal{S} \cap \mathcal{L}^2_\beta$ be a Bessel-Wright wavelet, we have the Planchrel formula*

$$\int_0^\infty |f(x)|^2 x^{2\beta+1} dx = \frac{1}{C^\alpha_{R^t_{(\alpha,\beta)}(g)}} \int_0^\infty \int_0^\infty \left| (R_\alpha)_b^{-1} S_g^{\alpha,\beta}(R_\alpha(f))(a,b) \right|^2 b^{2\beta+1} db \frac{da}{a}$$

Proof Using Theorem (8) and formula (10), we get

$$\int_0^\infty |f(x)|^2 x^{2\beta+1}dx = \frac{1}{C^\alpha_{R^t_{\alpha,\beta}(g)}} \int_0^\infty \int_0^\infty \left| S^\beta_{R^t_{(\alpha,\beta)}(g)}(f)(a,b)\right|^2 b^{2\beta+1}db\frac{da}{a}$$

$$= \frac{1}{C^\alpha_{R^t_{\alpha,\beta}(g)}} \int_0^\infty \int_0^\infty \left|(R_\alpha)^{-1}_b S^{\alpha,\beta}_g (R_\alpha(f))(a,b)\right|^2 b^{2\beta+1}db\frac{da}{a}.$$

■

Theorem 10 (Inversion Formulas) *Let $g \in \mathcal{L}^2_\beta$ be a Bessel-Wright wavelet, then we have*

- *For $f \in \mathcal{S}$, we have*

$$f(x) = \frac{1}{C^\alpha_{R^t_{(\alpha,\beta)}(g)}} \int_0^\infty \left(\int_0^\infty (R_\alpha)^{-1}_b S^{\alpha,\beta}_{R^t_{(\alpha,\beta)}(g)} (R_\alpha(f))(a,b) R^t_{(\alpha,\beta)}\left(g^\alpha_{a,b}\right)(x) b^{2\beta+2}db\right)\frac{da}{a}$$

  for almost all $x \geq 0$.
- *For $f \in \mathcal{S}$, we have*

$$R^{-1}_\alpha(f)(x) = \frac{1}{C^\alpha_{R^t_{(\alpha,\beta)}(g)}} \int_0^\infty \left(\int_0^\infty (R_\alpha)^{-1}_b S^\beta_{R^t_{(\alpha,\beta)}(g)}(f)(a,b) R^t_{(\alpha,\beta)}\left(g^\alpha_{a,b}\right)(x) b^{2\beta+2}db\right)\frac{da}{a}$$

  for almost all $x \geq 0$.

Proof Starting from Theorem 8 and Proposition 8, we get

$$f(x) = \frac{1}{C^\alpha_{R^t_{(\alpha,\beta)}(g)}} \int_0^\infty \left(\int_0^\infty S^\beta_{R^t_{\alpha,\beta}(g)}(f)(a,b) R^t_{(\alpha,\beta)}\left(g^\alpha_{a,b}\right)(x) b^{2\beta+1}db\right)\frac{da}{a}$$

$$= \frac{1}{C^\alpha_{R^t_{(\alpha,\beta)}(g)}} \int_0^\infty \left(\int_0^\infty (R_\alpha)^{-1}_b S^{\alpha,\beta}_{R^t_{(\alpha,\beta)}(g)} (R_\alpha(f))(a,b) R^t_{(\alpha,\beta)}\left(g^\alpha_{a,b}\right)(x) b^{2\beta+2}db\right)\frac{da}{a}$$

■

3.5 The Heat Kernel

The generalized Gaussian function introduced in [5], depending of parameter α, is defined by:

$$\psi_\alpha(x) = \sum_{n=0}^\infty (-1)^n \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+n)} \left(\frac{x^2}{2}\right)^n, \ \alpha > -1.$$

Note that when $\alpha = 0$, we find the classical Gaussian function $\psi(x) = e^{-\frac{x^2}{2}}$.

Proposition 10 *The generalized Gaussian function satisfies the following*

1. The function $(x, t) \mapsto \frac{1}{(2t)^{\beta+1}} \psi_\alpha \left(\frac{x}{\sqrt{2t}} \right)$ *is the solution of the heat equation*

$$\Delta^x_{(\alpha,\beta)} u\,(x, t) = \frac{\partial}{\partial t} u\,(x, t)\,.$$

2.

$$\mathcal{F}^x_{(\alpha,\beta)} e^{-tx^2} = \frac{1}{(2t)^{\beta+1}} \psi_\alpha \left(\frac{x}{\sqrt{2t}} \right).$$

3. The function $x \mapsto \psi_\alpha\,(x) \in \mathcal{S}$.

Proof Refer to [5]. ■

3.6 The Wave Kernel

As application, we introduce the generalized wave kernel function

$$\mathcal{W}_{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+1)}{2^\beta} \sum_{n=0}^{\infty} \frac{\Gamma(2n+2\beta+2)}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)} \left(-\frac{x^2}{2} \right)^n, \ \alpha, \beta > -1.$$

Corollary 2 *We have*

$$\mathcal{W}_{(\alpha,\beta)}(x) = \mathcal{F}_{(\alpha,\beta)} \left(e^{-x} \right) (x)\,. \tag{11}$$

In particular $\mathcal{W}_{(\alpha,\beta)} \in \mathcal{S}$.

Proof To prove this assertion we write

$$\begin{aligned}
\mathcal{F}_{(\alpha,\beta)} \left(e^{-x} \right) (x) &= c_\beta \int_0^\infty e^{-t} j_{(\alpha,\beta)}(tx) t^{2\beta+1} dt \\
&= c_\beta \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)} \left(\frac{x}{2} \right)^{2n} \left[\int_0^\infty e^{-t} t^{2\beta+2n+1} dt \right] \\
&= c_\beta \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2n+2\beta+2)\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)} \left(\frac{x}{2} \right)^{2n} \\
&= \frac{\Gamma(\alpha+1)}{2^\beta} \sum_{n=0}^{\infty} \frac{\Gamma(2n+2\beta+2)}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)} \left(-\frac{x^2}{2} \right)^n = \mathcal{W}_{(\alpha,\beta)}(x).
\end{aligned}$$

The fact that $\psi \in \mathcal{S}$ leads to the result. ■

Proposition 11 *The function* $(x,t) \mapsto \mathcal{W}_{(\alpha,\beta)}(tx)$ *is the solution of the wave equation*

$$\Delta_{(\alpha,\beta)}u\,(x,t) + \frac{\partial^2}{\partial t^2}u\,(x,t) = 0$$

Proof Now, recall that

$$\begin{aligned}
\Delta^x_{(\alpha,\beta)}\mathcal{W}_{(\alpha,\beta)}(tx) &= \Delta^x_{(\alpha,\beta)}\mathcal{F}^x_{(\alpha,\beta)}e^{-tx} \\
&= c_\beta \int_0^\infty e^{-tu}\Delta^x_{(\alpha,\beta)}j_{(\alpha,\beta)}(ux)u^{2\beta+1}du \\
&= c_\beta \int_0^\infty -u^2 e^{-tu} j_{(\alpha,\beta)}(ux)u^{2\beta+1}du \\
&= -c_\beta \int_0^\infty \frac{\partial^2}{\partial t^2}e^{-tu} j_{(\alpha,\beta)}(ux)u^{2\beta+1}du \\
&= -\frac{\partial^2}{\partial t^2}\mathcal{F}^x_{(\alpha,\beta)}e^{-tx}.
\end{aligned}$$

■

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On a Method of Solving Integral Equation of Carleman Type on the Pair of Segments

L. A. Khvostchinskaya

Abstract The method is considered of solving integral equations of Carleman type on the pair of adjacent and disjoint segments. The problem is reduced to boundary problem of Riemann with piecewise constant matrix and four and five singular points. The solution is expressed via the solution of a differential equation of Fuchs class in which it was possible to define all the parameters.

Keywords Integral equations of Carleman type · The canonical matrix · Riemann boundary value problem · Differential equation of the Fuchs class

In 1823, N. Abel considered and solved an integral equation

$$\int_a^x \frac{\varphi(t)}{\sqrt{x-t}}dt = f(x), x > a,$$

which describes the movement of a material point by gravity in a vertical plane along a curve. Abel integral equations

$$\frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt = f(x), \quad 0 < \alpha < 1, \quad x > a,$$

arise when solving inverse problems in solid state physics (determining the potential energy from the oscillation period or restoring the scattering field from the effective glow in classical mechanics). Abel integral equation with constant limits

$$\int_a^b \frac{\varphi(t)}{|x-t|^{1-\alpha}}dt = f(x), \quad 0 < \alpha < 1, \quad a < x < b, \tag{1}$$

L. A. Khvostchinskaya (✉)
Belarusian State Agrarian Technical University, Minsk, Belarus
e-mail: khv.ludmila.ark@gmail.com

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_19

it was decided by Carleman [1]. The unique solution to Eq. (1) is given by the formula [2]

$$\varphi(x) = \frac{tg\frac{\pi\alpha}{2}}{2\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{(x-t)^\alpha} - \frac{\sin^2\frac{\pi\alpha}{2}}{\pi^2}\frac{d}{dx}\int_a^x \left(\frac{b-t}{t-a}\right)^{\frac{\alpha}{2}} \frac{dt}{(x-t)^\alpha}\cdot$$
$$\cdot\frac{d}{dt}\int_a^t \frac{d\tau}{(t-\tau)^{1-\alpha}}\int_\tau^b \left(\frac{s-a}{b-s}\right)^{\frac{\alpha}{2}} \frac{f(s)\,ds}{(s-\tau)^\alpha}. \quad (2)$$

Consider the integral equation of Carleman type

$$\int_{L_1} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_1}} + \int_{L_2} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_2}} = f(x), \quad (3)$$

where α_1, α_2 are given real numbers, $0 < \alpha_k < 1$, $k = 1, 2$, $\alpha_1 \neq \alpha_2$, in the following two cases:

1. on a pair of adjacent segments $L_1 = [a_1, a_2]$, $L_2 = [a_2, a_3]$,
2. on a pair of disjoint segments $L_1 = [a_1, b_1]$, $L_2 = [a_2, b_2]$, $b_1 \neq a_2$.

The solution $\varphi(z)$ of problems (2) will be sought in the class of functions satisfying the Hölder condition inside the segments and that are integrable at the ends of segments, $f(x) = f_k(x)$, $x \in L_k$, $k = 1, 2$, are corresponding Holder functions.

Solution of Eq. (3) in the case $a_3 = \infty$ was constructed in [3] explicitly and expressed in terms of hypergeometric functions. It was also noted here that when $a_3 \neq \infty$ solution of Eq. (3) is much more complicated.

Equation (3) is a generalization of the Carleman equation (2). Let us reduce the integral equation (3) to the Riemann vector-matrix boundary value problem and construct a solution of this equation in each of the two cases, using the results of [4–7].

We construct the solution of the integral equation (3) on a pair of adjacent segments:

$$\int_{a_1}^{a_2} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_1}} + \int_{a_2}^{a_3} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_2}} = f(x), \quad a_1 < x < a_3. \quad (4)$$

We write Eq. (4) in the form of a system of three equations

$$\int_{a_1}^{a_2} \frac{\varphi_1(t)\,dt}{|x-t|^{\alpha_1}} + \int_{a_2}^{a_3} \frac{\varphi_2(t)\,dt}{(t-x)^{\alpha_2}} = f_1(x), \quad a_1 < x < a_2,$$

$$\int_{a_1}^{a_2} \frac{\varphi_1(t)\,dt}{(x-t)^{\alpha_1}} + \int_{a_2}^{a_3} \frac{\varphi_2(t)\,dt}{|x-t|^{\alpha_2}} = f_2(x), \quad a_2 < x < a_3, \quad (5)$$

$$\int_{a_1}^{a_2} \frac{\varphi_1(t)\,dt}{(x-t)^{\alpha_1}} + \int_{a_2}^{a_3} \frac{\varphi_2(t)\,dt}{(x-t)^{\alpha_2}} = 0, \quad a_3 < x < \infty.$$

We introduce two new unknown functions

$$\Phi_k(z) = \int_{a_k}^{a_{k+1}} \frac{\varphi_k(t)\,dt}{(t-z)^{\alpha_k}}, \quad k = 1, 2,$$

which are analytic in the complex plane z with the cut along the ray (a_1, ∞). Find the limiting values of these functions on the banks of the section.

For $a_1 < x < a_2$ we get $\Phi_1^{\pm}(x) = e^{\pm\pi i\alpha_1} \int_{a_1}^{x} \frac{\varphi(t)dt}{(x-t)^{\alpha_1}} + \int_{x}^{a_2} \frac{\varphi(t)dt}{(x-t)^{\alpha_1}}$, where we find

$$\int_{a_1}^{a_2} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_1}} = \frac{e^{\pi i\alpha_1}\Phi_1^+(x) + \Phi_1^-(x)}{1+e^{\pi i\alpha_1}}, \; \Phi_2^+(x) = \Phi_2^-(x) = \int_{a_2}^{a_3} \frac{\varphi(t)\,dt}{(t-x)^{\alpha_2}} \tag{6}$$

Similarly for $a_2 < x < a_3$ we get

$$\int_{a_2}^{a_3} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_2}} = \frac{e^{\pi i\alpha_2}\Phi_2^+(x) + \Phi_2^-(x)}{1+e^{\pi i\alpha_2}}, \; \Phi_1^+(x) = e^{-2\pi i\alpha_1}\Phi_1^-(x) =$$
$$= e^{-\pi i\alpha_1} \int_{a_1}^{a_2} \frac{\varphi(t)\,dt}{(x-t)^{\alpha_1}}. \tag{7}$$

For $a_3 < x < \infty$

$$\Phi_1^+(x) = e^{-\pi i\alpha_1}\Phi_1^-(x), \quad \Phi_2^+(x) = e^{-\pi i\alpha_2}\Phi_2^-(x). \tag{8}$$

Using formulas (6)–(8), we rewrite system (5) as boundary conditions for two functions $\Phi_1(z)$ and $\Phi_2(z)$ [8–10]:

$$\begin{cases} \Phi_1^+(x) = -e^{\pi i\alpha_1}\Phi_1^-(x) - \left(1+e^{-\pi i\alpha_1}\right)\Phi_2^-(x) + \left(1+e^{-\pi i\alpha_1}\right) f_1(x), \\ \Phi_2^+(x) = \Phi_2^-(x), \quad a_1 < x < a_2, \end{cases}$$

$$\begin{cases} \Phi_1^+(x) = e^{-2\pi i\alpha_1}\Phi_1^-(x), \quad a_2 < x < a_3, \\ \Phi_2^+(x) = -e^{-\pi i\alpha_1}\left(1+e^{-\pi i\alpha_2}\right)\Phi_1^-(x) - e^{-\pi i\alpha_2}\Phi_2^-(x) + \left(1+e^{-\pi i\alpha_2}\right) f_2(x), \end{cases}$$

$$\begin{cases} \Phi_1^+(x) = e^{-2\pi i\alpha_1}\Phi_1^-(x), \\ \Phi_2^+(x) = e^{-2\pi i\alpha_2}\Phi_2^-(x), \end{cases} \quad a_3 < x < \infty.$$

So we have obtained the Riemann boundary value problem for the vector function $\Phi(z) = \left(\Phi_1(z), \Phi_2(z)\right)$ with a piecewise constant matrix and four singular points a_1, a_2, a_3, ∞:

$$\Phi^+(x) = A_k\Phi^-(x) + F_k(x), \quad a_k < x < a_{k+1}, k = 1, 2, 3; a_4 = \infty, \tag{9}$$

$$A_1 = \begin{pmatrix} -e^{-\pi i\alpha_1} & -\left(1+e^{-\pi i\alpha_1}\right) \\ 0 & 1 \end{pmatrix}, \quad F_1(x) = \begin{pmatrix} \left(1+e^{-\pi i\alpha_1}\right) f_1(x) \\ 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} e^{-\pi i \alpha_1} & 0 \\ -e^{-\pi i \alpha_1}\left(1 + e^{-\pi i \alpha_2}\right) & -e^{-\pi i \alpha_2} \end{pmatrix}, \quad F_2(x) = \begin{pmatrix} 0 \\ \left(1 + e^{-\pi i \alpha_2}\right) f_2(x) \end{pmatrix},$$

$$A_3 = \begin{pmatrix} e^{-2\pi i \alpha_1} & 0 \\ 0 & e^{-2\pi i \alpha_2} \end{pmatrix}, \quad F_3(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The solution of problems (9) will be sought in the class of functions that are bounded as $z \to a_k$, $k = 1, 2, 3$, and disappearing at infinity.

In order to solve the inhomogeneous boundary value problem (9), it is necessary to construct a canonical matrix $X(z)$ corresponding homogeneous boundary value problem. The columns of the matrix $X(z)$ consist of linearly independent solutions of a homogeneous boundary problem

$$\Phi^+(x) = A_k \Phi^-(x), \quad a_k < x < a_{k+1}, \quad k = 1, 2, 3; \quad a_4 = \infty, \tag{10}$$

and orders p_1, p_2 first and second columns $X(z)$ at infinity satisfy inequality $p_1 \le p_2$. The matrix $X(z)$ has the following properties [11]:

1. $\det X(z) \neq 0$ for $\forall z \neq a_k$ $(k = 1, 2, 3)$;
2. the columns of the matrix $X(z)$ belong to the selected class of functions;
3. the order of the determinant $X(z)$ is equal to the sum of the orders of its columns.

If the matrix $X(z)$ multiply on the left by a constant nondegenerate second-order upper triangular matrix T, then the matrix $X(z)T$ will also be canonical, since the orders of the determinant and the columns of the matrix will not change.

The canonical matrix $\boldsymbol{X}(x)$ of homogeneous boundary value problem (10) is a solution of a system of differential equations of Fuchs class with four singular points a_1, a_2, a_3, ∞ [12]:

$$\frac{dX}{dz} = X \sum_{k=1}^{3} \frac{U_k}{z - a_k}, \tag{11}$$

moreover, differential matrices U_k like matrices $W_k = \frac{1}{2\pi i} \ln A_{k-1} A_k^{-1}, k = 1, \ldots, 4, A_0 = A_4 = E$. Matrices $V_k = A_{k-1} A_k^{-1}$, $k = 1, \ldots, 4$, form a monodromy group of a differential equation (11) [13–15].

Find differential matrices U_k systems (11) by the "logarithmization method of matrix product" of the 2nd order [4].

Let V_1, V_2 be constant non-degenerate matrices of the 2nd order, $V_3 = V_1 V_2$. Equality $\ln (V_1 V_2) = \ln V_1 + \ln V_2$ is valid only for transitive matrices. Denote by α_k, β_k the characteristic numbers of matrices V_k and by $\rho_k = \frac{1}{2\pi i} \ln \alpha_k$, $\sigma_k = \frac{1}{2\pi i} \ln \beta_k$ the characteristic numbers of matrices $W_k = \frac{1}{2\pi i} \ln V_k$, $k = 1, 2, 3$. Fix any branches of logarithms $\rho_1, \sigma_1, \rho_2, \sigma_2$ so that $|Re(\rho_k - \sigma_k)| < 1$, $k = 1, 2$. Then the branches of logarithms for ρ_3, σ_3 should be consistent and selected from the condition $\rho_1 + \sigma_1 + \rho_2 + \sigma_2 = \rho_3 + \sigma_3$.

If $\rho_3 \neq \sigma_3$, then the matrix $S = \begin{pmatrix} \rho_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$ uniquely accurate to a similarity transformation using a diagonal matrix can be represented as the sum of two matrices $S = S_1 + S_2$, where $S_k \sim W_k$, $k = 1, 2$. The last equality can be written as

$$\begin{pmatrix} \rho_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \frac{\rho_1\sigma_1-(\rho_3-\rho_2)(\rho_3-\sigma_2)}{\sigma_3-\rho_3} & \frac{(\rho_3-\rho_1)(\sigma_3-\sigma_1)-\rho_2\sigma_2}{\sigma_3-\rho_3}c \\ \frac{\rho_2\sigma_2-(\rho_3-\sigma_1)(\sigma_3-\rho_1)}{c(\sigma_3-\rho_3)} & \frac{(\sigma_3-\rho_2)(\sigma_3-\sigma_2)-\rho_1\sigma_1}{\sigma_3-\rho_3} \end{pmatrix} +$$

$$+ \begin{pmatrix} \frac{\rho_2\sigma_2+(\rho_3-\rho_1)(\rho_3-\sigma_1)}{\sigma_3-\rho_3} & \frac{\rho_2\sigma_2-(\rho_3-\rho_1)(\sigma_3-\sigma_1)}{\sigma_3-\rho_3}c \\ \frac{(\rho_3-\sigma_1)(\sigma_3-\rho_1)-\rho_2\sigma_2}{c(\sigma_3-\rho_3)} & \frac{(\sigma_3-\rho_1)(\sigma_3-\sigma_1)-\rho_2\sigma_2}{\sigma_3-\rho_3} \end{pmatrix} \quad (12)$$

where c is an arbitrary constant. If $\rho_3 = \rho_1 + \rho_2, \sigma_3 = \sigma_1 + \sigma_2$, then matrices V_1, V_2 are reduced by a single similarity transformation to a triangular form and simpler matrix representations take place S:

$$\begin{pmatrix} \rho_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \rho_1 & c \\ 0 & \sigma_1 \end{pmatrix} + \begin{pmatrix} \rho_2 & -c \\ 0 & \sigma_2 \end{pmatrix}, \quad (13)$$

$$\begin{pmatrix} \rho_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \rho_1 & 0 \\ c & \sigma_1 \end{pmatrix} + \begin{pmatrix} \rho_2 & 0 \\ -c & \sigma_2 \end{pmatrix}. \quad (14)$$

Let $V_1,\ V_2,\ V_3$ be constant non-degenerate matrices of the 2nd order, $V_4 = V_1V_2V_3$. Denote by $\alpha_k,\ \beta_k$ the characteristic numbers of matrices V_k and by $\rho_k = \frac{1}{2\pi i}\ln\alpha_k$, $\sigma_k = \frac{1}{2\pi i}\ln\beta_k$ the characteristic numbers of matrices $W_k = \frac{1}{2\pi i}\ln U_k$, $k = 1, \ldots, 4$, where the branches of logarithms satisfy the conditions $|Re\,(\rho_k - \sigma_k)| < 1$ and

$$\sum_{k=1}^{3}(\rho_k + \sigma_k) = \rho_4 + \sigma_4. \quad (15)$$

If $\rho_4 \neq \sigma_4$, then the matrix W_4 is reduced to diagonal Jordan form $S = \begin{pmatrix} \rho_4 & 0 \\ 0 & \sigma_4 \end{pmatrix}$. Representation of the matrix S as the sum of three matrices $S = S_1 + S_2 + S_3$, where $S_k \sim W_k$, $k = 1, 2, 3$, we get from the formulas (12)–(14). We write the product of matrices $V_1 \cdot V_2 \cdot V_3$ in the form of multiplication of two matrices as follows:

$$V_4 = V_1 \cdot V_2 \cdot V_3 = V_1 \cdot (V_2 \cdot V_3) = V_1 \cdot V_{23},$$

$$V_4 = V_1 \cdot V_2 \cdot V_3 = (V_1 \cdot V_2) \cdot V_3 = V_{12} \cdot V_3.$$

Therefore, we need to find the characteristic numbers α_{12}, β_{12} and α_{23}, β_{23} respectively matrices V_{12}, V_{23} and numbers $\rho_{12} = \frac{1}{2\pi i}\ln\alpha_{12}$, $\sigma_{12} = \frac{1}{2\pi i}\ln\beta_{12}$,

$\rho_{23} = \frac{1}{2\pi i}\ln\alpha_{23}$, $\sigma_{23} = \frac{1}{2\pi i}\ln\beta_{23}$, whose branches are chosen from the conditions $\rho_{12}+\sigma_{12} = \rho_1+\sigma_1+\rho_2+\sigma_2$, $|Re\,(\rho_{12}-\sigma_{12})| < 1$, $\rho_{23}+\sigma_{23} = \rho_2+\sigma_2+\rho_3+\sigma_3$, $|Re\,(\rho_{23}-\sigma_{23})| < 1$.

We write the matrices V_1, V_2, V_3, V_4 monodromy groups of problem (10) and their characteristic numbers λ_k, μ_k. $\left(k = \overline{1,4}\right)$:

$$V_1 = A_1^{-1} = \begin{pmatrix} -e^{\pi i\alpha_1} & -\left(1+e^{\pi i\alpha_1}\right) \\ 0 & 1 \end{pmatrix},$$

$$V_2 = A_1A_2^{-1} = \begin{pmatrix} -e^{\pi i\alpha_1} + \left(1+e^{\pi i\alpha_1}\right)\left(1+e^{\pi i\alpha_2}\right) & e^{\pi i\alpha_2}\left(1+e^{-\pi i\alpha_1}\right) \\ -e^{\pi i\alpha_1}\left(1+e^{\pi i\alpha_2}\right) & -e^{\pi i\alpha_2} \end{pmatrix},$$

$$V_3 = A_2A_3^{-1} = \begin{pmatrix} 1 & 0 \\ -e^{\pi i\alpha_1}\left(1+e^{-\pi i\alpha_2}\right) & -e^{\pi i\alpha_2} \end{pmatrix},$$

$$V_4 = A_3, \lambda_1 = -e^{\pi i\alpha_1}, \mu_1 = 1;$$

$$\lambda_2 = -1, \mu_2 = -e^{\pi i(\alpha_1+\alpha_2)}; \lambda_3 = 1, \mu_3 = -e^{-\pi i\alpha_2}; \lambda_4 = e^{-2\pi i\alpha_1},$$

$$\mu_4 = e^{-2\pi i\alpha_2}.$$

Next we find the numbers $\rho_k = \frac{1}{2\pi i}\ln\lambda_k$, $0 \le Re\lambda_k < 1$, $\sigma_k = \frac{1}{2\pi i}\ln\mu_k$, $0 \le Re\sigma_k < 1$, $k = \overline{1,4}$, $0 \le Re\lambda_k < 1$, $\rho_1 = \frac{1+\alpha_1}{2}$, $\sigma_1 = 0$; $\rho_2 = \frac{1}{2}$, $\sigma_2 = \frac{\alpha_1+\alpha_2+1}{2}$; $\rho_3 = 0$, $\sigma_3 = \frac{1+\alpha_2}{2}$; $\rho_4 = 1-\alpha_1$, $\sigma_4 = 1-\alpha_2$, $\Delta = \sum_{k=1}^{4}(\rho_k+\sigma_k) = 4$.

The behavior of the solution of problem (9) at infinity determine the numbers $\rho = \rho_4 - 1 = -\alpha_1$, $\sigma = \sigma_4 - 2 = -\alpha_2 - 1$, if $\alpha_1 > \alpha_2$ and $\rho = \rho_4 - 2 = -\alpha_1 - 1$, $\sigma = \sigma_4 - 1 = -\alpha_2$, if $\alpha_1 < \alpha_2$.

Numbers ρ_k, σ_k $(k = 1, 2, 3)$, ρ, σ satisfy the Fuchs relation:

$$\sum_{k=1}^{3}(\rho_k+\sigma_k) + \rho + \sigma = 1. \tag{16}$$

The total index κ and partial indices æ$_1$, æ$_2$ of the problem (9) are respectively equal æ $= -\Delta = -4$, æ$_1$ = æ$_2$ $= -2$, those problem (9) will be solvable if four solvability conditions are satisfied.

We also find the characteristic numbers λ_{12}, μ_{12} and λ_{23}, μ_{23} of the matrices

$$V_{12} = V_1 \cdot V_2 = A_2^{-1} = \begin{pmatrix} e^{2\pi i\alpha_1} & 0 \\ -e^{\pi i(\alpha_1+\alpha_2)}\left(1-e^{-\pi i\alpha_2}\right) & -e^{\pi i\alpha_2} \end{pmatrix},$$

$$\lambda_{12} = -e^{\pi i\alpha_1}, \mu_{12} = -e^{\pi i\alpha_2};$$

$$V_{23} = V_2 \cdot V_3 = A_1 \cdot A_3^{-1} = \begin{pmatrix} -e^{\pi i \alpha_1} & -e^{2\pi i \alpha_2}\left(1 + e^{-\pi i \alpha_1}\right) \\ 0 & e^{2\pi i \alpha_2} \end{pmatrix},$$

$$\lambda_{23} = -e^{\pi i \alpha_1}, \mu_{23} = e^{2\pi i \alpha_2}.$$

Branches of logarithms of numbers $\rho_{k,k+1} = \frac{1}{2\pi i} \ln \lambda_{k,k+1}$ and $\sigma_{k,k+1} = \frac{1}{2\pi i} \ln \mu_{k,k+1}$ should be conditions

$$\rho_{12}+\sigma_{12} = \rho_1+\sigma_1+\rho_2+\sigma_2 = 1+\alpha_1+\frac{1+\alpha_2}{2} \;\Rightarrow\; \rho_{12} = 1+\alpha_1,\; \sigma_{12} = \frac{1+\alpha_2}{2},$$

$$\rho_{23}+\sigma_{23} = \rho_2+\sigma_2+\rho_3+\sigma_3 = 1+\alpha_2+\frac{1+\alpha_1}{2} \;\Rightarrow\; \rho_{23} = \frac{1+\alpha_1}{2},\; \sigma_{12} = 1+\alpha_2.$$

Comparing formulas (15) and (16), we notice that $\rho_4 + \sigma_4 = 1 - \rho - \sigma$, those

$\rho_4 = 1 - \rho, \sigma_4 = -\sigma$, if $\alpha_1 > \alpha_2$,
$\rho_4 = 1 - \sigma, \sigma_4 = -\rho$, if $\alpha_1 < \alpha_2$.

Denote by $S = \begin{pmatrix} -\min(\rho, \sigma) & 0 \\ 0 & 1 - \max(\rho, \sigma) \end{pmatrix} = \begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & 1+\alpha_2 \end{pmatrix}$.

Imagine the matrix S as the sum of three matrices using the representations (13) and (14):

$$S = S_1 + S_2 + S_3 = S_1 + S_{23} = S_{12} + S_3, \tag{17}$$

where

$$S_k \sim \frac{1}{2\pi i} \ln V_k,\, S_{12} \sim \frac{1}{2\pi i} \ln V_{12},\, S_{23} \sim \frac{1}{2\pi i} \ln V_{23}. S_1 + S_{23} = S \;\Rightarrow$$

$$\begin{pmatrix} \frac{1+\alpha_1}{2} & c \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1+\alpha_1}{2} & -c \\ 0 & \alpha_2 + 1 \end{pmatrix} = \begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & 1+\alpha_2 \end{pmatrix}, S_{12} + S_3 = S \;\Rightarrow$$

$$\begin{pmatrix} \alpha_1 + 1 & 0 \\ d & \frac{\alpha_2+1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -d & \frac{1+\alpha_2}{2} \end{pmatrix} = \begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & 1+\alpha_2 \end{pmatrix},$$

where c, d are arbitrary constants. From (17) it follows that

$$S_2 = S_{23} - S_3 = S_{12} - S_1 = \begin{pmatrix} \frac{1+\alpha_1}{2} & -c \\ d & \frac{1+\alpha_2}{2} \end{pmatrix}.$$

Since $S_2 \sim \frac{1}{2\pi i} \ln V_2$, that $\det S_2 = \rho_2 \cdot \sigma_2$, or $\frac{1+\alpha_1}{2} \cdot \frac{1+\alpha_2}{2} + c \cdot d = \frac{1}{2} \cdot \frac{1+\alpha_1+\alpha_2}{2} \Rightarrow c \cdot d = -\frac{1}{4}\alpha_1 \cdot \alpha_2$. Matrices S_k $(k = 1, 2, 3)$ are differential matrices of system (11), which takes the form

$$\frac{dX}{dz} = X \left[\frac{\begin{pmatrix} (1+\alpha_1)/2 & c \\ 0 & 0 \end{pmatrix}}{z - a_1} + \frac{\begin{pmatrix} (1+\alpha_1)/2 & -c \\ -\alpha_1 \cdot \alpha_2 / 4c & (1+\alpha_2)/2 \end{pmatrix}}{z - a_2} + \right.$$

$$\left. + \frac{\begin{pmatrix} 0 & 0 \\ \alpha_1 \cdot \alpha_2 / 4c & (1+\alpha_2)/2 \end{pmatrix}}{z - a_3} \right], \tag{18}$$

where c is an arbitrary constant.

Let be $X(z) = \begin{pmatrix} u(z) & u_1(z) \\ v(z) & v_1(z) \end{pmatrix}$. Substituting this matrix into Eq. (18), we obtain the following system of differential equations connecting the functions $u(z)$ and $u_1(z)$:

$$\begin{aligned} u' &= \rho_1 \left(\frac{1}{z-a_1} + \frac{1}{z-a_2} \right) u + d \left(\frac{1}{z-a_3} - \frac{1}{z-a_2} \right) u_1, \\ u_1' &= c \left(\frac{1}{z-a_1} - \frac{1}{z-a_2} \right) u + \sigma_2 \left(\frac{1}{z-a_3} + \frac{1}{z-a_3} \right) u_1 \end{aligned} \tag{19}$$

Functions $v(z)$ and $v_1(z)$are also solutions of the system (19). Express the function from the first equation of system (19)

$$u_1 = \frac{(z-a_2)(z-a_3)}{d(a_3-a_2)} \left[u' - \rho_1 \left(\frac{1}{z-a_1} + \frac{1}{z-a_2} \right) u \right]$$

and substitute it into the second equation. We obtain a second-order differential equation whose fundamental system of solutions are functions $u(z)$and$v(z)$. This is a differential equation of Fuchs class with four singular points a_1, a_2, a_3, ∞:

$$\begin{aligned} & u'' - \frac{1}{2} \left(\frac{\alpha_1+1}{z-a_1} + \frac{\alpha_1+\alpha_2}{z-a_2} + \frac{\alpha_2-1}{z-a_3} \right) u' + \frac{1}{4} \left(\frac{2(\alpha_1+1)}{(z-a_1)^2} + \frac{\alpha_1+\alpha_2+1}{(z-a_2)^2} + \right. \\ & \left. + \frac{(4\alpha_1\alpha_2 + 3(\alpha_1-\alpha_2-1))z + (\alpha_1-\alpha_2-2\alpha_1\alpha_2+1)(a_1-a_3) + (\alpha_1-\alpha_2+1)a_2}{(z-a_1)(z-a_2)(z-a_3)} \right) u = 0 \end{aligned} \tag{20}$$

In the neighborhood of each singular point a_k $(k = 1, 2, 3)$ Eq. (20) has 2 linearly independent solutions, representable by series of the form

$$u_k(z) = (z-a_k)^{\rho_k} \sum_{n=0}^{\infty} c_n^{(k)} (z-a_k)^n,$$

$$v_k(z) = (z-a_k)^{\sigma_k} \sum_{n=0}^{\infty} d_n^{(k)} (z-a_k)^n, \tag{21}$$

whose coefficients are found directly from the recurrence relations after substituting the series in the equation. The canonical matrix of the problem (9) in the neighborhood of each singular point is given by the formula

$$X(z) = D_k \begin{pmatrix} u_k \; (z-a_2)(z-a_3)\left[u_k' - \frac{\alpha_1+1}{2}\left(\frac{1}{z-a_1}+\frac{1}{z-a_2}\right)u_k\right] \\ v_k \; (z-a_2)(z-a_3)\left[v_k' - \frac{\alpha_1+1}{2}\left(\frac{1}{z-a_1}+\frac{1}{z-a_2}\right)v_k\right] \end{pmatrix},$$

$k = 1, 2, 3$, where D_k are matrices transforming the matrices V_k to a Jordan form. The solution of the boundary value problem (9) is found by the formula

$$\Phi(z) = \frac{1}{2\pi i} X(z) \sum_{k=1}^{3} \int_{a_k}^{a_{k+1}} \left[X^+(x)\right]^{-1} F_k(x) \frac{dx}{x-z} =$$

$$= \frac{1}{2\pi i} X(z) \left[\int_{a_1}^{a_2} \left[X^+(x)\right]^{-1} F_1(x) \frac{dx}{x-z} + \int_{a_2}^{a_3} \left[X^+(x)\right]^{-1} F_2(x) \frac{dx}{x-z}\right].$$

Considering that $X^+(x) = A_k X^-(x), a_k < x < a_{k+1}, k = 1, 2, 3$, and applying the Sokhotsky formulas, as well as formulas (6) and (7), we find the integrals

$$\int_{a_k}^{a_{k+1}} \frac{\varphi(t)\,dt}{|x-t|^{\alpha_k}} = g_k(x), \quad k = 1, 2. \tag{22}$$

Reversing equations (22) using formulas (2), we obtain a unique solution to the integral equation (4) when two matrix solvability conditions are satisfied

$$\int_{a_1}^{a_2} \left[X^+(x)\right]^{-1} F_1(x)\, x^k dx + \int_{a_2}^{a_3} \left[X^+(x)\right]^{-1} F_2(x)\, x^k dx = 0, k = 1, 2.$$

We now consider the Carleman integral equation on a pair disjoint segments.

$$\int_{a_1}^{b_1} \frac{\varphi(t)}{|x-t|^{\alpha_1}} dt + \int_{a_2}^{b_2} \frac{\varphi(t)}{|x-t|^{\alpha_2}} dt = f(x), x \in [a_1, b_1] \bigcup [a_2, b_2], \tag{23}$$

where α_1, α_2 are given real numbers, $0 < \alpha_k < 1, k = 1, 2, \alpha_1 \neq \alpha_2, a_1 < b_1 < a_2 < b_2$.

We introduce two new unknown functions

$$\Phi_k(z) = \int_{a_k}^{b_k} \frac{\varphi_k(t)}{(t-z)^{\alpha_k}} dt, \quad k = 1, 2, \quad \varphi_k(t) = \varphi(t), \quad t \in [a_k, b_k],$$

which are analytic in the complex plane z with a cut along the ray (a_1, ∞). Find the limiting values of these functions on the banks of the section.

For $a_1 < x < b_1$ we get

$$\Phi_1^{\pm}(x) = e^{\pm\pi i\alpha_1}\int_{a_1}^{x}\frac{\varphi(t)\,dt}{(x-t)^{\alpha_1}} + \int_{x}^{b_1}\frac{\varphi(t)\,dt}{(x-t)^{\alpha_1}},$$

where we find

$$\int_{a_1}^{b_1}\frac{\varphi(t)\,dt}{|x-t|^{\alpha_1}} = \frac{e^{\pi i\alpha_1}\Phi_1^{+}(x) + \Phi_1^{-}(x)}{1+e^{\pi i\alpha_1}},$$

$$\Phi_2^{+}(x) = \Phi_2^{-}(x) = \int_{a_2}^{b_2}\frac{\varphi(t)\,dt}{(t-x)^{\alpha_2}}. \qquad (24)$$

Similarly for $a_2 < x < b_2$ we get

$$\int_{a_2}^{b_2}\frac{\varphi(t)\,dt}{|x-t|^{\alpha_2}} = \frac{e^{\pi i\alpha_2}\Phi_2^{+}(x) + \Phi_2^{-}(x)}{1+e^{\pi i\alpha_2}},\ \Phi_1^{+}(x) = e^{-2\pi i\alpha_1}\Phi_1^{-}(x) =$$

$$= e^{-\pi i\alpha_1}\int_{a_1}^{b_1}\frac{\varphi(t)\,dt}{(x-t)^{\alpha_1}} \qquad (25)$$

For $b_1 < x < a_2$

$$\Phi_1^{+}(x) = e^{-2\pi i\alpha_1}\Phi_1^{-}(x),\ \Phi_2^{+}(x) = \Phi_2^{-}(x).$$

For $b_2 < x < \infty$

$$\Phi_1^{+}(x) = e^{-2\pi i\alpha_1}\Phi_1^{-}(x),\ \Phi_2^{+}(x) = e^{-2\pi i\alpha_2}\Phi_2^{-}(x).$$

We write the system of boundary conditions for two functions $\Phi_1(z)$ and $\Phi_2(z)$:

$$\begin{cases}\Phi_1^{+}(x) = -e^{\pi i\alpha_1}\Phi_1^{-}(x) - \left(1+e^{-\pi i\alpha_1}\right)\Phi_2^{-}(x) + \left(1+e^{-\pi i\alpha_1}\right)f_1(x),\\ \Phi_2^{+}(x) = \Phi_2^{-}(x), \quad a_1 < x < b_1,\end{cases}$$

$$\begin{cases}\Phi_1^{+}(x) = e^{-2\pi i\alpha_1}\Phi_1^{-}(x),\\ \Phi_2^{+}(x) = \Phi_2^{-}(x),\end{cases} \quad b_1 < x < a_2.$$

$$\begin{cases}\Phi_1^{+}(x) = e^{-2\pi i\alpha_1}\Phi_1^{-}(x), \quad a_2 < x < b_2,\\ \Phi_2^{+}(x) = -e^{-\pi i\alpha_1}\left(1+e^{-\pi i\alpha_2}\right)\Phi_1^{-}(x) - e^{-\pi i\alpha_2}\Phi_2^{-}(x) + \left(1+e^{-\pi i\alpha_2}\right)f_2(x),\end{cases}$$

$$\begin{cases}\Phi_1^{+}(x) = e^{-2\pi i\alpha_1}\Phi_1^{-}(x),\\ \Phi_2^{+}(x) = e^{-2\pi i\alpha_2}\Phi_2^{-}(x),\end{cases} \quad b_2 < x < \infty.$$

So we have obtained the Riemann boundary value problem for the vector function $\Phi(z) = (\Phi_1(z), \Phi_2(z))$ with a piecewise constant matrix and five singular points $a_1, b_1, a_2, b_2, \infty$:

$$\Phi^+(x) = A_k\Phi^-(x) + F_k(x), \quad x \in l_k, \quad k = 1, 2, 3, 4; \tag{26}$$

$$l_1 \in (a_1, b_1), l_2 \in (b_1, a_2), l_3 \in (a_2, b_2), l_4 \in (b_2, \infty),$$

$$A_1 = \begin{pmatrix} -e^{-\pi i\alpha_1} & -\left(1+e^{-\pi i\alpha_1}\right) \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} e^{-2\pi i\alpha_1} & 0 \\ 0 & 1 \end{pmatrix},$$

$$F_1(x) = \begin{pmatrix} \left(1+e^{-\pi i\alpha_1}\right) f_1(x) \\ 0 \end{pmatrix}, A_3 = \begin{pmatrix} e^{-2\pi i\alpha_1} & 0 \\ -e^{-\pi i\alpha_1}\left(1+e^{-\pi i\alpha_2}\right) & -e^{-\pi i\alpha_2} \end{pmatrix},$$

$$A_4 = \begin{pmatrix} e^{-2\pi i\alpha_1} & 0 \\ 0 & e^{-2\pi i\alpha_2} \end{pmatrix}, F_3(x) = \begin{pmatrix} 0 \\ \left(1+e^{-\pi i\alpha_2}\right) f_2(x) \end{pmatrix},$$

$$F_2(x) = F_4(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f_k(x) = f(x), \quad x \in [a_k, b_k], \quad k = 1, 2.$$

Find the characteristic numbers $\lambda_k, \quad \mu_k, \quad k = \overline{1,5}$, of the monodromy matrices $V_k = A_{k-1}A_k^{-1}$, $A_0 = A_5 = E$ and numbers $\rho_k = \frac{1}{2\pi i}\ln\lambda_k$, $\sigma_k = \frac{1}{2\pi i}\ln\mu_k$, $0 \le Re\rho_k < 1$, $0 \le Re\sigma_k < 1$, and characteristic numbers and corresponding logarithms of matrices V_1V_2, V_3V_4, $V_1V_2V_3$, $V_2V_3V_4$:

$$V_1 = A_1^{-1}, V_2 = A_1A_2^{-1}, \lambda_1 = \lambda_2 = -e^{\pi i\alpha_1}, \mu_1 = \mu_2 = 1;$$

$$\rho_1 = \rho_2 = (1+\alpha_1)/2, \sigma_1 = \sigma_2 = 0,$$

$$V_3 = A_2A_3^{-1}, V_4 = A_3A_4^{-1}, \lambda_3 = \lambda_4 = 1, \mu_3 = \mu_4 = -e^{-\pi i\alpha_2},$$

$$\rho_3 = \rho_4 = 0, \rho_3 = \rho_4 = (1+\alpha_2)/2,$$

$$V_5 = A_4, \lambda_5 = e^{-2\pi i\alpha_1}, \mu_5 = e^{-2\pi i\alpha_2}, \rho_5 = 1-\alpha_1, \sigma_5 = 1-\alpha_2,$$

$$V_{12} = V_1 \cdot V_2 = A_2^{-1}, \lambda_{12} = e^{2\pi i\alpha_1}, \mu_{12} = 1, \rho_{12} = \alpha_1, \ \sigma_{12} = 1,$$

$$V_{34} = V_3 \cdot V_4 = A_2 \cdot A_4^{-1}, \lambda_{34} = 1, \mu_{34} = e^{2\pi i\alpha_2}, \ \rho_{34} = 1, \ \sigma_{34} = \alpha_2,$$

$$V_{123} = V_1 \cdot V_2 \cdot V_3 = A_3^{-1}, \lambda_{123} = e^{2\pi i\alpha_1}, \mu_{123} = -e^{-\pi i\alpha_2},$$

$$\rho_{123} = 1+\alpha_1, \sigma_{123} = (1+\alpha_2)/2,$$

$$V_{234} = V_2 \cdot V_3 \cdot V_4 = A_1A_4^{-1}, \lambda_{234} = -e^{-\pi i\alpha_1}, \mu_{234} = e^{2\pi i\alpha_2},$$

$$\rho_{234} = (1+\alpha_1)/2, \sigma_{234} = 1+\alpha_2.$$

The branches of the logarithms of monodromy matrix products are chosen from the conditions

$$\rho_{12} + \sigma_{12} = \rho_1 + \sigma_1 + \rho_2 + \sigma_2, \rho_{123} + \sigma_{123} = \rho_1 + \sigma_1 + \rho_2 + \sigma_2 + \rho_3 + \sigma_3$$

etc.

The behavior of the solution of problem (26) at infinity determine the numbers

$\rho = \rho_5 - 1 = -\alpha_1, \sigma = \sigma_5 - 2 = -\alpha_2 - 1$, if $\alpha_1 > \alpha_2$ and
$\rho = \rho_5 - 2 = -\alpha_1 - 1, \sigma = \sigma_5 - 1 = -\alpha_2$, if $\alpha_1 < \alpha_2$.

The numbers ρ_k,σ_k($k = 1, 2, 3, 4$),ρ,σ satisfy the Fuchs relation:

$$\sum_{k=1}^{4} (\rho_k + \sigma_k) + \rho + \sigma = 1.$$

The total index κ and partial indices æ$_1$, æ$_2$ of the problem (9) are respectively equal æ $= -\sum_{k=1}^{5} (\rho_k + \sigma_k) = -4$, æ$_1$ = æ$_2$ $= -2$, those problem (24) has a unique solution.

The canonical matrix $\boldsymbol{X}(x)$ of homogeneous boundary value problem

$$X^+(x) = A_k X^-(x), x \in l_k, \quad k = 1, 2, 3, 4,$$

is a solution of a system of differential equations of Fuchs class

$$\frac{dX}{dz} = X \sum_{k=1}^{4} \frac{U_k}{z - a_k}, \tag{27}$$

where $U_k \tilde{} \frac{1}{2\pi i} \ln V_k$, $k = 1, \ldots, 4$. Denote by

$$S = \begin{pmatrix} -\min(\rho, \sigma) & 0 \\ 0 & 1 - \max(\rho, \sigma) \end{pmatrix} = \begin{pmatrix} 1 + \alpha_1 & 0 \\ 0 & 1 + \alpha_2 \end{pmatrix}$$

and imagine the matrix S as the three sums of matrices:

$$S = S_1 + S_{234} = S_{123} + S_4 = S_{12} + S_{34}, \tag{28}$$

where

$$S_k \sim \frac{1}{2\pi i} \ln V_k,\ S_{12} \sim \frac{1}{2\pi i} \ln V_{12},\ S_{34} \sim \frac{1}{2\pi i} \ln V_{34},$$

$$S_{234} \sim \frac{1}{2\pi i} \ln V_{234},\ S_{123} \sim \frac{1}{2\pi i} \ln V_{123}.$$

Knowing the characteristic numbers and their logarithms of monodromy matrices and their products, we write three representations of the matrix S: $S_1 + S_{234} = S \;\Rightarrow\; \begin{pmatrix} (1+\alpha_1)/2 & c_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} (1+\alpha_1)/2 & -c_1 \\ 0 & 1+\alpha_2 \end{pmatrix} = \begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & 1+\alpha_2 \end{pmatrix}$,

$$S_{12} + S_{34} = S \;\Rightarrow\; \begin{pmatrix} \alpha_1 & 0 \\ c_2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -c_2 & \alpha_2 \end{pmatrix} = \begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & 1+\alpha_2 \end{pmatrix}, \tag{29}$$

$$S_{123} + S_4 = S \;\Rightarrow\; \begin{pmatrix} 1+\alpha_1 & c_3 \\ 0 & (1+\alpha_2)/2 \end{pmatrix} + \begin{pmatrix} 0 & -c_3 \\ 0 & (1+\alpha_2)/2 \end{pmatrix} = \\ = \begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & 1+\alpha_2 \end{pmatrix},$$

where c_1, c_2, c_3 are arbitrary constants. From formulas (28), (29) it follows that

$$S_2 = S_{12} - S_1 = S_{234} - S_{34} = \begin{pmatrix} (\alpha_1 - 1)/2 & -c_1 \\ c_2 & 1 \end{pmatrix},$$

$$S_3 = S_{123} - S_{12} = S_{34} - S_4 = \begin{pmatrix} 1 & c_3 \\ -c_2 & (\alpha_2 - 1)/2 \end{pmatrix}.$$

So $\det S_2 = \rho_2 \cdot \sigma_2 = 0$ and $\det S_3 = \rho_3 \cdot \sigma_3 = 0$, constants c_1, c_2, c_3 are related by: $c_1 c_2 = \frac{1-\alpha_1}{2}$, $c_2 c_3 = \frac{1-\alpha_2}{2}$, where do we find that $c_1 = \frac{1-\alpha_1}{2c_2}$, $c_3 = \frac{1-\alpha_2}{2c_2}$.

Matrices S_k $(k = 1, 2, 3, 4)$ are differential matrices of system (27), which takes the form

$$\frac{dX}{dz} = X \left[\frac{\begin{pmatrix} (1+\alpha_1)/2 & (1-\alpha_1)/2c \\ 0 & 0 \end{pmatrix}}{z - a_1} + \frac{\begin{pmatrix} (\alpha_1 - 1)/2 & (\alpha_1 - 1)/2c \\ c & 1 \end{pmatrix}}{z - a_2} + \right. \\ \left. + \frac{\begin{pmatrix} 1 & (1-\alpha_2)/2c \\ -c & (\alpha_2 - 1)/2 \end{pmatrix}}{z - a_3} + \frac{\begin{pmatrix} 0 & (\alpha_2 - 1)/2c \\ 0 & (\alpha_2 + 1)/2 \end{pmatrix}}{z - a_4} \right],$$

where c is an arbitrary constant. The elements of the matrix are solutions of a differential equation of Fuchs class with five singular points $a_1, b_1, a_2, b_2, \infty$:

$$u'' - \frac{1}{2}\left(\frac{\alpha_1+1}{z-a_1} + \frac{\alpha_1-1}{z-a_2} + \frac{\alpha_2-1}{z-a_3} + \frac{\alpha_2+1}{z-a_4}\right)u' + \tag{30}$$

$$+\frac{1}{4}\left(\frac{2(\alpha_1+1)}{(z-a_1)^2} + \frac{2(\alpha_1-1)}{(z-a_1)(z-a_2)} + + \frac{(\alpha_1+1)(\alpha_2-1)-4\alpha_1}{(z-a_1)(z-a_3)} + \right.$$

$$+\frac{(\alpha_1+1)(\alpha_2+1)}{(z-a_1)(z-a_4)} + + \frac{(\alpha_1+1)(\alpha_2-1)}{(z-a_2)(z-a_3)} + \frac{(\alpha_1+1)(\alpha_2+1)-4\alpha_2}{(z-a_2)(z-a_4)} +$$

$$\left. + \frac{4\alpha_2}{(z-a_3)(z-a_4)}\right)u = 0.$$

In the neighborhood of each singular point equation (30) has 2 linearly independent solutions, representable by series of the form (22).

The canonical matrix $X(z)$ of the problem (26) in the neighborhood of each singular point is given by the formula

$$X(z) = D_k \begin{pmatrix} u_k & (z-b_1)(z-a_2)\left[u_k' - \frac{1}{2}\left(\frac{\alpha_1+1}{z-a_1} + \frac{\alpha_1-1}{z-b_1} + \frac{2}{z-a_2}\right)u_k\right] \\ v_k & (z-b_1)(z-a_2)\left[v_k' - \frac{1}{2}\left(\frac{\alpha_1+1}{z-a_1} + \frac{\alpha_1-1}{z-b_1} + \frac{2}{z-a_2}\right)v_k\right] \end{pmatrix},$$

$k = 1, 2, 3, 4$, where D_k are matrices transforming the matrices V_k to a Jordan form.

The only solution to problem (26) is found by the formula

$$\Phi(z) = \frac{1}{2\pi i}X(z)\left[\int_{a_1}^{b_1}\left[X^+(x)\right]^{-1}F_1(x)\frac{dx}{x-z} + \int_{a_2}^{b_2}\left[X^+(x)\right]^{-1}F_3(x)\frac{dx}{x-z}\right].$$

Using the Sokhotsky formulas and formulas (25) and (26), we find the integrals $\int_{a_k}^{b_k}\frac{\varphi(t)dt}{|x-t|^{\lambda_k}} = g_k(x), \quad k = 1, 2$. Reversing the last equations, we find the only solution to the integral equation (24) when two matrix (four scalar) resolvability conditions are satisfied:

$$\int_{a_1}^{b_1}\left[X^+(x)\right]^{-1}F_1(x)x^k dx + \int_{a_2}^{b_2}\left[X^+(x)\right]^{-1}F_3(x)x^k dx = 0, k = 1, 2.$$

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Transmutation Operators Boundary Value Problems

Sergei M. Sitnik, Oleg Yaremko, and Natalia Yaremko

Abstract Transmutation operators method is used to solve and study boundary value problems. In this paper several ways to obtain transformation operators are considered: the finite integral transforms, Neumann series, the Fourier transforms, and reflection techniques. The finite integral transform technique leads to solution in the form of a composition of the Fourier sine transform and inverse finite integral transform. The Neumann series technique implies decomposition of the solution in power series of the shift operator. The Fourier transform technique provides transition to the Fourier images and comparison with the model boundary value problem. Reflection technique involves a consistent approach to the solution as a reflection from the borders. In all cases, the solution of the boundary value problem is obtained as an expansion in the solutions of the model boundary value problem. In some cases, the sum of a series can be calculated in elementary functions. New formulas have been found for solving the Dirichlet problem in a three-dimensional layer.

Keywords Transmutation operators · Boundary value problems · Integral transforms · Laplace equation · Poisson operator

MSC S44A05

1 Introduction

The aim of this article is to develop the theory of transmutation operators and apply it to solving boundary value problems for the Laplace equation in domains with plane symmetry. The classical transmutation operators are introduced by

S. M. Sitnik
Belgorod State National Research University (BelGU), Belgorod, Russia
e-mail: sitnik@bsu.edu.ru

O. Yaremko (✉) · N. Yaremko
Penza State University, Penza, Russia
e-mail: ct@pnzgu.ru; kaf-geom@yandex.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_20

K. Weierstrass, S. D. Poisson N. Y. Sonin and are used in mathematical physics [2–4, 6–8, 10, 14, 17, 18]. S.M. Sitnik [17] describes the general definition of the transmutation operator, see Definition 1 in [15].

Definition 1 An operator J is called the transmutation operator if for operators A, B the following condition holds

$$JA = BJ.$$

If the solution $y = B^{-1}x$ of the model problem $By = x$ is known, then the solution of the new problem$Az = x$ can be found using the transmutation operator J by the formula $z = J^{-1}B^{-1}Jx$. If we select

$$A = \frac{d^2}{dx^2}, B = B_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha + 1}{x},$$

B_α—the Bessel operator, then the transmutation operator $J = P_0$ is the Poisson operator [17]

$$P_0 [f(x)] = \frac{2}{\pi} \int_0^1 \frac{f(\varepsilon x)}{\sqrt{1-\varepsilon^2}} d\varepsilon.$$

The transmutation operator has the form $P_0 = H^{-1}F_c$, here H is the Hankel transform, and F_c is the Fourier cosine transform.

In the article, we clarify the concept of a transmutation operator in order to solve boundary value problems for potential theory. For this, we consider two boundary value problems for the Laplace equation

$$\begin{cases} u''_{xx} + u''_{yy} = 0, 0 < x, -\infty < y < \infty; \\ \Gamma u(0, y) = g(y); \end{cases} \quad \begin{cases} \tilde{u''}_{xx} + \tilde{u''}_{yy} = 0, 0 < x, -\infty < y < \infty; \\ \tilde{\Gamma} \tilde{u}(0, y) = g(y). \end{cases}$$

Below in Definition 2 we define the transmutation operator associated with boundary conditions. The transmutation operator establishes an isomorphism of these boundary value problems.

Definition 2 Let two boundary operators $\tilde{\Gamma}$, Γ be given. An operator J is called the transmutation operator if the following conditions hold:

(1) the transmutation operator J and operator $\frac{d^2}{dx^2}$ are permutable,
(2) $\tilde{\Gamma} J = \Gamma$.

In contrast to the general case [17], Definition 2 introduces special transmutation operators that take into account boundary conditions.The introduced operators are permutable with the Laplace operator, they transform the harmonic function into a harmonic function and change the type of boundary conditions.For example, the Dirichlet problem in a semi-plane is transformed into a boundary value problem with non-local boundary conditions. The transmutation operators introduced in the article (see Definition 2) establish a functional connection between the different boundary-value problems of the potential theory. Moreover, the properties of the solution of a new boundary value problem are determined by the properties of the solution of the model boundary value problem. The transmutation operator allows us to obtain the solution of a boundary value problem in the form of Neumann series, more convenient when implemented on a computer.The members of the Neumann series are powers of the shift operator, therefore, the calculations are cyclical.In addition, the usage of transmutation operators allows us to clarify the structure of potential field and present it as a sum of field reflections from domain boundary. Further, in Sect. 2 we present four ways to construct the transmutation operators: The finite integral transforms technique, Reflection method, the Fourier transform technique, Neumann series technique. The main results and conclusions are formulated in Sects. 3 and 4.

2 Materials and Methods

2.1 The Finite Integral Transforms Technique

The transmutation operators technique is based on the study of a pair of Sturm-Liouville problems. The transmutation operator establishes an isomorphism of the singular and regular Sturm-Liouville problems [5, 13]. For the most important cases in applications, an explicit expression for the transmutation operators is found.

2.1.1 Sturm–Liouville Problem with Dirichlet Boundary Conditions

Let's consider the Sturm–Liouville problem on finding nontrivial solutions on the interval $(0, \pi)$

$$\begin{cases} y'' + \lambda^2 y = 0, \\ y(0) = 0,\, y(\pi) = 0. \end{cases}$$

The eigenvalues have the form $\lambda_k = k, k = 1, 2, 3, \ldots$, and the corresponding eigenfunctions are $y_k(x) = \sin kx, k = 1, 2, 3, \ldots$ Let the function $y = f(x)$ be defined on the segment $[0, \pi]$ and $\hat{f}(k)$ be its the Fourier integral transform

$$\hat{f}(k) = \int_0^{\pi} \sin kx f(x)\, dx. \tag{1}$$

Then the function$y = f(x)$ can be represented

$$f(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \hat{f}(k) \sin kx. \tag{2}$$

For the function $y = F(x)$ on the interval $[0, \infty)$ we consider the Fourier sin transforms on the real semi-axis, direct:

$$\hat{F}(\lambda) = \int_0^\infty \sin\lambda x F(x)\, dx,$$

inverse:

$$F(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \hat{F}(\lambda)\, d\lambda.$$

Let the function $y = f(x)$ on the interval $[0, \pi]$ corresponds to the function $\hat{F}(\lambda)$ by formula (2):

$$f(x) = \frac{2}{\pi}\sum_{k=1}^{\infty} \hat{F}(k)\sin kx,\ x \in [0, \pi].$$

The mapping $J : F \to f$ is a transmutation operator

$$J[F](x) \equiv f(x) = \frac{2}{\pi}\sum_{k=1}^{\infty} \hat{F}(k)\sin kx.$$

Let the function $F(x)$ be sufficiently smooth and decreases sufficiently rapidly at infinity so that all arising integrals and series converge. We will transform the function $\hat{F}(k)$:

$$\hat{F}(k) = \int_0^\infty \sin kx F(x)\, dx =$$

$$\begin{aligned}
&= \textstyle\sum_{j=0}^\infty \left(\int_{2\pi j}^{2\pi j+\pi} \sin kx F(x)\, dx + \int_{2\pi j+\pi}^{2\pi j+2\pi} \sin kx F(x)\, dx\right) = \\
&= \textstyle\sum_{j=0}^\infty \int_0^\pi \sin kx F(x+2\pi j)\, dx + \int_\pi^{2\pi} \sin kx F(x+2\pi j)\, dx = \\
&= \textstyle\sum_{j=0}^\infty \int_0^\pi \sin kx F(x+2\pi j)\, dx - \int_0^\pi \sin kx F(2\pi - x + 2\pi j)\, dx = \\
&= \textstyle\sum_{j=0}^\infty \int_0^\pi \sin kx \left(F(x+2\pi j) - F(2\pi - x + 2\pi j)\right) dx = \\
&= \textstyle\int_0^\pi \sin kx \sum_{j=0}^\infty \left(F(x+2\pi j) - F(2\pi - x + 2\pi j)\right) dx.
\end{aligned}$$

We find the original $y = f(x)$ by formula (2). The transmutation operator J has the form:

$$\begin{aligned}
J[F](x) &= f(x) = \tfrac{2}{\pi}\textstyle\sum_{k=1}^\infty \hat{F}(k)\sin kx = \\
&= \textstyle\sum_{j=0}^\infty \left(F(x+2\pi j) - F(2\pi - x + 2\pi j)\right).
\end{aligned} \tag{3}$$

To apply the transmutation operator (3), we consider the Dirichlet problem for the strip

$$\begin{cases} u''_{xx} + u''_{yy} = 0, 0 < x < \pi, -\infty < y < \infty; \\ u(0, y) = g(y), u(\pi, y) = 0, \end{cases} \tag{4}$$

and the Dirichlet problem for the semi-plane

$$\begin{cases} \tilde{u}''_{xx} + \tilde{u}''_{yy} = 0, 0 < x, -\infty < y < \infty; \\ \tilde{u}(0, y) = g(y). \end{cases} \tag{5}$$

Using the transmutation operator (3), we establish relation of problems (4) and (5)

$$u(x, y) = J\left[\tilde{u}(x, y)\right] = \sum_{j=0}^{\infty} \left(\tilde{u}(x + 2\pi j, y) - \tilde{u}(2\pi - x + 2\pi j, y)\right). \tag{6}$$

Based on Poisson's formula for a semi-plane

$$\tilde{u}(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y - \eta)^2} g(\eta)\, d\eta,$$

and on identity from [12], we get

$$\sum_{j=0}^{\infty} \left(\frac{x + 2\pi j}{(x + 2\pi j)^2 + (y - \eta)^2} - \frac{2\pi - x + 2\pi j}{(2\pi - x + 2\pi j)^2 + (y - \eta)^2} \right) = \frac{1}{2} \frac{\sin x}{ch(y - \eta) - \cos x}. \tag{7}$$

Formula (7) is established for solving the Dirichlet problem in the strip [13]

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin x}{ch(y - \eta) - \cos x} g(\eta)\, d\eta.$$

2.1.2 Sturm–Liouville Problem with Neumann Boundary Conditions

Sturm–Liouville problem with Neumann boundary conditions is to find non-trivial solutions on the interval $(0, \pi)$

$$\begin{cases} y'' + \lambda^2 y = 0, \\ y'(0) = 0, y'(\pi) = 0. \end{cases}$$

The eigenvalues have the form $\lambda_k = k, k = 0, 1, 2, 3, \ldots$, and the corresponding eigenfunctions are $y_k(x) = \cos kx, k = 0, 1, 2, 3, \ldots$ Let a function $y = f(x)$ be given on a segment $[0, \pi]$ and $\hat{f}(k)$ be its the finite Fourier transform

$$\hat{f}(k) = \int_0^{\pi} \cos kx f(x)\,dx. \tag{8}$$

Then the conversion formula has the form:

$$f(x) = \frac{2}{\pi} \sum_{k=0}^{\infty} \hat{f}(k) \cos kx. \tag{9}$$

Let the function $F(x)$ be defined on the real semi-axis, and $\hat{F}(\lambda)$ be its Fourier cosine transform:

$$\hat{F}(\lambda) = \int_0^{\infty} \cos \lambda x F(x)\,dx.$$

As a result, we get the transmutation operator $J : F \to f$:

$$J[F](x) \equiv f(x) = \frac{2}{\pi} \sum_{k=0}^{\infty} \hat{F}(k) \cos kx, x \in [0, \pi]. \tag{10}$$

Simplify the function $\hat{F}(k)$

$$\begin{aligned} &\hat{F}(k) = \textstyle\int_0^{\infty} \sin kx F(x)\,dx = \\ &= \textstyle\int_0^{\pi} \cos kx \sum_{j=0}^{\infty} (F(x + 2\pi j) + F(2\pi - x + 2\pi j))\,dx, \end{aligned}$$

and back to (10):

$$\begin{aligned} &J[F](x) = f(x) = \tfrac{2}{\pi} \textstyle\sum_{k=0}^{\infty} \hat{F}(k) \cos kx = \\ &= \textstyle\sum_{j=0}^{\infty} (F(x + 2\pi j) + F(2\pi - x + 2\pi j)). \end{aligned} \tag{11}$$

Formula (11) defines the required transmutation operator. We will apply it to the Neumann problem in the strip

$$\begin{cases} u''_{xx} + u''_{yy} = 0, 0 < x < \pi, -\infty < y < \infty; \\ u'(0, y) = g(y), u'(\pi, y) = 0, \end{cases} \tag{12}$$

Let a function $U(x, y)$ be the solution of Neumann problem for a semi-plane

$$\begin{cases} U''_{xx} + U''_{yy} = 0, 0 < x, -\infty < y < \infty; \\ U'(0, y) = g(y). \end{cases} \tag{13}$$

By using (11), we obtain a new formula for solving problem (12):

$$u(x,y) = J[U(x,y)] = \sum_{j=0}^{\infty} \left(U(x+2\pi j, y) - U(2\pi - x + 2\pi j, y)\right). \tag{14}$$

By integrating identity (7), we get

$$\sum_{j=0}^{\infty} \left(\frac{1}{2} \ln \frac{(x+2\pi j)^2 + (y-\eta)^2}{(2\pi j)^2} + \frac{1}{2} \ln \frac{(2\pi - x + 2\pi j)^2 + (y-\eta)^2}{(2\pi + 2\pi j)^2} \right) =$$

$$= \frac{1}{2} \ln\left(ch(y-\eta) - \cos x\right).$$

As a result, we obtain a solution to the Neumann problem in the strip:

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \ln\left(ch(y-\eta) - \cos x\right) g(\eta)\, d\eta.$$

2.1.3 Sturm–Liouville Mixed Boundary Value Problem

The Sturm–Liouville problem about finding non-trivial solutions on the interval $[0, \pi]$

$$\begin{cases} y'' + \lambda^2 y = 0, \\ y(0) = 0,\ y'(\pi) = 0. \end{cases}$$

has eigenvalues $\lambda_k = k$, $k = 1, 2, 3, \ldots$ and corresponding eigenfunctions

$$y_k(x) = \sin\left(\left(k - \frac{1}{2}\right) x\right), k = 1, 2, 3, \ldots$$

Let the function $y = f(x)$ be given on segment $[0, \pi]$ and $\hat{f}(k)$ be its finite Fourier transform on segment $[0, \pi]$

$$\hat{f}(k) = \int_0^{\pi} \sin\left(k - \frac{1}{2}\right) x f(x)\, dx,$$

then

$$f(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \hat{f}(k) \sin\left(k - \frac{1}{2}\right) x.$$

Let the function $y = F(x)$ be given on the interval $[0, \infty)$ and $\hat{F}(\lambda)$ be its Fourier sine transform

$$\hat{F}(\lambda) = \int_0^\infty \sin \lambda x F(x)\, dx,$$

then

$$F(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \hat{F}(\lambda)\, d\lambda.$$

The function $y = f(x)$ on the interval $[0, \pi]$ corresponds to the function $F(x)$ by the rule:

$$f(x) = J[F](x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \hat{F}(k) \sin\left(k - \frac{1}{2}\right) x \tag{15}$$

The transmutation operator J is given by formula (15). Formula (15) can be simplified:

$$J[F(x)] = f(x) = \sum_{j=0}^{\infty} (-1)^j \left(F(x + 2\pi j) + F(2\pi - x + 2\pi j)\right). \tag{16}$$

We will apply the constructed transmutation operator (16) for the mixed boundary value problem in the strip

$$\begin{cases} u''_{xx} + u''_{yy} = 0, 0 < x < \pi, -\infty < y < \infty; \\ u(0, y) = g(y), u'(\pi, y) = 0, \end{cases} \tag{17}$$

and Dirichlet problem for the semi-plane (5). By using (16), we obtain a new formula for solving problem (17)

$$u(x, y) = J\left[\tilde{u}(x, y)\right] = \sum_{j=0}^{\infty} (-1)^j \left(\tilde{u}(x + 2\pi j, y) + \tilde{u}(2\pi - x + 2\pi j, y)\right). \tag{18}$$

Based on the identity of [12]

$$\sum_{j=0}^{\infty} (-1)^j \left(\frac{x + 2\pi j}{(x + 2\pi j)^2 + (y - \eta)^2} + \frac{2\pi - x + 2\pi j}{(2\pi - x + 2\pi j)^2 + (y - \eta)^2}\right) = \frac{\sin \frac{x}{2} ch \frac{y-\eta}{2}}{ch(y - \eta) - \cos x},$$

we get a new formula for solving a mixed boundaries [15] value problem in the strip [12]

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \frac{x}{2} ch \frac{y-\eta}{2}}{ch(y - \eta) - \cos x} g(\eta)\, d\eta.$$

2.1.4 Sturm–Liouville Problem with Dirichlet Boundary Conditions on Composite Real Semi-Axis

Let's consider the Sturm–Liouville singular problem about finding nontrivial solutions on composite real semi-axis $E_{1+} = (0, l) \cup (l, \infty)$,

$$\lambda^2 y_j + a_j^2 y''_{jxx} = 0, \quad x \in E_{1+}, \ j = 1, 2; \tag{19}$$

with boundary conditions

$$y_1(0) = 0, \ |y_2(x)| < \infty \tag{20}$$

and inner boundary conditions

$$y_1(l) = y_2(l), \quad \lambda_1 y'_1(l) = \lambda_2 y'_2(l). \tag{21}$$

The eigenvalues of problem (19)–(21) are the interval $(0, \infty)$, and eigenfunctions are, [16]

$$y_1(x, \lambda) = Jm\left[\left(e^{i\lambda\frac{x}{a_1}} - \frac{k-1}{k+1}e^{i\lambda\frac{2l-x}{a_1}}\right)\left(1 - \frac{k-1}{k+1}e^{i\lambda\frac{2l}{a_1}}\right)^{-1}\right], 0 < x < l,$$

$$y_2(x, \lambda) = \frac{2}{k+1} Jm\left[e^{i\lambda\frac{x-l}{a_2}} e^{i\lambda\frac{l}{a_1}}\left(1 - \frac{k-1}{k+1}e^{i\lambda\frac{2l}{a_1}}\right)^{-1}\right], l < x, k = \frac{\lambda_2}{\lambda_1}\frac{a_1}{a_2}.$$

Formulas can be represented as

$$y_1(x, \lambda) = \sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j\left(\sin\left(\frac{x+2lj}{a_1}\right) - \frac{k-1}{k+1}\sin\left(\frac{2l-x+2lj}{a_1}\right)\right), 0 < x < l,$$

$$y_2(x, \lambda) = \frac{2}{k+1}\sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j \sin\left(\frac{x-l}{a_2} + \frac{l+2lj}{a_1}\right), l < x. \tag{22}$$

The decomposition theorem on eigenfunctions is valid

$$f_1(x) = \frac{2}{\pi}\int_0^{\infty} y_1(x, \lambda) F(\lambda)\, d\lambda, 0 < x < l;$$

$$f_2(x) = \frac{2}{\pi}\int_0^{\infty} y_2(x, \lambda) F(\lambda)\, d\lambda, l < x. \tag{23}$$

where $F(\lambda)$ is the spectral function. Let the function $y = \tilde{f}(x)$ be define on the real semi-axis, and the function$F(\lambda)$ be its Fourier sine transform

$$F(\lambda) = \int_0^{\infty} \sin(\lambda\xi)\, \tilde{f}(\xi)\, d\xi.$$

The transmutation operator J is defined by formulas (23), i.e. $J : \tilde{f} \to f$,

$$f(x) = f_1(x)\,(\theta(l - x) \cdot \theta(x)) + f_2(x)\,\theta(x - l).$$

We obtain transformation operator from (22):

$$f_1(x) = \sum_{j=0}^{\infty} \left(\frac{k-1}{k+1}\right)^j \left(\tilde{f}\left(\frac{x + 2lj}{a_1}\right) - \frac{k-1}{k+1}\tilde{f}\left(\frac{2l - x + 2lj}{a_1}\right)\right), 0 < x < l;$$

$$f_2(x) = \frac{2}{k+1}\sum_{j=0}^{\infty} \left(\frac{k-1}{k+1}\right)^j \tilde{f}\left(\frac{x-l}{a_2} + \frac{l + 2lj}{a_1}\right), l < x; \tag{24}$$

2.2 Reflection Method

In this section a transmutation operator is constructed as infinite sum of reflections from the domain boundaries. As a result, the solution of the basic boundary value problem is obtained on the base of the model boundary value problem.

2.2.1 Non-local Boundary Value Problem on the Strip

Let the function $\tilde{u}(x, y)$ be a solution of the Dirichlet model problem (5) and let the function $u(x, y)$ be a solution of boundary value problem with non-local boundary conditions for the Laplace equation in the strip

$$\begin{cases} u''_{xx} + u''_{yy} = 0, \\ u(0, y) = f(y), \\ u'(0, y) = -u'(l, y). \end{cases} \tag{25}$$

We will apply the method of successive reflections from the boundaries $x = 0$ and $x = l$. As a zero-order approximation, we choose the solution of model problem (5), i.e. $u_0(x, y) = \tilde{u}(x, y)$. We will look for the first-order approximation in the form

$$u_1(x, y) = \tilde{u}(x, y) + v_0(x, y),$$

here $v_0(x, y)$ is a harmonic function in the right semi-plane

$$\tilde{u}'(0, y) - \tilde{u}'(l, y) = -v_0'(0, y) + v_0'(l, y).$$

Then $v_0(x, y) = \tilde{u}(l - x, y)$. So, the first-order approximation is

$$u_1(x, y) = \tilde{u}(x, y) + \tilde{u}(l - x, y).$$

Repeating the algorithm we find the second-order approximation $u_2(x, y)$ and a sequence of approximations

$$u_2(x, y) = \tilde{u}(x, y) + \tilde{u}(l - x, y) - \tilde{u}(l + x, y).$$

$$u_3(x, y) = \tilde{u}(x, y) + \tilde{u}(l - x, y) - \tilde{u}(l + x, y) + \tilde{u}(2l + x, y).$$

$$u_4(x, y) = \tilde{u}(x, y) + \tilde{u}(l - x, y) - \tilde{u}(l + x, y) + \tilde{u}(2l + x, y) - \tilde{u}(2l - x, y).$$

$$\dots$$

$$u_{2n}(x, y) = u_{2n-2}(x, y) + (-1)^n (\tilde{u}(x + nl, y) - \tilde{u}(-x + nl, y)).$$

$$u_{2n-1}(x, y) = u_{2n-2}(x, y) + (-1)^n \tilde{u}(x + nl, y).$$

As a limit we obtain the exact solution to problem (25)

$$u(x, y) = \tilde{u}(x, y) + \sum_{j=1}^{\infty} (-1)^j (\tilde{u}(x + lj, y) - \tilde{u}(-x + lj, y)).$$

2.2.2 Boundary Value Problem with Inner Boundary Conditions in a Strip

Let's consider the Dirichlet problem for the Laplace equation in the strip:

$$S_1 = \{(x, y) : x \in (0, l) \cup (l, L), y \in (-\infty, \infty)\}$$

$$\begin{aligned} u''_{1xx} + u''_{1yy} &= 0, 0 < x < l, -\infty < y < \infty, \\ u''_{2xx} + u''_{2yy} &= 0, l < x < L, -\infty < y < \infty \end{aligned}$$

with boundary conditions

$$y_1(0) = 0, |y_2(x)| < \infty$$

$$\begin{aligned} u_1(0, y) &= f(y), -\infty < y < \infty; \\ u_2(L, y) &= 0, -\infty < y < \infty \end{aligned} \tag{26}$$

and inner boundary conditions on the straight line $x = l$

$$u_1(l, y) = u_2(l, y), -\infty < y < \infty;$$
$$\lambda_1 u_1'(l, y) = \lambda_2 u_2'(l, y), -\infty < y < \infty.$$

The solution to problem (26) will be found by the reflection method. The zero-order approximation will be the solution of the model problem (4), i.e.

$$u_1^0(x, y) = \tilde{u}_0(x, y), 0 < x < l, \quad u_2^0(x, y) = \tilde{u}_0(x, y), l < x < L.$$

First- order approximation has the form

$$u_1^1(x, y) = \tilde{u}_0(x, y) + \tfrac{1-k}{1+k}\tilde{u}_0(2l - x, y), 0 < x < l;$$
$$u_2^1(x, y) = \tfrac{2}{1+k}\tilde{u}_0(x, y), l < x < L, k = \tfrac{\lambda_2}{\lambda_1}.$$

Let the function $\tilde{u}_1(x, y)$ be a solution of the model problem (4) with the boundary condition $\tilde{u}_1(0, y) = \tilde{u}_0(2l, y)$, then the second-order approximation will be

$$u_1^1(x, y) = u_1^0(x, y) + \tfrac{k-1}{k+1}\left(\tilde{u}_1(x, y) - \tfrac{k-1}{k+1}\tilde{u}_1(2l - x, y)\right), 0 < x < l;$$
$$u_2^1(x, y) = u_2^0(x, y) + \tfrac{2}{k+1}\tilde{u}_1(x, y), l < x < L.$$

If $u_1^n(x, y), u_2^n(x, y)$ are an approximations of order n, then the $(n + 1)$—order approximations are

$$u_1^{n+1}(x, y) = u_1^n(x, y) + \tfrac{k-1}{k+1}\left(\tilde{u}_{n+1}(x, y) - \tfrac{k-1}{k+1}\tilde{u}_{n+1}(2l - x, y)\right), 0 < x < l;$$
$$u_2^{n+1}(x, y) = u_2^n(x, y) + \tfrac{2}{k+1}\tilde{u}_{n+1}(x, y), l < x < L,$$

where $u_{n+1}(x, y)$ is the solution of model problem (4) with the boundary condition

$$\tilde{u}_{n+1}(0, y) = \tilde{u}_n(2l, y).$$

If $n \to \infty$ we get

$$u_1(x, y) = \sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j\left(\tilde{u}_j(x, y) - \frac{k-1}{k+1}\tilde{u}_j(x, y)\right), 0 < x < l; \tag{27}$$

$$u_2(x, y) = \frac{2}{k+1}\sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j\tilde{u}_j(x, y), l < x. \tag{28}$$

2.3 The Fourier Transform Technique

Let the function $u(x, y)$ be a solution of Laplace equation with periodicity boundary conditions in the strip

$$S = \{(x, y) : x \in (0, l), y \in (-\infty, \infty)\}$$

$$\begin{cases} u''_{xx} + u''_{yy} = 0, \\ u(0, y) - u(l, y) = f(y), \\ u'(0, y) - u'(l, y) = 0. \end{cases} \tag{29}$$

And let $F(\lambda)$ be the Fourier transform of function $f(y)$, i.e.

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda\eta} f(\eta)\, d\eta,$$

then the solution to problem (27) takes form

$$u(x, y) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-|\lambda|x} - e^{-|\lambda|(l-x)}}{1 - e^{-|\lambda|l}} e^{i\lambda y} F(\lambda)\, d\lambda.$$

Expand the kernel in a series of powers $e^{-|\lambda|l}$

$$\frac{e^{-|\lambda|x} - e^{-|\lambda|(l-x)}}{1 - e^{-|\lambda|l}} = \sum_{k=0}^{\infty} \left(e^{-|\lambda|(x+lj)} - e^{-|\lambda|(l-x+lj)} \right).$$

Then we get

$$u(x, y) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(e^{-|\lambda|(x+lj)} e^{i\lambda y} F(\lambda)\, d\lambda - e^{-|\lambda|(l-x+lj)} e^{i\lambda y} F(\lambda)\, d\lambda \right).$$

The Inverse Fourier transform gives:

$$u(x, y) = \frac{1}{2} \sum_{j=0}^{\infty} \left(\tilde{u}(x + lj) - \tilde{u}(l - x + lj) \right).$$

Taking into account the formulae from [12]

$$\frac{1}{\pi} \sum_{j=0}^{\infty} \left(\frac{x + lj}{(x + lj)^2 + y^2} - \frac{l - x + lj}{(l - x + lj)^2 + y^2} \right) = \frac{1}{l} \frac{\sin \frac{2\pi x}{l}}{ch \frac{2\pi y}{l} - \cos \frac{2\pi x}{l}}.$$

we get a solution to the problem with boundary conditions of periodicity

$$u(x, y) = \frac{1}{2l} \int_{-\infty}^{\infty} \frac{\sin \frac{2\pi x}{l}}{ch \frac{2\pi(\eta - y)}{l} - \cos \frac{2\pi x}{l}} f(\eta)\, d\eta.$$

2.4 Neumann Series Technique

In the section the transmutation operator is searched as the Neumann series sum [9] of shift or generalized shift operators.

2.4.1 Solution of the Laplace Equation with Non-local Boundary Conditions in the Strip

Let the function $u(x, y)$ be a solution of the Laplace equation with non-local boundary conditions in the strip $S = \{(x, y) : x \in (0, l), y \in (-\infty, \infty)\}$

$$\begin{cases} u(0, y) = f(y), -\infty < y < \infty; \\ u'(0, y) = u'(l, y), -\infty < y < \infty. \end{cases} \tag{30}$$

The solution to problem (30) will be sought in the form

$$u(x, y) = A_1 \tilde{u}(x, y) + A_2 \tilde{u}(l - x, y),$$

where A_1, A_2 are unknown operators, $\tilde{u}$ is the solution of the model problem (5). We get the system of equations for operators A_1, A_2

$$\begin{cases} A_1 + A_2 = 0, \\ A_1 + A_2 T_l = I, \end{cases}$$

here T_l is the shift operator $T_l : u(x, y) \to u(x + l, y)$ and I is an identity operator. The solution to the system of operator equations is

$$A_1 = (I - T_l)^{-1}, \quad A_2 = -(I - T_l)^{-1}.$$

By using Neumann series

$$(I - T_l)^{-1} = \sum_{j=0}^{\infty} T_l^j,$$

we get

$$u(x,y)=\sum_{j=0}^{\infty}\left(\tilde{u}(x+lj,y)-\tilde{u}(l-x+lj,y)\right),0<x<l,-\infty<y<\infty. \tag{31}$$

Based on formula (31), we obtain the solution of the non-local problem (30)

$$u(x,y)=\frac{1}{l}\int_{-\infty}^{\infty}\frac{\sin\frac{2\pi x}{l}}{ch\frac{2\pi(\eta-y)}{l}-\cos\frac{2\pi x}{l}}f(\eta)\,d\eta. \tag{32}$$

Formula (32) is obtained for the first time.

2.4.2 Solution of the Laplace Equation with Generalized Non-local Boundary Conditions in a Strip

Let the function $u(x,y)$ be a solution of the Laplace equation in a strip $S=\{(x,y):x\in(0,l),y\in(-\infty,\infty)\}$ with non-local boundary conditions

$$\begin{cases} u(0,y)=f(y), \\ ku'(0,y)=u'(l,y),-1\le k\le 1. \end{cases} \tag{33}$$

We will seek a solution to the problem in the form

$$u(x,y)=A_1\tilde{u}(x,y)+A_2\tilde{u}(l-x,y).$$

From the boundary conditions (33) we have a system of equations

$$\begin{cases} kA_1-kA_2T_l-A_1T_l+A_2=0, \\ A_1+A_2T_l=I. \end{cases}$$

The formal solution to the system of equations has the form

$$A_1=(I-kT_l)\left(I-2kT_l+T_l^2\right)^{-1},$$
$$A_2=(T_l-kI)\left(I-2kT_l+T_l^2\right)^{-1}.$$

We apply formulas for the generating functions of Chebyshev polynomials [11] of first and second kind

$$\sum_{n=0}^{\infty}T_n(k)t^n=\frac{1-tk}{1-2tk+t^2};$$
$$\sum_{n=0}^{\infty}U_n(k)t^n=\frac{1}{1-2tk+t^2}.$$

As a result, we get for operators A_1, A_2 [11]

$$A_1 = (I - kT_l)\left(I - 2kT_l + T_l^2\right)^{-1} = \sum_{j=0}^{\infty} T_j(k)\, T_l^j,$$
$$A_2 = (T_l - kI)\left(I - 2kT_l + T_l^2\right)^{-1} = \sum_{j=0}^{\infty} \left[-\tfrac{1}{k} T_j(k) + \tfrac{1-k^2}{k} U_j(k)\right] T_l^j.$$

Thus, we have

$$u(x, y) = \sum_{j=0}^{\infty} T_j(k)\, \tilde{u}(x + lj, y) + \sum_{j=0}^{\infty} \left[-\frac{1}{k} T_j(k) + \frac{1-k^2}{k} U_j(k)\right] \tilde{u}(l - x + lj, y).$$

Using the recurrent relation [11], we obtain

$$T_{j+2}(k) = kT_{j+1}(k) - (1 - k^2)U_j(k),$$

then

$$-\frac{1}{k} T_j(k) + \frac{1-k^2}{k} U_j(k) = -\frac{1}{k} T_j(k) + T_{j+1}(k) - \frac{1}{k} T_{j+2}(k).$$

The recurrent relation for Chebyshev polynomials of the first kind has the form

$$T_{j+2}(k) = 2kT_{j+1}(k) - T_j(k),$$

then

$$-\frac{1}{k} T_j(k) + \frac{1-k^2}{k} U_j(k) = -T_{j+1}(k).$$

As a result, we have the solution to the boundary value problem

$$u(x, y) = \tilde{u}(x, y) + \sum_{j=1}^{\infty} T_j(k)\left(\tilde{u}(x + lj, y) - \tilde{u}(-x + lj, y)\right).$$

3 Results

All proposed and developed methods from Sect. 2 are successfully applied to solving boundary value problems with non-classical boundary conditions. The proposed techniques allow us to find a formula, see (38), for solving the Dirichlet problem with inner boundary conditions for the semi-plane. We illustrate the proof of formula (38) by using the Neumann series expansion method. Formula (38) is a new result for the theory of potentials. To solve the Dirichlet problem with inner boundary conditions for the strip, the reflection method is most effective, the new result is represented in (39). Using the finite Fourier transforms method, a new result is obtained for the three-dimensional Dirichlet problem in a flat layer, see formula (41). We will apply the Neumann expansion method in solving problem of the Laplace equation in semi-plane $E_{1+} = \{ \quad (x, y) : y \in R, x \in (0, l) \cup (l, \infty) \}$

$$u''_{jyy} + a_j^2 u''_{jxx} = 0, \quad (x, y) \in E_{1+}, j = 1, 2; \tag{34}$$

with boundary condition

$$u_1 (0, y) = f (y) \tag{35}$$

and inner boundary conditions

$$u_1 (l, y) = u_2 (l, y), \quad \lambda_1 u'_{1x} (l, y) = \lambda_2 u'_{2x} (l, y). \tag{36}$$

We will seek a solution to problem (34)–(36) in the form

$$u_1 (x, y) = c_1 \tilde{u} \left(\frac{x}{a_1}, y \right) + c_2 \tilde{u} \left(\frac{2l - x}{a_1}, y \right), 0 < x < l;$$

$$u_2 (x, y) = c_3 \tilde{u} \left(\frac{x - l}{a_2} + \frac{l}{a_1}, y \right), l < x.$$

From (34)–(36) we get the system of equations

$$\begin{cases} c_1 + c_2 T = I, \\ c_1 + c_2 = c_3, \\ c_1 - c_2 = k c_3, \end{cases} \tag{37}$$

where $k = \frac{\lambda_2}{\lambda_1} \frac{a_1}{a_2}$ and T is the shift operator $T \left[\tilde{u} (x, y) \right] = \tilde{u} \left(x + \frac{2l}{a_1} \right)$. The solution of the system of equations (37) is obtained as an expansion in a series of Neumann operators in powers of the operator $\frac{k-1}{k+1} \cdot T$

$$c_1 = \sum_{j=0}^{\infty} \left(\frac{k-1}{k+1} \right)^j T^j, c_2 = -\frac{k-1}{k+1} \sum_{j=0}^{\infty} \left(\frac{k-1}{k+1} \right)^j T^j, c_3 = \frac{2}{k+1} \sum_{j=0}^{\infty} \left(\frac{k-1}{k+1} \right)^j T^j,$$

where T^j is the power of operator T i.e.

$$T^j \left[\tilde{u} (x, y) \right] = \tilde{u} \left(x + \frac{2lj}{a_1} \right), j = 0, 1, 2, \ldots$$

As a result, we obtain the formulas for solution to problem (34)–(36)

$$u_1(x,y)=\sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j\left(\tilde{u}\left(\frac{x+2lj}{a_1},y\right)-\frac{k-1}{k+1}\tilde{u}\left(\frac{2l-x+2lj}{a_1},y\right)\right),0<x<l;$$

$$u_2(x,y)=\frac{2}{k+1}\sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j\tilde{u}\left(\frac{x-l}{a_2}+\frac{l+2lj}{a_1},y\right),l<x. \tag{38}$$

The finite integral transforms method leads to formula (38) also. The reflection method is effective in the Dirichlet problem for the Laplace equation in the strip

$$S_1=\{(x,y):x\in(0,l)\cup(l,L),y\in(-\infty,\infty)\}.$$

Let $\tilde{u}(x,y)$be the solution of the model problem (4). The generalized shift operator T is defined by the rule $T\tilde{u}(0,y)=\tilde{u}(2l,y)$, then formulas (27)–(28) take the form

$$u_1(x,y)=\sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^j\left(T^j\tilde{u}(x,y)-\frac{k-1}{k+1}T^j\tilde{u}(2l-x,y)\right),0<x<l;$$

$$u_2(x,y)=\frac{2}{k+1}\sum_{j=0}^{\infty}\left(\frac{k-1}{k+1}\right)^jT^j\tilde{u}(x,y),l<x. \tag{39}$$

The Fourier transform method and the Neumann series method are less effective, since solution of problem (34)–(36) is obtained in the form of multiple series and obtained formulas are difficult to apply in practice. The transmutation operators method has shown its effectiveness in solving model boundary value problems. Boundary value problems for the Laplace equation in the semi-plane and in the strip with inner boundary conditions can be investigated by the transmutation operators method. The method effectively works in the three-dimensional case. For example, consider the Dirichlet problem for the three-dimensional Laplace equation in the layer $0<x<\pi,-\infty<y_1,y_2<\infty$

$$\begin{cases}u''_{xx}+u''_{y_1y_1}+u''_{y_2y_2}=0,0<x<\pi,-\infty<y_1,y_2<\infty;\\ u(0,y_1,y_2)=g(y_1,y_2),u(\pi,y_1,y_2)=0,\end{cases} \tag{40}$$

We apply The finite Fourier integral transforms technique from Sect. 2.1 and we have

$$u(x,y_1,y_2)=\sum_{j=0}^{\infty}\left(\tilde{u}(x+2\pi j,y_1,y_2)-\tilde{u}(2\pi-x+2\pi j,y_1,y_2)\right),$$

there

$$\tilde{u}(x, y_1, y_2) = \frac{1}{2\pi^2} \int_0^{2\pi} \frac{x g(\eta_1, \eta_2)}{\left(x^2 + (y_1 - \eta_1)^2 + (y_2 - \eta_2)^2\right)^{\frac{3}{2}}} d\eta_1 d\eta_2.$$

To transform the formula for $u(x, y_1, y_2)$ we use the integral

$$\int_0^{2\pi} \frac{dt}{x + iy\sin t} = \frac{2\pi}{\sqrt{x^2 + y^2}}, x > 0,$$

it is obtained by the residue method, [1]. Find the derivative for the real part of the integral with respect to x, we get

$$-\frac{1}{2\pi} \frac{d}{dx} \left[\int_0^{2\pi} \frac{x dt}{x^2 + y^2 \sin^2 t} \right] = \frac{x}{\left(x^2 + y^2\right)^{\frac{3}{2}}}, x > 0.$$

From (7) we obtain the solution of the Dirichlet problem (40)

$$u(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_0^{2\pi} \frac{1 - \cos x ch(\sin t |y - \eta|)}{(ch(\sin t |y - \eta|) - \cos x)^2} dt g(\eta_1, \eta_2) d\eta_1 d\eta_2, \tag{41}$$

where $|y - \eta|^2 = |y_1 - \eta_1|^2 + |y_2 - \eta_2|^2$.

4 Conclusions

The universality of transmutation operators method gives the possibility of its application for any dimension problems with non-local boundary conditions. The method advantage is the easily implementation form on a computer due to the cyclical nature of the corresponding algorithm. Further, the transmutation operators method can be developed for boundary value problems with axial and central symmetry. The method can also be useful in the theory of integral transforms with discontinuous trigonometric kernels and for calculating integrals, summing series.

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Solution of Inverse Problems for Differential Operators with Delay

Vjacheslav Yurko

Abstract Non-self-adjoint second-order differential operators with a constant delay are studied. We establish properties of the spectral characteristics and investigate the inverse problem of recovering operators from their spectra. For this nonlinear inverse problem the uniqueness theorem is proved and an algorithm for constructing the global solution is provided.

Keywords Differential operators · Retarded argument · Inverse spectral problems · Uniqueness and algorithms

AMS Mathematics Subject Classification (2010) 34A55, 34K10, 34K29, 47E05, 34B10, 34L40

1 Introduction

Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. Nowadays such problems attract much attention of mathematicians because of their applications in various fields of science and engineering, e.g. quantum mechanics, geophysics, chemistry, nanotechnology. The most complete results in the inverse problem theory were obtained for *differential* operators (see [1–4]). However, inverse problems for *nonlocal* operators are not so well-studied, although such operators are often more adequate for modeling physical processes (see [5, 6]). This paper concerns a class of nonlocal Sturm-Liouville operators with deviating argument.

V. Yurko (✉)
Faculty of Mathematical Physics, Department of Mathematics, Saratov State University, Saratov, Russia
e-mail: yurkova@info.sgu.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_21

Let $\{\mu_{nj}\}_{n\geq 0}$ be the eigenvalues of the boundary value problems $\mathcal{L}_j,\ j=1,2,$ of the form

$$-y''(x)+q(x)y(x-a)=\lambda y(x), \quad 0<x<\pi, \tag{1}$$
$$y'(0)-hy(0)=y'(\pi)+H_j y(\pi)=0,$$

Here $a\in(0,\pi)$, h and H_j are complex numbers ($H_1\neq H_2$), $q(x)$ is a complex-valued function, $q(x)\in L(a,\pi)$ and $q(x)=0$ a.e. on $(0,a)$. In this paper we study the inverse spectral problem of recovering potential $q(x)$ and the coefficients h, H_1, H_2, provided that the spectra $\{\mu_{nj}\}_{n\geq 0}$, $j=1,2$, are given. We pay attention to the essentially nonlinear case when $a\in[\pi/3,\pi/2)$ (the case $a\geq\pi/2$ is linear; the case $a<\pi/3$ is nonlinear and requires separate investigations). In this paper we obtain a global constructive procedure for the solution of the inverse problem and establish its uniqueness. The main results of the paper are Theorem 1 and Algorithm 1 (see Sect. 3 below). Note that some particular results on inverse problems for operators with delay were obtained in [7–11].

2 Auxiliary Propositions

Let $S(x,\lambda)$, $C(x,\lambda)$ be solutions of Eq. (1) satisfying the initial conditions

$$C(0,\lambda)=S'(0,\lambda)=1,\ \ S(0,\lambda)=C'(0,\lambda)=0.$$

Denote $\varphi(x,\lambda)=C(x,\lambda)+hS(x,\lambda)$. For each fixed x, the functions $C^{(\nu)}(x,\lambda)$, $S^{(\nu)}(x,\lambda)$ and $\varphi^{(\nu)}(x,\lambda)$, $\nu=0,1$, are entire in λ of order $1/2$. Denote

$$\mathcal{P}_j(\lambda):=\varphi'(\pi,\lambda)+H_j\varphi(\pi,\lambda), \quad j=1,2.$$

The eigenvalues $\{\mu_{nj}\}_{n\geq 0}$ of the boundary value problem $\mathcal{L}_j$ coincide with the zeros of the entire function $\mathcal{P}_j(\lambda)$. The function $\mathcal{P}_j(\lambda)$ is called the characteristic function for $\mathcal{L}_j$.

Let $\lambda=\rho^2$. The functions $C(x,\lambda)$ and $S(x,\lambda)$ are the unique solutions of the following integral equations

$$C(x,\lambda)=\cos\rho x+\int_a^x G(x,t,\lambda)C(t-a,\lambda)\,dt,\ \ S(x,\lambda)=\frac{\sin\rho x}{\rho}+\int_a^x G(x,t,\lambda)S(t-a,\lambda)\,dt,$$

where $G(x,t,\lambda)=\dfrac{q(t)\sin\rho(x-t)}{\rho}$. Therefore,

$$C(x,\lambda)=\cos\rho x+C_1(x,\lambda)+C_2(x,\lambda), \quad S(x,\lambda)=\frac{\sin\rho x}{\rho}+S_1(x,\lambda)+S_2(x,\lambda), \tag{2}$$

where

$$C_1(x,\lambda) = \int_a^x G(x,t,\lambda)\cos\rho(t-a)\,dt,\ \ S_1(x,\lambda) = \frac{1}{\rho}\int_a^x G(x,t,\lambda)\sin\rho(t-a)\,dt, \tag{3}$$

for $x \ge a$, and $C_1(x,\lambda) = S_1(x,\lambda) = 0$ for $x \in [0,a]$. Similarly,

$$C_2(x,\lambda) = \int_{2a}^x G(x,t,\lambda)C_1(t-a,\lambda)\,dt,\ \ S_2(x,\lambda) = \int_{2a}^x G(x,t,\lambda)S_1(t-a,\lambda)\,dt, \tag{4}$$

for $x \ge 2a$, and $C_2(x,\lambda) = S_2(x,\lambda) = 0$ for $x \in [0,2a]$. In particular, this yields

$$\left.\begin{aligned} C_1(\pi,\lambda) &= \frac{A\sin\rho(\pi-a)}{\rho} - \frac{1}{2\rho}\int_a^\pi q(t)\sin\rho(2t-\pi-a)\,dt,\\ S_1(\pi,\lambda) &= -\frac{A\cos\rho(\pi-a)}{\rho^2} + \frac{1}{2\rho^2}\int_a^\pi q(t)\cos\rho(2t-\pi-a)\,dt, \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} C_1'(\pi,\lambda) &= A\cos\rho(\pi-a) + \frac{1}{2}\int_a^\pi q(t)\cos\rho(2t-\pi-a)\,dt,\\ S_1'(\pi,\lambda) &= \frac{A\sin\rho(\pi-a)}{\rho} + \frac{1}{2\rho}\int_a^\pi q(t)\sin\rho(2t-\pi-a)\,dt, \end{aligned}\right\} \tag{6}$$

where $A := \dfrac{1}{2}\displaystyle\int_a^\pi q(t)\,dt$. Substituting (3) into (4), we calculate

$$\left.\begin{aligned} C_2(\pi,\lambda) &= -\frac{A_1\cos\rho(\pi-2a)}{4\rho^2} + \frac{1}{8\rho^2}\int_{-(\pi-2a)}^{\pi-2a} Q_+(\xi)\cos\rho\xi\,d\xi,\\ S_2(\pi,\lambda) &= -\frac{A_1\sin\rho(\pi-2a)}{4\rho^3} + \frac{1}{8\rho^3}\int_{-(\pi-2a)}^{\pi-2a} Q_-(\xi)\sin\rho\xi\,d\xi, \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} C_2'(\pi,\lambda) &= \frac{A_1\cos\rho(\pi-2a)}{4\rho} + \frac{1}{8\rho}\int_{-(\pi-2a)}^{\pi-2a} Q_+(\xi)\sin\rho\xi\,d\xi,\\ S_2'(\pi,\lambda) &= -\frac{A_1\sin\rho(\pi-2a)}{4\rho^2} - \frac{1}{8\rho^2}\int_{-(\pi-2a)}^{\pi-2a} Q_-(\xi)\cos\rho\xi\,d\xi, \end{aligned}\right\} \tag{8}$$

where

$$A_1 = \int_{2a}^\pi q(t)dt\int_a^{t-a} q(s)ds,\ \ Q_1(t) = q(t)\int_a^{t-a} q(s)ds,\ \ Q_2(t) = q(t)\int_{t+a}^\pi q(s)ds,$$

$$Q_3(t) = \int_{t+a}^\pi q(s)q(s-t)ds,\ \ Q_\mp(\xi) = Q_1(\xi/2+\pi/2+a) - Q_2(\xi/2+\pi/2) \mp Q_3(\xi/2+\pi/2).$$

Since $\mathcal{P}_j(\lambda) := \varphi'(\pi,\lambda)+H_j\varphi(\pi,\lambda),\ j=1,2$, and $\varphi(x,\lambda)=C(x,\lambda)+hS(x,\lambda)$, it follows from (2), (5)–(6) and (7)–(8) that

$$\mathcal{P}_j(\lambda) = -\rho\sin\rho\pi+(h+H_j)\cos\rho\pi+A\cos\rho(\pi-a)+o(\exp(|Im\,\rho|\pi)),\ |\rho|\to\infty. \tag{9}$$

Using (9) by the well-known arguments (see, for example [3]) we obtain

$$\sqrt{\mu_{nj}} = n+(h+H_j+A\cos na)/(\pi n)+o(1/n), \quad n\to\infty. \tag{10}$$

Moreover, the specification of the spectrum $\{\mu_{nj}\}_{n\geq 0}$, uniquely determines the characteristic function via

$$\mathcal{P}_j(\lambda) = \pi(\mu_{0j}-\lambda)\prod_{n=1}^{\infty}\frac{\mu_{nj}-\lambda}{n^2}, \quad j=1,2. \tag{11}$$

Denote $\Delta_k(\lambda) := \varphi^{(k)}(\pi,\lambda),\ k=0,1$. Then $\mathcal{P}_j(\lambda)=\Delta_1(\lambda)+H_j\Delta_0(\lambda),\ j=1,2$; hence

$$\Delta_0(\lambda)=\frac{\mathcal{P}_1(\lambda)-\mathcal{P}_2(\lambda)}{H_1-H_2}, \quad \Delta_1(\lambda)=\frac{\mathcal{P}_1(\lambda)H_2-\mathcal{P}_2(\lambda)H_1}{H_2-H_1}. \tag{12}$$

Since $\varphi(x,\lambda)=C(x,\lambda)+hS(x,\lambda)$, it follows from (2), (5)–(6) and (7)–(8) that

$$\Delta_0(\lambda)=\cos\rho\pi+\frac{h\sin\rho\pi}{\rho}+\frac{A\sin\rho(\pi-a)}{\rho}-\frac{hA\cos\rho(\pi-a)}{\rho^2}+\frac{d_0(\rho)}{2\rho}, \tag{13}$$

$$\Delta_1(\lambda)=-\rho\sin\rho\pi+h\cos\rho\pi+A\cos\rho(\pi-a)+\frac{hA\sin\rho(\pi-a)}{\rho}+\frac{d_1(\rho)}{2}, \tag{14}$$

where

$$d_0(\rho)=-\int_a^{\pi}q(t)\sin\rho(2t-\pi-a)\,dt+\frac{h}{\rho}\int_a^{\pi}q(t)\cos\rho(2t-\pi-a)\,dt-\frac{A_1\cos\rho(\pi-2a)}{2\rho}$$
$$-\frac{hA_1\sin\rho(\pi-2a)}{2\rho^2}+\frac{1}{4\rho}\int_{-(\pi-2a)}^{(\pi-2a)}Q_+(\xi)\cos\rho\xi\,d\xi+\frac{h}{4\rho^2}\int_{-(\pi-2a)}^{(\pi-2a)}Q_-(\xi)\sin\rho\xi\,d\xi, \tag{15}$$

$$d_1(\rho)=\int_a^{\pi}q(t)\cos\rho(2t-\pi-a)\,dt+\frac{h}{\rho}\int_a^{\pi}q(t)\sin\rho(2t-\pi-a)\,dt+\frac{A_1\sin\rho(\pi-2a)}{2\rho}$$
$$-\frac{hA_1\cos\rho(\pi-2a)}{2\rho^2}+\frac{1}{4\rho}\int_{-(\pi-2a)}^{(\pi-2a)}Q_+(\xi)\sin\rho\xi\,d\xi-\frac{h}{4\rho^2}\int_{-(\pi-2a)}^{(\pi-2a)}Q_-(\xi)\cos\rho\xi\,d\xi. \tag{16}$$

3 Solution of the Inverse Problem

Let the spectra $\{\mu_{nj}\}_{n\geq 0},\ j=1,2$, be given. Our goal is to find the potential $q(x)$ and the coefficients h, H_1, H_2. First of all, by (11) we construct the characteristic functions $\mathcal{P}_j(\lambda),\ j=1,2$. Then, using (10) we calculate

$$H_1 - H_2 = \pi \lim_{n\to\infty}(\sqrt{\mu_{n1}} - \sqrt{\mu_{n2}})n. \tag{17}$$

Now we can construct the function $\Delta_0(\lambda)$ with the help of (12). Using (13) we can find the coefficients h and A. Indeed, it follows from (13) that $A\sin an = (-1)^{n+1}(\Delta_0(n^2) - (-1)^n)n + o(1)$, as $n\to\infty$, and consequently,

$$A = \lim_{n_k\to\infty}(-1)^{n_k+1}(\sin an_k)^{-1}(\Delta_0(n_k^2) - (-1)^{n_k})n_k, \tag{18}$$

where n_k are such that $|\sin an_k| > \delta > 0$. Using (13) again we infer

$$h = \lim_{n\to\infty}\Big((2n+1/2)\Delta_0((2n+1/2)^2) - A\sin(2n+1/2)(\pi-a)\Big). \tag{19}$$

Furthermore, using (10) we calculate the coefficients H_1 and H_2, and then we can construct the function $\Delta_1(\lambda)$ by (12). Since A and h are known, we can find the functions $d_k(\rho),\ k=0,1$, with the help of (13) and (14).

In order to simplify calculations we assume that $q(x)$ and $q'(x)$ are absolutely continuous on $[a,\pi]$. The general case requires slightly different calculations. Integration by parts in (15)–(16) yields

$$2\rho d_0(\rho) = B_0\cos\rho(\pi-a) + \int_a^\pi g(t)\cos\rho(2t-\pi-a)dt - A_1\cos\rho(\pi-2a)$$
$$-\frac{hA_1\sin\rho(\pi-2a)}{\rho} + \frac{1}{2}\int_{-(\pi-2a)}^{(\pi-2a)} Q_+(\xi)\cos\rho\xi\,d\xi + \frac{h}{2\rho}\int_{-(\pi-2a)}^{(\pi-2a)} Q_-(\xi)\sin\rho\xi\,d\xi, \tag{20}$$

$$2\rho d_1(\rho) = B_1\sin\rho(\pi-a) + \int_a^\pi g(t)\sin\rho(2t-\pi-a)dt + A_1\sin\rho(\pi-2a)$$
$$-\frac{hA_1\cos\rho(\pi-2a)}{\rho} + \frac{1}{2}\int_{-(\pi-2a)}^{(\pi-2a)} Q_+(\xi)\sin\rho\xi\,d\xi - \frac{h}{2\rho}\int_{-(\pi-2a)}^{(\pi-2a)} Q_-(\xi)\cos\rho\xi\,d\xi, \tag{21}$$

where $g(x) = -q'(x) + 2hq(x)$, $B_0 = q(\pi) - q(a)$, $B_1 = q(\pi) + q(a)$. Using (20)–(21) we can find B_0, B_1 and A_1. Indeed, it follows from (20)–(21) that for real ρ, $|\rho|\to\infty$,

$$2\rho d_0(\rho) = B_0\cos\rho(\pi-a) - A_1\cos\rho(\pi-2a) + o(1), \tag{22}$$

$$2\rho d_1(\rho) = B_1 \sin\rho(\pi - a) + A_1 \sin\rho(\pi - 2a) + o(1). \tag{23}$$

Taking in (23) $\rho_n = n\pi/(\pi - a)$, we get for $n \to \infty$:

$$2\rho_n d_1(\rho_n) = A_1 \sin(\alpha n\pi) + o(1),\ \alpha = (\pi - 2a)/(\pi - a) < 1,$$

and consequently,

$$A_1 = 2 \lim_{m_k\to\infty} \Big(\rho_{m_k} d_1(\rho_{m_k})(\sin\alpha m_k\pi)^{-1}\Big), \tag{24}$$

where m_k are such that $|\sin\alpha m_k\pi| > \delta > 0$. Using (22)–(23) we infer

$$\left.\begin{aligned} B_1 &= \lim_{n\to\infty}\Big(2\rho_{n1}d_1(\rho_{n1}) - A_1\sin\rho_{n1}(\pi - 2a)\Big),\ \rho_{n1} = (2n + 1/2)\pi/(\pi - a), \\ B_0 &= \lim_{n\to\infty}\Big(2\rho_{n0}d_0(\rho_{n0}) + A_1\cos\rho_{n0}(\pi - 2a)\Big),\ \rho_{n0} = 2n\pi/(\pi - a). \end{aligned}\right\} \tag{25}$$

Since B_0 and B_1 are known, we calculate $q(a)$ and $q(\pi)$ by the formulas $q(\pi) = (B_1 + B_0)/2$ and $q(a) = (B_1 - B_0)/2$. Let us now construct the functions

$$\left.\begin{aligned} d_0^*(\rho) &= 2\rho d_0(\rho) - B_0\cos\rho(\pi - a) + A_1\cos\rho(\pi - 2a) + \frac{hA_1\sin\rho(\pi - 2a)}{\rho}, \\ d_1^*(\rho) &= 2\rho d_1(\rho) - B_1\sin\rho(\pi - a) - A_1\sin\rho(\pi - 2a) + \frac{hA_1\cos\rho(\pi - 2a)}{\rho}. \end{aligned}\right\} \tag{26}$$

It follows from (22)–(23) that

$$d_0^*(\rho) = \int_a^\pi g(t)\cos\rho(2t - \pi - a)dt + \frac{1}{2}\int_{-(\pi-2a)}^{(\pi-2a)} Q_+(\xi)\cos\rho\xi\, d\xi + \frac{h}{2\rho}\int_{-(\pi-2a)}^{(\pi-2a)} Q_-(\xi)\sin\rho\xi\, d\xi,$$

$$d_1^*(\rho) = \int_a^\pi g(t)\sin\rho(2t - \pi - a)dt + \frac{1}{2}\int_{-(\pi-2a)}^{(\pi-2a)} Q_+(\xi)\sin\rho\xi\, d\xi + \frac{h}{2\rho}\int_{-(\pi-2a)}^{(\pi-2a)} Q_-(\xi)\cos\rho\xi\, d\xi.$$

Integration by parts yields

$$2\rho d_0^*(\rho) = b_0\sin\rho(\pi - a) + \omega_0\sin\rho(\pi - 2a) - \int_{-(\pi-a)}^{(\pi-a)} g_0(\xi)\sin\rho\xi\, d\xi - \int_{-(\pi-2a)}^{(\pi-2a)} G(\xi)\sin\rho\xi\, d\xi, \tag{27}$$

$$2\rho d_1^*(\rho) = b_1\cos\rho(\pi - a) + \omega_1\cos\rho(\pi - 2a) + \int_{-(\pi-a)}^{(\pi-a)} g_0(\xi)\cos\rho\xi\, d\xi + \int_{-(\pi-2a)}^{(\pi-2a)} G(\xi)\cos\rho\xi\, d\xi, \tag{28}$$

where $G(\xi) = Q'_+(\xi) - hQ_-(\xi)$, $g_0(\xi) = g_1((\xi + \pi + a)/2)/2$, $g_1(x) = g'(x)$, $b_0 = g(a) + g(\pi)$, $b_1 = g(a) - g(\pi)$, $\omega_0 = Q_+(\pi - 2a) + Q_+(-(\pi - 2a))$,

$\omega_1 = Q_+(\pi - 2a) - Q_+(-(\pi - 2a))$. Using (27)–(28) by similar arguments as above we can find b_0, b_1, ω_0 and ω_1:

$$\left.\begin{aligned} \omega_0 &= 2 \lim_{m_k\to\infty} \Big(\rho_{m_k} d_0^*(\rho_{m_k})(\sin\alpha m_k\pi)^{-1}\Big), \\ \omega_1 &= 2 \lim_{r_k\to\infty} \Big(\rho_{r_k} d_1^*(\rho_{r_k})(\cos\alpha(2r_k+1/2)\pi)^{-1}\Big), \end{aligned}\right\} \tag{29}$$

$$\left.\begin{aligned} b_0 &= \lim_{n\to\infty} \Big(2\rho_n^0 d_0^*(\rho_n^0) - \omega_0 \sin\rho_n^0(\pi - 2a)\Big),\ \ \rho_n^0 = (2n+1/2)\pi/(\pi - a), \\ b_1 &= \lim_{n\to\infty} \Big(2\rho_n^1 d_1^*(\rho_n^1) - \omega_1 \cos\rho_n^1(\pi - 2a)\Big),\ \ \rho_n^1 = 2n\pi/(\pi - a), \end{aligned}\right\} \tag{30}$$

where r_k are such that $|\cos\alpha(2r_k + 1/2)\pi| > \delta > 0$.

Since b_0 and b_1 are known, we calculate $g(a)$ and $g(\pi)$ by the formulas $g(\pi) = (b_0 - b_1)/2$ and $g(a) = (b_0 + b_1)/2$, and consequently, we can find $q'(a)$ and $q'(\pi)$ via $q'(a) = -g(a) + 2hq(a)$, $q'(\pi) = -g(\pi) + 2hq(\pi)$. Let us now construct the functions

$$\left.\begin{aligned} D_0(\rho) &= 2\rho d_0^*(\rho) - b_0 \sin\rho(\pi - a) - \omega_0 \sin\rho(\pi - 2a), \\ D_1(\rho) &= 2\rho d_1^*(\rho) - b_1 \cos\rho(\pi - a) - \omega_1 \cos\rho(\pi - 2a). \end{aligned}\right\} \tag{31}$$

It follows from (27)–(28) that

$$D_0(\rho) = -\int_{-(\pi-a)}^{(\pi-a)} R(\xi)\sin\rho\xi\, d\xi, \quad D_1(\rho) = \int_{-(\pi-a)}^{(\pi-a)} R(\xi)\sin\rho\xi\, d\xi, \tag{32}$$

where

$$R(\xi) = g_0(\xi) + G(\xi), \tag{33}$$

and $G(\xi) \equiv 0$ for $\xi \notin (-(\pi - 2a), \pi - 2a)$. Using (32) we construct the function $R(\xi)$. Since $G(\xi) \equiv 0$ for $\xi \notin (-(\pi - 2a), \pi - 2a)$, we find the function $g_0(\xi)$ for $\xi \notin (-(\pi - 2a), \pi - 2a)$ via $g_0(\xi) = R(\xi)$. This yields

$$q''(x) - 2hq'(x) = -2R_1(x), \quad x \in [a, 3a/2] \cup [\pi - a/2, \pi], \tag{34}$$

where $R_1(x) := R(2x - \pi - a)$. Since $q(a), q'(a), q(\pi)$ and $q'(\pi)$ are known, we can construct the potential $q(x)$ for $x \in [a, 3a/2] \cup [\pi - a/2, \pi]$ by solving the linear equation (34).

Moreover, it follows from (33) that

$$\begin{aligned} q''(x) - 2hq'(x) &= -2R_1(x) + Q_1'(x + a/2) - Q_2'(x - a/2) + Q_3'(x - a/2) \\ &-2hQ_1(x + a/2) + 2hQ_2(x - a/2) + 2hQ_3(x - a/2),\ \ x \in [3a/2, \pi - a/2]. \end{aligned} \tag{35}$$

Since $q(x)$ is known for $x \in [a, 3a/2] \cup [\pi - a/2, \pi]$, then Eq. (35) is linear with respect to $q(x)$, and the solution exists. In particular, if $a \in [2\pi/5, \pi/2)$, then the right-hand side in (35) is the known function. Solving linear equation (35), we can find $q(x)$ for $x \in [3a/2, \pi - a/2]$. Thus, we have proved the following theorem.

Theorem 1 *The specification of the spectra* $\{\mu_{nj}\}_{n\geq 0}$, $j = 1, 2$, *uniquely determines the potential* $q(x)$ *and the coefficients* h, H_1, H_2. *The solution of the inverse problem can be found by the following algorithm.*

Algorithm 1 *Let the spectra* $\{\mu_{nj}\}_{n\geq 0}$, $j = 1, 2$, *be given.*

(1) Construct the characteristic functions $\mathcal{P}_j(\lambda)$, $j = 1, 2$ *by (11).*
(2) Find $H_1 - H_2$ *via (17).*
(3) Calculate the function $\Delta_0(\lambda)$ *using (12).*
(4) Calculate A *and* h *with the help of (13), for example, by (18)–(19).*
(5) Find H_1 *and* H_2 *using (10).*
(6) Construct the function $\Delta_1(\lambda)$ *by (12).*
(7) Find the functions $d_j(\rho)$, $j = 0, 1$, *with the help of (13) and (14).*
(8) Calculate B_0, B_1 *and* A_1, *using (20)–(21), for example, by (24)–(25).*
(9) Find $q(\pi) = (B_1 + B_0)/2$ *and* $q(a) = (B_1 - B_0)/2$.
(10) Construct the functions $d_j^*(\rho)$, $j = 0, 1$, *by (26).*
(11) Calculate ω_0, ω_1, b_0 *and* b_1, *using (27)–(28), for example, by (29)–(30).*
(12) Find $g(a) = (b_0 + b_1)/2$ *and* $g(\pi) = (b_0 - b_1)/2$.
(13) Calculate $q'(a) = -g(a) + 2hq(a)$ *and* $q'(\pi) = -g(\pi) + 2hq(\pi)$.
(14) Construct the functions $D_j(\rho)$, $j = 0, 1$, *by (31).*
(15) Find the function $R(\xi)$ *using (32).*
(16) Calculate the potential $q(x)$ *for* $x \in [a, 3a/2] \cup [\pi - a/2, \pi]$ *by solving Eq. (34).*
(17) Calculate the potential $q(x)$ *for* $x \in [3a/2, \pi - a/2]$ *using (35) and knowledge* $q(x)$ *for* $x \in [a, 3a/2] \cup [\pi - a/2, \pi]$.

Acknowledgements This work was supported in part by Grant 1.1660.2017/4.6 of the Russian Ministry of Education and Science and by Grant 19-01-00102 of Russian Foundation for Basic Research.

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Part III
Transmutations for Partial and Fractional Differential Equations

Transmutations of the Composed Erdélyi-Kober Fractional Operators and Their Applications

M. Al-Kandari, L. A-M. Hanna, and Yu. F. Luchko

Abstract This chapter provides a survey of an important class of transmutations for the composed Erdélyi-Kober fractional operators and some of their applications. The transmutations are given in a closed form as the generalized Obrechkoff-Stiltjes integral transforms. They translate the composed Erdélyi-Kober fractional operators to multiplication with a power function. These transmutations can be applied for treating the linear fractional integro-differential equations containing both the right- and the left-hand sided Erdélyi-Kober fractional derivatives. The equations of this type are subject of active research in fractional calculus of variations and by determination of the scale-invariant solutions of the partial differential equations of fractional order to mention only few of many relevant research areas.

1 Introduction

Fractional Calculus (FC) as a theory of integrals and derivatives of non-integer order became very popular within the last few decades both in mathematical research and in applications. The main definitions of fractional derivatives and integrals and their properties have been introduced more than two centuries ago. However, most of the mathematical models in form of the fractional differential and integral equations used nowadays in physics, chemistry, engineering, biology, medicine, and even in life and social sciences are very recent.

In the literature, several different definitions of the fractional integrals and derivatives including the Riemann-Liouville fractional integrals and derivatives, the Grünwald-Letnikov derivatives, the Erdélyi-Kober integrals and derivatives, and the Caputo-Djrbashian derivatives are actively used. In this chapter, we deal with

M. Al-Kandari · L. A-M. Hanna
Department of Mathematics, Kuwait University, Kuwait, Kuwait

Y. F. Luchko (✉)
Department of Mathematics, Physics, and Chemistry, Beuth Technical University of Applied Sciences, Berlin, Germany
e-mail: luchko@beuth-hochschule.de

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_22

the Erdélyi-Kober left- and right-hand sided fractional integrals and derivatives as well as with their suitable compositions that we call the composed Erdélyi-Kober fractional operators. Theory of these operators started with the papers [5, 10] by Erdélyi and Kober, respectively, who introduced and studied some particular cases of the operators named after them. The general case of the Erdélyi-Kober fractional operators was treated in [8, 26, 28, 29, 34] to mention only few of the relevant publications. The case of the Caputo-type Erdélyi-Kober fractional operators was considered in [18]. In [8, 34], compositions of the left- or right-hand sided Erdélyi-Kober fractional integrals and derivatives were investigated in detail. The compositions of the left- and right-hand sided Erdélyi-Kober fractional integrals and derivatives were studied in [13, 14, 34]. In [13], some operational rules for these compositions were deduced and in [2, 7, 15] they were used to determine the scale-invariant solutions of some partial differential equations of fractional order.

The transmutations of the composed Erdélyi-Kober fractional operators are provided in form of the generalized Obrechkoff-Stiltjes integral transform that contains both the Obrechkoff and the Stiltjes integral transforms as its particular cases. The Obrechkoff transform was introduced in [23]. In [3, 4], it was used as a transmutation that translates the hyper-Bessel differential operator to a multiplication with a power function. Because the hyper-Bessel operator is a particular case of a composition of the Erdélyi-Kober fractional derivatives, these results are included in our schema.

Worth mentioning is another kind of transmutations that involve the Erdélyi-Kober fractional derivatives, namely, the so called Sonin transmutations. This time, the Erdélyi-Kober fractional derivatives are employed as transmutations that translate the Bessel differential operator to the second order derivative. For a detailed description of these transmutations, their generalizations, and applications we refer the interested readers to [27]; in this chapter we do not repeat these results.

The technique employed for deriving results of this chapter is mainly based on the Mellin integral transform. For elements of the Mellin integral transform and its applications we refer the reader to [22]. A survey of some applications of the Mellin transform technique in FC was provided in [17].

As to the applications of the transmutations of the composed Erdélyi-Kober fractional operators, we mention an operational treatment of the fractional differential equations containing both the left- and the right-hand sided Erdélyi-Kober derivatives. Such equations were deduced as a suitably modified Euler-Lagrange equation in the fractional calculus of variations (see, e.g., [1, 21]). Whereas on the finite intervals these equations can be solved by the method of power series extension [9], the case of infinite intervals is still open. Another important application of this technique is for determination of the scale-invariant solutions of some partial differential equations of fractional order [2, 7, 15].

The rest of this chapter is organized as follows. In the second section, some basic facts concerning the Mellin integral transform and the Mellin-Barnes integrals are presented. The third section deals with the integral transforms of the Mellin convolution type and their properties. In the next section, the generalized Obrechkoff-Stiltjes integral transform is introduced as a generalization of the Obrechkoff integral transform and the Stiltjes integral transform. The Obrechkoff-Stiltjes integral transform turns out to play a role of a transmutation operator for the Erdélyi-Kober fractional operators that are introduced in the last section. This transmutation translates the Erdélyi-Kober fractional operators into a multiplication with a power function. As a consequence, application of the transmutation operator reduces the integro-differential equations with the Erdélyi-Kober fractional operators to some algebraic equations and thus allows to determine their solutions in explicit form. This solution technique is discussed in the last section, too.

2 The Mellin Integral Transform

The Mellin integral transform is one of the most used and important integral transforms in mathematics and its applications. In particular, almost all known special functions including the generalized hypergeometric functions, the Mittag-Leffler function, the Wright function, and the Fox H-function can be interpreted as the inverse Mellin integral transforms of some quotients of products of the Gamma-functions (the Mellin-Barnes integrals). Moreover, many integral transforms including the Riemann-Liouville and the Erdélyi-Kober fractional integrals have the Mellin convolution form and can be represented as the Mellin-Barnes integrals.

The Mellin integral transform of a function $f = f(x), x > 0$ at the point $s \in \mathbb{C}$ is defined as the following improper integral (in the case it is convergent one):

$$f^*(s) = \mathcal{M}\{f(x); s\} = \int_0^{+\infty} f(x)x^{s-1}\, dx. \tag{1}$$

Let us denote by $L^c(a, b)$ the space of functions that are continuous on the interval (a, b) with a possible exception of finite many points and for that the improper integral $\int_a^b |f(x)|\, dx$ converges. The Mellin integral transform is well defined in particular for the functions f that satisfy the following sufficient conditions: $f \in L^c(\epsilon, E), 0 < \epsilon < E < \infty$, $f \in C(0, \epsilon]$ and $f \in C[E, +\infty)$, $|f(t)| \leq Mt^{-\gamma_1}$ for $0 < t < \epsilon$ and $|f(t)| \leq Mt^{-\gamma_2}$ for $t > E$, where M is a constant and $\gamma_1 < \gamma_2$. If the conditions formulated above are fulfilled, the Mellin integral transform $f^* = f^*(s)$ exists and is an analytical function in the strip $\gamma_1 < \Re(s) < \gamma_2$ of the complex plane.

The inverse Mellin integral transform is defined by the following improper integral in the sense of the Cauchy principal value:

$$f(x) = \mathcal{M}^{-1}\{f^*(s); x\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f^*(s)x^{-s}\, ds,\ \gamma = \Re(s). \tag{2}$$

In particular, the formula (2) is valid in a point $x > 0$ for the functions f that are piecewise differentiable in an ϵ-neighborhood of the point x, continuous at the point x, and satisfy the inclusion $f(x)x^{\gamma-1} \in L^c(0, +\infty)$. In the case, f has a jump at the point x, but satisfies all other conditions mentioned above, the left-hand side of the formula (2) has to be replaced by $(f(x-0) + f(x+0))/2$.

It is worth mentioning that the Mellin integral transform can be interpreted as the Fourier integral transform for the complex frequencies:

$$\mathcal{M}\{f(x); s\} = \int_0^{+\infty} f(x)x^{s-1}\, dx = \int_{-\infty}^{+\infty} f(e^x)e^{ix(-is)}\, dx = \mathcal{F}\{f(e^x); -is\}.$$

Employing this relation, both the inverse Mellin integral transform and the convolution for the Mellin integral transform can be obtained from the formulas for the inverse Fourier integral transform and the convolution for the Fourier integral transform by the same variables substitutions.

The Mellin convolution is provided by the following integral:

$$(f \overset{\mathcal{M}}{*} g)(x) = \int_0^{+\infty} f(x/t)g(t)\, \frac{dt}{t}. \tag{3}$$

According to the results presented in [30], the Mellin convolution $h = f \overset{\mathcal{M}}{*} g$ is well defined and satisfies the inclusion $h(x)\, x^{\gamma-1} \in L(0, \infty)$ and the convolution property

$$\mathcal{M}\{(f \overset{\mathcal{M}}{*} g)(x); s\} = \mathcal{M}\{f(x); s\} \cdot \mathcal{M}\{g(x); s\} \tag{4}$$

for the functions satisfying the inclusions $f(x)\, x^{\gamma-1} \in L(0, \infty)$ and $g(x)\, x^{\gamma-1} \in L(0, \infty)$.

Combining the convolution property (4) and the formula (2) for the inverse Mellin integral transform, we get the well-known and important Parseval equality for the Mellin integral transform

$$\int_0^{+\infty} f(x/t)g(t)\frac{dt}{t} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)g^*(s)\, x^{-s} ds. \tag{5}$$

The basic properties of the Mellin integral transform are as follows (by $\rightarrow$ we denote the correspondence between a function and its Mellin transform):

$$f(ax) \rightarrow a^{-s} f^*(s),\ a > 0, \tag{6}$$

$$x^p f(x) \rightarrow f^*(s+p), \tag{7}$$

$$f(x^p) \rightarrow \frac{1}{|p|} f^*(s/p),\ p \neq 0, \tag{8}$$

$$f^{(n)}(x) \to \frac{\Gamma(n+1-s)}{\Gamma(1-s)} f^*(s-n),\ \text{if}\ \lim_{x\to 0} x^{s-k-1} f^{(k)}(x) = 0, \tag{9}$$

$$n = 1, 2, \ldots,\quad k = 0, 1, \ldots, n-1,$$

$$\left(x\frac{d}{dx}\right)^n f(x) \to (-s)^n f^*(s),\ n = 1, 2, \ldots, \tag{10}$$

$$\left(\frac{d}{dx}x\right)^n f(x) \to (1-s)^n f^*(s),\ n = 1, 2, \ldots. \tag{11}$$

It is a very remarkable and important fact that the Mellin integral transforms of practically all known elementary and special functions are in form of quotients of products of the Gamma-functions [22, 25]. In the further discussions we need the closed form formulas for the Mellin integral transforms of some elementary and special functions that are presented below:

$$e^{-x^p} \to \frac{1}{|p|}\Gamma(s/p),\ \Re(s/p) > 0, \tag{12}$$

$$\frac{(1-x^p)_+^{\alpha-1}}{\Gamma(\alpha)} \to \frac{\Gamma(s/p)}{|p|\Gamma(s/p+\alpha)},\ \Re(\alpha) > 0,\ \Re(s/p) > 0, \tag{13}$$

where $(x)_+ \equiv H(x)$, $H(x)$ is the Heaviside function,

$$\frac{(x^p-1)_+^{\alpha-1}}{\Gamma(\alpha)} \to \frac{\Gamma(1-\alpha-s/p)}{|p|\Gamma(1-s/p)},\ 0 < \Re(\alpha) < 1 - \Re(s/p), \tag{14}$$

$$\Gamma(\rho)(1+x)^{-\rho} \to \Gamma(s)\Gamma(\rho-s),\ 0 < \Re(s) < \Re(\rho), \tag{15}$$

$$\frac{1}{\pi(1-x)} \to \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(s+1/2)\Gamma(1/2-s)},\ 0 < \Re(s) < 1, \tag{16}$$

$$K_\nu(2\sqrt{x}) \to \frac{1}{2}\Gamma(s+\nu/2)\Gamma(s-\nu/2),\ \Re(s) > |\Re(\nu)|/2, \tag{17}$$

$K_\nu(x)$ is the Macdonald function (37),

$$e^x \operatorname{erfc}(\sqrt{x}) \to \frac{1}{\pi}\Gamma(s+1/2)\Gamma(s)\Gamma(1/2-s),\ 0 < \Re(s) < 1/2, \tag{18}$$

$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$, $\operatorname{erf}(x)$ is the probability integral,

$$\Psi(a,b;x) \to \frac{1}{\Gamma(a)\Gamma(a-b+1)}\Gamma(s)\Gamma(s+1-b)\Gamma(a-s),\ \max\{0, \Re(b-1)\} < \Re(s) < \Re(a), \tag{19}$$

$\Psi(a, b; z)$ is the Tricomi function (39),

$$e^{x/2}K_\nu\left(\frac{x}{2}\right) \to \frac{1}{\sqrt{\pi}}\cos(\pi\nu)\Gamma(s+\nu)\Gamma(s-\nu)\Gamma(1/2-s),\ |\Re(\nu)| < \Re(s) < 1/2, \tag{20}$$

$K_\nu(x)$ is the Macdonald function (37),

$$\pi 2^{\mu+1}\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)|1-x|^{\mu/2}P_\nu^\mu(\sqrt{x}) \to \Gamma(s)\Gamma(s+1/2) \times \Gamma((1+\nu-\mu)/2-s)\Gamma(-(\mu+\nu)/2-s), \tag{21}$$

$$0 < \Re(s) < \min\{(1+\Re(\nu-\mu))/2, -\Re(\nu+\mu)/2\},$$

$P_\nu^\mu(x)$ is the Legendre function of the first kind (40),

$$J_\nu^2(\sqrt{x}) + Y_\nu^2(\sqrt{x}) \to \frac{2\cos(\nu\pi)}{\pi^{5/2}}\Gamma(s)\Gamma(s+\nu)\Gamma(s-\nu)\Gamma(1/2-s),\ |\Re(\nu)| < \Re(s) < 1/2, \tag{22}$$

$J_\nu(x)$ is the Bessel function of the first kind (35), and $Y_\nu(x)$ is the Neumann function (38),

$$H^{m,n}_{p,q}\left(x\middle|\begin{matrix}(\alpha_p, a_p)\\(\beta_q, b_q)\end{matrix}\right) \to \frac{\prod_{j=1}^m \Gamma(\beta_j + b_j s)\prod_{j=1}^n \Gamma(1-\alpha_j - a_j s)}{\prod_{j=n+1}^p \Gamma(\alpha_j + a_j s)\prod_{j=m+1}^q \Gamma(1-\beta_j - a_j s)} \tag{23}$$

$$-\min_{1\le j\le m}\Re(\beta_j)/b_j < \Re(s) < \min_{1\le j\le n}(1-\Re(\alpha_j))/a_j$$

and

(1) $\sigma > 0$ or
(2) $\sigma = 0,\ \delta\Re(s) < \frac{q-p}{2} - 1 + \Re\left(\sum_{j=1}^p \alpha_j - \sum_{j=1}^q \beta_j\right)$,

where

$$\sigma = \sum_{j=1}^n a_j - \sum_{j=n+1}^p a_j + \sum_{j=1}^m b_j - \sum_{j=m+1}^q b_j,\ \delta = \sum_{j=1}^q b_j - \sum_{j=1}^p a_j.$$

In (23), $H^{m,n}_{p,q}$ stands for the Fox H-function defined by (41).

In the rest of this section, we provide definitions of the special functions mentioned above and some of their important properties.

The Euler Gamma-function is defined by the following improper integral:

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx,\ \Re(s) > 0. \tag{24}$$

For $\Re(s) > 0$, it is an analytic function that can be extended to $\Re(s) \leq 0$, $s \neq 0, -1, -2, \ldots$ by analytic continuation of the integral at the right-hand side of (24). The standard way for the analytic continuation is to employ the reduction formula

$$\Gamma(s+1) = s\Gamma(s),\ \Re(s) > 0 \tag{25}$$

that immediately follows from (24) by means of integration by parts.

Other important properties of the Gamma-function are the supplement formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)},\ s \in \mathbb{C} \tag{26}$$

and the asymptotic formulas [6, 22, 34]:

$$\Gamma(s) = \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s}(1 + O(s^{-1})),\ |\arg(s)| < \pi,\ |s| \to \infty, \tag{27}$$

$$\frac{\Gamma(s+\alpha)}{\Gamma(s+\beta)} = s^{\alpha-\beta}(1 + O(s^{-1})),\ |\arg(s)| < \pi,\ \alpha,\ \beta \in \mathbb{C},\ |s| \to \infty, \tag{28}$$

$$|\Gamma(x+iy)| = \sqrt{2\pi}|y|^{x-\frac{1}{2}} e^{-\pi|y|/2}(1 + O(|y|^{-1})),\ x,\ y \in \mathbb{R},\ |y| \to \infty. \tag{29}$$

The Pochhammer symbol $(z)_n$ is defined by

$$(z)_n = \prod_{k=0}^{n-1}(z+k) = \frac{\Gamma(z+n)}{\Gamma(z)}. \tag{30}$$

The Euler Beta-function is defined by the improper integral

$$B(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}dx,\ \Re(s) > 0,\ \Re(t) > 0. \tag{31}$$

It is related to the Gamma-function by the formula

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}. \tag{32}$$

The generalized hypergeometric function ${}_pF_q(z)$ is one of the most general and used special functions. One of its definitions is in form of the following series (in the case it is convergent)

$${}_pF_q\left[(a)_p;\ (b)_q;\ z\right] \equiv {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} z\right] \equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{z^n}{n!}. \tag{33}$$

The series at the right-hand side of (33) is absolutely convergent in the whole complex plane when $p \le q$.

When $p = q+1$, it converges only for $|z| < 1$. When

$$\Re\left[\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right] > 0,$$

the series (33) converges for $z = 1$ and when

$$\Re\left[\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right] > -1,$$

it converges for $|z| = 1,\ z \ne 1$. For other values of z, ${}_{q+1}F_q(z)$ can be defined as an analytic continuation of the series (33). One of the ways for the analytic continuation is employing the following Mellin-Barnes integral representation

$$ {}_{q+1}F_q\left[(a)_{q+1};\ (b)_q;\ z\right] = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{q+1}\Gamma(a_j)}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\prod_{j=1}^{q+1}\Gamma(a_j-s)\Gamma(s)}{\prod_{j=1}^{q}\Gamma(b_j-s)}(-z)^{-s}ds, \tag{34}$$

where $0 < \Re(s) = \gamma < \min\limits_{1\le j\le q+1}\Re(a_j);\ |\arg(-z)| < \pi$.

For $p > q+1$, the series at the right-hand side of (33) is divergent everywhere with exception of the point $z = 0$.

The Bessel functions $J_\nu(z)$, $I_\nu(z)$, the Macdonald function $K_\nu(z)$, and the Neumann function $Y_\nu(z)$ are all particular cases of the function ${}_0F_1$:

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^{\nu} {}_0F_1\left[\nu+1;\ -\frac{z^2}{4}\right] = \sum_{n=0}^{\infty}\frac{(-1)^n (z/2)^{2k+\nu}}{\Gamma(\nu+n+1)n!}, \tag{35}$$

$$I_\nu(z) = \sum_{n=0}^{\infty}\frac{(z/2)^{2k+\nu}}{\Gamma(\nu+n+1)n!} = e^{-\pi i\nu/2}J_\nu(iz), \tag{36}$$

$$K_\nu(z) = \frac{\pi}{2\sin(\pi\nu)}\left[I_{-\nu}(z) - I_\nu(z)\right],\ \nu \ne 0, \pm1, \pm2, \ldots, \tag{37}$$

$$K_n(z) = \lim_{\nu\to n} K_\nu(z),\ n = 0, \pm1, \pm2, \ldots,$$

$$Y_\nu(z) = \frac{\cos(\pi\nu)J_\nu(z) - J_{-\nu}(z)}{\sin(\pi\nu)}\ \ \nu \ne 0, \pm1, \pm2, \ldots, \tag{38}$$

$$Y_n(z) = \lim_{\nu\to n} Y_\nu(z),\ n = 0, \pm1, \pm2, \ldots.$$

The Tricomi function $\Psi(a, b; z)$ can be defined in terms of ${}_1F_1(z)$, whereas the Legendre function of the first kind $P_\nu^\mu(z)$ is a special case of ${}_2F_1(z)$:

$$\Psi(a, b; z) = \frac{\Gamma(1-c)}{\Gamma(1+a-c)} {}_1F_1(a; c; z) + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c} {}_1F_1(1+a-c; 2-c; z), \tag{39}$$

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1(-\nu, \nu+1; 1-\mu; (1-z)/2),$$
$$|\arg(z-1)| < \pi; \ \mu \neq 1, 2, \ldots. \tag{40}$$

The Fox H-function was initially introduced as an extension of the hypergeometric function ${}_pF_q(z)$ for the case $p > q+1$. It is defined by the following Mellin-Barnes integral:

$$H^{m,n}_{p,q}\left(z \middle| \begin{matrix} (\alpha_p, a_p) \\ (\beta_q, b_q) \end{matrix}\right) = H^{m,n}_{p,q}\left(z \middle| \begin{matrix} (\alpha, a)_{1,p} \\ (\beta, b)_{1,q} \end{matrix}\right) = \frac{1}{2\pi i} \int_L \Phi(s) z^{-s}\, ds, \tag{41}$$

where $z \neq 0$, $0 \leq m \leq q$, $0 \leq n \leq p$, $\alpha_j \in \mathbf{C}$, $a_j > 0$, $1 \leq j \leq p$, $\beta_j \in \mathbf{C}$, $b_j > 0$, $1 \leq j \leq q$,

$$\Phi(s) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j + b_j s) \prod_{j=1}^{n} \Gamma(1 - \alpha_j - a_j s)}{\prod_{j=n+1}^{p} \Gamma(\alpha_j + a_j s) \prod_{j=m+1}^{q} \Gamma(1 - \beta_j - a_j s)}, \tag{42}$$

an empty product, if it occurs, is taken to be one. The contour L is an infinite contour in the complex plane that separates the left poles $s = (-\beta_j - k)/b_j$, $j = 1, 2, \ldots, m$, $k = 0, 1, 2, \ldots$ of the numerator of $\Phi(s)$ from the right poles $s = (1 - \alpha_j + k)/a_j$, $j = 1, 2, \ldots, n$, $k = 0, 1, 2, \ldots$. It may be of three different types: $L_{-\infty}$, $L_{+\infty}$ or $L_{i\infty}$ (in particular, even a rectilinear line $L = (\gamma - i\infty, \gamma + i\infty)$). For more details regarding the contour L and properties of the Fox H-function see, e.g., [25].

3 Integral Transforms of the Mellin Convolution Type

As we could see in the previous section, the Mellin integral transforms of the elementary and special functions of the hypergeometric type are all in form of quotients of products of the Gamma-functions. In particular, this important property opens a gateway for construction of a unified theory of the Mellin convolution type integral transforms with the special functions in the kernels. The idea is to study them in form of the Mellin-Barnes integrals given by the right-hand side of the Parseval equality (5). In the Mellin-Barnes integral, one has to deal with the Mellin integral transforms of the kernel functions that are provided in form of quotients of products of the Gamma-functions. Using the formulas (27), (28), and (29) for the asymptotic behavior of the Gamma-function, the asymptotic behavior of the

Mellin integral transforms of different kernel functions can be then obtained in a uniform way. Accordingly, it is very convenient to study the Mellin convolution type integral transforms with the special functions in the kernels in some special spaces of functions that are introduced in terms of their Mellin integral transforms.

The basic space of functions $\mathcal{M}^{-1}(L)$ used for this approach was introduced and investigated in [32, 33], and [34].

Definition 3.1 The space of functions $\mathcal{M}^{-1}(L)$ consists of all functions $f = f(x),\ x > 0$ that can be represented as the inverse Mellin integral transform

$$f(x) = \mathcal{M}^{-1}\{f^*(s); x\} = \frac{1}{2\pi i}\int_\sigma f^*(s)\,x^{-s}\,ds,\quad x > 0,\ \sigma = \{s \in \mathbb{C} : \Re(s) = 1/2\} \tag{43}$$

of the functions $f^* \in L(\sigma)$.

The space of functions $\mathcal{M}^{-1}(L)$ equipped with the norm

$$\| f \|_{\mathcal{M}^{-1}(L)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |f^*(1/2 + it)|\,dt \tag{44}$$

is a Banach space.

For a function $f \in \mathcal{M}^{-1}(L)$, the following useful properties are fulfilled:

(1) $x^{-1}f(x^{-1}) \in \mathcal{M}^{-1}(L)$. Vice versa, if $x^{-1}f(x^{-1}) \in \mathcal{M}^{-1}(L)$ then $f \in \mathcal{M}^{-1}(L)$.
(2) $x^{1/2}f(x)$ is uniformly bounded, continuous on $(0, +\infty)$, and the relation $x^{1/2}f(x) = o(1)$ when $x \to +\infty$ and $x \to 0$ holds true.
(3) If $g \in \mathcal{M}^{-1}(L)$ then $x^{1/2}f(x)g(x) \in \mathcal{M}^{-1}(L)$.
(4) If $x^{-1/2}g(x) \in L(\mathbb{R}_+)$ then $f \overset{\mathcal{M}}{*} g \in \mathcal{M}^{-1}(L)$.

In this chapter, we deal with the integral transforms of the Mellin convolution type with the kernels from a class $\mathcal{K}$ defined as follows:

Definition 3.2 ([32]) A function $k : (0, \infty) \to \mathbb{R}$ is said to belong to the class $\mathcal{K}$ of kernels if it satisfies the following conditions:

(1) $k \in L(\epsilon, E)$ for any $\epsilon,\ E$, such that $0 < \epsilon < E < \infty$,
(2) the integral

$$k^*(s) = \mathcal{M}\{k(u); s\} = \int_0^\infty k(u)u^{s-1}du,\ s \in \sigma,\ \sigma = \{s \in \mathbb{C} : \Re(s) = 1/2\} \tag{45}$$

converges for any $s \in \sigma$,
(3) for almost all $\epsilon,\ E > 0$ and $t \in \mathbb{R}$

$$\left| \int_\epsilon^E k(u)u^{it-1/2}du \right| < C_k, \tag{46}$$

where the constant $C_k > 0$ does not depend on ϵ, E, and t.

If for a function $k \in \mathcal{K}$, there exists a kernel $\hat{k}^* \in \mathcal{K}$ such that the equality

$$k^*(s)\hat{k}^*(1-s) = 1,\ k,\ \hat{k} \in \mathcal{K} \tag{47}$$

holds true almost everywhere on the line $\Re(s) = 1/2$, we say that $k \in \mathcal{K}^* \subset \mathcal{K}$. The kernel $\hat{k} \in \mathcal{K}^*$ satisfying (47) is called a conjugate kernel of the kernel $k \in \mathcal{K}^*$.

It is easy to verify that if $x^{-1/2}k(x) \in L(0,\infty)$, then $k \in \mathcal{K}$ but $k \notin \mathcal{K}^*$.

In what follows, we consider the integral transforms of the Mellin convolution type in form [11, 13, 14, 16, 19, 31, 32, 34]

$$g(x) = (Kf)(x) = \int_0^\infty k\left(\frac{x}{u}\right) f(u)\frac{du}{u} \tag{48}$$

with the kernels from the space $\mathcal{K}$ and their inverse transforms

$$f(x) = (\hat{K}g)(x) = \int_0^\infty \hat{k}\left(\frac{x}{u}\right) g(u)\frac{du}{u} \tag{49}$$

with the kernels from the space $\mathcal{K}^*$ and some of their properties in the space of functions $\mathcal{M}^{-1}(L)$. The results that we use in the further discussions are formulated in Theorems 3.1 and 3.2 (for the proofs we refer the interested reader to [31, 32], and [34]).

Theorem 3.1 ([31]) *Let $f \in \mathcal{M}^{-1}(L)$, $k \in \mathcal{K}$, k^* be given by (45) and f^* be defined as in (43). Then the Parseval formula*

$$\int_0^\infty k\left(\frac{x}{u}\right) f(u)\frac{du}{u} = \frac{1}{2\pi i}\int_\sigma k^*(s)f^*(s)x^{-s}ds \tag{50}$$

holds true.

Theorem 3.2 ([31]) *Let $k \in \mathcal{K}^*$ and $\hat{k} \in \mathcal{K}^*$ be its conjugate kernel. Then the integral transform*

$$g(x) = (Kf)(x) = \int_0^\infty k\left(\frac{x}{u}\right) f(u)\frac{du}{u}$$

is an automorphism in the space $\mathcal{M}^{-1}(L)$ and its inverse transform is given by

$$f(x) = (\hat{K}g)(x) = \int_0^\infty \hat{k}\left(\frac{x}{u}\right) g(u)\frac{du}{u}.$$

Corollary 3.1 *Let $k \in \mathcal{K}^*$, $\hat{k} \in \mathcal{K}^*$ be its conjugate kernel, and $|k^*(s)| = 1, s \in \sigma$. Then the integral transforms (48) and (49) are isometric automorphisms in the space $\mathcal{M}^{-1}(L)$.*

As an example, it is an easy exercise to verify that the sine-Fourier transform

$$(\mathcal{F}_s f)(x) = \sqrt{2/\pi} \int_0^\infty \sin(xy) f(y)\, dy \tag{51}$$

is an isometric automorphism in the space $\mathcal{M}^{-1}(L)$ and its inverse transform has the same form.

As we already mentioned, the Mellin integral transforms of the functions of the hypergeometric type typically have the form

$$k^*(s) = \frac{\prod_{i=1}^m \Gamma(\alpha_i + a_i s) \prod_{i=1}^n \Gamma(\beta_i - b_i s)}{\prod_{i=1}^l \Gamma(\gamma_i + c_i s) \prod_{i=1}^k \Gamma(\delta_i - d_i s)},\ \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C},\ a_i, b_i, c_i, d_i > 0. \tag{52}$$

For $s \in \sigma$ ($\Re(s) = 1/2$), the asymptotic behavior of the right-hand side of (52) is given by the formula

$$k^*(s) = |s|^{-\gamma} e^{-\pi c |\Im(s)|} \left(C + O(|s|^{-1}\right),\ |\Im(s)| \to \infty, \tag{53}$$

where

$$\gamma = \frac{1}{2}(m+n-k-l) - \sum_{i=1}^m (\alpha_i + \frac{1}{2} a_i) - \sum_{i=1}^n (\beta_i - \frac{1}{2} b_i) + \sum_{i=1}^l (\gamma_i + \frac{1}{2} c_i) + \sum_{i=1}^k (\delta_i - \frac{1}{2} d_i), \tag{54}$$

$$c = \frac{1}{2}(\sum_{i=1}^n a_i + \sum_{i=1}^m b_i - \sum_{i=1}^l c_i - \sum_{i=1}^k d_i). \tag{55}$$

The formula (53) is a simple consequence of the formula (29) for the asymptotic behavior of the Gamma-function.

Thus, the Mellin integral transforms of the kernel functions can increase or decrease when $\Im(s) \to \infty$ and we have to take their asymptotic behavior into consideration while defining the spaces of functions for the Mellin convolution type integral transforms with the hypergeometric type functions in the kernel.

Definition 3.3 The space of functions $\mathcal{M}_{c,\gamma}^{-1}(L)$ consists of all functions $f = f(x),\ x > 0$ that can be represented as the inverse Mellin integral transforms

$$f(x) = \mathcal{M}^{-1}\{f^*(s); x\} = \frac{1}{2\pi i} \int_\sigma f^*(s)\, x^{-s}\, ds,\ \ x > 0,\ \sigma = \{s \in \mathbb{C} :\ \Re(s) = 1/2\} \tag{56}$$

of the functions $f^* = f^*(s)$ that satisfy the inclusion

$$f^*(s)|s|^\gamma e^{\pi c|\Im(s)|} \in L(\sigma) \tag{57}$$

under the condition

$$2\mathrm{sign}(c) + \mathrm{sign}(\gamma) \geq 0,\ c,\ \gamma \in \mathbb{R}. \tag{58}$$

For $|s| \to \infty,\ s \in \sigma$, $|\Im(s)|$ behaves like $|s|$ and thus the integral at the right-hand side of (56) converges if $c > 0,\ \gamma \in \mathbb{R}$ or if $c = 0,\ \gamma \geq 0$, i.e., under the condition (58).

Evidently, the space of functions $\mathcal{M}^{-1}_{c,\gamma}(L)$ is a subspace of $\mathcal{M}^{-1}(L)$ and the family of these subspaces is partially ordered, i.e., the inclusion

$$\mathcal{M}^{-1}_{c_1,\gamma_1}(L) \subset \mathcal{M}^{-1}_{c_2,\gamma_2}(L) \tag{59}$$

holds true if and only if

$$2\mathrm{sign}(c_1 - c_2) + \mathrm{sign}(\gamma_1 - \gamma_2) \geq 0. \tag{60}$$

Equipped with the norm

$$\|f\|_{\mathcal{M}^{-1}_{c,\gamma}(L)} = \frac{1}{2\pi}\int_\sigma e^{\pi c|\Im(s)|}|s^\gamma f^*(s)\,ds| \tag{61}$$

$\mathcal{M}^{-1}_{c,\gamma}(L)$ becomes a Banach space.

For $c = 0$, we get an important particular case of the space of functions $\mathcal{M}^{-1}_{c,\gamma}(L)$ that is denoted by $\mathcal{M}^{-1}_{\gamma}(L)$. In particular, this space of functions will be employed for investigation of the mapping properties of the Erdélyi-Kober fractional operators.

4 The Generalized Obrechkoff-Stieltjes Integral Transform

In this section, a closed form representation for transmutations of the composed Erdélyi-Kober fractional operators is introduced and discussed.

We start with a generalization of the Obrechkoff transform [4, 23, 34] in form of a Mellin convolution type integral transform

$$(\mathcal{O}f)(x) = \int_0^\infty H^{0,n}_{n,0}\left(\frac{x}{u}\,\middle|\,\begin{matrix}(\alpha,a)_{1,n}\\ -\end{matrix}\right) f(u)\frac{du}{u},\ u > 0, \tag{62}$$

where $H^{0,n}_{n,0}$ is a particular case of the Fox H-function (41).

The formula (41) and the integral representation (24) of the Gamma-function lead to the following identity for the kernel function $H^{0,n}_{n,0}$ in form of a multiple integral:

$$H^{0,n}_{n,0}\left(z\middle|\begin{matrix}(\alpha,a)_{1,n}\\ -\end{matrix}\right) = \frac{z^{\frac{\alpha_n-1}{a_n}}}{a_n}\int_0^\infty \ldots \int_0^\infty \exp\left(-\sum_{i=1}^{n-1} u_i - z^{-\frac{1}{a_n}} \prod_{i=1}^{n-1} u_i^{-\frac{a_i}{a_n}}\right) \times \tag{63}$$
$$\prod_{i=1}^{n-1} u_i^{-a_i\frac{1-\alpha_n}{a_n}-\alpha_i} du_1 \ldots du_{n-1}.$$

Denoting the right-hand side of the formula (63) by $\Phi(z|(\alpha_i, a_i)_{1,n})$, we can rewrite the generalized Obrechkoff transform (62) as

$$(\mathcal{O}f)(x) = \int_0^\infty \Phi_n\left(\frac{x}{u} \mid (\alpha_i, a_i)_{1,n}\right) f(u)\frac{du}{u}. \tag{64}$$

Applying now Theorem 3.1, the known asymptotic behavior of the H-function (see e.g. [34]), and the formula (23) we conclude that the generalized Obrechkoff transform (62) maps the space of functions $\mathcal{M}^{-1}(L)$ into a subspace of $\mathcal{M}^{-1}(L)$ and the following representation holds true:

$$(\mathcal{O}f)(x) = \frac{1}{2\pi i}\int_\sigma \prod_{i=1}^{n} \Gamma(1-\alpha_i - a_i s) f^*(s) x^{-s} ds. \tag{65}$$

Now we continue with the generalized Stieltjes integral transform that was introduced in [11, 34] in the following form:

$$(\mathcal{S}^\alpha_\beta f)(x) = \frac{x^{\frac{\alpha}{\beta}}}{\beta}\int_0^\infty \frac{f(u)u^{\frac{1-\alpha}{\beta}}}{x^{\frac{1}{\beta}} + u^{\frac{1}{\beta}}}\frac{du}{u},\ u > 0,\ \beta > 0. \tag{66}$$

For $\alpha = 0$, $\beta = 1$, the generalized Stieltjes integral transform (66) is reduced to the conventional Stieltjes transform.

Employing Theorem 3.1 and the formulas (7), (8), and (15), we get the following representation of the generalized Stieltjes transform (66) in the space of functions $\mathcal{M}^{-1}(L)$:

$$(\mathcal{S}^\alpha_\beta f)(x) = \frac{1}{2\pi i}\int_\sigma \Gamma(1-\alpha-\beta s)\Gamma(\alpha+\beta s) f^*(s) x^{-s} ds. \tag{67}$$

Motivated by the representations (65) and (67) of the generalized Obrechkoff and Stieltjes transforms, the generalized Obrechkoff-Stieltjes transform was introduced in [13] in form of the following Mellin-Barnes integral:

$$(\mathcal{OS} f)(x) = \frac{1}{2\pi i}\int_{\sigma}\prod_{j=1}^{n}\Gamma(1-\alpha_j-a_js)\prod_{j=1}^{m}\Gamma(\beta_j+b_js)f^*(s)x^{-s}ds. \tag{68}$$

The formula (23) for the Mellin integral transform of the Fox H-function, Theorem 3.1, and the asymptotic behavior of the H-function (see e.g. [34]) allow us to represent the generalized Obrechkoff-Stieltjes transform (68) in the space of fucntions $\mathcal{M}^{-1}(L)$ as an integral transform of the Mellin convolution type with the Fox H-function in the kernel:

$$(\mathcal{OS} f)(x) = \int_0^{\infty} H^{m,n}_{n,m}\left(\frac{x}{u}\middle| \begin{matrix}(\alpha,a)_{1,n}\\ (\beta,b)_{1,m}\end{matrix}\right) f(u)\frac{du}{u},\ u>0. \tag{69}$$

As can be easily seen from the relations (65), (67), and (68), both the generalized Obrechkoff transform and the generalized Stieltjes transform are particular cases of the generalized Obrechkoff-Stieltjes transform. The Mellin transform formulas (17)–(22) lead to other interesting and important particular cases of the generalized Obrechkoff-Stieltjes transform (69):

1. the modified Meijer transform ($m=2$, $n=0$, $b_1=b_2=1$, $\beta_1=\frac{\nu}{2}$, $\beta_2=-\nu/2$):

$$(\mathcal{OS} f)(x) = \int_0^{\infty} K_\nu\left(2\sqrt{\frac{x}{u}}\right) f(u)\frac{du}{u},\ x>0, \tag{70}$$

2. the integral transform with the Macdonald function (37) in the kernel ($m=2$, $n=1$, $b_1=b_2=a_1=1$, $\beta_1=\nu$, $\beta_2=-\nu$, $\alpha_1=1/2$):

$$(\mathcal{OS} f)(x) = \int_0^{\infty} e^{\frac{x}{2u}} K_\nu\left(\frac{x}{2u}\right) f(u)\frac{du}{u},\ x>0, \tag{71}$$

3. the integral transform with the probability integral in the kernel ($m=2$, $n=1$, $b_1=b_2=a_1=1$, $\beta_1=1/2$, $\beta_2=0$, $\alpha_1=1/2$):

$$(\mathcal{OS} f)(x) = \int_0^{\infty} e^{\frac{x}{u}}\,\mathrm{erfc}\left(\sqrt{\frac{x}{u}}\right) f(u)\frac{du}{u},\ x>0, \tag{72}$$

4. the integral transform with the Tricomi function (39) in the kernel ($m=2$, $n=1$, $b_1=b_2=a_1=1$, $\beta_1=0$, $\beta_2=1-b$, $\alpha_1=1-a$):

$$(\mathcal{OS} f)(x) = \int_0^{\infty} \Psi\left(a;b;\frac{x}{u}\right) f(u)\frac{du}{u},\ x>0, \tag{73}$$

5. the integral transform with the Legendre function of the first kind (40) in the kernel ($m = 2, n = 2, b_1 = b_2 = a_1 = a_2 = 1, \beta_1 = 0, \beta_2 = 1/2, \alpha_1 = 1 - (1 + \nu - \mu)/2, \alpha_2 = 1 + (\nu + \mu)/2$):

$$(\mathcal{OS}f)(x) = \int_0^\infty \left|1 - \frac{x}{u}\right|^{\mu/2} P_\nu^\mu\left(\sqrt{\frac{x}{u}}\right) f(u)\frac{du}{u},\ x > 0, \tag{74}$$

6. the integral transform with the sum of squares of the Bessel function (35) and the Neumann function (38) in the kernel ($m = 3, n = 1, b_1 = b_2 = b_3 = a_1 = 1, \beta_1 = 0, \beta_2 = \nu, \beta_3 = -\nu, \alpha_1 = 1/2$):

$$(\mathcal{OS}f)(x) = \int_0^\infty \left(J_\nu^2\left(\sqrt{\frac{x}{u}}\right) + Y_\nu^2\left(\sqrt{\frac{x}{u}}\right)\right) f(u)\frac{du}{u},\ x > 0. \tag{75}$$

In analogy to the Obrechkoff integral transform (62), the generalized Obrechkoff-Stieltjes transform can be represented in form of a multiple integral with an exponential function and power multipliers in the kernel (see the formula (64) for the corresponding representation of the Obrechkoff integral transform).

For illustration of the method, we first consider the modified Laplace transform that is a particular case of the Obrechkoff transform. Employing the integral representation (24) of the Gamma-function and changing the order of integration in the double integral, for a function f from the space $\mathcal{M}^{-1}(L)$ we get the following chain of equalities:

$$\begin{aligned}(\mathcal{L}f)(x) &= \frac{1}{2\pi i}\int_\sigma \Gamma(1-s)f^*(s)x^{-s}ds = \frac{1}{2\pi i}\int_\sigma \int_0^\infty e^{-t}t^{-s}dt\ f^*(s)x^{-s}ds \\ &= \int_0^\infty e^{-t}\frac{1}{2\pi i}\int_\sigma f^*(s)(xt)^{-s}ds\,dt = \int_0^\infty e^{-t}f(xt)\,dt.\end{aligned} \tag{76}$$

The same method can be used for the generalized Obrechkoff-Stieltjes integral transform (68) if $\min\limits_{1\le i\le n}(1-\alpha_i)/a_i > \max\limits_{1\le i\le m} -\beta_j/b_i$:

$$(\mathcal{OS}f)(x) = \frac{1}{2\pi i}\int_\sigma \prod_{i=1}^m \Gamma(\beta_i + b_i s)\prod_{i=1}^n \Gamma(1-\alpha_i - a_i s)f^*(s)x^{-s}ds \tag{77}$$

$$= \frac{1}{2\pi i}\int_\sigma \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^m u_i\right)\prod_{i=1}^m u_i^{\beta_i + b_i s - 1}du_1\ldots du_m$$

$$\times \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^n v_i\right)\prod_{i=1}^n v_i^{-\alpha_i - a_i s}dv_1\ldots dv_n\ f^*(s)x^{-s}ds$$

$$= \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^m u_i - \sum_{i=1}^n v_i\right) \prod_{i=1}^m u_i^{\beta_i - 1} \prod_{i=1}^n v_i^{-\alpha_i} du_1 \dots du_m\, dv_1 \dots dv_n$$

$$\times \frac{1}{2\pi i} \int_\sigma f^*(s) \left(x \prod_{i=1}^m u_i^{-b_i} \prod_{i=1}^n v_i^{a_i}\right)^{-s} ds$$

$$= \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^m u_i - \sum_{i=1}^n v_i\right) \prod_{i=1}^m u_i^{\beta_i - 1} \prod_{i=1}^n v_i^{-\alpha_i}$$

$$\times f\left(x \prod_{i=1}^m u_i^{-b_i} \prod_{i=1}^n v_i^{a_i}\right) du_1 \dots du_m\, dv_1 \dots dv_n.$$

As we will see in the next section, the generalized Obrechkoff-Stieltjes integral transform is a transmutation of the composed Erdélyi-Kober fractional operators.

5 Composed Erdélyi-Kober Fractional Operators and Their Transmutations

In this section, we first introduce the composed Erdélyi-Kober fractional operators related to the generalized Obrechkoff-Stieltjes integral transform (69). It turns out that the composed Erdélyi-Kober fractional operators and the generalized Obrechkoff-Stieltjes transform are connected by some operational relations, i.e., the generalized Obrechkoff-Stieltjes transform is a transmutation of the composed Erdélyi-Kober fractional operators that translates them into multiplication with a power function. In this sense, the generalized Obrechkoff-Stieltjes transform and the composed Erdélyi-Kober fractional operators can be interpreted as a far-reaching generalization of the well-known Laplace integral transform and the differential operators of integer order.

In the Fractional Calculus literature, finite compositions of the right-hand sided or the left-hand sided Erdélyi-Kober fractional integrals or derivatives were already considered (see e.g. [8] or [34]). Their transmutation operators can be represented via the generalized Obrechkoff integral transform (see the examples at the end of this section). In this section, we introduce and study the composed Erdélyi-Kober fractional operators, i.e., the compositions of both the right-hand sided and the left-hand sided Erdélyi-Kober fractional integrals and derivatives. This type of operators has very different properties compared to those of compositions of only right-hand sided or only left-hand sided Erdélyi-Kober fractional integrals or derivatives. As a particular case of the composed Erdélyi-Kober fractional operator let us mention the Hilbert integral transform (see the end of this section for details):

$$(\mathcal{H}f)(x) = \frac{1}{\pi}\int_0^\infty \frac{f(t)}{t-x}\,dt. \tag{78}$$

We start with a discussion of some basic properties of the Erdélyi-Kober fractional integrals and derivatives that are among most used and important definitions of the fractional calculus operators. These operators and their numerous applications both for mathematical and applied problems were discussed in a number of publications [8, 18, 24, 34]. In what follows, we focus on the left-hand sided Erdélyi-Kober fractional integrals and derivatives because the properties of the right-hand sided Erdélyi-Kober fractional integrals and derivatives are very similar to ones of the left-hand sided operators and can be derived from them by simple variables substitutions.

The left- and right-hand sided Erdélyi-Kober fractional integrals of order δ and α, respectively, are given by the relations

$$(I_\beta^{\gamma,\delta} f)(x) = \frac{\beta}{\Gamma(\delta)} x^{-\beta(\gamma+\delta)}\int_0^x (x^\beta - u^\beta)^{\delta-1} u^{\beta(\gamma+1)-1} f(u)du,\ \delta,\beta > 0, \gamma \in \mathbb{R}, \tag{79}$$

$$(K_\beta^{\tau,\alpha} f)(x) = \frac{\beta}{\Gamma(\alpha)} x^{\beta\tau}\int_x^\infty (u^\beta - x^\beta)^{\alpha-1} u^{-\beta(\tau+\alpha-1)-1} f(u)du,\ \alpha,\beta > 0, \tau \in \mathbb{R}. \tag{80}$$

For $\delta = 0$ or $\alpha = 0$, respectively, these operators are reduced to the identity operators:

$$(I_\beta^{\gamma,0} f)(x) = f(x),\ (K_\beta^{\tau,0} f(x) = f(x).$$

For $\beta = 1$, the Erdélyi-Kober fractional integrals (79) and (80) can be represented in terms of the Riemann-Liouville fractional integrals with the power functions weights [26, 34]:

$$(I_1^{\gamma,\delta} f)(x) = (x^{-\gamma-\delta} I_{0+}^\delta u^\gamma f)(x) = \frac{1}{\Gamma(\delta)} x^{-\gamma-\delta}\int_0^x (x-u)^{\delta-1} u^\gamma f(u)du, \tag{81}$$

$$(K_1^{\tau,\alpha} f)(x) = (x^\tau I_-^\alpha u^{-\tau-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} x^\tau \int_x^\infty (u-x)^{\alpha-1} u^{-\tau-\alpha} f(u)du. \tag{82}$$

Below we list the main properties of the left-hand sided Erdelyi-Kober fractional integral (79) that will be used in the further discussions (for the proofs see, e.g., [8]):

$$(I_\beta^{\gamma,\delta} x^{\lambda\beta} f)(x) = x^{\lambda\beta}(I_\beta^{\gamma+\lambda,\delta} f)(x), \tag{83}$$

$$(I_\beta^{\gamma,\delta} I_\beta^{\gamma+\delta,\alpha} f)(x) = (I_\beta^{\gamma,\delta+\alpha} f)(x), \tag{84}$$

$$(I_\beta^{\gamma,\delta} I_\beta^{\alpha,\eta} f)(x) = (I_\beta^{\alpha,\eta} I_\beta^{\gamma,\delta} f)(x). \tag{85}$$

For the theory of the Erdélyi-Kober fractional integrals in several different spaces of functions and some of their applications we refer to e.g., [8, 26], and [34].

We introduce now the left- and right-hand sided Erdélyi-Kober fractional derivatives [8, 18, 34]. Let $n-1 < \delta \le n,\ n \in \mathbb{N}$ and $m-1 < \alpha \le m,\ m \in \mathbb{N}$. The integro-differential operators

$$(D_\beta^{\gamma,\delta} f)(x) = \prod_{i=1}^{n}\left(\gamma + i + \frac{1}{\beta} x \frac{d}{dx}\right)(I_\beta^{\gamma+\delta,n-\delta} f)(x), \tag{86}$$

$$(P_\beta^{\tau,\alpha} f)(x) = \prod_{i=0}^{m-1}\left(\tau + i - \frac{1}{\beta} x \frac{d}{dx}\right)(K_\beta^{\tau+\alpha,m-\alpha} f)(x) \tag{87}$$

are called the left- and right-hand sided Erdélyi-Kober fractional derivatives of order δ or α, respectively.

In the formulas (86) and (87), the operators $I_\beta^{\gamma,\delta}$ and $K_\beta^{\tau,\alpha}$ are the left- and right-hand sided Erdélyi-Kober fractional integrals defined by (79) and (80), respectively.

The left- and the right-hand sided Erdélyi-Kober fractional derivatives are the left-inverse operators to the left- and the right-hand sided Erdélyi-Kober fractional integrals, respectively [18].

Of course, the Erdélyi-Kober fractional derivatives are not right-inverse operators to the Erdélyi-Kober fractional integrals (see [18] for the closed form formulas for the compositions of the Erdélyi-Kober fractional integrals and the Erdélyi-Kober fractional derivatives).

In analogy to the case of the fractional derivatives in the Riemann-Liouville and Caputo sense, a Caputo-type modification of the Erdélyi-Kober fractional derivatives was introduced in [7] and analyzed in details in [18]. These fractional derivatives are similar to the conventional Erdélyi-Kober fractional derivatives, but allow a traditional form of initial conditions while considering initial value problems for the fractional differential equations with the Erdélyi-Kober fractional derivatives.

Let $n-1<\delta\le n,\ n\in\mathbb{N}$, $m-1<\alpha\le m,\ m\in\mathbb{N}$, and $\beta>0$. The integro-differential operator

$$({}_*D_\beta^{\gamma,\delta}f)(x)=(I_\beta^{\gamma+\delta,n-\delta}\prod_{i=0}^{n-1}\left(1+\gamma+i+\frac{1}{\beta}u\frac{d}{du}\right)f(u))(x),\ x>0 \tag{88}$$

is called the left-hand sided Caputo-type modification of the Erdélyi-Kober fractional derivative of order δ. The Caputo-type modification of the right-hand sided Erdélyi-Kober fractional derivative of order α is defined by the integro-differential operator

$$({}_*P_\beta^{\tau,\alpha}f)(x)=(K_\beta^{\tau+\alpha,m-\alpha}\prod_{i=0}^{m-1}\left(\tau+i-\frac{1}{\beta}u\frac{d}{du})f(u)\right)(x),\ x>0. \tag{89}$$

In the formulas (88), (89), the operators $I_\beta^{\gamma,\delta}$ and $K_\beta^{\tau,\alpha}$ are the left- and right-hand sided Erdélyi-Kober fractional integrals of order δ or α, respectively.

The Caputo-type modifications of the Erdélyi-Kober fractional derivatives are the left-inverse operators to the corresponding Erdélyi-Kober fractional integrals [18], but not the right-inverse ones. A closed form formula for the composition of the left-hand sided Erdélyi-Kober fractional integral and the corresponding Caputo-type modification of the Erdélyi-Kober fractional derivative was derived in [18].

Now we introduce the suitable compositions of the left- and right-hand sided Erdélyi-Kober fractional integrals and derivatives we deal with in this chapter.

Definition 5.1 Let $a_i>0,\ \alpha_i\in\mathbb{R},\ i=1,\dots,n;\ b_i>0,\ \beta_i\in\mathbb{R},\ i=1,\dots,m.$ The integro-differential operators

$$(\mathcal{L}_\eta f)(x)=\begin{cases}x^\eta(\prod_{i=1}^m P_{1/b_i}^{\beta_i-b_i\eta,b_i\eta}\prod_{i=1}^n I_{1/a_i}^{-\alpha_i,a_i\eta}f)(x), & \eta>0,\\ x^\eta(\prod_{i=1}^n D_{1/a_i}^{-\alpha_i+a_i\eta,-a_i\eta}\prod_{i=1}^m K_{1/b_i}^{\beta_i,-b_i\eta}f)(x), & \eta<0,\end{cases} \tag{90}$$

where $I_\beta^{\gamma,\delta}$ and $K_\beta^{\tau,\alpha}$ are the left- and the right-hand sided Erdélyi-Kober fractional integrals and $D_\beta^{\gamma,\delta}$ and $P_\beta^{\tau,\alpha}$ are the left- and the right-hand sided Erdélyi-Kober fractional derivatives are called the composed Erdélyi-Kober fractional operators.

In analogy to Definition 5.1, one can define the Caputo-type modifications of the composed Erdélyi-Kober fractional operators (see [18] for details).

Employing Theorem 3.1, formula (8), and the relations (13), (14), we arrive at the following Melling-Barnes representations of the Erdélyi-Kober fractional operators in the space of functions $\mathcal{M}_\gamma^{-1}(L)$ (under suitable restrictions on the parameter γ):

$$(I_\beta^{\gamma,\delta} f)(x) = \int_\sigma \frac{\Gamma(1+\gamma-s/\beta)}{\Gamma(1+\gamma+\delta-s/\beta)} f^*(s) x^{-s} ds, \tag{91}$$

$$(K_\beta^{\tau,\alpha} f)(x) = \int_\sigma \frac{\Gamma(\tau+s/\beta)}{\Gamma(\tau+\alpha+s/\beta)} f^*(s) x^{-s} ds, \tag{92}$$

$$(D_\beta^{\gamma,\delta} f)(x) = \int_\sigma \frac{\Gamma(1+\gamma+\delta-s/\beta)}{\Gamma(1+\gamma-s/\beta)} f^*(s) x^{-s} ds, \tag{93}$$

$$(P_\beta^{\tau,\alpha} f)(x) = \int_\sigma \frac{\Gamma(\tau+\alpha+s/\beta)}{\Gamma(\tau+s/\beta)} f^*(s) x^{-s} ds. \tag{94}$$

The mapping properties of the Erdélyi-Kober fractional integrals and derivatives follow from the relations (91)–(94), the definition of the space $\mathcal{M}_\gamma^{-1}(L)$, and the asymptotic formula (29) for the Gamma-function: the right-hand sided Erdélyi-Kober fractional integral (91) maps the space $\mathcal{M}_\gamma^{-1}(L)$ into $\mathcal{M}_{\gamma+\delta}^{-1}(L)$ and the left-hand sided Erdélyi-Kober fractional integral (92) maps the space $\mathcal{M}_\gamma^{-1}(L)$ into $\mathcal{M}_{\gamma+\alpha}^{-1}(L)$. For $\gamma \geq \delta$, the right-hand sided Erdélyi-Kober fractional derivative (93) maps the space $\mathcal{M}_\gamma^{-1}(L)$ into $\mathcal{M}_{\gamma-\delta}^{-1}(L)$. Finally, the left-hand sided Erdélyi-Kober fractional derivative (94) maps the space $\mathcal{M}_\gamma^{-1}(L)$ into $\mathcal{M}_{\gamma-\alpha}^{-1}(L)$ under the condition $\gamma \geq \alpha$.

The representations (91)–(94) and the mapping properties mentioned above lead to the following result:

Theorem 5.1 ([13]) *Let the condition*

$$\gamma + \eta \left(\sum_{i=1}^n a_i - \sum_{i=1}^m b_i \right) > 0 \tag{95}$$

be satisfied.

Then the composed Erdélyi-Kober fractional operator maps the space $\mathcal{M}_\gamma^{-1}(L)$ *into the space* $\mathcal{M}_{\gamma_\eta}^{-1}(L)$ *with* $\gamma_\eta = \gamma + \eta \left(\sum_{i=1}^n a_i - \sum_{i=1}^m b_i\right) > 0$ *and can be represented as the Mellin-Barnes integral*

$$(\mathcal{L}_\eta f)(x) = \frac{x^\eta}{2\pi i} \int_\sigma \prod_{i=1}^n \frac{\Gamma(1-\alpha_i-a_i s)}{\Gamma(1-\alpha_i+a_i\eta-a_i s)} \prod_{i=1}^m \frac{\Gamma(\beta_i+b_i s)}{\Gamma(\beta_i-b_i\eta+b_i s)} f^*(s) x^{-s} ds. \tag{96}$$

The representation (96) explains the idea behind derivation of the closed form formula for the transmutation operator of the composed Erdélyi-Kober fractional operator (90). Indeed, let us denote the kernel of the generalized Obrechkoff-Stieltjes transform (68) by $\Phi(s)$, i.e.,

$$\Phi(s) = \prod_{i=1}^{n} \Gamma(1 - \alpha_i - a_i s) \prod_{i=1}^{m} \Gamma(\beta_i + b_i s). \tag{97}$$

Then it is easy to verify that the kernel of the Mellin-Barnes representation (96) of the composed Erdélyi-Kober fractional operator satisfies the relation

$$\prod_{i=1}^{n} \frac{\Gamma(1 - \alpha_i - a_i s)}{\Gamma(1 - \alpha_i + a_i \eta - a_i s)} \prod_{i=1}^{m} \frac{\Gamma(\beta_i + b_i s)}{\Gamma(\beta_i - b_i \eta + b_i s)} = \frac{\Phi(s)}{\Phi(s - \eta)}. \tag{98}$$

The representation (98) is a basis for proving that the generalized Obrechkoff-Stieltjes transform is a transmutation for the composed Erdélyi-Kober fractional operator.

Theorem 5.2 ([13]) *Let $f \in \mathcal{M}_\gamma^{-1}(L)$ and the condition (95) be satisfied.*

Then the generalized Obrechkoff-Stieltjes transform (69) is a transmutation for the composed Erdélyi-Kober fractional operator (90) that translates it into multiplication by a power function:

$$(\mathcal{OS}(\mathcal{L}_\eta f))(x) = x^\eta (\mathcal{OS} f)(x). \tag{99}$$

We reproduce here just a basic idea of the proof of this theorem given in [13]. According to Theorem 5.1 and under the condition (95), the composed Erdélyi-Kober fractional operator (90) can be represented in the space of functions $\mathcal{M}_\gamma^{-1}(L)$ as a Mellin-Barnes integral:

$$(\mathcal{L}_\eta f)(x) = \frac{x^\eta}{2\pi i} \int_\sigma \frac{\Phi(s)}{\Phi(s - \eta)} f^*(s) x^{-s} ds,$$

where $\Phi(s)$ is defined by (97). Using the shift property (7) of the Mellin transform and the representation (68) of the generalized Obrechkoff-Stieltjes transform we have then a simple chain of equalities:

$$(\mathcal{OS}(\mathcal{L}_\eta f))(x) = \frac{1}{2\pi i} \int_\sigma \Phi(s) (\mathcal{L}_\eta f)^*(s) x^{-s} ds = \frac{1}{2\pi i} \int_\sigma \Phi(s) \frac{\Phi(s+\eta)}{\Phi(s)} f^*(s+\eta) x^{-s} ds$$

$$= \frac{1}{2\pi i} \int_\sigma \Phi(s+\eta) f^*(s+\eta) x^{-s} ds = \frac{x^\eta}{2\pi i} \int_\sigma \Phi(s) f^*(s) x^{-s} ds = x^\eta (\mathcal{OS} f)(x),$$

that proves Theorem 5.2.

The explicit form of the transmutations and the operational relation (99) can be used for analytical treatment of a class of linear integro-differential equations containing the operator (90) of the type

$$\sum_{i=0}^{n} a_i \, (\mathcal{L}_\eta^i y)(x) = f(x), \tag{100}$$

where $a_i, \; i = 0, \ldots, n$ are some coefficients, $\mathcal{L}_\eta^i$ means a composition of i operators $\mathcal{L}_\eta$, and $\mathcal{L}_\eta^0$ is interpreted as the identity operator: $\mathcal{L}_\eta^0 \equiv Id$.

The solution algorithm follows the standard procedure: First, the transmutation operator is applied to the integro-differential equation (100) that translates it into an algebraic equation for the generalized Obrechkoff-Stieltjes transform of the unknown solution. Solving this equation, we get the generalized Obrechkoff-Stieltjes transform of the solution. Finally, the inversion formula

$$f(x) = (\mathcal{OS}g)^{-1}(x) = \frac{1}{2\pi i} \int_\sigma \frac{1}{\prod_{j=1}^{n} \Gamma(1 - \alpha_j - a_j s) \prod_{j=1}^{m} \Gamma(\beta_j + b_j s)} g^*(s) x^{-s} ds \tag{101}$$

for the generalized Obrechkoff-Stieltjes transform leads to an explicit formula for a solution of the integro-differential equation (100) in the space of functions $\mathcal{M}_{c,\gamma}^{-1}(L)$ under suitable conditions posed on the parameters c and γ.

One example of an equation in form (100) with $n = 1$ is the equation

$$y(x) - \lambda x^\eta (P_{\frac{\eta}{\alpha}}^{1+\gamma-\alpha,\alpha} I_1^{-\eta,\eta} y)(x) = \sum_{i=1}^{n} c_i x^{2\eta - i}, \; \eta > \alpha, \; \lambda, \; c_i \in \mathbb{R}, \; j{=}i, \ldots, n, \tag{102}$$

that was deduced and solved in [15] to obtain the scale-invariant solutions of a space-time fractional partial differential equation (for details see [15]). In the Eq. (102), the composed Erdélyi-Kober fractional operator has the form

$$(\mathcal{L}_\eta f)(x) = x^\eta (P_{\frac{\eta}{\alpha}}^{1+\gamma-\alpha,\alpha} I_1^{-\eta,\eta} f)(x).$$

The operator $\mathcal{L}_\eta$ is a composition of the Erdélyi-Kober left-hand sided fractional integral $I_1^{-\eta,\eta}$ of order η and the Erdélyi-Kober right-hand sided fractional derivative $P_{\eta/\alpha}^{1+\gamma-\alpha,\alpha}$ of order α. Because $\eta > \alpha$, $\mathcal{L}_\eta$ can be interpreted as an "integral operator" and therefore no initial conditions for the Eq. (102) are required.

In general, the composed Erdélyi-Kober fractional operator (90) has characteristic properties of a fractional integral for $\eta > 0$ and $\sum_{i=1}^{n} a_i > \sum_{i=1}^{m} b_i$ and for $\eta < 0$ and $\sum_{i=1}^{n} a_i < \sum_{i=1}^{m} b_i$. If $\eta > 0$ and $\sum_{i=1}^{n} a_i < \sum_{i=1}^{m} b_i$ or $\eta < 0$ and $\sum_{i=1}^{n} a_i > \sum_{i=1}^{m} b_i$, the operator (90) can be interpreted as a fractional derivative. Finally, if $\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} b_i$, the composed Erdélyi-Kober fractional operator (90) can be considered to be a generalization of the Hilbert transform (an example illustrated this situation will be presented at the end of this section).

For the composed Erdélyi-Kober fractional integral (90) ($\sum_{i=1}^{n} a_i > \sum_{i=1}^{m} b_i$ if $\eta > 0$ or $\sum_{i=1}^{n} a_i < \sum_{i=1}^{m} b_i$ if $\eta < 0$), the inverse operator (composed Erdélyi-Kober fractional derivative) is defined as follows:

$$(\mathcal{D}_\eta f)(x) = \begin{cases} x^{-\eta}\left(\prod_{i=1}^{n} D_{1/a_i}^{-\alpha_i - a_i\eta, a_i\eta} \prod_{i=1}^{m} K_{1/b_i}^{\beta_i, b_i\eta} f\right)(x), & \eta > 0, \\ x^{-\eta}\left(\prod_{i=1}^{m} P_{1/b_i}^{\beta_i + b_i\eta, -b_i\eta} \prod_{i=1}^{n} I_{1/a_i}^{-\alpha_i, -a_i\eta} f\right)(x), & \eta < 0. \end{cases} \tag{103}$$

Using the properties of the Erdélyi-Kober fractional integrals and derivatives (see, e.g., [8, 11, 26, 34]), it can be easy shown that the composed Erdélyi-Kober fractional derivative $\mathcal{D}_\eta$ is a left-inverse operator to the composed Erdélyi-Kober fractional integral $\mathcal{L}_\eta$, i.e.

$$(\mathcal{D}_\eta \mathcal{L}_\eta f)(x) = f(x). \tag{104}$$

However, $\mathcal{D}_\eta$ is not a right-inverse operator to $\mathcal{L}_\eta$ and their composition has a complicated form. To demonstrate this, let us restrict ourselves to the parameter values $\eta > 0$ and $m = 0$. In this case, the operators $\mathcal{L}_\eta$ and $\mathcal{D}_\eta$ are called the multiple Erdélyi-Kober fractional integrals and derivatives, respectively [11, 20]. They have the form

$$(\mathcal{L}_\eta f)(x) = x^{\eta}\left(\prod_{i=1}^{n} I_{1/a_i}^{-\alpha_i, a_i\eta} f\right)(x), \tag{105}$$

$$(\mathcal{D}_\eta f)(x) = x^{-\eta}\left(\prod_{i=1}^{n} D_{1/a_i}^{-\alpha_i - a_i\eta, a_i\eta} f\right)(x). \tag{106}$$

For these operators, the composition $\mathcal{L}_\eta \mathcal{D}_\eta$ takes the following form in the corresponding space of functions (see [11] or [34] for details):

$$(\mathcal{L}_\eta \mathcal{D}_\eta f)(x) = f(x) - \sum_{i=1}^{n} \sum_{k=1}^{\eta_i} C_{ik} \left(\lim_{x\to 0} (A_{ik} f)(x)\right) x^{\eta - \frac{k-\alpha_i}{a_i}}, \tag{107}$$

where

$$C_{ik} = \frac{\prod_{j=i+1}^{n} \Gamma(1-\alpha_j - \frac{a_j}{a_i}(k-\alpha_i)) \prod_{j=1}^{i-1} \Gamma(1-\alpha_j - \frac{a_j}{a_i}(k-\alpha_i)+\eta_j)}{\prod_{j=1}^{n} \Gamma(1-\alpha_j - \frac{a_j}{a_i}(k-\alpha_i)+a_j\eta)},$$

$$(A_{ik}y)(x) = x^{-\eta+\frac{k-\alpha_i}{a_i}} \prod_{j=1}^{\eta_i-k} \left(k+j-\alpha_i - a_i\eta + a_i x\frac{d}{dx}\right) \prod_{l=i+1}^{n} \prod_{j=1}^{\eta_i} \left(j-\alpha_l - a_l\eta + a_l x\frac{d}{dx}\right)$$

$$\left(\prod_{j=1}^{n} I_{1/a_j}^{-\alpha_j,\eta_j-a_j\mu} f\right)(x), \quad \eta_i = \begin{cases} [a_i\eta]+1, a_i\eta \notin \mathbb{N}, \\ a_i\eta, \ a_i\eta \in \mathbb{N}. \end{cases}$$

Remark 5.3 In [8], a more general definition of the multiple Erdélyi-Kober fractional integral was given in the following form:

$$(\mathcal{I}f)(x) = x^{\beta_0}(I_{(\beta_i),n}^{(\gamma_i),(\delta_i)} f)(x) = x^{\beta_0}\left(\prod_{i=1}^{n} I_{\beta_i}^{\gamma_i,\delta_i} f\right)(x) \tag{108}$$

$$= x^{\beta_0}\int_0^1 H_{n,n}^{n,0}\left[t \middle| \begin{matrix} (\gamma_i+\delta_i+1-\frac{1}{\beta_i}, \frac{1}{\beta_i}) \\ (\gamma_i+1-\frac{1}{\beta_i}, \frac{1}{\beta_i}) \end{matrix}\right] f(xt)dt \ ,$$

where $\gamma_i \in \mathbb{R}, \delta_i \geq 0, \beta_i > 0, \ i = 1,\ldots,n, \beta_0 > 0$ and $\delta_i\beta_i > 0, \ i = 1,\ldots,n$ can be different, not obligatory all equal to $\beta_0 = \eta$, as it is the case for the operator (105): $(a_i\eta) \times (1/a_i) = \eta, i = 1,\ldots,n$. However, the multiple Erdélyi-Kober fractional integral (108) does not in general possess the operational property (99) because its kernel cannot be represented in the form (98), the only exception being the case of the operator (105).

Returning back to the equations with the composed Erdélyi-Kober fractional derivatives (103), the natural formulation of the Cauchy problem for the equations with these fractional derivatives should incorporate the initial conditions given in form of the projector of the corresponding composed Erdélyi-Kober fractional integrals:

$$\sum_{i=0}^{n} a_i(\mathcal{D}_\eta^i y)(x) = f(x), \tag{109}$$

$$(F\mathcal{D}_\eta^k y)(x) = \gamma_k(x), \ k = 0,1,\ldots,n-1, \ \gamma_k(x) \in \ker \mathcal{D}_\eta,$$

where $F = \mathrm{Id} - \mathcal{L}_\eta\mathcal{D}_\eta$ is the projector of the operator $\mathcal{L}_\eta$. One example of the projector for a particular case of the operator $\mathcal{L}_\eta$-the multiple Erdélyi-Kober fractional integral-is presented in the formula (107).

The Cauchy problem (109) can be solved by applying the transmutation operator in form of the generalized Obrechkoff-Stieltjes integral transform. The basis for the solution method is again Theorem 5.2 that remains valid in the space $\mathcal{M}_\gamma^{-1}(L)$ if we replace η with $-\eta$ and $\mathcal{L}_\eta$ with $\mathcal{D}_\eta$. Thus we arrive at the transmutation formula

$$(\mathcal{OS}(\mathcal{D}_\eta f))(x) = x^{-\eta}(\mathcal{OS}f)(x)$$

that is valid in the space of functions $\mathcal{M}_\gamma^{-1}(L)$. Otherwise, in the suitable "classical" spaces of functions, the right-hand side of the last relation will include some additional terms that arise from the initial conditions of the Cauchy problem (109) (for details see [11, 12, 34])).

Particular cases of the multiple Erdélyi-Kober fractional derivative (106) are the hyper-Bessel differential operator ($a_i = \frac{1}{\beta},\ \alpha_i = -\gamma_i,\ \eta_i = 1,\ 1 \le i \le n,\ \eta = \beta$ in (106)):

$$(Bf)(x) = x^{-\beta} \prod_{i=1}^{n} \left(\gamma_i + \frac{1}{\beta} x \frac{d}{dx} \right) f(x) \tag{110}$$

and the Riemann-Liouville fractional derivative ($n = 1,\ a_1 = 1,\ \alpha_1 = 0$, and $\eta_1 = \begin{cases} [\eta] + 1, & \eta \notin \mathbb{N}, \\ \eta, & \eta \in \mathbb{N} \end{cases}$ in (106)):

$$(D_{0+}^{\eta} f)(x) = \left(\frac{d}{dx} \right)^{\eta_1} (I_{0+}^{\eta_1 - \eta} f)(x), \quad \eta_1 = \begin{cases} [\eta] + 1, & \eta \notin \mathbb{N}, \\ \eta, & \eta \in \mathbb{N}, \end{cases} \tag{111}$$

where $(I_{0+}^{\alpha} f)(x)$ is the right-hand sided Riemann-Liouville fractional integral

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} f(t)\, dt. \tag{112}$$

For the Riemann-Liouville fractional derivative (111), the representation (107) has the well-known form

$$(\mathcal{L}_\eta \mathcal{D}_\eta f)(x) = f(x) - \sum_{k=1}^{\eta_1} \frac{x^{\eta - k}}{\Gamma(\eta - k + 1)} \lim_{x \to 0} (D_{0+}^{\eta - k} f)(x). \tag{113}$$

For the hyper-Bessel differential operator (110), the formula (107) can be rewritten in the form

$$\begin{aligned} (\mathcal{L}_\eta \mathcal{D}_\eta f)(x) = f(x) - \sum_{i=1}^{n} x^{-\beta\gamma_i} \beta^{i-n} \prod_{j=i+1}^{n} (\gamma_j - \gamma_i)^{-1} \\ \times \lim_{x \to 0} \left(x^{\beta\gamma_i} \prod_{j=i+1}^{n} (\beta\gamma_j + x \frac{d}{dx}) f(x) \right), \end{aligned} \tag{114}$$

$$\gamma_1 < \gamma_2 < \cdots < \gamma_n < \gamma_1 + 1.$$

Thus, in the Cauchy problems of type (109) for the Riemann-Liouville fractional derivative or for the hyper-Bessel differential operator, the initial conditions should be formulated as in the formulas (113) or (114), respectively.

Finally, we present here some of the important particular cases of the composed Erdélyi-Kober fractional operators (90) and their transmutations in form of the generalized Obrechkoff-Stieltjes transform (69).

The simplest particular cases of the generalized Obrechkoff-Stieltjes transform (69) are the modified Borel-Dzrbasjan transforms [11, 34]:

$$(\mathcal{BD}_{+}^{\beta,b} f)(x) = \frac{1}{b}\int_0^{\infty} \left(\frac{x}{t}\right)^{\frac{\beta}{b}} \exp\left(-\left(\frac{x}{t}\right)^{\frac{1}{b}}\right) f(t)\frac{dt}{t},\ x>0, \tag{115}$$

$$(\mathcal{BD}_{-}^{\alpha,a} f)(x) = \frac{1}{a}\int_0^{\infty} \left(\frac{x}{t}\right)^{\frac{\alpha-1}{a}} \exp\left(-\left(\frac{x}{t}\right)^{-\frac{1}{a}}\right) f(t)\frac{dt}{t},\ x>0. \tag{116}$$

The relations (7), (12), and Theorem 3.1 lead to the following Mellin-Barnes integral representations of the modified Borel-Dzrbasjan transforms:

$$(\mathcal{BD}_{+}^{\beta,b} f)(x) = \frac{1}{2\pi i}\int_{\sigma} \Gamma(\beta+bs) f^*(s) x^{-s} ds, \tag{117}$$

$$(\mathcal{BD}_{-}^{\alpha,a} f)(x) = \frac{1}{2\pi i}\int_{\sigma} \Gamma(1-\alpha-as) f^*(s) x^{-s} ds. \tag{118}$$

The composed Erdélyi-Kober fractional operators of type (90) with the modified Borel-Dzrbasjan transforms (115), (116) as the transmutation operators, respectively, have the following form:

$$(\mathcal{L}_\eta f)(x) = x^{\eta}\frac{1}{2\pi i}\int_{\sigma} \frac{\Gamma(\beta+bs)}{\Gamma(\beta-b\eta+bs)} f^*(s)x^{-s}ds = \begin{cases} x^{\eta}(K_{1/b}^{\beta,-b\eta} f)(x),\ \eta<0, \\ x^{\eta}(P_{1/b}^{\beta-b\eta,b\eta} f)(x),\ \eta>0, \end{cases} \tag{119}$$

$$(\mathcal{L}_\eta f)(x) = x^{\eta}\frac{1}{2\pi i}\int_{\sigma} \frac{\Gamma(1-\alpha-as)}{\Gamma(1-\alpha+a\eta-as)} f^*(s)x^{-s}ds = \begin{cases} x^{\eta}(I_{1/a}^{-\alpha,a\eta} f)(x),\ \eta>0, \\ x^{\eta}(D_{1/a}^{a\eta-\alpha,-a\eta} f)(x),\ \eta<0, \end{cases} \tag{120}$$

where $I_{\beta}^{\gamma,\delta}$, $K_{\beta}^{\tau,\alpha}$, $D_{\beta}^{\gamma,\delta}$, $P_{\beta}^{\tau,\alpha}$ are the Erdélyi-Kober fractional integrals and derivatives.

As the next example, we consider the multiple Erdélyi-Kober fractional integrals (105) and derivatives (106). Their Mellin-Barnes integral representations have the form

$$(\mathcal{L}_\eta f)(x) = \frac{x^{\eta}}{2\pi i}\int_{\sigma} \frac{\prod_{j=1}^{n}\Gamma(1-\alpha_j-a_js)}{\prod_{j=1}^{n}\Gamma(1-\alpha_j-a_j(s-\eta))} f^*(s)x^{-s}\,ds \tag{121}$$

that corresponds to the following operators for $\eta > 0$ and $\eta < 0$, respectively:

$$(\mathcal{L}_\eta f)(x) = \begin{cases} x^\eta \left(\prod_{j=1}^n I_{1/a_j}^{-\alpha_j, a_j\eta} f\right)(x), \ \eta > 0, \\ x^\eta \left(\prod_{j=1}^n D_{1/a_j}^{-\alpha_j + a_j\eta, -a_j\eta} f\right)(x), \ \eta < 0. \end{cases}$$

In particular, for $\eta = -\beta < 0$, $a_j = -\frac{1}{\eta} = \frac{1}{\beta}$, $1 \le j \le n$, and $-\alpha_j = \gamma_j$, $1 \le j \le n$, the multiple Erdélyi-Kober fractional derivative (121) is reduced to the hyper-Bessel differential operator

$$(\mathcal{L}_\eta f)(x) = x^{-\beta} \left(\prod_{j=1}^n D_\beta^{\gamma_j - 1, 1} f\right)(x) = x^{-\beta} \prod_{j=1}^n \left(\gamma_j + \frac{1}{\beta} x \frac{d}{dx}\right) f(x). \tag{122}$$

According to Theorem 5.2, the transmutation operator for the multiple Erdélyi-Kober fractional integral (105) and derivative (106) is the generalized Obrechkoff integral transform (62).

As the last example, we consider the following particular case of the composed Erdélyi-Kober fractional operator:

$$(\mathcal{L}_\eta f)(x) = x^\eta \frac{1}{2\pi i} \int_\sigma \frac{\Gamma(\alpha + \beta s)\Gamma(1 - \alpha - \beta s)}{\Gamma(\alpha - \beta\eta + \beta s))\Gamma(1 - \alpha + \beta\eta - \beta s)} f^*(s) x^{-s} ds. \tag{123}$$

Using the formula (90), we immediately get the following representation of the operator (123):

$$(\mathcal{L}_\eta f)(x) = \begin{cases} x^\eta (P_{1/\beta}^{\alpha - \beta\eta, \beta\eta} I_{1/\beta}^{-\alpha, \beta\eta} f)(x), \quad\quad \eta > 0, \\ x^\eta (D_{1/\beta}^{-\alpha + \beta\eta, -\beta\eta} K_{1_\beta}^{\alpha, -\beta\eta} f)(x), \ \eta < 0. \end{cases}$$

Another, even more interesting representation of the operator (123) can be obtained starting from the supplement formula (26) for the Euler Γ-function:

$$\frac{\Gamma(\alpha + \beta s)\Gamma(1 - \alpha - \beta s)}{\Gamma(\alpha + \beta(s - \eta))\Gamma(1 - \alpha - \beta(s - \eta))} = \frac{\sin(\pi(\alpha - \beta\eta + \beta s))}{\sin(\pi(\alpha + \beta s))}$$

$$= \frac{\cos(\pi\beta\eta)\sin(\pi(\alpha + \beta s)) - \sin(\pi\beta\eta)\cos(\pi(\alpha + \beta s))}{\sin(\pi(\alpha + \beta s))} = \cos(\pi\beta\eta) - \sin(\pi\beta\eta)$$

$$\times \frac{\cos(\pi(\alpha + \beta s))}{\sin(\pi(\alpha + \beta s))} = \cos(\pi\beta\eta) - \sin(\pi\beta\eta) \frac{\Gamma(\alpha + \beta s)\Gamma(1 - \alpha - \beta s)}{\Gamma(1/2 + \alpha + \beta s)\Gamma(1/2 - \alpha - \beta s)}.$$

Substituting the last representation into the formula (123), we arrive at the following relation:

$$(\mathcal{L}_\eta f)(x) = x^\eta(\cos(\pi\beta\eta)f(x) - \sin(\pi\beta\eta)(\mathcal{H}^\alpha_\beta f)(x)), \tag{124}$$

where the operator

$$(\mathcal{H}^\alpha_\beta f)(x) = \frac{1}{2\pi i}\int_\sigma \frac{\Gamma(\alpha+\beta s)\Gamma(1-\alpha-\beta s)}{\Gamma(1/2+\alpha+\beta s)\Gamma(1/2-\alpha-\beta s)} f^*(s)x^{-s}ds = \frac{1}{\pi}\int_0^\infty \frac{f(xt^\beta)t^{-\alpha}}{t-1}dt \tag{125}$$

can be interpreted as the generalized Hilbert transform (the integral at the right-hand side of the formula (125) has to be considered in the sense of the principal value). The last representation follows from the Parseval formula (5) for the Mellin integral transform, the shift property (7), and the Mellin transform formula (16). For $\alpha = 0$ and $\beta = 1$, the operator (125) is reduced to the classical Hilbert transform (78).

Theorem 5.2 and the formula (67) ensure that the generalized Stieltjes transform (66) is a transmutation of the operator (124) that translates it into multiplication with a power function.

Acknowledgements The authors acknowledge the support of the Kuwait University for their joint research project No. SM01/17 “Operational method in fractional calculus” funded by the Kuwait University.

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Distributed Order Equations in Banach Spaces with Sectorial Operators

Vladimir E. Fedorov and Aliya A. Abdrakhmanova

Abstract We study the Cauchy problem for a class of solved with respect to the distributed Gerasimov–Caputo derivative inhomogeneous equations in Banach spaces with a linear unbounded operator, generating an analytic in a sector resolving family of operators. The unique solvability theorem for the Cauchy problem was proved, the form of the solution is found. These results were applied to the research of the Cauchy problem and the Showalter–Sidorov problem for linear inhomogeneous equations in Banach spaces with degenerate operator at the distributed order derivative. In the case of the generation by the pair of operators (at unknown function and its distributed order derivative) of an analytic resolving family of the corresponding degenerate homogeneous equation, we obtain the theorems of the existence of a unique solution to such problems, and derive the form of the solution. Abstract results for the degenerate equation are used for research of initial-boundary value problems unique solvability for a class of distributed order in time equations with polynomials of self-adjoint elliptic differential operator with respect to the spatial variables.

Keywords Distributed order differential equation · Fractional Gerasimov–Caputo derivative · Differential equation in a Banach space · Degenerate evolution equation · Cauchy problem · Initial boundary value problem

V. E. Fedorov (✉)
Chelyabinsk State University, Chelyabinsk, Russia

South Ural State University, Chelyabinsk, Russia
e-mail: kar@csu.ru

A. A. Abdrakhmanova
Ufa State Aviation Technical University, Ufa, Russia

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_23

1 Introduction

The last 2–3 decades the increased interest of researchers arised with respect to differential equations with distributed fractional derivatives (see the works of A.M. Nakhushev [1, 2], M. Caputo [3, 4], A.V. Pskhu [5, 6]). Such equations appear in various applied problems in describing certain physical or technical processes [3, 4, 7–10]. Among the mathematical investigations of the distributed order equation we note the works of A.V. Pskhu [5, 6] on the solvability and qualitative properties of both ordinary differential equations of distributed order, and the diffusion equation of distributed order in time, the paper of S. Umarov and R. Gorenflo [11], devoted to the unique solvability study of multipoint problems, including the Cauchy problem, to the equation with the distributed Gerasimov–Caputo derivative in time and with pseudodifferential operators with respect to the spatial variables, the papers of A.N. Kochubei [12, 13], R. Gorenflo, Y. Luchko, M. Stojanović [14], in which the theory of solvability is constructed for initial-boundary value problems to the diffusion and diffusion-wave equations of distributed order in time, the works of T. Atanacković, S. Pilipović, B. Stanković, D. Zorica [15–17] on the research of distributed order diffusion-wave equation by means of the theory of abstract Volterra equations, the works of E. Bazhlekova, I. Bazhlekov [18, 19] on the subordination principle for distributed order differential equations.

In this paper we study linear distributed order equations in Banach spaces by means of Laplace transform theory and apply the obtained results to research of initial boundary value problems for distributed order in time partial differential equations.

In the second section we consider the Cauchy problem for distributed order equation with the Gerasimov–Caputo derivative

$$\int_b^c \omega(\alpha) D_t^\alpha z(t) d\alpha = Az(t) + g(t), \quad t > 0. \tag{1}$$

with $c \in (0, 1]$ and with the linear closed unbounded operator A, generating an analytic in a sector resolving family of operators to the corresponding homogeneous equation, namely, $A \in \mathscr{A}^c(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$ [20, 21]. The unique solvability theorems for the Cauchy problem were proved, the form of the solution is found. In Sect. 3 analogous results were obtained for the case of $c \in (1, 2)$. The research of this form equations was ended by applications of abstract results to the study of initial boundary value problems for equations with polynomials of elliptic differential operators with respect to spatial variables in Sect. 4.

In the fifth section of the work the equation

$$\int_a^b \omega(\alpha) D_t^\alpha Lx(t) d\alpha = Mx(t) + f(t), \quad t > 0, \tag{2}$$

is studied with linear closed and densely defined in $\mathscr{X}$ operators $L, M : \mathscr{X} \to \mathscr{Y}$ under the assumptions $\ker L \neq \{0\}$, $(L, M) \in \mathscr{H}_c(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$. (The class $\mathscr{H}_\alpha(\theta_0, a_0)$ of operators pairs was introduced in [22] and used in [23] for research of initial problems unique solvability of the fractional order equation $D_t^\alpha Lx(t) = Mx(t) + f(t)$.) For the Cauchy problem and the Showalter–Sidorov problem to Eq. (2) we obtain the theorems of a unique solution existence, and derive the form of the solution. Here we apply the theorem on the Cauchy problem for Eq. (1). Abstract results for Eq. (2) are used for research of the same class of initial-boundary value problems, but in the case of degenerating of the differential operator with respect to the spatial variables under the distributed time-derivative.

This work is the continuation of the papers [24, 25], in which the solvability of Eqs. (1) and (2) was studied in the case of bounded operator A, and papers [22, 23, 26–33] on evolution equations, with a degenerate operator at the highest order fractional derivative.

2 Nondegenerate Equation at $c \in (0, 1]$

In this section we study the existence and the uniqueness of the Cauchy problem classical solution to equations, solved with respect to the distributed derivative with upper order integration limit not greater than one.

2.1 Homogeneous Equation at $c \in (0, 1]$

At $\beta > 0, t > 0$ denote $g_\beta(t) := t^{\beta-1}/\Gamma(\beta)$, where $\Gamma(\cdot)$ is the Euler function,

$$J_t^\beta h(t) := \int_0^t g_\beta(t-s)h(s)ds = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s)ds.$$

Let $m - 1 < \alpha \leq m \in \mathbf{N}$, D_t^m is the usual m-th order derivative, D_t^α is the Gerasimov–Caputo fractional derivative (see in details, for example, in [21]), i.e.

$$D_t^\alpha h(t) := D_t^m J_t^{m-\alpha} \left(h(t) - \sum_{k=0}^{m-1} h^{(k)}(0) g_{k+1}(t) \right).$$

Let $\overline{\mathbf{R}}_+ := \mathbf{R}_+ \cup \{0\}$, $\mathscr{Z}$ be a Banach space. The Laplace transform of the function $h : \overline{\mathbf{R}}_+ \to \mathscr{Z}$ is denoted by $\hat{L}[h]$. The formula for the Laplace transform of the

Gerasimov–Caputo fractional derivative has the form

$$\hat{L}[D_t^\alpha h](\lambda) = \lambda^\alpha \hat{L}[h](\lambda) - \sum_{k=0}^{m-1} \lambda^{\alpha-k-1} h^{(k)}(0). \tag{3}$$

Hereafter the fractional power will be understood as its principal branch.

Denote by $\mathscr{L}(\mathscr{Z})$ the Banach space of all linear continuous operators from $\mathscr{Z}$ to $\mathscr{Z}$, and by $\mathscr{C}l(\mathscr{Z})$ the set of all linear closed operators with dense domains in $\mathscr{Z}$, acting into $\mathscr{Z}$. For $A \in \mathscr{C}l(\mathscr{Z})$ endow its domain D_A with the graph norm $\|\cdot\|_{D_A} := \|\cdot\|_{\mathscr{Z}} + \|A\cdot\|_{\mathscr{Z}}$, then it will be the Banach space, denoted by D_A also.

Consider the Cauchy problem

$$z(0) = z_0 \tag{4}$$

to the distributed order equation

$$\int_b^c \omega(\alpha) D_t^\alpha z(t) d\alpha = Az(t), \quad t > 0, \tag{5}$$

where D_t^α is the Gerasimov–Caputo fractional derivative, $0 \leq b < c \leq 1$, $\omega : (b, c) \to \mathbf{C}$. By a solution of (4), (5) we mean a function $z \in C(\overline{\mathbf{R}}_+; \mathscr{Z}) \cap C(\mathbf{R}_+; D_A)$, such that there exists $\int_b^c \omega(\alpha) D_t^\alpha z(t) d\alpha \in C(\mathbf{R}_+; \mathscr{Z})$ and equalities (4) and (5) are fulfilled.

Denote by $\rho(A)$ the resolvent set of the operator A. In notation of [21] an operator $A \in \mathscr{C}l(\mathscr{Z})$ belongs to the class $\mathscr{A}^\alpha(\theta_0, a_0)$ at some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, if there exists a resolving operators family $\{Z(t) \in \mathscr{L}(\mathscr{Z}) : t \in \overline{\mathbf{R}}_+\}$ for the fractional order equation $D_t^\alpha z(t) = Az(t)$, having a holomorphic extension in the sector

$$\Sigma_{\theta_0} := \{t \in \mathbf{C} : |\arg t| < \theta_0 - \pi/2, t \neq 0\}$$

and for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists a constant $C(\theta, a)$, such that for all $t \in \Sigma_\theta$ $\|Z(t)\|_{\mathscr{L}(\mathscr{Z})} \leq C(\theta, a)e^{a\mathrm{Re}t}$. In according to Theorem 2.14 [21] (see more general Theorem 2.1 [20] also) at $\alpha \in (0, 2)$ $A \in \mathscr{A}^\alpha(\theta_0, a_0)$, if and only if the following conditions are satisfied:

1. for all $\lambda \in S_{\theta_0,a_0} := \{\mu \in \mathbf{C} : |\arg(\mu - a_0)| < \theta_0, \mu \neq a_0\}$ we have $\lambda^\alpha \in \rho(A)$;
2. for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists a constant $K = K(\theta, a) > 0$, such that for all $\mu \in S_{\theta,a}$

$$\|R_{\mu^\alpha}(A)\|_{\mathscr{L}(\mathscr{Z})} \leq \frac{K(\theta, a)}{|\mu^{\alpha-1}(\mu - a)|}, \tag{6}$$

where $R_{\mu^\alpha}(A) = (\mu^\alpha I - A)^{-1}$.

We shall consider an operator A from the class $\mathscr{A}^c(\theta_0, a_0)$, where c is from (5). Denote $\Gamma = \Gamma_+ \cup \Gamma_-$, $\Gamma_\pm = \{\mu \in \mathbf{C} : \mu = a + re^{\pm i\theta},\ r \in (0, \infty)\}$ at $a > a_0$, $\theta \in (\pi/2, \theta_0)$,

$$W_d^h(\lambda) := \int_d^h \omega(\alpha)\lambda^\alpha d\alpha,$$

$$Z_0(t) := \frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda t}}{\lambda} W_b^c(\lambda)\left(W_b^c(\lambda)I - A\right)^{-1} d\lambda.$$

Denote by $E(K, \beta; \mathscr{Z})$ the set of functions $z : \overline{\mathbf{R}}_+ \to \mathscr{Z}$, such that $\|z(t)\|_{\mathscr{Z}} \leq Ke^{\beta t}$ for all $t \in \overline{\mathbf{R}}_+$. Besides, we shall use the denotation

$$E(\mathscr{Z}) := \bigcup_{K>0}\bigcup_{\beta\geq 0} E(K, \beta; \mathscr{Z}).$$

Theorem 1 *Let $0 \leq b < c \leq 1$, $A \in \mathscr{A}^c(\theta_0, a_0)$, $z_0 \in D_A$, and $W_b^c(\lambda)$ be holomorphic function on S_{θ_1,a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, satisfying the conditions*

$$\forall\lambda \in S_{\theta_1,a_1} \quad (W_b^c(\lambda))^{1/c} \in S_{\theta_0,a_0}, \tag{7}$$

$$\exists C_1, C_2 > 0 \quad \exists\varepsilon \in (0, c) \quad \forall\lambda \in S_{\theta_1,a_1} \quad C_1|\lambda|^\varepsilon \leq |W_b^c(\lambda)| \leq C_2|\lambda|^c. \tag{8}$$

Then the function $z(t) = Z_0(t)z_0$ is a unique solution of the Cauchy problem (4), (5) *in the space $E(\mathscr{Z})$.*

Proof Take the contour Γ with the constants $a = a_1 + \delta$, $\theta = \theta_1$, where θ_1, a_1 are from condition (7), $\delta > 0$ is a small number. Then for $\lambda \in \Gamma$ we have $W_b^c(\lambda) \in \rho(A)$ and

$$\left\|\left(W_b^c(\lambda)I - A\right)^{-1}\right\|_{\mathscr{L}(\mathscr{Z})} \leq \frac{K_1}{|W_b^c(\lambda)|} \leq \frac{K_2}{|\lambda|^\varepsilon} \tag{9}$$

with some constants $K_1 = K_1(\theta_1, a_1)$, $K_2 = K_1/C_1$, since $A \in \mathscr{A}^c(\theta_0, a_0)$. Therefore,

$$\left\|W_b^c(\lambda)\left(W_b^c(\lambda)I - A\right)^{-1}\right\|_{\mathscr{L}(\mathscr{Z})} \leq K_1, \tag{10}$$

and at $t > 0$ the integral

$$\int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} W_b^c(\lambda) \left(W_b^c(\lambda)I - A\right)^{-1} d\lambda,$$

converges, $Z_0(t)z_0 \in D_A$, $Z_0(t)$ and

$$AZ_0(t)z_0 = \int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} W_b^c(\lambda) \left(W_b^c(\lambda)I - A\right)^{-1} d\lambda A z_0$$

are holomorphic in $\Sigma_{\theta_1} := \{t \in \mathbf{C} : |\arg t| < \theta_1 - \pi/2, t \neq 0\}$.

Let $R > \delta$,

$$\Gamma_R = \bigcup_{k=1}^{3} \Gamma_{k,R}, \quad \Gamma_{1,R} = \{\lambda \in \mathbf{C} : \lambda = a + Re^{i\varphi},\ \varphi \in (-\theta_1, \theta_1)\},$$

$$\Gamma_{2,R} = \{\lambda \in \mathbf{C} : \lambda = a + re^{i\theta_1},\ r \in [0, R]\},$$

$$\Gamma_{3,R} = \{\lambda \in \mathbf{C} : \lambda = a + re^{-i\theta_1},\ r \in [0, R]\},$$

Γ_R be the closed loop, oriented counter-clockwise. Consider also the contours

$$\Gamma_{4,R} = \{\lambda \in \mathbf{C} : \lambda = a + re^{i\theta_1},\ r \in [R, \infty)\},$$

$$\Gamma_{5,R} = \{\lambda \in \mathbf{C} : \lambda = a + re^{-i\theta_1},\ r \in [R, \infty)\},$$

then $\Gamma = \Gamma_{4,R} \cup \Gamma_{5,R} \cup \Gamma_R \setminus \Gamma_{1,R}$.

For $t > 0$, $z_0 \in D_A$

$$Z_0(t)z_0 = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} W_b^c(\lambda) \left(W_b^c(\lambda)I - A\right)^{-1} z_0 d\lambda =$$

$$= \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} d\lambda z_0 + \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} \left(W_b^c(\lambda)I - A\right)^{-1} A z_0 d\lambda.$$

For $t \in [0, 1]$, $\lambda \in \Gamma$ by (9)

$$\left\| \frac{e^{\lambda t}}{\lambda} \left(W_b^c(\lambda)I - A\right)^{-1} A z_0 \right\|_{\mathscr{Z}} \leq \frac{e^a K_2 \|Az_0\|_{\mathscr{Z}}}{|\lambda|^{1+\varepsilon}},$$

therefore, for some $C > 0$

$$\left\| \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} \left(W_b^c(\lambda)I - A\right)^{-1} Az_0 d\lambda \right\|_{\mathscr{Z}} \leq C.$$

Consequently, the integral converges uniformly with respect to $t \in [0, 1]$ and

$$Z_0(0)z_0 = z_0 + \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{1}{\lambda} \left(W_b^c(\lambda)I - A\right)^{-1} Az_0 d\lambda =$$

$$= z_0 + \lim_{R\to\infty} \frac{1}{2\pi i} \left(\int\limits_{\Gamma_R} - \int\limits_{\Gamma_{1,R}} + \int\limits_{\Gamma_{4,R}} + \int\limits_{\Gamma_{5,R}} \right) \frac{1}{\lambda} \left(W_b^c(\lambda)I - A\right)^{-1} Az_0 d\lambda = z_0,$$

since by the Cauchy Theorem

$$\int\limits_{\Gamma_R} \frac{1}{\lambda} \left(W_b^c(\lambda)I - A\right)^{-1} Az_0 d\lambda = 0,$$

and

$$\left\| \int\limits_{\Gamma_{1,R}} \frac{e^{\lambda t}}{\lambda} \left(W_b^c(\lambda)I - A\right)^{-1} Az_0 d\lambda \right\|_{\mathscr{Z}} \leq \frac{C\|Az_0\|_{\mathscr{Z}}}{R^{\varepsilon}},$$

$$\left\| \int\limits_{\Gamma_{s,R}} \frac{e^{\lambda t}}{\lambda} \left(W_b^c(\lambda)I - A\right)^{-1} Az_0 d\lambda \right\|_{\mathscr{Z}} \leq \frac{C\|Az_0\|_{\mathscr{Z}}}{R^{\varepsilon}}, \quad s = 4, 5.$$

Consequently, $Z_0(\cdot)z_0 \in C(\overline{\mathbf{R}}_+; \mathscr{Z})$, the function $z(t) = Z_0(t)z_0$ satisfies Cauchy condition (4).

By the construction, due to (6)

$$\|Z_0(t)z_0\|_{\mathscr{Z}} \leq \frac{K_1}{2\pi} \int\limits_{\Gamma} \frac{e^{t\mathrm{Re}\lambda}|d\lambda|}{|\lambda|} \|z_0\|_{\mathscr{Z}} \leq C_3 e^{(a_1+\delta)t},$$

because

$$\int\limits_{\Gamma} \frac{e^{t\mathrm{Re}\lambda}|d\lambda|}{|\lambda|} \le Ce^{(a_1+\delta)t} \int\limits_{-\infty}^{0} e^x dx = Ce^{(a_1+\delta)t}, \quad t \ge 1.$$

Here $C = \min\{|\lambda| : \lambda \in \Gamma\}$. Therefore, we can take

$$C_3 = \max\left\{\frac{K_1 C\|z_0\|_{\mathscr{Z}}}{2\pi}, \max_{t\in[0,1]} e^{-(a_1+\delta)t}\|Z_0(t)z_0\|_{\mathscr{Z}}\right\}.$$

Thus, $Z_0(\cdot)z_0 \in E(\mathscr{Z})$.

Under the condition $\mathrm{Re}\mu > a_1 + \delta$ we have the equality

$$\hat{L}[z](\mu) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{W_b^c(\lambda)}{\lambda(\mu-\lambda)} \left(W_b^c(\lambda)I - A\right)^{-1} z_0 d\lambda.$$

Due to (10) this integral converges and

$$\lim_{R\to\infty} \frac{1}{2\pi i}\int\limits_{\Gamma_{s,R}} \frac{W_b^c(\lambda)}{\lambda(\mu-\lambda)} \left(W_b^c(\lambda)I - A\right)^{-1} z_0 d\lambda = 0, \quad s = 1, 4, 5.$$

Therefore, by the Cauchy integral formula

$$\hat{L}[z](\mu) = \lim_{R\to\infty} \frac{1}{2\pi i}\int\limits_{\Gamma_R} \frac{W_b^c(\lambda)}{\lambda(\mu-\lambda)} \left(W_b^c(\lambda)I - A\right)^{-1} z_0 d\lambda =$$

$$= \frac{W_b^c(\mu)}{\mu} \left(W_b^c(\mu)I - A\right)^{-1} z_0,$$

$$\hat{L}[Az](\mu) = \frac{W_b^c(\mu)}{\mu} \left(W_b^c(\mu)I - A\right)^{-1} Az_0.$$

Hence, $\hat{L}[z](\mu) \in D_A$, $A\hat{L}[z](\mu) = \hat{L}[Az](\mu)$, $\hat{L}[z](\mu)$ and $\hat{L}[Az](\mu)$ have holomorphic extensions on S_{θ_1,a_1}.

Further, using formula (3) for the Laplace transform, we can write

$$\hat{L}\left[\int\limits_b^c \omega(\alpha)D_t^\alpha z(t)d\alpha\right](\mu) = \frac{(W_b^c(\mu))^2}{\mu} \left(W_b^c(\mu)I - A\right)^{-1} z_0 - \frac{W_b^c(\mu)}{\mu} z_0 =$$

$$= \frac{W_b^c(\mu)}{\mu} \left(W_b^c(\mu)I - A\right)^{-1} Az_0 = \hat{L}[Az](\mu).$$

Here the commutation of an operator and its resolvent was taken into account. We can apply the inverse Laplace transform on the both parts of the last equality and obtain equality (5) in all continuity points of function z, i.e. for all $t \geq 0$. It was proved, that $Az \in C(\mathbf{R}_+; \mathscr{Z})$, hence the left-hand side of the equation is continuous also and the function z is a solution of problem (4), (5).

If there are two solutions z_1, z_2 of problem (4), (5) from the class $E(\mathscr{Z})$, then their difference $y = z_1 - z_2 \in E(\mathscr{Z})$ is a solution of Eq. (5) and satisfy the initial condition $y(0) = 0$. Applying the Laplace transform to the both sides of Eq. (5) gives the equality $W_b^c(\lambda)\hat{L}[y](\lambda) = A\hat{L}[y](\lambda)$. Therefore, for $\lambda \in S_{\theta_1,a_1}$ we have $\hat{L}[y](\lambda) \equiv 0$. It means that $y \equiv 0$. □

Remark 1 If $\omega \equiv 1$, then $W(\lambda) = \frac{\lambda^c - \lambda^b}{\ln \lambda}$. It is evident, that the condition (7) is satisfied for $\theta_1 = \theta_0$ and some $a_1 \geq a_0$. Condition (8) will be discussed in the end of Sect. 3.1.

Remark 2 In the proof of Theorem 1 it is shown that the solution $z(t) = Z_0(t)z_0$ of problem (4), (5) has holomorphic extension to the sector Σ_{θ_1}.

Remark 3 It can be proved that under the conditions of Theorem 1 for $z_0 \in D_{A^2}$ we have $Z_0(\cdot)z_0 \in C(\overline{\mathbf{R}}_+; D_A)$ and Eq. (5) is satisfied at $t = 0$.

Remark 4 By the Banach–Steinhaus Theorem we have also that for every $z_0 \in \mathscr{Z}$ $Z_0(\cdot)z_0 \in C(\overline{\mathbf{R}}_+; \mathscr{Z})$ and $Z_0(0)z_0 = z_0$.

2.2 Inhomogeneous Equation at $c \in (0, 1]$

A solution of problem (4) for the equation

$$\int_b^c \omega(\alpha) D_t^\alpha z(t) d\alpha = Az(t) + g(t), \quad t > 0, \tag{11}$$

where $0 \leq b < c \leq 1$, $\omega : (a, b) \to \mathbf{C}$, $g \in C(\overline{\mathbf{R}}_+; \mathscr{Z})$, is a function $z \in C(\overline{\mathbf{R}}_+; \mathscr{Z}) \cap C(\mathbf{R}_+; D_A)$, such that there exists $\int_b^c \omega(\alpha) D_t^\alpha z(t) d\alpha \in C(\mathbf{R}_+; \mathscr{Z})$ and equalities (4) and (11) are valid.

Denote

$$Z(t) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \left(W_b^c(\lambda) I - A\right)^{-1} d\lambda. \tag{12}$$

Lemma 1 *Let $0 \leq b < c \leq 1$, $A \in \mathscr{A}^c(\theta_0, a_0)$, $W_b^c(\lambda)$ be the holomorphic function on S_{θ_1,a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, satisfying conditions (7), (8),*

$g \in C(\overline{\mathbf{R}}_+; D_A) \cap E(D_A)$. *Then the function*

$$z_g(t) = \int\limits_0^t Z(t-s)g(s)ds$$

is a unique solution to the Cauchy problem $z(0) = 0$ *for Eq.* (11) *in* $E(\mathscr{Z})$.

Proof It is easy to show that the integral (12) converges uniformly with respect to t on every compact set from the sector Σ_{θ_1}, therefore, $Z(t)$ can be holomorphically extended onto this set.

For $t \in [0, 1]$ we have

$$\int\limits_\Gamma \frac{e^{t\mathrm{Re}\lambda}}{|\lambda|^\varepsilon}|d\lambda| \le C \int\limits_{-\infty}^0 \frac{e^{tx}dx}{|x|^\varepsilon} = Ct^{\varepsilon-1} \int\limits_0^{+\infty} \frac{e^{-y}dy}{y^\varepsilon} = C\Gamma(1-\varepsilon)t^{\varepsilon-1}.$$

Therefore,

$$\|Z(t)\|_{\mathscr{L}(\mathscr{Z})} \le C \int\limits_\Gamma \frac{e^{t\mathrm{Re}\lambda}}{|\lambda|^\varepsilon}|d\lambda| = O(t^{\varepsilon-1}) \text{ as } t \to 0+,$$

$\|z_g(t)\|_{\mathscr{Z}} \le Ct^\varepsilon \to 0$ as $t \to 0+$. Thus, zero initial condition (4) is fulfilled.

Note also, that for $g \in E(K_g, \beta_g, D_A)$, $t \ge t_0 > 0$,

$$\|z_g(t)\|_{\mathscr{Z}} \le C \int\limits_0^t \int\limits_{-\infty}^0 \frac{e^{(a_1+\delta+x)(t-s)}dx}{|x-\delta_1|^\varepsilon} e^{\beta_g s}ds \le Ce^{\beta t} \int\limits_{-\infty}^0 \int\limits_0^t e^{(x-\delta_1)(t-s)}ds \frac{dx}{|x-\delta_1|^\varepsilon} \le$$
$$\le Ce^{\beta t} \int\limits_{-\infty}^0 \frac{(1-e^{(x-\delta_1)t})dx}{|x-\delta_1|^{1+\varepsilon}} \le Ce^{\beta t} \int\limits_0^{+\infty} \frac{dy}{(y+\delta_1)^{1+\varepsilon}} = \frac{Ce^{\beta t}}{\varepsilon\delta_1^\varepsilon},$$

where $\delta_1 > 0$ is a small number, $\beta = \max\{a_1 + \delta + \delta_1, \beta_g\}$. Taking into account the previous paragraph, we obtain that $z_g \in E(\mathscr{Z})$.

We have the convolution $z_g = Z * g$, hence $\hat{L}[z_g] = \hat{L}[Z]\hat{L}[g]$. Reasoning as in the proof of Theorem 1, we obtain $\hat{L}[Z](\mu) = \left(W_b^c(\mu)I - A\right)^{-1}$, because from (9) it follows that

$$\left\| \frac{1}{\mu-\lambda} \left(W_b^c(\lambda)I - A\right)^{-1} \right\|_{\mathscr{L}(\mathscr{Z})} \le \frac{C}{|\lambda|^{1+\varepsilon}}.$$

Since $g \in C(\overline{\mathbf{R}}_+; D_A)$ and the operator A is closed, for $t \geq 0$ we have $z_g(t) \in D_A$, and $Az_g(t) = z_{Ag}(t)$. Hence

$$\hat{L}\left[\int_b^c \omega(\alpha) D_t^\alpha z_g d\alpha\right](\mu) = W_b^c(\mu)\left(W_b^c(\mu)I - A\right)^{-1}\hat{L}[g](\mu) =$$

$$= \hat{L}[g](\mu) + \left(W_b^c(\mu)I - A\right)^{-1}\hat{L}[Ag](\mu).$$

Acting by the inverse Laplace transform on the both sides of this equality, obtain

$$\int_a^b \omega(\alpha) D_t^\alpha z_g(t) d\alpha = g(t) + (Z * Ag)(t) = g(t) + Az_g(t).$$

The proof of the solution uniqueness reduces in the obvious way to the proof of the uniqueness for the homogeneous equation. □

From Theorem 1 and Lemma 1 the next statement follows immediately.

Theorem 2 *Let* $0 \leq b < c \leq 1$, $A \in \mathscr{A}^c(\theta_0, a_0)$, $z_0 \in D_A$, $W_b^c(\lambda)$ *be the holomorphic function on* S_{θ_1, a_1} *with some* $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, *satisfying the conditions* (7), (8), $g \in C(\overline{\mathbf{R}}_+; D_A) \cap E(D_A)$. *Then the function*

$$z(t) = Z_0(t)z_0 + \int_0^t Z(t-s)g(s)ds$$

is a unique solution to problem (4), (11) *in* $E(\mathscr{Z})$.

3 Nondegenerate Equation at $c > 1$

The unique solvability issues for the Cauchy problem to distributed order differential equation in a Banach space with upper order integration limit greater than one is studied in this section.

3.1 Homogeneous Equation at $c > 1$

Consider the Cauchy problem

$$z(0) = z_0, \quad z'(0) = z_1 \tag{13}$$

to the distributed order equation

$$\int_b^c \omega(\alpha) D_t^\alpha z(t) d\alpha = Az(t), \quad t > 0, \tag{14}$$

where D_t^α is the Gerasimov–Caputo fractional derivative, $1 < c \le 2$, $0 \le b < c$, $\omega : (b, c) \to \mathbf{C}$. By a solution of problem (13), (14) we mean a function $z \in C^1(\overline{\mathbf{R}}_+; \mathscr{Z}) \cap C(\mathbf{R}_+; D_A)$, such that there exist $\int_b^c \omega(\alpha) D_t^\alpha z(t) d\alpha \in C(\mathbf{R}_+; \mathscr{Z})$ and equalities (13) and (14) are fulfilled.

Denote $b_1 := \max\{b, 1\}$,

$$Z_0(t) := \frac{1}{2\pi i} \int_\Gamma \frac{e^{\lambda t}}{\lambda} W_b^c(\lambda) \left(W_b^c(\lambda) I - A\right)^{-1} d\lambda,$$

$$Z_1(t) := \frac{1}{2\pi i} \int_\Gamma \frac{e^{\lambda t}}{\lambda^2} W_{b_1}^c(\lambda) \left(W_b^c(\lambda) I - A\right)^{-1} d\lambda$$

with $\Gamma = \Gamma_+ \cup \Gamma_-$, $\Gamma_\pm = \{\mu \in \mathbf{C} : \mu = a_1 + \delta + re^{\pm i\theta_1},\ r \in (0, \infty)\}$ at the constants $a_1 > a_0$, $\theta_1 \in (\pi/2, \theta_0)$ from the conditions of the next theorem, $\delta > 0$.

Theorem 3 *Let* $c \in (1, 2)$, $A \in \mathscr{A}^c(\theta_0, a_0)$, $z_0, z_1 \in D_A$, $W_b^c(\lambda)$ *and* $W_{b_1}^c(\lambda)$ *be holomorphic functions on* S_{θ_1,a_1} *with some* $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \ge a_0$, *satisfying the conditions*

$$\forall \lambda \in S_{\theta_1,a_1} \quad (W_b^c(\lambda))^{1/c} \in S_{\theta_0,a_0}, \tag{15}$$

$$\exists C_1, C_2 > 0 \quad \exists \varepsilon \in (0, c-1) \quad \forall \lambda \in S_{\theta_1,a_1} \quad C_1|\lambda|^{1+\varepsilon} \le |W_b^c(\lambda)| \le C_2|\lambda|^c, \tag{16}$$

$$\exists C_3 > 0 \quad \forall \lambda \in S_{\theta_1,a_1} \quad |W_b^{b_1}(\lambda)| \le C_3|\lambda|. \tag{17}$$

Then the function $z(t) = Z_0(t)z_0 + Z_1(t)z_1$ *is a unique solution to problem* (13), (14) *in the space* $E(\mathscr{Z})$.

Proof We have $W_{b_1}^c(\lambda) = W_b^c(\lambda) - W_b^{b_1}(\lambda)$, therefore,

$$\frac{|W_{b_1}^c(\lambda)|}{|W_b^c(\lambda)|} \le 1 + C_1^{-1} C_3 |\lambda|^{-\varepsilon} \le C_4 \quad \forall \lambda \in S_{\theta_1,a_1}.$$

Hereafter $W_b^{b_1} \equiv 0$, if $b \geq 1$, $W_b^{b_1} = W_b^1$ for $b < 1$; $b_0 = b$. Thus,

$$\left\| \frac{W_{b_k}^c(\lambda)}{\lambda^{k+1}} \left(W_b^c(\lambda)I - A\right)^{-1} \right\|_{\mathscr{L}(\mathscr{Z})} \leq \frac{C}{|\lambda|^{k+1}}, \tag{18}$$

since $A \in \mathscr{A}^c(\theta_0, a_0)$, and at $t > 0$ the integrals $Z_k(t)$ converge for $k = 0, 1$. Moreover, for $t > 0$, $z_0, z_1 \in D_A$ we have $Z_0(t)z_0, Z_1(t)z_1 \in D_A$,

$$AZ_k(t)z_k = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda^{k+1}} W_{b_k}^c(\lambda) \left(W_b^c(\lambda)I - A\right)^{-1} d\lambda A z_k,$$

$Z_0(t)$ and $Z_1(t)$ are holomorphic in $\Sigma_{\theta_1} := \{t \in \mathbf{C} : |\arg t| < \theta_1 - \pi/2\}$, Z_1 is differentiable in $t = 0$, $Z_1(0) = 0$ since the right-hand side of (18) at $k = 1$ is $O(|\lambda|^{-2})$ as $|\lambda| \to \infty$.

For $R > \delta$ we shall use the contours $\Gamma_{k.R}$, $k = 1, 2, 3, 4, 5$, and

$$\Gamma_R = \bigcup_{k=1}^{3} \Gamma_{k,R},$$

as in the proof of Theorem 1. For $t > 0$, $z_0, z_1 \in D_A$

$$Z_0(t)z_0 = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} d\lambda z_0 + \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} \left(W_b^c(\lambda)I - A\right)^{-1} A z_0 d\lambda,$$

$$Z_1'(t)z_1 = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} W_{b_1}^c(\lambda) \left(W_b^c(\lambda)I - A\right)^{-1} z_1 d\lambda =$$

$$= \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} d\lambda z_1 + \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} \left(W_b^c(\lambda)I - A\right)^{-1} A z_1 d\lambda -$$

$$- \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} W_b^{b_1}(\lambda) \left(W_b^c(\lambda)I - A\right)^{-1} z_1 d\lambda.$$

For $t \in [0, 1]$, $k = 0, 1$, $\lambda \in \Gamma$ by conditions (16), (17)

$$\left\| \frac{e^{\lambda t}}{\lambda} (W_b^{b_1})^k \left(W_b^c(\lambda)I - A\right)^{-1} \right\|_{\mathscr{L}(\mathscr{Z})} \leq \frac{C}{|\lambda|^{2+\varepsilon-k}}.$$

Consequently, the integrals converge uniformly with respect to $t \in [0, 1]$ and

$$Z_0(0)z_0 = z_0 + \lim_{R\to\infty} \frac{1}{2\pi i}\left(\int\limits_{\Gamma_R} - \int\limits_{\Gamma_{1,R}} + \int\limits_{\Gamma_{4,R}} + \int\limits_{\Gamma_{5,R}}\right) \frac{1}{\lambda}\left(W_b^c(\lambda)I - A\right)^{-1} Az_0 d\lambda = z_0,$$

$$Z_1'(0)z_1 = z_1 + \lim_{R\to\infty} \frac{1}{2\pi i}\left(\int\limits_{\Gamma_R} - \int\limits_{\Gamma_{1,R}} + \int\limits_{\Gamma_{4,R}} + \int\limits_{\Gamma_{5,R}}\right) \frac{1}{\lambda}\left(W_b^c(\lambda)I - A\right)^{-1} Az_1 d\lambda -$$

$$- \lim_{R\to\infty} \frac{1}{2\pi i}\left(\int\limits_{\Gamma_R} - \int\limits_{\Gamma_{1,R}} + \int\limits_{\Gamma_{4,R}} + \int\limits_{\Gamma_{5,R}}\right) \frac{1}{\lambda} W_b^{b_1}(\lambda)\left(W_b^c(\lambda)I - A\right)^{-1} z_1 d\lambda = z_1.$$

Moreover, for $t \geq 0$, $z_0 \in D_A$ we have

$$Z_0'(t)z_0 = \frac{1}{2\pi i}\int\limits_{\Gamma} e^{\lambda t} W_b^c(\lambda)\left(W_b^c(\lambda)I - A\right)^{-1} z_0 d\lambda =$$

$$= \frac{1}{2\pi i}\int\limits_{\Gamma} e^{\lambda t} z_0 d\lambda + \frac{1}{2\pi i}\int\limits_{\Gamma} e^{\lambda t}\left(W_b^c(\lambda)I - A\right)^{-1} Az_0 d\lambda = 0.$$

Thus, the function $z \in C^1(\overline{\mathbf{R}}_+; \mathscr{Z}) \cap C(\mathbf{R}_+; D_A)$ satisfies Cauchy conditions (13).

By the construction, it can be shown as in the proof of Theorem 1 that due to estimate (18)

$$\|Z_k(t)\|_{\mathscr{L}(\mathscr{Z})} \leq C\int\limits_{\Gamma} \frac{e^{t\mathrm{Re}\lambda}|d\lambda|}{|\lambda|^{k+1}} \leq C_k^0 e^{(a_1+\delta)t}, \quad k = 0, 1,$$

$$\|Z_k'(t)\|_{\mathscr{L}(\mathscr{Z})} \leq C\int\limits_{\Gamma} \frac{e^{t\mathrm{Re}\lambda}|d\lambda|}{|\lambda|^{k}} \leq C_k^1 e^{(a_1+\delta)t}, \quad k = 0, 1.$$

Consequently, $\|z(t)\|_{\mathscr{Z}} \leq e^{(a_1+\delta)t}(C_0^0\|z_0\|_{\mathscr{Z}} + C_1^0\|z_1\|_{\mathscr{Z}})$, i.e. $z \in E(\mathscr{Z})$.

Under the condition $\mathrm{Re}\mu > a_1 + \delta$ we have the equality

$$\hat{L}[z](\mu) = \sum_{k=0}^{1} \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{W_{b_k}^c(\lambda)}{\lambda^{k+1}(\mu - \lambda)}\left(W_b^c(\lambda)I - A\right)^{-1} z_k d\lambda.$$

Due to (18) these integrals converge and at $k = 0, 1$

$$\lim_{R\to\infty} \sum_{k=0}^{1} \frac{1}{2\pi i} \int\limits_{\Gamma_{s,R}} \frac{W^c_{b_k}(\lambda)}{\lambda^{k+1}(\mu-\lambda)} \left(W^c_b(\lambda)I - A\right)^{-1} z_k d\lambda = 0, \quad s = 1, 4, 5.$$

Therefore, by the Cauchy integral formula

$$\hat{L}[z](\mu) = \lim_{R\to\infty} \sum_{k=0}^{1} \frac{1}{2\pi i} \int\limits_{\Gamma_R} \frac{W^c_{b_k}(\lambda)}{\lambda^{k+1}(\mu-\lambda)} \left(W^c_b(\lambda)I - A\right)^{-1} z_k d\lambda =$$

$$= \frac{W^c_b(\mu)}{\mu} \left(W^c_b(\mu)I - A\right)^{-1} z_0 + \frac{W^c_{b_1}(\mu)}{\mu^2} \left(W^c_b(\mu)I - A\right)^{-1} z_1.$$

Analogously we can obtain the equality at $z_0, z_1 \in D_A$

$$\hat{L}[Az](\mu) = \frac{W^c_b(\mu)}{\mu} \left(W^c_b(\mu)I - A\right)^{-1} Az_0 + \frac{W^c_{b_1}(\mu)}{\mu^2} \left(W^c_b(\mu)I - A\right)^{-1} Az_1.$$

Consequently, $\hat{L}[z](\mu) \in D_A$, $A\hat{L}[z](\mu) = \hat{L}[Az](\mu)$, $\hat{L}[z](\mu)$ and $\hat{L}[Az](\mu)$ have holomorphic extensions on S_{θ_1,a_1}.

Formula (3) for the Laplace transform implies that

$$\hat{L}\left[\int\limits_a^b \omega(\alpha) D^\alpha_t z(t) d\alpha\right](\mu) =$$

$$= \sum_{k=0}^{1} \frac{W^c_{b_k}(\mu)}{\mu^{k+1}} W^c_b(\mu) \left(W^c_b(\mu)I - A\right)^{-1} z_k - \sum_{k=0}^{1} \frac{W^c_{b_k}(\mu)}{\mu^{k+1}} z_k =$$

$$= A \sum_{k=0}^{1} \frac{W^c_{b_k}(\mu)}{\mu^{k+1}} \left(W^c_b(\mu)I - A\right)^{-1} z_k = A\hat{L}[z](\mu).$$

The rest of the reasoning is the same as in the proof of Theorem 1. □

Remark 5 We obtained that the families $\{Z_k(t) \in \mathscr{L}(\mathscr{Z}) : t \in (0, 1]\}$, $\{Z'_k(t) \in \mathscr{L}(\mathscr{Z}) : t \in (0, 1]\}$, $k = 0, 1$, are uniformly bounded. The density of D_A in $\mathscr{Z}$ and the Banach–Steinhaus Theorem imply that for every $z_0, z_1 \in \mathscr{Z}$

$$\lim_{t\to 0+} Z_k^{(k)}(t) z_k = z_k, \quad \lim_{t\to 0+} Z_k^{(1-k)}(t) z_k = 0, \quad k = 0, 1.$$

Therefore, $Z_k \in C^1(\overline{\mathbf{R}}_+; \mathscr{L}(\mathscr{Z}))$, $k = 0, 1$

Proposition 1 ([25]) *Let a function* $\omega : (b, c) \to \mathbf{R}$ *be bounded, and for some* $\gamma \in (0, c-b)$ *in the left* γ*-neighborhood of the point* c *it do not change the sign and*

$$\exists k_1 > 0 \quad \forall \alpha \in (c-\gamma, c) \quad |\omega(\alpha)| \geq k_1.$$

Then for $c \in (0, 1]$ *conditions* (8) *with arbitrary* $\varepsilon \in (0, c)$*, for* $c \in (1, 2)$ *conditions* (16) *with arbitrary* $\varepsilon \in (0, c-1)$ *and condition* (17) *hold.*

Corollary 1 ([25]) *Let* $\omega \in C([a, b]; \mathbf{R})$ *and* $\omega(b) \neq 0$*. Then for* $c \in (0, 1]$ *conditions* (8) *with arbitrary* $\varepsilon \in (0, c)$*, for* $c \in (1, 2)$ *conditions* (16) *with* $\varepsilon \in (0, c-1)$ *and condition* (17) *hold.*

3.2 Inhomogeneous Equation at c > 1

By a solution of problem (13) for the equation

$$\int_a^b \omega(\alpha) D_t^\alpha z(t) d\alpha = Az(t) + g(t), \quad t > 0, \tag{19}$$

where $c \in (1, 2)$, $b \in [0, c)$, $\omega : (a, b) \to \mathbf{C}$, $g \in C(\overline{\mathbf{R}}_+; \mathscr{Z})$, we shall call a function $z \in C^1(\overline{\mathbf{R}}_+; \mathscr{Z}) \cap C(\mathbf{R}_+; D_A)$, such that there exists $\int_a^b \omega(\alpha) D_t^\alpha z(t) d\alpha \in C(\mathbf{R}_+; \mathscr{Z})$ and equalities (13) and (19) are valid.

As before, denote

$$Z(t) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \left(W_b^c(\lambda) I - A\right)^{-1} d\lambda. \tag{20}$$

Lemma 2 *Let* $c \in (1, 2)$*,* $A \in \mathscr{A}^c(\theta_0, a_0)$*,* $W_b^c(\lambda)$ *and* $W_{b_1}^c(\lambda)$ *be holomorphic functions on* S_{θ_1, a_1} *with some* $\theta_1 \in (\pi/2, \theta_0]$*,* $a_1 \geq a_0$*, satisfying* (15)–(17)*,* $g \in C(\overline{\mathbf{R}}_+; D_A) \cap E(D_A)$*. Then the function*

$$z_g(t) = \int_0^t Z(t-s) g(s) ds$$

is a unique solution to the Cauchy problem $z(0) = z'(0) = 0$ *for Eq.* (19) *in* $E(\mathscr{Z})$*.*

Proof The integrals

$$Z^{(k)}(t) := \frac{1}{2\pi i} \int_{\Gamma} \lambda^k e^{\lambda t} \left(W_b^c(\lambda)I - A\right)^{-1} d\lambda, \quad k = 0, 1,$$

can be holomorphically extended onto Σ_{θ_1}. A more difficult question is the behavior of this functions at zero. Let us consider it.

For $t \in [0, 1]$ we have

$$\|Z(t)\|_{\mathscr{L}(\mathscr{Z})} \leq C \int_{\Gamma} \frac{|d\lambda|}{|\lambda|^{1+\varepsilon}}.$$

Hence, integral (20) converges uniformly with respect to $t \in [0, 1]$, and there exists the limit

$$\lim_{t \to 0+} Z(t) = \frac{1}{2\pi i} \int_{\Gamma} \left(W_b^c(\lambda)I - A\right)^{-1} d\lambda = Z(0) = 0,$$

since $1+\varepsilon > 1$ (see the proof of Theorem 3). Reasoning as in the proof of Lemma 1, we can show, that $\|Z'(t)\|_{\mathscr{L}(\mathscr{Z})} = O(t^{\varepsilon-1})$ as $t \to 0+$. Consequently,

$$z_g'(t) = 0 + \int_0^t Z'(t-s)g(s)ds,$$

$\|z_g'(t)\|_{\mathscr{Z}} \leq Ct^{\varepsilon} \to 0$ as $t \to 0+$. Thus, zero initial conditions (13) are fulfilled.

The remaining part of the proof does not differ from the same in the proof of Lemma 1. □

Theorem 4 *Let $c \in (1, 2)$, $A \in \mathscr{A}^c(\theta_0, a_0)$, $W_b^c(\lambda)$ and $W_{b_1}^c(\lambda)$ be holomorphic functions on S_{θ_1,a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, satisfying* (15)–(17)*, $g \in C(\overline{\mathbf{R}}_+; D_A) \cap E(D_A)$, $z_0, z_1 \in D_A$. Then the function*

$$z(t) = Z_0(t)z_0 + Z_1(t)z_1 + \int_0^t Z(t-s)g(s)ds$$

is a unique solution to problem (13), (19) *in $E(\mathscr{Z})$.*

4 A Class of Initial Boundary Value Problems

Let $P_n(\lambda) = \sum\limits_{i=0}^{n} c_i\lambda^i$, $Q_p(\lambda) = \sum\limits_{j=0}^{p} d_j\lambda^j$, $c_i, d_j \in \mathbf{R}$, $i = 0, 1, \dots, n$, $j = 0, 1, \dots, p$, $c_n \neq 0$, $d_p \neq 0$, $n < p$. Let $\Omega \subset \mathbf{R}^d$ be a bounded region with a smooth boundary $\partial\Omega$, operators pencil $\Lambda, B_1, B_2, \dots, B_r$ be regularly elliptic [34], where

$$(\Lambda u)(s) = \sum_{|q|\le 2r} a_q(s)D_s^q u(s), \quad a_q \in C^\infty(\overline{\Omega}),$$

$$(B_l u)(s) = \sum_{|q|\le r_l} b_{lq}(s)D_s^q u(s), \quad b_{lq} \in C^\infty(\partial\Omega),\ l = 1, 2, \dots, r,$$

$D_s^q = D_{s_1}^{q_1} D_{s_2}^{q_2} \dots D_{s_d}^{q_d}$, $D_{s_i}^{q_i} = \partial^{q_i}/\partial s_i^{q_i}$, $i = 1, 2, \dots, d$, $q = (q_1, q_2, \dots, q_d) \in \mathbf{N}_0^d$. Define the operator $\Lambda_1 \in \mathscr{C}l(L_2(\Omega))$ with domain $D_{\Lambda_1} = H^{2r}_{\{B_l\}}(\Omega)$ [34] by the equality $\Lambda_1 u = \Lambda u$. Let Λ_1 be self-adjoint operator and it has a bounded from the right-hand side spectrum. Then the spectrum $\sigma(\Lambda_1)$ of the operator Λ_1 is real, discrete and condensed at $-\infty$. Let $0 \notin \sigma(\Lambda_1)$, $\{\varphi_k : k \in \mathbf{N}\}$ is an orthonormal in $L_2(\Omega)$ system of the operator Λ_1 eigenfunctions, numbered in according to nonincreasing of the corresponding eigenvalues $\{\lambda_k : k \in \mathbf{N}\}$, taking into account their multiplicity.

Consider the initial-boundary value problem

$$u(s, 0) = u_0(s), \ \frac{\partial u}{\partial t}(s, 0) = u_1(s), \quad s \in \Omega, \tag{21}$$

$$B_l\Lambda^k u(s, t) = 0, \quad k = 0, 1, \dots, p-1, \quad l = 1, 2, \dots, r, \quad (s, t) \in \partial\Omega \times \mathbf{R}_+, \tag{22}$$

$$\int_b^c \omega(\alpha)D_t^\alpha P_n(\Lambda)u(s, t)d\alpha = Q_p(\Lambda)u(s, t) + f(s, t), \quad (s, t) \in \Omega \times \mathbf{R}_+, \tag{23}$$

where D_t^α is the Gerasimov–Caputo fractional derivative, $c \in (1, 2)$, $b \in [0, c)$, $\omega : (b, c) \to \mathbf{R}$, $f : \Omega \times \overline{\mathbf{R}}_+ \to \mathbf{R}$.

Set

$$\mathscr{Z} = \{v \in H^{2rn}(\Omega) : \ B_l\Lambda^k v(x) = 0,\ k = 0, 1, \dots, n-1,\ l = 1, 2, \dots, r,\ x \in \partial\Omega\}.$$

Under the condition of the existence of the inverse operator

$$[P_n(\Lambda_1)]^{-1} : L_2(\Omega) \to \mathscr{Z},$$

define in the Banach space $\mathscr{Z}$ the operator $Az = [P_n(\Lambda_1)]^{-1} Q_p(\Lambda)z$ with domain

$$D_A = \{v \in H^{2rp}(\Omega) : B_l \Lambda^k v(x) = 0, k = 0, 1, \dots, p-1,\ l = 1, 2, \dots, r,\ x \in \partial\Omega\}.$$

Theorem 5 ([23]) *Let $p > n$, $(-1)^{p-n}(d_p/c_n) < 0$, the spectrum $\sigma(\Lambda_1)$ be bounded from the right-hand side, do not contain zeros of the polynomial $P_n(\lambda)$, $0 \notin \sigma(\Lambda_1)$. Then for $\alpha \in [1, 2)$ there exist $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, such that $A \in \mathscr{A}^\alpha(\theta_0, a_0)$. If, moreover, $\max\limits_{k\in\mathbf{N}}\{Q_p(\lambda_k)/P_n(\lambda_k)\} < 1$, then $A \in \mathscr{A}^\alpha(\theta_0, a_0)$ at $\alpha \in (0, 1)$. Furthermore, for every $\alpha \in (0, 2)$ $\sigma(A) = \{\mu \in \mathbf{C} : \mu = Q_p(\lambda_k)/P_n(\lambda_k)\}$.*

Theorem 6 *Let $p > n$, $(-1)^{p-n}(d_p/c_n) < 0$, the spectrum $\sigma(\Lambda_1)$ be bounded from the right-hand side, do not contain zeros of the polynomial $P_n(\lambda)$, $0 \notin \sigma(\Lambda_1)$, $c \in (1, 2)$, $W_b^c(\lambda)$ and $W_{b_1}^c(\lambda)$ be holomorphic functions on S_{θ_1,a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, satisfying (15)–(17), $f \in C(\overline{\mathbf{R}}_+; D_A) \cap E(D_A)$. Then for all $u_0, u_1 \in D_A$ there exists a unique solution of problem (21)–(23) in $E(\mathscr{Z})$.*

Proof By the choosing of the space $\mathscr{Z}$ and the operator A problem (21)–(23) is reduced to problem (13), (19). Theorems 5 and 4 imply the required. □

For $c \in (0, 1]$ in (23) we consider the problem with the initial condition

$$u(s, 0) = u_0(s),\ \ s \in \Omega. \tag{24}$$

By the analogous way with the obvious changing due to Theorems 5 and 2 we obtain the corresponding unique solvability theorem.

Theorem 7 *Let $p > n$, $(-1)^{p-n}(d_p/c_n) < 0$, the spectrum $\sigma(\Lambda_1)$ be bounded from the right-hand side, do not contain zeros of the polynomial $P_n(\lambda)$, $0 \notin \sigma(\Lambda_1)$, $\max\limits_{k\in\mathbf{N}}\{Q_p(\lambda_k)/P_n(\lambda_k)\} < 1$, $c \in (0, 1]$, $W_b^c(\lambda)$ be holomorphic function on S_{θ_1,a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, satisfying (7), (8), $f \in C(\overline{\mathbf{R}}_+; D_A) \cap E(D_A)$. Then for all $u_0 \in D_A$ there exists a unique solution of problem (22)–(24) in $E(\mathscr{Z})$.*

At $n = 0$, $P_0(\lambda) = 1$, $p = 1$, $Q_1(\lambda) = \lambda$ Theorems 6 and 7 (if $\max\limits_{k\in\mathbf{N}} \lambda_k < 1$) imply the unique solvability of the initial-boundary value problems

$$\int\limits_b^c \omega(\alpha) D_t^\alpha u(s, t) d\alpha = \Lambda u(s, t) + f(s, t), \quad (s, t) \in \Omega \times \mathbf{R}_+, \tag{25}$$

$$B_l u(s, t) = 0, \quad l = 1, 2, \dots, r, \quad (s, t) \in \partial\Omega \times \mathbf{R}_+,$$

with the initial conditions (21) or (24) at $c \in (1, 2)$, or $c \in (0, 1]$ respectively. If here $r = 1$, $\Lambda = \Delta = \sum\limits_{i=1}^{s} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and, for example, $B_1 = I$, then $\max\limits_{k\in\mathbf{N}} \lambda_k < 0$, and (25) is the ultraslow diffusion equation with the Dirichlet boundary condition.

For $P_1(\lambda) = 2 + \lambda$, $Q_2(\lambda) = \lambda + \lambda^2$, $d = 1$, $\Omega = (0, \pi)$, $r = 1$, $A = \frac{\partial^2}{\partial s^2}$, $B_1 = I$ we obtain $\lambda_k = -k^2$, $\varphi_k(s) = \sin ks$, $k \in \mathbf{N}$. Then (22), (23) at $f \equiv 0$ has the form

$$\int_b^c \omega(\alpha) D_t^\alpha \left(2 + \frac{\partial^2}{\partial s^2}\right) u(s,t) d\alpha = \frac{\partial^2 u}{\partial s^2}(s,t) + \frac{\partial^4 u}{\partial s^4}(s,t), \quad (s,t) \in (0,\pi) \times \mathbf{R}_+,$$

$$u(0,t) = u(\pi,t) = \frac{\partial^2 u}{\partial s^2}(0,t) = \frac{\partial^2 u}{\partial s^2}(\pi,t) = 0, \quad t \in \mathbf{R}_+.$$

It is evident, that

$$\max_{k\in\mathbf{N}} \frac{Q_2(\lambda_k)}{P_1(\lambda_k)} = \max_{k\in\mathbf{N}} \frac{k^4 - k^2}{2 - k^2} = 0 < 1.$$

5 Degenerate Distributed Order Equation

Let us consider initial problems for equations in Banach spaces with a degenerate linear operator at the distributed order derivative.

5.1 *The Case* $c \in (0, 1]$

Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces, $\mathscr{L}(\mathscr{X};\mathscr{Y})$ be the Banach space of linear continuous operators, acting from $\mathscr{X}$ into $\mathscr{Y}$, $\mathscr{C}l(\mathscr{X};\mathscr{Y})$ be the set of all linear closed densely defined in the space $\mathscr{X}$ operators, acting into $\mathscr{Y}$, $\mathscr{L}(\mathscr{X};\mathscr{X}) := \mathscr{L}(\mathscr{X})$, $\mathscr{C}l(\mathscr{X};\mathscr{X}) := \mathscr{C}l(\mathscr{X})$.

Let $L, M \in \mathscr{C}l(\mathscr{X};\mathscr{Y})$ have domains D_L, D_M, $\ker L \neq \{0\}$. Since L and M are closed operators, we can consider D_L and D_M as the Banach spaces with the graph norms of the operator L and M respectively. Let us consider the distributed order equation

$$\int_b^c \omega(\alpha) D_t^\alpha Lx(t) d\alpha = Mx(t) + f(t), \quad t > 0, \tag{26}$$

where D_t^α is the Gerasimov–Caputo fractional derivative, $0 \le b < c \le 1$, $\omega : (b,c) \to \mathbf{C}$, $f \in C(\overline{\mathbf{R}}_+;\mathscr{Y})$. Equation (26) is called degenerate, because it is supposed that $\ker L \neq \{0\}$.

A function $x : \mathbf{R}_+ \to D_L \cap D_M$ is called a solution of Eq. (26), if $Mx \in C(\mathbf{R}_+; \mathscr{Y})$, there exists $\int\limits_a^b \omega(\alpha) D_t^\alpha Lx(t) d\alpha \in C(\mathbf{R}_+; \mathscr{Y})$ and equality (26) is valid. A solution x of (26) is called a solution to the Cauchy problem

$$x(0) = x_0 \tag{27}$$

for Eq. (26), if $x \in C(\overline{\mathbf{R}}_+; \mathscr{X})$ satisfies condition (27).

By $\rho^L(M)$ the set of $\mu \in \mathbf{C}$ is denoted, for which the mapping

$$\mu L - M : D_L \cap D_M \to \mathscr{Y}$$

is injective, and $R^L_\mu(M) := (\mu L - M)^{-1} L \in \mathscr{L}(\mathscr{X})$, $L^L_\mu(M) := L(\mu L - M)^{-1} \in \mathscr{L}(\mathscr{Y})$.

Definition 1 ([22]) Let $L, M \in \mathscr{C}l(\mathscr{X}; \mathscr{Y})$. A pair of operators (L, M) belongs to the class $\mathscr{H}_\alpha(\theta_0, a_0)$, if the following two conditions are valid:

1. there exist $\theta_0 \in (\pi/2, \pi)$ and $a_0 \geq 0$, such that for all $\lambda \in S_{\theta_0, a_0}$ we have $\lambda^\alpha \in \rho^L(M)$;
2. for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists a constant $K = K(\theta, a) > 0$, such that for all $\mu \in S_{\theta, a}$

$$\max\{\|R^L_{\mu^\alpha}(M)\|_{\mathscr{L}(\mathscr{X})}, \|L^L_{\mu^\alpha}(M)\|_{\mathscr{L}(\mathscr{Y})}\} \leq \frac{K(\theta, a)}{|\mu^{\alpha-1}(\mu - a)|}.$$

Remark 6 If there exists the operator $L^{-1} \in \mathscr{L}(\mathscr{Y}; \mathscr{X})$ and $\alpha \in (0, 2)$, then $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$, when and only when $L^{-1}M \in \mathscr{A}^\alpha(\theta_0, a_0)$ and $ML^{-1} \in \mathscr{A}^\alpha(\theta_0, a_0)$.

It is easy to show, that the subspaces $\ker R^L_\mu(M) = \ker L$, $\operatorname{im} R^L_\mu(M)$, $\ker L^L_\mu(M)$, $\operatorname{im} L^L_\mu(M)$ do not depend on the parameter $\mu \in \rho^L(M)$. Denote $\ker R^L_\mu(M) = \mathscr{X}^0$, $\ker L^L_\mu(M) = \mathscr{Y}^0$. By $\mathscr{X}^1$ ($\mathscr{Y}^1$) the closure of the subspace $\operatorname{im} R^L_\mu(M)$ ($\operatorname{im} L^L_\mu(M)$) in the norm of $\mathscr{X}$ ($\mathscr{Y}$) is denoted. By L_k (M_k) we denote the restriction L (M) on $D_{L_k} := D_L \cap \mathscr{X}^k$ ($D_{M_k} := D_M \cap \mathscr{X}^k$), $k = 0, 1$. Introduce also the denotations $S = L_1^{-1} M_1 : D_S \to \mathscr{X}^1$, $D_S = \{x \in D_{M_1} : M_1 x \in \operatorname{im} L_1\}$; $T = M_1 L_1^{-1} : D_T \to \mathscr{Y}^1$, $D_T = \{y \in \operatorname{im} L_1 : L_1^{-1} y \in D_{M_1}\}$.

We shall use the properties of the operators pairs from the class $\mathscr{H}_\alpha(\theta_0, a_0)$ in the case of reflexive Banach spaces $\mathscr{X}$ and $\mathscr{Y}$, which were proved in the work [22].

Theorem 8 ([22]) *Let Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ be reflexive, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$. Then*

1. $\mathscr{X} = \mathscr{X}^0 \oplus \mathscr{X}^1$, $\mathscr{Y} = \mathscr{Y}^0 \oplus \mathscr{Y}^1$;
2. *projection P (Q) on the subspace $\mathscr{X}^1$ ($\mathscr{Y}^1$) along $\mathscr{X}^0$ ($\mathscr{Y}^0$) has the form* $P = s\text{-}\lim\limits_{n\to\infty} nR^L_n(M)$ ($Q = s\text{-}\lim\limits_{n\to\infty} nL^L_n(M)$);

3. $L_0 = 0$, $M_0 \in \mathscr{C}l(\mathscr{X}^0; \mathscr{Y}^0)$, $L_1, M_1 \in \mathscr{C}l(\mathscr{X}^1; \mathscr{Y}^1)$;
4. *there exist inverse operators* $L_1^{-1} \in \mathscr{C}l(\mathscr{Y}^1; \mathscr{X}^1)$, $M_0^{-1} \in \mathscr{L}(\mathscr{Y}^0; \mathscr{X}^0)$;
5. $\forall x \in D_L$ $Px \in D_L$ *and* $LPx = QLx$;
6. $\forall x \in D_M$ $Px \in D_M$ *and* $MPx = QMx$;
7. D_S *is dense in the space* $\mathscr{X}$, D_T *is dense in* $\mathscr{Y}$;
8. *if* $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$, *or* $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$, *then the operator* $S \in \mathscr{C}l(\mathscr{X}^1)$, $S \in \mathscr{A}^\alpha(\theta_0, a_0)$;
9. *if* $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$, *or* $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$, *then* $T \in \mathscr{C}l(\mathscr{Y}^1)$, $T \in \mathscr{A}^\alpha(\theta_0, a_0)$.

As before, we define the contour $\Gamma = \Gamma_+ \cup \Gamma_-$ with $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, $\delta > 0$, where θ_0, a_0 are from Definition 1, the constants θ_1, a_1 are from the next theorem conditions, and operators

$$X_0(t) := \frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda t}}{\lambda} W_b^c(\lambda) R^L_{W_b^c(\lambda)}(M)d\lambda, \quad X(t) := \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} R^L_{W_b^c(\lambda)}(M)d\lambda.$$

Denote by $E(\mathscr{X}; P)$ the set of all functions $x : \overline{\mathbf{R}}_+ \to \mathscr{X}$, such that $Px \in E(\mathscr{X}^1)$.

Theorem 9 *Let Banach spaces* $\mathscr{X}$ *and* $\mathscr{Y}$ *be reflexive,* $c \in (0, 1]$, *a pair* $(L, M) \in \mathscr{H}_c(\theta_0, a_0)$, $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$ *or* $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$, $W_b^c(\lambda)$ *be holomorphic function on* S_{θ_1, a_1} *with some* $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, *satisfying the conditions* (7), (8), $f \in C(\overline{\mathbf{R}}_+; \mathscr{Y})$, $L_1^{-1}Qf \in C\big(\overline{\mathbf{R}}_+; D_S\big) \cap E(D_S)$, $x_0 \in \mathscr{X}$, *such that* $Px_0 \in D_S$, *and*

$$(I - P)x_0 = -M_0^{-1}(I - Q)f(0). \tag{28}$$

Then the function

$$x(t) = X_0(t)x_0 + \int_0^t X(t - s)L_1^{-1}Qf(s)ds - M_0^{-1}(I - Q)f(t) \tag{29}$$

is a unique solution to Cauchy problem (26), (27) *from the class* $E(\mathscr{X}; P)$.

Proof By means of Theorem 8 problem (26), (27) can be reduced to the two Cauchy problems

$$\int_b^c \omega(\alpha)D_t^\alpha v(t)d\alpha = Sv(t) + L_1^{-1}Qf(t), \quad t > 0, \tag{30}$$

$$v(0) = Px_0, \tag{31}$$

and

$$0 = w(t) + M_0^{-1}(I - Q)f(t), \quad t > 0, \tag{32}$$

$$w(0) = (I - P)x_0 \tag{33}$$

on the subspaces $\mathscr{X}^1$ and $\mathscr{X}^0$ respectively. Here $v(t) := Px(t)$, $w(t) := (I - P)x(t)$.

Equation (32) has the unique solution $w(t) = -M_0^{-1}(I - Q)f(t)$. Therefore, condition (33) is equivalent to (28).

Problem (30), (31) is uniquely solvable by Theorem 2, and its solution has the form

$$v(t) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} W_b^c(\lambda)(W_b^c(\lambda)I - S)^{-1}d\lambda Px_0 +$$

$$+\int\limits_0^t \frac{1}{2\pi i}\int\limits_{\Gamma} e^{\lambda(t-s)}(W_b^c(\lambda)I - S)^{-1}d\lambda L_1^{-1}Qf(s)ds =$$

$$= X_0(t)x_0 + \int\limits_0^t X(t-s)L_1^{-1}Qf(s)ds,$$

since $(I - P)x_0 \in \ker X_0(t)$, $t \geq 0$, and $(W_b^c(\lambda)I - S)^{-1} = (W_b^c(\lambda)L_1 - M_1)^{-1}L_1$. □

Theorem 10 *Let Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ be reflexive, $c \in (0, 1]$, a pair $(L, M) \in \mathscr{H}_c(a_0, \theta_0)$, $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$ or $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$, $W_b^c(\lambda)$ be holomorphic function on S_{θ_1,a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, satisfying the conditions (7), (8), $f \in C(\overline{\mathbf{R}}_+; \mathscr{Y})$, $Qf \in C\left(\overline{\mathbf{R}}_+; D_T\right) \cap E(D_T)$, $x_0 \in D_M$, such that condition (28) be fulfilled. Then function (29) is a unique solution to Cauchy problem (26), (27) from the class $E(\mathscr{X}; P)$.*

Proof In this case instead of (30) obtain the equation

$$\int\limits_b^c \omega(\alpha)D_t^\alpha y(t)d\alpha = Ty(t) + Qf(t), \tag{34}$$

where $y(t) = L_1Px(t)$. By Theorem 8 if $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$, or $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$, then $T \in \mathscr{A}^\alpha(a_0, \theta_0)$, hence there exists a unique solution of the

Cauchy problem $y(0) = L_1 P x_0 \in D_T$ to Eq. (34). It has the form

$$y(t) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{e^{\lambda t}}{\lambda} W_b^c(\lambda)(W_b^c(\lambda) I - T)^{-1} d\lambda L_1 P x_0 +$$
$$+ \int\limits_0^t \frac{1}{2\pi i} \int\limits_{\Gamma} e^{\lambda(t-s)} (W_b^c(\lambda) I - T)^{-1} d\lambda Q f(s) ds.$$

Consequently,

$$Px(t) = L_1^{-1} y(t) = X_0(t) x_0 + \int\limits_0^t X(t-s) L_1^{-1} Q f(s) ds,$$

since $(W_b^c(\lambda) I - T)^{-1} = L_1 (W_b^c(\lambda) L_1 - M_1)^{-1}$ and the operator L_1 is closed. Finally, $x(t) = Px(t) - M_0^{-1}(I - Q) f(t)$ has form (29). □

Consider the Showalter–Sidorov problem

$$(Lx)(0) = y_0 \tag{35}$$

for Eq. (26). The solution of problem (26), (35) is a solution x of Eq. (26), such that $Lx \in C(\overline{\mathbf{R}}_+; \mathscr{X})$ satisfies condition (35).

Theorem 11 *Let Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ be reflexive, $c \in (0, 1]$, a pair $(L, M) \in \mathscr{H}_c(a_0, \theta_0)$, $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$ or $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$, $W_b^c(\lambda)$ be holomorphic function on S_{θ_1, a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \ge a_0$, satisfying conditions (7), (8), $f \in C(\overline{\mathbf{R}}_+; \mathscr{Y})$, $L_1^{-1} Q f \in C\big(\overline{\mathbf{R}}_+; D_S\big) \cap E(D_S)$, $x_0 \in D_M$, such that $P x_0 \in D_S$. Then function (29) is a unique solution to problem (26), (35) in $E(\mathscr{X}; P)$.*

Theorem 12 *Let Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ be reflexive, $c \in (0, 1]$, a pair $(L, M) \in \mathscr{H}_c(a_0, \theta_0)$, $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$ or $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$, $W_b^c(\lambda)$ be holomorphic function on $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \ge a_0$, satisfying conditions (7), (8), $f \in C(\overline{\mathbf{R}}_+; \mathscr{Y})$, $Qf \in C\big(\overline{\mathbf{R}}_+; D_T\big) \cap E(D_T)$, $x_0 \in D_M$. Then function (29) is a unique solution to problem (26), (35) in $E(\mathscr{X}; P)$.*

Proof Under the conditions of Theorem 11 (or Theorem 12) equations system (30), (32) (or (34), (32) respectively) has initial condition only for (30) (or (34)). Equation (32) is uniquely solvable. Reasoning as in the proof of Theorems 9 and 10 we shall obtain the required. □

5.2 The Case $c \in (1, 2)$

A function $x : \mathbf{R}_+ \to D_L \cap D_M$ is called a solution of the degenerate equation

$$\int_b^c \omega(\alpha) D_t^\alpha Lx(t)d\alpha = Mx(t) + f(t), \quad t > 0, \tag{36}$$

$c \in (1, 2)$, $b \in [0, c)$, $\omega : (b, c) \to \mathbf{C}$, $f \in C(\overline{\mathbf{R}}_+; \mathscr{Y})$, if $Mx \in C(\mathbf{R}_+; \mathscr{Y})$, there exists $\int_a^b \omega(\alpha) D_t^\alpha Lx(t)d\alpha \in C(\mathbf{R}_+; \mathscr{Y})$ and equality (36) is valid. A solution x of (36) is called a solution to the Cauchy problem

$$x(0) = x_0, \quad x'(0) = x_1 \tag{37}$$

for Eq. (36), if $x \in C^1(\overline{\mathbf{R}}_+; \mathscr{X})$ satisfies condition (37).

Let the contour $\Gamma = \Gamma_+ \cup \Gamma_-$ be the same, as before, $b_1 := \max\{b, 1\}$,

$$X_0(t) := \frac{1}{2\pi i} \int_\Gamma \frac{e^{\lambda t}}{\lambda} W_b^c(\lambda) R^L_{W_b^c(\lambda)}(M)d\lambda, \quad X_1(t) := \frac{1}{2\pi i} \int_\Gamma \frac{e^{\lambda t}}{\lambda^2} W_{b_1}^c R^L_{W_b^c(\lambda)}(M)d\lambda,$$

$$X(t) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R^L_{W_b^c(\lambda)}(M)d\lambda.$$

Theorem 13 *Let Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ be reflexive, $c \in (1, 2)$, a pair $(L, M) \in \mathscr{H}_c(a_0, \theta_0)$, $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$ or $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$, $W_b^c(\lambda)$ and $W_{b_1}^c(\lambda)$ be holomorphic functions on S_{θ_1, a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, satisfying conditions (15)–(17), $(I - Q)f \in C^1(\overline{\mathbf{R}}_+; \mathscr{Y})$, $L_1^{-1}Qf \in C\big(\overline{\mathbf{R}}_+; D_S\big) \cap E(D_S)$, $x_0, x_1 \in \mathscr{X}$, such that $Px_0, Px_1 \in D_S$, and*

$$(I - P)x_0 = -M_0^{-1}(I - Q)f(0), \quad (I - P)x_1 = -M_0^{-1}((I - Q)f)'(0). \tag{38}$$

Then the function

$$x(t) = X_0(t)x_0 + X_1(t)x_1 + \int_0^t X(t-s)L_1^{-1}Qf(s)ds - M_0^{-1}(I - Q)f(t) \tag{39}$$

is a unique solution to the Cauchy problem (36), (37) from the class $E(\mathscr{X}; P)$.

Proof As in the proof of Theorem 9 we can reduce problem (36), (37) to the two Cauchy problems (30), (31), and (32), (33) on the subspaces $\mathscr{X}^1$ and $\mathscr{X}^0$ respectively. Instead of Theorem 2, we need to apply Theorem 4. □

Theorem 14 *Let Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ be reflexive, $c \in (1, 2)$, a pair $(L, M) \in \mathscr{H}_c(a_0, \theta_0)$, $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$ or $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$, $W_b^c(\lambda)$ and $W_{b_1}^c(\lambda)$ be holomorphic functions on S_{θ_1,a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, satisfying conditions* (15)–(17)*, $(I - Q)f \in C^1(\overline{\mathbf{R}}_+; \mathscr{Y})$, $Qf \in C\big(\overline{\mathbf{R}}_+; D_T\big) \cap E(D_T)$, $x_0, x_1 \in D_M$, such that conditions* (38) *be fulfilled. Then function* (39) *is a unique solution to Cauchy problem* (36), (37) *from the class $E(\mathscr{X}; P)$.*

Proof This statement can be proved similar to Theorem 10, but using Theorem 4 instead of Theorem 2. Note that, as in the proof of Theorem 10, for $x_k \in D_M$ we have $L_1 P x_k \in D_T$, $k = 0, 1$. □

The solution of the Showalter–Sidorov problem

$$(Lx)(0) = y_0, \quad (Lx)'(0) = y_1 \tag{40}$$

for Eq. (36) is a solution x of Eq. (36), such that $Lx \in C^1(\overline{\mathbf{R}}_+; \mathscr{X})$ satisfies conditions (40). As in the previous subsection, it is not difficult to obtain the next two assertions.

Theorem 15 *Let Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ be reflexive, $c \in (1, 2)$, $(L, M) \in \mathscr{H}_c(a_0, \theta_0)$, $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$ or $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$, $W_b^c(\lambda)$ and $W_{b_1}^c(\lambda)$ be holomorphic functions on S_{θ_1,a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, satisfying conditions* (15)–(17)*, $f \in C^1(\overline{\mathbf{R}}_+; \mathscr{Y})$, $L_1^{-1}Qf \in C\big(\overline{\mathbf{R}}_+; D_S\big) \cap E(D_S)$, $x_0, x_1 \in D_M$, such that $Px_0, Px_1 \in D_S$. Then function* (39) *is a unique solution to problem* (36), (40) *in $E(\mathscr{X}; P)$.*

Theorem 16 *Let Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ be reflexive, $c \in (1, 2)$, $(L, M) \in \mathscr{H}_c(a_0, \theta_0)$, $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$ or $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$, $W_b^c(\lambda)$ and $W_{b_1}^c(\lambda)$ be holomorphic functions on S_{θ_1,a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, satisfying conditions* (15)–(17)*, $f \in C^1(\overline{\mathbf{R}}_+; \mathscr{Y})$, $Qf \in C\big(\overline{\mathbf{R}}_+; D_T\big) \cap E(D_T)$, $x_0, x_1 \in D_M$. Then function* (39) *is a unique solution to problem* (36), (40) *in $E(\mathscr{X}; P)$.*

6 Applications to Boundary Value Problems

Consider the initial-boundary value problem

$$u(s, 0) = u_0(s), \ \frac{\partial u}{\partial t}(s, 0) = u_1(s), \quad s \in \Omega, \tag{41}$$

$$B_l \Lambda^k u(s, t) = 0, \quad k = 0, 1, \ldots, p-1, \quad l = 1, 2, \ldots, r, \quad (s, t) \in \partial\Omega \times \mathbf{R}_+, \tag{42}$$

$$\int_b^c \omega(\alpha) D_t^\alpha P_n(\Lambda) u(s, t) d\alpha = Q_p(\Lambda) u(s, t) + f(s, t), \quad (s, t) \in \Omega \times \mathbf{R}_+, \tag{43}$$

from Sect. 4. Recall that $n < p$. In contrast to that situation we suppose, that P_n has zeros among $\{\lambda_k\} = \sigma(\Lambda_1)$. Now we set

$$\mathscr{X} = \{u \in H^{2rn}(\Omega) : B_l \Lambda^k u(s) = 0,\ k = 0, 1, \dots, n-1,\ l = 1, 2, \dots, r,\ x \in \partial\Omega\}, \tag{44}$$

$$D_M = \{u \in H^{2rp}(\Omega) : B_l \Lambda^k u(s) = 0,\ k = 0, 1, \dots, p-1,\ l = 1, 2, \dots, r,\ x \in \partial\Omega\}, \tag{45}$$

$$\mathscr{Y} = L_2(\Omega), \quad L = P_n(\Lambda), \quad M = Q_p(\Lambda). \tag{46}$$

Then $L \in \mathscr{L}(\mathscr{X};\mathscr{Y})$, $M \in \mathscr{C}l(\mathscr{X};\mathscr{Y})$ and problem (41)–(43) is presented in form (36), (37).

Theorem 17 ([22]) *Let the spaces and the operators have forms* (44)–(46)*, the spectrum $\sigma(\Lambda_1)$ do not contain common zeros of the polynomials $P_n(\lambda)$ and $Q_p(\lambda)$, $0 \notin \sigma(\Lambda_1)$. Then the operator $L_1 : \mathscr{X}^1 \to \mathscr{Y}^1$ is the homeomorphism, for $\alpha \in [1, 2)$ there exist $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, such that $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$. If moreover*

$$\max_{P_n(\lambda_k)\neq 0} \frac{Q_p(\lambda_k)}{P_n(\lambda_k)} < 1,$$

then at $\alpha \in (0, 1)$ $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$.

Remark 7 Under assumptions of Theorem 17 we have

$$\sigma^L(M) = \left\{\mu \in \mathbf{C} : \mu = \frac{Q_p(\lambda_k)}{P_n(\lambda_k)},\ P_n(\lambda_k) \neq 0\right\},$$

$\mathscr{X}^0 = \mathscr{Y}^0 = \text{span}\{\varphi_k : P_n(\lambda_k) = 0\}$; $\mathscr{X}^1$ is the closure of $\text{span}\{\varphi_k : P_n(\lambda_k) \neq 0\}$ in the norm of the space $\mathscr{X}$; $\mathscr{Y}^1$ is the closure of the same set in $L_2(\Omega)$.

Theorem 18 *Let $\sigma(\Lambda_1)$ do not contain common zeros of the polynomials $P_n(\lambda)$ and $Q_p(\lambda)$, $0 \notin \sigma(\Lambda_1)$, $c \in (1, 2)$, $W_b^c(\lambda)$, $W_{b_1}^c(\lambda)$ be holomorphic functions on S_{θ_1,a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, satisfying conditions* (15)–(17)*, a function $f : \overline{\mathbf{R}}_+ \to L_2(\Omega)$ be such that $\langle f, \varphi_l\rangle_{L_2(\Omega)} \in C^1(\overline{\mathbf{R}}_+; \mathbf{R})$, if $P_n(\lambda_l) = 0$;*

$$\sum_{P_n(\lambda_k)\neq 0} \frac{\langle f, \varphi_k\rangle_{L_2(\Omega)}}{P_n(\lambda_k)}\varphi_k \in C(\overline{\mathbf{R}}_+; D_M), \tag{47}$$

$u_0, u_1 \in D_M$; if $P_n(\lambda_l) = 0$, then

$$Q_p(\lambda_l)\,\langle u_k, \varphi_l\rangle_{L_2(\Omega)} = -D_t^k|_{t=0}\,\langle f(\cdot, t), \varphi_l\rangle_{L_2(\Omega)}\,,\ k = 0, 1. \tag{48}$$

Then there exists a unique solution of problem (41)–(43) *from the class $E(\mathscr{X}; P)$.*

Proof Due to Theorem 17 $(L, M) \in \mathcal{H}_\alpha(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, $L_1 \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y}^1)$, $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$. Therefore, $D_S = D_{M_1}$. Conditions (48) mean (38) for this case. It remains to apply Theorem 13. □

Consider the Showalter–Sidorov initial conditions

$$P_n(\Lambda)u(s, 0) = y_0(s), \quad \frac{\partial P_n(\Lambda)u}{\partial t}(s, 0) = y_1(s), \quad s \in \Omega. \tag{49}$$

The respective unique solvability statement need not the matching condition (48).

Theorem 19 *Let $\sigma(\Lambda_1)$ do not contain common zeros of the polynomials $P_n(\lambda)$ and $Q_p(\lambda)$, $0 \notin \sigma(\Lambda_1)$, $c \in (1, 2)$, $W_b^c(\lambda)$, $W_{b_1}^c(\lambda)$ be holomorphic functions on S_{θ_1,a_1} with some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$, satisfying conditions (15)–(17), a function $f : \overline{\mathbf{R}}_+ \to L_2(\Omega)$ be such that $\langle f, \varphi_l \rangle_{L_2(\Omega)} \in C^1(\overline{\mathbf{R}}_+; \mathbf{R})$, if $P_n(\lambda_l) = 0$, and condition (47) be satisfied, $y_0 = P_n(\Lambda)u_0$, $y_1 = P_n(\Lambda)u_1$ for some $u_0, u_1 \in D_M$. Then there exists a unique solution of problem (42), (43), (49) from the class $E(\mathcal{X}; P)$.*

Let $n = 1$, $P_1(\lambda) = 1 + \lambda$, $p = 2$, $Q_2(\lambda) = \lambda + 2\lambda^2$, $\Omega = (0, \pi)$, $d = 1$. Then $\lambda_k = -k^2$, $\varphi_k(s) = \sin ks$, $k \in \mathbf{N}$, problem (41)–(43) has the form

$$\int \omega(\alpha) D_t^\alpha (u + u_{ss})\, d\alpha = u_{ss} + 2u_{ssss}, \quad (s, t) \in (0, \pi) \times \mathbf{R}_+,$$

$$u(0, t) = u(\pi, t) = u_{ss}(0, t) = u_{ss}(\pi, t) = 0, \quad t \in \mathbf{R}_+,$$

$$u(s, 0) = u_0(s), \quad u_t(s, 0) = u_1(s), \quad s \in (0, \pi).$$

The eigenvalue $\lambda_1 = -1$ is a zero of P_1, hence the equation is degenerate. Note that there are no common zeros of P_1 and Q_2.

Acknowledgements The work is supported by Act 211 of Government of the Russian Federation, contract 02.A03.21.0011, and by the Ministry of Education and Science of the Russian Federation, task No 1.6462.2017/BCh.

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Transformation Operators for Fractional Order Ordinary Differential Equations and Their Applications

Mark M. Malamud

Abstract The survey is concerned with triangular transformation operators for fractional order $\alpha = n - \varepsilon$ ordinary differential equations. We discuss the existence of transformation operators in the case of holomorphic coefficients. Similarity between such operators and the simplest fractional differentiation D_0^α is discussed too.

Applications to the unique determination of the operator from n spectra of boundary value problems are given. Applications to the completeness property of certain boundary value problems for such equations are considered.

1 Introduction

The paper continues the previous review [35] and is devoted to the following fractional order ordinary differential equation

$$l_{n-\varepsilon}(D)f := D^{n-\varepsilon} f + \sum_{j=1}^{n-1} q_j(x) D^{n-\varepsilon-j-1} f + \int_0^x M(x,t)(J^\varepsilon f)(t)\, dt = \lambda f. \tag{1}$$

Here $D^{k-\varepsilon}$ denotes fractional order differentiation,

$$D^{k-\varepsilon} f(x) = f^{(k-\varepsilon)}(x) = (D^k J^\varepsilon f)(x) = \frac{d^k}{dx^k} J^\varepsilon f, \qquad k \in \mathbb{N} \cup 0, \quad \varepsilon \in [0,1), \tag{2}$$

M. M. Malamud (✉)
Peoples' Friendship University of Russia (RUDN University), Moscow, Russian Federation
e-mail: malamud-mm@rudn.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*,
Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_24

and J^α denotes the Riemann-Liouville fractional integration:

$$J^\alpha f = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \qquad \alpha \in \mathbb{R}_+ = (0,\infty), \quad \text{and} \quad J^0 := I, \tag{3}$$

where I denotes the identical operator in $L^p[0,b]$. The operator J^α is well defined on $L^p[0,b]$ for each $p \in [1,\infty]$ and $b > 0$. Moreover, it is a Volterra operator, i.e. it is compact operator with zero spectrum, $\sigma(J^\alpha) = \{0\}$. It is well known also (see e.g. [6, 8, 11, 21, 44, 51]) that the family $\{J^\alpha\}_{\alpha\in\mathbb{R}_+}$ forms a continuous semigroup, $J^\alpha J^\beta = J^{\alpha+\beta}$. Moreover, $s - \lim_{\varepsilon\downarrow} J^\varepsilon = I$.

The operator $l_{n-\varepsilon}(D)$ is well defined on the Sobolev space $W_1^{n-\varepsilon}[0,b]$ (see Sect. 2 for definition).

Denote by $A(D)$ the class of operators (1) with coefficients q_j and kernels $M(x,t)$ being restrictions of entire functions in one and two variables, respectively.

With this notation one of our main results (in a weak form) reads as follows.

Theorem 1 *Let $l_{n-\varepsilon}(D) \in A(D)$ and let $y(x,\lambda)$ be the solution of Eq.* (1) *satisfying the initial conditions*

$$y^{(j-\varepsilon-1)}(0,\lambda) = h_j, \quad j \in \{1,\dots,n\}. \tag{4}$$

Then there exists a unique kernel $R(x,t)$ analytic in both variables in Ω

$$y(x,\lambda) = (I+R)w(x,\lambda) := w(x,\lambda) + \int_0^x R(x,t)w(t,\lambda)\, dt \tag{5}$$

in which $w(x,\lambda)$ is a solution of the Cauchy problem for the simplest fractional order equation

$$D_x^{n-\varepsilon} w(x,\lambda) = \lambda w(x,\lambda) \tag{6}$$

satisfying the same initial conditions (4).

To prove this result we first investigate the operators $l^0_{n-\varepsilon}(D)$ being restrictions of the operators $l_{n-\varepsilon}(D)$ to the subspace $W^{n-\varepsilon}_{1,0}[0,b]$ of functions from $W_1^{n-\varepsilon}[0,b]$ vanishing at zero. Namely, we first show (and this result is a special case of Theorem 1) the similarity in $L^2[0,b]$ between operators $l^0_{n-\varepsilon}(D)$ and $D_0^{n-\varepsilon} := D^{n-\varepsilon} \upharpoonright W^{n-\varepsilon}_{1,0}[0,b]$:

$$l^0_{n-\varepsilon}(D) = (I+R)^{-1} D_0^{n-\varepsilon}(I+R) \tag{7}$$

where $I+R$ is given by (5). This result allows one to define functional calculus for the operator $l^0_{n-\varepsilon}(D)$ by setting $\varphi(l^0_{n-\varepsilon}(D)) := (I+R)^{-1}\varphi(D_0^{n-\varepsilon})(I+R)$ where a function φ is such that $\varphi(D_0^{n-\varepsilon})$ is well defined. This functional calculus for $l_{n-\varepsilon}^0(D)$ is wider than the ordinary Riesz-Dunford calculus (see [9]) and includes functions non-holomorphic at zero. For instance, fractional powers of the operator $l_{n-\varepsilon}^0(D)$ can be defined as follows

$$(l_{n-\varepsilon}^0(D))^\alpha = (I+R)^{-1} D_0^{(n-\varepsilon)\alpha}(I+R), \qquad \alpha \in \mathbb{R}. \tag{8}$$

where the powers $D_0^{(n-\varepsilon)\alpha}$ for positive αth are well defined by $D_0^{(n-\varepsilon)\alpha} := J^{-(n-\varepsilon)\alpha}$.

Setting $K := (l_{n-\varepsilon}^0(D))^{-1}$ we obtain a Volterra operator

$$K : f \to \int_0^x k(x,t) f(t)\, dt \tag{9}$$

with a "good" kernel $k(x,t)$. With this notation the definition of (8) for negative αth becomes

$$K^\alpha = (I+R)^{-1} J^{(n-\varepsilon)\alpha}(I+R), \qquad \alpha \in \mathbb{R}_+. \tag{10}$$

It can easily be shown that both definitions (8) and (10) do not depend on a choice of transformation operator.

Transformation operators are applied in (see [32]) to prove the unique determination of operator (1) (with $M(x,t) = 0$) from n spectra of boundary value problems (a generalization of the classical Borg-Marchenko theorem on unique determination of the Sturm-Liouville operator $-d^2/dx^2 + q$ from two spectra). Following [36] we also apply them to investigate completeness property of certain boundary value problems for Eq. (1).

The paper is organized as follows. In Sect. 2 we discuss relation (7), i.e. the similarity between operators $l_{n-\varepsilon}(D)$ and $D_0^{n-\varepsilon}$. We show that relation (7) with smooth kernel $R(x,t)$ is satisfied if and only if the kernel $\widetilde{R}(x,t) := R(x, x-t)$ is a solution of the incomplete Cauchy problem for equation

$$\sum_{j=1}^{n} \binom{n-\varepsilon}{j} D_1^j D_2^{n-j} \widetilde{R}(x,t) + \sum_{j=1}^{n-1} q_j(x) \sum_{i=0}^{n-1-j} \binom{n-\varepsilon-1-j}{i} \times D_1^i D_2^{n-1-i-j} \widetilde{R}(x,t) = F(\widetilde{R}(x,t)) \tag{11}$$

where the right-hand side $F(\widetilde{R}(x,t))$ contains certain integro-differential terms (see the problem (18)–(19) below). This equation plays a crucial role in the sequel. For instance, solvability of the incomplete Cauchy problem (18)–(19) implies similarity (7).

Emphasize that the left hand side of Eq. (11) can be obtained from the corresponding equation for nth order operator $l_n(D)$ (with $\varepsilon = 0$) just by replacing n

by $n-\varepsilon$ in binomial coefficients. While for $\varepsilon \neq 0$ the right-hand side of Eq. (11) contains partial $(n+1)$th derivatives, it is natural to define a principle symbol of this operator by setting $L_{n-\varepsilon}(\xi) := \sum_{j=1}^{n} \binom{n-\varepsilon}{j} \xi_1^j \xi_2^{n-j}$.

In Sect. 3 we apply the main result of Sect. 2 to prove the similarity of integral Volterra operator of the form (9) with a kernel $k(x,t)$ being analytical in one variable, to the fractional integration J^{α}, $\alpha \in (0,\infty)$, on $L^p[0,l]$, $p \in [1,\infty]$. To this end we find explicit conditions for a kernel $k(x,t)$ of Volterra operator (9) to admit a representation $K = l_{n-\varepsilon}^0(D)^{-1}$ with some differential operator $l_{n-\varepsilon}(D)$ from the class $A(D)$ or from more general class. In particular, we discuss here the problem of similarity for (weak) perturbations of the fractional integration J^{α}, $\alpha \in (0,\infty)$, on $L^2[0,l]$, by Volterra integral operators of the form

$$K = J^{\alpha}(I + K_1), \quad \text{where} \quad K_1 : f \to \int_0^x k_1(x,t) f(t)\,dt. \tag{12}$$

Clearly, if K of the form (12) is similar to J^{α}, it is unicellular and $\operatorname{Lat} K = \operatorname{Lat} J^{\alpha}$.

In Sect. 4 we discuss existence of a triangular transformation operator for Eq. (1). Note that the similarity between operators $l_{n-\varepsilon}^0(D)$ and $D_0^{n-\varepsilon}$ is used for proving existence of transformation operators for Eq. (1). In particular, we prove here Theorem 1 in more general form assuming that a kernel $M(x, x-t)$ is holomorphic in x for each t instead of its homomorphy in both variables. The proof is reduced to the proof of solvability of Goursat problem (19), (51) for Eq. (11).

The necessary conditions for representation (5) to exist is also discussed here. The proof (for odd n) is used the factorization of the principal symbol $L_n(\zeta) = \zeta_1 Q_{n-1}(\zeta)$ where $Q_{n-1}(\zeta)$ is already elliptic polynomial with constant coefficients, and is heavily relied on the result of regularity up to a boundary of elliptic boundary value problems with "good" coefficients.

In Sect. 5 we apply representation (5) to prove uniqueness results for Eq. (1). More precisely, following [32] we apply transformation operators to prove the unique determination of operator (1) (with $M(x,t) = 0$) from n spectra of boundary value problems (a generalization of the classical Borg-Marchenko theorem on the unique determination of the Sturm-Liouville operator $-d^2/dx^2 + q$ by spectra of two boundary value problems). The proof is reduced to the uniqueness of either Goursat or Cauchy problem (97) for Eq. (11).

Following [34] we also briefly discuss here a problem of the unique determination of a potential matrix of first order system of ODE on a finite interval by its monodromy matrix.

In Sect. 6 following [36] we discuss completeness property for boundary value problems for Eq. (1) with splitting boundary conditions. The proof of the corresponding completeness result in [36] is based on existence of transformation operators for Eq. (1). The second main ingredient of the proof is Theorem 14, a "fractional version" of the classical Birkhoff result on the asymptotic behaviour of solutions of nth order Eq. (1) with $\varepsilon = 0$. Emphasize that this generalization (Theorem 14) is valid for Eq. (1) with arbitrary L^1-coefficients q_j.

Notations Through the paper X_1, X_2, and X denote Banach spaces, $\mathcal{B}(X_1, X_2)$ denotes the set of bounded linear operators from X_1 to X_2; $\mathcal{B}(X) = \mathcal{B}(X, X)$. $L^p\big([0,1]; \mathbb{C}^n\big) = L^p[0,1] \otimes \mathbb{C}^n$

2 Similarity of Fractional Order Ordinary Differential Operators

Let $0 < \alpha = n - \varepsilon$, $n \in \mathbb{N}$, $\varepsilon \in [0,1)$. Recall that the operator of fractional derivative of order α is given by (see [8, 44]),

$$f^{(\alpha)}(x) = D^\alpha f(x) = (D^n J^\varepsilon f)(x) = \frac{d^n}{dx^n} J^\varepsilon f. \tag{13}$$

Next we denote by $W_p^\alpha[0,l]$ the Sobolev space of functions $f \in L^1[0,l]$ having fractional derivatives $f^{(\alpha)} \in L^p[0,l]$. The functions $f \in W_p^\alpha[0,l]$ are characterized by means of the following integral representation

$$f(x) = \sum_{j=1}^{n} c_j \frac{x^{\alpha-j}}{\Gamma(\alpha-j+1)} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f_1(t)\, dt, \tag{14}$$

where $c_j = f^{(\alpha-j)}(0)$, $j \in \{1,\dots,n\}$, and $f_1(\cdot) = D^\alpha f(\cdot) = f^{(\alpha)}(\cdot) \in L^p[0,l]$.

Let us denote also by $W_{p,0}^\alpha[0,l]$ the subspace of those $f \in W_p^\alpha[0,l]$ for which the polynomial in (14) is absent, that is, $c_j = f^{(\alpha-j)}(0) = 0$, $j \in \{1,\dots,n\}$.

Here assuming that $\varepsilon \in [0,1)$ we consider in $L^1[0,l]$ the operators $D^\alpha = D^{n-\varepsilon}$ and $l_{n-\varepsilon}(D)$:

$$l_{n-\varepsilon}(D) f = D^{n-\varepsilon} f + \sum_{j=1}^{n-1} q_j(x) D^{n-\varepsilon-j-1} f + \int_0^x M(x,t)(J^\varepsilon f)(t) dt \tag{15}$$

with domain $W_1^{n-\varepsilon}[0,1]$. As above we denote by $D_0^{n-\varepsilon}$ and $l_{n-\varepsilon}^0(D)$ their restrictions to the subspace $W_{1,0}^{n-\varepsilon}[0,1]$. We also put $\Omega = \{0 < t < x < l\}$.

Definition 1 It is said that an operator of triangular from

$$(I + R) f = f(x) + \int_0^x R(x,t) f(t) dt \tag{16}$$

with sufficiently smooth kernel $R(x,t)$ intertwines unbounded operators $l^0_{n-\varepsilon}(D)$ and $D_0^{n-\varepsilon}$ if $I+R$ maps $W_{1,0}^{n-\varepsilon}[0,l]$ onto $W_{1,0}^{n-\varepsilon}[0,l]$ and satisfies the identity

$$l^0_{n-\varepsilon}(D)(I+R)f = (I+R)D_0^{n-\varepsilon} f \quad \text{for all} \quad f\in W_{1,0}^{n-\varepsilon}[0,l]. \tag{17}$$

Proposition 1 ([32]) *Let $R(x,t)\in C^{n+1}(\overline{\Omega})$ and let $\widetilde{M}(x,t):=M(x,x-t)$. Then an operator $I+R$ of the form (16) intertwines the operators $D_0^{n-\varepsilon}$ and $l^0_{n-\varepsilon}(D)$ if and only if the function $\widetilde{R}(x,t):=R(x,x-t)$ is a solution of the following fractional order equation in partial derivatives*

$$\begin{aligned}
&\sum_{j=1}^{n}\binom{n-\varepsilon}{j} D_1^j D_2^{n-j}\widetilde{R}(x,t) + \sum_{j=1}^{n-1} q_j(x)\sum_{i=0}^{n-1-j}\binom{n-\varepsilon-1-j}{i} D_1^i D_2^{n-1-i-j}\widetilde{R}(x,t)\\
&\qquad + \widetilde{M}(x,t) + \int_0^t \widetilde{M}(x,s)\widetilde{R}(x-s,t-s)\,ds\\
&= \int_0^t d\xi \int_0^1 \frac{(1-\beta)^{\varepsilon}\beta^{n-\varepsilon}}{\Gamma(\varepsilon)\Gamma(1-\varepsilon)} D_1^{n+1}\widetilde{R}\big(\xi + x - t + (t-\xi)\beta, \xi\big)\,d\beta\\
&\quad + \sum_{j=1}^{n-1} q_j(x)\int_0^t d\xi \int_0^1 \frac{(1-\beta)^{\varepsilon}\beta^{n-1-j-\varepsilon}}{\Gamma(\varepsilon)\Gamma(1-\varepsilon)} D_1^{n-j}\widetilde{R}\big(\xi + x - t + (t-\xi)\beta, \xi\big)\,d\beta\\
&\quad + \int_0^t \widetilde{M}(x,s)\,ds \int_0^{t-s} d\xi \int_0^1 \frac{(1-\beta)^{-\varepsilon}\beta^{\varepsilon}}{\Gamma(\varepsilon)\Gamma(1-\varepsilon)} D_1\widetilde{R}\big(x-s-(t-s-\xi)\beta, \xi\big)\,d\beta,
\end{aligned} \tag{18}$$

subject to the following initial conditions

$$\begin{aligned}
&\sum_{j=1}^{r}\binom{n-\varepsilon}{j} D_1^j D_2^{r-j}\widetilde{R}(x,0) + \sum_{j=1}^{r-1} q_j(x)\sum_{i=0}^{r-1-j}\binom{n-\varepsilon-1-j}{i} D_1^i D_2^{r-1-i-j}\widetilde{R}(x,0)\\
&\qquad = -q_r(x), \qquad r\in\{1,\dots,n-1\}.
\end{aligned} \tag{19}$$

Theorem 2 ([32]) *Let q_j be entire analytical functions, $j\in\{1,\dots,n-1\}$, $M(x,t)\in C(\overline{\Omega})$, and let $\widetilde{M}(x,t_0)=M(x,x-t_0)$ be also entire analytical functions in x for all $t_0\in[0,l]$, and let $l_{n-\varepsilon}(D)$ be the fractional order integro-differential operator of the form (15). Then:*

- (i) *There exists a (non-unique) triangular operator $I+R$ of the form (16) intertwining operators $D_0^{n-\varepsilon}$ and $l^0_{n-\varepsilon}(D)$, i.e. identity (17) holds. Moreover, the kernel $\widetilde{R}(x,t):=R(x,x-t)$ is a solution of the incomplete Cauchy problem (18)–(19);*
- (ii) *For each $t_0\in[0,1)$ the function $\widetilde{R}(x,t_0):=R(x,x-t_0)$ admits a holomorphic continuation to an entire function in x. Moreover, $R(x,t)$ admits a holomorphic continuation to an entire function in both variables, whenever $M(x,t)$ admits.*

Sketch of the Proof Due to Proposition 1 we should prove existence of the problem (18)–(19) using analyticity of q_j and $M(x,t)$. It can be shown (see [32] and [31] for $\varepsilon = 0$) that conditions (19) are transformed into the conditions

$$D_2^{r-1} u(x,0) = -(n-\varepsilon)^{-1} q_r(x) + \varphi_r(x), \quad r \in \{1,\dots,n-1\}. \tag{20}$$

where $\varphi_1(x) = 0$ and $\varphi_r(x)$ with $r \geq 2$ is expressed via $\{q_j\}_1^{r-1}$ by means of operations of summation, multiplication, differentiation, and integration. Thus we should prove solvability of the incomplete Cauchy problem (18), (20).

Imposing additional condition

$$\widetilde{R}(l,t) = \widetilde{R}(x,t)|_{x=l} = 0. \tag{21}$$

we arrive at the problem (18), (20), (21). It is natural to call this problem a Goursat problem.

It is easily reduced to a (complete) Cauchy problem. Indeed, setting $D_1\widetilde{R}(x,t) = u(x,t)$ and using condition (21) we find that $\widetilde{R}(x,t) = \int_l^x u(\zeta,t)d\zeta$. Inserting this expression into (18) one arrives at the integro-differential PDE of order $n-1$ for the function u. At the same time boundary conditions (20) turn in to the conditions for u and we arrive at the Cauchy problem for the function $u(x,t)$.

So, we should prove the existence of the (non-characteristic) Cauchy problem for partial differential equation of order n with fractional order terms in the right hand side. It is shown in [32] by using the method of successive approximations that this problem has a unique solution $u(x,t)$. Hence we find a solution $\widetilde{R}(x,t)$ to the problem (18)–(19). □

Corollary 1 *Let the operator $l^0_{n-\varepsilon}(D)$ satisfy the conditions of Theorem 2. Then they are similar in each $L^p[0,b]$, $p \in [1,\infty]$. More precisely, each triangular operator $I+R$ of the form (16) intertwining operators $D_0^{n-\varepsilon}$ and $l^0_{n-\varepsilon}(D)$ maps $W_{1,0}^{n-\varepsilon}[0,l]$ onto $W_{1,0}^{n-\varepsilon}[0,l]$ and the following similarity identity holds*

$$l^0_{n-\varepsilon}(D)f = (I+R)^{-1} D_0^{n-\varepsilon}(I+R)f, \qquad f \in W_{1,0}^{n-\varepsilon}[0,l]. \tag{22}$$

Remark 1 Passing in (18) to the limit as $\varepsilon \to 0$ and noting that $\Gamma(0) = \lim_{\varepsilon\to 0}\Gamma(\varepsilon) = \infty$, we conclude that the right-hand side of Eq. (18) vanishes and it turns into the equation

$$\sum_{j=1}^{n} \binom{n}{j} D_1^j D_2^{n-j}\widetilde{R}(x,t) + \sum_{j=1}^{n-1} q_j(x) \sum_{i=0}^{n-1-j} \binom{n-1-j}{i} D_1^i D_2^{n-1-i-j}\widetilde{R}(x,t) + \widetilde{M}(x,t) + \int_0^t \widetilde{M}(x,s)\widetilde{R}(x-s,t-s)\,ds \tag{23}$$

obtained earlier in [31, formula (67)]. This equation is equivalent to

$$D_x^n R(x,t) + \sum_{j=1}^{n-1} q_j(x) D_x^{n-1-j} R(x,t) = (-1)^n D_t^n R(x,t) + M(x,t) + \int_t^x M(x,s)R(s,t)\,ds \tag{24}$$

with $R(x,t) = \widetilde{R}(x, x-t)$ and $M(x, x-t) = \widetilde{M}(x,x)$. The latter is a well-known equation for the kernel of transformation operator of nth order ordinary differential equation with $M(x,t) = 0$ (see [31, 42], and also [24, Chapter 5.6]). Moreover, setting $u(x,t) := D_x \widetilde{R}(x,t)$ and passing to the limit as $\varepsilon \to 0$ one transforms the initial conditions (19) into the following conditions

$$\sum_{j=1}^{r} \binom{n}{j} \left[D_1^{j-1} D_2^{r-j} u(x,t)|_{t=0} \right] + \sum_{j=1}^{r-1} q_j(x) \int_1^x D_2^{r-j-1} u(\xi, 0)\,d\xi$$
$$+ \sum_{j=1}^{r-2} \binom{n-j-1}{i} q_j(x) \left[\sum_{i=1}^{r-1-j} D_x^{i-1} D_t^{r-j-i-1} u(x,t)|_{t=0} \right] = -q_r(x), \quad r \in \{1, \ldots, n-1\}, \tag{25}$$

coinciding with conditions (70) in [31]:

Emphasize that the left hand side of Eq. (18) can be obtained from the nth order equation (23) just by replacing n by $n - \varepsilon$ in binomial coefficients. Moreover, boundary conditions (19) are obtained from their limit form (25) (with $\varepsilon = 0$) in the same manner: just by replacing n by $n - \varepsilon$ in binomial coefficients. So, the main distinguishing between Eqs. (18) and (23) is appeared in the right hand side of (18) containing fractional derivatives. Note however that despite of presence of $(n+1)$th order derivative $D_1^{n+1}\widetilde{R}$ in the right hand side of (18) it is natural to consider the polynomial

$$L_{n-\varepsilon}(\xi) = \sum_{j=1}^{n} \binom{n-\varepsilon}{j} \xi_1^j \xi_2^{n-j}$$

as a principal symbol of the fractional order differential operator $L_{n-\varepsilon}(D)$ generated by Eq. (18).

Remark 2 In particular, for $n = 2$, $\varepsilon = 0$, and $M(x,t) = 0$, the boundary value problem (18)–(19) is equivalent to the following well-known incomplete Cauchy problem for hyperbolic (string) equation

$$(D_x^2 - D_t^2) R(x,t) + q(x) R(x,t) = 0, \tag{26}$$

$$\frac{d}{dx} R(x,x) = -2^{-1} q(x). \tag{27}$$

(see [24, 25, 38]).

3 Similarity of Volterra Operators

Here we apply Theorem 2 to prove similarity of the Volterra operator

$$K : f \to \int_0^x k(x,t) f(t)\, dt \tag{28}$$

with a "good" kernel $k(x,t)$ to the operator J^α with $\alpha \neq 1$. Roughly speaking the conditions of similarity read as follows:

(i) $k(x, x-t)$ should be an entire function in x for all $t \in [0,l)$;
(ii) the behavior of $k(x,t)$ at the diagonal $x = t$ coincides with that of the kernel $\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}$ of the operator J^α.

The precise statement on similarity of Volterra operators reads as follows.

Theorem 3 ([30, 32]) *Let $\alpha = n - \varepsilon$, $\varepsilon \in [0,1)$. Suppose that the kernel $k(x,t)$ of the Volterra operator (28) satisfies the following conditions:*

(i) *for all $t \in [0,l]$ the derivatives*

$$D_x^{j-\varepsilon} k(x,t) \in C(\overline{\Omega}) \quad \textit{exist}, \quad j \in \{0, 1, \dots, n\}; \tag{29}$$

(ii) *for all $x \in [0,l]$ the derivative $D_t^n k_1(x,t) \in C(\overline{\Omega})$ exist, where $k_1(x,t) = D_x^{n-\varepsilon} k(x,t) \in C(\overline{\Omega})$;*
(iii) *$D_t^j k_1(x, x-t_0)$ is an entire function in x for all $j \in \{1, \dots, n\}$, and for all $t_0 \in [0,l]$;*
(iv) *$[D_x^{j-\varepsilon} k(x,t)]|_{t=x} = 0$, $j \in \{0, n-2\}$, and $[D_x^{n-1-\varepsilon} k(x,t)]|_{t=x} = 1$.*

Then K is similar in $L^p[0,l]$, $p \in [1,\infty]$, to the operator of fractional integration J^α.

Moreover, if $D_s^{n-\varepsilon} k(s,t)|_{t=s} = 0$, then:

(a) *there exists a Volterra operator R of the form (28) with a kernel $R(x,t)$ and such that $I + R$ intertwines the operators K and J^α, i.e. $K(I+R) = (I+R)J^\alpha$,*
(b) *for each $t_0 \in [0,1)$ the function $\widetilde{R}(x,t_0) := R(x, x-t_0)$ admit a holomorphic continuation to an entire function in x. Moreover, $R(x,t)$ admit a holomorphic continuation to an entire function in both variables whenever $k(x,t)$ does.*

Sketch of the Proof Denote by D_0^α the restriction of the operator D^α (see (13)) to the domain $\mathcal{D}(D_0^\alpha) = W_{p,0}^{n-\varepsilon}[0,l]$. It is easily seen that $(J^\alpha)^{-1} = D_0^\alpha$. Next we show that the inverse to the Volterra operator K is a fractional order integro-differential operator of the form

$$K^{-1}f = l^0_{n-\varepsilon,1}(D)f = D^{n-\varepsilon}f + \sum_{j=1}^{n-1} q_j(x)D^{n-\varepsilon-j}f + \int_0^x N(x,t)(J^\varepsilon f)(t)dt \tag{30}$$

with domain $\mathcal{D}(l^0_{n-\varepsilon,1}(D)) = W^{n-\varepsilon}_{1,0}[0,l]$. It allows us to reduce the problem to the investigation of similarity between the operators $l^0_{n-\varepsilon,1}(D)$ and D^α_0.

Since, by hypothesis, $D^{j-\varepsilon}_x k(x,t) \in C[t,l]$ for all $t \in [0,l)$, $j \in \{0,\ldots,n\}$, it follows that $k(\cdot,t) \in W^{n-\varepsilon}_1[t,l]$. Therefore, in view of (14) and condition (iv) of the theorem, we have the representation

$$k(x,t) = \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} + \int_t^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} k_1(s,t)\,ds \tag{31}$$

in which $k_1(s,t) = D^{n-\varepsilon}_s k(s,t)$.

We now find the inverse of the operator K (28). For this we apply the operator $D^{n-\varepsilon}_x$ to the equality

$$\int_0^x k(x,t)f(t)\,dt = g(x), \qquad g \in W^{n-\varepsilon}_{1,0}[0,l]. \tag{32}$$

Using (31) we obtain

$$\begin{aligned}
D^{n-\varepsilon}_x \int_0^x k(x,t)f(t)dt &= D^n_x \int_0^x \frac{(x-s)^{\varepsilon-1}}{\Gamma(\varepsilon)}\,ds \int_0^s k(s,t)f(t)\,dt \\
&= D^n_x \int_0^x f(t)\,dt \int_t^x \frac{(x-s)^{\varepsilon-1}}{\Gamma(\varepsilon)} k(s,t)\,ds = D^n_x \int_0^x f(t)\,dt \int_t^x \frac{(x-s)^{\varepsilon-1}}{\Gamma(\varepsilon)} \frac{(s-t)^{n-\varepsilon-1}}{\Gamma(n-\varepsilon)}\,ds \\
&\quad + D^n_x \int_0^x f(t)\,dt \int_t^x \frac{(x-s)^{\varepsilon-1}}{\Gamma(\varepsilon)}\,ds \int_t^s \frac{(s-\xi)^{n-\varepsilon-1}}{\Gamma(n-\varepsilon)} k_1(\xi,t)\,d\xi \\
&= D^n_x \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)\,dt + D^n_x \int_0^x f(t)\,dt \int_t^x k_1(\xi,t)\,d\xi \int_\xi^x \frac{(x-s)^{\varepsilon-1}}{\Gamma(\varepsilon)} \frac{(s-\xi)^{n-\varepsilon-1}}{\Gamma(n-\varepsilon)}\,ds \\
&= f(x) + D^n_x \int_0^x f(t)\,dt \int_t^x \frac{(x-\xi)^{n-1}}{(n-1)!} k_1(\xi,t)\,d\xi = f(x) + \int_0^x k_1(x,t)f(t)\,dt.
\end{aligned} \tag{33}$$

Thus it follows from (32) and (33) that

$$f(x) + \int_0^x k_1(x,t)f(t)dt = D^{n-\varepsilon}_x g(x) = g^{(n)}_1(x), \tag{34}$$

where $g_1(x) = J^{\varepsilon} g(x)$. Let

$$(I+P)f = f(x) + \int_0^x P(x,t) f(t) dt \tag{35}$$

be the inverse of the operator $I + K_1$. It is easily seen that the kernels $P(x,t)$ and $k_1(x,t)$ are related by the equation

$$P(x,t) + k_1(x,t) + \int_t^x P(x,s) k_1(s,t) ds = 0. \tag{36}$$

Setting $\widetilde{P}(x,t) := P(x, x-t)$ and $\widetilde{k}_1(x,t) := k_1(x, x-t)$, we write (36) in the form

$$\widetilde{P}(x,t) = -\widetilde{k}_1(x,t) - \int_0^t \widetilde{P}(x,s) \widetilde{k}_1(x-s, t-s)\, ds. \tag{37}$$

Since $k_1(x, t_0)$ is an entire function for all $t_0 < l$, solving (37) by the method of successive approximations we can represent $P(x,t)$ as the sum of a uniformly convergent series of entire functions. Consequently, $\widetilde{P}(x,t_0)$ is also an entire function for all $t_0 \in [0,l)$. Differentiating (37) repeatedly with respect to t, we then obtain

$$\begin{aligned} D_2^j \widetilde{P}(x,t) = -D_2^j \widetilde{k}_1(x,t) - \sum_{i=0}^{j-1} D_t^{j-1-i} [\widetilde{P}(x,t) D_2^i \widetilde{k}_1(x-t, 0)] \\ - \int_0^t \widetilde{P}(x,s) D_2^i \widetilde{k}_1(x-s, t-s) ds. \end{aligned} \tag{38}$$

Hence we can easily deduce by induction on j that $D_2^j \widetilde{P}(x,t_0) \in A_R[t_0, l]$ for all $t_0 \in [0,l] (1 \le j \le n)$ and $D_2^n \widetilde{P}(x,t) \in C(\overline{\Omega})$. We now apply the operator $I + P$ to (34) and integrate by parts to obtain

$$\begin{aligned} f(x) = (I+P) g_1^{(n)}(x) &= g_1^{(n)}(x) + \int_0^x P(x,t) g_1^{(n)}(t) dt \\ &= g_1^{(n)}(x) + \sum_{j=1}^{n} q_j(x) g_1^{(n-j)}(x) + \int_0^x N(x,t) g_1(t) dt, \end{aligned} \tag{39}$$

where

$$\begin{aligned} q_j(x) = (-1)^j [D_2^{j-1} P(x,t)|_{t=x}] = D_2^{j-1} \widetilde{P}(x,0),\ 1 \le j \le n-1, \\ \text{and} \quad D_2^n P(x,t) = N(x,t). \end{aligned} \tag{40}$$

Thus, the inverse K^{-1} of K is of the form (30).

Next to “kill” the coefficient $q_1(x) \neq 0$ we introduce the multiplication operator

$$\Phi f(x) = \varphi_0(x) f(x), \qquad \varphi_0(x) = \exp\left(\frac{1}{\varepsilon - n}\int_0^x q_1(s)ds\right) \tag{41}$$

and prove the equality

$$\Phi^{-1} l^0_{n-\varepsilon,1}(D)\Phi f = l^0_{n-\varepsilon,2}(D) f \qquad \text{for all} \qquad f \in W^{n-\varepsilon}_{1,0}[0,l], \tag{42}$$

in which

$$l^0_{n-\varepsilon,2}(D) f := D^{n-\varepsilon} f + \sum_{j=2}^{n} r_j(x) D^{n-\varepsilon-j} f + \int_0^x M(x,t)(J^\varepsilon f)(t)dt \tag{43}$$

We achieve this representation by applying following analogue of Leibniz’s formula (see [44], Section 15.2, formula (15.11)):

$$D^{n-j-\varepsilon}\big(\varphi(x) f(x)\big) = \sum_{i=0}^{\infty} \binom{n-j-\varepsilon}{i} \varphi^{(i)}(x) D^{n-j-\varepsilon-i} f(x), \quad j \in \{1,\ldots,n\}. \tag{44}$$

By Theorem 2 (see below), the operator $l^0_{n-\varepsilon,2}(D)$ is similar to the operator D^α_0 in $L^p[0,1]$ for each $p \in [0,\infty]$. This implies the similarity of inverses. □

Remark 3 If $n = 1$, the conditions (iv) are reduced to the solo condition $[D_x^{-\varepsilon} k(x,t)]|_{t=x} = 1$.

Corollary 2 *Let*

$$k(x,t) = \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} P(x,t) \quad \text{and} \quad P(x,x) = 1, \tag{45}$$

and suppose that the derivatives

$$D_x^n P(x,t) := Q(x,t) \in C(\overline{\Omega}) \quad \text{and} \quad D_t^j Q(x,t) := Q_j(x,t) \in C(\overline{\Omega}), \quad j \in \{0,\ldots,n\},$$

exist. Assume, in addition, that for each $t_0 \in [0,1)$ the functions $\widetilde{Q}_j(x,t_0) := Q_j(x, x-t_0)$, $(0 \le j \le n)$, admit holomorphic continuations to entire functions in x. Then the operator K of the form (28) is similar in $L^p[0,l]$, $p \in [1,\infty]$, to the operator of fractional integration J^α.

Example 1 Let the kernel $k(x,t)$ has the form

$$k(x,t) = \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\left[1 + \sum_{j=1}^{m} \varphi_j(x)\psi_j(x-t)\right], \tag{46}$$

where $\sum_{j=1}^{m} \varphi_j(x)\psi_j(0) = 0$, $\psi_j(t) \in C^{2n}[0,l]$, $j \in \{0,\ldots,m\}$, and $\varphi_j(x)$ is an entire function in x for all $j \in \{1,\ldots,m\}$. Then $k(x,t)$ satisfies the conditions of Theorem 3, and hence the Volterra operator K of the form (28) is similar to the operator J^α.

Corollary 3 *Assume that $k(x,t)$ is a kernel of the form (45) with $P(x,t)$ admitting a holomorphic continuation to entire function in both variables. Then for any $n \in \mathbb{N}$ there exists a Volterra operator K_n with a kernel $k_n(x,t)$ of the form*

$$k_n(x,t) = \frac{(x-t)^{\alpha/n-1}}{\Gamma(\alpha/n)} P_n(x,t) \qquad \text{and} \qquad P_n(x,x) = 1, \tag{47}$$

and such that:

(i) *$P_n(x,t)$ admits a holomorphic continuation to entire function in both variables;*
(ii) *K_n is an nth root of K, i.e. $K_n^n = K$.*

An improvement of Theorem 3 was obtained by Ignat'ev [15] (see Theorem 6 below). His method being an improvement of the method of Khachatryan allows him to weaken the constraints imposed in Theorem 3 in the case $\alpha > 2$.

Remark 4

(i) The problem of similarity between operators K and J^α has a long history going back to the works by G. Kalish [16] and L.A. Sakhnovich [41, 42] who treated the case of integers $\alpha = n \in \mathbb{N}$. Theorem 3 strengthens their result from [16, 41, 42], where the kernel $k(x,t)$ was required to be analytic in both variables instead of being finitely smooth in one of them, as required in Theorem 3. In the case of non-integer α Theorem 3 was proved in [32], while a weaker result was announced in [30].

    Other results on similarity of Volterra operators and its applications to Riesz basis property can be found in [12, 13] (see also references therein and in [32]).

(ii) Corollary 3 generalizes the classical result of Volterra and Peres [48] and coincides with it for $\alpha = n$. It was substantially employed by G. Kalish [16] in his proof of similarity between operators K and J^n.

4 Triangular Transformation Operators

4.1 Sufficient Conditions for Existence of Transformation Operators

Here we discuss triangular transformation operators for solutions to the equation $l_{n-\varepsilon}(D)y(x,\lambda) = \lambda y(x,\lambda)$, where $l_{n-\varepsilon}(D)$ is the fractional order integro-differential operator of the form (15).

To this end we denote by $E_1$and E_2 the subspaces of $W_1^{n-\varepsilon}[0,l]$ defined by the relations

$$E_1 := \{f \in W_1^{n-\varepsilon}[0,l] : f^{(j-\varepsilon)}(0) = 0, \quad 1 \le j \le n-1\}, \tag{48}$$

$$E_2 := \{f \in W_1^{n-\varepsilon}[0,l] : f^{(j-\varepsilon)}(0) = h_j f^{(-\varepsilon)}(0),\ 1 \le j \le n-1\}. \tag{49}$$

Clearly, $E_j \supset W_{1,0}^{n-\varepsilon}[0,l]$ and $\dim(E_j/W_{1,0}^{n-\varepsilon}[0,l]) = 1$, $j \in \{1,2\}$.

Proposition 2 ([32]) *Let $l_{n-\varepsilon}(D)$ be a fractional order differential operator of the form (15), let $I + R_1$ be a triangular operator of the form (16) with kernel $R_1(x,t)$, and let $\widetilde{R}_1(x,t) := R_1(x,x-t)$. Then in order that the operator $I + R_1$ maps E_1onto E_2 and intertwines the operators $l_{n-\varepsilon}(D)|_{E_2}$ and $D^{n-\varepsilon}|_{E_1}$, that is in order that the following equality should hold:*

$$l_{n-\varepsilon}(D)(I+R_1)f = (I+R_1)D^{n-\varepsilon}f \quad \textit{for all} \quad f \in E_1, \tag{50}$$

it is necessary and sufficient that $\widetilde{R}_1(x,t)$ satisfies Eq. (18), conditions (19), and the further conditions

$$[D_2^{n-1}\widetilde{R}_1(x,t)]|_{t=x} = 0 \tag{51}$$

and

$$\sum_{i=0}^{j-1}\binom{j-\varepsilon}{i}\left[D_1^i D_2^{j-i-1}\widetilde{R}_1(x,t)\right]\Big|_{x=t=0} = h_j, \qquad j \in \{1,\dots,n-1\}. \tag{52}$$

Now we are ready to state our main result on existence of a triangular transformation operator.

Theorem 4 ([32]) *Let q_j be entire analytical functions, $j \in \{1,\dots,n-1\}$, $M(x,t) \in C(\overline{\Omega})$, $\Omega = \{0 < t < x < l\}$, and let $\widetilde{M}(x,t_0) = M(x,x-t_0)$ be also entire in x for all $t_0 \in [0,l]$. Suppose further that $y(x,\lambda)$ is a solution to the* *following Cauchy problem*

$$l_{n-\varepsilon}(D)y(x,\lambda) := D^{n-\varepsilon}y(x,\lambda) + \sum_{j=1}^{n-1} q_j(x)D^{n-\varepsilon-j-1}y(x,\lambda)$$

$$+ \int_0^x M(x,t)(J^{\varepsilon}y(t,\lambda))\,dt = \lambda y(x,\lambda), \tag{53}$$

$$y^{(j-\varepsilon-1)}(0,\lambda) = h_j, \quad j \in \{1,\dots,n\}. \tag{54}$$

Then there exists a unique kernel $R_1(x,t) \in C^n(\overline{\Omega})$ *such that* $R_1(x, x-t_0)$ *is an entire function for all* $t_0 \in [0,l]$*, and*

$$y(x,\lambda) = (I+R_1)w(x,\lambda) := w(x,\lambda) + \int_0^x R_1(x,t)w(t,\lambda)\,dt \tag{55}$$

in which $w(x,\lambda)$ *is a solution of the Cauchy problem for the simplest fractional order equation*

$$D_x^{n-\varepsilon}w(x,\lambda) = \lambda w(x,\lambda) \tag{56}$$

with the initial data

$$w^{(j-1-\varepsilon)}(0,\lambda) = h_{j,0}, \qquad j \in \{1,\dots,n\}, \tag{57}$$

where if $h_{j,0} = 0$, $j \in \{1,\dots,p-1\}$ *and* $h_{p,0} \neq 0$*, then* $h_{j,0} = h_j$ *for all* $j \le p$*. If, in addition,* $M(x,t)$ *is analytic in both variables, then* $R_1(x,t)$ *is also analytic in both variables* x *and* $t \in \overline{\Omega}$*.*

Proof Let $I + R$ be an operator intertwining operators $l^0_{n-\varepsilon}(D)$ and $D_0^{n-\varepsilon}$, i.e. Eq. (17) holds. By Proposition 1 $R(x,t)$ is a solution of the incomplete Cauchy problem (18)–(19) and by Theorem 2 a solution to this problem exists.

Next we define the transformation operator $I + R_1$ by setting

$$I + R_1 = (I+R)(I+\Phi) \quad \text{where} \quad \Phi : f \to \int_0^x \varphi(x-t)f(t)\,dt$$

is a convolution operator with smooth kernel $\varphi \in C^{n+1}[0,1]$. Since the operator Φ commutes with J^{α}, one gets that the operator $I + R_1$ intertwines the operators $l^0_{n-\varepsilon}(D)$ and J^{α}, i.e.

$$l^0_{n-\varepsilon}(D)(I+R_1)f = (I+R_1)D_0^{n-\varepsilon}f \quad \text{for all} \quad f \in W_{1,0}^{n-\varepsilon}[0,l]. \tag{58}$$

By Proposition 1, $R_1(x,t)$ is also a solution to the problem (18)–(19). Next we find $\varphi \in C^{n+1}[0,1]$ to satisfy the conditions (51)–(52) from the following second order integral equation on unknown function φ:

$$R_1(x,t) = R(x,t) + \varphi(x-t) + \int_t^x R(x,s)\varphi(s-t)\,ds$$

Rewriting this equation in the form

$$\widetilde{R}_1(x,t) = \widetilde{R}(x,t) + \varphi(x-t) + \int_0^t \widetilde{R}(x,s)\varphi(t-s)\,ds$$

we show that such a function does exist. □

Remark 5 Here $w(x,\lambda)$ admits a representation:

$$w(x,\lambda) = \sum_{j=1}^{n} h_{n+1-j,0}\, x^{j-1-\varepsilon} E_{\alpha,j-\varepsilon}(\lambda x^{\alpha}) \tag{59}$$

where $E_{\alpha,\mu}$ is the classical Mittag-Leffler function (see [8, Chapter 3]),

$$E_{\alpha,\mu}\big(zx^{\alpha}\big) = \sum_{k=0}^{\infty} \frac{(zx^{\alpha})^k}{\Gamma(k\alpha+\mu)}, \qquad \mu > 0. \tag{60}$$

It is known [8] that $E_{\alpha,\mu}\big(zx^{\alpha}\big)$ is an entire function in z of order $1/\alpha$ and type x not depending on μ, hence so is $w(x,z)$.

Note that the theory of triangular transformation operators for Sturm-Liouville equation goes back to the classical paper by Marchenko [37] (see also monographs [24, 25, 38]). Namely, he proved representation (55) with $w(x,\lambda) = \cos x\sqrt{\lambda}$ for the solution of the following Cauchy problem

$$-y'' + q(x)y = \lambda y, \qquad y(0) = 1, \quad y'(0) = h, \quad x \in \mathbb{R}_+, \tag{61}$$

for Sturm-Liouville equation with $L^1_{\rm loc}$-potential q and applied it to different problems of spectral theory of such operators (asymptotic behaviour of spectral functions, uniqueness of reconstruction of a potential q from spectral function, etc.).

To construct a transformation operator for the problem (61) Marchenko writing equation (61) as $y'' + \lambda y = q(x)y$ with the right hand side $q(x)y$ and by using the method of variation of constants reduced the problem (61) to the equivalent integral Sturm-Liouville equation

$$y(x,\lambda) = \cos(x\sqrt{\lambda}) + h\frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}} + \int_0^x \frac{\sin((x-t)\sqrt{\lambda})}{\sqrt{\lambda}} q(t)y(t,\lambda)\,dt. \tag{62}$$

Then he has obtained a triangular representation (55) for $y(x,\lambda)$ by applying the method of successive approximations. Later on inserting representation (55) for

$y(x, \lambda)$ into (62) he obtained an integral equation for the kernel $R(x, t)$ expressing it via a potential q.

Gelfand and Levitan (see [25]) proposed a slightly different approach to deduce representation (55) for the solution $y(x, \lambda)$ of the problem (61). Namely, starting with formula (55) for $y(x, \lambda)$ they shown that the kernel $R(x, t)$ satisfies the certain Goursat problem for the string equation (26). Then proving the (unique) solvability to this problem they arrived at representation (55).

L. Sakhnovich [42] extended the Gel'fand-Levitan method to the case of nth order equation (15). Again starting with representation (55) for the solution $y(x, \lambda)$ of the problem (53)–(54) (with $\varepsilon = 0$ and $M(x, t) = 0$) he shown that $R(x, t)$ satisfies a certain Goursat problem for a partial differential equation (24). Assuming that the coefficients q_j are entire functions and $M(x, t) = 0$, and applying the Cauchy-Kovalevskii theorem, he proved solvability of this problem. This leads to representation (55) for a solution to Cauchy problem of Eq. (15) (with $\varepsilon = 0$, entire coefficients q_j, and $M(x, t) = 0$).

Our proof of Theorem 4 is a further generalization of the above mentioned proofs from [25] and [41] regarding the cases $\alpha = 2$ and $(\mathbb{N} \ni)\alpha = n \geq 3$, respectively, while a deduction of the respective Goursat problem (18)–(19), (51) for the kernel $R(x, t)$ of transformation operator is much more complicated.

On the other hand, Marchenko's method was extended by I.G. Khachatryan [18] to the case of nth order equation. Namely, writing down Eq. (53) (with $\varepsilon = 0$ and $M = 0$) in the form

$$D^n y(x, \lambda) - \lambda y(x, \lambda) = \sum_{j=1}^{n-1} q_j(x) D^{n-j-1} y(x, \lambda)$$

and applying the method of variation of constants, he reduced the Cauchy problem for Eq. (53) to the equivalent integral equation similar to (62). He improved the result of Sakhnovich by showing that formula (55) remains valid provided that the coefficients q_j are holomorphic in certain polygon (see below).

Next contribution to the subject was done by Ignat'ev [15]. Assuming that $\alpha > 2$ he simplified the original proof of Theorem 4 by generalization the reasoning of Hachatryan [18]. The proposed method allows him to weaken the constraints imposed in Theorem 4 on the analyticity domain of the kernel $M(x, t)$ and the coefficients $\{q_j\}_1^{n-1}$ of Eq. (53).

Following [15] we assume that $\alpha > 2$, $l = 1$, and denote by D_a the quadrangle in the complex plane with vertices

$$\{0, a, a(1-w)^{-1}, a(1-w^{-1})^{-1}\}, \qquad w = \exp(2\pi i/\alpha), \tag{63}$$

and put

$$V := \{(x, \xi) : \ x \in [0, 1], \ \xi \in D_{1-x}\}.$$

A Volterra integral operator is said to be an operator of class A if it can be expressed as

$$(Nf)(x) = \int_0^x N(x-t,t)f(t)dt \tag{64}$$

where $N(x,\xi)$ is a continuous function defined on V and, for any fixed $x \in [0,1]$, is analytic with respect to ξ on the domain D_{1-x}.

Theorem 5 ([15]) *Suppose that $\alpha > 2$, and the coefficients of the integro-differential operator (53) are analytic on D_1, the functions $q_j^{(n-j-1)}$, $j \in \{1,\dots,n-1\}$ are continuous on D_1, and M is a Volterra operator of class A. Let also $y(x,\lambda)$ and $w(x,\lambda)$ be the solutions of Eqs. (53) and (56), respectively, satisfying the common initial conditions*

$$y^{(\alpha-k-1)}(x,\lambda)|_{x=0} = w^{(\alpha-k-1)}(x,\lambda)|_{x=0} = \delta_{k,n-1}, \quad k \in \{1,\dots,n-1\}$$

Then $y(x,\lambda)$ admits a representation (55) with $R(x,t)$ being of class A.

As a corollary of this result Ignat'ev obtained the following improvement of Theorem 3 on similarity to the operator J^α with $\alpha > 2$.

Theorem 6 ([15]) *Suppose that $\alpha > 2$ and $J^\alpha(I+JN)$, where N is a Volterra operator of class A. Then there exists a Volterra operator R of class A such that $K = (I+R)J^\alpha(I+R)^{-1}$ in the spaces $L^p[0,b]$ for $p \in [1,\infty]$.*

4.2 Necessary Conditions

Analyticity of coefficients $q_j(\cdot)$ of Eq. (53) is in sense necessary for Eq. (53) (with $M(x,t)=0$) to admit a triangular transformation operator. First we present results for nth order equation assuming that $\varepsilon = 0$.

Theorem 7 ([31]) *Let $l_n(D)$ be nth order operator of the form (53) with $\varepsilon = M(x,t) = 0$ and let $q_j \in A(0,b)(C^\infty(0,l))$ for $j \in \{1,\dots,[n/2]\}$. Assume also that Eq. (53) admits a triangular transformation operator, i.e. that representation (55) holds with a kernel $R_1(x,t) \in C^n(\overline{\Omega})$.*

Then $q_j \in A(0,b)(C^\infty(0,l))$ for $j \in \{[n/2]+1,\dots,n-1\}$.

***Proof** (Sketch of the Proof)* Let transformation operator $I+R_1$ exist. Then the kernel $R_1(x,t)$ satisfies Eq. (23), conditions (25) and one more condition (51). First we show that there exists another solution $\widehat{R}(x,t)$ of Eq. (23) satisfying conditions (25) and one more condition

$$\widehat{R}(l,t) = 0 \tag{65}$$

instead of condition (51). Setting $u(x,t) := D_x \widehat{R}(x,t)$ we get from (65) that

$$\widehat{R}(x,t) = D_1^{-1} u := \int_l^x u(\xi, t)\, d\xi.$$

Assuming that n is odd, $n = 2m + 1$ and inserting this expression in Eq. (23) we conclude that $u(x,t)$ satisfies the elliptic equation

$$\sum_{j=1}^{n} \binom{n}{j} D_1^{j-1} D_2^{n-j} u(x,t) = -\sum_{j=1}^{n-1} q_j(x) \sum_{i=0}^{n-1-j} \binom{n-1-j}{i} D_1^{i-1} D_2^{n-1-i-j} u(x,t). \tag{66}$$

and initial conditions (25). Since the coefficients of the principle part of $2m$th equation (66) are real, the operator is properly elliptic.

Next we express the second part of the coefficients q_j, $j \in \{m+1, \dots, 2m\}$ by means of the first one and functions $D_2^j u(x,t)|_{t=0}$, $j \in \{m, \dots, 2m\}$, and insert these expressions into Eq. (66). Summing up we arrive at the Dirichlet problem at $t = 0$ for the nonlinear integro-differential properly elliptic equation. Assuming that $q_j \in C^\infty(0,l)$ for $j \in \{1, \dots, m\}$ one gets that the right hand side of (66) is in $C^1(\overline{\Omega})$. Since the Dirichlet problem satisfies the covering condition, we apply the regularity result for elliptic problem and obtain that $u(x,t) \in C^{n+1}(\Omega_1)$ where $\Omega_1 = \{0 \le t < x < l\}$. Repeating this reasoning we conclude that $u(x,t) \in C^\infty(\Omega_1)$.

Now returning to the conditions (25) we obtain that $q_j \in C^\infty(0,l)$ for $j \in \{m+1, \dots, 2m\}$.

If $q_j \in C^\infty(0,l)$ for $j \in \{1, \dots, m\}$, we use generalization of the Morrey-Nirenberg approach to prove analyticity up to the boundary of the solutions to Dirichlet problem for the above elliptic non-linear integro-differential equation. □

Remark 6 Special cases of Theorem 7 have earlier been established by V.I. Macaev [29] and L.A. Sakhnovich [43]. More precisely, V. Macaev established absence of triangular transformation operators for two-term equation $y^{(n)} + qy = \lambda y$ with a coefficient q being analytical on $[0, 1/2)$ and vanishing on $[1/2, 1)$. Clearly, $q \notin A(0,1)$ and this result is immediate from Theorem 7. The detailed discussion of this question can be found in the monographs [17] and [47].

The following result shows that Theorem 7 is sharp.

Theorem 8 ([31]) *Let $q_j \in A[0,l]$ for $j \in \{1, \dots, [n/2] - 1\}$. Then there exists an operator $l_n(D)$ admitting triangular transformation operator of the form (55) with $q_j \notin A[0,l]$ for $j \in \{[n/2], \dots, n-1\}$.*

For non-integer $\alpha = n - \varepsilon$, $\varepsilon \in (0,1)$, a weaker result is known.

Theorem 9 ([32, Theorem 4]) *Let $l \le \infty$ and let for Eq. (53) there exists triangular transformation operator $I + R_1$ of the form (55) with $R_1(x,t) \in C^{n+\varepsilon}(\overline{\Omega})$.*

Assume also that $q_j(\cdot) \in C^\infty(0,l)$ *for* $j \in \{1,\dots,[n/2]\}$ *and* $\widetilde{M}(x,t_0) := M(x, x-t_0) \in C^\infty[t_0,l]$ *for each* $t_0 \in [0,l]$. *Then* $q_j(x) \in C^\infty(0,l)$ *for* $j \in \{[n/2]+1,\dots,n-1\}$.

Apparently Theorem 7 remains valid for fractional order equations with $\varepsilon \in (0,1)$. In other words, in Theorem 9 inclusions $q_j(\cdot) \in C^\infty(0,b)$ (both for the assumption and the conclusion) can be replaced by $q_j(\cdot) \in A(0,b)$.

Example 2 For two-term equation

$$y^{(n-\varepsilon)} + q(x)y^{-\varepsilon} = \lambda y \tag{67}$$

the inclusion $q \in A(\mathbb{R})$ is sufficient for Eq. (67) to admit triangular transformation operator, while condition $q \in C^\infty(0,b)$ is necessary.

If $\varepsilon = 0$, then the stronger inclusion $q \in A(0,b)$ is necessary.

5 Uniqueness Results

5.1 Fractional Order Equations

Here we employ transformation operators to prove unique recovery of Eq. (53) with $M(x,t) = 0$ by n spectra of boundary value problems.

So, consider Eq. (53) with $M(x,t) = 0$, namely

$$l_{n-\varepsilon}(D)y_1 = D_x^{n-\varepsilon} y_1(x,\lambda) + \sum_{j=1}^{n-1} q_j(x) D_x^{n-1-j-\varepsilon} y_1(x,\lambda) = \lambda y_1(x,\lambda), \tag{68}$$

and impose the following boundary conditions

$$U_i y_1 = \sum_{j=1}^{n} h_{ij} y_1^{(j-1-\varepsilon)}(0,\lambda) = 0, \qquad i \in \{1,\dots,n-1\}, \tag{69}$$

$$V_r y_l = \sum_{j=1}^{n} H_{ir} y_l^{(j-1)}(l,\lambda) = 0. \tag{70}$$

Assuming the forms $\{U_i\}_l^{n-1}$ to be linearly independent and fixing $r \in \{1,\dots,n\}$, we denote by $S_r\big(\{U_i\}_1^{n-1}; V_r; \{q_j\}_l^{n-1}\big)$ the spectrum of the problem (68)–(70) taking multiplicity into account.

Next, let Δ_j be the minor of the matrix $\|h_{il} h_{i2} \dots h_{in}\|_{i=1}^{n-1}$ obtained by deleting the jth column.

Finally, alongside Eq. (68) we consider the analogous equation with entire coefficients $\{\widetilde{q}_j(x)\}_1^{n-1}$,

$$\widetilde{l}_{n-\varepsilon}(D)y_2 = D_x^{n-\varepsilon} y_2(x,\lambda) + \sum_{j=1}^{n-1} \widetilde{q}_j(x) D_x^{n-1-j-\varepsilon} y_2(x,\lambda) = \lambda y_2(x,\lambda), \tag{71}$$

subject to the boundary conditions

$$\widetilde{U}_i y_1 = \sum_{j=1}^{n} \widetilde{h}_{ij} y_1^{(j-1-\varepsilon)}(0,\lambda) = 0, \quad i \in \{1,\dots,n-1\}, \tag{72}$$

$$V_r y_l = \sum_{j=1}^{n} H_{ir} y_l^{(j-1)}(l,\lambda) = 0. \tag{73}$$

Similarly, assuming the forms $\{\widetilde{U}_i\}_1^{n-1}$ to be linearly independent, we denote by $S_r\big(\{\widetilde{U}_i\}_1^{n-1}; V_r; \{\widetilde{q}_j\}_l^{n-1}\big)$ the spectrum of the problem (71)–(73) counting multiplicities. Finally, denote by $\widetilde{\Delta}_j$ the minor of the matrix $\|\widetilde{h}_{i1}\widetilde{h}_{i2}\dots\widetilde{h}_{in}\|_{i=1}^{n-1}$ obtained by deleting the jth column.

Now we are ready to state the main result of this section.

Theorem 10 ([32, Theorem 3]) *Let $n \geq 2$ and $l_{n-\varepsilon}(D)$, $\widetilde{l}_{n-\varepsilon}(D) \in A(D)$. Suppose also that the forms $\{V_r\}_1^n$ are linearly independent, and the following n spectra*

$$S_r\big(\{U_i\}_1^{n-1}; V_r; \{q_j\}_1^{n-1}\big) = S_r\big(\{\widetilde{U}_i\}_1^{n-1}; V_r; \{\widetilde{q}_j\}_1^{n-1}\big), \qquad r \in \{1,\dots,n\}, \tag{74}$$

for Eqs. (68) and (71) coincide. Here if $\Delta_j = 0$ $(1 \le j \le p-1)$ and $\Delta_p \neq 0$, then $\widetilde{\Delta}_j = 0$ (for all $j \le p-1$), and $\widetilde{\Delta}_p \neq 0$. Then

$$q_j(x) = \widetilde{q}_j(x) \qquad \textit{for} \qquad j \in \{1,\dots,n-1\},$$

and there exists a constant $c_0 \neq 0$ such that $\widetilde{\Delta}_j = c_0\Delta_j$ for j $\in \{1,\dots,n-1\}$.

Proof ***(Sketch of the Proof)*** Since the forms $\{U_j\}_1^{n-1}$ are linearly independent, it follows from (69) that

$$y_1^{(j-1-\varepsilon)}(0,\lambda) = C_1\Delta_j, \qquad C_1 \neq 0, \quad j \in \{1,\dots,n\}. \tag{75}$$

Similarly, we get

$$y_2^{(j-1-\varepsilon)}(0,\lambda) = C_2\widetilde{\Delta}_j, \qquad C_2 \neq 0, \quad j \in \{1,\dots,n\}. \tag{76}$$

Without loss of generality we can assume that $C_2\widetilde{\Delta}_p = C_1\Delta_p(\neq 0)$.

On the other hand, it follows from (55), that each $y_j(\cdot,\lambda)$ admits a triangular representation

$$y_j(x,\lambda) = (I+R_j)w(x,\lambda) := w(x,\lambda) + \int_0^x R_j(x,t)w(t,\lambda)\,dt, \qquad j \in \{1,2\}, \tag{77}$$

where both representations hold with the same $w(\cdot,\lambda)$ being the solution of the Cauchy problem of the simplest equation (56) with $\beta_j = 0$ for $j < p$ and $\beta_p = C_1\Delta_p = C_2\widetilde{\Delta}_p \neq 0$.

Each of the boundary conditions $V_r y_1 = 0$ generates the characteristic determinant of the problem (68)–(70). Taking representation (77) into account this determinant can be written in the form

$$F_r(\lambda) = \sum_{k=1}^{n} A_{kr} w^{(k-1)}(b,\lambda) + \int_0^b Q_r(t)w(t,\lambda)\,dt = 0, \tag{78}$$

where

$$Q_r(t) = \sum_{j=1}^{n} H_{jr} D_1^{j-1} R_1(b,t), \qquad A_{kr} = \sum_{j=k}^{n} a_{jk} H_{jr}, \tag{79}$$

and a_{jk} $(1 \le k \le j \le n)$ are linearly expressed by means of the values of the kernel $R_1(x,t)$ and its partial derivatives at the point (b,b).

Further, it is shown in a standard way that the spectrum $S_r\left(\{U_i\}_1^{n-1}; V_r; \{q_j\}_1^{n-1}\right)$ of the problem (68)–(70) coincides with zeros of the entire function $F_r(\cdot)$ (the characteristic determinant) counting multiplicities. Being an entire function of order $(n-\varepsilon)^{-1} < 1$ the function $F_r(\cdot)$ is uniquely determined by its zeros up to a multiplicative constant.

Next we show that the family $\{F_r(\cdot)\}_1^n$ determines uniquely the functions

$$D_1^j R_1(b,t) := D_x^j R_1(x,t)\restriction_{x=b}, \qquad t \in [0,b], \qquad j \in \{1,\dots,n\}. \tag{80}$$

Similar reasoning with respect to the problem (71)–(73) together with condition (74) of the theorem yields

$$D_1^j R_1(b,t) = D_1^j R_2(b,t), \qquad t \in [0,b], \qquad j \in \{1,\dots,n\}. \tag{81}$$

Starting with these relations it is established that

$$D_1^k D_2^j \widetilde{R}_1(b,0) = D_1^k D_2^j \widetilde{R}_2(b,0), \qquad j,k \in \mathbb{N} \cup 0.$$

Using analyticity of both kernels we arrive at the equality $R_1(x,t) = R_2(x,t)$ in $\overline{\Omega}$. This relation implies the required uniqueness. □

If a part of coefficients $\{q_j\}_1^{n-1}$ is known, the number of spectra required for the unique determination of Eq. (68) can be reduced.

Theorem 11 ([32]) *Assume that* $l_{n-\varepsilon}(D)$ *and* $\widetilde{l}_{n-\varepsilon}(D)$ *be the operators of the form* (68) *and* (71)*, respectively,* $n \geq 3$*, and*

$$V_r(y) = y^{(r-1)}(l), \qquad r \in \{1, \dots, n\}. \tag{82}$$

Let also $\{U_j\}_1^{n-1}$, $\{\widetilde{U}_j\}_1^{n-1}$ *be the linear forms given by* (69) *and* (72)*, respectively. Assume also that the following* $k+1$ *spectra for operators* $l_{n-\varepsilon}(D)$ *and* $\widetilde{l}_{n-\varepsilon}(D)$ *coincide:*

$$S_r\big(\{U_i\}_1^{n-1}; V_r; \{q_j\}_1^{n-1}\big) = S_r\big(\{\widetilde{U}_i\}_1^{n-1}; V_r; \{\widetilde{q}_j\}_1^{n-1}\big), \qquad r \in \{1, \dots, k+1\}, \tag{83}$$

and $q_j(x) = \widetilde{q}_j(x)$ *for* $j \in \{1, \dots, n-k-1\}$*. Then*

$$l_{n-\varepsilon}(D) = \widetilde{l}_{n-\varepsilon}(D), \qquad \textit{i.e.} \quad q_j(x) = \widetilde{q}_j(x) \quad \textit{for} \quad j \in \{1, \dots, n-1\}.$$

Proof Here we propose another approach. Namely, we reduce the problem to Goursat problem for linear fractional order partial differential equation. To this end starting with representations (77) we set $I + R := (I + R_2)(I + R_1)^{-1}$. Clearly, R is a triangular operator with the kernel $R(\cdot, \cdot)$ given by

$$R(x,t) = R_2(x,t) + P_1(x,t) + \int_t^x R_2(x,s)P_1(s,t)ds \tag{84}$$

where $I + P_1 = (I + R_1)^{-1}$. It follows that

$$y_2(x,\lambda) = (I+R)y_1(x,\lambda) = y_1(x,\lambda) + \int_0^x R(x,t)y_1(t,\lambda)dt \tag{85}$$

It can be shown similarly to the deduction of Eq. (18) that the kernel $\widetilde{R}(x,t) := R(x, x-t)$ is a solution of the following nth order equation with partial derivatives

$$\begin{aligned}
&\sum_{j=1}^{n} \binom{n-\varepsilon}{j} D_1^j D_2^{n-j} \widetilde{R}(x,t) + \sum_{j=1}^{n-1} [\widetilde{q}_j(x) - q_j(t)] \sum_{i=0}^{n-1-j} \binom{n-\varepsilon-1-j}{i} D_1^i D_2^{n-1-i-j} \widetilde{R}(x,t) \\
&\quad = \int_0^t d\xi \int_0^1 \frac{(1-\beta)^{\varepsilon}\beta^{n-\varepsilon}}{\Gamma(\varepsilon)\Gamma(1-\varepsilon)} D_1^{n+1} \widetilde{R}\big(\xi + x - t + (t-\xi)\beta, \xi\big)\, d\beta \\
&\quad + \sum_{j=1}^{n-1} [\widetilde{q}_j(x) - q_j(t)] \int_0^t d\xi \int_0^1 \frac{(1-\beta)^{\varepsilon}\beta^{n-1-j-\varepsilon}}{\Gamma(\varepsilon)\Gamma(1-\varepsilon)} D_1^{n-j} \widetilde{R}\big(\xi + x - t + (t-\xi)\beta, \xi\big)\, d\beta
\end{aligned} \tag{86}$$

subject to the following initial conditions

$$D_2^{j-1} D_1 \widetilde{R}(x,0) = \varphi_j(x), \qquad j \in \{1, \dots, n-1\}. \tag{87}$$

Here $\varphi_j(\cdot)$ is expressed via coefficients $q_k(\cdot)$, $\widetilde{q}_k(\cdot)$, and the derivatives $D_1^{j-k-1}\widetilde{R}(x,0)$, $1 \le k \le j$, of the kernel $\widetilde{R}(x,t)$ with $t = 0$ by means of operations of addition, multiplication, and differentiation. For instance, $\varphi_1(x) = [q_1(x) - \widetilde{q}_1(x)](n-\varepsilon)^{-1}$.

As it is shown in the proof of Theorem 10, the coincidence of the $k+1$ spectra (83) yields

$$y_1^{(j)}(b,\lambda) = y_2^{(j)}(b,\lambda) \quad \text{for} \quad j \in \{0, 1, \dots, k\}. \tag{88}$$

In turn, combining these equalities with representation (85) implies

$$D_1^j \widetilde{R}(x,t)|_{x=b} = 0, \quad j \in \{0, 1, \dots, k\}. \tag{89}$$

Further, it easily follows from equalities $q_j(x) = \widetilde{q}_j(x)$, $j \in \{1, \dots, n-k-1\}$, and initial conditions (69), (72), that

$$D_2^{j-1} D_1 \widetilde{R}(x,0) = 0, \quad j \in \{1, \dots, n-k-1\}. \tag{90}$$

So, to prove the uniqueness result it suffices to show that the Goursat problem (86), (89), (90), has only trivial solution $R(x,t) \equiv 0$.

Setting $P(x,t) = D^{k+1}\widetilde{R}(x,t)$ and taking boundary conditions (89) into account, one gets

$$\widetilde{R}(x,t) = \int_b^x \frac{(x-\xi)^k}{k!} P(\xi,t)\, d\xi. \tag{91}$$

Inserting this expression for $\widetilde{R}(x,t)$ into Eq. (86), and initial conditions (90) we arrive at certain fractional order integro-differential equation

$$\widetilde{L}P = 0 \tag{92}$$

for $P(x,t)$ and initial conditions

$$D_2^{j-1} P(x,t)|_{t=0} = 0, \qquad j \in \{1, \dots, n-k-1\}. \tag{93}$$

Using these relations and Eq. (92) we show by induction that

$$D_1^k D_2^j P(x,t)|_{x=l,t=0} = 0, \qquad j,k \in \mathbb{N} \cup \{0\}. \tag{94}$$

Since the kernel $P(x,t)$ is analytical in certain domain $\Omega' \supset \overline{\Omega}$, relations (94) yield $P(x,t) \equiv 0$ in Ω. In turn, this implies $R_1(x,t) \equiv R_2(x,t)$, $\{x,t\} \in \overline{\Omega}$. The required relations

$$q_j(x) = \widetilde{q}_j(x), \qquad j \in \{n-k, \dots, n-1\}, \tag{95}$$

are extracted now from initial conditions (19) for $\widetilde{R}_1(x,t) := R_1(x, x-t)$ and $\widetilde{R}_2(x,t) := R_2(x, x-t)$. □

Example 3 Let $l_{n-\varepsilon}(D)$ and $\widetilde{l}_{n-\varepsilon}(D)$ be two-terms fractional $(n-\varepsilon)$-order operators,

$$l_{n-\varepsilon}(D) = D^{n-\varepsilon} + q \quad \text{and} \quad \widetilde{l}_{n-\varepsilon}(D) = D^{n-\varepsilon} + \widetilde{q}$$

where q and $\widetilde{q}$ are entire coefficients. Then $q = \widetilde{q}$ whenever two spectra coincide.

Remark 7 Note that in the case of integer $\alpha = n (\Longleftrightarrow \varepsilon = 0)$ and $k = n-1$ the second part of the proof of Theorem 11 is immediate from the Cauchy-Kovalevskii theorem. Namely, the Cauchy-Kovalevskii theorem applied to the non-characteristic Cauchy problem (86), (90) yields uniqueness: $\widetilde{R}(x,t) \equiv 0$ in Ω, and hence $R_1(x,t) \equiv R_2(x,t)$ in Ω.

However, the following more general uniqueness result holds.

Theorem 12 ([32]) *Let $l_n(D)$ and $\widetilde{l}_n(D)$ be operators of the form* (68) *and* (71) *with $\varepsilon = 0$ and smooth coefficients $q_j, \widetilde{q}_j \in C^{n-j}[0,b]$, $j \in \{1, \dots, n-1\}$, and let the boundary forms $\{U_j\}_1^{n-1}$, $\{\widetilde{U}_j\}_1^{n-1}$, V_r, be the same as in Theorem 11. Assume also that the following conditions hold:*

(i) *Equations* (68) *and* (71) *admit triangular transformation operators;*
(ii) *the spectra* (83) *coincide for $r \in \{1, \dots, k+1\}$ with some $k \le n-1$;*
(iii) *$q_j(x) = \widetilde{q}_j(x)$ for $j \in \{1, \dots, n-k-1\}$.*

Then $q_j(x) = \widetilde{q}_j(x)$ for $j \in \{1, \dots, n-1\}$.

Proof Let us explain the scheme of the proof assuming that $k = n-1$. In accordance with condition (i) the solutions $y_j(x,\lambda)$ admit representations (77), and hence $y_2(x,\lambda)$ and $y_1(x,\lambda)$ are related by formula (85) with $I + R = (I + R_2)(I + R_1)^{-1}$. Now the kernel $\widetilde{R}(x,t) = R(x, x-t)$ satisfies the partial differential equation

$$\sum_{j=1}^{n} \binom{n}{j} D_1^j D_2^{n-j} \widetilde{R}(x,t) = \sum_{j=1}^{n-1} \left[q_j(t) - \widetilde{q}_j(x)\right] \sum_{i=0}^{n-1-j} \binom{n-1-j}{i} D_1^i D_2^{n-1-i-j} \widetilde{R}(x,t) \tag{96}$$

and the following boundary conditions

$$D_1^j \widetilde{R}(x,t)|_{x=b} = 0, \quad j \in \{0,1,\ldots,n-1\}. \tag{97}$$

As in the proof of Theorem 11 it suffices to show that the non-characteristic Cauchy problem (96)–(97) has only trivial solution.

Let $L_n(\zeta) = L_n(x,\zeta)$ be the principle (homogeneous) symbol of the operator $L_n(x,\zeta)$ in the left hand side of Eq. (96). Clearly, $L_n(\zeta)$ is a polynomial (with constant coefficients) in ζ with constant (not depending on x) coefficients. It is easily seen that for each fixed ζ the polynomial $L_n(\zeta+\tau N)$ in τ has no multiple roots whenever $\zeta+\tau N \neq 0$. Therefore the Calderon theorem (see [14]) ensures local triviality of the solution of the Cauchy problem (96)–(97).

To prove global uniqueness we reduce the problem to the equivalent Cauchy problem for elliptic equation. To this end we note that for odd n the polynomial $L_n(\zeta)$ admits the factorization $L_n(\zeta) = \zeta_1 Q_{n-1}(\zeta)$ where $Q_{n-1}(\zeta)$ is already elliptic polynomial with real (constant) coefficients.

Setting $P(x,t) = D_1\widetilde{R}(x,t)$ and inserting this expression into (96)–(97) we arrive at the following Cauchy problem

$$Q_{n-1}(D)P(x,t) = \sum_{j=1}^{n-1}[q_j(t)-\widetilde{q}_j(x)] \sum_{i=0}^{n-1-j} \binom{n-1-j}{i} D_1^{i-1} D_2^{n-1-i-j} P(x,t), \tag{98}$$

$$D_1^j P(x,t)|_{x=b} = 0, \qquad j \in \{0,1,\ldots,n-2\}, \tag{99}$$

for elliptic operator $Q_1(D)$. Here

$$D_1^{-1} D_2^{n-1-j} \widetilde{R}(x,t) := \int_b^x D^{n-1-j} P(\xi,t)d\xi. \tag{100}$$

Now the Kalderon uniqueness result [14] (in fact, its generalization to elliptic equation containing integral terms of the form (100)) yields $P(x,t) \equiv 0$, hence $\widetilde{R}(x,t) = R(x,t) \equiv 0$ and $R_1(x,t) \equiv R_2(x,t)$. Since each of the operators $l_n^0(D)$ and $\widetilde{l}_n^0(D)$ is similar to the operator D_n^0 (see identity (58)), one gets

$$l_n^0(D) = (I+R_1)^{-1}D_n^0(I+R_1) = (I+R_2)^{-1}D_n^0(I+R_2) = \widetilde{l}_n^0(D). \tag{101}$$

This identity yields (in fact, is equivalent to) $q_j(x) = \widetilde{q}_j(x)$ for $j \in \{1,\ldots,n-1\}$.

The case of even $n = 2n_1$ is more cumbersome because the principle symbol admits now the following factorization: $L_n(\zeta) = \zeta_1 Q_{n-2}(\zeta)[\zeta_1^2 + 2\zeta_1\zeta_2]$ where $Q_{n-2}(\zeta)$ is elliptic polynomial of degree $n-2$ with real coefficients. The rest of reasonings is similar to the case of odd n. □

Remark 8 Theorem 3 can be regarded as a generalization of the well known result of Borg and Marchenko (see [37, 38]), [24, 25] on the unique determination of the Sturm-Liouville operator from two spectra. For equations of integer orders (that is, for $\varepsilon = 0$) and $H_{jk} = \delta_{jk}$ (δ_{jk} is the Kronecker delta) Theorem 10 was proved independently by the author and Yurko [49]. The method of [49] does not use transformation operators technique. Note also that I. Khachatryan [19] applied triangular transformation operators to investigate the scattering problem for nth order equation with analytic coefficients.

5.2 *First Order Systems of Ordinary Equations*

Here we consider some recent uniqueness results on first order systems of ODE. Let B be a non-singular diagonal $n \times n$-matrix

$$B = \mathrm{diag}(b_1 I_{n_1}, \ldots, b_r I_{n_r}) \in \mathbb{C}^{n\times n}, \qquad n = n_1 + \ldots + n_r, \tag{102}$$

with pairwise different complex numbers, $b_j \neq b_k$ for $j \neq k$.

Consider a system of differential equations

$$\mathcal{P}y := -iB^{-1}y' + Q(x)y = \lambda y, \quad y = \mathrm{col}(y_1, \ldots, y_n), \tag{103}$$

on the interval $[0, 1]$ with a summable potential matrix $Q \in L^1[0, 1] \otimes \mathbb{C}^{n\times n}$.

Systems of the form (103) play important role in theoretical and practical problems. For instance, for $n = 2m$, $B = \mathrm{diag}(-I_m, I_m)$ and $Q_{11} = Q_{22} = 0$ the system (103) is equivalent to $1D$ stationary Dirac system (see [26, Chapter 7]). Further nth order ordinary differential equation is reduced to system (103) with $r = n$ and $b_j = \exp(2\pi ij/n)$, $j \leq n$ (see [33]). Note also that system (103) appears in the Lax representation for n-waves equation arisen in non-linear optic [40, Chapter 3].

Alongside with Eq. (103) consider vector equation

$$\widetilde{\mathcal{P}}y := -iB^{-1}\widetilde{y}' + \widetilde{Q}(x)\widetilde{y} = \lambda\widetilde{y}, \quad \widetilde{y} = \mathrm{col}(\widetilde{y}_1, \ldots, \widetilde{y}_n), \tag{104}$$

with a summable potential matrix $\widetilde{Q} \in L^1[0, l] \otimes \mathbb{C}^{n\times n}$.

Assume also that with respect to the orthogonal decomposition $\mathbb{C}^n = \oplus_{j=1}^r \mathbb{C}^{n_j}$ the potential matrices $Q(\cdot)$ and $\widetilde{Q}(\cdot)$ have zero diagonals,

$$Q = (Q_{jk})_{j,k=1}^r, \quad \widetilde{Q} = (\widetilde{Q}_{jk})_{j,k=1}^r, \quad Q_{jk}, \widetilde{Q}_{jk} : [0, 1] \to \mathbb{C}^{n_j\times n_k},$$
$$Q_{jj}(x) = \widetilde{Q}_{jj}(x) = 0, \quad x \in [0, l], \quad j \in \{1, \ldots, r\}. \tag{105}$$

Theorem 13 ([34]) *Let $T \in \mathbb{C}^{n\times n}$, $\det T \neq 0$, and let $Q = (Q_{jk})_{j,k=1}^r$ and $\widetilde{Q} = (\widetilde{Q}_{jk})_{j,k=1}^r$ be potential matrices of the form (105). Let also $W(\cdot,\lambda)$ and $\widetilde{W}(\cdot,\lambda)$ be $n \times n$ fundamental matrices of solutions of Eqs. (103) and (104), respectively, satisfying the common initial condition*

$$W(0,\lambda) = \widetilde{W}(0,\lambda) = T, \qquad \lambda \in \mathbb{C}. \tag{106}$$

Then $Q(x) = \widetilde{Q}(x)$ for a.e. $x \in [0,1]$ whenever

$$W(\lambda) := W(l,\lambda) = \widetilde{W}(l,\lambda) =: \widetilde{W}(\lambda).$$

If the spectrum of the matrix B is simple ($p := n_1 = \ldots = n_r = 1$), i.e. $r = n$, this theorem was proved by another method by Z.L. Leibenzon [22].

Let $W(\cdot) = (W_{jk}(\cdot))_{j,k=1}^r$ be the block-matrix representation of the monodromy matrix $W(\cdot)$ with respect to the orthogonal decomposition $\mathbb{C}^n = \oplus_{j=1}^r \mathbb{C}^{n_j}$. In the self-adjoint case $\big(B = B^*,\ Q(\cdot) = Q^*(\cdot)\big)$ it is shown in [34] that for $n_1 = \ldots = n_r$ and a special choice of the matrix T in (106), a potential matrix $Q(\cdot)$ is uniquely determined from a certain (not arbitrary) system of $r(r-1)/2$ matrix functions

$$M_{jk}(\cdot) := W_{jk}(\cdot)W_{kk}^{-1}(\cdot), \qquad j,k \in \{1,\ldots,r\}.$$

In particular, $Q(\cdot)$ is uniquely determined by $(r-1)(r+2)/2 = r(r-1)/2 + r - 1$ block-matrix entries of the monodromy matrix $W(\cdot)$.

Note that despite of existence triangular transformation operators for Eq. (103) in selfadjoint case the proof of just mentioned result as well as Theorem 13 have not used this fact. More precisely, it is shown in [33] that Eq. (103) with $B = B^*$ and $Q(\cdot) = Q^*(\cdot) \in L^\infty[0,1] \otimes C^{n\times n}$ admit transformation operators. This result was applied there to prove much weaker result: unique determination of the potential matrix $Q(\cdot) = Q^*(\cdot)$ by $(r-1)$ columns of the monodromy matrix $W(\lambda)$. It is also shown in [33, Theorem 3.4] that if $Q(\cdot)$ and $\widetilde{Q}$ admit a continuation to a holomorphic entire function and $\arg b_j \neq \arg b_k$ for all $j \neq k$, then $Q(\cdot)$ is uniquely determined by a column of the monodromy matrix $W(\lambda)$, i.e. by its r matrix entries.

Note also that in [23] triangular transformation operators was applied to solve inverse spectral problem for selfadjoint equation (103) on the half-line, i.e. in $L^2(\mathbb{R}_+) \otimes \mathbb{C}^{n\times n}$. The Gel'fand-Levitan type equation was obtained and investigated there. The case of 2×2-Dirac operators was earlier treated in [26]. We also refer the reader to the survey [10] and the monograph [50] precisely treated different kind of inverse problems.

Numerous applications of transformation operators to different problems of mathematical physics can be found in [17] and [47].

We finalize this section by considering one more uniqueness problem for a canonical system

$$J\frac{dy}{dt} = \lambda \mathcal{H}(t)y(t), \qquad J = -J^* = -J^{-1}, \qquad y = \mathrm{col}(y_1,\ldots,y_n), \tag{107}$$

on a finite interval $[0, l]$ with $n \times n$ Hamiltonian $\mathcal{H}(\cdot) \geq 0$. Denote by $W(x, \lambda)$ the fundamental $n \times n$ matrix solution of Eq. (107) satisfying the initial condition $W(0, \lambda) = I_n$. The matrix function $W(\lambda) := W(l, \lambda)$ is called the monodromy matrix $W(\lambda)$. *The problem of unique recovery of the Hamiltonian by monodromy matrix* $W(\lambda)$ has attracted a lot of attention.

In the definite case ($J = iI_n$) the most complete result was obtained by G. Kisilevskii [20] (see also [11]). Complete solution to this problem in indefinite case was obtained by de Brange [7] for $n = 2$ and $iJ = \text{diag}(1, -1)$ for real normed $\big(\text{tr}\mathcal{H}(t) \equiv 1\big)$ Hamiltonian. For $n > 2$ and $J \neq iI_n$ some partial uniqueness results are also known (see e.g. [4, 34], and references therein).

6 Completeness of Root Functions of BVPs for Fractional Order Ordinary Differential Equations

Let $\alpha = n - \varepsilon$, where $n \in \mathbb{N}, \ 0 \leq \varepsilon < 1$. Now we consider the equation

$$l_\alpha(D)y = D_x^{n-\varepsilon} y + \sum_{k=2}^{n} p_{n-k}(x) D_x^{n-k-\varepsilon} y = \lambda y \tag{108}$$

subject to the splitting boundary conditions:

$$U_j(y) = \sum_{k=0}^{n-1} \alpha_{jk} y^{(k-\varepsilon)}(0) = 0, \qquad 1 \leq j \leq l, \tag{109}$$

$$U_j(y) = \sum_{k=0}^{n-1} \beta_{jk} y^{(k)}(1) = 0, \qquad l+1 \leq j \leq n. \tag{110}$$

Let us denote by L the operator generated by the differential expression $l_\alpha(D)$ and the boundary conditions (109), (110).

The classical Birkhoff theorem [5] (see also [39, Chapter 2, Theorem 1]) treats the existence of solutions of n-th order differential equation on a finite interval with exponential asymptotics. The following theorem extends the Birkhoff result to the case of fractional order differential equation of the form (108). At first we consider the simplest differential equation

$$z^{(\alpha)}(x, \lambda) = \lambda z(x, \lambda). \tag{111}$$

of fractional order $\alpha = n - \varepsilon$ $(n \in \mathbb{Z}_+ \setminus \{0\}, \quad 0 \le \varepsilon < 1)$. Denote by $\{e_j(x;\lambda)\}_1^n$ the fundamental system of solutions of Eq. (111). And let us denote

$$n_1 := [(1-n+\varepsilon)/2], \quad n_2 := [(n-\varepsilon)/2] \text{ and } \omega_j = \exp(2\pi i j/\alpha), \quad j \in \{n_1, n_1+1, \dots, n_2\}.$$

Next we set

$$\beta = 2\pi \min\left\{\left\{\frac{n-\varepsilon}{4}\right\}, 1 - \left\{\frac{n-\varepsilon}{4}\right\}\right\}$$

where, as usual, $\{x\}$ stands for the fractional part of a number $x \in \mathbb{R}$.

Consider the following sectors in the complex plane:

$$\begin{aligned} \tilde{S}_\beta^- &= \{\lambda \in \mathbb{C} : -\pi < \arg\lambda < -\beta\}; \\ S_\beta^- &= \{\lambda \in \mathbb{C} : -\beta < \arg\lambda < 0\}; \\ S_\beta^+ &= \{\lambda \in \mathbb{C} : 0 < \arg\lambda < \beta\}; \\ \tilde{S}_\beta^+ &= \{\lambda \in \mathbb{C} : \beta < \arg\lambda < \pi\}. \end{aligned}$$

In each of these sectors we can put the numbers $\{\omega_j\}_{n_1}^{n_2}$ in order so that

$$\Re(\omega_{j_1}\lambda^{1/\alpha}) > \Re(\omega_{j_2}\lambda^{1/\alpha}) > \cdots > \Re(\omega_{j_q}\lambda^{1/\alpha}) > 0 > \Re(\omega_{j_{q+1}}\lambda^{1/\alpha}) > \cdots > \Re(\omega_{j_n}\lambda^{1/\alpha}).$$

Here $\lambda^{1/\alpha}$ stands for the branch of the corresponding multifunction in $\lambda \in \mathbb{C} \setminus \mathbb{R}$ fixed by the initial condition $1^{1/\alpha} := 1$.

Theorem 14 ([36]) *Let $p_j(x) \in C[0,1]$ $(2 \le j \le n)$ and let S be one of the sectors S_β^+, S_β^-, $\tilde{S}_\beta^+$ and $\tilde{S}_\beta^-$. Then there exists a fundamental system of the solutions $\{y_k(x,\lambda)\}_1^n$ of Eq. (108) holomorphic with respect to $\lambda \in S_\beta(R_0) := \{\lambda \in S : |\lambda| > R_0\}$ with sufficiently large R_0 and satisfying the following asymptotic relations*

$$y_k(x;\lambda) = (1 + O(|\lambda|^{-\frac{1}{\alpha}}))e_k(x;\lambda), \qquad k \in \{1,\dots,n\},$$

and

$$D_x^{\nu-\varepsilon}(y_k(x;\lambda)) = (1 + O(|\lambda|^{-\frac{1}{\alpha}}))D_x^{\nu-\varepsilon}e_k(x;\lambda), \quad \nu \in \{0,1,\dots,n-1\}.$$

This theorem is substantially used in proving the following completeness result. Note that the resolvent of the operator L generated in the space $L_1[0,1]$ (or in $L_2[0,1]$) by problem (108)–(110) with $\varepsilon > 0$ can have exponential growth in any direction in the complex plane (as in the case $\varepsilon = 0$). Therefore the classical tests (like Keldysh theorem, Macaev theorem, or Lidskii-Keldysh theorem, etc.) cannot be applied to prove completeness since in each of these tests it is assumed (explicitly or tacitly) that there are rays in the complex plane on which the resolvent decays (the so called the rays of minimal growth).

Theorem 15 ([36]) *Let $p_j(x)$ $(2 \le j \le n)$ be entire analytic functions of $x \in \mathbb{R}$ and $2l \ge n$, $n \geqslant 3$. Then the system of root subspaces of the splitting boundary value problem* (108)–(110) *is complete and minimal in $L^1[0, 1]$.*

This theorem generalizes the result of A.A. Shkalikov [46]. The proof is heavily relied on Theorem 14 (the Birkhoff type result) and triangular transformation operators. Namely, first using Theorem 14 we reduce the proof of completeness of the system of root vectors of the problem (108)–(110) to the proof of completeness of the corresponding Cauchy problem. The latter is established by using triangular transformation operators (55).

Next we consider in $L_1[0, 1]$ the differential equation of order $2 - \varepsilon$, $\varepsilon \in (0, 1)$,

$$y^{(2-\varepsilon)} + q(x)y^{(-\varepsilon)} = \lambda y \tag{112}$$

subject to the boundary conditions

$$hy^{(-\varepsilon)}(0, \lambda) + y^{(1-\varepsilon)}(0, \lambda) = 0, \tag{113}$$

$$a_{21}y^{(-\varepsilon)}(0, \lambda) + a_{22}y^{(1-\varepsilon)}(0, \lambda) + a_{23}y^{(-\varepsilon)}(1, \lambda) + a_{24}y^{(1-\varepsilon)}(1, \lambda) = 0. \tag{114}$$

The following theorem is the main result.

Theorem 16 ([1]) *Let q be entire analytic function. Then the system of root functions of the problem* (112)–(114) *is complete in $L^1[0, 1]$.*

Finally, we consider in $L^1[0, 1]$ the differential equation of order $(1 - \varepsilon)$, $\varepsilon \in (0, 1)$,

$$y^{(1-\varepsilon)} + q(x)y^{(-\varepsilon)} = \lambda y, \tag{115}$$

subject to the boundary condition

$$y^{(-\varepsilon)}(0, \lambda) + hy^{(-\varepsilon)}(1, \lambda) = 0, \quad h \neq 0. \tag{116}$$

Theorem 17 ([3]) *Let q be entire analytic function. Then the system of root functions of the problem* (115), (116) *is complete in $L^1[0, 1]$.*

Detailed discussion of different boundary value problems for fractional order equations including equations (112) and (115) can be found in the recent survey [3].

Recently Riesz basis property of the systems of root vectors of boundary value problems for Dirac type operators with summable potential matrices have been investigated by different methods in [27, 28] and [45]. The corresponding result in [27, 28] was obtained by applying the technique of triangular transformation operators for Dirac type equations.

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Strong Solutions of Semilinear Equations with Lower Fractional Derivatives

Marina V. Plekhanova and Guzel D. Baybulatova

Abstract We find conditions of a unique strong solution existence for the Cauchy problem to solved with respect to the highest fractional Gerasimov–Caputo derivative semilinear fractional order equation in a Banach space with nonlinear operator, depending on the lower Gerasimov–Caputo derivatives. Then the generalized Showalter–Sidorov problem for semilinear fractional order equation in a Banach space with a degenerate linear operator at the highest order fractional derivative is researched in the sense of strong solution. The nonlinear operator in this equation depends on time and on lower fractional derivatives. The corresponding unique solvability theorem was applied to study of linear degenerate fractional order equation with depending on time linear operators at lower fractional derivatives. Applications of the abstract results are demonstrated on examples of initial-boundary value problems to partial differential equations with time-fractional derivatives.

Keywords Fractional order differential equation · Fractional Gerasimov–Caputo derivative · Degenerate evolution equation · Cauchy problem · Generalized Showalter–Sidorov problem · Initial boundary value problem

1 Introduction

Consider the semilinear equation of fractional order

$$D_t^{\alpha} Lx(t) = Mx(t) + N(t, D_t^{\alpha_1}x(t), D_t^{\alpha_2}x(t), \dots D_t^{\alpha_n}x(t)), \quad t \in (t_0, T), \tag{1}$$

where $L \in \mathscr{L}(\mathscr{X}; \mathscr{Y})$ (linear and continuous operator from a Banach space $\mathscr{X}$ into a Banach space $\mathscr{Y}$), $M \in \mathscr{C}l(\mathscr{X}; \mathscr{Y})$ (linear closed operator with dense domain

M. V. Plekhanova (✉)
South Ural State University, Chelyabinsk, Russia

Chelyanisk State University, Chelyabinsk, Russia
e-mail: kar@csu.ru

G. D. Baybulatova
Chelyanisk State University, Chelyabinsk, Russia

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_25

D_M in the space $\mathscr{X}$ and with the image in $\mathscr{Y}$), $n \in \mathbf{N}$, $N : \mathbf{R} \times \mathscr{X}^n \to \mathscr{Y}$ is a nonlinear operator, D_t^{α}, $D_t^{\alpha_1}$, $D_t^{\alpha_2}, \ldots, D_t^{\alpha_n}$ are the fractional Gerasimov–Caputo derivatives, $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n \le m-1 < \alpha \le m \in \mathbf{N}$. The equation is supposed to be degenerate, i. e. $\ker L \ne \{0\}$.

Equations, which are not solved with respect to highest integer order time-derivative are often found among non-classical equations of mathematical physics [1–5]. Interest in fractional order equations is associated with a lot of results of the fractional calculus successful applications in the mechanics of viscoelastic fluids [6], in the physics of real processes in fractal structures [7] and in many other areas.

The unique solvability of initial problems for linear degenerate fractional order equations was studied by many authors [5, 8–17]. In the papers [18–22] semilinear degenerate fractional order equations with nonlinear operator, depending on lower derivatives of integer orders, were investigated.

In the present work we research such equations with lower derivatives of fractional orders. Firstly conditions of a unique strong solution existence for the Cauchy problem to nondegenerate equation (1), i. e. with $\mathscr{X} = \mathscr{Y}$, $L = I$ (the identical operator). Then the unique solvability of the generalized Showalter–Sidorov problem

$$(Px)^{(k)}(t_0) = x_k, \quad k = 0, \ldots, m-1, \tag{2}$$

for degenerate semilinear fractional order Eq. (1) is researched. (The projection P will be defined further.) It is reduced to the Cauchy problem for the nondegenerate equation on a subspace and to equation with a nilpotent operator at the highest order derivative on its complement. Then this result was applied to the study of the linear degenerate fractional order equation

$$D_t^{\alpha} Lx(t) = Mx(t) + \sum_{k=1}^{n} N_k(t) D_t^{\alpha_k} x(t), \quad t \in (t_0, T),$$

with depending on t linear continuous operators N_k, $k = 1, 2, \ldots, n$, at the lower fractional derivatives. Applications of the abstract results are demonstrated on examples of initial-boundary value problems to partial differential equations with time-fractional derivatives.

2 Equations Solved with Respect to the Highest Derivative

2.1 Linear Equation

Let $\mathscr{Z}$ be a Banach space. Denote $g_\delta(t) = \Gamma(\delta)^{-1} t^{\delta-1}$, $\tilde{g}_\delta(t) = \Gamma(\delta)^{-1}(t-t_0)^{\delta-1}$, $J_t^{\delta} h(t) = (g_\delta * h)(t) = \int\limits_{t_0}^{t} g_\delta(t-s)h(s)ds$ for $\delta > 0, t > t_0$. Let $m-1 < \alpha \le m \in \mathbf{N}$,

D_t^m is the usual derivative of the order $m \in \mathbf{N}$, J_t^0 is the identical operator. The Gerasimov–Caputo derivative of a function h is (see [23, p. 11])

$$D_t^\alpha h(t) = D_t^m J_t^{m-\alpha}\left(h(t) - \sum_{k=0}^{m-1} h^{(k)}(t_0)\tilde{g}_{k+1}(t)\right), \quad t \geq t_0.$$

Consider the Cauchy problem

$$z^{(k)}(t_0) = z_k, \quad k = 0, 1, \ldots, m-1, \tag{3}$$

for the inhomogeneous differential equation

$$D_t^\alpha z(t) = Az(t) + f(t), \quad t \in [t_0, T], \tag{4}$$

where $A \in \mathscr{L}(\mathscr{Z}) := \mathscr{L}(\mathscr{Z}; \mathscr{Z})$, the function $f : [t_0, T] \to \mathscr{Z}$ is given for some $T > t_0$.

A strong solution of problem (3), (4) is a function $z \in C^{m-1}([t_0, T]; \mathscr{Z})$, such that

$$g_{m-\alpha} * \left(z - \sum_{k=0}^{m-1} z^{(k)}(t_0)\tilde{g}_{k+1}\right) \in W_q^m(t_0, T; \mathscr{Z})$$

and equalities (3), (4) are true. Here we use some $q > 1$.

For $\alpha, \beta > 0$ denote the Mittag-Leffler function $E_{\alpha,\beta}(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}$.

Theorem 1 ([19]) *Let $A \in \mathscr{L}(\mathscr{Z})$, $q > (\alpha - m + 1)^{-1}$, $f \in L_q(t_0, T; \mathscr{Z})$. Then for any $z_k \in \mathscr{Z}$, $k = 0, 1, \ldots, m-1$, there exists a unique strong solution of problem* (3), (4). *Moreover, it has the form*

$$z(t) = \sum_{k=0}^{m-1}(t-t_0)^k E_{\alpha,k+1}(A(t-t_0)^\alpha)z_k + \int_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s)ds. \tag{5}$$

2.2 *Semilinear Equation*

Let $m - 1 < \alpha \leq m \in \mathbf{N}$, $n \in \mathbf{N}$, an operator $B : (t_0, T) \times \mathscr{Z}^n \to \mathscr{Z}$ be nonlinear. Suppose that an operator B is the Caratheodory mapping, i. e. for every $z_1, z_2, \cdots, z_n \in \mathscr{Z}$ it defines measurable mapping on (t_0, T) and for almost all

$t \in (t_0, T)$ it is continuous in $z_1, z_2, \cdots, z_n \in \mathscr{Z}$. Consider Cauchy problem (3) for the nonlinear differential equation

$$D_t^{\alpha} z(t) = Az(t) + B(t, D_t^{\alpha_1} z(t), D_t^{\alpha_2} z(t), \ldots, D_t^{\alpha_n} z(t)) \tag{6}$$

where $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq m-1$.

A strong solution of problem (3), (6) is a function $z \in C^{m-1}([t_0, T]; \mathscr{Z})$, such that $g_{m-\alpha} * \left(z - \sum\limits_{k=0}^{m-1} z^{(k)}(t_0)\tilde{g}_{k+1} \right) \in W_q^m(t_0, T; \mathscr{Z})$, conditions (3) is satisfied and almost everywhere on (t_0, T) equality (6) is valid.

Lemma 1 *Let $l - 1 < \beta \leq l \in \mathbf{N}$, $t > t_0$. Then*

$$\exists C_{l,\beta} > 0 \quad \forall h \in C^l([t_0, t]; \mathscr{Z}) \quad \|D_t^{\beta} h\|_{C([t_0,t];\mathscr{Z})} \leq C_{l,\beta} \|h\|_{C^l([t_0,t];\mathscr{Z})}.$$

Proof For the function $f(t) = h(t) - \sum\limits_{k=0}^{l-1} h^{(k)}(t_0)\tilde{g}_{k+1}(t)$ we have $f^{(k)}(t_0) = 0$, $k = 0, 1, \ldots, l-1$. So at $\beta < l$

$$\|D_t^l J_t^{l-\beta} f(t)\|_{\mathscr{Z}} = \left\| D_t^l \int\limits_0^{t-t_0} \frac{s^{l-\beta-1} f(t-s)}{\Gamma(l-\beta)} ds \right\|_{\mathscr{Z}} =$$

$$= \|J_t^{l-\beta} f^{(l)}(t)\|_{\mathscr{Z}} \leq \frac{(t-t_0)^{l-\beta}}{\Gamma(l-\beta+1)} \|f^{(l)}\|_{C([t_0,t];\mathscr{Z})} \leq \frac{(t-t_0)^{l-\beta}}{\Gamma(l-\beta+1)} \|h\|_{C^l([t_0,t];\mathscr{Z})}.$$

In the case $\beta = l$ the statement is obvious. □

Lemma 2 *Let $A \in \mathscr{L}(\mathscr{Z})$, $q > (\alpha - m + 1)^{-1}$, $B : (t_0, T) \times \mathscr{Z}^n \to \mathscr{Z}$ be Caratheodory mapping, at all $y_1, y_2, \ldots, y_n \in Z$ and almost everywhere on (t_0, T) estimate*

$$\|B(t, y_1, y_2, \ldots, y_n)\|_{\mathscr{Z}} \leq a(t) + c \sum_{k=1}^{n} \|y_k\|_{\mathscr{Z}} \tag{7}$$

be true for some $a \in L_q(t_0, T; \mathbf{R})$, $c > 0$. Then a function $z \in C^{m-1}([t_0, T]; \mathscr{Z})$ is a strong solution of problem (3), (6), if and only if the equality

$$z(t) = \sum_{k=0}^{m-1} (t-t_0)^k E_{\alpha,k+1}(A(t-t_0)^{\alpha}) z_k +$$

$$+ \int\limits_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) B(s, D_t^{\alpha_1} z(s), D_t^{\alpha_2} z(s), \ldots, D_t^{\alpha_n} z(s)) ds \tag{8}$$

holds.

Proof Let $z \in C^{m-1}([t_0, T]; \mathscr{Z})$ be a solution of problem (3), (6), then $D_t^{\alpha_k} z \in C([t_0, T]; \mathscr{Z})$, $k = 1, 2, \ldots, n$, due to Lemma 1, and the inequality (7) implies that $B(\cdot, D_t^{\alpha_1} z(\cdot), D_t^{\alpha_2} z(\cdot), \ldots, D_t^{\alpha_n} z(\cdot)) \in L_q(t_0, T; \mathscr{Z})$. By Theorem 1 the solution satisfies Eq. (8).

Let a function $z \in C^{m-1}([t_0, T]; \mathscr{Z})$ satisfies Eq. (8), then reasoning as in the proof of Theorem 1 (see [19]) we obtain directly that the function z is a strong solution of problem (3), (6). □

Denote $\bar{z} = (z_1, z_2, \ldots, z_n)$. A mapping $B : (t_0, T) \times \mathscr{Z}^n \to \mathscr{Z}$ is called uniformly Lipschitz continuous in $\bar{z}$, if there exists $l > 0$, such that for almost all $t \in (t_0, T)$ and for all $\bar{z}, \bar{y} \in \mathscr{Z}^n$

$$\|B(t, \bar{z}) - B(t, \bar{y})\|_{\mathscr{Z}} \le l \sum_{k=1}^{n} \|z_k - y_k\|_{\mathscr{Z}}.$$

Remark 1 If $B : (t_0, T) \times \mathscr{Z}^n \to \mathscr{Z}$ is Caratheodory mapping, uniformly Lipschitz continuous in $\overline{z}$, and for some $\overline{x} \in \mathscr{Z}^n$ $B(\cdot, \overline{x}) \in L_q(t_0, T; \mathscr{Z})$, then condition (7) is valid at $a(t) = \|B(t, \overline{x})\|_{\mathscr{Z}} + l \sum\limits_{k=1}^{n} \|x_k\|_{\mathscr{Z}}$, $c = l$.

Theorem 2 *Suppose that* $A \in \mathscr{L}(\mathscr{Z})$, $q > (\alpha - m + 1)^{-1}$, $B : (t_0, T) \times \mathscr{Z}^n \to \mathscr{Z}$ *is Caratheodory mapping, uniformly Lipschitz continuous in* $\overline{z}$, *at all* $y_1, y_2, \ldots, y_n \in Z$ *and almost everywhere on* (t_0, T) *inequality* (7) *is valid for some* $a \in L_q(t_0, T; \mathbf{R})$, $c > 0$; $z_0, z_1, \ldots, z_{m-1} \in \mathscr{Z}$. *Then problem* (3), (6) *has a unique strong solution on* (t_0, T).

Proof By Lemma 2, it suffices to prove that Eq. (8) has a unique solution $z \in C^{m-1}([t_0, T]; \mathscr{Z})$. In Banach space $C^{m-1}([t_0, T]; \mathscr{Z})$ we define an operator G by the equality

$$G(y)(t) = \sum_{k=0}^{m-1} (t - t_0)^k E_{\alpha,k+1}(A(t - t_0)^{\alpha}) z_k +$$

$$+ \int_{t_0}^{t} (t - s)^{\alpha-1} E_{\alpha,\alpha}(A(t - s)^{\alpha}) B(s, D_t^{\alpha_1} y(s), D_t^{\alpha_2} y(s), \ldots, D_t^{\alpha_n} y(s))\, ds.$$

By Lemma 1 and Theorem 1 we have $G : C^{m-1}([t_0, T]; \mathscr{Z}) \to C^{m-1}([t_0, T]; \mathscr{Z})$.

Denote as G^r the r-th power of the operator G, $r \in \mathbf{N}$, and if $T - t_0 < 1$ we replace $T - t_0$ by 1 in the following reasoning. For $t \in [t_0, T]$, $r \in \mathbf{N}$,

$y, z \in C^{m-1}([t_0, T]; \mathscr{Z})$ we shall prove by the induction the inequality

$$\|G^r(y) - G^r(z)\|_{C^{m-1}([t_0,t];\mathscr{Z})} \leq \\ \leq \frac{(KCnlm)^r (T-t_0)^{(m-1)r}(t-t_0)^{\alpha-m+r}\|y-z\|_{C^{m-1}([t_0,t];\mathscr{Z})}}{(\alpha-m+1)^r r!}, \tag{9}$$

where $K = \max\limits_{k=0,1,\dots,m-1} E_{\alpha,\alpha-k}((T-t_0)^\alpha \|A\|_{\mathscr{L}(\mathscr{Z})})$, $C = \max\limits_{k=1,2,\dots,n} C_{m_k,\alpha_k}$, $m_k - 1 < \alpha_k \leq m_k \in \mathbf{N}$, C_{m_k,α_k} are the constants from Lemma 1, $k = 1, 2, \dots, n$.

Indeed, for $r = 1$, $k = 0, 1, \dots, m-1$ we have

$$\|[G(y)]^{(k)}(t) - [G(z)]^{(k)}(t)\|_{\mathscr{Z}} \leq E_{\alpha,\alpha-k}((t-t_0)^\alpha \|A\|_{\mathscr{L}(\mathscr{Z})})\times$$

$$\times \int_{t_0}^{t} (t-s)^{\alpha-k-1}\|B(s, D_t^{\alpha_1}y(s), \dots, D_t^{\alpha_n}y(s)) - B(s, D_t^{\alpha_1}z(s), \dots, D_t^{\alpha_n}z(s))\|_{\mathscr{Z}} ds \leq$$

$$\leq KCnl\|y-z\|_{C^{m-1}([t_0,t];\mathscr{Z})}\frac{(t-t_0)^{\alpha-k}}{\alpha-k},$$

$$\|G(y) - G(z)\|_{C^{m-1}([t_0,t];\mathscr{Z})} \leq \frac{KCnl}{\alpha-m+1}\|y-z\|_{C^{m-1}([t_0,t];\mathscr{Z})}\sum_{k=0}^{m-1}(t-t_0)^{\alpha-k} \leq$$

$$\leq \frac{KCnlm(T-t_0)^{m-1}}{\alpha-m+1}\|y-z\|_{C^{m-1}([t_0,t];\mathscr{Z})}(t-t_0)^{\alpha-m+1}.$$

Suppose that inequality (9) holds. Then we have

$$\|[G^{r+1}(y)]^{(k)}(t) - [G^{r+1}(z)]^{(k)}(t)\|_{\mathscr{Z}} \leq$$

$$\leq Kl\int_{t_0}^{t}(t-s)^{\alpha-k-1}\sum_{k=1}^{n}\|D_t^{\alpha_k}(G^r(y) - G^r(z))(s)\|_{\mathscr{Z}} ds \leq$$

$$\leq KCnl(T-t_0)^{\alpha-1}\int_{t_0}^{t}\|G^r(y) - G^r(z)\|_{C^{m-1}([t_0,s];\mathscr{Z})} ds \leq$$

$$\leq \frac{(KCnl)^{r+1}m^r(T-t_0)^{(m-1)(r+1)}}{(\alpha-m+1)^r r!(\alpha-m+r+1)}\|y-z\|_{C^{m-1}([t_0,s];\mathscr{Z})}(t-t_0)^{\alpha-m+r+1},$$

$$\|G^{r+1}(y) - G^{r+1}(z)\|_{C^{m-1}([t_0,t];\mathscr{Z})} \leq$$

$$\leq \frac{(KCnlm)^{r+1}(T-t_0)^{(m-1)(r+1)}}{(\alpha-m+1)^{r+1}(r+1)!}\|y-z\|_{C^{m-1}([t_0,s];\mathscr{Z})}(t-t_0)^{\alpha-m+r+1}.$$

From (9) it follows that for $r \in \mathbf{N}$

$$\|G^r(y) - G^r(z)\|_{C^{m-1}([t_0,T];\mathscr{Z})} \leq \frac{(KCnlm)^r (T-t_0)^{\alpha+m(r-1)} \|y-z\|_{C^{m-1}([t_0,T];\mathscr{Z})}}{(\alpha - m + 1)^r r!}.$$

Therefore, if r is large enough, G^r is a strict contraction in $C^{m-1}([t_0, T]; \mathscr{Z})$, hence this mapping has a unique fixed point in this space by the fixed point theorem. So it is a unique strong solution of (3), (6) on (t_0, T). □

Theorem 3 *Let $A \in \mathscr{L}(\mathscr{Z})$, $q > (\alpha - m + 1)^{-1}$, $B_k : (t_0, T) \to \mathscr{L}(\mathscr{Z})$, $k = 1, 2, \ldots, n$, be measurable and essentially bounded on (t_0, T), $z_0, z_1, \ldots, z_{m-1} \in \mathscr{Z}$. Then problem (3) to the linear equation*

$$D_t^{\alpha} z(t) = Az(t) + \sum_{k=1}^{n} B_k(t) D_t^{\alpha_k} z(t) \tag{10}$$

has a unique strong solution on (t_0, T).

3 Degenerate Equations

3.1 Degenerate Semilinear Equation

Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces, $L \in \mathscr{L}(\mathscr{X}; \mathscr{Y})$, $M \in \mathscr{C}l(\mathscr{X}; \mathscr{Y})$, D_M be a domain of an operator M, endowed by the graph norm $\|\cdot\|_{D_M} := \|\cdot\|_{\mathscr{X}} + \|M\cdot\|_{\mathscr{Y}}$. Define L-resolvent set $\rho^L(M) := \{\mu \in \mathbf{C} : (\mu L - M)^{-1} \in \mathscr{L}(\mathscr{Y}; \mathscr{X})\}$ of an operator M and its L-spectrum $\sigma^L(M) := \mathbf{C}\backslash\rho^L(M)$, and denote $R_\mu^L(M) := (\mu L - M)^{-1} L$, $L_\mu^L := L(\mu L - M)^{-1}$.

An operator M is called (L, σ)-*bounded*, if

$$\exists a > 0 \quad \forall \mu \in \mathbf{C} \quad (|\mu| > a) \Rightarrow (\mu \in \rho^L(M)).$$

Under the condition of (L, σ)-boundedness of operator M we have the projections

$$P := \frac{1}{2\pi i} \int_\gamma R_\mu^L(M)\, d\mu \in \mathscr{L}(\mathscr{X}), \quad Q := \frac{1}{2\pi i} \int_\gamma L_\mu^L(M)\, d\mu \in \mathscr{L}(\mathscr{Y}),$$

where $\gamma = \{\mu \in \mathbf{C} : |\mu| = r > a\}$ (see [24, p. 89, 90]). Put $\mathscr{X}^0 := \ker P$, $\mathscr{X}^1 := \operatorname{im} P$, $\mathscr{Y}^0 := \ker Q$, $\mathscr{Y}^1 := \operatorname{im} Q$. Denote by L_k (M_k) the restriction of the operator L (M) on $\mathscr{X}^k$ ($D_{M_k} := D_M \cap \mathscr{X}^k$), $k = 0, 1$.

Theorem 4 ([24, p. 90, 91]) *Let an operator M be (L, σ)-bounded. Then*

(i) $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$, $M_0 \in \mathscr{C}l(\mathscr{X}^0; \mathscr{Y}^0)$, $L_k \in \mathscr{L}(\mathscr{X}^k; \mathscr{Y}^k)$, $k = 0, 1$;
(ii) *there exist operators* $M_0^{-1} \in \mathscr{L}(\mathscr{Y}^0; \mathscr{X}^0)$, $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$.

Denote $\mathbf{N}_0 := \{0\} \cup \mathbf{N}$, $G := M_0^{-1} L_0$. For $p \in \mathbf{N}_0$ operator M is called (L, p)-bounded, if it is (L, σ)-bounded, $G^p \neq 0$, $G^{p+1} = 0$.

Remark 2 The number $p \in \mathbf{N}_0$ characterizes the maximal length of M-adjoint vectors chains (see [24]).

Let $n \in \mathbf{N}$, $N : (t_0, T) \times \mathscr{X}^n \to \mathscr{Y}$ be nonlinear operator. As before $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq m-1$, $r-1 < \alpha_n \leq r \in \mathbf{N}$. Consider the equation

$$D_t^\alpha Lx(t) = Mx(t) + N(t, D_t^{\alpha_1} x(t), D_t^{\alpha_2} x(t), \ldots, D_t^{\alpha_n} x(t)) + f(t). \tag{11}$$

Its strong solution on (t_0, T) is a function $x \in C^r([t_0, T]; \mathscr{X}) \cap L_q(t_0, T; D_M)$, such that $Lx \in C^{m-1}([t_0, T]; \mathscr{Y})$,

$$J_t^{m-\alpha} \left(Lx - \sum_{k=0}^{m-1} (Lx)^{(k)}(t_0) \tilde{g}_{k+1} \right) \in W_q^m(t_0, T; \mathscr{Y}), \quad q \in (1, \infty),$$

and almost everywhere on (t_0, T) equality (11) holds.

A solution of the generalized Showalter–Sidorov problem

$$(Px)^{(k)}(t_0) = x_k, \quad k = 0, 1, \ldots, m-1, \tag{12}$$

to Eq. (11) is a solution of the equation, such that conditions (12) are true. Note here that $Px = L_1^{-1} L_1 Px = L_1^{-1} QLx$, and the smoothness of Px is not less the smoothness of Lx, since $L_1^{-1} Q \in \mathscr{L}(\mathscr{Y}; \mathscr{X})$.

Theorem 5 *Let $q > (\alpha - m + 1)^{-1}$, $p \in \mathbf{N}_0$, an operator M be (L, p)-bounded, $N : (t_0, T) \times \mathscr{X}^n \to \mathscr{Y}$ be Caratheodory mapping, uniformly Lipschitz continuous in $\overline{x} \in \mathscr{X}^n$, at all $y_1, y_2, \ldots, y_n \in Z$ and almost everywhere on (t_0, T) the inequality*

$$\|N(t, y_1, y_2, \ldots, y_n)\|_{\mathscr{Z}} \leq a(t) + c \sum_{k=1}^{n} \|y_k\|_{\mathscr{Z}} \tag{13}$$

be true for some $a \in L_q(t_0, T; \mathbf{R})$, $c > 0$, $N(t, y_1, y_2, \ldots, y_n) \in \mathscr{Y}^1$. Suppose also that $Qf \in L_q(t_0, T; \mathscr{Y})$, for all $k = 0, 1, \ldots, p$ there exist $(D_t^\alpha G)^k M_0^{-1}(I - Q)f \in C^r([t_0, T]; \mathscr{X})$; $x_0, x_1, \ldots, x_{m-1} \in \mathscr{X}^1$. Then problem (11), (12) has a unique strong solution.

Proof Due to the theorem conditions the mapping

$$x(\cdot) \to N(\cdot, D_t^{\alpha_1} x(\cdot), D_t^{\alpha_2} x(\cdot), \ldots, D_t^{\alpha_n} x(\cdot))$$

acts from $C^r([t_0, T]; \mathscr{X})$ into the space $L_q(t_0, T; \mathscr{Y})$.

By condition $\mathrm{im}N \subset \mathscr{Y}^1$ we have $(I-Q)N \equiv 0$, $QN \equiv N$. Equation (11) after action of the operator $M_0^{-1}(I-Q)$ has a form $D_t^\alpha Gw(t) = w(t)+M_0^{-1}(I-Q)f(t)$, where $w(t) = (I-P)x(t)$. Since the operator G is nilpotent, the unique solution of this equation has the form

$$w(t) = -\sum_{k=0}^{p}(D_t^\alpha G)^k M_0^{-1}(I-Q)f(t).$$

Note that $w \in C^r([t_0, T]; \mathscr{X})$ and there exist derivatives $D_t^\alpha L(D_t^\alpha G)^k M_0^{-1}(I-Q)f \in L_q(t_0, T; \mathscr{Y})$ for $k = 0, 1, \dots, p$, because

$$D_t^\alpha L(D_t^\alpha G)^k M_0^{-1}(I-Q)f = M_0 D_t^\alpha G(D_t^\alpha G)^k M_0^{-1}(I-Q)f =$$

$$= M_0(D_t^\alpha G)^{k+1} M_0^{-1}(I-Q)f \in L_q(t_0, T; \mathscr{Y})$$

by this theorem conditions, and if $k = p$, then $(D_t^\alpha G)^{p+1} = (D_t^\alpha)^{p+1}G^{p+1} = 0$.

It remains to prove the existence and the uniqueness of the strong solution to the Cauchy problem

$$D_t^\alpha v(t) = S_1 v(t) + L_1^{-1}N(t, D_t^{\alpha_1}(v(t)+w(t)), \dots, D_t^{\alpha_n}(v(t)+w(t))) + L_1^{-1}Qf(t),$$
$$v^{(k)}(t_0) = x_k, \ \ k = 0, 1, \dots, m-1,$$

where $v(t) = Px(t)$, $S_1 = L_1^{-1}M_1 \in \mathscr{L}(\mathscr{X}^1)$ due to Theorem 4. This problem is obtained from (11), (12) after the action of the continuous operator $L_1^{-1}Q$. Here the operator

$$B(t, v_0, v_1, \dots, v_n) = L_1^{-1}N(t, v_0 + D_t^{\alpha_1}w(t), \dots, v_n + D_t^{\alpha_n}w(t)) + L_1^{-1}Qf(t)$$

satisfies the conditions of Theorem 2, and the proof is completed. □

3.2 Degenerate Multi-Term Linear Equation

Let $n \in \mathbf{N}$, $N_k : (t_0, T) \to \mathscr{L}(\mathscr{X}; \mathscr{Y})$, $k = 1, 2, \dots, n$, $0 \le \alpha_1 < \alpha_2 < \dots < \alpha_n \le m-1$, $r-1 < \alpha_n \le r \in \mathbf{N}$. Consider the degenerate multi-term linear equation

$$D_t^\alpha Lx(t) = Mx(t) + \sum_{k=1}^{n} N_k(t) D_t^{\alpha_k} x(t) + f(t), \quad t \in (t_0, T). \tag{14}$$

The definitions of its strong solution on (t_0, T) and of the solution to the generalized Showalter–Sidorov problem

$$(Px)^{(k)}(t_0) = x_k, \quad k = 0, 1, \ldots, m-1, \tag{15}$$

for Eq. (14) do not differ from the analogous definitions for semilinear equation (11).

Theorem 6 *Let $q > (\alpha - m + 1)^{-1}$, $p \in \mathbf{N}_0$, an operator M be (L, p)-bounded, $N_k : (t_0, T) \to \mathscr{L}(\mathscr{X}; \mathscr{Y})$, $k = 1, 2, \ldots, n$, be measurable and essentially bounded on (t_0, T), $\mathrm{im} N_k(t) \subset \mathscr{Y}^1$ for almost all $t \in (t_0, T)$, $Qf \in L_q(t_0, T; \mathscr{Y})$, for all $k = 0, 1, \ldots, p$ there exist $(D_t^\alpha G)^k M_0^{-1}(I - Q)f \in C^r([t_0, T]; \mathscr{X})$; $x_0, x_1, \ldots, x_{m-1} \in \mathscr{X}^1$. Then problem (14), (15) has a unique strong solution.*

Proof By the construction the operator

$$N(t, x_1, x_2, \ldots, x_n) := N_1(t)x_1 + N_2(t)x_2 + \cdots + N_n(t)x_n$$

is Caratheodory mapping due to the theorem conditions on N_k, $k = 1, 2, \ldots, n$. Let

$$l := \max_{k=1,2,\ldots,n} \operatorname*{ess\,sup}_{t \in (t_0, T)} \|N_k(t)\|_{\mathscr{L}(\mathscr{X};\mathscr{Y})},$$

then the operator N is uniformly Lipschitz continuous in $\bar{x}$ with the constant l and it satisfies inequality (13) with $a \equiv 0$, $c = l$. Thus, by Theorem 5 we obtain the required. □

4 Application

Consider the initial-boundary value problem

$$\frac{\partial^k w}{\partial t^k}(s, t_0) = v_k(s), \quad k = 0, 1, \ldots, m-1, \quad s \in (0, \pi), \tag{16}$$

$$w(0, t) = w(\pi, t) = 0, \quad t \in (t_0, T). \tag{17}$$

for the model equation

$$D_t^\alpha \left(\frac{\partial^2 w}{\partial s^2} + \gamma w \right) = \delta w + \sum_{k=1}^{n} \delta_k(t) D_t^{\alpha_k} \left(\frac{\partial^2 w}{\partial s^2} + \gamma_k(t) w \right), \; s \in (0, \pi), \; t \in (t_0, T), \tag{18}$$

where $\alpha, \alpha_k, \gamma, \delta \in \mathbf{R}$, $\gamma_k, \delta_k : (t_0, T) \to \mathbf{R}$, $k = 1, 2, \ldots, n$, $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n \le m - 1 < \alpha \le m \in \mathbf{N}$.

Define Banach spaces $\mathscr{X} = \{v : H^2(0, \pi) : v(0) = v(\pi) = 0\}$, $\mathscr{Y} = L_2(0, \pi)$, and operators

$$L = \frac{\partial^2}{\partial s^2} + \gamma, \quad M = \delta I, \quad N_k(t) = \delta_k(t)\left(\frac{\partial^2}{\partial s^2} + \gamma_k(t)\right), \ k = 1, 2, \ldots, n.$$

Theorem 7 *Let $\gamma \neq b^2$ for all $b \in \mathbf{N}$, $v_k \in \mathscr{X}$, $k = 0, 1, \ldots, m-1$, $\gamma_d, \delta_d : (t_0, T) \to \mathbf{R}$ are measurable and essentially bounded, $d = 1, 2, \ldots, n$. Then there exists a unique strong solution of problem* (16)–(18) *on* (t_0, T).

Proof Since $\gamma \neq b^2$, $b \in \mathbf{N}$, L has a continuous inverse operator $L^{-1} : \mathscr{Y} \to \mathscr{X}$. Then Eq. (18) has the form (10), where $\mathscr{Z} = \mathscr{X}$, the operator $A = L^{-1}M : \mathscr{X} \to \mathscr{X}$ is the continuous as composition of continuous operators, $B_d(t) = L^{-1}N_d(t)$, $d = 1, 2, \ldots, n$. For $w \in \mathscr{X}$ we have

$$\|B_d(t)w\|^2_{L_2(0,\pi)} = |\delta_d(t)|^2 \sum_{k=1}^{\infty} \left|\frac{\gamma_d(t) - k^2}{\gamma - k^2}\right|^2 |w_k|^2 \leq$$

$$\operatorname*{ess\,sup}_{t \in (t_0,T)} |\gamma_d(t)|^2 c^{-2} \left(1 + \operatorname*{ess\,sup}_{t \in (t_0,T)} |\gamma_d(t)|\right)^2 \|w\|^2_{\mathscr{X}},$$

where $w_k := \langle w, \sin ks\rangle_{L_2(0,\pi)}$, $c := \min\limits_{k \in \mathbf{N}} |1 - k^{-2}\gamma|$. By Theorem 3 we obtain the required statement. □

If there exists $b \in \mathbf{N}$, such that $\gamma = b^2$, then $\ker L \neq \{0\}$, and Eq. (18) is degenerate. We can obtain the unique solvability theorem, for example, for the case of the Showalter–Sidorov initial conditions

$$\frac{\partial^k}{\partial t^k}\left(\frac{\partial^2 w}{\partial s^2} + \gamma w\right)(s, t_0) = y_k(s), \quad k = 0, 1, \ldots, m-1, \quad s \in (0, \pi), \tag{19}$$

in the partial case $\gamma_d(t) \equiv \gamma$, $d = 1, 2, \ldots, n$.

Theorem 8 *Let for all $d = 1, 2, \ldots, n$ and for almost all $t \in (t_0, T)$ $\gamma_d(t) \equiv \gamma = b^2$ at some $b \in \mathbf{N}$, $y_k \in L_2(0, \pi)$,*

$$\int_0^{\pi} y_k(s) \sin(bs) ds = 0, \quad k = 0, 1, \ldots, m-1, \tag{20}$$

$\delta_d : (t_0, T) \to \mathbf{R}$ are measurable and essentially bounded, $d = 1, 2, \ldots, n$. Then there exists a unique strong solution of problem (17)–(19) *on* (t_0, T).

Proof Here we have $(L, 0)$-bounded operator M due to [10, Theorem 8], hence $\operatorname{im} L = \mathscr{Y}^1$, and the generalized Showalter–Sidorov problem (15) is equivalent to the Showalter–Sidorov problem $(Lx)^{(k)}(t_0) = y_k = Lx_k \in \mathscr{Y}^1$, $k = 0, 1, \dots, m-1$. Hence problem (17)–(19) has form (14), (15). Moreover, since $\gamma_k \equiv \gamma$, we have $N_k = \delta_k(t)L$, therefore, $\mathrm{im}N_k(t) \subset \mathrm{im}L = \mathscr{Y}^1$, $k = 1, 2, \dots, n$.

It remains to note, that conditions (20) mean that for all $k = 0, 1, \dots, m-1$ $y_k \in \mathscr{Y}^1 = \mathrm{im}L_1$. By Theorem 6 obtain the required. □

Acknowledgements The work is supported by Act 211 of Government of the Russian Federation, contract 02.A03.21.0011, and by the Ministry of Education and Science of the Russian Federation, task No 1.6462.2017/BCh.

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Mean Value Theorems and Properties of Solutions of Linear Differential Equations

I. P. Polovinkin and M. V. Polovinkina

Abstract This paper describes an accompanying distributions technique that allows to obtain mean value formulas for linear homogeneous partial differential equations. One of these formulas can be interpreted as a generalization of the Asgeirsson principle for the string vibration equation into the case of an arbitrary natural order. In addition, this mean value formula is an exact difference scheme for a two-dimensional linear homogeneous equation with a symbol factorized up to linear factors.

Keywords Mean value formula · Accompanying distribution · Difference scheme · Hyperbolic equation

1 Introduction

In different fields and applied problems, the notions of "mean value formula" and "mean value theorem" often refer to somewhat different facts. Nevertheless, various results for different types of equations have the common point that they involve the mean value of a smooth function over a certain manifold. In this paper we discuss mean value formulas for linear partial differential equations.

Mean value theorems are most widely known for elliptic equations. The basic result for using in applications is the following classical result (by Gauss): a continuous in a domain $\Omega \subset R^n$ function is harmonic in Ω if and only if for every point $x \in \Omega$ and every r such that the ball $|\xi - x| \le r$ is contained in Ω its value at the point x is equal to the mean of the function over this sphere [1]. This statement is called the mean value theorem for the Laplace equation. Generalizations of this result have been established for solutions of second-order elliptic equations (see the works of V. A. Il'in and E. I. Moiseev [2, 3]). The well-known Asgeirsson theorem

I. P. Polovinkin (✉) · M. V. Polovinkina
Voronezh State University, Voronezh, Russia
e-mail: sitnik@bsu.edu.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_26

for an ultra-hyperbolic equation is a sort of generalization of the mean value theorem for harmonic functions, and also a generalization for the Green's function formula for the linear constant coefficient wave equation (see [4]). A mean value theorems for some classes of equations were proved by A. V. Bitsadze and A. M. Nachushev in 1974 [5]. In particular, this includes the mean value theorem for the wave equation, which was inverted by I.P. Polovinkin in 1991 [6] for smooth functions.

In this paper we expound a uniform viewpoint on mean value theorems for linear elliptic and hyperbolic partial differential equations that allows to obtain new mean value formulas. This was proposed by L. Zalcman [7] and generalized by means of Hermander's [8] methods in [9] and then in [10] for a class of singular differential equations with the Bessel operator. Furthermore we deduce a difference mean value formula for a factorable linear two-dimensional hyperbolic equation as an example of the method.

2 Accompanying Distributions

We denote by $\mathcal{D}$ the space of compactly supported test functions of variables $x=(x_1, \ldots, x_n)\in R^n$, by $\mathcal{S}$ the Schwartz space of rapidly decaying test functions, and by $\mathcal{D}'$, $\mathcal{S}'$ corresponding spaces of distributions.

Denote by $\hat{f}$ the Fourier transform of a distribution $f \in \mathcal{S}$. We use the same symbol $\hat{f}(w)$ for designating the Fourier-Laplace transform of a compactly supported distribution f. In this case $\hat{f}(w)$ is an entire analytic function of a complex variable $w \in \mathbb{C}^n$. We use a symbol $f^\vee$ for designating the inverse Fourier transform of a distribution $f \in \mathcal{S}$. Furthermore, let

$$D_j = -i\frac{\partial}{\partial x_j},\ \ j = 1, \ldots, n,\ \ D = (D_1, \ldots, D_n),\ x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

In what follows, we assume that the multi-index α has nonnegative integral coordinates. Denote by $S_R(x_0)$ the sphere in R^n. By $\delta(x - x_0)$ we denote the Dirac measure supported at a point x_0, and by $\delta_{S_R(x_0)}(x - x_0)$ the Dirac measure supported on the sphere $S_R(x_0)$.

Definition W Let $P(w)$ be a polynomial of degree m. Consider the equation

$$P(D)u \equiv \sum_{|\beta|\le m} a_\beta D^\beta u = 0. \tag{1}$$

A compactly supported distribution Φ is called an *accompanying distribution* for Eq. (1) if the relation

$$\langle \Phi, u\rangle = 0 \tag{2}$$

holds for any solution $u(x) \in C^{\infty}(R^n)$. The relation (2) is called a *mean value formula* for Eq. (1). The distribution Φ is said to be an *accompanying distribution* for the operator $P(D)$.

Theorem A (Criterion of the Accompanying Distribution in Terms of the Fourier-Laplace Transform) *A compactly supported distribution Φ is an accompanying distribution for Eq. (1) in R^n, if and only if the function*

$$\frac{\hat{\Phi}(w)}{P(-w)}, \ w \in C^n \tag{3}$$

is an entire analytic function.

Theorem B (Properties of Means for Operators that can be Factorized) *Let $P(D) = P_1(D)P_2(D)$, where P_1 and P_2 are polynomials. Let Φ_l be a compactly supported distribution accompanying the operator $P_l(D)$, $l = 1, 2$. The distribution*

$$\Phi = \Phi_1 * \Phi_2$$

is accompanying for the operator $P(D) = P_1(D)P_2(D)$.

Theorems A and B were proved in [9]. They were proved in [7] under the assumption that the distribution Φ is a finite complex Borel measure supported in the closed unit ball in R^n.

Theorem C *If a distribution Φ is accompanying for the operator P then the distribution $\Phi_0 = \Phi + \lambda \left(\hat{\Phi}(\xi)/P^*(\xi)\right)^{\vee}(x)$ is accompanying for the operator $P + \lambda$.*

3 Accompanying Distributions for Singular Operators

In [10] definition W and theorems A, B, C were extended to the case of singular differential equations with the Bessel operator. Below we describe that briefly.

Let $R_N^+ = \{x = (x', x''),\ x'=(x_1, \ldots, x_n),\ x''=(x_{n+1}, \ldots, x_N),\ x_1>0, \ldots, x_n >0\}$. We denote by Ω^+ a domain adjacent to the hyperplanes $x_1 = 0, \ldots, x_n = 0$. The boundary of Ω^+ consists of two parts: Γ^+ in R_N^+ and Γ_0 in the hyperplanes $x_1 = 0, \ldots, x_n = 0$.

Let Ω_σ^+ be an interior subdomain of Ω^+ adjacent to the boundary Γ_0 and such that all its points are located at a distance at least σ from the part of the boundary Γ^+ of the domain Ω^+. Then Ω_σ^+ is called a symmetrically interior (s-interior) subdomain of the domain Ω^+.

We denote by $C_{ev}^l(\Omega^+)$ the linear space of functions possessing the following properties.

1. Every function $\varphi \in C^l_{ev}(\Omega^+)$, together with all its partial derivatives of order up to l, is continuous in Ω^+. If every function φ has continuous partial derivatives of any order in Ω^+, we set $l = \infty$.
2. Even extensions of a function $\varphi \in C^l_{ev}(\Omega^+)$ with respect to x' must remain in the class $C^l(\Omega)$, where Ω is the union of the domain Ω^+ and its symmetric image Ω^- with respect to $x' = 0$.

We say that functions admitting smooth even extension relative to the corresponding variables are *even* with respect to these variables.

We denote by $C^l_{ev,0}(\Omega^+)$ the linear space of functions $\varphi \in C^l_{ev}(\Omega^+)$ vanishing outside some s-interior subdomain of Ω^+.

If Ω^+ and R^+_N coincide, we omit the symbol (R^+_N).

Let $\gamma = (\gamma_1, \dots, \gamma_n)$, $(x')^\gamma = \prod\limits_{i=1}^{n} x_i^{\gamma_i}$, $\gamma_i > 0$.

We denote by $L^\gamma_{p,loc}(\Omega^+)$ the linear space of functions such that

$$\int\limits_{\Omega^+_\delta} |f(x)|^p \ (x')^\gamma \, dx < +\infty$$

for any s-interior subdomain Ω^+_δ of the domain Ω^+. Let $\Omega \subseteq R^N$ be the union of the set Ω^+ and the set Ω^- obtained from Ω^+ by symmetry with respect to $x' = 0$. We denote by $\mathcal{D}_{ev}(\Omega^+)$ $(\mathcal{E}_{ev}(\Omega^+))$ the set of all restrictions of even functions with respect to $x' = 0$ in the space $\mathcal{D}(\Omega)$ $(\mathcal{E}_{ev}(\Omega^+))$ onto the set Ω^+. The topology in $\mathcal{D}_{ev}(\Omega^+)$ $(\mathcal{E}(\Omega^+))$ is induced by the topology in $\mathcal{D}(\Omega)$ $(\mathcal{E}_{ev}(\Omega))$. By definition, $\mathcal{D}_{ev} = \mathcal{D}_{ev}(R^+_N)$. We denote by $\mathcal{S}_{ev}$ the linear space of functions $\varphi(x) \in C^\infty_{ev}(R^+_N)$, that, together with all their derivatives, decrease faster than any power of $|x|^{-1}$ as $|x| \to \infty$. The topology in $\mathcal{S}_{ev}$ is introduced in the same way as in the space $\mathcal{S}$ [8, 11, 12]. The dual space for $\mathcal{D}_{ev}(\Omega^+)$ $(\mathcal{E}_{ev}(\Omega^+),\ \mathcal{S}_{ev})$ equipped with the weak topology is denoted by $\mathcal{D}'_{ev}(\Omega^+)$ $(\mathcal{E}'_{ev}(\Omega^+),\ \mathcal{S}'_{ev})$. The following relations hold: $\mathcal{D}_{ev} \subset \mathcal{S}_{ev} \subset \mathcal{S}'_{ev} \subset \mathcal{D}'_{ev}$. In all three cases, the action of a distribution f on a test function φ is denoted by

$$\langle f(x), \varphi(x) \rangle_\gamma = \langle f(x), \varphi(x) \rangle .$$

We identify each function $f(x) \in L^\gamma_{1,loc}(\Omega^+)$ with the functional $f \in \mathcal{D}'_{ev}(\Omega^+)$, called *regular*, acting by

$$\langle f(x), \varphi(x) \rangle = \int\limits_{\Omega^+} f(x)\varphi(x) \ (x')^\gamma \, dx.$$

The remaining functionals in $\mathcal{D}'_{ev}(\Omega^+)$ are said to be *singular*.

Let $\beta = (\beta', \beta'')$ be a multi-index with non-negative integer components, $\beta' = (\beta_1, \beta_2, \dots, \beta_n)$, $\beta'' = (\beta_{n+1}, \dots, \beta_N)$. We denote by $B^{\beta'}_{x'}$ the operator defined by

$$B_{x'}^{\beta'} u = B_{x_1}^{\beta_1} B_{x_2}^{\beta_2} \dots B_{x_n}^{\beta_n} u,$$

where $B_{x_i} = B_{x_i,\gamma_i}$ is the Bessel operator acting relative to x_i by the formula

$$B_{x_i} u = B_{x_i,\gamma_i} u = \frac{\partial^2 u}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial u}{\partial x_i} = x_i^{-\gamma_i}\frac{\partial}{\partial x_i}\left(x_i^{\gamma_i}\frac{\partial u}{\partial x_i}\right).$$

Let $D_{x''}^{\beta''}$ be the operator acting by

$$D_{x''}^{\beta''} f(x', x'') = \partial^{|\beta''|} f(x', x'') \,/\, \partial x_{n+1}^{\beta_{n+1}} \dots \partial x_N^{\beta_N},$$

where $|\beta''| = \beta_{n+1} + \dots + \beta_N$.

We define the operator $P{=}P(B_{x'}, D_{x''})$ with the symbol $P(-\zeta_1^2, \dots, -\zeta_n^2, -i\zeta_{n+1}, \dots, -i\zeta_N)$ together with the formal-adjoint operator P^* by formulas

$$Pu = \sum_{2|\beta'|+|\beta''|\le m} b_\beta B_{x'}^{\beta'} D_{x''}^{\beta''} u, \tag{4}$$

$$P^* u = \sum_{2|\beta'|+|\beta''|\le m} b_\beta B_{x'}^{\beta'} (-D_{x''})^{\beta''}.$$

The *mixed generalized shift* is defined by

$$f \to (T^y f)(x) = \prod_{i=1}^{n} T_{x_i}^{y_i} f(x', x'' - y''),$$

where each of the generalized shifts $T_{x_i}^{y_i}$ has the form (see [13])

$$(T_{x_i}^{y_i} f)(x) = \frac{\Gamma(\frac{\gamma_i+1}{2})}{\sqrt{\pi}\;\Gamma\left(\frac{\gamma_i}{2}\right)} \times$$

$$\times \int_0^\pi f\left(x_1, \dots, x_{i-1}, \sqrt{x_i^2 + y_i^2 - 2x_i y_i \cos\alpha}, x_{i+1}, \dots, x_N\right) \sin^{\gamma_i - 1}\alpha \; d\alpha,$$

$$i = 1, \dots, n,$$

where $\prod_{k=1}^{n} T_{x_k}^{y_k}$ is understood as the superposition of operators.

The generalized convolution of functions $f,\ g \in L_p^{\gamma}(R_N^+)$ is defined by

$$(f * g)_\gamma(x) = \int\limits_{R_N^+} f(y) T_x^y g(x) (y')^\gamma dy.$$

If $f \in \mathcal{D}'_{ev},\ g \in \mathcal{E}'_{ev}$, then the generalized convolution $(f * g)_\gamma$ of such distributions is defined by

$$\langle (f * g)_\gamma(x), \varphi(x) \rangle_\gamma = \langle f(y), \langle g(x), T_x^y \varphi(x) \rangle_\gamma \rangle_\gamma,\ \varphi(x) \in \mathcal{D}_{ev}.$$

The direct and the inverse mixed Fourier–Bessel transforms are introduced by

$$F_{B,\gamma}[\varphi(x', x'')](\xi) = \int\limits_{R_N^+} \varphi(x) \prod_{k=1}^{n} j_{\nu_k}(\xi_k x_k) e^{-ix''\cdot\xi''} (x')^\gamma\, dx =$$

$$= (2\pi)^{N-n} 2^{2|\nu|} \prod_{k=1}^{n} \Gamma^2(\nu_k + 1) F_{B,\gamma}^{-1}[\psi(x', -x'')](\xi),$$

where

$$x' \cdot \xi' = x_1\xi_1 + \ldots + x_n\xi_n,\ \ x'' \cdot \xi'' = x_{n+1}\xi_{n+1} + \ldots + x_N\xi_N, \quad |\nu| = \nu_1 + \ldots + \nu_n,$$

$$j_{\nu_k}(z_k) = \frac{2^{\nu_k}\Gamma(\nu_k + 1)}{z_k^{\nu_k}} J_{\nu_k}(z_k) = \Gamma(\nu_k + 1) \sum_{m=1}^{\infty} \frac{(-1)^m z_k^{2m}}{2^{2m} m! \Gamma(m + \nu_k + 1)}$$

$\Gamma(\cdot)$ is the Euler gamma-function, $J_{\nu_k}(\cdot)$ is the Bessel function of the first kind of order $\nu_k = (\gamma_k - 1)/2,\ k = 1, \ldots, n.$

Definition W′ A distribution $\Phi \in \mathcal{E}'_{ev}(R_N^+)$ is called an accompanying distribution of the equation

$$P\,u = 0,$$

if for any solution $u(x) \in C^\infty(R^n)$

$$\langle \Phi, u \rangle_\gamma = 0.$$

Theorem A′ *A distribution Φ is an accompaniment of an operator P if and only if the Fourier–Bessel image $F_{B,\gamma}[\Phi](\xi)$ of $\Phi \in \mathcal{E}'_{ev}(R_N^+)$ is divisible by the symbol of the formally adjoint of P, i.e., $F_{B,\gamma}[\Phi](\xi) = \widehat{\psi}(\xi) P(-\xi_1^2, \ldots, -\xi_n^2, -i\xi'')$, where $\psi(\xi)$ is an entire function.*

Theorem B′ *Let an operator P can be factorized in the form $P = P_1 P_2$, where P_1, P_2, are operators of the form (4), but of less order. Let Φ_l be a compactly supported accompanying distribution of the operator P_l, $l = 1, 2$. Then $\Phi = \Phi_1 * \Phi_2$ is an accompanying distribution of the operator P.*

Theorem C′ *If Φ is an accompanying distribution of an operator P , then*

$$\Phi_0 = \Phi + \lambda\, F_{B,\gamma}^{-1}[F_{B,\gamma}[\Phi](\xi)/P^*(\xi)](x)$$

is an accompanying distribution of the operator $P + \lambda$.

The technique of accompanying distributions can be used to obtain an analogue of the Asgeirsson theorem for the B-ultrahyperbolic equation (see [14]).

4 Some Examples of Applying the Method

The above method is able to lead either to well-known results or to new ones. We give some examples of its application (see [9]).

Example 1 For the string vibration equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \tag{5}$$

one has the following assertion (a mean-value theorem, or the one-dimensional Asgeirsson principle): a function $u(x, y) \in C^2(R^2)$ is a solution of Eq. (5), if and only if the relation

$$u(M_1) + u(M_3) - u(M_2) - u(M_4) = 0, \tag{6}$$

holds for each rectangle formed by lines $x \pm y{=}const$, where $M_k{=}(x_k, y_k)$, $(k{=}1, 2, 3, 4)$ are the successively numbered vertices of this rectangle.

By subjecting the solutions of Eq. (5) to the more restrictive condition $u(x) \in C^\infty(R^2)$, one can readily find that the necessity of relation (6) is equivalent to the following assertion: the distribution

$$\sum_{k=1}^{4} (-1)^{k-1} \delta(M - M_k), \tag{7}$$

where $M = (x, y) \in R^2$, is accompanying for Eq. (5).

Example 2 The well-known classical (Gauss) mean-value theorem for the Laplace equation has the following form: a function $u \in C(R^n)$ is a solution to the Laplace equation

$$\Delta u \equiv \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = 0,\ \ x \in R^n, \tag{8}$$

if and only if the relation

$$u(x_0) = \frac{1}{|S_n| R^{n-1}} \int_{S_R(x_0)} u(x)\, dS_x, \tag{9}$$

holds for each $R > 0$ and for each point $x_0 \in R^n$, where dS_x is the surface area element of the sphere $S_R(x_0)$ with the center at the point x_0 and radius R (the arc length element of the circumference when $n = 2$), $|S_n|$ is the surface area of the unit sphere in R^n (the length of the unit circumference when $n = 2$). Hence it readily follows that the distribution

$$\delta(x - x_0) - \frac{1}{|S_n| R^{n-1}} \delta_{S_R(x_0)}(x) \tag{10}$$

is accompanying for the Laplace operator Δ.

Example 3 Consider the equation

$$L[u] \equiv \frac{\partial^4 u}{\partial x^4} - \frac{\partial^4 u}{\partial y^4} = 0. \tag{11}$$

The operator L on the right-hand side of this equation obviously decomposes into multipliers:

$$L = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

It follows from formulas (7) and (10) that a distribution of the form

$$\sum_{k=1}^{4} (-1)^{k-1} \left(\delta(M - M_k) - \delta_{S_R(M_k)}(M)\right),$$

where $M_k = (x_k, y_k)$ $(k = 1, 2, 3, 4)$ are sequentially enumerated vertices of the rectangle composed of the lines $x \pm y = const$, is accompanying for the operator L. In turn, this implies that the solution of Eq. (11) satisfies the following mean value formula:

$$\sum_{k=1}^{4}(-1)^{k-1}\left(u(M_k)-\frac{1}{2\pi R}\oint\limits_{S_R(M_k)} u(\xi)\,dS_\xi\right)=0.$$

Example 4 Theorem B allows us to find accompanying distributions for linear homogeneous differential operators with constant coefficients and two independent variables, since such operators are factorized into multipliers whose accompanying distributions are known. Let us pass to a detailed presentation of this fact.

Consider a linear homogeneous differential operator

$$P\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)=\sum_{|\alpha|=m} a_\alpha\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)^\alpha$$

with constant coefficients $a_\alpha,\ \alpha=(\alpha_1,\ \alpha_2)$. It is well known that such an operator is factorable into simplest multipliers:

$$\begin{aligned}P\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)&=\left(a_1\frac{\partial}{\partial x}+b_1\frac{\partial}{\partial y}\right)^{l_1}\times\cdots\times\\&\times\left(a_s\frac{\partial}{\partial x}+b_s\frac{\partial}{\partial y}\right)^{l_s}\times\\&\times\left(A_1\frac{\partial^2}{\partial x^2}+2B_2\frac{\partial^2}{\partial x\partial y}+C_1\frac{\partial^2}{\partial y^2}\right)^{q_1}\times\cdots\times\\&\times\left(A_r\frac{\partial^2}{\partial x^2}+2B_r\frac{\partial^2}{\partial x\partial y}+C_r\frac{\partial^2}{\partial y^2}\right)^{q_r},\end{aligned}\tag{12}$$

$$B_k^2-A_kC_k<0,\ k=1\ldots r,\ l_1+\ldots l_s+2(q_1+\ldots q_r)=m.$$

Decomposition (12)contains two types of multipliers:
the multiplier

$$\left(a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}\right)\tag{13}$$

of the first order and elliptic multipliers of the form

$$\left(A\frac{\partial^2}{\partial x^2}+2B\frac{\partial^2}{\partial x\partial y}+C\frac{\partial^2}{\partial y^2}\right),\ B^2-AC<0.\tag{14}$$

Now let us consider separately accompanying distributions for each of the types (13) and (14).

It is easy to see that the following distribution can be an accompanying distribution for operator (13):

$$\delta(M - M_1) - \delta(M - M_2), \tag{15}$$

where $M = (x, y)$, $M_k = (x_k, y_k)$, $k = 1, 2$, are arbitrary points of the plane connected by the relation

$$bx_1 - ay_1 = bx_2 - ay_2. \tag{16}$$

This directly follows from the fact that a solution of the homogeneous equation corresponding to operator (13) is a plane wave.

For operator (14), by the change

$$\xi = \sqrt{AC - B^2}x, \ \eta = Ay - Bx, \tag{17}$$

the corresponding elliptic equation can be reduced to the Laplace equation

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0. \tag{18}$$

From the Gauss formula of mean value for the Laplace operator and change (17), we obtain that the distribution

$$\delta(M - M_0) - \frac{1}{2\pi R}\delta_{S_R(M_0)}(M - M_0), \tag{19}$$

where

$$M = (\sqrt{AC - B^2}x, Ay - Bx), \ M_0 = (\sqrt{AC - B^2}x_0, Ay_0 - Bx_0),$$

is accompanying for operator (14).

This implies that for an arbitrary homogeneous differential operator with constant coefficients and two independent variables, we can find an accompanying distribution that is the convolution of accompanying distributions of multipliers in decomposition (12), i.e., the convolution of distributions of the form (15) and (19).

5 Mean Value Formula for a Two-Dimensional Hyperbolic Equation

Now we consider the equation

$$\prod_{j=1}^{m} \left(a_j\partial/\partial x + b_j\partial/\partial t + c_j\right) u = 0 \tag{20}$$

with constant coefficients a_j, b_j, c_j, $j = 1, \ldots, m$.

A line given by the equation

$$b_j x - a_j t = const \tag{21}$$

is called the j-th type characteristic of Eq. (20), j = 1, . . . , m. We assume that all of the characteristics of Eq. (20) are simple.

In accordance with Theorem B, we would be able to construct an accompanying distribution for Eq. (20), if we constructed one for every factor. Consider the equation

$$a_j\partial u/\partial x + b_j\partial u/\partial t + c_j u = 0. \tag{22}$$

Let a function $w_j(x, t)$ be a certain regular solution of Eq. (22):

$$a_j\partial w_j/\partial x + b_j\partial w_j/\partial t + c_j w_j = 0. \tag{22$'$}$$

Changing the function $u(x, t)$ to a new unknown function $v(x, t)$ by the formula

$$u(x, t) = w_j(x, t)v(x, t), \tag{23}$$

we lead Eq. (22) to the equation

$$w_j(a_j\partial v/\partial x + b_j\partial v/\partial t) = 0. \tag{24}$$

If $w_j(x, t)$ is non vanishing, then we may reduce (24) by $w_j(x, t)$ and obtain an equation

$$a_j\partial v/\partial x + b_j\partial v/\partial t = 0. \tag{25}$$

Let $Z = (x, t)$. We introduce the compactly supported distributions

$$\Phi_j(Z) = \delta(Z - Q_j^0) - \delta(Z - Q_j^1), \quad j = 1, \ldots, m, \tag{26}$$

where every pair of points

$$Q_j^{\alpha_j} = (\xi_j^{\alpha_j}, \tau_j^{\alpha_j}),\ \alpha_j \in \{0; 1\}, \quad j = 1, \dots, m, \tag{27}$$

lies on the j-th type characteristic. It is obvious that every $\Phi_j(Z)$ is an accompanying distribution of the operator

$$\left(a_j \partial/\partial x + b_j \partial/\partial t\right), \quad j = 1, \dots, m. \tag{28}$$

With respect to (23) we obtain

$$u(Q_j^1)/w_j(Q_j^1) - u(Q_j^0)/w_j(Q_j^0) = 0. \tag{29}$$

If $1/w_j(x,t) \in C^\infty(R^2)$, then (29) leads to the fact that the distribution

$$\Phi_j(Z) = (\delta(Z - Q_j^0) - \delta(Z - Q_j^1))/w_j(x,t), \quad j = 1, \dots, m, \tag{30}$$

is accompanying for Eq. (22) and the operator

$$\left(a_j \partial/\partial x + b_j \partial/\partial t + c_j\right), \quad j = 1, \dots, m. \tag{31}$$

As an example of $w_j(x,t)$ we can use a function

$$w_j(x,t) = e^{\mu_j x + \nu_j t}, \tag{32}$$

where

$$a_j \mu_j + b_j \nu_j + c_j = 0. \tag{33}$$

Then the accompanying distribution (30) for the operator (31) can be written in the form

$$\Phi_j(Z) = (\delta(Z - Q_j^0) - \delta(Z - Q_j^1))e^{-\mu_j x - \nu_j t}, \quad j = 1, \dots, m. \tag{34}$$

By Theorem B, the distribution

$$\begin{aligned} \Phi(Z) = \Phi_1(Z) * \Phi_2(Z) * \dots * \Phi_m(Z) = \\ = (e^{-\mu_1 x - \nu_1 t}(\delta(Z - Q_1^0) - \delta(Z - Q_1^1))) * \dots * \\ * (e^{-\mu_m x - \nu_m t}(\delta(Z - Q_m^0) - \delta(Z - Q_m^1))) \end{aligned} \tag{35}$$

is accompanying for Eq. (20).

Taking into account the equality

$$(\rho_1(Z)\delta(Z - Z_1)) * (\rho_2(Z)\delta(Z - Z_2)) = \rho_1(Z_1)\rho_2(Z_2)\delta(Z - Z_1 - Z_2), \qquad (36)$$

we describe distribution (35) and the corresponding mean value formula. We denote by the symbol "$\oplus$" the modulo 2 sum of Boolean variables: $\alpha \oplus \beta = 0$, if $\alpha = \beta$ and $\alpha \oplus \beta = 1$, if $\alpha \neq \beta$. Let $\alpha = (\alpha_1, \ldots, \alpha_m),\ l = \alpha_1 \oplus \cdots \oplus \alpha_m$. Denote by Θ the set of all ordered collections $\alpha = (\alpha_1, \ldots, \alpha_m),\ \alpha_j \in \{0; 1\},\ j = 1, \ldots, m$. Let

$$A_l^\alpha = \sum_{j=1}^{m} Q_j^{\alpha_j}. \qquad (37)$$

Then accompanying distribution (35) for Eq. (20) can be written as

$$\Phi(Z) = \sum_{\alpha\in\Theta} (-1)^l\, \delta(Z - A_l^\alpha) \prod_{j=1}^{m} \exp(-\mu_j \xi_j^{\alpha_j} - \nu_j \tau_j^{\alpha_j}), \qquad (38)$$

and the corresponding mean value formula for Eq. (20) can be written as

$$\sum_{\alpha\in\Theta} (-1)^l u(A_l^\alpha) \prod_{j=1}^{m} \exp(-\mu_j \xi_j^{\alpha_j} - \nu_j \tau_j^{\alpha_j}) = 0. \qquad (39)$$

Note that two points $A_l^\alpha = (\eta_l^\alpha, \omega_l^\alpha)$ and $A_s^\gamma = (\eta_s^\gamma, \omega_s^\gamma)$, defined by (37), lie on the j-th characteristic if and only if, their upper multi-indexes

$$\alpha = (\alpha_1, \ldots, \alpha_m), \gamma = (\gamma_1, \ldots, \gamma_m),\ \alpha_j, \gamma_j \in \{0; 1\},\ j = 1, \ldots, m,$$

are connected with relations

$$\alpha_k = \gamma_k,\ k = 1, \ldots, j-1, j+1, \ldots, m,\ \gamma_j = \neg\alpha_j,\ s = \neg l = l \oplus 1, \qquad (40)$$

where "$\neg$" is the negation operation. Let's prove that fact. The assertion that two points $Q_j^0 = (\xi_j^0, \tau_j^0)$ and $Q_j^1 = (\xi_j^1, \tau_j^1)$ lie on a j-th characteristic means that their coordinates are connected by the equality

$$b_j \xi_j^0 - a_j \tau_j^0 = b_j \xi_j^1 - a_j \tau_j^1. \qquad (41)$$

Then at the point $A_l^\alpha = (\eta_l^\alpha, \omega_l^\alpha)$, where $\alpha = (\alpha_1, \dots, \alpha_m)$, we have

$$b_j \eta_l^\alpha - a_j \omega_l^\alpha = b_j(\xi_1^{\alpha_1} + \xi_2^{\alpha_2} + \dots + \xi_j^{\alpha_j} + \dots + \xi_m^{\alpha_m}) -$$

$$-a_j(\tau_1^{\alpha_1} + \tau_2^{\alpha_2} + \dots + \tau_j^{\alpha_j} + \dots + \tau_m^{\alpha_m})$$

$$= b_j(\xi_1^{\alpha_1} + \xi_2^{\alpha_2} + \dots + \xi_j^{\neg\alpha_j} + \dots + \xi_m^{\alpha_m}) -$$

$$- a_j(\tau_1^{\alpha_1} + \tau_2^{\alpha_2} + \dots + \tau_j^{\neg\alpha_j} + \dots + \tau_m^{\alpha_m}) = b_j \eta_s^\gamma - a_j \omega_s^\gamma, \tag{42}$$

and this completes our proof.

The set of all points A_l^α, defined by formula (37), together with all segments of characteristics, connecting pairs of points A_l^α and A_l^γ, satisfying equalities (40), can be interpreted as a graph with the vertices A_l^α and the edges coinciding with the segments of the characteristics.

This graph is isomorphic to a m-dimensional cube with edge length 1. Denote by $\widetilde{A}_l^\alpha$ vertices of this cube. Now we can interpret the numbers $\alpha_1, \dots, \alpha_m$ as usual coordinates in space R^m. Let A_l^α and A_l^γ satisfy equalities (40). An edge of the graph joining vertices A_l^α and A_l^γ is corresponded to an edge of the cube with vertices $\widetilde{A}_l^\alpha$ and $\widetilde{A}_l^\gamma$.

Let $G = G(A_l^\alpha)$ be a graph on the plane with vertices $A_l^\alpha \in R^2$, where $\alpha = (\alpha_1, \dots, \alpha_m)$, $l = \alpha_1 \oplus \dots \oplus \alpha_m$ $\alpha_j \in \{0; 1\}$, multi-index α is running the set Θ. Let any two points A_l^α, A_s^γ and only them, satisfying conditions (40), be connected by a segment of j-th type characteristic. We declare these segments to be edges of the graph. We say that a graph of this kind is a *characteristic graph*. Let's see if formula (39) holds for any characteristic graph G. We proved that formula (39) holds for a graph with vertices defined by (37). If we prove that vertices of any characteristic graph are representable in form (37) we will confirm our assumption.

We know that every graph of this kind is isomorphic to a m-dimensional cube. We denote this cube by $\widetilde{G}$. Let H be an isomorphism that maps the cube $\widetilde{G}$ onto the graph G. For example, the isomorphism H can be a linear mapping (projection). Let $\widetilde{Q}_k^{\alpha_k} = (0, \dots, \alpha_k, \dots, 0)$ (k-th coordinate is equal to α_k, the rest ones are equal to zero moreover $\widetilde{Q}_k^0 = (0, \dots, 0)$ for all $k = 1, \dots, m$). Then it is sufficient to set $Q_k^{\alpha_k} = H(\widetilde{Q}_k^{\alpha_k})$, $k = 1, \dots, m$, $\alpha_k = 0, 1$, that leads to equalities (37).

Now we have to remember that direct using of the accompanying distributions technics leads to mean value formulas for solutions in C^∞. Nevertheless, the action of the accompanying distribution is able to be extended to arbitrary regular solutions. So we proved the next main result.

Theorem *Let $u(Z) = u(x, t)$ be a regular solution of Eq. (20) and G be an arbitrary characteristic graph with vertices A_l^α, which are represented by form (37),*

$Q_j^{\alpha_j} = (\xi_j^{\alpha_j}, \tau_j^{\alpha_j})$. *Then the mean value formula (39) holds. This formula is an exact difference ratio.*

For $m = 3$ and $m = 4$, $c_j = 0$, this theorem was proved in [15] and [16] respectively.

We considered the case when all of the characteristics for Eq. (20) were simple. Let us now decline the requirement of simple characteristics. Then formally the mean-value formula will look like (39), but we have to get $2k$ points on every characteristic instead of two ones, where k is the multiplicity of the characteristic. Furthermore let us assume that we are dealing with alone m-multiple characteristic. Then Eq. (20) takes the form

$$(a\,\partial/\partial x + b\,\partial/\partial t + c)^m\, u = 0. \tag{43}$$

Making the substitutions $\eta = x/a,\ \xi = (b\,x - a\,t)/a$, we reduce this equation to the form

$$\left(\frac{\partial}{\partial \eta} + c\right)^m u = 0. \tag{44}$$

This is an ordinary differential equation. Its general solution is

$$u(\eta) = P_{m-1}(\eta)\exp\left(-c\eta\right), \tag{45}$$

where $P_{m-1}(\eta)$ is a polynomial of degree $m-1$ in η with coefficients depending on variable ξ. In addition let $c = 0$. Then formula (39) becomes "inclusions-exceptions formula" for the polynomial of degree $m-1$ (see [17])

$$u(0) = \sum_{k=1}^{m} (-1)^{k-1} \sum_{i_1<\cdots<i_k} u(x_{i_1} + \cdots + x_{i_k}), \tag{46}$$

where $x_1, \ldots, x_m$ are arbitrary points of the real line.

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Transmutations for Multi-Term Fractional Operators

Arsen V. Pskhu

Abstract In this paper, we construct a transmutation operator for fractional multi-term differential operators. The constructed operator intertwines multi-term differential operators and the operator of first order differentiation, and allows us to find explicit representations of solutions for initial and boundary value problems for fractional multi-term evolution type differential equations. As an example, we find solutions to a boundary value problem for the multi-term fractional diffusion equation in an unbounded domain.

Keywords Fractional derivative · Transmutation operator · Multi-term equation

2010 Mathematics Subject Classification 35R11, 35A22, 34A08, 34A25

1 Introduction

Consider the operator

$$\mathcal{D}_t^{\sigma,\lambda} = \sum_{k=1}^{n} \lambda_k \frac{\partial^{\sigma k}}{\partial t^{\sigma_k}} \tag{1.1}$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\sigma_k \in (0, 1)$, $\lambda_k > 0$ and $\partial^{\sigma_k}/\partial t^{\sigma_k}$ stands for the fractional derivative of order σ_k with respect to t with starting point at $t = 0$.

Our purpose is to construct a transmutation operator T that intertwines the multi-term operator (1.1) and the operator of first order differentiation $D_t = \frac{\partial}{\partial t}$, i.e. an operator T satisfying

$$\mathcal{D}_t^{\sigma,\lambda} T = T D_t. \tag{1.2}$$

A. V. Pskhu (✉)
Institute of Applied Mathematics and Automation KBSC RAS, Nalchik, Russia
e-mail: pskhu@list.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_27

Having constructed the operator T, one can derive solutions to evolution type equations of the form

$$\mathcal{D}_t^{\sigma,\lambda} u = Lu$$

in terms of solutions to the first order equation

$$D_t v = Lv.$$

Here, L is a linear operator. As an example, we find an explicit form of solutions to a boundary value problem for the multi-term fractional diffusion equation.

As it is well known, the method of transmutations provides us with powerful tools to study operators with complicated structure in terms of simpler operators. In this connection, we refer to [1–8], which contain the principal points of the transmutation theory and comprehensive bibliographies on the topic.

The operator (1.1) can be treated as a fractional differentiation operator of distributed (segment) order [9]

$$\int_a^b \left(\lambda(\xi, t) \frac{\partial^\xi}{\partial t^\xi} \right) d\mu(\xi) \tag{1.3}$$

with a measure concentrated on a discrete set. It can also be used for finding approximate solutions of equations with operators of the form (1.3) [10]. Differential equations of fractional order including those with the operators (1.1) and (1.3) have been widely used in modeling, physics, and mechanics [11–15].

We also mention [16–25] that reflect the variety of approaches to the study of multi-term fractional differential equations, and contain an extensive bibliography on the subject.

The paper has the following structure. In Sects. 2 and 3, we give the definitions of the fractional differentiation operators and related special functions necessary for what follows. In Sect. 4, we construct a transmutation operator satisfying (1.2). In Sect. 5 some applications are indicated.

2 Fractional Differentiation

The fractional differentiation is given by the Dzhrbashyan–Nersesyan operator [26] (see also [27]). The Dzhrbashyan–Nersesyan operator of order $\zeta \in (0, 1)$ associated with an ordered pair $\{\xi, \eta\}$, is defined by

$$\frac{\partial^\zeta}{\partial t^\zeta} = D_{0t}^{\{\xi,\eta\}} = D_{0t}^{\eta-1} D_{0t}^{\xi}, \qquad \xi, \eta \in (0, 1], \qquad \zeta = \xi + \eta - 1, \tag{2.1}$$

where $D_{0t}^{\eta-1}$ and D_{0t}^{ξ} are the Riemann–Liouville fractional integral and fractional derivative, respectively, with starting point at $t = 0$ [11, p. 11], [28, §2.1]:

$$D_{0t}^{\eta-1} g(t) = \int_0^t g(s) \frac{(t-s)^{-\eta}}{\Gamma(1-\eta)} ds \qquad D_{0t}^{\xi} g(t) = \frac{\partial}{\partial t} \int_0^t g(s) \frac{(t-s)^{-\xi}}{\Gamma(1-\xi)} ds.$$

It is also assumed that $D_{0t}^{0} g(t) = g(t)$.

From the definition (2.1) and the composition law for the Riemann–Liouville derivatives (see, e.g., [28]), it follows that

$$D_{0t}^{\{\xi,\eta\}} g(t) = D_{0t}^{\zeta} g(t) - \frac{t^{-\eta}}{\Gamma(1-\eta)} \left[D_{0t}^{\xi-1} g(t) \right]_{t=0}. \tag{2.2}$$

It should be noted that the operator (2.1) associated with the pairs $\{\zeta, 1\}$ and $\{1, \zeta\}$, coincides with the Riemann–Liouville derivative and the Caputo derivative, respectively, i.e.:

$$D_{0t}^{\{\zeta,1\}} g(t) = D_{0t}^{\zeta} g(t) \qquad \text{and} \qquad D_{0t}^{\{1,\zeta\}} g(t) = \partial_{0t}^{\zeta} g(t) = \int_0^t g'(s) \frac{(t-s)^{-\zeta}}{\Gamma(1-\zeta)} ds.$$

It means that the operator (1.1) covers the special cases of operators containing the Riemann–Liouville and Caputo derivatives.

3 Auxiliary Assertions

In what follows, the symbol C denotes positive constants, which are different in different cases; if necessary, the parameters on which it can depend are indicated in parentheses: $C = C(\alpha, \beta, \ldots)$.

Consider the function [23]

$$w_\mu(s,t) = S_n^\mu\left(t; -\lambda_1 s, \ldots, -\lambda_n s; -\sigma_1, \ldots, -\sigma_n\right)$$

where [18]

$$S_n^\mu(t; -\lambda_1 s, \ldots, -\lambda_n s; -\sigma_1, \ldots, -\sigma_n) = (h_1 * h_2 * \ldots * h_n)(t), \tag{3.1}$$

$$h_k = h_k(t) \equiv t^{\mu_k - 1} \phi\left(-\sigma_k, \mu_k, -\lambda_k s t^{-\sigma_k}\right), \qquad \mu = \sum_{k=1}^n \mu_k,$$

$$\phi(\delta,\ \varepsilon;\ z) = \sum_{m=0}^\infty \frac{z^m}{m!\,\Gamma(m\delta + \varepsilon)} \qquad (\delta > -1)$$

is the Wright function [29, 30]; $(h * g)(t)$ denotes the Laplace convolution of the functions $h(t)$ and $g(t)$:

$$(h * g)(t) = \int_0^t h(t-\eta)\, g(\eta)\, d\eta.$$

It should be noted that the function (3.1) (as well as the function $w_\mu(x, y)$) does not depend on the distribution of the numbers μ_k but depends only on their sum μ [23].

In the following lemma, we formulate the properties of the function $w_\mu(s, t)$ that we need in our further considerations. The proof of the statement can be found in [24, Lemma 1] (see also [23]).

Lemma 3.1 *Let*

$$\sigma_* = \max\{\sigma_1, \dots, \sigma_n\}, \qquad \lambda_* = \max_{k:\, \sigma_k = \sigma_*} \{\lambda_k\}.$$

Then the following assertions hold.

1. *If* $\mu \geq 0$, $s > 0$ *and* $t > 0$, *then*

$$w_\mu(s, t) > 0. \tag{3.2}$$

2. *The inequality*

$$w_\mu(s, t) \leq C s^{-\theta} t^{\mu + \theta\sigma_* - 1} \exp\left(-\rho s^{\frac{1}{1-\sigma_*}} t^{-\frac{\sigma_*}{1-\sigma_*}}\right), \qquad C = C(\mu, \lambda, \sigma, \theta, \rho), \tag{3.3}$$

holds for arbitrary $\mu \in \mathbb{R}$, θ *and* ρ *such that*

$$\rho < (1-\sigma_*)\sigma_*^{\frac{\sigma_*}{1-\sigma_*}} \lambda_*^{\frac{1}{1-\sigma_*}} \qquad \text{and} \qquad \theta \geq \begin{cases} 0, & (-\mu) \notin \mathbb{N} \cup \{0\}, \\ -1, & (-\mu) \in \mathbb{N} \cup \{0\}. \end{cases}$$

3. *The following relation holds:*

$$D_{0t}^{\nu} w_\mu(s, t) = w_{\mu-\nu}(s, t). \tag{3.4}$$

4. *If* $\mu \geq 0$ *and* $t > 0$, *then*

$$\lim_{s \to +0} w_\mu(s, t) = \frac{t^{\mu-1}}{\Gamma(\mu)}. \tag{3.5}$$

5. The relation

$$\left(\frac{\partial}{\partial s} + \sum_{k=1}^{n} \lambda_k D_{0t}^{\sigma_k}\right) D_{0t}^{-\nu} w_\mu(x, y) = \frac{s^{\nu-1}}{\Gamma(\nu)} \frac{t^{\mu-1}}{\Gamma(\mu)} \tag{3.6}$$

holds for arbitrary $\mu \geq 0$ and $\nu \geq 0$.

4 Transmutation Operator

Further, we use the following notations

$$\sigma_* = \max\{\sigma_1, \ldots, \sigma_n\}, \qquad \alpha_* = \max\{\alpha_1, \ldots, \alpha_n\},$$

and, as above, we always assume that

$$\sigma_k \in (0, 1), \quad \alpha_k,\ \beta_k \in (0, 1], \quad \sigma_k = \alpha_k + \beta_k - 1, \quad \lambda_k > 0, \quad k = \overline{1, n}.$$

Moreover, without loss of generality we assume that all pairs $\{\alpha_k, \beta_k\}$ are pairwise distinct, i.e. $(\alpha_k - \alpha_j)^2 + (\beta_k - \beta_j)^2 = 0$ if and only if $k = j$.

Consider the operator

$$T^{\lambda}_{\alpha,\beta} u(t) = \left(T^{\lambda}_{\alpha,\beta} u\right)(t) = \int_0^\infty u(s) \sum_{k:\, \alpha_k = \alpha_*} \lambda_k\, w_{1-\beta_k}(s, t)\, ds. \tag{4.1}$$

The summation in (4.1) is over all k such that $\alpha_k = \alpha_*$. In (4.1), $u(x)$ is assumed to be locally integrable, i.e.

$$u(t) \in L(0, a) \quad \text{for any} \quad a > 0,$$

and satisfying

$$\lim_{t\to\infty} u(t) \exp\left(-t^{\varepsilon}\right) = 0 \quad \text{for some} \quad \varepsilon < \frac{1}{1 - \sigma_*}. \tag{4.2}$$

We can now formulate the main results of the work.

Theorem 4.1 *Let $u(t)$ satisfy (4.2), $u(t) \in C[0, \infty) \cap C^1(0, \infty)$ and*

$$\sigma_* > \max_{k:\, \alpha_k < \alpha_*} \{\sigma_k\}. \tag{4.3}$$

Then

$$\lim_{t\to 0} D_{0t}^{\alpha_*-1} T^{\lambda}_{\alpha,\beta} u(t) = u(0) \tag{4.4}$$

and

$$\mathcal{D}_t^{\sigma,\lambda} T_{\alpha,\beta}^{\lambda} u(t) = T_{\alpha,\beta}^{\lambda} u'(t). \tag{4.5}$$

Proof Let $K(s,t)$ denote the kernel in (4.1):

$$K(s,t) = \sum_{k:\,\alpha_k=\alpha_*} \lambda_k\, w_{1-\beta_k}(s,t).$$

By (3.4) and (3.5), we get

$$D_{0t}^{\alpha_*-1} K(s,t) = \sum_{k:\,\alpha_k=\alpha_*} \lambda_k w_{1-\sigma_k}(s,t) =$$

$$= \sum_{k=1}^{n} \lambda_k w_{1-\sigma_k}(s,t) - \sum_{k:\alpha_k<\alpha_*} \lambda_k w_{1-\sigma_k}(s,t) = W_1(s,t) - W_2(s,t).$$

Therefore, we have

$$D_{0t}^{\alpha_*-1} T_{\alpha,\beta}^{\lambda} u(t) = \int_0^\infty u(s)\,[W_1(s,t) - W_2(s,t)]\,dt = I_1 - I_2.$$

It follows from (3.3) that

$$|I_2| \le C t^{\theta\sigma_*} \sum_{k:\alpha_k<\alpha_*} t^{-\sigma_k} \qquad (0 \le \theta < 1).$$

By virtue of (4.3) this means that $\lim_{t\to 0} I_2 = 0$. Next, for an arbitrary $\varepsilon > 0$ we can rewrite I_1 as

$$I_1 = \left(\int_0^\varepsilon + \int_\varepsilon^\infty\right) [u(s) - u(0)]\, W_1(s,t)\,dt + u(0) \int_0^\infty W_1(s,t)\,dt = J_1^\varepsilon + J_2^\varepsilon + J_3.$$

By (3.2) and (3.6) it is easy to check that

$$0 \le W_1(s,t) = \left(\sum_{k=1}^{n} \lambda_k D_{0t}^{\sigma_k}\right) w_1(s,t) = -\frac{\partial}{\partial s} w_1(s,t).$$

Together with (3.3) and (3.5), this yields

$$\left|J_1^\varepsilon\right| \le \sup_{t\in(0,\varepsilon)} |u(s) - u(0)|\,[w_1(0,t) - w_1(\varepsilon,t)], \qquad \lim_{t\to 0} J_2^\varepsilon = 0,$$

and

$$J_3 = u(0) \int_0^\infty \left[-\frac{\partial}{\partial s} w_1(s,t) \right] dt = u(0).$$

Taking into account the arbitrary choice of ε, this leads to (4.4).

Let us prove (4.5). By (2.2) and (4.4), we get

$$\mathcal{D}_t^{\sigma,\lambda} T_{\alpha,\beta}^{\lambda} u(t) = \sum_{k=1}^{n} \lambda_k D_{0t}^{\beta_k - 1} D_{0t}^{\alpha_k} T_{\alpha,\beta}^{\lambda} u(t) =$$

$$= \sum_{k=1}^{n} \lambda_k \left[D_{0t}^{\sigma_k} T_{\alpha,\beta}^{\lambda} u(t) - \frac{t^{-\beta_k}}{\Gamma(1-\beta_k)} (D_{0t}^{\alpha_k - 1} T_{\alpha,\beta}^{\lambda} u)_{t=0} \right] =$$

$$= \sum_{k=1}^{n} \lambda_k D_{0t}^{\sigma_k} T_{\alpha,\beta}^{\lambda} u(t) - u(0) \sum_{k:\alpha_k=\alpha_*} \frac{t^{-\beta_k}}{\Gamma(1-\beta_k)}. \tag{4.6}$$

Consider the first summand in the right hand side of the last equality. The properties of $w_\mu(s,t)$ and the requirements imposed on $u(t)$ allow us to bring the operator

$$\mathcal{D}^* = \sum_{k=1}^{n} \lambda_k D_{0t}^{\sigma_k}$$

inside the integral:

$$\sum_{k=1}^{n} \lambda_k D_{0t}^{\sigma_k} T_{\alpha,\beta}^{\lambda} u(t) = \int_0^\infty u(s) \mathcal{D}^* K(s,t)\, dt. \tag{4.7}$$

By (3.5) and (3.6), we get

$$\mathcal{D}^* K(s,t) = \sum_{k:\,\alpha_k=\alpha_*} \lambda_k \mathcal{D}^* w_{1-\beta_k}(s,t) = -\frac{\partial}{\partial s} \sum_{k:\,\alpha_k=\alpha_*} \lambda_k w_{1-\beta_k}(s,t)$$

and

$$\int_0^\infty u(s) \mathcal{D}^* K(s,t)\, dt = \int_0^\infty u(s) \left(-\frac{\partial}{\partial s} \sum_{k:\,\alpha_k=\alpha_*} \lambda_k w_{1-\beta_k}(s,t) \right) dt =$$

$$= T_{\alpha,\beta}^{\lambda} u'(t) + u(0) \sum_{k:\alpha_k=\alpha_*} \frac{t^{-\beta_k}}{\Gamma(1-\beta_k)}.$$

Combining the last equality with (4.6) and (4.7), we obtain (4.5). □

Remark 4.2 For the case $n = 1$, we have

$$\sigma = \{\sigma_1\}, \qquad \sigma_1 = \alpha_1 + \beta_1 - 1, \qquad \lambda = \{\lambda_1\},$$

and

$$w_\mu(s,t) = S_1^\mu\,(t; -s; -\sigma_1) = t^{\mu-1}\phi\left(-\sigma_1, \mu; -\frac{s}{t^{\sigma_1}}\right)$$

[Without loss of generality, we put $\lambda_1 = 1$.] Thus, for $n = 1$ the operator $T^\lambda_{\alpha,\beta}$ can be written as

$$T^\lambda_{\alpha,\beta}u(t) = \int_0^\infty u(s)\,t^{-\beta_1}\phi\left(-\sigma_1, 1-\beta_1; -\frac{s}{t^{\sigma_1}}\right)ds = A^{\sigma_1, 1-\beta_1}u(t).$$

That is, in this case, $T^\lambda_{\alpha,\beta}$ coincides with the Stankovich transform $A^{\sigma_1,1-\beta_1}$, which intertwines the fractional differentiation operator $\frac{\partial^{\sigma_1}}{\partial t^{\sigma_1}} = D^{\beta_1-1}_{0t}D^{\alpha_1}_{0t}$ and the operator of first order differentiation $\frac{\partial}{\partial t}$ [31–33].

5 Application

As an example of application of our results, we construct a solution of a boundary value problem in an unbounded domain for the multi-term fractional diffusion equation.

Let $\mathbb{R}_+$ denote the set of positive real numbers, $p, q \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ and $p + q = m$, and let $S_{p,q}$ be the subset of $\mathbb{R}^m$ defined by

$$S_{p,q} = \mathbb{R}^p_+ \times \mathbb{R}^q = \{x = (x_1, .., x_m) \in \mathbb{R}^m : x_j > 0, j = 1, \ldots, p\}.$$

It is evident that $S_{0,m}$ coincides with $\mathbb{R}^m$, and $S_{m,0}$ is the positive orthant $\mathbb{R}^m_+$.

Next, for a point $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$, let $\xi^{[k]}$ denote the symmetric reflection of ξ with respect to the hyper-plane perpendicular to ξ_k-axis, i.e

$$\xi^{[k]} = (\xi_1, \ldots, \xi_{k-1}, -\xi_k, \xi_{k+1}, \ldots, \xi_m).$$

For a function $f(\xi)$, $\xi \in \mathbb{R}^m$, we introduce the operator $J^{[k]}_\xi$ defined by

$$J^{[k]}_\xi f(\xi) = f(\xi) - f(\xi^{[k]}).$$

It is well known (see, i.e., [34]) that the function

$$\Gamma(x,\xi,t) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x-\xi|^2}{4t}\right)$$

is the fundamental solution of the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta_x\right) v(x,t) = 0 \tag{5.1}$$

where

$$\Delta_x = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}$$

is the Laplace operator with respect to x, $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$. Moreover, it is easy to check that the function

$$G(x,\xi,t) = J_\xi^{[1]} J_\xi^{[2]} \ldots J_\xi^{[p]} \Gamma(x,\xi,t)$$

is the Green function of the boundary value problem:

$$v(x,0) = \tau(x) \qquad x \in S_{p,q}, \tag{5.2}$$

$$v(x,t)\big|_{x\in\partial S_{p,q}} = 0 \qquad t \in \mathbb{R}_+, \tag{5.3}$$

for Eq. (5.1). As usual, $\partial S_{p,q}$ stands for the boundary of $S_{p,q}$. [Note: $\partial S_{p,q} = \emptyset$ for $p = 0$, and in this case the condition (5.3) is superfluous.] In addition, under some restrictions on the function $\tau(x)$, the solution of the problem (5.1), (5.2) and (5.3) can be given by [34]

$$v(x,t) = \int_{S_{p,q}} \tau(\xi)\, G(x,\xi,t)\, d\xi.$$

Now, our goal is to find a solution of the multi-term fractional diffusion equation

$$\left(\mathcal{D}_t^{\sigma,\lambda} - \Delta_x\right) u(x,t) = 0, \tag{5.4}$$

in the domain

$$\Omega = S_{p,q} \times \mathbb{R}_+,$$

with the initial and boundary conditions

$$\lim_{t\to 0} D_{0t}^{\alpha_*-1} u(x,t) = \tau(x) \qquad x \in S_{p,q}, \tag{5.5}$$

$$u(x,t)\big|_{x\in\partial S_{p,q}} = 0 \qquad t \in \mathbb{R}_+. \tag{5.6}$$

[As above, we consider (5.5) only, if $p = 0$.]

Let $v(x,t)$ be a solution of the problem (5.1), (5.2) and (5.3), and let $v(x,t)$ and $\Delta_x v(x,t)$ satisfy (4.2) uniformly with respect to $x \in S_{p,q}$. The formula (4.5) now yields

$$\mathcal{D}_t^{\sigma,\lambda} T_{\alpha,\beta}^{\lambda} v(x,t) = T_{\alpha,\beta}^{\lambda} \frac{\partial}{\partial t} v(x,t) = -T_{\alpha,\beta}^{\lambda} \Delta_x v(x,t) = -\Delta_x T_{\alpha,\beta}^{\lambda} v(x,t).$$

Next, it follows from (4.4) that

$$\lim_{t\to 0} D_{0t}^{\alpha_*-1} T_{\alpha,\beta}^{\lambda} v(x,t) = v(x,0) = \tau(x).$$

Thus, we can claim that the function

$$u(x,t) = T_{\alpha,\beta}^{\lambda} v(x,t) \tag{5.7}$$

is the solution of the Eq. (5.4) and satisfies the conditions (5.5) and (5.6).

Moreover, we can rewrite (5.7) as

$$u(x,t) = \int_{S_{p,q}} \tau(\xi)\, G_{\alpha,\beta}^{\lambda}(x,\xi,t)\, d\xi. \tag{5.8}$$

where

$$G_{\alpha,\beta}^{\lambda}(x,\xi,t) = T_{\alpha,\beta}^{\lambda} G(x,\xi,t)\, d\xi$$

is the Green function of the problem (5.4), (5.5) and (5.6).

More precisely, in much the same way as in [24], we can prove the following statement.

Theorem 5.1 *Let (4.3) be satisfied; $\tau(x) \in C(\overline{S}_{p,q})$ if $m = 1$ or $\beta_k = 1$ for all $k = 1,\ldots,n$; $\tau(x)$ be Hölder continuous if $m \geq 2$ and $\beta_k < 1$ for some k; and $\tau(x)$ satisfy*

$$\lim_{|x|\to\infty} \tau(x) \exp\left(-\rho|x|^{\delta}\right) = 0$$

for some $\rho > 0$ and $\delta < \frac{2}{2-\sigma_*}$. Then the function (5.8) is a solution of the problem (5.4), (5.5) and (5.6).

Acknowledgement The reported study was funded by RFBR according to the research project no 18–51–45005.

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Fractional Bessel Integrals and Derivatives on Semi-axes

E. L. Shishkina and S. M. Sitnik

Abstract In this paper we study fractional powers of the Bessel differential operator. The fractional powers are defined explicitly in the integral form without use of integral transforms in its definitions. Some general properties of the fractional powers of the Bessel differential operator are proved and some are listed. Among them are different variations of definitions, relations with the Mellin and Hankel transforms, group property, evaluation of resolvent integral operator in terms of the Wright or generalized Mittag–Leffler functions. At the end, some topics are indicated for further study and possible generalizations. Also the aim of the paper is to attract attention and give references to not widely known results on fractional powers of the Bessel differential operator. This class of fractional operators is in close connection with transmutation theory and classic transmutational operators. We also study connections of Bessel fractional operators with different kinds of integral transforms.

Keywords Fractional Bessel operator · Hypergeometric function · Hankel transform

1 Introduction

We study the differential Bessel operator in the form

$$B_\nu := D^2 + \frac{\nu}{x} D, \qquad \nu \geq 0, \quad D := \frac{d}{dx}, \tag{1}$$

E. L. Shishkina (✉)
University of Information Technology and Management in Rzeszow, Rzeszow, Poland
e-mail: shishkina@amm.vsu.ru

S. M. Sitnik
Belgorod State National Research University (BelGU), Belgorod, Russia
e-mail: sitnik@bsu.edu.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*,
Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_28

and its fractional powers $(B_\nu)^\alpha$, $\alpha \in \mathbb{R}$. This operator has essential role in the theory of differential equations both as a radial part of the Laplace operator and also as involved in partial differential equations with Bessel operators. Such equations were called B-elliptic, B-hyperbolic and B-parabolic by I.A. Kipriyanov and intensively studied by his scientific school and many others researchers, now the term "Laplace–Bessel equations" is also used. For equations with Bessel operators and related topics cf. [1–3].

Of course fractional powers of the Bessel operator (1) were studied in many papers. But in the most of them fractional powers were defined implicitly as a power function multiplication under Hankel transform. This definition via integral transforms leads to many restrictions. Just imagine that for the classical Riemann–Liouville fractional integrals we have to work only with its definitions via Laplace or Mellin transforms and nothing more without explicit integral representations. If it would be true, then 99% of classical "Bible" [4] and other books on fractional calculus would be empty as they mostly use explicit integral definitions! But for fractional powers of the Bessel operator at most papers implicit definitions via Hankel transform are still used.

Of course such situation is not natural and in some papers different approaches to step closer to explicit formulas were studied. Let us mention that in [5] explicit formulas were derived as compositions of Erdélyi–Kober fractional integrals [4] on distribution spaces, in this monograph results on fractional powers of Bessel and related operators are gathered of McBride's and earlier papers. An important step was done in [6] in which explicit definitions were derived in terms of the Gauss hypergeometric functions with different applications to PDE, we also use basic formulas from [6] in this paper. The most general study was fulfilled by I. Dimovski and V. Kiryakova [7–10] for the more general class of hyper-Bessel differential operators related to the Obrechkoff integral transform. They constructed explicit integral representations of the fractional powers of these operators by using Meijer G-functions as kernels, and also intensively and successfully used for this the theory of transmutations. Note that in this and others fields of theoretical and applied mathematics, the methods of transmutation theory are very useful and productive and for some problems are even irreplaceable (see e.g. [11]). In [12, 13] simplified representations for fractional powers of the Bessel operator were derived with Legendre functions as kernels, and based on them general definitions were simplified and unified with standard fractional calculus notation as in [4], and also important generalized Taylor formulas were proved which mix integer powers of Bessel operators (instead of derivatives in the classical Taylor formula) with fractional power of the Bessel operator as integral remainder term, cf. also [14, 15].

This class of fractional operators is in close connection with transmutation theory and classic transmutational operators such as Sonine and Poisson ones [16, 17]. We also study connections of Bessel fractional operators with different kinds of integral transforms: Hankel, Mellin, Erd'elyi–Kober, Mejer, integral transforms with Wittaker, Wright, Mittag–Leffler and hypergeometric kernels.

2 Definitions

2.1 Special Functions and Integral Transforms

In this subsection we give definitions of some special functions. Special functions enable us to introduce integral transforms connected with these functions such that a new problem can be attacked within a known framework, usually in the context of differential equations and their generalizations.

Let start with normalized Bessel functions.

The symbol j_α is used for the normalized Bessel function:

$$j_\alpha(t) = \frac{2^\alpha \Gamma(\alpha+1)}{t^\alpha} J_\alpha(t), \qquad j_\alpha(0) = 1, \qquad j'_\alpha(0) = 0, \tag{2}$$

where $J_\alpha(t)$ is the Bessel function of the first kind of order α (see [18]):

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

Function J_α first defined by the mathematician Daniel Bernoulli and then generalized by Friedrich Bessel (see [18]).

Using formulas 9.1.27 from [19] we obtain that the function $j_\nu(t)$ is an eigenfunction of a linear operator B_ν:

$$(B_\nu)_t\, j_{\frac{\nu-1}{2}}(\tau t) = -\tau^2 j_{\frac{\nu-1}{2}}(\tau t). \tag{3}$$

We also will need some other normalized Bessel functions. Normalized Bessel functions of the second kind y_α is

$$y_\alpha(t) = \frac{2^\alpha \Gamma(\alpha+1)}{t^\alpha} Y_\alpha(t), \tag{4}$$

where Y_α is the Bessel functions of the second kind. Function Y_α for non-integer α is related to J_α by:

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}.$$

In the case of integer order n, the function Y_n is defined by taking the limit as a non-integer α tends to n,

$$Y_n(x) = \lim_{\alpha \to n} Y_\alpha(x).$$

Normalized modified Bessel functions of the first and second kind $I_\alpha(x)$ and $K_\alpha(x)$ are defined by

$$i_\alpha(t) = \frac{2^\alpha \Gamma(\alpha+1)}{t^\alpha} I_\alpha(t), \qquad k_\alpha(t) = \frac{2^\alpha \Gamma(\alpha+1)}{t^\alpha} K_\alpha(t), \tag{5}$$

where modified Bessel functions of the first and second kind $I_\alpha(x)$ and $K_\alpha(x)$ are

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}, \qquad K_\alpha(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)},$$

when α is not an integer and when α is an integer, then the limit is used.

Next we consider generalized hypergeometric functions which have many particular special functions as special cases, such as elementary functions, Bessel functions, and the classical orthogonal polynomials.

A generalized hypergeometric function is defined as a power series

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

The functions of the form ${}_0F_1(;a;z)$ are called confluent hypergeometric limit functions and are closely related to Bessel functions J_α and I_α. The relationships are

$$J_\alpha(x) = \frac{(\frac{x}{2})^\alpha}{\Gamma(\alpha+1)} {}_0F_1\left(;\alpha+1;-\tfrac{x^2}{4}\right),$$

$$I_\alpha(x) = \frac{(\frac{x}{2})^\alpha}{\Gamma(\alpha+1)} {}_0F_1\left(;\alpha+1;\tfrac{x^2}{4}\right)$$

or

$${}_0F_1\left(;\alpha+1;-\tfrac{x^2}{4}\right) = j_\alpha(x), \qquad {}_0F_1\left(;\alpha+1;\tfrac{x^2}{4}\right) = i_\alpha(x).$$

Beside we need function ${}_1F_2(;a;z)$. It is known (see [20]) that for $\alpha > 0$, $\xi \geq 0$, $t > 0$

$$\int_0^t \left(t^2-u^2\right)^{\alpha-1} u^{1-\gamma} J_\gamma(u\xi)dt = \frac{\xi^\gamma t^{2\alpha}}{2^{\gamma+1}\alpha\Gamma(\gamma+1)} {}_1F_2\left(1;\alpha+1,\gamma+1;-\frac{t^2\xi^2}{4}\right)$$

and for $\gamma < 2, \alpha > 0, \xi \geq 0, t > 0$

$$\int_0^t \left(t^2 - u^2\right)^{\alpha-1} u^{1-\gamma} I_\gamma(u\xi)dt = \frac{\xi^\gamma t^{2\alpha}}{2^{\gamma+1}\alpha\Gamma(\gamma+1)}\, {}_1F_2\left(1; \alpha+1, \gamma+1; \frac{t^2\xi^2}{4}\right).$$

Next we present Whittaker functions which are appear in kernel of integral transform connected with fractional Bessel integral.

Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ are special solutions of Whittaker's equation

$$\frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1/4 - \mu^2}{z^2}\right) w = 0.$$

They are modified forms of the of Kummer's confluent hypergeometric functions were introduced by Edmund Taylor Whittaker by

$$M_{\kappa,\mu}(z) = \exp(-z/2)\, z^{\mu+\frac{1}{2}} M\left(\mu - \kappa + \frac{1}{2}, 1 + 2\mu; z\right),$$

$$W_{\kappa,\mu}(z) = \exp(-z/2)\, z^{\mu+\frac{1}{2}} U\left(\mu - \kappa + \frac{1}{2}, 1 + 2\mu; z\right), \tag{6}$$

where

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!} = {}_1F_1(a; b; z).$$

and

$$U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(a+1-b)} M(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a+1-b, 2-b, z).$$

are Kummer's functions.

The Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ are the same as those with opposite values of μ, in other words considered as a function of μ at fixed κ and z they are even functions. When κ and z are real, the functions give real values for real and imaginary values of μ.

2.2 Integral Transforms

In this subsection we give definitions of integral transforms which can be used in dealing with differential equations with fractional Bessel derivatives on semi-axes.

The Mellin transform of a function $f : \mathbb{R}_+ \to \mathbb{C}$ is the function f^* defined by

$$f^*(s) = \mathcal{M}f(s) = \int\limits_0^\infty x^{s-1} f(x)dx,$$

where $s = \sigma + i\tau \in \mathbb{C}$, provided that the integral exists.

Following to [21] as space of originals we choose the space P_a^b, $-\infty < a < b < \infty$ which is the linear space of $\mathbb{R}_+ \to \mathbb{C}$ functions such that $x^{s-1}f(x) \in L_1(\mathbb{R}_+)$ for every $s \in \{p \in \mathbb{C} : a \le \operatorname{Re} p \le b\}$.

If additionally $f^*(c + i\tau) \in L_1(\mathbb{R})$ with respect to τ then complex inversion formula holds:

$$\left\{\mathcal{M}^{-1}\varphi\right\}(x) = f(x) = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} x^{-s}\varphi(s)\, ds.$$

For functions $f \in L_1^\nu(\mathbb{R}_+)$ the Hankel transform of order $\frac{\nu-1}{2} > -\frac{1}{2}$ is

$$F_\nu[f](\xi) = \widehat{f}(\xi) = \int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\, f(x)x^\nu\, dx.$$

Let $f \in L_1^\nu(\mathbb{R}_+)$ and of bounded variation in a neighborhood of a point x of continuity of f. Then for $\nu > 0$ the inversion formula

$$F_\nu^{-1}[\widehat{f}](x) = f(x) = \frac{2^{1-\nu}}{\Gamma^2\left(\frac{\nu+1}{2}\right)} \int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\, \widehat{f}(\xi)\xi^\nu\, d\xi$$

holds.

For functions f the integral transforms involving Bessel function $K_{\frac{\nu-1}{2}}$, $\nu \ge 1$ as kernel is the Meijer transform defined by

$$\mathcal{K}_\nu[f](\xi) = F(\xi) = \int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\, f(x)x^\nu\, dx.$$

Let $f \in L_1^{loc}(\mathbb{R}_+)$ and $f(t) = o\left(t^{\beta-\frac{\nu}{2}}\right)$ as $t \to +0$ where $\beta > \frac{\nu}{2} - 2$ if $\nu > 1$ and $\beta > -1$ if $\nu = 1$. Furthermore let $f(t) = O(e^{at})$ as $t \to +\infty$. Then its Meijer exists a.e. for $\operatorname{Re}\xi > a$ (see [21, p. 94]).

If $0 < \nu < 2$ and $F(\xi)$ is analytic on the half-plane $H_a = \{p \in \mathbb{C} : \mathrm{p} \ge a$, $a \le 0$ and $s^{\frac{\nu}{2}-1}F(\xi) \to$, $|\xi| \to +\infty$, uniformly with respect to $\arg s$ then for any number c, $c > a$ the inverse transform $\mathcal{K}_\nu^{-1}$ is

$$\mathcal{K}_\nu^{-1}[\widehat{f}](x) = f(x) = \frac{1}{\pi i} \int\limits_{c-i\infty}^{c+i\infty} \widehat{f}(\xi) i_{\frac{\nu-1}{2}}(x\xi)\xi^\nu d\xi.$$

Generalized Whittaker transform is

$$(W^k_{\rho,\gamma} f)(x) = \int\limits_0^\infty (xt)^k e^{\frac{x^2t^2}{2}} W_{\rho,\gamma}(x^2t^2) f(t)dt$$

with $\rho, \gamma \in \mathbb{C}$ and $k \in \mathbb{R}$, containing the Whittaker function (6) in the kernel.

2.3 *Fractional Bessel Integrals and Derivatives on Semi-axes*

In this section we give definition of the fractional Bessel integrals on semi-axes following to [5, 6, 12, 13, 22, 23].

Definition 1 Let f is integrable by $(0, \infty)$ with the weight $\rho(x)$, $\alpha > 0$. The integrals

$$(B^{-\alpha}_{\nu,0+} f)(x)=(IB^\alpha_{\gamma,0+} f)(x) = \frac{1}{\Gamma(2\alpha)} \int\limits_0^x \left(\frac{y}{x}\right)^\nu \left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{y^2}{x^2}\right) f(y)dy \tag{7}$$

and

$$(B^{-\alpha}_{\nu,-} f)(x) = (IB^\alpha_{\gamma,-} f)(x) = \frac{1}{\Gamma(2\alpha)} \int\limits_x^\infty \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{x^2}{y^2}\right) f(y)dy \tag{8}$$

are called **left-sided fractional Bessel integral** and **right-sided fractional Bessel integral** on semi-axis $[0, \infty)$ of order α, accordingly. In the case of the integral (7) the weight $\rho(x) = x^{4\alpha+\nu}$ and in the case of the integral (7) the weight $\rho(x) = x^{4\alpha}$.

In Definition 1 function ${}_2F_1(a, b; c; z)$ is the hypergeometric function defined for $|z| < 1$ by the power series

$${}_2F_1(a, b; c; z) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}.$$

For complex argument z with $|z| \geq 1$ function ${}_2F_1(a,b;c;z)$ can be analytically continued along any path in the complex plane that avoids the branch points 1 and infinity.

Using formula 22 p. 64 from [24] of the form

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) $$

we can rewrite (7) as

$$ (B^{-\alpha}_{\nu,0+}f)(x) = \frac{1}{\Gamma(2\alpha)}\int\limits_0^x \left(\frac{x^2-y^2}{2y}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{x^2}{y^2}\right) f(y)dy. \tag{9} $$

In [12, 13, 15] was shown that formulas (7) and (8) can be simplified using formula 15.4.7 p. 561 from [19]

$$ {}_2F_1(a,b;2b;z) = $$

$$ = 2^{2b-1}\Gamma\left(b+\frac{1}{2}\right) z^{\frac{1}{2}-b}(1-z)^{\frac{1}{2}\left(b-a-\frac{1}{2}\right)} P^{\frac{1}{2}-b}_{a-b-\frac{1}{2}}\left[\left(1-\frac{z}{2}\right)\frac{1}{\sqrt{1-z}}\right]. $$

So, we can write for $\alpha > 0$

$$ (B^{-\alpha}_{\nu,0+}f)(x) = \frac{\sqrt{\pi}}{2^{2\alpha-1}\Gamma(\alpha)}\int\limits_0^x (x^2-y^2)^{\alpha-\frac{1}{2}}\left(\frac{y}{x}\right)^{\frac{\nu}{2}} P^{\frac{1}{2}-\alpha}_{\frac{\nu}{2}-1}\left[\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)\right] f(y)dy $$

and

$$ (B^{-\alpha}_{\nu,-}f)(x) = \frac{\sqrt{\pi}}{2^{2\alpha-1}\Gamma(\alpha)}\int\limits_x^\infty (y^2-x^2)^{\alpha-\frac{1}{2}}\left(\frac{y}{x}\right)^{\frac{\nu}{2}} P^{\frac{1}{2}-\alpha}_{\frac{\nu}{2}-1}\left[\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)\right] f(y)dy $$

where $f(x) \in L_1(0,\infty)$. Here the kernels of the fractional Bessel integrals on semi-axes are expressed using two-parameter Legendre functions instead of three-parameter Gauss hypergeometric functions.

Next we give some known facts proved in [22, 23].

2.3.1 Basic Properties of the Fractional Bessel Integrals on Semi-axes

1. For $\nu = 0$ we have

$$(B_{0,0+}^{-\alpha} f)(x) = \frac{1}{\Gamma(2\alpha)} \int_0^x (x-y)^{2\alpha-1} f(y)dy = (I_{0+}^{2\alpha} f)(x), \tag{10}$$

$$(B_{0,-}^{-\alpha} f)(x) = \frac{1}{\Gamma(2\alpha)} \int_x^\infty (y-x)^{2\alpha-1} f(y)dy = (I_{-}^{2\alpha} f)(x), \tag{11}$$

where $I_{0+}^{2\alpha}$ is the left-sided Riemann-Liouville fractional integrals (see formula 5.1 on p. 94 in [4]) and $I_{-}^{2\alpha}$ is the Liouville fractional integral (see formula 5.3 on p. 94 in [4]).

2. When α=1 if $\lim\limits_{x\to+0} g(x)$=0, $\lim\limits_{x\to+0} g'(x)$=0 the left-sided fractional Bessel integral on semi-axis is the left inverse to the differential Bessel operator

$$(B_{\nu,a+}^{-1} B_\nu g(x))(x) = g(x)$$

and when α=1 if $\lim\limits_{x\to+\infty} g(x)$=0, $\lim\limits_{x\to+\infty} g'(x)$=0 the right-sided fractional Bessel integral $B_{\nu,-}^{-1}$ is the left inverse to the differential Bessel operator

$$(B_{\nu,-}^{-1} B_\nu g(x))(x) = g(x).$$

3. The formula for integration by parts is valid on proper functions:

$$\int_0^\infty f(x)(B_{\nu,0+}^{-\alpha} g)(x) x^\nu dx = \int_0^\infty g(x)(B_{\nu,-}^{-\alpha} f)(x) x^\nu dx. \tag{12}$$

Definition 2 Let $\alpha > 0$. The **left-sided fractional Bessel derivative** and **right-sided fractional Bessel derivative** on semi-axis $[0, \infty)$ of order α are defined by the next equalities, accordingly

$$(B_{\gamma,0+}^{\alpha} f)(x) = (DB_{\gamma,0+}^{\alpha} f)(x) = B_\gamma^n (IB_{\gamma,0+}^{n-\alpha} f)(x), \qquad n = [\alpha] + 1 \tag{13}$$

and

$$(B_{\gamma,-}^{\alpha} f)(x) = (DB_{\gamma,-}^{\alpha} f)(x) = B_\gamma^n (IB_{\gamma,-}^{n-\alpha} f)(x), \qquad n = [\alpha] + 1. \tag{14}$$

In [5] spaces adapted to work with operators of the form $B^{\alpha}_{\gamma,0+}$ and $B^{\alpha}_{\gamma,-}$, $\alpha\in\mathbb{R}$ were introduced:

$$F_p=\left\{\varphi\in C^\infty(0,\infty):x^k\frac{d^k\varphi}{dx^k}\in L^p(0,\infty)\,\text{for}\,k=0,1,2,\dots\right\},\qquad 1\le p<\infty,$$

$$F_\infty=\left\{\varphi\in C^\infty(0,\infty):x^k\frac{d^k\varphi}{dx^k}\to 0\,\text{as}\,x\to 0+\,\text{ and as}\,x\to\infty\,\text{for}\,k=0,1,2,\dots\right\}$$

and

$$F_{p,\mu}=\left\{\varphi:x^{-\mu}\varphi(x)\in F_p\right\},\qquad 1\le p\le\infty,\qquad \mu\in\mathbb{C}.$$

We present here two theorems that are special cases of theorems from [5].

Theorem 1 *Let $\alpha\in\mathbb{R}$. For all p,μ and $\nu>0$ such that $\mu\neq\frac{1}{p}-2m$, $\gamma\neq\frac{1}{p}-\mu-2m+1$, $m=1,2\dots$ the operator $B^{\alpha}_{\gamma,0+}$ is a continuous linear mapping from F_p,μ into $F_{p,\mu-2\alpha}$. If also $2\alpha\neq\mu-\frac{1}{p}+2m$ and $\gamma-2\alpha\neq\frac{1}{p}-\mu-2m+1$, $m=1,2\dots$, then $B^{\alpha}_{\gamma,0+}$ a homeomorphism from F_p,μ onto $F_{p,\mu-2\alpha}$ with inverse $B^{-\alpha}_{\gamma,0+}$.*

Theorem 2 *Let $\alpha\in\mathbb{R}$. For all p,μ and $\gamma>0$ such that $\mu\neq\frac{1}{p}-2m+1$, $\gamma\neq\frac{1}{p}-\mu-2m$, $m=1,2\dots$ the operator $B^{\alpha}_{\gamma,-}$ is a continuous linear mapping from $F_{q,-\mu+2\alpha}$ into $F_{q,\mu}$, where $\frac{1}{q}=1-\frac{1}{p}$. If also $2\alpha\neq\mu-\frac{1}{p}+2m-1$ and $\gamma+2\alpha\neq\mu-\frac{1}{p}+2m$, $m=1,2\dots$, then $B^{\alpha}_{\gamma,-}$ a homeomorphism from $F_{q,-\mu+2\alpha}$ onto $F_{q,-\mu}$ with inverse $B^{-\alpha}_{\gamma,-}$.*

3 Factorisation

Following [6] and [5] we present next results.

Let $\mathrm{Re}\,(2\eta+\mu)+2>1/p$, and $\varphi\in F_{p,\mu}$. For $\mathrm{Re}\,\alpha>0$, we define $I^{\eta,\alpha}_2\varphi$ by formula

$$I^{\eta,\alpha}_2\varphi(x)=\frac{2}{\Gamma(\alpha)}\,x^{-2\eta-2\alpha}\int\limits_0^x(x^2-u^2)^{\alpha-1}u^{2\eta+1}\varphi(u)du. \tag{15}$$

Let $\mathrm{Re}\,(2\eta-\mu)>-1/p$, and $\varphi\in F_{p,\mu}$. For $\mathrm{Re}\,\alpha>0$, we define $K^{\eta,\alpha}_2\varphi$ by formula

$$K^{\eta,\alpha}_2\varphi(x)=\frac{2}{\Gamma(\alpha)}\,x^{2\eta}\int\limits_x^\infty(u^2-x^2)^{\alpha-1}u^{1-2(\eta+\alpha)}\varphi(u)du. \tag{16}$$

The definitions are extended to $\mathrm{Re}\,\alpha \leq 0$ by means of the formulas

$$I_2^{\eta,\alpha}\varphi = (\eta+\alpha+1)I_2^{\eta,\alpha+1}\varphi + \frac{1}{2}I_2^{\eta,\alpha+1}\,x\frac{d\varphi}{dx} \tag{17}$$

and

$$K_2^{\eta,\alpha}\varphi = (\eta+\alpha)K_2^{\eta,\alpha+1}\varphi - \frac{1}{2}K_2^{\eta,\alpha+1}\,x\frac{d\varphi}{dx}. \tag{18}$$

Theorem 3 *The next factorizations of* (7) *and* (8) *are valid*

$$(B_{\nu,0+}^{-\alpha}\varphi)(x)=\frac{1}{\Gamma(2\alpha)}\int\limits_0^x \left(\frac{u}{x}\right)^{\nu}\left(\frac{x^2-u^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)\varphi(u)du =$$

$$= \left(\frac{x}{2}\right)^{2\alpha}\ I_2^{\frac{\nu-1}{2},\alpha}I_2^{0,\alpha}\varphi, \tag{19}$$

$$(B_{\nu,-}^{-\alpha}f)(x) = \frac{1}{\Gamma(2\alpha)}\int\limits_x^{\infty}\left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)f(y)dy =$$

$$= 2^{-2\alpha}K_2^{\frac{1-\nu}{2},\alpha}K_2^{0,\alpha}x^{2\alpha}\varphi \tag{20}$$

where

$$I_2^{0,\alpha}\varphi(x) = \frac{2}{\Gamma(\alpha)}\,x^{-2\alpha}\int\limits_0^x (x^2-u^2)^{\alpha-1}u\varphi(u)du,$$

$$I_2^{\frac{\nu-1}{2},\alpha}\varphi(x) = \frac{2}{\Gamma(\alpha)}\,x^{1-\nu-2\alpha}\int\limits_0^x (x^2-u^2)^{\alpha-1}u^{\nu}\varphi(u)du,$$

$$K_2^{0,\alpha}\varphi(x) = \frac{2}{\Gamma(\alpha)}\int\limits_x^{\infty}(u^2-x^2)^{\alpha-1}u^{1-2\alpha}\varphi(u)du.$$

$$K_2^{\frac{1-\nu}{2},\alpha}\varphi(x) = \frac{2}{\Gamma(\alpha)}\,x^{1-\nu}\int\limits_x^{\infty}(u^2-x^2)^{\alpha-1}u^{\nu-2\alpha}\varphi(u)du.$$

Proof We have

$$(B_{\nu,0+}^{-\alpha}\varphi)(x)=\frac{1}{\Gamma(2\alpha)}\int\limits_0^x\left(\frac{u}{x}\right)^{\nu}\left(\frac{x^2-u^2}{2x}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)\varphi(u)du=$$

$$=2^{-2\alpha}x^{2\alpha}I_2^{\frac{\nu-1}{2},\alpha}I_2^{0,\alpha}\varphi=$$

$$=\frac{2^{1-2\alpha}x^{2\alpha}}{\Gamma(\alpha)}I_2^{\frac{\nu-1}{2},\alpha}y^{-2\alpha}\int\limits_0^y(y^2-u^2)^{\alpha-1}u\varphi(u)du=$$

$$=\frac{2^{2-2\alpha}x^{2\alpha}}{\Gamma^2(\alpha)}x^{-\nu+1-2\alpha}\int\limits_0^x(x^2-y^2)^{\alpha-1}y^{\nu-2\alpha}dy\int\limits_0^y(y^2-u^2)^{\alpha-1}u\varphi(u)du=$$

$$=\frac{2^{2-2\alpha}}{\Gamma^2(\alpha)}x^{1-\nu}\int\limits_0^x u\varphi(u)du\int\limits_u^x(y^2-u^2)^{\alpha-1}(x^2-y^2)^{\alpha-1}y^{\nu-2\alpha}dy.$$

Let find

$$\int\limits_u^x(y^2-u^2)^{\alpha-1}(x^2-y^2)^{\alpha-1}y^{\nu-2\alpha}dy=\{y^2=t\}=\frac{1}{2}\int\limits_{u^2}^{x^2}(t-u^2)^{\alpha-1}(x^2-t)^{\alpha-1}t^{\frac{\nu-1}{2}-\alpha}dt=$$

$$=\frac{\sqrt{\pi}\,\Gamma(\alpha)}{2^{2\alpha}\Gamma\left(\alpha+\frac{1}{2}\right)}\left(x^2-u^2\right)^{2\alpha-1}u^{-2\alpha+\nu-1}{}_2F_1\left(\alpha+\frac{1-\nu}{2},\alpha;2\alpha;1-\frac{x^2}{u^2}\right).$$

Using formula

$${}_2F_1(a,b;c;z)=(1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$$

we obtain

$${}_2F_1\left(\alpha+\frac{1-\nu}{2},\alpha;2\alpha;1-\frac{x^2}{u^2}\right)={}_2F_1\left(\alpha,\alpha+\frac{1-\nu}{2};2\alpha;1-\frac{x^2}{u^2}\right)=$$

$$=\left(\frac{x^2}{u^2}\right)^{-\alpha}{}_2F_1\left(\alpha,\alpha+\frac{\nu-1}{2};2\alpha;1-\frac{u^2}{x^2}\right)=\left(\frac{x^2}{u^2}\right)^{-\alpha}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)$$

and

$$\int_u^x (y^2-u^2)^{\alpha-1}(x^2-y^2)^{\alpha-1}y^{\nu-2\alpha}dy =$$

$$=\frac{\sqrt{\pi}\,\Gamma(\alpha)}{2^{2\alpha}\Gamma\left(\alpha+\frac{1}{2}\right)}\left(x^2-u^2\right)^{2\alpha-1}u^{-2\alpha+\nu-1}\left(\frac{x^2}{u^2}\right)^{-\alpha}{}_2F_1\left(\alpha,\alpha+\frac{\nu-1}{2};2\alpha;1-\frac{u^2}{x^2}\right)=$$

$$=\frac{\sqrt{\pi}\,\Gamma(\alpha)}{2^{2\alpha}\Gamma\left(\alpha+\frac{1}{2}\right)}\left(x^2-u^2\right)^{2\alpha-1}u^{\nu-1}x^{-2\alpha}{}_2F_1\left(\alpha,\alpha+\frac{\nu-1}{2};2\alpha;1-\frac{u^2}{x^2}\right).$$

Finally

$$(B_{\nu,0+}^{-\alpha}\varphi)(x)=\frac{2^{2(1-2\alpha)}\sqrt{\pi}}{\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)}x^{1-\nu-2\alpha}\int_0^x\left(x^2-u^2\right)^{2\alpha-1}u^{\nu}$$
$${}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)\varphi(u)du.$$

Applying the duplication formula

$$\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)=2^{1-2\alpha}\sqrt{\pi}\,\Gamma(2\alpha)$$

we obtain

$$(B_{\nu,0+}^{-\alpha}\varphi)(x)=\frac{2^{1-2\alpha}}{\Gamma(2\alpha)}x^{1-\nu-2\alpha}\int_0^x\left(x^2-u^2\right)^{2\alpha-1}u^{\nu}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)\varphi(u)du=$$

$$=\frac{1}{\Gamma(2\alpha)}\int_0^x\left(\frac{x^2-u^2}{2x}\right)^{2\alpha-1}\left(\frac{u}{x}\right)^{\nu}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)\varphi(u)du.$$

which gives (19).

Now we proof (20). We have

$$B_{\nu,-}^{-\alpha}\varphi=2^{-2\alpha}K_2^{\frac{1-\nu}{2},\alpha}K_2^{0,\alpha}x^{2\alpha}\varphi=$$

$$=\frac{2^{1-2\alpha}}{\Gamma(\alpha)}K_2^{\frac{1-\nu}{2},\alpha}\int_y^\infty(u^2-y^2)^{\alpha-1}u\varphi(u)du=$$

$$
=\frac{2^{2-2\alpha}}{\Gamma^2(\alpha)}x^{1-\nu}\int\limits_x^\infty(y^2-x^2)^{\alpha-1}y^{\nu-2\alpha}dy\int\limits_y^\infty(u^2-y^2)^{\alpha-1}u\varphi(u)du=
$$

$$
=\frac{2^{2-2\alpha}}{\Gamma^2(\alpha)}x^{1-\nu}\int\limits_x^\infty u\varphi(u)du\int\limits_x^u(u^2-y^2)^{\alpha-1}(y^2-x^2)^{\alpha-1}y^{\nu-2\alpha}dy.
$$

For inner integral we have

$$
\int\limits_x^u(y^2-x^2)^{\alpha-1}(u^2-y^2)^{\alpha-1}y^{\nu-2\alpha}dy=
$$

$$
=\frac{1}{2}\frac{2^{1-2\alpha}\sqrt{\pi}\Gamma(\alpha)}{\Gamma\left(\alpha+\frac{1}{2}\right)}\left(u^2-x^2\right)^{2\alpha-1}x^{\nu-1}u^{-2\alpha}{}_2F_1\left(\alpha,\alpha+\frac{\nu-1}{2};2\alpha;1-\frac{x^2}{u^2}\right)
$$

and

$$
B_{\nu,-}^{-\alpha}\varphi=\frac{2^{1-2\alpha}}{\Gamma^2(\alpha)}\frac{2^{1-2\alpha}\sqrt{\pi}\Gamma(\alpha)}{\Gamma\left(\alpha+\frac{1}{2}\right)}x^{1-\nu}\times
$$

$$
\times\int\limits_x^\infty\left(u^2-x^2\right)^{2\alpha-1}x^{\nu-1}u^{-2\alpha}{}_2F_1\left(\alpha,\alpha+\frac{\nu-1}{2};2\alpha;1-\frac{x^2}{u^2}\right)u\varphi(u)du=
$$

$$
=\frac{2^{1-2\alpha}}{\Gamma(2\alpha)}\int\limits_x^\infty\left(u^2-x^2\right)^{2\alpha-1}u^{1-2\alpha}{}_2F_1\left(\alpha,\alpha+\frac{\nu-1}{2};2\alpha;1-\frac{x^2}{u^2}\right)\varphi(u)du=
$$

$$
=\frac{1}{\Gamma(2\alpha)}\int\limits_x^\infty\left(\frac{u^2-x^2}{2u}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{x^2}{u^2}\right)\varphi(u)du.
$$

Which coincides with formula (20).

The proof is complete. □

4 Resolvent for Fractional Powers of the Bessel Differential Operator

We consider resolvents for integral operators at standard setting, cf. [25]. For any linear operator A on some Banach space Φ let us consider the equation

$$(A-\lambda I)\, g = f; \qquad \lambda \in C; \qquad f, g \in \Phi, \tag{21}$$

and its solution as resolvent operator due to the well-known formula from [25]

$$g = R_\lambda f = (A-\lambda I)^{-1} f = -(\lambda I - A)^{-1} f = -\frac{1}{\lambda}\left(I - \frac{1}{\lambda}A\right)^{-1} f$$

$$= -\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{1}{\lambda}A\right)^{k} f = -\frac{1}{\lambda} f - \frac{1}{\lambda}\left(\sum_{k=1}^{\infty}\frac{A^k}{\lambda^k} f\right). \tag{22}$$

Note that if integral representations are known for all powers A^k, then an integral representation for the resolvent is readily following from (21), of course if the series are convergent. In this way it is possible to get resolvent operators for the Riemann–Liouville fractional integrals, known as the Hille–Tamarkin formula [4] (in fact first proved by M.M. Dzhrbashyan in [26]), and also for the Erdélyi–Kober fractional integrals but we omit it here.

Theorem 4 *For a resolvent operator of* $(B_{\nu,-}^{-\alpha})$ *the next formula is valid*

$$R_\lambda f = -\frac{1}{\lambda} f - \frac{1}{\lambda^2}\int_x^{+\infty} f(y)\left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} dy \int_0^1 t^{\alpha-1}(1-t)^{\alpha-1}$$

$$\times\left(1-\left(1-\frac{x^2}{y^2}\right)t\right)^{-\alpha-\frac{\nu-1}{2}} E_{(\alpha,\alpha),(\alpha,\alpha)}\left(\frac{1}{\lambda}\left(\frac{1}{4}\frac{t(1-t)(y^2-x^2)^2}{y^2-(y^2-x^2)t}\right)^{\alpha}\right) dt,$$

with the Wright or generalized (multi-index) Mittag–Leffler function

$$E_{(1/\rho_i),(\mu_i)}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+k/\rho_1)\ldots\Gamma(\mu_m+k/\rho_m)}, \tag{23}$$

cf. [10, 27–32].

Proof Let us consider

$$(B_{\nu,-}^{-\alpha} f)(x) = \frac{1}{\Gamma(2\alpha)} \int\limits_x^{+\infty} \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{x^2}{y^2}\right) f(y)dy.$$

Using the group property or index law, we have

$$(B_{\nu,-}^{-\alpha} f)^k = B_{\nu,-}^{-\alpha k} f.$$

Then from (22) we obtain

$$R_\lambda f = -\frac{1}{\lambda} f - \frac{1}{\lambda}\left(\sum_{k=1}^{\infty} \frac{1}{\lambda^k} B_{\nu,-}^{-\alpha k} f\right) = -\frac{1}{\lambda} f - \frac{1}{\lambda}\Bigg(\sum_{k=1}^{\infty} \frac{1}{\lambda^k \Gamma(2\alpha k)}$$

$$\times \int\limits_x^{+\infty} \left(\frac{y^2-x^2}{2y}\right)^{2\alpha k-1} {}_2F_1\left(\alpha k+\frac{\nu-1}{2}, \alpha k; 2\alpha k; 1-\frac{x^2}{y^2}\right) f(y)dy\Bigg)$$

$$= -\frac{1}{\lambda} f - \frac{1}{\lambda}\left(\int\limits_x^{+\infty} f(y)dy \sum_{k=1}^{\infty}\left[\frac{1}{\lambda^k\Gamma(2\alpha k)}\left(\frac{y^2-x^2}{2y}\right)^{2\alpha k-1}\right.\right.$$

$$\left.\left.\times {}_2F_1\left(\alpha k+\frac{\nu-1}{2}, \alpha k; 2\alpha k; 1-\frac{x^2}{y^2}\right)\right]\right).$$

Using the integral representation for the hypergeometric function for $c-a-b>0$:

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int\limits_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt,$$

we obtain

$$R_\lambda f = -\frac{1}{\lambda} f - \frac{1}{\lambda} \int\limits_x^{+\infty} f(y)dy \int\limits_0^1 \sum_{k=1}^{\infty} \frac{1}{\lambda^k \Gamma^2(\alpha k)} \left(\frac{y^2-x^2}{2y}\right)^{2\alpha k-1} t^{\alpha k-1}(1-t)^{\alpha k-1}$$

$$\times \left(1-\left(1-\frac{x^2}{y^2}\right)t\right)^{-\alpha k-\frac{\nu-1}{2}} dt$$

$$= \{k = p+1\} = -\frac{1}{\lambda} f - \frac{1}{\lambda} \int\limits_{x}^{+\infty} f(y) dy \int\limits_{0}^{1} \sum_{p=0}^{\infty} \frac{1}{\lambda^{p+1} \Gamma^2(\alpha(p+1))} \left(\frac{y^2 - x^2}{2y} \right)^{2\alpha(p+1)-1}$$

$$\times t^{\alpha(p+1)-1} (1-t)^{\alpha(p+1)-1} \left(1 - \left(1 - \frac{x^2}{y^2} \right) t \right)^{-\alpha(p+1) - \frac{\nu-1}{2}} dt$$

$$= -\frac{1}{\lambda} f - \frac{1}{\lambda} \int\limits_{x}^{+\infty} f(y) \left(\frac{y^2 - x^2}{2y} \right)^{2\alpha-1} dy \int\limits_{0}^{1} t^{\alpha-1} (1-t)^{\alpha-1} \left(1 - \left(1 - \frac{x^2}{y^2} \right) t \right)^{-\alpha - \frac{\nu-1}{2}}$$

$$\times \sum_{p=0}^{\infty} \frac{1}{\lambda^{p+1} \Gamma^2(\alpha(p+1))} \left(\frac{y^2 - x^2}{2y} \right)^{2\alpha p} t^{\alpha p} (1-t)^{\alpha p} \left(1 - \left(1 - \frac{x^2}{y^2} \right) t \right)^{-\alpha p} dt$$

$$= -\frac{1}{\lambda} f - \frac{1}{\lambda^2} \int\limits_{x}^{+\infty} f(y) \left(\frac{y^2 - x^2}{2y} \right)^{2\alpha-1} dy \int\limits_{0}^{1} t^{\alpha-1} (1-t)^{\alpha-1} \left(1 - \left(1 - \frac{x^2}{y^2} \right) t \right)^{-\alpha - \frac{\nu-1}{2}}$$

$$\times \sum_{p=0}^{\infty} \frac{1}{\Gamma^2(\alpha + \alpha p)} \left[\frac{1}{\lambda} \left(\frac{1}{4} \frac{t(1-t)(y^2 - x^2)^2}{y^2 - (y^2 - x^2)t} \right)^{\alpha} \right]^p dt. \quad (24)$$

The function in (24) is a special case of the Wright generalized hypergeometric function defined above as (23). So it follows

$$\sum_{p=0}^{\infty} \frac{1}{\Gamma^2(\alpha + \alpha p)} \left[\frac{1}{\lambda} \left(\frac{1}{4} \frac{t(1-t)(y^2 - x^2)^2}{y^2 - (y^2 - x^2)t} \right)^{\alpha} \right]^p$$

$$= E_{(\alpha,\alpha),(\alpha,\alpha)} \left(\frac{1}{\lambda} \left(\frac{1}{4} \frac{t(1-t)(y^2 - x^2)^2}{y^2 - (y^2 - x^2)t} \right)^{\alpha} \right),$$

and we finally derive

$$R_\lambda f = -\frac{1}{\lambda} f - \frac{1}{\lambda^2} \int\limits_{x}^{+\infty} f(y) \left(\frac{y^2 - x^2}{2y} \right)^{2\alpha-1} dy \int\limits_{0}^{1} t^{\alpha-1} (1-t)^{\alpha-1}$$

$$\times \left(1 - \left(1 - \frac{x^2}{y^2} \right) t \right)^{-\alpha - \frac{\nu-1}{2}} E_{(\alpha,\alpha),(\alpha,\alpha)} \left(\frac{1}{\lambda} \left(\frac{1}{4} \frac{t(1-t)(y^2 - x^2)^2}{y^2 - (y^2 - x^2)t} \right)^{\alpha} \right) dt.$$

□

5 Integral Transforms

Integral transform maps the original space into or onto the image space. Wherein usually difficult operations in the original space are converted in general into simple operations in the image space. For example, the Fourier transform converts a derivative of order n into multiplication by the n power of the variable with some constant. This is the reason that the Fourier transform is beneficial to use for solution to differential equations. Since the Hankel transform applied to a Bessel operator of order n gives multiplication of a Hankel image of a function by the $2n$ power of the variable with some constant this transform is used instead of the Fourier transform when differential equation with the Bessel operator is solved. But the action of Hankel transform to the fractional Bessel derivatives of order α on semi-axes gives multiplication of the 2α power of the variable by not a Hankel image of a function with some constant (see Theorem 7). In this section we collect some integral transforms which can be used to solve differential equations the fractional Bessel derivatives on semi-axes.

5.1 The Mellin Transform

Using the following formula 2.21.1.11 from [33, p. 265] of the form

$$\int\limits_0^z x^{\alpha-1}(z-x)^{c-1}\,{}_2F_1\left(a,b;c;1-\frac{x}{z}\right)dx=z^{c+\alpha-1}\Gamma\left[\begin{matrix}c, & \alpha, & c-a-b+\alpha\\ c-a+\alpha, & & c-b+\alpha\end{matrix}\right], \tag{25}$$

$$z>0,\ \mathrm{Re}\,c>0,\ \mathrm{Re}\,(c-a-b+\alpha)>0,$$

we prove next theorems.

Theorem 5 *Let $\alpha>0$. Mellin transforms of the $IB^{\alpha}_{\nu,-}$ and the $IB^{\alpha}_{\nu,0+}$ are*

$$\mathcal{M}IB^{\alpha}_{\nu,-}f(s)=\frac{1}{2^{2\alpha}}\,\Gamma\left[\begin{matrix}\frac{s}{2}, & \frac{s}{2}-\frac{\nu-1}{2}\\ \alpha+\frac{s}{2}-\frac{\nu-1}{2}, & \alpha+\frac{s}{2}\end{matrix}\right]f^*(2\alpha+s),\qquad s>\nu-1,\qquad IB^{\alpha}_{\nu,-}f\in P^b_a, \tag{26}$$

$$\mathcal{M}IB^{\alpha}_{\nu,0+}f(s)=\frac{1}{2^{2\alpha}}\Gamma\left[\begin{matrix}\frac{\nu-s+1}{2}-\alpha, & 1-\frac{s}{2}-\alpha\\ 1-\frac{s}{2}, & \frac{\nu-s+1}{2}\end{matrix}\right]f^*(2\alpha+s),\qquad 2\alpha+s<2,\qquad IB^{\alpha}_{\nu,0+}f\in P^b_a. \tag{27}$$

Proof Let start from the definitions

$$((IB^{\alpha}_{\nu,-}f)(x))^*(s) = \int\limits_0^{\infty} x^{s-1}(IB^{\alpha}_{\nu,-}f)(x)dx =$$

$$= \frac{1}{\Gamma(2\alpha)}\int\limits_0^{\infty} x^{s-1}dx \int\limits_x^{+\infty}\left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)f(y)dy$$

$$= \frac{1}{\Gamma(2\alpha)}\int\limits_0^{\infty} f(y)(2y)^{1-2\alpha}dy\int\limits_0^{y}(y^2-x^2)^{2\alpha-1}\,{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)x^{s-1}dx.$$

Using (25) let us find inner integral for $s>\nu-1$

$$\int\limits_0^{y}(y^2-x^2)^{2\alpha-1}\,{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)x^{s-1}dx$$

$$=\frac{y^{4\alpha+s-2}}{2}\,\Gamma\left[\begin{array}{ccc}2\alpha, & \frac{s}{2}, & \frac{s}{2}-\frac{\nu-1}{2}\\ \alpha+\frac{s}{2}-\frac{\nu-1}{2}, & & \alpha+\frac{s}{2}\end{array}\right].$$

We obtain

$$((IB^{\alpha}_{\nu,-}f)(x))^*(s) = \frac{1}{2^{2\alpha}}\,\Gamma\left[\begin{array}{cc}\frac{s}{2}, & \frac{s}{2}-\frac{\nu-1}{2}\\ \alpha+\frac{s}{2}-\frac{\nu-1}{2}, & \alpha+\frac{s}{2}\end{array}\right]\int\limits_0^{\infty} f(y)y^{2\alpha+s-1}dy =$$

$$= \frac{1}{2^{2\alpha}}\,\Gamma\left[\begin{array}{cc}\frac{s}{2}, & \frac{s}{2}-\frac{\nu-1}{2}\\ \alpha+\frac{s}{2}-\frac{\nu-1}{2}, & \alpha+\frac{s}{2}\end{array}\right]f^*(2\alpha+s).$$

Similarly we have

$$((IB^{\alpha}_{\nu,0+}f)(x))^*(s) = \int\limits_0^{\infty} x^{s-1}(B^{-\alpha}_{\nu,0+}f)(x)dx =$$

$$= \frac{1}{\Gamma(2\alpha)}\int\limits_0^{\infty} x^{s-1}dx\int\limits_0^{x}\left(\frac{y}{x}\right)^{\nu}\left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1}\,{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)f(y)dy =$$

$$= \frac{1}{\Gamma(2\alpha)}\int\limits_0^{\infty} f(y)y^{\nu}dy\int\limits_y^{\infty}\left(\frac{1}{x}\right)^{\nu}\left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1}\,{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)x^{s-1}dx.$$

Let find inner integral

$$\int\limits_{y}^{\infty}\left(\frac{1}{x}\right)^{\nu}\left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)x^{s-1}dx=$$

$$=2^{1-2\alpha}\int\limits_{y}^{\infty}\left(\frac{1}{x}\right)^{2\alpha-s+\nu}\left(x^2-y^2\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)dx=\left\{\frac{1}{x}=t\right\}=$$

$$=2^{1-2\alpha}\int\limits_{0}^{1/y}t^{\nu-2\alpha-s}\left(1-t^2y^2\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-t^2y^2\right)dt=\{ty=z\}=$$

$$=2^{1-2\alpha}y^{2\alpha+s-\nu-1}\int\limits_{0}^{1}z^{\nu-2\alpha-s}\left(1-z^2\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-z^2\right)dz=\{z^2=s\}=$$

$$=\frac{1}{2^{2\alpha}}y^{2\alpha+s-\nu-1}\int\limits_{0}^{1}s^{\frac{\nu-s-1}{2}-\alpha}(1-s)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-s\right)ds.$$

Using (25) we get for $2\alpha+s<2$

$$\int\limits_{0}^{1}s^{\frac{\nu-s-1}{2}-\alpha}(1-s)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-s\right)ds$$
$$=\frac{1}{2^{2\alpha}}y^{2\alpha+s-\nu-1}\Gamma\left[\begin{array}{cc}2\alpha, & \frac{\nu-s+1}{2}-\alpha,\ 1-\frac{s}{2}-\alpha\\ 1-\frac{s}{2}, & \frac{\nu-s+1}{2}\end{array}\right]$$

and

$$((B_{\nu,0+}^{-\alpha}f)(x))^*(s)=\frac{1}{2^{2\alpha}}\Gamma\left[\begin{array}{cc}\frac{\nu-s+1}{2}-\alpha, & 1-\frac{s}{2}-\alpha\\ 1-\frac{s}{2}, & \frac{\nu-s+1}{2}\end{array}\right]\int\limits_{0}^{\infty}f(y)y^{2\alpha+s-1}dy=$$

$$=\frac{1}{2^{2\alpha}}\Gamma\left[\begin{array}{cc}\frac{\nu-s+1}{2}-\alpha, & 1-\frac{s}{2}-\alpha\\ 1-\frac{s}{2}, & \frac{\nu-s+1}{2}\end{array}\right]f^*(2\alpha+s).$$

This complete the proof. □

In order to obtain formulas for Mellin transform of fractional Bessel derivatives on semi-axes we should proof next statement.

Lemma 1 *Let $B_\nu^n f \in P_a^b$ then for $n \in \mathbb{N}$*

$$\mathcal{M}B_\nu^n f(s) = 2^{2n}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\nu}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\nu}{2} \end{array}\right] f^*(s-2n). \tag{28}$$

Proof Using formulas for Mellin transform from [34] we get

$$\mathcal{M}f'(s) = (1-s)\mathcal{M}f(s-1), \qquad \mathcal{M}\frac{1}{x}f(s) = \mathcal{M}f(s-1),$$

$$\mathcal{M}\frac{1}{x}f'(s) = (\mathcal{M}f'(t-1))(s) = (2-s)\mathcal{M}f(s-2),$$

$$\mathcal{M}f''(s) = (2-s)(1-s)\mathcal{M}f(s-2),$$

$$\mathcal{M}B_\nu f(s) = (2-s)(1-s)f^*(s-2)+\nu(2-s)f^*(s-2) = (2-s)(1-s+\nu)f^*(s-2).$$

So

$$\mathcal{M}B_\nu f(s) = (2-s)(1-s+\nu)f^*(s-2). \tag{29}$$

Applying the formula (29) n times we obtain

$$\mathcal{M}B_\nu^n f(s) = (2-s)(4-s)\ldots(2n-s)(1-s+\nu)(3-s+\nu)\ldots(2n-1-s+\nu)f^*(s-2n).$$

Since

$$(2-s)(4-s)\ldots(2n-s) = 2^n\left(1-\frac{s}{2}\right)\left(2-\frac{s}{2}\right)\ldots\left(n-\frac{s}{2}\right) = 2^n\left(1-\frac{s}{2}\right)_n = \frac{2^n\Gamma\left(n+1-\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}$$

and

$$(1-s+\nu)(3-s+\nu)\ldots(2n-1-s+\nu)$$
$$= 2^n\left(\frac{1-s+\nu}{2}\right)\left(\frac{1-s+\nu}{2}+1\right)\ldots\left(\frac{1-s+\nu}{2}+n-1\right) =$$
$$= 2^n\left(\frac{1-s+\nu}{2}\right)_n = \frac{2^n\Gamma\left(\frac{1-s+\nu}{2}+n\right)}{\Gamma\left(\frac{1-s+\nu}{2}\right)},$$

then

$$\mathcal{M}B_\nu^n f(s) = 2^{2n}\frac{\Gamma\left(n+1-\frac{s}{2}\right)\Gamma\left(\frac{1-s+\nu}{2}+n\right)}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{1-s+\nu}{2}\right)}f^*(s-2n) =$$

$$= 2^{2n}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\nu}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\nu}{2}\end{array}\right] f^*(s-2n).$$

It completes the proof □

Theorem 6 *Let* $\alpha > 0$, $n = [\alpha]+1$. *Mellin transforms of the* $DB_{\nu,-}^\alpha$ *and the* $DB_{\nu,0+}^\alpha$ *are*

$$\mathcal{M}DB_{\nu,-}^\alpha f(s) = 2^{2\alpha}\,\Gamma\left[\begin{array}{cc} \frac{s}{2}, & \frac{s}{2}-\frac{\nu-1}{2} \\ \frac{s}{2}-\alpha-\frac{\nu-1}{2}, & \frac{s}{2}-\alpha\end{array}\right] f^*(s-2\alpha),\quad s-2n>\nu-1,\quad IB_{\nu,-}^{n-\alpha}f\in P_a^b, \tag{30}$$

$$\mathcal{M}DB_{\nu,0+}^\alpha f(s) = 2^{2\alpha}\Gamma\left[\begin{array}{cc} 1-\frac{s}{2}+\alpha, & \frac{\nu-s+1}{2}+\alpha \\ 1-\frac{s}{2}, & \frac{\nu-s+1}{2}\end{array}\right] f^*(s-2\alpha),\ 2\alpha-2n+s<2,\ IB_{\nu,0+}^{n-\alpha}f\in P_a^b. \tag{31}$$

Proof Applying (26) and (28) we obtain

$$((DB_{\nu,-}^\alpha f)(x))^*(s) = ((B_\nu^n(IB_{\nu,-}^{n-\alpha}f(x))^*(s) =$$

$$= 2^{2n}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\nu}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\nu}{2}\end{array}\right]((IB_{\nu,-}^{n-\alpha}f(x))^*(s-2n) =$$

$$= 2^{2\alpha}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\nu}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\nu}{2}\end{array}\right]\Gamma\left[\begin{array}{cc} \frac{s}{2}-n, & \frac{s}{2}-n-\frac{\nu-1}{2} \\ \frac{s}{2}-\alpha-\frac{\nu-1}{2}, & \frac{s}{2}-\alpha\end{array}\right] f^*(s-2\alpha). \tag{32}$$

Using the formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},\qquad z\notin\mathbb{Z}$$

in the numerator we get

$$\Gamma\left(1+n-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-n\right) = \frac{\pi}{\sin(\frac{s}{2}-n)\pi} = \frac{(-1)^n\pi}{\sin(\frac{s}{2})\pi},$$

$$\Gamma\left(\frac{1-s+\nu}{2}+n\right)\Gamma\left(\frac{s-\nu+1}{2}-n\right) = \Gamma\left(1-\frac{1-s+\nu}{2}-n\right)\Gamma\left(\frac{1-s+\nu}{2}+n\right) =$$

$$= \frac{\pi}{\sin(\frac{1-s+\nu}{2}+n)\pi} = \frac{(-1)^n\pi}{\sin(\frac{1-s+\nu}{2})\pi}.$$

So

$$\frac{(-1)^n\pi}{\Gamma\left(\frac{1-s+\nu}{2}\right)\sin(\frac{1-s+\nu}{2})\pi}=(-1)^n\Gamma\left(\frac{1+s-\nu}{2}\right),$$

$$\frac{(-1)^n\pi}{\Gamma\left(1-\frac{s}{2}\right)\sin(\frac{s}{2})\pi}=(-1)^n\Gamma\left(\frac{s}{2}\right).$$

Substituting the found expressions in (32) we obtain (30).

Similarly, using (27) and (28) we

$$((DB^{\alpha}_{\nu,0+}f)(x))^*(s)=((B^n_\nu(IB^{n-\alpha}_{\nu,0+}f(x))^*(s)=$$

$$=2^{2n}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\nu}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\nu}{2}\end{array}\right]((IB^{n-\alpha}_{\nu,0+}f(x))^*(s-2n)=$$

$$=2^{2\alpha}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\nu}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\nu}{2}\end{array}\right]\Gamma\left[\begin{array}{cc} 1-\frac{s}{2}+\alpha, & \frac{\nu-s+1}{2}+\alpha \\ 1-\frac{s}{2}+n, & \frac{\nu-s+1}{2}+n\end{array}\right]f^*(s-2\alpha)=$$

$$=2^{2\alpha}\Gamma\left[\begin{array}{cc} 1-\frac{s}{2}+\alpha, & \frac{\nu-s}{2}+\alpha \\ 1-\frac{s}{2}, & \frac{\nu-s+1}{2}\end{array}\right]f^*(s-2\alpha).$$

□

5.2 *The Hankel Transform*

Theorem 7 *Let $B^{-\alpha}_{\nu,0+}\varphi$, $B^{-\alpha}_{\nu,-}\varphi\in L^\nu_1(\mathbb{R}_+)$, then*

$$F_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi)=\xi^{-2\alpha}\int\limits_0^\infty\varphi(t)\left[\cos(\alpha\pi)j_{\frac{\nu-1}{2}}(\xi t)-\sin(\alpha\pi)y_{\frac{\nu-1}{2}}(\xi t)\right]t^\nu dt,$$

$$4\alpha-2<\nu<4-2\alpha, \tag{33}$$

$$F_\nu[(B^{-\alpha}_{\nu,-}\varphi)](\xi)=\xi^{-2\alpha}\int\limits_0^\infty j^1_{\frac{\nu-1}{2},\alpha}(t\xi)\varphi(t)t^\nu dt, \tag{34}$$

where

$$j^1_{\frac{\nu-1}{2},\alpha}(t\xi)=\frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{(t\xi)^{\frac{\nu-1}{2}}}J^1_{\frac{\nu-1}{2},\alpha}(t\xi),$$

$$J^1_{\frac{\nu-1}{2},\alpha}(t\xi)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\alpha+n+1)\Gamma\left(\frac{\nu+1}{2}+\alpha+n\right)}\left(\frac{t\xi}{2}\right)^{2n+\frac{\nu-1}{2}+2\alpha}.$$

Proof Using factorization formula (19) and denoting $g(x)=I_2^{0,\alpha}\varphi(x)$ we obtain

$$F_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi)=\int_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\,(B^{-\alpha}_{\nu,0+}\varphi)(x)x^\nu\,dx=$$

$$=\frac{1}{2^{2\alpha}}\int_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\;I_2^{\frac{\nu-1}{2},\alpha}I_2^{0,\alpha}\varphi(x)x^{2\alpha+\nu}\,dx=\frac{1}{2^{2\alpha}}\int_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\;I_2^{\frac{\nu-1}{2},\alpha}g(x)x^{2\alpha+\nu}\,dx=$$

$$=\frac{1}{2^{2\alpha-1}\Gamma(\alpha)}\int_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\,x\,dx\int_0^x(x^2-u^2)^{\alpha-1}u^\nu g(u)du=$$

$$=\frac{1}{2^{2\alpha-1}\Gamma(\alpha)}\int_0^\infty u^\nu g(u)du\int_u^\infty(x^2-u^2)^{\alpha-1}j_{\frac{\nu-1}{2}}(x\xi)\,x\,dx.$$

Let consider inner integral

$$\int_u^\infty(x^2-u^2)^{\alpha-1}j_{\frac{\nu-1}{2}}(x\xi)\,x\,dx=\frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}}}\int_u^\infty(x^2-u^2)^{\alpha-1}J_{\frac{\nu-1}{2}}(x\xi)\,x^{1-\frac{\nu-1}{2}}\,dx.$$

Using the formula 2.12.4.17 from [20] of the form

$$\int_a^\infty x^{1-\rho}(x^2-a^2)^{\beta-1}J_\rho(cx)dx=2^{\beta-1}a^{\beta-\rho}c^{-\beta}\Gamma(\beta)J_{\rho-\beta}(ac),$$

$$a,c,\beta>0;\qquad(2\beta-\rho)<3/2$$

we obtain for $4\alpha-\nu<2$

$$\int\limits_u^\infty (x^2-u^2)^{\alpha-1} J_{\frac{\nu-1}{2}}(x\xi)\, x^{1-\frac{\nu-1}{2}}\, dx = 2^{\alpha-1} u^{\alpha-\frac{\nu-1}{2}} \xi^{-\alpha} \Gamma(\alpha) J_{\frac{\nu-1}{2}-\alpha}(u\xi)$$

and

$$F_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi) = \frac{2^{\frac{\nu-1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}+\alpha}} \int\limits_0^\infty u^{\alpha+\frac{\nu+1}{2}} J_{\frac{\nu-1}{2}-\alpha}(u\xi) g(u) du =$$

$$= \frac{2^{\frac{\nu+1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha)\xi^{\frac{\nu-1}{2}+\alpha}} \int\limits_0^\infty u^{\frac{\nu+1}{2}-\alpha} J_{\frac{\nu-1}{2}-\alpha}(u\xi) du \int\limits_0^u (u^2-y^2)^{\alpha-1} y\varphi(y) dy =$$

$$= \frac{2^{\frac{\nu+1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha)\xi^{\frac{\nu-1}{2}+\alpha}} \int\limits_0^\infty y\varphi(y) dy \int\limits_y^\infty (u^2-y^2)^{\alpha-1} u^{\frac{\nu+1}{2}-\alpha} J_{\frac{\nu-1}{2}-\alpha}(u\xi) du.$$

Let calculate inner integral using the formula 2.12.4.17 from [20] of the form

$$\int\limits_a^\infty x^{1+\rho}(x^2-a^2)^{\beta-1} J_\rho(cx) dx = 2^{\beta-1} a^{\beta+\rho} c^{-\beta} \Gamma(\beta)[\cos(\beta\pi) J_{\rho+\beta}(ac) - \sin(\beta\pi) Y_{\rho+\beta}(ac)],$$

$$a, c, \beta > 0; \qquad (2\beta+\rho) < 3/2$$

we obtain

$$\int\limits_y^\infty (u^2-y^2)^{\alpha-1} u^{\frac{\nu+1}{2}-\alpha} J_{\frac{\nu-1}{2}-\alpha}(u\xi) du = 2^{\alpha-1} y^{\frac{\nu-1}{2}} \xi^{-\alpha} \Gamma(\alpha)[\cos(\alpha\pi) J_{\frac{\nu-1}{2}}(\xi y) - \sin(\alpha\pi) Y_{\frac{\nu-1}{2}}(\xi y)]$$

for $2\alpha+\nu<4$ and

$$F_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi) =$$

$$= \frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}+2\alpha}} \int\limits_0^\infty y^{\frac{\nu+1}{2}} \varphi(y)[\cos(\alpha\pi) J_{\frac{\nu-1}{2}}(\xi y) - \sin(\alpha\pi) Y_{\frac{\nu-1}{2}}(\xi y)] dy =$$

$$= \xi^{-2\alpha} \int\limits_0^\infty \varphi(t) \left[\cos(\alpha\pi) j_{\frac{\nu-1}{2}}(\xi t) - \sin(\alpha\pi) y_{\frac{\nu-1}{2}}(\xi t)\right] t^\nu dt.$$

So (33) is proved.

Now let consider (34). Let $g(x)=K_2^{0,\alpha}x^{2\alpha}\varphi(x)$. Using factorization (20) we get

$$F_\nu[(B_{\nu,-}^{-\alpha}\varphi)](\xi)=2^{-2\alpha}\int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\,x^\nu\,K_2^{\frac{1-\nu}{2},\alpha}K_2^{0,\alpha}x^{2\alpha}\varphi(x)dx=$$

$$=2^{-2\alpha}\int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\,x^\nu\,K_2^{\frac{1-\nu}{2},\alpha}g(x)dx=\frac{2^{1-2\alpha}}{\Gamma(\alpha)}\int\limits_0^\infty j_{\frac{\nu-1}{2}}(x\xi)\,xdx\int\limits_x^\infty(u^2-x^2)^{\alpha-1}u^{\nu-2\alpha}g(u)du=$$

$$=\frac{2^{1-2\alpha}}{\Gamma(\alpha)}\int\limits_0^\infty g(u)u^{\nu-2\alpha}du\int\limits_0^u j_{\frac{\nu-1}{2}}(x\xi)(u^2-x^2)^{\alpha-1}xdx.$$

Using the formula 2.12.4.7 from [20] of the form

$$\int\limits_0^a x^{1-\rho}(a^2-x^2)^{\beta-1}J_\rho(cx)dx=\frac{2^{1-\rho}a^{\beta-\rho}}{c^\beta\Gamma(\rho)}s_{\rho+\beta-1,\beta-\rho}(ac),$$

$$a>0;\qquad \mathrm{Re}\,\beta>0$$

we obtain for inner integral

$$\int\limits_0^u(u^2-x^2)^{\alpha-1}j_{\frac{\nu-1}{2}}(x\xi)\,x\,dx=\frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}}}\int\limits_0^u(u^2-x^2)^{\alpha-1}J_{\frac{\nu-1}{2}}(x\xi)\,x^{1-\frac{\nu-1}{2}}\,dx=$$

$$=\frac{\Gamma(\alpha)}{2\Gamma(\alpha+1)}u^{2\alpha}\,{}_1F_2\left(1;\alpha+1,\frac{\nu+1}{2};-\frac{u^2\xi^2}{4}\right).$$

So

$$F_\nu[(B_{\nu,-}^{-\alpha}\varphi)](\xi)=\frac{1}{2^{2\alpha}\Gamma(\alpha+1)}\int\limits_0^\infty{}_1F_2\left(1;\alpha+1,\frac{\nu+1}{2};-\frac{u^2\xi^2}{4}\right)g(u)u^\nu du=$$

$$=\frac{1}{2^{2\alpha}\Gamma(\alpha+1)}\int\limits_0^\infty{}_1F_2\left(1;\alpha+1,\frac{\nu+1}{2};-\frac{u^2\xi^2}{4}\right)u^\nu K_2^{0,\alpha}u^{2\alpha}\varphi(u)du=$$

$$= \frac{1}{2^{2\alpha-1}\Gamma(\alpha)\Gamma(\alpha+1)}\int\limits_0^\infty {}_1F_2\left(1;\alpha+1,\frac{\nu+1}{2};-\frac{u^2\xi^2}{4}\right)u^\nu du\int\limits_u^\infty (t^2-u^2)^{\alpha-1}t\varphi(t)dt =$$

$$= \frac{1}{2^{2\alpha-1}\Gamma(\alpha)\Gamma(\alpha+1)}\int\limits_0^\infty t\varphi(t)dt\int\limits_0^t (t^2-u^2)^{\alpha-1}\,{}_1F_2\left(1;\alpha+1,\frac{\nu+1}{2};-\frac{u^2\xi^2}{4}\right)u^\nu du.$$

Using Wolfram Mathematica we obtain

$$\int\limits_0^t (t^2-u^2)^{\alpha-1}\,{}_1F_2\left(1;\alpha+1,\frac{\nu+1}{2};-\frac{u^2\xi^2}{4}\right)u^\nu du =$$

$$= \frac{\Gamma(\alpha)\Gamma\left(\frac{\nu+1}{2}\right)}{2\Gamma\left(\alpha+\frac{\nu+1}{2}\right)}t^{2\alpha+\nu-1}\,{}_1F_2\left(1;\alpha+1,\alpha+\frac{\nu+1}{2};-\frac{t^2\xi^2}{4}\right)$$

and

$$F_\nu[(B_{\nu,-}^{-\alpha}\varphi)](\xi) =$$

$$= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{2^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{\nu+1}{2}\right)}\int\limits_0^\infty \varphi(t)\,t^{2\alpha+\nu}\,{}_1F_2\left(1;\alpha+1,\alpha+\frac{\nu+1}{2};-\frac{t^2\xi^2}{4}\right)dt.$$

Since

$${}_1F_2\left(1;\alpha+1,\alpha+\frac{\nu+1}{2};-\frac{t^2\xi^2}{4}\right) = \Gamma(\alpha+1)\Gamma\left(\alpha+\frac{\nu+1}{2}\right)$$
$$\sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(\alpha+n+1)\Gamma\left(\alpha+\frac{\nu+1}{2}+n\right)}\left(\frac{t\xi}{2}\right)^{2n}$$

and the Wright function through which the Hankel transform of $B_{\nu,-}^{-\alpha}\varphi$ is expressed in [22] is given by

$$J^1_{\frac{\nu-1}{2},\alpha}(t\xi) = \sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(\alpha+n+1)\Gamma\left(\frac{\nu+1}{2}+\alpha+n\right)}\left(\frac{t\xi}{2}\right)^{2n+\frac{\nu-1}{2}+2\alpha}$$

we obtain

$$F_\nu[(B^{-\alpha}_{\nu,-}\varphi)](\xi)=\frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}}}\xi^{-2\alpha}\int\limits_0^\infty\varphi(t)\,t^{\frac{\nu+1}{2}}\,J^1_{\frac{\nu-1}{2},\alpha}(t\xi)dt=\xi^{-2\alpha}\int\limits_0^\infty j^1_{\frac{\nu-1}{2},\alpha}(t\xi)\varphi(t)t^\nu dt.$$

The (34) is proved.

□

Since $F_\nu[(B^n_\nu\varphi)](\xi){=}(-1)^n\xi^{2n}F_\nu[\varphi](\xi)$ we obtain for $B^\alpha_{\nu,0+}\varphi$, $B^\alpha_{\nu,-}\varphi{\in}L^\nu_1(\mathbb{R}_+)$

$$F_\nu[(B^\alpha_{\nu,0+}\varphi)(x)](\xi){=}F_\nu[(B^n_\nu B^{-(n-\alpha)}_{\nu,0+}\varphi)(x)](\xi){=}(-1)^n\xi^{2n}F_\nu[B^{-(n-\alpha)}_{\nu,0+}\varphi(x)](\xi){=}$$

$$=(-1)^n\xi^{2\alpha}\int\limits_0^\infty\varphi(t)\left[\cos((n-\alpha)\pi)\,j_{\frac{\nu-1}{2}}(\xi t)-\sin((n-\alpha)\pi)\,y_{\frac{\nu-1}{2}}(\xi t)\right]t^\nu dt,$$

$$n=[\alpha]+1,\qquad 4(n-\alpha)-2<\nu<4-2(n-\alpha)$$

and

$$F_\nu[(B^\alpha_{\nu,-}\varphi)(x)](\xi)=F_\nu[(B^n_\nu B^{-(n-\alpha)}_{\nu,-}\varphi)(x)](\xi){=}(-1)^n\xi^{2n}F_\nu[B^{-(n-\alpha)}_{\nu,-}\varphi(x)](\xi)=$$

$$=(-1)^n\xi^{2\alpha}\int\limits_0^\infty j^1_{\frac{\nu-1}{2},n-\alpha}(t\xi)\varphi(t)t^\nu dt,\qquad n=[\alpha]+1.$$

5.3 The Meijer Transform

Theorem 8 *The Meijer transforms of $B^{-\alpha}_{\nu,0+}$, $B^{-\alpha}_{\nu,-}$ for proper functions are*

$$\mathcal{K}_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi)=\xi^{-2\alpha}\mathcal{K}_\nu\varphi(\xi),\tag{35}$$

$$\mathcal{K}_\nu[(B^{-\alpha}_{\nu,-}\varphi)(x)](\xi)=\tag{36}$$

$$=\frac{\Gamma\left(\frac{1-\nu}{2}\right)\Gamma^2\left(\frac{\nu+1}{2}\right)}{2^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{\nu+1}{2}\right)}\int\limits_0^\infty\varphi(t)t^{2\alpha+\nu}\,{}_1F_2\left(1;\alpha+1,\alpha+\frac{\nu+1}{2};\frac{t^2\xi^2}{4}\right)dt-$$

$$-\frac{\pi 2^{\nu-2\alpha-2}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{3-\nu}{2}\right)\cos\left(\frac{\pi\nu}{2}\right)}\xi^{1-\nu}\int\limits_0^\infty\varphi(t)t^{2\alpha+1}\,{}_1F_2\left(1;\alpha+1,\alpha+\frac{3-\nu}{2};\frac{t^2\xi^2}{4}\right)dt.$$

Proof We start with (35). Let $g(x) = I_2^{0,\alpha}\varphi(x)$. Then using the factorization (19) we obtain

$$\mathcal{K}_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi) = \int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\,(B^{-\alpha}_{\nu,0+}\varphi)(x)x^\nu\,dx =$$

$$= \frac{1}{2^{2\alpha}}\int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\;I_2^{\frac{\nu-1}{2},\alpha} I_2^{0,\alpha}\varphi(x)x^{2\alpha+\nu}\,dx = \frac{1}{2^{2\alpha}}\int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\;I_2^{\frac{\nu-1}{2},\alpha} g(x)x^{2\alpha+\nu}\,dx =$$

$$= \frac{1}{2^{2\alpha-1}\Gamma(\alpha)}\int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\,x\,dx\int\limits_0^x (x^2-u^2)^{\alpha-1}u^\nu g(u)du =$$

$$= \frac{1}{2^{2\alpha-1}\Gamma(\alpha)}\int\limits_0^\infty u^\nu g(u)du\int\limits_u^\infty (x^2-u^2)^{\alpha-1}k_{\frac{\nu-1}{2}}(x\xi)\,x\,dx.$$

Let consider the inner integral. Using the formula 2.16.3.7 from [20] of the form

$$\int\limits_a^\infty x^{1\pm\rho}(x^2-a^2)^{\beta-1}K_\rho(cx)dx = 2^{\beta-1}a^{\beta\pm\rho}c^{-\beta}\Gamma(\beta)K_{\rho\pm\beta}(ac), \qquad a,c,\beta>0 \tag{37}$$

we get

$$\int\limits_u^\infty (x^2-u^2)^{\alpha-1}k_{\frac{\nu-1}{2}}(x\xi)\,x\,dx = \frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}}}\int\limits_u^\infty (x^2-u^2)^{\alpha-1}K_{\frac{\nu-1}{2}}(x\xi)\,x^{1-\frac{\nu-1}{2}}\,dx=$$

$$= \frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}}}\cdot 2^{\alpha-1}u^{\alpha-\frac{\nu-1}{2}}\xi^{-\alpha}\Gamma(\alpha)K_{\frac{\nu-1}{2}-\alpha}(u\xi)$$

and

$$\mathcal{K}_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi)=\frac{2^{\frac{\nu-1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}+\alpha}}\int\limits_0^\infty u^{\alpha+\frac{\nu+1}{2}}K_{\frac{\nu-1}{2}-\alpha}(u\xi)g(u)du=$$

$$=\frac{2^{\frac{\nu+1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha)\xi^{\frac{\nu-1}{2}+\alpha}}\int\limits_0^\infty u^{\frac{\nu+1}{2}-\alpha}K_{\frac{\nu-1}{2}-\alpha}(u\xi)du\int\limits_0^u(u^2-t^2)^{\alpha-1}t\varphi(t)dt=$$

$$=\frac{2^{\frac{\nu+1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha)\xi^{\frac{\nu-1}{2}+\alpha}}\int\limits_0^\infty t\varphi(t)dt\int\limits_t^\infty(u^2-t^2)^{\alpha-1}u^{\frac{\nu+1}{2}-\alpha}K_{\frac{\nu-1}{2}-\alpha}(u\xi)du.$$

Using again (37) we can write

$$\int\limits_t^\infty(u^2-t^2)^{\alpha-1}u^{\frac{\nu+1}{2}-\alpha}K_{\frac{\nu-1}{2}-\alpha}(u\xi)du=2^{\alpha-1}t^{\frac{\nu-1}{2}}\xi^{-\alpha}\Gamma(\alpha)K_{\frac{\nu-1}{2}}(t\xi)$$

and

$$\mathcal{K}_\nu[(B^{-\alpha}_{\nu,0+}\varphi)(x)](\xi)=\frac{2^{\frac{\nu+1}{2}-\alpha}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha)\xi^{\frac{\nu-1}{2}+\alpha}}\cdot 2^{\alpha-1}\xi^{-\alpha}\Gamma(\alpha)\int\limits_0^\infty\varphi(t)K_{\frac{\nu-1}{2}}(t\xi)t^{\frac{\nu+1}{2}}$$

$$dt=\xi^{-2\alpha}\int\limits_0^\infty\varphi(t)k_{\frac{\nu-1}{2}}(t\xi)t^\nu dt=$$

$$=\xi^{-2\alpha}\mathcal{K}_\nu\varphi.$$

Now let prove (36). Let $g(x)=K_2^{0,\alpha}x^{2\alpha}\varphi(x)$. Then using the factorization (20) we obtain

$$\mathcal{K}_\nu[(B^{-\alpha}_{\nu,-}\varphi)(x)](\xi)=\int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\,(B^{-\alpha}_{\nu,0-}\varphi)(x)x^\nu\,dx=$$

$$=2^{-2\alpha}\int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\,x^\nu\,K_2^{\frac{1-\nu}{2},\alpha}K_2^{0,\alpha}x^{2\alpha}\varphi(x)\,dx=$$

$$= \frac{2^{1-2\alpha}}{\Gamma(\alpha)} \int\limits_0^\infty k_{\frac{\nu-1}{2}}(x\xi)\, x dx \int\limits_x^\infty (u^2 - x^2)^{\alpha-1} u^{\nu-2\alpha} g(u) du =$$

$$= \frac{2^{1-2\alpha}}{\Gamma(\alpha)} \int\limits_0^\infty u^{\nu-2\alpha} g(u) du \int\limits_0^u (u^2 - x^2)^{\alpha-1} k_{\frac{\nu-1}{2}}(x\xi)\, x dx.$$

Let consider the inner integral

$$\int\limits_0^u (u^2-x^2)^{\alpha-1} k_{\frac{\nu-1}{2}}(x\xi)\, x dx = \frac{2^{\frac{\nu-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\frac{\nu-1}{2}}} \int\limits_0^u (u^2-x^2)^{\alpha-1} K_{\frac{\nu-1}{2}}(x\xi)\, x^{1-\frac{\nu-1}{2}}\, dx.$$

Using the formula 2.16.3.3 from [20] of the form

$$\int\limits_0^a x^{1-\rho}(a^2 - x^2)^{\beta-1} K_\rho(cx) dx = \frac{\pi 2^{\beta-2} a^{\beta-\rho}}{c^\beta \sin\rho\pi}\Gamma(\beta) I_{\beta-\rho}(ac) +$$

$$+ \frac{a^{2\beta} c^\nu}{2^{\rho+2}\beta}\Gamma(-\rho)\, {}_1F_2\left(1; \rho+1, \beta; \frac{a^2c^2}{4}\right), \qquad a, \beta > 0, \rho < 1$$

we obtain for $\nu < 3$

$$\int\limits_0^u (u^2 - x^2)^{\alpha-1} K_{\frac{\nu-1}{2}}(x\xi)\, x^{1-\frac{\nu-1}{2}}\, dx = \frac{\xi^{\frac{\nu-1}{2}}\Gamma\left(\frac{1-\nu}{2}\right)}{2^{\frac{\nu+3}{2}}\alpha} u^{2\alpha}\, {}_1F_2\left(1; \alpha+1, \frac{\nu}{2} + \frac{1}{2}; \frac{u^2\xi^2}{4}\right) -$$

$$- \frac{\pi 2^{\alpha-2}\Gamma(\alpha)}{\xi^\alpha \cos\left(\frac{\pi\nu}{2}\right)} u^{\alpha+\frac{1-\nu}{2}} I_{\alpha+\frac{1-\nu}{2}}(u\xi)$$

and

$$\int\limits_0^u (u^2 - x^2)^{\alpha-1} k_{\frac{\nu-1}{2}}(x\xi)\, x dx = \frac{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}{2\alpha} u^{2\alpha}\, {}_1F_2\left(1; \alpha+1, \frac{\nu}{2} + \frac{1}{2}; \frac{u^2\xi^2}{4}\right) -$$

$$- \frac{\pi 2^{\frac{\nu-1}{2}+\alpha-2}\Gamma(\alpha)\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\alpha+\frac{\nu-1}{2}} \cos\left(\frac{\pi\nu}{2}\right)} u^{\alpha+\frac{1-\nu}{2}} I_{\alpha+\frac{1-\nu}{2}}(u\xi)$$

So

$$\mathcal{K}_\nu[(B_{\nu,-}^{-\alpha}\varphi)(x)](\xi)=$$

$$=\frac{2^{1-2\alpha}}{\Gamma(\alpha)}\left[\frac{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}{2\alpha}\int\limits_0^\infty u^\nu\,{}_1F_2\left(1;\alpha+1,\frac{\nu}{2}+\frac{1}{2};\frac{u^2\xi^2}{4}\right)g(u)du-\right.$$

$$\left.-\frac{\pi 2^{\frac{\nu-1}{2}+\alpha-2}\Gamma(\alpha)\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\alpha+\frac{\nu-1}{2}}\cos\left(\frac{\pi\nu}{2}\right)}\int\limits_0^\infty u^{\frac{\nu+1}{2}-\alpha}I_{\alpha+\frac{1-\nu}{2}}(u\xi)g(u)du\right]=$$

$$=\frac{2^{2-2\alpha}}{\Gamma^2(\alpha)}\left[\frac{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}{2\alpha}\int\limits_0^\infty u^\nu\,{}_1F_2\left(1;\alpha+1,\frac{\nu}{2}+\frac{1}{2};\frac{u^2\xi^2}{4}\right)du\int\limits_u^\infty(t^2-u^2)^{\alpha-1}t\varphi(t)dt-\right.$$

$$\left.-\frac{\pi 2^{\frac{\nu-1}{2}+\alpha-2}\Gamma(\alpha)\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\alpha+\frac{\nu-1}{2}}\cos\left(\frac{\pi\nu}{2}\right)}\int\limits_0^\infty u^{\frac{\nu+1}{2}-\alpha}I_{\alpha+\frac{1-\nu}{2}}(u\xi)du\int\limits_u^\infty(t^2-u^2)^{\alpha-1}t\varphi(t)dt\right]=$$

$$=\frac{2^{2-2\alpha}}{\Gamma^2(\alpha)}\left[\frac{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}{2\alpha}\int\limits_0^\infty t\varphi(t)dt\int\limits_0^t(t^2-u^2)^{\alpha-1}u^\nu\,{}_1F_2\left(1;\alpha+1,\frac{\nu}{2}+\frac{1}{2};\frac{u^2\xi^2}{4}\right)du-\right.$$

$$\left.-\frac{\pi 2^{\frac{\nu-1}{2}+\alpha-2}\Gamma(\alpha)\Gamma\left(\frac{\nu+1}{2}\right)}{\xi^{\alpha+\frac{\nu-1}{2}}\cos\left(\frac{\pi\nu}{2}\right)}\int\limits_0^\infty t\varphi(t)dt\int\limits_0^t(t^2-u^2)^{\alpha-1}u^{\frac{\nu+1}{2}-\alpha}I_{\alpha+\frac{1-\nu}{2}}(u\xi)du\right].$$

Using Wolfram Mathematica we obtain

$$\int\limits_0^t(t^2-u^2)^{\alpha-1}u^\nu\,{}_1F_2\left(1;\alpha+1,\frac{\nu}{2}+\frac{1}{2};\frac{u^2\xi^2}{4}\right)du=\frac{\Gamma(\alpha)\Gamma\left(\frac{\nu+1}{2}\right)}{2\Gamma\left(\alpha+\frac{\nu+1}{2}\right)}t^{2\alpha+\nu-1}$$

$${}_1F_2\left(1;\alpha+1,\alpha+\frac{\nu+1}{2};\frac{t^2\xi^2}{4}\right)$$

and

$$\int\limits_0^t(t^2-u^2)^{\alpha-1}u^{\frac{\nu+1}{2}-\alpha}I_{\alpha+\frac{1-\nu}{2}}(u\xi)du=\frac{2^{\frac{\nu-3}{2}-\alpha}\Gamma(\alpha)}{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{3-\nu}{2}\right)}t^{2\alpha}\xi^{\alpha-\frac{\nu}{2}+\frac{1}{2}}$$

$${}_1F_2\left(1;\alpha+1,\alpha+\frac{3-\nu}{2};\frac{t^2\xi^2}{4}\right),\qquad 2\alpha<\nu+3.$$

Finally

$$\mathcal{K}_\nu[(B_{\nu,-}^{-\alpha}\varphi)(x)](\xi) =$$

$$= \frac{\Gamma\left(\frac{1-\nu}{2}\right)\Gamma^2\left(\frac{\nu+1}{2}\right)}{2^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{\nu+1}{2}\right)}\int\limits_0^\infty \varphi(t)t^{2\alpha+\nu}\,{}_1F_2\left(1;\alpha+1,\alpha+\frac{\nu+1}{2};\frac{t^2\xi^2}{4}\right)dt-$$

$$-\frac{\pi 2^{\nu-2\alpha-2}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{3-\nu}{2}\right)\cos\left(\frac{\pi\nu}{2}\right)}\xi^{1-\nu}\int\limits_0^\infty \varphi(t)t^{2\alpha+1}\,{}_1F_2\left(1;\alpha+1,\alpha+\frac{3-\nu}{2};\frac{t^2\xi^2}{4}\right)dt.$$

□

Since $\mathcal{K}_\nu[(B_\nu^n\varphi)](\xi) = \xi^{2n}\mathcal{K}_\nu[\varphi](\xi)$ we obtain for proper functions

$$\mathcal{K}_\nu[(B_{\nu,0+}^{\alpha}\varphi)(x)](\xi) = \xi^{2\alpha}\mathcal{K}_\nu\varphi(\xi).$$

5.4 Generalized Whittaker Transform

Theorem 9 *The generalized Whittaker transform of $B_{\nu,0+}^{-\alpha}$ for proper functions is*

$$\left(W_{\rho,\frac{\nu-1}{4}}^{\frac{\nu-1}{2}} B_{\nu,0+}^{-\alpha} f\right)(x) = C(\nu,\alpha,\rho)x^{-2\alpha}\left(W_{\rho+\alpha,\frac{\nu-1}{4}}^{\frac{\nu-1}{2}} f\right)(x),$$

where

$$C(\nu,\alpha,\rho) = \frac{\Gamma\left(\frac{\nu+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\alpha-\rho\right)}{2^{2\alpha}\Gamma\left(\frac{\nu+1}{4}-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\rho\right)}.$$

Proof We have

$$\left(W_{\rho,\frac{\nu-1}{4}}^{\frac{\nu-1}{2}} B_{\nu,0+}^{-\alpha} f\right)(x) = \frac{1}{\Gamma(2\alpha)}\int\limits_0^\infty (xt)^{\frac{\nu-1}{2}} e^{\frac{x^2t^2}{2}} W_{\rho,\frac{\nu-1}{4}}(x^2t^2)dt\times$$

$$\times\int\limits_0^t \left(\frac{y}{t}\right)^\nu \left(\frac{t^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{t^2}\right)f(y)dy =$$

$$= \frac{x^{\frac{\nu-1}{2}}}{2^{2\alpha-1}\Gamma(2\alpha)} \int\limits_0^\infty f(y) y^\nu dy \int\limits_y^\infty t^{\frac{\nu-1}{2}-\nu-2\alpha+1} e^{\frac{x^2t^2}{2}} (t^2-y^2)^{2\alpha-1} W_{\rho,\frac{\nu-1}{4}}(x^2t^2)$$

$${}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{y^2}{t^2}\right) dt.$$

Using formula

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$$

we obtain

$${}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{y^2}{t^2}\right) = \left(\frac{y}{t}\right)^{1-\nu-2\alpha} {}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{t^2}{y^2}\right)$$

and

$$\left(W^{\frac{\nu-1}{2}}_{\rho,\frac{\nu-1}{4}} B^{-\alpha}_{\nu,0+} f\right)(x) =$$

$$= \frac{x^{\frac{\nu-1}{2}}}{2^{2\alpha-1}\Gamma(2\alpha)} \int\limits_0^\infty f(y) y^{1-2\alpha} dy \int\limits_y^\infty t^{\frac{\nu-1}{2}} e^{\frac{x^2t^2}{2}} (t^2-y^2)^{2\alpha-1} W_{\rho,\frac{\nu-1}{4}}(x^2t^2)$$

$${}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{t^2}{y^2}\right) dt.$$

Let consider an inner integral. We have

$$\int\limits_y^\infty t^{\frac{\nu-1}{2}} e^{\frac{x^2t^2}{2}} (t^2-y^2)^{2\alpha-1} W_{\rho,\frac{\nu-1}{4}}(x^2t^2)\, {}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{t^2}{y^2}\right) dt = \{t^2 \to t,\ y^2 = p\} =$$

$$= \frac{1}{2} \int\limits_p^\infty t^{\frac{\nu-1}{4}-\frac{1}{2}} e^{\frac{x^2t}{2}} (t-p)^{2\alpha-1} W_{\rho,\frac{\nu-1}{4}}(x^2t)\, {}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{t}{p}\right) dt.$$

Using formula 2.21.8.2 from [33] of the form

$$\int\limits_p^\infty t^{\frac{a+b-c-1}{2}}(t-p)^{c-1}e^{\frac{\sigma t}{2}}W_{\rho,\frac{a+b-c}{2}}(\sigma t)\,{}_2F_1\left(a,b;c;1-\frac{t}{p}\right)dt=$$

$$=\frac{p^{\frac{a+b-1}{2}}}{\sigma^{\frac{c}{2}}}\frac{\Gamma(c)\,\Gamma\left(\frac{a-b-c+1}{2}-\rho\right)\Gamma\left(\frac{b-a-c+1}{2}-\rho\right)}{\Gamma\left(\frac{a+b-c+1}{2}-\rho\right)\Gamma\left(\frac{c-a-b+1}{2}-\rho\right)}e^{\frac{\sigma p}{2}}W_{\rho+\frac{c}{2},\frac{a-b}{2}}(\sigma p),$$

$$p,\operatorname{Re}c>0,\operatorname{Re}(c+2\rho)<1-|\operatorname{Re}(a-b)|;\ |\arg\sigma|<\frac{3\pi}{2}$$

we obtain

$$a=\alpha+\frac{\nu-1}{2},\ b=\alpha,\ c=2\alpha,\ \sigma=x^2,\ 2\alpha+2\rho<1-\left|\frac{\nu-1}{2}\right|$$

and

$$\frac{1}{2}\int\limits_p^\infty t^{\frac{\nu-1}{4}-\frac{1}{2}}e^{\frac{x^2t}{2}}(t-p)^{2\alpha-1}W_{\rho,\frac{\nu-1}{4}}(x^2t)\,{}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{t}{p}\right)dt=$$

$$=\frac{1}{2}\frac{p^{\alpha+\frac{\nu-3}{4}}}{x^{2\alpha}}\frac{\Gamma(2\alpha)\,\Gamma\left(\frac{\nu+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\alpha-\rho\right)}{\Gamma\left(\frac{\nu+1}{4}-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\rho\right)}e^{\frac{x^2p}{2}}W_{\rho+\alpha,\frac{\nu-1}{4}}(x^2p)=$$

$$=\frac{1}{2}\frac{y^{2\alpha+\frac{\nu-3}{2}}}{x^{2\alpha}}\frac{\Gamma(2\alpha)\,\Gamma\left(\frac{\nu+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\alpha-\rho\right)}{\Gamma\left(\frac{\nu+1}{4}-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\rho\right)}e^{\frac{x^2y^2}{2}}W_{\rho+\alpha,\frac{\nu-1}{4}}(x^2y^2)=$$

$$=A(\nu,\alpha,\rho)x^{-2\alpha}y^{2\alpha+\frac{\nu-3}{2}}e^{\frac{x^2y^2}{2}}W_{\rho+\alpha,\frac{\nu-1}{4}}(x^2y^2),$$

where

$$A(\nu,\alpha,\rho)=\frac{1}{2}\frac{\Gamma(2\alpha)\,\Gamma\left(\frac{\nu+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\alpha-\rho\right)}{\Gamma\left(\frac{\nu+1}{4}-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\rho\right)}$$

Then

$$\left(W^{\frac{\nu-1}{2}}_{\rho,\frac{\nu-1}{4}} B^{-\alpha}_{\nu,0+} f\right)(x) =$$

$$= A(\nu,\alpha,\rho)\frac{x^{\frac{\nu-1}{2}-2\alpha}}{2^{2\alpha-1}\Gamma(2\alpha)}\int\limits_0^\infty f(y) y^{\frac{\nu-1}{2}} e^{\frac{x^2y^2}{2}} W_{\rho+\alpha,\frac{\nu-1}{4}}(x^2y^2)dy =$$

$$= C(\nu,\alpha,\rho)x^{-2\alpha}\int\limits_0^\infty f(y)(xy)^{\frac{\nu-1}{2}} e^{\frac{x^2y^2}{2}} W_{\rho+\alpha,\frac{\nu-1}{4}}(x^2y^2)dy =$$

$$= C(\nu,\alpha,\rho)x^{-2\alpha}\left(W^{\frac{\nu-1}{2}}_{\rho+\alpha,\frac{\nu-1}{4}} f\right)(x),$$

where

$$C(\nu,\alpha,\rho) = \frac{\Gamma\left(\frac{\nu+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\alpha-\rho\right)}{2^{2\alpha}\Gamma\left(\frac{\nu+1}{4}-\rho\right)\Gamma\left(\frac{3-\nu}{4}-\rho\right)}.$$

□

It is worth mentioning that the mapping property of the transmutation operators allowing one to obtain the images of the powers of the independent variable without knowledge of the transmutation operator itself [35] can be used for further study of fractional powers of Bessel operator.

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The Fractional Derivative Expansion Method in Nonlinear Dynamics of Structures: A Memorial Essay

Marina V. Shitikova

Abstract The history of formulation of the efficient method for studying the nonlinear dynamic response of structures, damping features of which depend on natural frequencies of vibrations, is presented. This technique is the modified version of the method of multiple scales.

This memorial essay is dedicated to the bright memory of two great scientists, Ali Hasan Nayfeh and Yury Rossikhin, who had gone away one after another in 2 days, March 27 and 29, 2017.

1 Introduction

In March 2017, together with Professor Yury Rossikhin we were working on a review paper devoted to the latest results in applications of fractional-order operators in mechanics of solids and structures, which should be an extension of our two previous reviews published in 1997 and 2010 in *Applied Mechanics Reviews* [20, 28]. The article was practically prepared, and I took in my hands the 1973 book by Professor Ali Nayfeh [10] in order to find the phrase, which he liked to repeat in his numerous conference talks dealing with the method of multiple scales and which we wanted to utilize as an epigraph for a nonlinear dynamics section in our review:

> The method of multiple scales is so popular that it is being rediscovered just about every 6 months.

I found rather quickly this phrase in page 232 of [10], since we have in our home library its Russian translation published in 1976 in Moscow, and we began with Yury to recall our meetings and discussions with Professor Nayfeh. It was in the late evening of March 27, 2017... At that moment we did not know that he is not with us any more...

M. V. Shitikova (✉)
Research Center of Wave Dynamics, Voronezh State Technical University, Voronezh, Russia
e-mail: mvs@vgasu.vrn.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_29

It is unbelievable for me up today that it had happened in the late evening on March 27, maybe in those minutes when we were talking about him. And 2 days later, on March 29, 2017 the second great loss crashed in my life: Professor Yury Rossikhin had passed away after sudden heart attack. Thus, the mechanics community lost two distinguished representatives during three March days.

In this memorial essay, I am going to tell a story how a generalization of the method of multiple time scales, and to be more precisely, of its one version, which was named by Sturrock [42] and Nayfeh [9, 10] as the derivative expansion method, was suggested and constructed by Yury Rossikhin and myself [21, 22] via considering fractional derivatives in nonlinear dynamics of structures. It could not be happened without the influence of the Professor Nayfeh contribution in the field.

2 Nonlinear Vibrations of Suspension Bridges and the Method of Multiple Time Scales

In 1977, when I was a first-year student of Voronezh Civil Engineering Institute, during one of the introductory lectures Professor Nikolai M. Kirsanov, who was an outstanding expert in metal structures and particularly in suspension combined systems, showed us a record film about the collapse of the Tacoma Narrows Suspension Bridge, which impressed me a lot. At that time I could not imagine that I would take part in studies of the Golden Gate Bridge some day. But certainly that lecture with the film run was a starting point.

Seven years later being a PhD student, I collected and studied literature devoted to dynamics of suspension bridges, since it was the topic of my future PhD thesis. It was not an easy task in those times in Russia, and to find some interesting and useful papers (especially by Western researchers), which could be included in the thesis list of references, it was a need to spend a lot of time in the Lenin Library in Moscow, the largest Russian library.

Thus, among other papers in the field, my attention was attracted by the articles by Abdel-Ghaffar and Rubin [2, 3], wherein nonlinear undamped free coupled vertical-torsional vibrations of suspension bridges were examined using the multiple scales method with the reference to the book by Nayfeh and Mook [14]. It was the first time when I got acquainted with the name of Professor Nayfeh and the method of multiple scales.

Unfortunately, this book [14] was unavailable even in the Lenin Library, but I have found that other two books by Nayfeh published in 1973 [10] and 1981 [11] had been translated into Russian in 1976 and 1984, respectively. Just these two books influenced greatly researchers especially young ones, including myself, to study the perturbation technique with its further utilization.

Reference to [10] shows that the method of multiple scales has been applied to a wide variety of problems in physics, engineering, and applied mathematics. Moreover, it is very efficient for studying the problems resulting in modal interaction in dynamical and structural systems subjected to different cases of internal and combinational resonances [12, 13].

2.1 Nonlinear Undamped Vibrations of Suspension Bridges

It is known for suspension bridges (Fig. 1) that some natural modes belonging to different types of vibrations could be coupled with each other, i.e., the excitation of one natural mode gives rise to another one [2]. Two modes interact more often that not, although the possibility for interaction of a greater number of modes is not ruled out.

Consider the case when only two modes predominate in the vibrational process, namely: the vertical n-th mode with linear natural frequency ω_{0n}, and the torsional m-th mode with the natural frequency Ω_{0m}. Under such an assumption the functions $\eta(z,t)$ and $\varphi(z,t)$ could be approximately defined as

$$\eta(z,t) \sim v_n(z)x_{1n}(t), \quad \varphi(z,t) \sim \Theta_m(z)x_{2m}(t) \tag{1}$$

where $x_{1n}(t)$ and $x_{2m}(t)$ are the generalized displacements, and $v_n(z)$ and $\Theta_m(z)$ are natural shapes of the two interacting modes of vibrations.

The modes interaction could be observed under the conditions of the two-to-one internal resonance, when the linear natural frequency ω_{0n} is approximately twice as large than the linear natural frequency Ω_{0m}, i.e.,

$$\omega_0 = 2\Omega_0 + \varepsilon\sigma, \tag{2}$$

or the one-to-one internal resonance

$$\omega_0 = \Omega_0 + \varepsilon^2\sigma, \tag{3}$$

where σ is a detuning parameter.

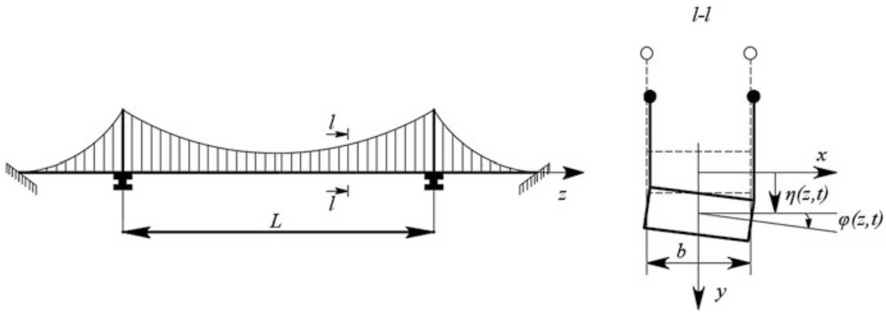

Fig. 1 Suspension bridge scheme

In this case the resolving set of equations in dimensionless form was written in Abdel-Ghaffar and Rubin [2] in terms of the generalized displacements as

$$\ddot{x}_1 + \omega_0^2 x_1 + a_{11}x_1^2 + a_{22}x_2^2 + (b_{11}x_1^2 + b_{22}x_2^2)x_1 = 0, \tag{4}$$

$$\ddot{x}_2 + \Omega_0^2 x_2 + a_{12}x_1x_2 + (c_{11}x_1^2 + c_{22}x_2^2)x_2 = 0, \tag{5}$$

where dots denote differentiation with respect to time, all coefficients are given in [2], and the indices n and m denoting the numbers of the interacting modes are omitted for ease of presentation.

An approximate solution of Eqs. (4) and (5) for small amplitudes weakly varying with time can be represented by an expansion in terms of different time scales in the following form [10]:

$$x_1(t) = \varepsilon x_{11}(T_0, T_1, T_2, \ldots) + \varepsilon^2 x_{12}(T_0, T_1, T_2, \ldots) + \varepsilon^3 x_{13}(T_0, T_1, T_2, \ldots) + \ldots \tag{6}$$

$$x_2(t) = \varepsilon x_{21}(T_0, T_1, T_2, \ldots) + \varepsilon^2 x_{22}(T_0, T_1, T_2, \ldots) + \varepsilon^3 x_{23}(T_0, T_1, T_2, \ldots) + \ldots \tag{7}$$

where

$$T_n = \varepsilon^n t \quad (n = 0, 1, 2, \ldots) \tag{8}$$

are new independent variables. The time scale T_0 is the fast one characterizing motions with the natural frequencies ω_0 and Ω_0, the time scale T_1 is slower than T_0, while the time scale T_2 is slower than T_1. In general, T_n is slower than T_{n-1}. All slow scales characterize the modulations of the amplitudes and phases.

Equations (6), (7), and (8) show that the problem has been transformed from the ordinary differential equations to partial differential equations. Since the equations of motion involve the time-derivatives, then it a need to represent them in terms of new time scales. Thus, as it is written in Nayfeh [10], using the chain rule, the time derivative is transformed according to

$$d/dt = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots, \tag{9}$$

where $D_n = \partial/\partial T_n$.

Then the second-order time-derivative could be found as

$$d^2/dt^2 = \left(D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots\right)^2 = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) + \ldots \tag{10}$$

Equations (8) through (10) formulate one version of the method of multiple scales; namely, the many-variable version. This technique has been developed by Sturrock [42] and Nayfeh [9]. Equations show that a uniformly valid expansion is obtained by expanding the derivatives as well as the dependent variables in powers of the small parameter. Hence Sturrock [42] and Nayfeh [10] called this technique *the derivative expansion method*.

Substituting Eqs. (6)–(10) in Eqs. (4) and (5) and equating coefficients of like powers of ϵ, the equations for determining $x_{i1}, x_{i2}, \ldots x_{im}$ $(i = 1, 2)$ could be obtained. Thus, restricting ourselves by three-term expansions in (4) and (5), i.e. considering the scales T_0, T_1, and T_2, yields

- to order ε:

$$D_0^2 x_{11} + \omega_0^2 x_{11} = 0, \quad D_0^2 x_{21} + \Omega_0^2 x_{21} = 0; \tag{11}$$

- to order ε^2:

$$D_0^2 x_{12} + \omega_0^2 x_{12} = -2D_0 D_1 x_{11} - a_{11} x_{11}^2 - a_{22} x_{21}^2, \tag{12}$$

$$D_0^2 x_{22} + \Omega_0^2 x_{22} = -2D_0 D_1 x_{21} - a_{12} x_{11} x_{21}; \tag{13}$$

- to order ε^3:

$$\begin{aligned} D_0^2 x_{13} + \omega_0^2 x_{13} = &-2D_0 D_1 x_{12} - (D_1^2 + 2D_0 D_2) x_{11} - 2a_{11} x_{11} x_{12} \\ &-2a_{22} x_{21} x_{22} - b_{11} x_{11}^3 - b_{22} x_{21}^2 x_{11}, \end{aligned} \tag{14}$$

$$\begin{aligned} D_0^2 x_{23} + \Omega_0^2 x_{23} = &-2D_0 D_1 x_{22} - (D_1^2 + 2D_0 D_2) x_{21} - a_{12}(x_{11} x_{22} + x_{12} x_{21}) \\ &-c_{22} x_{21}^3 - c_{11} x_{11}^2 x_{21}. \end{aligned} \tag{15}$$

The solutions of these equations

$$x_{11} = A_1(T_1, T_2) \exp(i\omega_0 T_0) + \bar{A}_1 \exp(-i\omega_0 T_0), \tag{16}$$

$$x_{21} = A_2(T_1, T_2) \exp(i\Omega_0 T_0) + \bar{A}_2 \exp(-i\Omega_0 T_0) \tag{17}$$

contain arbitrary complex functions A_1 and A_2 of the time scales T_1 and T_2, and $\bar{A}_1$ and $\bar{A}_2$ are the complex conjugates of A_1 and A_2, respectively.

In order to determine these functions, additional conditions, which were named as *solvability conditions* [12] and which are equivalent to the elimination of circular terms, need to be imposed.

Substituting expressions (16) and (17) into the right-hand sides of Eqs. (12) and (13) yields

$$\begin{aligned} D_0^2 x_{12} + \omega_0^2 x_{12} = &-2i\omega_0 D_1 A_1 \exp(i\omega_0 T_0) - a_{11} A_1^2 \exp(2i\omega_0 T_0) \\ &-a_{11} A_1 \bar{A}_1 - a_{22} A_2^2 \exp(2i\Omega_0 T_0) - a_{22} A_2 \bar{A}_2 + cc, \end{aligned} \tag{18}$$

$$D_0^2 x_{22} + \Omega_0^2 x_{22} = -2i\Omega_0 D_1 A_2 \exp(i\Omega_0 T_0)$$
$$-a_{12} A_1 A_2 \exp[i(\omega_0 + \Omega_0)T_0] - a_{12} A_1 \bar{A}_2 \exp[i(\omega_0 - \Omega_0)T_0] + cc, \quad (19)$$

where cc is the complex conjugate part to the preceding terms.

Reference to Eqs. (18) and (19) show that they could describe the two-to-one internal resonance (2). However, this type of the internal resonance was mentioned but was not considered by Abdel-Ghaffar and Rubin [2]. It has been done lately by Rossikhin and Shitikova [18].

Further considering that the functions $\exp(i\omega_0 T_0)$ and $\exp(i\Omega_0 T_0)$ entering into the right-hand sides of Eqs. (18) and (19) produce secular terms, therefore the following solvability conditions should be imposed:

$$D_1 A_1(T_1, T_2) = 0, \quad D_1 A_2(T_1, T_2) = 0, \quad (20)$$

i.e., the functions A_1 and A_2 are dependent on T_2 only.

Substituting now Eqs. (16)–(19) with due account for (20) into the right-hand sides of Eqs. (14) and (15) results in the system of equations for determining x_{13} and x_{23}. Such a set of equations has been obtain in Abdel-Ghaffar and Rubin [2], and some particular cases have been considered including the case of the one-to-one internal resonance (3) [2, 3].

The systematic quantitative and qualitative analysis of the both internal resonances, i.e. (2) and (3), which could occur in suspension bridges, has been carried out in [18], wherein two first integrals of the system of Eqs. (4) and (5) have been obtained. The first of them defines the energy of the system, while the second one gives the stream-function, therefore the hydrodynamical analogy has been suggested. Using the data of the Golden Gate Bridge provided in [3], the phase portraits have been constructed for several cases of the internal resonance which could be appropriate for nonlinear vibrations of this bridge.

By using the hydrodynamic analogy (phase fluid flow) a qualitative method of analysis of the suspension bridge non-linear free vibrations has been proposed. This procedure allows one to determine the types of oscillatory process, to investigate the stability of each vibrational regime, to determine the character of amplitude and phase difference dependences from initial conditions, etc. Theorems allowing one to detect various types of vibrational regimes: periodic, aperiodic and stationary—have been proved. These vibrational regimes correspond to the three types of energy exchange between the vertical and torsional modes: two-sided energy exchange (a periodic energy exchange from one subsystem to another), one-sided energy interchange (one subsystem transfers energy to another), and energy exchange does not occur. Two-sided energy exchange corresponds to both amplitude and phase modulated aperiodic motions, one-sided energy interchange is appropriate to both amplitude and phase aperiodic motions or pure amplitude modulated aperiodic motions. In the absence of energy exchange, there are stationary vibrations or pure phase modulated motions. The class of soliton-like solutions describing the complete one-sided energy transfer has been found.

2.2 Nonlinear Damped Free Vibrations of Suspension Bridges

The natural extension of studies presented in Rossikhin and Shitikova [18] was to consider damped vibrations, since all engineering structures possess the intrinsic structural damping and work in the surrounding media which quench oscillations. For this purpose, damping terms proportional to the first-order time-derivative in displacements have been added to Eqs. (4) and (5) in [19], as it is traditionally accepted in dynamics of structures [8].

The similar approach was realized 10 years later in [7] with the only difference that the method of derivative expansions has been applied directly to the governing equations of suspension bridge motion without preliminary expansion in terms of eigenmodes.

The experimental data obtained by Abdel-Ghaffar and Housner [1], Abdel-Ghaffar and Scanlan [4], and Baranov et al. [5] during ambient vibration studies of the Vincent-Thomas Suspension Bridge, the Golden Gate Bridge, and the pedestrian suspension bridge over the Sura River in Penza, Russia, respectively, show that different vibrational modes feature different amplitude damping factors, and the order of smallness of these coefficients tells about low damping capacity of suspension combined systems, resulting in prolonged energy transfer from one partial subsystem to another. Besides, as natural frequencies of vibrations increase, the corresponding damping ratios decrease.

It has been shown in Rossikhin and Shitikova [19] for both types of the internal resonance that when damping features of the system are prescribed by the first derivative of the displacement with respect to time, then the damping coefficient does not depend on the natural frequency of vibrations. It means that this result is in conflict with the experimental data presented in [1, 4, 5].

Thus, this raised the question of whether it is possible to create such a model which could lead the theoretical investigations in line with the experiment. The answer for this question was found in 1997 by Professor Yury Rossikhin who suggested to introduce fractional derivatives for describing the processes of internal friction proceeding in suspension combined systems at free vibrations. He possessed the encyclopedical lore in fractional calculus viscoelasticity, and the history of fractional calculus applications in dynamic problems of mechanics of solids and structures was described by him in the retrospective paper [17].

There exist several definitions of fractional order derivatives, and the most useful in hereditary mechanics is the Riemann-Liouville fractional derivative [39]

$$D_+^\gamma f = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_{-\infty}^{t} \frac{f(t')dt'}{(t-t')^\gamma} = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^{\infty} \frac{f(t-t')dt'}{t'^\gamma}, \qquad (21)$$

where $\Gamma(1-\gamma)$ is the Gamma-function, and $0 < \gamma < 1$ is the order of the fractional derivative.

Then equations of motion of damped nonlinear vibrations of suspension bridge take the form

$$\ddot{x}_1 + \beta D_{+}^{\gamma_1} x_1 + \omega_0^2 x_1 + a_{11} x_1^2 + a_{22} x_2^2 + (b_{11} x_1^2 + b_{22} x_2^2) x_1 = 0, \tag{22}$$

$$\ddot{x}_2 + \beta D_{+}^{\gamma_2} x_2 + \Omega_0^2 x_2 + a_{12} x_1 x_2 + (c_{11} x_1^2 + c_{22} x_2^2) x_2 = 0, \tag{23}$$

where β is the damping coefficient.

However, the Riemann-Liouville definition could not be incorporated in the method of multiple time scales used for solving the problem of undamped vibrations. The Grunwald-Letnikov definition [20] is widely used in different numerical procedures. Another representation of the fractional derivative was needed, and Professor Rossikhin had found another approach.

The matter is fact that it was shown in Samko et al. [39] (see Chapter 2, Paragraph 5, point 7^0) that the fractional order of the operator of differentiation $\left(\frac{d}{dt}\right)^{\gamma}$ is equal to the Marchaud fractional derivative, which, in its turn, equal to the Riemann-Liouville derivative D_{+}^{γ} for sufficiency "good" functions

$$\left(\frac{d}{dt}\right)^{\gamma} f = \frac{1}{\Gamma(-\gamma)} \int_0^{\infty} \frac{f(t-t') - f(t)}{t'^{1+\gamma}} \, dt' = D_{+}^{\gamma} f, \tag{24}$$

or with due account for the equality $\gamma \Gamma(\gamma) = \Gamma(1+\gamma)$

$$\left(\frac{d}{dt}\right)^{\gamma} f = \frac{\gamma}{\Gamma(1-\gamma)} \int_0^{\infty} \frac{f(t) - f(t-t')}{t'^{1+\gamma}} \, dt' = D_{+}^{\gamma} f. \tag{25}$$

In order to utilize the definition (24), the expansion of the fractional derivative in terms of new time scales was written by Rossikhin and Shitikova [21, 22] as

$$\begin{aligned} D_{+}^{\gamma} &= (d/dt)^{\gamma} = (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots)^{\gamma} \\ &= D_{0+}^{\gamma} + \varepsilon \gamma D_{0+}^{\gamma-1} D_1 + \frac{1}{2} \varepsilon^2 \gamma \left[(\gamma - 1) D_{0+}^{\gamma-2} D_1^2 + 2 D_{0+}^{\gamma-1} D_2 \right] + \ldots, \end{aligned} \tag{26}$$

where D_{0+}^{γ} is obtained from (21) by replacing t by T_0.

Thus, Eqs. (21), (24)–(26) together with Eqs. (8)–(10) formulate the modified version of the method of multiple scales. We called this technique, following Professor Nayfeh, *the fractional derivative expansion method*.

The set of Eqs. (22) and (23) describes the two processes, which are related to each other and go on concurrently: the energy-exchange mechanism between vertical and torsional modes, and the process of energy dissipation during this interaction. Since further investigations will be carried out by the method of multiple scales and these two processes should proceed at the same time scale, then there is a need to assume that the viscosity coefficient β could be represented as $\beta = \varepsilon^k \mu$, where μ is a finite value, $k = 1$ and 2 for the 2:1 and 1:1 internal resonances, respectively. At other orders of smallness of the viscosity coefficient, energy dissipation will occur either too fast or too slow relative to the process of energy exchange.

It has been noted in [22] that the fractional derivative is the immediate extension of an ordinary derivative. In fact, when $\gamma \to 1$, $D_{+}^{\gamma}x$ tends to $\dot{x}$, i.e., at $\gamma \to 1$ the fractional derivative goes over into the ordinary derivative, and the mathematical model of the suspension bridge transforms into the Kelvin-Voigt model, wherein the elastic element behaves nonlinearly, but the viscous element behaves linearly. When $\gamma \to 0$, the fractional derivative $D_{+}^{\gamma}x$ tends to $x(t)$. To put it otherwise, the introduction of the new fractional parameter along with the parameter β allows one to change not only the magnitude of viscosity at the cost of an increase or decrease in the parameter β, but also the character of viscosity at the sacrifice of variations in the fractional parameter.

With due account for the additional fractional derivative terms in Eqs. (22) and (23) and considering the expansion of the fractional derivative (26), Eqs. (12)–(15) take the following form:

- to order ε^2:

$$D_0^2 x_{12} + \omega_0^2 x_{12} = -2D_0D_1x_{11} - \mu(2-k)D_{0+}^{\gamma_1}x_{11} - a_{11}x_{11}^2 - a_{22}x_{21}^2, \tag{27}$$

$$D_0^2 x_{22} + \Omega_0^2 x_{22} = -2D_0D_1x_{21} - \mu(2-k)D_{0+}^{\gamma_2}x_{21} - a_{12}x_{11}x_{21}; \tag{28}$$

- to order ε^3:

$$\begin{aligned} D_0^2 x_{13} + \omega_0^2 x_{13} = & -2D_0D_1x_{12} - (D_1^2 + 2D_0D_2)x_{11} \\ & -\mu(2-k)D_{0+}^{\gamma_1}x_{12} - \mu(2-k)\gamma_1 D_{0+}^{\gamma_1 - 1}D_1x_{11} \\ & -\mu(k-1)D_{0+}^{\gamma_1}x_{11} - 2a_{11}x_{11}x_{12} - 2a_{22}x_{21}x_{22} \\ & -b_{11}x_{11}^3 - b_{22}x_{21}^2x_{11}, \end{aligned} \tag{29}$$

$$\begin{aligned} D_0^2 x_{23} + \Omega_0^2 x_{23} = & -2D_0D_1x_{22} - (D_1^2 + 2D_0D_2)x_{21} \\ & -\mu(2-k)D_{0+}^{\gamma_2}x_{22} - \mu(2-k)\gamma_2 D_{0+}^{\gamma_2 - 1}D_1x_{21} \\ & -\mu(k-1)D_{0+}^{\gamma_2}x_{21} - a_{12}(x_{11}x_{22} + x_{12}x_{21}) \\ & -c_{22}x_{21}^3 - c_{11}x_{11}^2x_{21}. \end{aligned} \tag{30}$$

To solve the sets of Eqs. (27)–(28) and (29)–(30), it is necessary to specify the action of the fractional derivative D_{0+}^{γ} on the functions x_{j1} and x_{j2} $j = 1, 2$, i.e., to calculate $D_{0+}^{\gamma}e^{i\omega t}$. It has been shown in [34] that

$$D_0^{\gamma}e^{i\omega t} = (i\omega)^{\gamma}e^{i\omega t}. \tag{31}$$

Note that since the process of vibrations starts at $t = 0$, then the fractional derivative should be defined on the segment $[0, t]$, i.e.,

$$D_0^\gamma x(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^t \frac{x(s)ds}{(t-s)^\gamma}. \tag{32}$$

Then instead of formula (31), the application of the fractional derivative (32) on the exponent results in the following relationship [34] (see Appendix):

$$D_0^\gamma e^{i\omega t} = (i\omega)^\gamma e^{i\omega t} + \frac{\sin \pi\gamma}{\pi} \int_0^\infty \frac{u^\gamma}{u + i\omega} e^{-ut} du. \tag{33}$$

However, the second term of (33), as it has been proved in [29], does not influence the solution constructed via the method of multiple time scales restricted to the first- and second-order approximations.

In other words, even the utilization of exact formula (33) in the problem under consideration produces completely equivalent results given by the approximate formula (31) if the solution is constructed via the method of multiple time scales within the considered orders of approximation. Thus, in further analysis we will utilize formula (31), as well as the following relationship:

$$i^\gamma = \cos \frac{\gamma\pi}{2} + i \sin \frac{\gamma\pi}{2} = e^{i\frac{\gamma\pi}{2}}. \tag{34}$$

The case of the 1:1 internal resonance at $k = 2$ and $\gamma_1 = \gamma_2 = \gamma$ was presented in [20, 22]. Representing the functions A_1 and A_2 in their polar form, i.e.,

$$A_1(T_2) = a_1(T_2) \exp\left[i\varphi_1(T_2)\right],$$

$$A_2(T_2) = a_2(T_2) \exp\left[i\varphi_2(T_2)\right],$$

the modulation equations have been obtained

$$\left(a_1^2\right)^{\cdot} + \mu\omega_0^{\gamma-1} a_1^2 \sin\left(\frac{1}{2}\pi\gamma\right) - \frac{1}{2}\Gamma_1 a_1^2 a_2^2 \sin\delta = 0, \tag{35}$$

$$\left(a_2^2\right)^{\cdot} + \mu\omega_0^{\gamma-1} a_2^2 \sin\left(\frac{1}{2}\pi\gamma\right) + \frac{1}{4}\Gamma_2 a_1^2 a_2^2 \sin\delta = 0, \tag{36}$$

$$\dot\varphi_1 - \frac{1}{2}\mu\omega_0^{\gamma-1} \cos\left(\frac{1}{2}\pi\gamma\right) - \lambda_1 a_1^2 - \lambda_2 a_2^2 + \frac{1}{4}\Gamma_1 a_2^2 \cos\delta = 0, \tag{37}$$

$$\dot\varphi_2 - \frac{1}{2}\mu\omega_0^{\gamma-1} \cos\left(\frac{1}{2}\pi\gamma\right) - \lambda_3 a_1^2 - \lambda_4 a_2^2 + \frac{1}{4}\Gamma_2 a_1^2 \cos\delta = 0, \tag{38}$$

where $\delta = 2(\varphi_2 - \varphi_1)$ is the phase difference, and a dot denotes differentiation with respect to T_2.

Multiplying (36) by the value $\Gamma_1 \Gamma_2^{-1}$ and then adding it to (35) yield

$$\dot{E} + \alpha E, \quad \alpha = \mu \omega_0^{\gamma-1} \sin\left(\frac{1}{2}\pi\gamma\right), \tag{39}$$

where $E = a_1^2 + \Gamma_1 \Gamma_2^{-1} a_2^2$ is the energy of the system.

Integrating (39) yields

$$E = E_0 \exp(-\alpha T_2), \tag{40}$$

where E_0 is the initial magnitude of the system's energy.

Equation (40) shows that owing to the fractional parameter γ dissipation of the system's energy depends on the natural frequency of vibrations. When $\gamma \to 1$, the damping value α tends to the viscosity coefficient μ, and from (40) it follows that

$$E = E_0 \exp(-\mu T_2) = 0. \tag{41}$$

Reference to (41) shows that in the case $\gamma = 1$ the damping coefficient is independent of the natural frequency ω_0, what is in conflict with the experimental data.

The cases of different fractional parameters and force driven vibrations of suspension bridges have been considered in [26, 30]

2.3 Correlation with Experiment

The experimental data obtained by Abdel-Ghaffar and Housner [1] and Abdel-Ghaffar and Scanlan [4] during ambient vibration studies of the Vincent-Thomas Suspension Bridge in Los Angeles and the Golden Gate Bridge in San Francisco, respectively, show that each natural mode of vibration has its own damping ratio, which decreases as the natural frequency increases (see Table 2 in [1] and Tables 2–5 in [4]).

As it is evident from (39), the damping coefficient α satisfies the enumerated properties. Moreover, using the tables of the papers cited above, the damping parameters μ and γ of the system can be selected so that all pairs of the magnitudes of α and ω_0 taken from the tables would be connected by the dependence (39).

Thus, summarizing the data for the Golden Gate Bridge presented in Tables 2–5 [4] for symmetric and antisymmetric vertical and torsional modes in one table, arranging the pairs of the magnitudes of α and ω_0 in order of increasing natural frequency ω_0, and then using two pairs of extreme values $\alpha_1^{aver} = 0.085\ \text{s}^{-1}$, $\omega_0 = 0.1221$ rad/s and $\alpha_m^{aver} = 0.007\ \text{s}^{-1}$, $\omega_{0n} = 1.6855$ rad/s from the pooled table, the magnitudes of γ and μ could be determined ($n = 37$; α^{aver} is the average value of α corresponding to the natural frequency ω_0). As a result we obtain

$$\gamma = 1 + \frac{\ln(\alpha_1 \alpha_n^{-1})}{\ln(\omega_{01} \omega_{0n}^{-1})} = 0.05; \quad \mu = \frac{\alpha_1 \omega_{01}^{1-\gamma}}{\sin(\pi\gamma/2)} = 0.15. \tag{42}$$

Figures 2 and 3 show the dispersion of the experimental values of the magnitude α, which varies in some interval at each fixed value of the natural frequency ω_0 and Ω_0 (the extreme points of each of such an interval are indicated in Figs. 2 and 3 for vertical and torsional modes of vibrations). The ω_0-dependence of α^{aver} (this curve is based on the data from [4]) and the ω_0-dependence of α (this curve is calculated with the use of (39) at $\gamma = 0.05$ and $\mu = 0.15$) are also presented in Figs. 2 and 3. It is seen that the theoretical curve of the ω_0-dependence of α and the curve of the ω_0-dependence of α^{aver} differ little from each other, and the experimental points are arranged within a rather narrow band containing these curves. Thus, the selected values of γ and μ may be considered as the parameters of the suspension bridge under discussion.

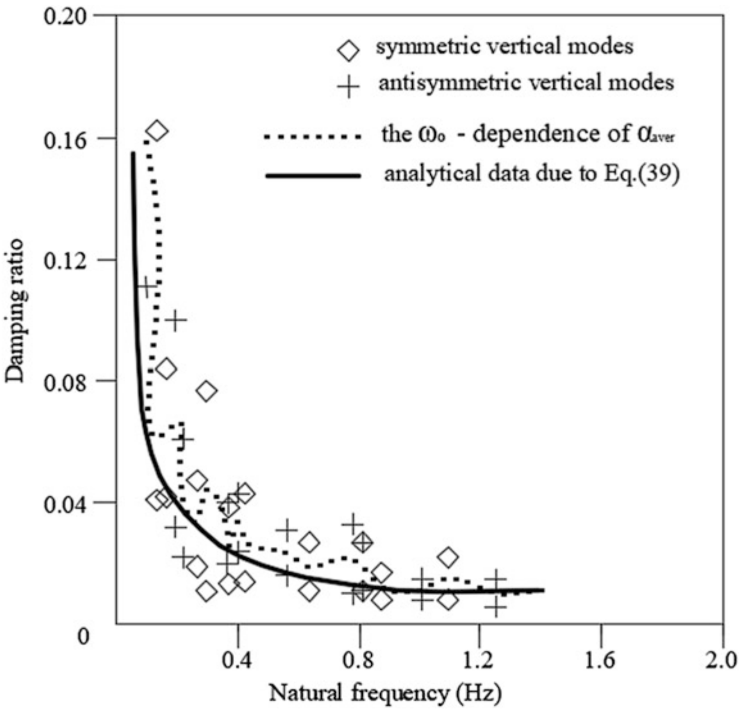

Fig. 2 Comparison of analytical and experimental results for vertical modes of the Golden Gate Suspension Bridge

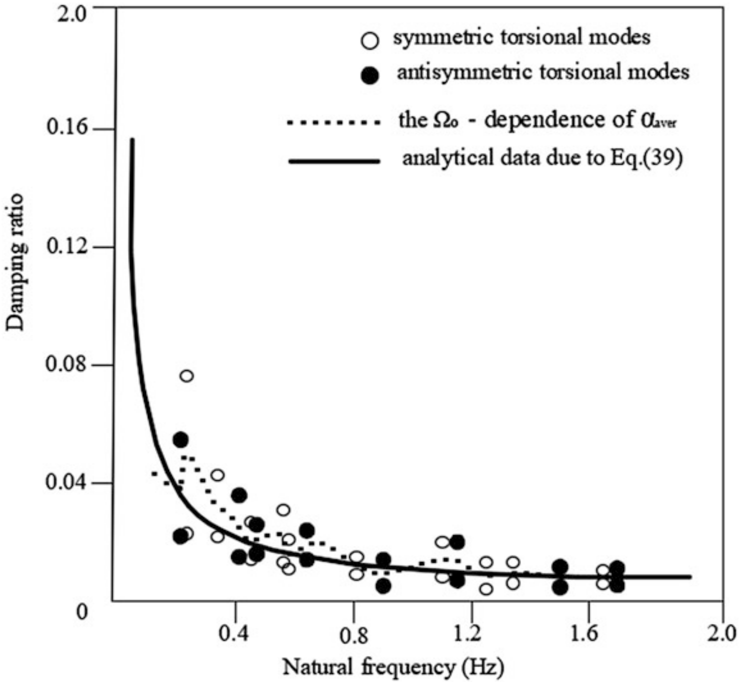

Fig. 3 Comparison of analytical and experimental results for torsional modes of the Golden Gate Suspension Bridge

Distinct to the traditional modeling the viscous resistance forces via first order time-derivatives [8], in the present research we adopt the fractional order time-derivatives D_+^γ, what allowed us to obtain the damping coefficients dependent on the natural frequency of vibrations. It has been demonstrated in [22, 26] that such an approach for modeling the damped non-linear vibrations of thin bodies provides the good agreement between the theoretical results and the experimental data through the appropriate choice of the fractional parameter (the order of the fractional derivative) and the viscosity coefficient.

3 Conclusion

It has been shown the necessity to formulate the modified version of the method of multiple scales, which could be called, following Professor Nayfeh, *the fractional derivative expansion method*. It is the efficient method for studying the nonlinear dynamic response of structures damping features of which depend on natural frequencies of vibrations.

For the first time it has been suggested by Rossikhin and Shitikova in 1997 to show that the damping features of a suspension combined system are adequately described by fractional derivatives. In so doing the fractional parameter (the order of the fractional derivative) fulfills the role of the structural parameter of the whole system and influences the character of the system's damping coefficient as a function of natural frequencies of linear vibrations. The obtained power relationship with a negative exponent between the damping coefficient of the system and its natural frequencies of linear vibrations correlates well with the experimental data describing the natural frequency dependence of the damping ratio. When the fractional parameter tends to one, i.e., when the fractional derivative transforms into the common derivative with respect to time, the system's damping coefficient does not depend on the natural frequencies of linear vibrations, which is in contradiction with the experimental data. Thus, the nonlinear viscoelastic models with fractional derivatives with respect to time are more preferred over the models with integral derivatives for describing damping features of a suspension combined system.

Further this method was applied for the analysis of different internal and combinational resonances in a two-degree-of-freedom mechanical system [23], thin plates [24, 25, 32, 35, 36, 38, 41] and shells [31, 33, 37, 40].

Ten years ago, in May 2008, when together with Professor Rossikhin I was at Virginia Tech attending the Mechanics Conference to Celebrate the 100th Anniversary of the Department of Engineering Science and Mechanics, we had a very fruitful discussion with Professor Nayfeh. He said us that until the appearance of our 1998 paper [22], he could not see the possibility of incorporation of fractional calculus in perturbation technique, and in particular within the method of multiple scales.

And then Professor Nayfeh suddenly to us gave us very profound advise: to pose the fractional derivative expansion technique on examples of linear and nonlinear oscillators arguing that it would be didactic material for students and engineers who are not familiar with such 'exotic' field of Mathematics as Fractional Calculus.

We had realized Professor Nayfeh suggestion and published two papers [27, 34], wherein free and forced vibrations of different types of linear and nonlinear fractional oscillators have been treated using the modified method of multiple time scales. I am very grateful to Professor Nayfeh for this advise, since these two papers together with two state-of-the-art articles [20, 28] are the inestimable basis of the lecture course on "Fractional calculus in mechanics".

Professor Nayfeh had the vision of supporting Fractional Calculus in Nonlinear Mechanics via approving as an Editor-in-Chief several special issues of *Nonlinear Dynamics* and *Journal of Vibration and Control*.

Acknowledgements Studies of the research team headed by Professor Yury Rossikhin in the field of fractional calculus application in mechanics of solids and structures have been supported by the Ministry of Science and Education of the Russian Federation, including the current Project No. 9.5138.2017/8.9.

Appendix

Following [27], formula (33) could be obtained from the Mellin-Fourier formula for the function $D_{0+}^{\gamma} e^{i\omega t}$. Really,

$$D_{0+}^{\gamma} e^{i\omega t} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{p^{\gamma} e^{pt}}{p - i\omega} dp. \tag{43}$$

To calculate the integral in the right-hand side of Eq. (43), we use the contour of integration L shown in Fig. 4.

Applying the theorem of theory of residues for the integral over the contour L, writing the contour integral in terms of the sum of integrals along the vertical segment of the straight line, along the arcs of the circumferences with radii C_R and C_ρ, and along the branches of the cut of the negative real semi-axis, then tending R to ∞ and ρ to 0, and considering Jordan lemma, we arrive at the relationship (33).

Changing in (33) $i\omega$ with $-i\omega$ yields

$$D_{0+}^{\gamma} e^{-i\omega t} = (-i\omega)^{\gamma} e^{-i\omega t} + \frac{\sin \pi\gamma}{\pi} \int_0^{\infty} \frac{u^{\gamma} e^{-ut} du}{u - i\omega}. \tag{44}$$

Adding Eqs. (33) and (44), we obtain

$$D_{0+}^{\gamma} (\cos \omega t) = \omega^{\gamma} \cos \left(\omega t + \frac{\pi}{2} \gamma \right) + \frac{\sin \pi\gamma}{\pi} \int_0^{\infty} \frac{u^{1+\gamma} e^{-ut} du}{u^2 + \omega^2}, \tag{45}$$

while subtracting Eq. (44) from (33) we have

$$D_{0+}^{\gamma} (\sin \omega t) = \omega^{\gamma} \sin \left(\omega t + \frac{\pi}{2} \gamma \right) - \frac{\sin \pi\gamma}{\pi} \omega \int_0^{\infty} \frac{u^{\gamma} e^{-ut} du}{u^2 + \omega^2}. \tag{46}$$

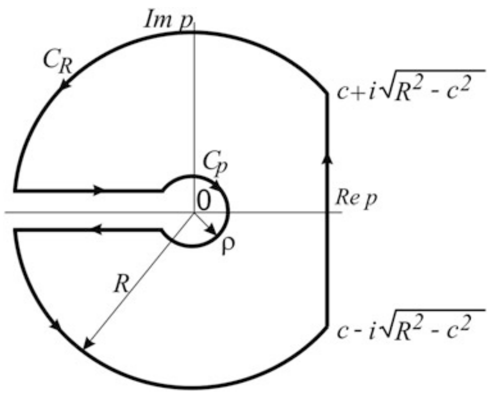

Fig. 4 Contour of integration

When $\gamma = 1$, these formulas go over into the corresponding formulas of conventional differentiation.

The derived Eqs. (33) and (44)–(46) are more preferred in engineering applications than those presented in Table 9.1 of [39], or in tables for semiderivatives in Chapter 7 [15], or in tables in [16], since allow one to estimate the accuracy of approximate solutions like those constructed in the given paper.

Reference to Eqs. (33) and (44)–(46) shows that it is possible to ignore the improper integrals in the following cases: (1) if γ differs a little from unit, (2) when the magnitude of the fractional parameter γ is rather small, (3) for rather large frequencies ω. The second terms in Eqs. (33) and (44)–(46) could be also neglected from some instants of time after beginning of vibratory motion.

It should be emphasized that in some engineering problems a combination of the conditions mentioned above can occur at a time, what allows one to use Eq. (31). Another very important case, what was discussed above, is dealing with the method of solution, as it takes place with the generalized method of multiple time scales proposed in [22], which allows one to obtain the valid results adopting approximate formulas [29].

Note that the formulas similar to Eqs. (33) and (44)–(46) were derived by Caputo [6] using the Gerasimov-Caputo fractional derivative representation. The estimates of improper integrals useful for engineering applications were suggested in [6] as well.

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Boundary Value Problem with Integral Condition for the Mixed Type Equation with a Singular Coefficient

Natalya Vladimirovna Zaitseva

Abstract We study the boundary value problem for the mixed type equation with a singular coefficient and nonlocal integral first-kind condition. We establish the uniqueness criterion and prove the solution existence and stability theorems. The solution of the problem is constructed explicitly and the proof of convergence of the series in the class of regular solutions is derived.

Keywords Mixed type equation · Singular coefficient · Nonlocal integral condition · Uniqueness · Existence · Stability · Fourier–Bessel series

MSC2010 35M12

1 Introduction

Let $D = \{(x, y) | \, 0 < x < l, -\alpha < y < \beta\}$ be a rectangular domain of coordinate plane Oxy, where l, α, β are given positive real numbers. We introduce denotation: $D_+ = D \cap \{y > 0\}$ and $D_- = D \cap \{y < 0\}$.

In the domain D we consider the elliptic-hyperbolic equation

$$Lu \equiv u_{xx} + (\operatorname{sgn} y)u_{yy} + \frac{p}{x}u_x = 0, \tag{1}$$

where $p \geq 1$ is a given positive real number.

Boundary value problems for mixed type equations are one of the most important topics of the modern theory of partial differential equations. Mathematical models of heat transfer in capillary-porous media, formation of a temperature field, movement of a viscous fluid and many others leads to the problems for equations of this type.

N. V. Zaitseva (✉)
N.I. Lobachevskii Institute of Mathematics and Mechanics, Kazan (Volga Region) Federal University, Kazan, Tatarstan, Russia
e-mail: Natalya.Chepanova@kpfu.ru

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_30

Interest in the degenerate equations is caused not only by the need to solve applied problems, but also by the intense development of the theory of mixed type equations. The first boundary value problem for degenerate partial differential equations of elliptic type with variable coefficients was initially studied in [1]. The research of equations which contains the Bessel differential operator holds a special place in this theory. The study of this class of equations was begun by Euler, Poisson, Darboux and was continued in the theory of generalized axisymmetric potential [1–4]. The equations of the three main classes containing the Bessel operator, according to the [5], are called B-elliptic, B-hyperbolic and B-parabolic, respectively. The boundary value problems for parabolic equations with the Bessel operator are studied in [6, 7], a rather complete review of the papers, devoted to boundary value problems for elliptic equations with singular coefficients is given in monograph [8]. An extensive study of B-hyperbolic equations is presented in [9]. The papers [10–16] are also devoted to the study of boundary value problems for singular equations.

In this paper we study the following nonlocal problem with first-kind integral condition when $p \geq 1$ for Eq. (1) in the domain D.

Statement of the Problem Let $p \geq 1$. We need to find function $u(x, y)$ which satisfies the following conditions:

$$u(x, y) \in C^1(\overline{D}) \cap C^2(D_+ \cup D_-), \tag{2}$$

$$Lu(x, y) \equiv 0, \quad (x, y) \in D_+ \cup D_-, \tag{3}$$

$$u(x, \beta) = \varphi(x), \quad u(x, -\alpha) = \psi(x), \quad 0 \leq x \leq l, \tag{4}$$

$$\int_0^l x^p u(x, y)\, dx = A = \text{const}, \quad -\alpha \leq y \leq \beta, \tag{5}$$

where A is a given real number, $\varphi(x)$, $\psi(x)$ are given smooth enough functions, which satisfy conditions

$$\int_0^l x^p \varphi(x)\, dx = \int_0^l x^p \psi(x)\, dx = A. \tag{6}$$

The boundary value problem (2)–(6) has nonlocal boundary conditions on the sides of the rectangle D. When $p \geq 1$ in the domain of ellipticity D_+ of Eq. (1), due to [1], the segment $x = 0$ is free of boundary condition in the class of bounded solutions. By dividing the variables it is easy to show that in the domain of hyperbollicity D_- of the Eq. (1) there is valid equation

$$u_x(0, y) = 0, \quad -\alpha \leq y \leq \beta. \tag{7}$$

Nonlocal problems for different classes of differential equations are studied in the works [17–24]. The integral condition (5) was introduced in [25] for the heat equation. The boundary value problems with (5)-type integral condition have been studied in [26–28].

2 Uniqueness

Let's represent the solution (1) as

$$x^{-p}\frac{\partial}{\partial x}\left(x^{p}\frac{\partial u}{\partial x}\right)+(\operatorname{sgn} y)u_{yy}=0.$$

Let's multiply it by x^p and integrate it over the x variable with fixed $y \in (-\alpha, 0) \cup (0, \beta)$ on interval from ε to $l-\varepsilon$, where $\varepsilon > 0$ is a number small enough. As a result we will get

$$\int_{\varepsilon}^{l-\varepsilon}\frac{\partial}{\partial x}\left(x^{p}\frac{\partial u}{\partial x}\right)dx+(\operatorname{sgn} y)\int_{\varepsilon}^{l-\varepsilon}x^{p}u_{yy}\,dx,$$

or

$$\left(x^{p}\frac{\partial u}{\partial x}\right)\Big|_{\varepsilon}^{l-\varepsilon}+(\operatorname{sgn} y)\frac{d^2}{dy^2}\int_{\varepsilon}^{l-\varepsilon}x^{p}u(x,y)\,dx=0.$$

At $\varepsilon \to 0$, due to the conditions (2) and (5) we will get the local boundary condition

$$u_x(l,y)=0,\quad -\alpha\le y\le\beta. \tag{8}$$

In what follows we will consider the problem (2)–(4), (8) instead of (2)–(6).

We will look for particular solutions of the Eq. (1) which are not equal to zero in the domain $D_+ \cup D_-$ and which satisfy the conditions (2) and (8) in the form $u(x, y) = X(x)Y(y)$. By substituting this product into the Eq. (1) and the condition (8), we will get the following spectral problem with respect to $X(x)$

$$X''(x)+\frac{p}{x}X'(x)+\lambda^2X(x)=0,\quad 0<x<l, \tag{9}$$

$$|X(0)|<+\infty,\quad X'(l)=0, \tag{10}$$

where λ^2 is a separation constant.

The general solution of Eq. (9) has the form

$$\widetilde{X}(x) = C_1 x^{\frac{1-p}{2}} J_{\frac{p-1}{2}}(\lambda x) + C_2 x^{\frac{1-p}{2}} Y_{\frac{p-1}{2}}(\lambda x),$$

where $J_\nu(\xi)$, $Y_\nu(\xi)$ are the first-kind and second-kind Bessel functions respectively, $\nu = (p-1)/2$, C_1, C_2 are arbitrary constants.

We put $C_2 = 0$ so the function satisfies the first condition from (10). Since the eigenfunctions of the spectral problem are determined to within a constant factor, we set $C_1 = 1$. Thus, the solution of the Eq. (9), which satisfies the first condition from (10), has the form

$$\widetilde{X}(x) = x^{\frac{1-p}{2}} J_{\frac{p-1}{2}}(\lambda x).$$

Let's note that this function satisfies the condition (7). By substituting the function $\widetilde{X}(x)$ into the second condition from (10) we will get

$$\lambda_0 = 0,$$

$$\widetilde{X}'(l) = \left(x^{\frac{1-p}{2}} J_{\frac{p-1}{2}}(\lambda x)\right)'_x \Big|_{x=l} = -l^{\frac{1-p}{2}} J_{\frac{p+1}{2}}(\lambda l),$$

and now we can obtain

$$J_{\frac{p+1}{2}}(\mu) = 0, \quad \mu = \lambda l. \tag{11}$$

It is known [29, p. 530] that function $J_\nu(\xi)$ with $\nu > -1$ has a countable set of real zeros. We denote the n–th root of the (11) equation by μ_n with given p and find the eigenvalues $\lambda_n = \mu_n / l$ of the problem (9) and (10). According to [30, p. 317] there is valid assimptotic formula for the zeros of the Eq. (11) when n is big enough

$$\mu_n = \lambda_n l = \pi n + \frac{\pi}{4} p + O\left(\frac{1}{n}\right). \tag{12}$$

Let's note that when $\lambda_0 = 0$ the spectral problem (9) and (10) has constant eigenfunction which we will take as one. Thus, the system of eigenfunctions of the problem (9) and (10) has the form

$$\widetilde{X}_0(x) = 1, \quad \lambda_0 = 0, \tag{13}$$

$$\widetilde{X}_n(x) = x^{\frac{1-p}{2}} J_{\frac{p-1}{2}}\left(\frac{\mu_n x}{l}\right) = x^{\frac{1-p}{2}} J_{\frac{p-1}{2}}(\lambda_n x), \quad n \in \mathbb{N}, \tag{14}$$

where eigenvalues λ_n are determined as zeros of the Eq. (11).

Let's note that the system of eigenfunctions (13) and (14) of the problem (9) and (10) is orthogonal in the space $L_2[0, l]$ with a weight x^p and also forms a complete system in this space [31, p. 343].

For further calculations we will use an orthonormal system of functions:

$$X_n(x) = \frac{1}{||\widetilde{X}_n(x)||}\widetilde{X}_n(x), \quad n = 0, 1, 2, \dots, \tag{15}$$

where

$$||\widetilde{X}_n(x)||^2 = \int_0^l \rho(x)\, \widetilde{X}_n^2(x)\, dx, \quad \rho(x) = x^p. \tag{16}$$

Let $u(x, y)$ be a solution of the problem (2)–(4), (8). Let's introduce the functions

$$u_n(y) = \int_0^l u(x, y)x^p X_n(x)\, dx, \;\; n = 0, 1, 2, \dots, \tag{17}$$

based on which we consider an auxiliary functions of the form

$$u_{n,\varepsilon}(y) = \int_\varepsilon^{l-\varepsilon} u(x, y)x^p X_n(x)\, dx, \;\; n = 1, 2, \dots, \tag{18}$$

where $\varepsilon > 0$ is a number small enough. Let's differentiate the Eq. (18) over the y variable twice with $y \in (-\alpha, 0) \cup (0, \beta)$ and with respect to Eq. (1), we will get the equation

$$u''_{n,\varepsilon}(y) = \int_\varepsilon^{l-\varepsilon} u_{yy}(x, y)x^p X_n(x)\, dx = -(\operatorname{sgn} y) \int_\varepsilon^{l-\varepsilon} \left(u_{xx} + \frac{p}{x}u_x\right) x^p X_n(x)\, dx =$$

$$= -(\operatorname{sgn} y) \int_\varepsilon^{l-\varepsilon} \frac{\partial}{\partial x}(x^p u_x)X_n(x)\, dx = -(\operatorname{sgn} y)\left[x^p u_x X_n(x)\Big|_\varepsilon^{l-\varepsilon} - \int_\varepsilon^{l-\varepsilon} x^p u_x X'_n(x)\, dx \right]. \tag{19}$$

From (18), due to Eq. (9), we can obtain

$$u_{n,\varepsilon}(y) = -\frac{1}{\lambda_n^2}\int\limits_{\varepsilon}^{l-\varepsilon} u(x,y)x^p\left[X_n''(x) + \frac{p}{x}X_n'(x)\right]dx =$$

$$= -\frac{1}{\lambda_n^2}\int\limits_{\varepsilon}^{l-\varepsilon} u(x,y)\frac{d}{dx}\left(x^p X_n'(x)\right)dx = -\frac{1}{\lambda_n^2}\left[u(x,y)x^p X_n'(x)\Big|_{\varepsilon}^{l-\varepsilon} - \int\limits_{\varepsilon}^{l-\varepsilon} x^p u_x X_n'(x)\,dx\right],$$

and, thus,

$$\int\limits_{\varepsilon}^{l-\varepsilon} x^p u_x X_n'(x)\,dx = \lambda_n^2 u_{n,\varepsilon}(y) + u(x,y)x^p X_n'(x)\Big|_{\varepsilon}^{l-\varepsilon}.$$

By substituting this expression into (19) we will have

$$u_{n,\varepsilon}''(y) = -(\operatorname{sgn} y)\left[x^p u_x X_n(x)\Big|_{\varepsilon}^{l-\varepsilon} - \lambda_n^2 u_{n,\varepsilon}(y) - u(x,y)x^p X_n'(x)\Big|_{\varepsilon}^{l-\varepsilon}\right].$$

By virtue of (2) in the last equation, we can pass to the limit as $\varepsilon \to 0$, from which, according to the conditions (8) and (10) we obtain the following differential equation that we will use to find the functions (17)

$$u_n''(y) - (\operatorname{sgn} y)\lambda_n^2 u_n(y) = 0, \quad y \in (-\alpha, 0) \cup (0, \beta). \tag{20}$$

It's general solution has the form

$$u_n(y) = \begin{cases} a_n e^{\lambda_n y} + b_n e^{-\lambda_n y}, & y > 0, \\ c_n \cos\lambda_n y + d_n \sin\lambda_n y, & y < 0, \end{cases} \tag{21}$$

where a_n, b_n, c_n, d_n are arbitrary constants which must be defined.

Now we will pick the constants a_n, b_n, c_n and d_n in (21) with respect to (2) such that the conjugation conditions $u_n(0+) = u_n(0-)$, $u_n'(0+) = u_n'(0-)$ are satisfied. Those conditions are satisfied when $a_n = (c_n + d_n)/2$, $b_n = (c_n - d_n)/2$, $n = 1, 2, \ldots$. By substituting the values found in (21) we will have

$$u_n(y) = \begin{cases} c_n \operatorname{ch}\lambda_n y + d_n \operatorname{sh}\lambda_n y, & y > 0, \\ c_n \cos\lambda_n y + d_n \sin\lambda_n y, & y < 0. \end{cases} \tag{22}$$

Now let's substitute (17) into the boundary conditions (4):

$$u_n(\beta) = \int_0^l \varphi(x) x^p X_n(x)\, dx = \varphi_n, \quad u_n(-\alpha) = \int_0^l \psi(x) x^p X_n(x)\, dx = \psi_n. \tag{23}$$

Based on (22) and (23) we can obtain a system for finding the constants c_n and d_n:

$$\begin{cases} c_n \mathrm{ch}\, \lambda_n \beta + d_n \mathrm{sh}\, \lambda_n \beta = \varphi_n, \\ c_n \cos \lambda_n \alpha - d_n \sin \lambda_n \alpha = \psi_n, \end{cases} \tag{24}$$

which has the unique solution

$$c_n = \frac{\varphi_n \sin \lambda_n \alpha + \psi_n \mathrm{sh}\, \lambda_n \beta}{\sin \lambda_n \alpha \, \mathrm{ch}\, \lambda_n \beta + \cos \lambda_n \alpha \, \mathrm{sh}\, \lambda_n \beta}, \quad d_n = \frac{\varphi_n \cos \lambda_n \alpha - \psi_n \mathrm{ch}\, \lambda_n \beta}{\sin \lambda_n \alpha \, \mathrm{ch}\, \lambda_n \beta + \cos \lambda_n \alpha \, \mathrm{sh}\, \lambda_n \beta}, \tag{25}$$

if for all $n \in \mathbb{N}$ the determinant of the system (24) is non-zero:

$$\triangle_n(\alpha, \beta) = \sin \lambda_n \alpha \, \mathrm{ch}\, \lambda_n \beta + \cos \lambda_n \alpha \, \mathrm{sh}\, \lambda_n \beta \neq 0. \tag{26}$$

By substituting the values we found (25) into (22) we will find the final form of the functions

$$u_n(y) = \begin{cases} \triangle_n^{-1}(\alpha, \beta) \left(\varphi_n \triangle_n(\alpha, y) + \psi_n \mathrm{sh}\, \lambda_n (\beta - y) \right), & y > 0, \\ \triangle_n^{-1}(\alpha, \beta) \left(\varphi_n \sin \lambda_n (\alpha + y) + \psi_n \triangle_n(-y, \beta) \right), & y < 0. \end{cases} \tag{27}$$

Similarly, we find

$$u_0(y) = \frac{\alpha \varphi_0 + \beta \psi_0}{\alpha + \beta} + \frac{\varphi_0 - \psi_0}{\alpha + \beta} y, \quad y \in (-\alpha, 0) \cup (0, \beta), \tag{28}$$

$$u_0(\beta) = l^{-\frac{p+1}{2}} \sqrt{p+1} \int_0^l \varphi(x) x^p dx = \varphi_0, \quad u_0(-\alpha) = l^{-\frac{p+1}{2}} \sqrt{p+1} \int_0^l \psi(x) x^p dx = \psi_0. \tag{29}$$

When the condition (26) is satisfied, the problem (2)–(4), (8) has the unique solution. Indeed, let $\varphi(x) = \psi(x) \equiv 0$ and $\triangle_n(\alpha, \beta) \neq 0$. Then it follows from (23) and (29) that $\varphi_n = \psi_n \equiv 0$, $n = 0, 1, 2, \ldots$, and it follows from (27) and (28) that $u_n(y) = 0$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Due to (17) we have

$\int\limits_0^l u(x,y)x^p X_n(x)\,dx = 0$. Hence, as the system (15) is complete in the space $L_2[0,l]$ with weight x^p, $u(x,y) = 0$ almost everywhere on the interval $x \in [0,l]$ and for all $y \in [-\alpha, \beta]$. As according to (2) function $u(x,y) \in C(\overline{D})$, then $u(x,y) \equiv 0$ in $\overline{D}$.

Let's suppose that for some values p, l, α, β and some $n = m$ the condition (26) is not satisfied. When $\varphi(x) = \psi(x) \equiv 0$ and $\triangle_m(\alpha,\beta) = 0$ the system (24) is equivalent to one of the equations (let it be the first one)

$$c_m \operatorname{ch} \lambda_m \beta + d_m \operatorname{sh} \lambda_m \beta = 0,$$

which has an infinite set of solutions $\left\{-d_m \frac{\operatorname{sh} \lambda_m \beta}{\operatorname{ch} \lambda_m \beta},\ d_m\right\}$. By substituting the values we found into (22) we get

$$u_m(y) = \begin{cases} \widetilde{d_m}\left(\operatorname{sh} \lambda_m y \operatorname{ch} \lambda_m \beta - \operatorname{sh} \lambda_m \beta \operatorname{ch} \lambda_m y\right), & y \geq 0, \\ \widetilde{d_m}\left(\operatorname{ch} \lambda_m \beta\ \sin \lambda_m y - \operatorname{sh} \lambda_m \beta\ \cos \lambda_m y\right), & y \leq 0, \end{cases}$$

where $\widetilde{d_m}$ is an arbitrary non-zero constant.

Thus the homogenous problem (2)–(4), (8) has the non-zero solution

$$u_m(x,y) = \begin{cases} \widetilde{d_m}\left(\operatorname{sh} \lambda_m y \operatorname{ch} \lambda_m \beta - \operatorname{sh} \lambda_m \beta \operatorname{ch} \lambda_m y\right) X_m(x), & y \geq 0, \\ \widetilde{d_m}\left(\operatorname{ch} \lambda_m \beta\ \sin \lambda_m y - \operatorname{sh} \lambda_m \beta\ \cos \lambda_m y\right) X_m(x), & y \leq 0, \end{cases} \tag{30}$$

where the functions $X_m(x)$ are determined by (15). It is easy to prove that the built function (30) satisfies all the conditions (2)–(4), (8) when $\varphi(x) = \psi(x) \equiv 0$.

Let's find out for which values of the parameters p, l, α, β the condition (26) is violated. We represent $\triangle_n(\alpha,\beta)$ as

$$\triangle_n(\alpha,\beta) = \sqrt{\operatorname{ch} 2\lambda_n \beta}\ \sin(\mu_n \widetilde{\alpha} + \gamma_n), \tag{31}$$

where $\mu_n = \lambda_n l$, $\widetilde{\alpha} = \alpha / l$, $\gamma_n = \arcsin \dfrac{\operatorname{sh} \lambda_n \beta}{\sqrt{\operatorname{ch} 2\lambda_n \beta}} \to \dfrac{\pi}{4}$ at $n \to +\infty$.

This representation shows that $\triangle_n(\alpha,\beta) = 0$, if $\sin(\mu_n \widetilde{\alpha} + \gamma_n) = 0$, that is, if

$$\widetilde{\alpha} = \frac{\pi k - \gamma_n}{\mu_n}, \quad k = 1, 2, \ldots. \tag{32}$$

Thus we proved

Theorem 1 *If the solution of the problem (2)–(4), (8) exists, then it is unique if and only if the condition (26) is satisfied for all* $n \in \mathbb{N}$.

3 Existence

As according to (31) the expression $\triangle_n(\alpha, \beta)$ has a countable set of zeros, we examine the values of this expression, included in the denominators of the formula (27) when n is big enough.

Lemma 1 *If $\widetilde{\alpha} = a/b$ is a rational number, a, b are mutually prime numbers and $p \neq \frac{1}{a}(4bd - b - 4r)$, $r = \overline{1, b-1}$, $d \in \mathbb{Z}$, then there exists constants $C_0 > 0$, $n_0 \in \mathbb{N}$ such that for all $n > n_0$ there is valid inequality*

$$|\triangle_n(\alpha, \beta)| \geq C_0 e^{\lambda_n \beta}. \tag{33}$$

Proof Let's substitute (14) into (31):

$$\triangle_n(\alpha, \beta) = \sqrt{\operatorname{ch} 2\lambda_n \beta}\, \sin\left(\pi n \widetilde{\alpha} + \frac{\pi}{4} p \widetilde{\alpha} + \gamma_n + O\left(\frac{1}{n}\right)\right).$$

Let $\widetilde{\alpha} = a/b$, $a, b \in \mathbb{N}$, $(a, b) = 1$. Let's divide na by b. According to the division theorem we have

$$na = bq + r, \quad q \in \mathbb{N}_0, \quad 1 \leq r \leq b - 1.$$

Then

$$\triangle_n(\alpha, \beta) = \sqrt{\operatorname{ch} 2\lambda_n \beta}\, (-1)^q \sin\left(\frac{\pi r}{b} + \frac{\pi a}{4b} p + \gamma_n + O\left(\frac{1}{n}\right)\right) =$$

$$= \frac{e^{\lambda_n \beta}}{\sqrt{2}} \sqrt{1 + \operatorname{ch} - 4\lambda_n \beta}\, (-1)^q \sin\left(\frac{\pi r}{b} + \frac{\pi a}{4b} p + \frac{\pi}{4} - \varepsilon_n + O\left(\frac{1}{n}\right)\right),$$

where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ at $n \to +\infty$. Thus there is a number n_0, such that for any $n > n_0$ there is valid inequality

$$|\triangle_n(\alpha, \beta)| \geq \frac{e^{\lambda_n \beta}}{2\sqrt{2}} \left|\sin\left(\frac{\pi r}{b} + \frac{\pi a}{4b} p + \frac{\pi}{4}\right)\right| = C_0 e^{\lambda_n \beta}.$$

In order to get $C_0 > 0$ it is necessary that

$$\frac{\pi r}{b} + \frac{\pi a}{4b} p + \frac{\pi}{4} \neq \pi d, \quad d \in \mathbb{Z},$$

hence

$$p \neq \frac{1}{a}(4bd - b - 4r), \quad d \in \mathbb{Z}. \tag{34}$$

The condition (34) is satisfied for any irrational value $p \geq 1$. □

Lemma 2 *If for $n > n_0$ the condition (33) is satisfied, then there are valid estimates*

$$|u_n(y)| \le C_1(|\varphi_n| + |\psi_n|), \quad y \in [-\alpha, \beta], \tag{35}$$

$$|u_n'(y)| \le C_2 n(|\varphi_n| + |\psi_n|), \quad y \in [-\alpha, \beta], \tag{36}$$

$$|u_n''(y)| \le C_3 n^2(|\varphi_n| + |\psi_n|), \quad y \in [-\alpha, 0), \tag{37}$$

$$|u_n''(y)| \le C_4 n^2(|\varphi_n| + |\psi_n|), \quad y \in (0, \beta], \tag{38}$$

where C_i are positive constants (here and further).

Proof From formula (27) with respect to (33) we can get

$$|u_n(y)| \le \frac{1}{|\triangle_n(\alpha, \beta)|}\,(|\varphi_n|(\operatorname{sh}\lambda_n\beta + \operatorname{ch}\lambda_n\beta) + |\psi_n|\operatorname{sh}\lambda_n\beta) \le$$

$$\le \frac{1}{C_0 e^{\lambda_n\beta}}\,(|\varphi_n|(\operatorname{sh}\lambda_n\beta + \operatorname{ch}\lambda_n\beta) + |\psi_n|\operatorname{sh}\lambda_n\beta) \le \widetilde{C_1}(|\varphi_n| + |\psi_n|), \quad y \ge 0,$$

$$|u_n(y)| \le \frac{1}{C_0 e^{\lambda_n\beta}}\,(|\varphi_n| + |\psi_n|(\operatorname{sh}\lambda_n\beta + \operatorname{ch}\lambda_n\beta)) \le \widetilde{C_2}(|\varphi_n| + |\psi_n|), \quad y \le 0,$$

where $\widetilde{C_i}$ are positive constants (here and further). By denoting $C_1 = \max\{\widetilde{C_1}, \widetilde{C_2}\}$ we get the estimate (35) for all $n > n_0$ and $y \in [-\alpha, \beta]$.

Let's calculate the derivative $u_n'(y)$ based on (27) and with respect to (33) and formula (12):

$$|u_n'(y)| \le \frac{n}{C_0 e^{\lambda_n\beta}}\,(|\varphi_n|(\operatorname{ch}\lambda_n\beta + \operatorname{sh}\lambda_n\beta) - |\psi_n|\operatorname{ch}\lambda_n\beta) \le \widetilde{C_3} n(|\varphi_n| + |\psi_n|), \quad y \ge 0,$$

$$|u_n'(y)| \le \frac{n}{C_0 e^{\lambda_n\beta}}\,(|\varphi_n| - |\psi_n|(\operatorname{sh}\lambda_n\beta + \operatorname{ch}\lambda_n\beta)) \le \widetilde{C_4} n(|\varphi_n| + |\psi_n|), \quad y \le 0.$$

Form those inequalities we can obtain the estimate (36) for all $n > n_0$ and $y \in [-\alpha, \beta]$, where $C_2 = \max\{\widetilde{C_3}, \widetilde{C_4}\}$.

The validity of the estimates (37) and (38) follows from the equalities (12), (20) and the estimate (35). □

Lemma 3 *For n big enough and for all $x \in [0, l]$ there are valid estimates:*

$$|X_n(x)| \le C_5, \quad |X_n'(x)| \le C_6 n, \quad |X_n''(x)| \le C_7 n^2.$$

Proof of this lemma can be found in [32].

Lemma 4 *If functions* $\varphi(x), \psi(x) \in C^2[0, l]$ *and there exists the derivatives* $\varphi'''(x)$*,* $\psi'''(x)$ *which has finite variation on* $[0, l]$*, and*

$$\varphi'(0) = \varphi''(0) = \psi'(0) = \psi''(0) = \varphi'(l) = \psi'(l) = 0,$$

then there are valid estimates:

$$|\varphi_n| \le C_8/n^4, \quad |\psi_n| \le C_9/n^4.$$

Proof of this lemma can be found in [32].

Based on the found particular solutions (15), (27) and (28), if the conditions (26) and (33) are satisfied, the solution of the problem (2)–(4), (8) is defined as a Fourier–Bessel series

$$u(x, y) = u_0(y)X_0(x) + \sum_{n=1}^{\infty} u_n(y)X_n(x). \tag{39}$$

We will consider the following series together with the series (39):

$$u_y(x, y) = u_0'(y)X_0(x) + \sum_{n=1}^{\infty} u_n'(y)X_n(x), \quad u_x(x, y) = \sum_{n=1}^{\infty} u_n(t)X_n'(x); \tag{40}$$

$$u_{yy}(x, y) = \sum_{n=1}^{\infty} u_n''(y)X_n(x), \quad u_{xx}(x, y) = \sum_{n=1}^{\infty} u_n(y)X_n''(x). \tag{41}$$

According to Lemmas 2 and 3, for any $(x, y) \in \overline{D}$ the series (39) and (40) are majorized, correspondingly, by the series $C_{10} \sum_{n=1}^{\infty} (|\varphi_n| + |\psi_n|)$, $C_{11} \sum_{n=1}^{\infty} n\,(|\varphi_n| + |\psi_n|)$, and the series (41) for any $(x, y) \in \overline{D_+} \cup \overline{D_-}$ are majorized by the series $C_{12} \sum_{n=1}^{\infty} n^2\,(|\varphi_n| + |\psi_n|)$, which, in turn, according to Lemma 4, are estimates by the number series $C_{13} \sum_{n=1}^{\infty} n^{-2}$. Consequently, by virtue of Weierstrass M-test, the series (39) and (40) converges uniformly in the bounded domain $\overline{D}$ and the series (41) converges uniformly in the bounded domains $\overline{D_+}$ and $\overline{D_-}$. Thus we have built the function $u(x, y)$ which is defined by the series (39) and satisfies all the (2)–(4), (8) problem conditions.

If for numbers $\widetilde{\alpha}$ in Lemma 1, for some natural $n = m = m_1, \ldots, m_k$, where $1 \le m_1 < \ldots < m_k \le n_0$, $k \in \mathbb{N}$, there is $\Delta_m(\alpha, \beta) = 0$ satisfied, then for the solvability of the problem (2)–(4), (8) it is necessary and sufficient to fulfill the conditions

$$\psi_m \operatorname{ch} \lambda_m \beta - \varphi_m \cos \lambda_m \alpha = 0, \quad m = m_1, \ldots, m_k. \tag{42}$$

In this case, the solution of the problem (2)–(4), (8) is determined by the series

$$u(x, y) = \left(\sum_{n=1}^{m_1-1} + \cdots + \sum_{n=m_{k-1}+1}^{m_k-1} + \sum_{n=m_k+1}^{\infty} \right) u_n(y) X_n(x) + \sum_{n=1} u_m(x, y), \tag{43}$$

where m takes the values $m_1, \ldots, m_k$, and the function $u_m(x, y)$ is determined by the formula (30). If the lower limit is greater than the upper limit in some sums, then these sums should be considered equal to zero.

Thus, we proved

Theorem 2 *Let functions $\varphi(x)$ and $\psi(x)$ satisfy the Lemma 4 conditions and the condition (33) is satisfied for $n > n_0$. Then there exists the unique solution $u(x, y)$ of the problem (2)–(4), (8) determined by the series (39), if $\triangle_n(\alpha, \beta) \neq 0$ for all $n = \overline{1, n_0}$; if $\triangle_m(\alpha, \beta) = 0$ with some $m = m_1, \ldots, m_k \leq n_0$, the problem has a solution determined by (43), if and only if the conditions (42) are satisfied.*

Theorem 3 *Let functions $\varphi(x)$ and $\psi(x)$ satisfy the Lemma 4 conditions and the conditions (6) and the inequality (33) is valid for all $n > n_0$. Then there exists the unique solution $u(x, y)$ of the problem (2)–(6) determined by the series (39), if $\triangle_n(\alpha, \beta) \neq 0$ for all $n = \overline{1, n_0}$; if $\triangle_m(\alpha, \beta) = 0$ with some $m = m_1, \ldots, m_k \leq n_0$, the problem has a solution determined by (43), if and only if the conditions (42) are satisfied.*

Proof Let $u(x, y)$ be a solution of the problem (2)–(4), (8) and functions $\varphi(x)$ and $\psi(x)$ satisfies the theorem conditions. Then the Eq. (1) is valid everywhere on set $D_+ \cup D_-$. Let's multiply the Eq. (1) by x^p and integrate it over the x variable with $y \in (-\alpha, 0) \cup (0, \beta)$ fixed on interval from ε to $l - \varepsilon$, where $\varepsilon > 0$ is small enough. As a result we will get

$$\left(x^p \frac{\partial u}{\partial x} \right) \Bigg|_{\varepsilon}^{l-\varepsilon} + (\operatorname{sgn} y) \int_{\varepsilon}^{l-\varepsilon} x^p u_{yy}(x, y)\, dx = 0. \tag{44}$$

By passing to the limit as $\varepsilon \to 0$ and with respect to conditions (2) and (8), we have

$$\int_0^l u_{yy}(x, t) x^p \, dx = 0.$$

By integrating the last equation over the y variable twice we have

$$\int_0^l u(x, y)x^p\,dx = K_1 y + K_2, \quad K_1,\ K_2 = \text{const.} \tag{45}$$

By putting $y = \beta$ and then $y = -\alpha$ in the Eq. (45) and with respect to the conditions (4) and (6) we get

$$\int_0^l u(x, \beta)x^p\,dx = \int_0^l \varphi(x)x^p\,dx = K_1\beta + K_2 = A,$$

$$\int_0^l u(x, -\alpha)x^p\,dx = \int_0^l \psi(x)x^p\,dx = -\alpha K_1 + K_2 = A,$$

and thus we can find the values of the constants $K_1 = 0$ and $K_2 = A$. Then from the formula (45) we have

$$\int_0^l u(x, y)x^p\,dx = A,$$

which means that the condition (5) is satisfied.

Now let $u(x, y)$ be a solution of the problem (2)–(6). Then from the Eq. (44) we can obtain

$$\left(x^p \frac{\partial u}{\partial x}\right)\Bigg|_{\varepsilon}^{l-\varepsilon} + (\operatorname{sgn} y)\frac{d^2}{dy^2}\int_{\varepsilon}^{l-\varepsilon} x^p u(x, y)\,dx = 0.$$

By passing to limit as $\varepsilon \to 0$ and according to conditions (2) and (5) we obtain the local second-kind boundary condition $u_x(l, y) = 0$.

Thus, we showed that when the conditions (6) are satisfied, the conditions (5) and (8) are equivalent. This means that the problems (2)–(6) and (2)–(4), (8) are also equivalent. $\square$

4 Stability

Theorem 4 *For the solution of the problem (2)–(6) there is valid estimate*

$$||u(x, y)|| \leq C_{14}(||\varphi(x)|| + ||\psi(x)||),$$

where $||f(x)||^2 = \int\limits_0^l \rho(x)|f(x)|^2 dx, \quad \rho(x) = x^p.$

Proof According to the formula (39) with respect to the estimate (35) we can calculate

$$||u||^2 = \int\limits_0^l x^p u^2(x, y)\, dx = \int\limits_0^l x^p \sum_{n=0}^{\infty} u_n(y) X_n(x) \sum_{m=0}^{\infty} u_m(y) X_m(x)\, dx =$$

$$= \sum_{n=0}^{\infty} u_n^2(y) = u_0^2(y) + \sum_{n=1}^{\infty} u_n^2(y) \leq C_{15}(\varphi_0^2 + \psi_0^2) + 2C_1^2 \sum_{n=1}^{\infty} \left(|\varphi_n|^2 + |\psi_n|^2\right) \leq$$

$$\leq C_{15}(\varphi_0^2 + \psi_0^2) + 2C_1^2 \left(\sum_{n=1}^{\infty} \varphi_n^2 + \sum_{n=1}^{\infty} \psi_n^2\right) = C_{14}\left(||\varphi||^2 + ||\psi||^2\right).$$

□

The author is very grateful to his colleagues from the Regional scientific and educational mathematical center of Kazan (Volga Region) Federal University Marat M. Arslanov, Viktor L. Selivanov, Marat K. Faizrahmanov.

The research was funded by project no. 0212/02.12.10179.001.

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