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Control Theory of Infinite-Dimensional Systems

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Preface

The workshop on "Control theory of infinite-dimensional systems" was held January 10–12, 2018, in Hagen and attracted from three continents more than 30 researchers who are active in the fields of control theory, operator theory, and systems theory. The program included 11 plenary talks by distinguished senior researchers and 11 contributed talks; we are particularly proud of the many young scientists from Africa and Europe who attended our meeting.

Held on the premises of the FernUniversität in Hagen, the workshop's goal was to bring together leading international experts and young scientists in the field of control theory of infinite-dimensional systems for a mutually beneficial exchange of new ideas and results. The contributions to our meeting covered a broad spectrum of topics and reflected various aspects of control theory, including well-posedness, controllability, and optimal control problems, as well as the stability of linear and nonlinear systems. Moreover, the fields of partial differential equations, semigroup theory, mathematical physics, graph and network theory, as well as numerical analysis, were addressed. By publishing this book, it is our great pleasure to collect outstanding contributions that share novel results in these highly active areas of research.

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Contents

Well-posedness and stability for interconnection structures of port-Hamiltonian type

Björn Augner

Abstract. We consider networks of infinite-dimensional port-Hamiltonian systems $\mathfrak{S}_i$ on 1D spatial domains. These subsystems of port-Hamiltonian type are interconnected via boundary control and observation and are allowed to be of distinct port-Hamiltonian orders $N_i \in \mathbb{N}$. Well-posedness and stability results for port-Hamiltonian systems of fixed order $N \in \mathbb{N}$ are thereby generalised to networks of such. The abstract theory is applied to some particular model examples.

Mathematics Subject Classification (2010). Primary: 93D15, 35B35. Secondary: 35G46, 37L15, 47B44, 47D06.

Keywords. Infinite-dimensional port-Hamiltonian systems, networks of PDE, feedback interconnection, contraction semigroups, stability analysis.

1. Introduction

A *port-based* modelling and analysis initially had been introduced in the 1960s to treat complex, multiphysics systems within a unified mathematical framework [23]. Each of these subsystems, may it be of mechanical, electrical or thermal type etc. is described by its inner dynamics, usually by a system of ODEs or PDEs, on the one hand, and *ports*, which enable the interconnection with other subsystems, on the other hand. For *port-Hamiltonian systems* the notion of an *energy* has been highlighted, similar to classical Hamiltonian systems. In contrast to the latter, however, the port-Hamiltonian formulation allows besides *conservative*, i.e. energy preserving, elements also for *dissipative*, i.e. energy dissipating, elements, e.g. frictional losses in mechanical systems or energy conversion in resistors within

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an electric circuit, where the energy leaves the system in form of heat while the latter is not included in the model.

For the description and analysis of port-Hamiltonian systems in a geometrical way, in [24] the concept of a *Dirac structure* had been introduced into the theory of port-Hamiltonian systems. These Dirac structures have the very convenient property that (suitable) interconnections of Dirac structures again give a Dirac structure (of higher dimension). The underlying models for the physical systems up to the 2000s had been primarily finite-dimensional, i.e. the inner dynamics of the subsystems interconnected via ports had usually been described by ODEs. Probably with the article [25] first attempts were made to extend the developed finite-dimensional theory of port-Hamiltonian systems to infinite-dimensional models, i.e. PDEs, and thereby filling in the gap between results on finite-dimensional systems and infinite-dimensional port-Hamiltonian systems. E.g. first in [14], it has been demonstrated that for linear infinite-dimensional port-Hamiltonian systems on an interval, i.e. evolution equations of the form

$$\frac{\partial x}{\partial t} = \sum_{k=0}^{N} P_k \frac{\partial^k (\mathcal{H} x)}{\partial \zeta^k}(t, \zeta), \quad t \geq 0, \, \zeta \in (0, l)$$

with $x(t, \cdot) \in L_2(0, l; \mathbb{K}^d)$ (where $K = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and for suitable $P_k \in \mathbb{K}^{d \times d}$ and $\mathcal{H} : [0, 1] \to \mathbb{K}^{d \times d}$, those boundary conditions (or, in a rather systems theoretic interpretation: linear closure relations) that lead to generation of a bounded (even contractive, when $L_2(0, l; \mathbb{K}^d)$ is equipped with an appropriate energy norm) C_0-semigroup can be characterised: Crucial is the dissipativity (w.r.t. the energy inner product), which can be checked solely via a matrix criterion on the boundary conditions [13], [14]. Next steps then have been sufficient conditions for asymptotic or uniform exponential stability of the system [26, 27, 5]. Then followed efforts to generalise these results to PDE–ODE-systems, i.e. feedback control via a finite-dimensional linear control system [18, 5], and non-linear boundary feedback [22, 19, 3]. Here, we want to push forward into a different direction and in a sort return to the beginnings of port-Hamiltonian modelling: What happens, if we consider a network of infinite-dimensional port-Hamiltonian subsystems instead of a single one, where the subsystems, just in the spirit of port-based modelling, are coupled via boundary control and observation of the distinct port-Hamiltonian subsystems? To what extend do the results on well-posedness (in the sense of semigroup theory) and stability extend to this network case? For special classes of PDE, especially the wave equation and several beam models, such an analysis is not new by any means, see e.g. [28, 12, 15, 8] and [17]. Also, note that the composition of boundary control systems has already been studied e.g. in [1] for coupling via static Kirchhoff laws. Concerning the notion of well-posedness note that in this manuscript always well-posedness in the sense of semigroup theory is meant. For the more restrictive notion of well-posedness also including input, output and feed-through maps, let us refer to the work of Staffans, Tucsnak, Weiss, etc., see e.g. the monograph [21] or the recent conference paper [29].

Before giving an outline of the organisation of this paper, let us emphasise that for systems with constant *Hamiltonian energy densities* $\mathcal{H}_i : [0, l_i] \to \mathbb{K}^{d_i \times d_i}$, already alternative approaches to well-posedness and stability are well-known. In particular, in that case it is often possible to determine (in an analytical way or via sufficiently good numerical approximation) the eigenvalues of the total system up to sufficient accuracy, and derive conclusions on well-posedness and stability. For non-constant $\mathcal{H}_i$ such an approach is not that easily accessible, in particular there are situations, in which stability properties of a port-Hamiltonian system are very sensitive to multiplicative perturbation by $\mathcal{H}_i$, see e.g. [9] for an astonishing counter example. Therefore, we deem the port-Hamiltonian approach as a legitimate way to describe and analyse such systems.

This manuscript is structured as follows. Section 2 serves as an introduction to the (mainly standard) notation we use throughout this paper, we recall some basic facts on (strongly continuous) semigroup theory, and the notion of a port-Hamiltonian system is introduced. In Section 3, we recall previous results on the well-posedness and stability of infinite-dimensional linear port-Hamiltonian systems on a one-dimensional domain. We do this with the background of particular interconnection schemes which have been considered up to now, and also comment on some of the techniques used to prove the corresponding results. The subsequent Sections 4, 5 and 6 constitute the main sections of this paper: First, in Section 4 we provide the general well-posedness result for multi-port Hamiltonian systems interconnected in a dissipative way: As for single port-Hamiltonian systems, a dissipative linear closure relation is already enough to have existence of unique (strong) solutions for all initial data, and the solution depends continuously on the initial datum, i.e. the initial datum to solution map is given by a strongly continuous semigroup of linear operators. Secondly, the focus lies on asymptotic and exponential stability for closed loop port-Hamiltonian systems, which we investigate in Section 5 under additional structural constraints, e.g. the port-Hamiltonian systems being serially interconnected in a chain. Then, Section 6 is devoted to systems consisting themselves of systems of port-Hamiltonian systems again which for complex structures of the total system might be a helpful point of view for stability considerations. We illustrate the results of the preceding sections by networks of first order port-Hamiltonian systems and of Euler–Bernoulli beam type. Finally, in Section 8 we rephrase the main aspects of this paper and comment on further open or related problems. After that, some technical results on the Euler–Bernoulli beam equation are collected in an appendix.

2. Preliminaries

2.1. Notation

Let us fix some notation. Throughout, the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ denotes real or complex numbers and all Banach or Hilbert spaces appearing are $\mathbb{K}$-Banach spaces or $\mathbb{K}$-Hilbert spaces, respectively. Without further notice, we assume that w.l.o.g. $\mathbb{K} = \mathbb{C}$ whenever we consider eigenvalues of operators. Note that this is no restriction since in case $\mathbb{K} = \mathbb{R}$ we may always consider the complexification of the involved operators and, e.g. for a generator A of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a real Banach space X, the complexification $A^{\mathbb{C}}$ of the operator A on the complexified Banach space $X^{\mathbb{C}}$ is the generator of a C_0-semigroup $\big(T^{\mathbb{C}}(t)\big)_{t\geq 0}$ on $X^{\mathbb{C}}$ and $T^{\mathbb{C}}(t)$ is just the complexification of $T(t)$ for all $t \geq 0$. For any Banach spaces X and Y, we denote by $\mathcal{B}(X,Y)$ the Banach space of bounded linear operators $T : X \to Y$, equipped with the operator norm $\|\cdot\| = \|\cdot\|_{\mathcal{B}(X,Y)}$. In the special case $X = Y$ we also write $\mathcal{B}(X) := \mathcal{B}(X,X)$. For any Banach space E, any compact set $K \subseteq \mathbb{R}^n$ and any open set $U \subseteq \mathbb{R}^n$, numbers $k \in \mathbb{N}_0 := \{0,1,\dots\}$ and $p \in [1,\infty]$ we denote by $C(K;E)$, $C^k(K;E)$, $L_p(\Omega;E)$ and $W_p^k(\Omega;E)$ (special case $p = 2$: $H^k(\Omega;E) := W_2^k(\Omega;E)$) the spaces of E-valued continuous functions, E-valued k-times continuously differentiable functions, the E-valued Bochner–Lebesgue spaces and the E-valued Bochner–Sobolev spaces of degree k, with norms

$$\|f\|_{C(K;E)} := \|f\|_\infty := \sup_{e\in K} \|f(e)\|_E \,,$$

$$\|f\|_{C^k(K;E)} := \|f\|_{C^k} := \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{C(K;E)} \,,$$

$$\|f\|_{L_p(U;E)} := \|f\|_p := \begin{cases} \left(\int_U \|f(x)\|_E^p \, \mathrm{d}x\right)^{1/p}, & p\in[1,\infty),\\ \operatorname{esssup}_{x\in U} \|f(x)\|_E \,, & p=\infty,\end{cases}$$

$$\|f\|_{W_p^k(U;E)} := \|f\|_{k,p} := \begin{cases} \left(\sum_{|\alpha|\leq k} \|\partial^\alpha f\|_p^p\right)^{1/p}, & p\in[1,\infty),\\ \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_\infty \,, & p=\infty.\end{cases}$$

The notions $C(K;E)$ and $C^k(K;E)$ also extend to closed subsets K of more general topological vector spaces F. For $p = 2$ and any Hilbert space E with inner product $\left(\cdot \mid \cdot\right)_E$, the spaces $L_2(\Omega;E)$ and $H^k(\Omega;E)$ are Hilbert spaces with standard inner products

$$\left(f \mid g\right)_{L_2(\Omega;E)} = \left(f \mid g\right)_{L_2} = \int_\Omega \big(f(x) \mid g(x)\big)_E \, \mathrm{d}x,$$

$$\left(f \mid g\right)_{H^k(\Omega;E)} = \left(f \mid g\right)_{H^k} = \sum_{|\alpha|\leq k} \int_\Omega \big(\partial^\alpha f(x) \mid \partial^\alpha g(x)\big)_E \, \mathrm{d}x.$$

Note that for $E = \mathbb{K}^d$ the $\mathbb{K}^d$-valued Bochner–Lebesgue and Bochner–Sobolev space are (up to an isomorphism) nothing but the d-fold product of the usual

Lebesgue spaces and the usual Sobolev spaces, resp., i.e.

$$L_p(\Omega;\mathbb{K}^d) \cong \prod_{j=1}^{d} L_p(\Omega;\mathbb{K}), \quad W_p^k(\Omega;\mathbb{K}^d) \cong \prod_{j=1}^{d} W_p^k(\Omega;\mathbb{K}),$$

$$H^k(\Omega;\mathbb{K}^d) \cong \prod_{j=1}^{d} H^k(\Omega;\mathbb{K}).$$

In particular, for $E = \mathbb{R}^d$ the strongly measurable functions are simply the measurable functions.

2.1.1. Some basic facts on semigroup theory. The focus of this manuscript lies on well-posedness (in the sense of semigroups) and stability for linear closure relation to boundary control and observation systems of infinite-dimensional port-Hamiltonian type. Therefore, let us recall some basic definitions and important theoretical results from semigroup theory that will be used heavily later on.

We start with the definition of a C_0-semigroup.

Definition 2.1 (C_0-semigroup). *Let X be a Banach space and $\big(T(t)\big)_{t\geq 0}$ be a family of bounded linear operators on X. Then $(T(t))_{t\geq 0}$ is called strongly continuous semigroup (for short, C_0-semigroup), if it has the following properties:*

1. *$T(0) = I$, the identity map on X,*
2. *$T(s+t) = T(s)T(t)$ for all $s,t \geq 0$, (semigroup property) and*
3. *$T(\cdot)x_0 \in C(\mathbb{R}_+;X)$ for every $x_0 \in X$, i.e. $\big(T(t)\big)_{t\geq 0}$ has continuous trajectories (strong continuity).*

A C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ is called (strongly continuous) contraction semigroup*, if the operator norm $\|T(t)\|_{\mathcal{B}(X)} \leq 1$ for all $t \geq 0$.*

The existence of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ is closely related to well-posedness of the abstract Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) \quad (t \geq 0), \quad x(0) = x_0 \tag{ACP}$$

in the sense of *existence* and *uniqueness* of solutions which *continuously depend on the initial datum.* Roughly speaking, the abstract Cauchy problem (ACP) has for every initial datum $x_0 \in D(A)$ a unique classical solution $x \in C^1(\mathbb{R}_+;X)$ with values in $D(A)$, and the solution continuously depends on the initial datum x_0, if and only if there is a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on X such that $Ax = \lim_{t\to 0+} \frac{T(t)x-x}{x}$ for every $x \in D(A) = \{x \in X : \text{ this limit exits}\}$, i.e. A is the *generator* of $\big(T(t)\big)_{t\geq 0}$, and then $x(t) := T(t)x_0$ defines the unique classical solution, for every $x_0 \in D(A)$. For a precise statement of this result, see e.g. Proposition II.6.6 in [10].

Linear operators A generating a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ can be exactly characterised by the general Hille–Yosida Theorem due to Feller, Miyadera and Phillips; see e.g. Theorem III.3.8 in [10]. In this paper, however, all appearing semigroups will be contraction semigroups on Hilbert spaces, so that the Hilbert space version of the Lumer–Phillips Theorem, a special case of the Hille–Yosida theorem, and with conditions which are much easier to handle, can be applied.

Theorem 2.2 (Lumer–Phillips). *Let X be a Hilbert space with inner product $(\cdot \mid \cdot)$. Further, let $A : D(A) \subseteq X \to X$ be a densely defined, closed linear operator. Then A generates a strongly continuous contraction semigroup if and only if*

1. *A is dissipative, i.e.* $\operatorname{Re}\left(Ax \mid x\right) \leq 0$ *for all $x \in X$, and*
2. $\operatorname{ran}(\lambda - A) = X$ *for some (then, all) $\lambda > 0$.*

Proof. We refer to Theorem II.3.15 in [10] for the general Banach space version thereof. □

To describe the long-time behaviour of the solutions to the abstract Cauchy problem (ACP) in terms of the C_0-semigroup associated to it, several notions of stability exist. Here, we are interested in *strong stability* and *uniform exponential stability* which are defined as follows. Note that these stability concepts coincide for finite dimensional Banach spaces X, but are distinct if $\dim X = \infty$.

Definition 2.3 (Stability concepts). *Let $\left(T(t)\right)_{t\geq 0}$ be a C_0-semigroup on some Banach space X.*

1. *The semigroup is called (asymptotically)* strongly stable *if for every $x \in X$ one has $T(t)x \to 0$ in X.*
2. *It is called* uniformly exponentially stable, *if there are constants $M \geq 1$ and $\omega < 0$ such that $\|T(t)\|_{\mathcal{B}(X)} \leq M\mathrm{e}^{\omega t},\ t \geq 0$.*

Stability properties of a C_0-semigroup can be tested via certain spectral properties and bounds on the resolvent operators, see the following two theorems which will be employed later on.

Theorem 2.4 (Arendt–Batty–Lyubich–Vũ). *Suppose $\left(T(t)\right)_{t\geq 0}$ is a bounded C_0-semigroup on some Banach space X, i.e. there is $M \geq 1$ such that $\|T(t)\|_{\mathcal{B}(X)} \leq M$ for all $t \geq 0$. Further assume that its generator A has compact resolvent, i.e. $(\lambda - A)^{-1} : X \to X$ is a compact operator for some (then, all) $\lambda \in \rho(A)$. In this case, $\left(T(t)\right)_{t\geq 0}$ is strongly stable if and only if the point spectrum satisfies $\sigma_p(A) \subseteq \mathbb{C}_0^- = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$.*

Proof. See Theorem V.2.21 in [10] for the general version of this Tauberian type theorem. □

Theorem 2.5 (Gearhart–Prüss–Huang). *Let $\left(T(t)\right)_{t\geq 0}$ be C_0-semigroup on some Hilbert space X. It is uniformly exponentially stable if and only if the following two properties hold true*

1. *$\sigma(A) \subseteq \mathbb{C}_0^-$, i.e. the spectrum lies in the complex left half-plane, and*
2. $\sup_{\beta \in \mathbb{R}} \left\|(\mathrm{i}\beta - A)^{-1}\right\| < \infty$, *i.e. the resolvent operators are uniformly bounded on the imaginary axis.*

Proof. See Theorem V.1.11 in [10]. □

Remark 2.6. For $\mathrm{i}\mathbb{R} \subseteq \rho(A)$, the condition $\sup_{\beta\in\mathbb{R}} \left\|(\mathrm{i}\beta - A)^{-1}\right\| < \infty$ is equivalent to the following property:

For every sequence $\left(x_n, \beta_n\right)_{n\geq 1} \subseteq D(A) \times \mathbb{R}$ with

1. $\sup_{n\geq 1} \|x_n\|_X < \infty$,
2. $|\beta_n| \to \infty$ as $n \to \infty$, and
3. $Ax_n - \mathrm{i}\beta_n x_n \to 0$ as $n \to \infty$,

it follows that $x_n \to 0$ in X as $n \to \infty$.

This characterisation proves helpful for stability analysis of port-Hamiltonian systems. For further details on semigroup theory, we refer to the monograph [10].

2.2. Basic definitions

Within this subsection, we introduce the notion of a (linear, infinite-dimensional) port-Hamiltonian system (in boundary control and observation form) as we use it later on for interconnection of several systems of port-Hamiltonian type to networks. Let us start with the basic definition of a single open-loop infinite-dimensional port-Hamiltonian system in boundary control and observation form.

Definition 2.7 (Port-Hamiltonian System). *We call a triple* $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *of linear operators an* (*open-loop, linear, infinite-dimensional*) port-Hamiltonian system (*in boundary control and observation form*) *of order* $N \in \mathbb{N}$, *if*

1. *The* (*maximal*) port-Hamiltonian operator

$$\mathfrak{A} : D(\mathfrak{A}) \subseteq L_2(0,1;\mathbb{K}^d) \to L_2(0,1;\mathbb{K}^d)$$

is a linear differential operator of the form

$$\mathfrak{A}x = \sum_{k=0}^{N} P_k \frac{\mathrm{d}^k}{\mathrm{d}\zeta^k}(\mathcal{H}x), \quad D(\mathfrak{A}) = \left\{x \in L_2\left(0,1;\mathbb{K}^d\right) : \mathcal{H}x \in H^N\left(0,1;\mathbb{K}^d\right)\right\}$$

where $\mathcal{H} \in L_\infty\left(0,1;\mathbb{K}^{d\times d}\right)$ *is coercive on* $L_2(0,1;\mathbb{K}^d)$, *i.e. there is* $m > 0$ *such that* $\left(\mathcal{H}(\zeta)\xi \mid \xi\right)_{\mathbb{K}^d} \geq m\,|\xi|^2_{\mathbb{K}^d}$, $\xi \in \mathbb{K}^d$, *a.e.* $\zeta \in (0,1)$, *and* $P_k \in \mathbb{K}^{d\times d}$ $(k = 1,\dots,N)$ *are matrices satisfying the anti-/symmetry relations* $P_k^* = (-1)^{k+1}P_k$ $(k = 1,\dots,N)$ *and such that the matrix* P_N, *i.e. the matrix corresponding to the principal part of the differential operator* $\mathfrak{A}$, *is invertible, whereas* $P_0 \in L_\infty(0,1;\mathbb{K}^{d\times d})$ *may depend on the spatial variable* $\zeta \in (0,1)$.

2. *The* boundary input map $\mathfrak{B}$ *and the* boundary output map $\mathfrak{C}$ *are linear* $\mathbb{K}^{Nd}$*-valued operators with common domain* $D(\mathfrak{A}) = D(\mathfrak{B}) = D(\mathfrak{C})$ *of the form*

$$\begin{pmatrix} \mathfrak{B}x \\ \mathfrak{C}x \end{pmatrix} = \begin{bmatrix} W_B \\ W_C \end{bmatrix} \tau(\mathcal{H}x), \quad x \in D(\mathfrak{A}),$$

$$\tau(y) = \left(y(1), y'(1), \dots, y^{(N-1)}(1), y(0), \dots, y^{(N-1)}(0)\right) \in \mathbb{K}^{Nd},$$

$$y \in H^N(0,1;\mathbb{K}^d)$$

for matrices $W_B, W_C \in \mathbb{K}^{Nd\times 2Nd}$ *such that* $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]$ *is invertible.*

Remark 2.8. More generally, we call a triple $\mathfrak{S}_1 = (\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{C}_1)$ with $\mathfrak{B}_1, \mathfrak{C}_1 : D(\mathfrak{B}_1) = D(\mathfrak{A})_1 = D(\mathfrak{C}_1) \to \mathbb{K}^k$ for some $k \in \{0, 1, \ldots, Nd\}$ a port-Hamiltonian system as well, if there are linear operators $\mathfrak{A}$, $\mathfrak{B} = (\mathfrak{B}_0, \mathfrak{B}_1)$ and $\mathfrak{C} = (\mathfrak{C}_0, \mathfrak{C}_1)$ such that $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a port-Hamiltonian system in the sense of Definition 2.7 and $\mathfrak{A}_1 = \mathfrak{A}|_{\ker \mathfrak{B}_0}$. This tacit convention makes it possible to consider a partial interconnection of port-Hamiltonian systems (of same order N) to be a port-Hamiltonian system itself.

Whenever $\mathfrak{S}$ is a port-Hamiltonian system on the space $L_2(0,1;\mathbb{K}^d)$, by coercivity of $\mathcal{H}$ the sesquilinear form

$$(\cdot \mid \cdot)_{\mathcal{H}} : \left(L_2(0,1;\mathbb{K}^d)\right)^2 \to \mathbb{K}, \quad (f \mid g)_{\mathcal{H}} := \int_0^1 \left(f(\zeta) \mid \mathcal{H}(\zeta) g(\zeta)\right)_{\mathbb{K}^d} \, \mathrm{d}\zeta$$

defines an inner product on $L_2(0,1;\mathbb{K}^d)$ and the corresponding norm $\|\cdot\|_{\mathcal{H}}$ is equivalent to the standard norm $\|\cdot\|_{L_2}$. We call $(\cdot \mid \cdot)_{\mathcal{H}}$ the *energy inner product* and set the *energy state space* X to be the Hilbert space $L_2(0,1;\mathbb{K}^d)$ equipped with inner product $(\cdot \mid \cdot)_X := (\cdot \mid \cdot)_{\mathcal{H}}$ (and, hence, the *energy norm* $\|\cdot\|_X = \|\cdot\|_{\mathcal{H}}$). Note that the operator $\mathfrak{A} : D(\mathfrak{A}) \subseteq X \to X$ is a closed operator as a conjunction of the continuous matrix multiplication operator $\mathcal{H}(\cdot)$ on X and the closed (thanks to P_N being invertible) differential operator

$$\sum_{k=0}^{N} P_k \frac{\mathrm{d}^k}{\mathrm{d}\zeta^k} : H^N(0,1;\mathbb{K}^d) \subseteq X \to X.$$

Remark 2.9. For now, let us consider an infinite-dimensional port-Hamiltonian system $\mathfrak{S}$ with $P_0 = 0$. Then for every $x, y \in X$ such that $\mathcal{H}x, \mathcal{H}y \in C_c^\infty(0,1;\mathbb{K}^d)$ it holds via integration by parts that

$$(\mathfrak{A}x \mid y)_X = -\,(x \mid \mathfrak{A}y)_X$$

i.e. the operator is formally skew-symmetric on the space X. For the case $P_0 \neq 0$ this holds exactly in the case that $P_0(\zeta)^* = -P_0(\zeta)$ for a.e. $\zeta \in (0,1)$.

When looking for dissipative closure relations of the type $\mathfrak{B}x = K\mathfrak{C}x$ for some matrix $K \in \mathbb{K}^{Nd \times Nd}$ it is convenient to have the property of passivity for the port-Hamiltonian system.

Definition 2.10 (Passive Systems). *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a port-Hamiltonian system in boundary control and observation form. The system $\mathfrak{S}$ is called*

- impedance passive*, if*

$$\operatorname{Re}\,(\mathfrak{A}x \mid x)_X \le (\mathfrak{B}x \mid \mathfrak{C}x)_{\mathbb{K}^{Nd}}, \quad x \in D(\mathfrak{A});$$

- scattering passive*, if*

$$\operatorname{Re}\,(\mathfrak{A}x \mid x)_X \le |\mathfrak{B}x|^2_{\mathbb{K}^{Nd}} - |\mathfrak{C}x|^2_{\mathbb{K}^{Nd}}, \quad x \in D(\mathfrak{A}).$$

Remark 2.11. 1. Note that both notions of passivity do not depend on the Hamiltonian energy density matrix function $\mathcal{H}$: A port-Hamiltonian system $\mathfrak{S}$ is impedance passive (scattering passive) if and only if the corresponding port-Hamiltonian system for $\mathcal{H} = I$ is impedance passive (scattering passive).
2. A port-Hamiltonian system is impedance passive (scattering passive) if and only if the symmetric part Sym $P_0(\zeta) := \frac{1}{2}(P_0(\zeta) + P_0(\zeta)^*)$ of P_0 is negative semi-definite for a.e. $\zeta \in (0,1)$ and W_B, W_C satisfy a certain matrix condition (including also the matrices P_k $(k \geq 1)$), see [14].

Besides the energy state space $X = L_2(0,1;\mathbb{K}^d)$ (equipped with the energy inner product), extended energy state spaces $\widehat{X} = X \times X_c$ for some finite dimensional Hilbert space X_c will be used as well, and its elements are denoted by $\widehat{x} = (x, x_c) \in X \times X_c$. Operators acting on elements of such product energy state spaces are denoted by a hat, e.g. $\widehat{A}$, $\widehat{\mathfrak{A}}$, $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{C}}$ and $\widehat{T}(t)$.

3. Examples and Previous Results

We give some examples of dissipative closure relations which had been considered previously in the literature. Additionally, we recall the main results on well-posedness and stability for these linear closure relations. Starting from open-loop passive port-Hamiltonian systems one can easily obtain dissipative operators when closing with a suitable closure relation and possibly interconnects the port-Hamiltonian system with either another port-Hamiltonian system or an impedance passive control and observation system. Below we list some particular examples for such static or dynamic closure relations.

Example 3.1 (Dissipative, static closure). Assume that $\mathfrak{S}$ is an impedance passive port-Hamiltonian system and let $K \in \mathbb{K}^{Nd \times Nd}$ be a matrix with negative semi-definite symmetric part

$$\text{Sym } K := \frac{1}{2}(K + K^*) \leq 0$$

(the simplest choice being $K = 0$) and define $A : D(A) \subseteq X \to X$ by

$$\begin{aligned} Ax &:= \mathfrak{A}x, \\ D(A) &:= \{x \in D(\mathfrak{A}) : \mathfrak{B}x = K\mathfrak{C}x\}. \end{aligned}$$

Then A is a dissipative operator on X, and, therefore, generates a strongly continuous contraction semigroup $\left(T(t)\right)_{t\geq 0}$ on X, see Theorem 3.2 below.

Proof. Dissipativity can be checked easily, using the impedance passivity of $\mathfrak{S}$ and the negative semi-definiteness of Sym (K). Then, the generator property follows from Theorem 3.2 below. □

The first result on well-posedness of infinite-dimensional port-Hamiltonian systems has been due to Y. Le Gorrec, H. Zwart and B. Maschke [14] who proved that for operators of port-Hamiltonian type a dissipative linear closure relation, i.e.

dissipative boundary conditions, is already enough for the corresponding abstract Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x(t) = Ax(t), \quad t \geq 0, \\ x(0) = x_0 \in X \end{cases}$$

to be well-posed, i.e. for every initial value $x_0 \in D(A)$, there is a unique classical solution $x \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(A))$ of this Cauchy problem, where $D(A)$ is equipped with the graph norm of A, and the solution depends continuously on the initial datum x_0 and has non-increasing energy $\frac{1}{2} \|x(t)\|_X^2$. In other words, if A is dissipative, then A generates a strongly continuous contraction semigroup $(T(t))_{t\geq 0}$ on X. This is

Theorem 3.2 (Le Gorrec, Zwart, Maschke (2005)). *Let* $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *be any port-Hamiltonian system and* $K \in \mathbb{K}^{Nd \times Nd}$*. Then, the operator*

$$A = \mathfrak{A}|_{\ker(\mathfrak{B} - K\mathfrak{C})}$$

generates a contractive C_0*-semigroup* $\big(T(t)\big)_{t\geq 0}$ *on* $X = \big(X, (\cdot \mid \cdot)_X\big)$ *if and only if* A *is dissipative on* X*.*

Proof. For the proof, see [14]. □

While well-posedness for itself is an important property, often one is not satisfied with well-posedness alone, but also looks for stability properties of the abstract Cauchy problem associated to A. In contrast to well-posedness – for which the case of general coercive $\mathcal{H} \in L_\infty(0, 1; \mathbb{K}^{d\times d})$ can be reduced to the special case $\mathcal{H} = I$, see Lemma 7.2.3 in [11] – stability properties of A may (and will, as Engel [9] showed) generally depend on the Hamiltonian density matrix function which can be seen as a multiplicative perturbation to the operator A for $\mathcal{H} = I$. However, as has been known for the wave equation, the Timoshenko beam equation and the Euler-Bernoulli beam equation, there are examples where one could expect that some classes of linear boundary feedback relations imply asymptotic stability, i.e. trajectory-wise for every initial datum $x_0 \in X$, or even uniform exponential stability, i.e. the energy decay can be bounded by an exponentially decaying function times initial energy, where the exponential decay rate is *independent* of the initial datum $x_0 \in X$. For the particular case of first order port-Hamiltonian systems such stability results have first been proved in the Ph.D. thesis [26] and the research article [27], showing that for first order port-Hamiltonian systems it is enough to damp at one end, whereas at the other end arbitrary conservative or dissipative boundary conditions can be imposed.

Theorem 3.3 (Villegas et al. (2009)). *Let* A *be a port-Hamiltonian operator closed with a dissipative boundary condition as in Theorem* 3.2*. Further assume that the order of the port-Hamiltonian system is* $N = 1$*, the Hamiltonian density matrix function* $\mathcal{H} : [0, 1] \to \mathbb{K}^{d\times d}$ *is Lipschitz continuous and one has the following estimate:*

$$\operatorname{Re}\, \big(Ax \mid x\big)_X \leq -\kappa \left|(\mathcal{H}x)(0)\right|^2, \quad x \in D(A)$$

where $\kappa > 0$ *does not depend on* $x \in D(A)$*. Then, the* C_0*-semigroup* $\big(T(t)\big)_{t\geq 0}$ *generated by* A *is uniformly exponentially stable.*

Proof. For the proof, see [27], where $\mathcal{H} \in C^1\big([0,1];\mathbb{K}^{d\times d}\big)$ had been assumed. However, the proof carries over to $\mathcal{H} \in \mathrm{Lip}\,\big([0,1];\mathbb{K}^{d\times d}\big)$, see [5]. □

Actually, up to now there are at least three approaches known to prove the stability result above:

1. The original proof in [27] is based on some *sideways-energy estimate* (as it is called in [6]) or *final observability estimate*
$$\|T(\tau)x_0\|_X \leq c\big\|[T(\cdot)x](0)\big\|_{L_2(0,\tau;\mathbb{K}^d)}, \quad x_0 \in X$$
for some sufficiently large $\tau > 0$ and some $c > 0$. This approach is very helpful when considering non-linear dissipative boundary feedback, cf. [3], however, it seems difficult to extend this result to higher order port-Hamiltonian systems, e.g. Euler–Bernoulli type systems.
2. A frequency domain approach, i.e., showing that the resolvent
$$(\mathrm{i}\beta - A)^{-1} \in \mathcal{B}(X)$$
exists for all $\beta \in \mathbb{R}$ and is uniformly bounded, employing Arendt–Batty–Lyubich–Vũ Theorem (asymptotic stability) and Gearhart–Prüss–Huang Theorem (uniform exponential stability) has been applied in [5]. This approach is suitable for interconnection with finite dimensional control systems [5], and as we later see, for linear interconnection with other port-Hamiltonian systems; see the Section 5.
3. Third, a multiplier approach leading to a Lyapunov function is possible as well. Again, this approach is suitable for non-linear feedback interconnection, especially of dynamic type [3].

Strictly speaking, there also is a fourth approach (actually, the oldest one!), but it only works under much stronger regularity assumptions, namely analyticity of $\mathcal{H}$; see [20].

The second example for a class of closure relations consists of dissipative or conservative feedback interconnection with a linear control system.

Example 3.4 (Interconnection of a PHS with a finite-dimensional controller). Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian system and
$$\Sigma_c = (A_c, B_c, C_c, D_c) \in \mathcal{B}(X_c) \times \mathcal{B}(U_c; X_c) \times \mathcal{B}(X_c; Y_c) \times \mathcal{B}(U_c; Y_c),$$
for some finite-dimensional Hilbert spaces X_c, U_c and Y_c, be a finite dimensional control system
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x_c(t) = A_c x_c(t) + B_c u_c(t), \\ y_c(t) = C_c x_c(t) + D_c u_c(t), \quad t \geq 0 \end{cases}$$

and impedance passive, i.e. $U_c = Y_c$ and

$$\operatorname{Re}\left(A_c x_c + B_c u_c \,\middle|\, x_c\right)_{X_c} \le \operatorname{Re}\left(u_c \,\middle|\, C_c x_c + D_c u_c\right)_{U_c}, \quad x_c \in X_c, u_c \in U_c.$$

Further assume that $U_c = Y_c = \mathbb{K}^{Nd}$. Then $\widehat{A} : D(\widehat{A}) \subseteq \widehat{X} \to \widehat{X}$,

$$\begin{aligned} \widehat{A}(x, x_c) &:= (\mathfrak{A}x, A_c x_c + B_c \mathfrak{C}x), \\ D(\widehat{A}) &:= \left\{(x, x_c) \in D(\mathfrak{A}) \times X_c : \mathfrak{B}x = -C_c x_c - D_c \mathfrak{C}x\right\}, \end{aligned}$$

resulting from the standard feedback interconnection

$$u_c = \mathfrak{C}x \quad \text{and} \quad \mathfrak{B}x = -y_c,$$

is a dissipative operator on the product Hilbert space $\widehat{X} = X \times X_c$, and thus generates a strongly continuous contraction semigroup on $\widehat{X}$.

Proof. Dissipativity follows from impedance passivity of both subsystems and some easy computation. For the assertion on semigroup generation, we need the following result. □

Theorem 3.5 (Villegas (2007), Augner, Jacob (2014)). *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an infinite-dimensional port-Hamiltonian system and $\Sigma_c = (A_c, B_c, C_c, D_c)$ be a finite dimensional linear control system. The operator*

$$\begin{aligned} \widehat{A}(x, x_c) &:= (\mathfrak{A}x, A_c x_c + B_c \mathfrak{C}x), \\ D(\widehat{A}) &:= \left\{(x, x_c) \in D(\mathfrak{A}) \times X_c : \mathfrak{B}x = -C_c x_c - D_c \mathfrak{C}x\right\} \end{aligned}$$

generates a contractive C_0-semigroup $\left(\widehat{T}(t)\right)_{t \ge 0}$ on the product Hilbert space $\widehat{X} = X \times X_c$ if and only if it is dissipative.

Proof. A result like this has probably first been stated in the Ph.D. thesis [26], however under some slightly more restrictive conditions on the infinite-dimensional port-Hamiltonian system and the finite dimensional linear control system Σ_c. For the general situation stated above, see [5]. □

As for the static feedback case, the generation theorem is based on the Lumer–Phillips Theorem which states that besides dissipativity of an operator a range condition, namely $\operatorname{ran}\left(\widehat{A} - \lambda I\right) = \widehat{X}$ for some (then, all) $\lambda > 0$ is sufficient (and necessary as well) for the operator $\widehat{X}$ to generate a strongly continuous contraction semigroup. Here, the range condition for $\widehat{A}$ is reduced to a range condition for some operator A_{cl} (with suitable static linear closure relations), i.e. the generation theorem for the dynamic case already relies on (the proof of) the generation theorem for the static case.

As for the static case, one can ask for sufficient (hopefully $\mathcal{H}$-independent) conditions on the damping via the controller such that the hybrid PDE–ODE systems is uniformly exponentially stable, i.e. its total energy decays uniformly exponentially to zero for all initial data $(x_0, x_{c,0})$.

Theorem 3.6 (Ramirez, Zwart, Le Gorrec (2013), Augner, Jacob (2014)). *Assume that $\mathfrak{S}$ is an impedance passive port-Hamiltonian system of order $N = 1$ and such that $\mathcal{H} : [0,1] \to \mathbb{K}^{d\times d}$ is Lipschitz continuous, and let $\mathfrak{S}$ be interconnected by standard feedback interconnection*

$$\mathfrak{B}x = -y_c, \quad u_c = \mathfrak{C}x$$

with a strictly impedance passive finite-dimensional control system

$$\Sigma_c = (A_c, B_c, C_c, D_c)$$

in the sense that for $x_c \in X_c$, $u_c \in U_c$,

$$\mathrm{Re}\, \left(A_c x_c + B_c u_c \mid x_c\right)_{X_c} \leq \mathrm{Re}\, \left(C_c x_c + D_c u_c \mid u_c\right)_{U_c} - \kappa \left|D_c u_c\right|^2,$$

and such that $\left(\mathrm{e}^{tA_c}\right)_{t\geq 0}$ is uniformly exponentially stable, $\ker D_c \subseteq \ker B_c$ *and*

$$\left|\mathfrak{B}x\right|^2 + \left|D_c \mathfrak{C}x\right|^2 \gtrsim \left|(\mathcal{H}x)(0)\right|^2, \quad x \in D(\mathfrak{A}),$$

then the C_0-semigroup $\left(\widehat{T}(t)\right)_{t\geq 0}$ is uniformly exponentially stable.

Proof. For the situation where Σ_c is strictly input passive, in particular $D_c > 0$ is positive definite, see [18]. The (slightly) generalised result can be found in [5]. □

The general idea for the proof of this dynamic feedback result is to consider the state variable x_c as a perturbation to the static boundary feedback one would have for $x_c = 0$, namely $\mathfrak{B}x = -D_c\mathfrak{C}x$. From the impedance passivity of $\mathfrak{S}$ and Σ_c one then obtains a dissipation estimate of the type

$$\mathrm{Re}\, \left(\mathfrak{A}x \mid x\right)_X \leq -\kappa \left(\left|\mathfrak{B}x\right|_U^2 + \left|D_c\mathfrak{C}x\right|_U^2\right) \lesssim -\kappa \left|(\mathcal{H}x)(0)\right|^2, \quad x \in \ker(\mathfrak{B} + D_c\mathfrak{C}).$$

Uniform exponential energy decay then can be expected from the static feedback result and the exponential stability of $(\mathrm{e}^{tA_c})_{t\geq 0}$ ensures that the perturbation does not hurt this property.

Besides dynamic feedback, the other direction of generalisation aims at higher order port-Hamiltonian systems. The first result in this perspective follows rather easily from the Arendt–Batty–Lyubich–Vũ Theorem and considerations on possible eigenfunctions with eigenvalues $\mathrm{i}\beta$ for some $\beta \in \mathbb{R}$, but only gives *asymptotic stability.*

Proposition 3.7 (Augner, Jacob (2014)). *Let A be a port-Hamiltonian operator of order $N \in \mathbb{N}$, resulting from linear closure of a port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ by a linear closure relation $\mathfrak{B}x = K\mathfrak{C}x$ and assume that $\mathcal{H} : [0,1] \to \mathbb{K}^{d\times d}$ is Lipschitz continuous, and*

$$\mathrm{Re}\, \left(Ax \mid x\right)_X \leq -\kappa \sum_{k=0}^{N-1} \left|(\mathcal{H}x)^{(k)}(0)\right|^2, \quad x \in D(A)$$

for some $\kappa > 0$. Then, the C_0-semigroup $(T(t))_{t\geq 0}$ generated by A is (asymptotically) strongly stable.

Proof. See [5]. □

In [5], it has also been shown that generally one cannot expect uniform exponential stability, namely there is a counter example (Schrödinger equation) where full dissipation at one end and a correct choice of conservative boundary conditions at the other end only lead to asymptotic stability, but not to uniform exponential stability. (For the counter example, one can compute the resolvents $(\mathrm{i}\beta - A)^{-1}$ and show that they are not uniformly bounded for $\beta \in \mathbb{R}$.) However, under further conditions on the boundary conditions at the conservative end, more can be said:

Theorem 3.8 (Augner, Jacob (2014)). *Let A be a (closed by linear static feedback) port-Hamiltonian operator A of order $N = 2$ and $\mathcal{H} \in \mathrm{Lip}([0,1];\mathbb{K}^{d\times d})$ be Lipschitz continuous and assume that*

$$\mathrm{Re}\,\left(Ax \mid x\right)_X \leq -\kappa\left(\left|(\mathcal{H}x)(0)\right|^2 + \left|(\mathcal{H}x)'(0)\right|^2 + \left|(\mathcal{H}x)(1)\right|^2\right)$$

for all $x \in D(A)$. Then the C_0-semigroup $\left(T(t)\right)_{t\geq 0}$ generated by A is uniformly exponentially stable.

Proof. See [5]. □

Remark 3.9. By the way, fully dissipative boundary conditions at *both ends*

$$\mathrm{Re}\,\left(Ax \mid x\right)_X \leq -\kappa\left(\sum_{k=0}^{N-1}\left|(\mathcal{H}x)^{(k)}(0)\right|^2 + \left|(\mathcal{H}x)^{(k)}(1)\right|^2\right), \quad x \in D(A)$$

for some $\kappa > 0$, inevitably lead to uniform exponential stability, for all port-Hamiltonian systems of arbitrary order $N \in \mathbb{N}$ and for Lipschitz continuous $\mathcal{H}$, see [2].

In this article, we are concerned with the case where a port-Hamiltonian system $\mathfrak{S}^1$ is interconnected with further port-Hamiltonian systems in a energy preserving or dissipative way, e.g.

Example 3.10 (Interconnection of impedance passive PHS). Let $\mathfrak{S}^1$ and $\mathfrak{S}^2$ be two impedance passive port-Hamiltonian systems with $N^1 d^1 = N^2 d^2$, i.e. the input and output spaces for $\mathfrak{S}^1$ and $\mathfrak{S}^2$ should have the same dimension, then the operator $A : D(A) \subseteq X \to X$ defined by

$$A(x^1, x^1) := (\mathfrak{A}^1 x^1, \mathfrak{A}^1 x^2),$$
$$D(A) = \left\{x = (x^1, x^2) \in D(\mathfrak{A}^1) \times D(\mathfrak{A}^2) :\, \mathfrak{B}^1 x^1 = -\mathfrak{C}^2 x^2,\, \mathfrak{B}^2 x^2 = \mathfrak{C}^1 x^1\right\}$$

is dissipative on the product Hilbert space $X = X^1 \times X^2$ and generates a strongly continuous contraction semigroup $\left(T(t)\right)_{t\geq 0}$ on X.

Proof. The dissipativity of the operator A can be checked using the impedance passivity of the two subsystems:

$$\begin{aligned}\mathrm{Re}\,\left(Ax \mid x\right)_X &= \sum_{j=1}^{2}\mathrm{Re}\,\left(\mathfrak{A}^j x^j \mid x^j\right)_{X^j} \\ &\leq \mathrm{Re}\,\left(\mathfrak{B}^1 x^1 \mid \mathfrak{C}^1 x^1\right) + \mathrm{Re}\,\left(\mathfrak{B}^2 x^2 \mid \mathfrak{C}^2 x^2\right) = 0, \quad x \in D(A).\end{aligned}$$

For the generation result, see Proposition 4.4 in the next section. □

Example 3.11 (Interconnection of scattering passive PHS). Assume $\mathfrak{S}^1, \dots, \mathfrak{S}^m$ are scattering passive port-Hamiltonian systems. Then

$$A(x^1, \dots, x^m) = (\mathfrak{A}^1 x^1, \dots, \mathfrak{A}^m x^m),$$
$$D(A) = \left\{ x = (x^1, \dots, x^m) \in \prod_{j=1}^m D(\mathfrak{A}^j),\ \mathfrak{B}^1 x^1 = 0,\ \mathfrak{B}^j x^j = \mathfrak{C}^{j-1} x^{j-1}\ (j \geq 1) \right\}$$

is dissipative on $X = \prod_{j=1}^m X^j$ and generates a strongly continuous contraction semigroup $\big(T(t)\big)_{t\geq 0}$ on X.

Proof. Dissipativity can be checked easily by using the scattering-passivity of the subsystems $\mathfrak{S}^j$. For the generation result, see Proposition 4.4 in the next section. □

We comment on stability properties later on.

4. Port-Hamiltonian Systems: Networks

After recalling some known results on different static or dynamic closure relations for port-Hamiltonian systems, let us focus on the main topic of this paper, namely the interconnection of several infinite-dimensional port-Hamiltonian subsystems to a *network* of port-Hamiltonian systems. Assume that $\mathcal{J} = \{1, 2, \dots, m\}$ is a finite index set. Here, the number $m \in \mathbb{N}$ is the number of infinite-dimensional port-Hamiltonian subsystems $\mathfrak{S}^j = \big(\mathfrak{A}^j, \mathfrak{B}^j, \mathfrak{C}^j\big)$ the network consists of. Moreover, $\mathcal{J}_c = \{1, 2, \dots, m_c\}$ denotes another index set, corresponding to a finite number of finite-dimensional *linear control systems* $\Sigma_c^j = \big(A_c^j, B_c^j, C_c^j, D_c^j\big)$, we may interconnect the port-Hamiltonian systems with, the case $m_c = 0$, i.e. $\mathcal{J}_c = \emptyset$ being allowed, but w.l.o.g. we may always assume that $m_c = m \in \mathbb{N}$.

We generally assume that $\mathfrak{S}^j = \big(\mathfrak{A}^j, \mathfrak{B}^j, \mathfrak{C}^j\big)$ $(j \in \mathcal{J})$ are (open-loop, linear, infinite-dimensional) port-Hamiltonian systems (on a one-dimensional spatial domain) on spaces $X^j = L_2\big(0, 1; \mathbb{K}^{d_j}\big)$ (all equipped with their respective energy norm $\|\cdot\|_{X^j} = \|\cdot\|_{\mathcal{H}^j}$, thus being Hilbert spaces for the energy inner product $\big(\cdot \mid \cdot\big)_{X^j} = \big(\cdot \mid \cdot\big)_{\mathcal{H}^j}$ and input and output spaces $U^j = Y^j = \mathbb{K}^{N^j d^j}$, and similarly $\Sigma_c^j = \big(A_c^j, B_c^j, C_c^j, D_c^j\big)$ $(j \in \mathcal{J}_c)$ are finite-dimensional linear control systems with finite dimensional state space X_c^j and finite-dimensional input and output space $U_c^j = Y_c^j$. We further set

$$\widehat{X} := X \times X_c := \prod_{j=1}^m X^j \times \prod_{j=1}^{m_c} X_c^j,$$
$$\widehat{U} := U \times U_c := \prod_{j=1}^m U^j \times \prod_{j=1}^{m_c} U_c^j = \prod_{j=1}^m Y^j \times \prod_{j=1}^{m_c} Y_c^j =: Y \times Y_c =: \widehat{Y}.$$

We equip these spaces with their respective product inner product and the induced norms, i.e.

$$\left(\widehat{x} \mid \widehat{y}\right)_{\widehat{X}} = \sum_{j\in\mathcal{J}} \left(x^j \mid y^j\right)_{X^j} + \sum_{j\in\mathcal{J}_c} \left(x_c^j \mid y_c^j\right)_{X_c^j}, \quad \widehat{x},\widehat{y} \in \widehat{X},$$

$$\|\widehat{x}\|_{\widehat{X}} = \sqrt{\sum_{j\in\mathcal{J}} \|x^j\|^2_{X^j} + \sum_{j\in\mathcal{J}_c} \|x_c^j\|^2_{X_c^j}}, \quad \widehat{x} \in \widehat{X},$$

and accordingly for $\widehat{U} = \widehat{Y}$. For $x \in \prod_{j\in\mathcal{J}} D(\mathfrak{A}^j)$ we write

$$\mathfrak{A}x := (\mathfrak{A}^1 x^1, \ldots, \mathfrak{A}^m x^m) \in X,$$
$$\mathfrak{B}x := (\mathfrak{B}^1 x^1, \ldots, \mathfrak{B}^m x^m) \in U,$$
$$\mathfrak{C}x := (\mathfrak{C}^1 x^1, \ldots, \mathfrak{C}^m x^m) \in Y.$$

This defines linear operators

$$\mathfrak{A} : D(\mathfrak{A}) = \prod_{j\in\mathcal{J}} D(\mathfrak{A}^j) \subseteq X \to X, \quad \mathfrak{B}, \mathfrak{C} : D(\mathfrak{B}) = D(\mathfrak{C}) = D(\mathfrak{A}) \subseteq X \to U = Y.$$

Also, we define $A_c \in \mathcal{B}(X_c)$, $B_c \in \mathcal{B}(U_c, X_c)$, $C_c \in \mathcal{B}(X_c, Y_c)$ and $D_c \in \mathcal{B}(U_c, Y_c)$ by

$$A_c x_c = (A_c^1 x_c^1, \ldots, A_c^{m_c} x_c^{m_c}), \quad x_c \in X_c,$$
$$B_c u_c = (B_c^1 u_c^1, \ldots, B_c^{m_c} u_c^{m_c}), \quad u_c \in U_c,$$
$$C_c x_c = (C_c^1 x_c^1, \ldots, C_c^{m_c} x_c^{m_c}), \quad x_c \in X_c,$$
$$D_c u_c = (D_c^1 u_c^1, \ldots, D_c^{m_c} u_c^{m_c}), \quad u_c \in X_c.$$

Further, let

$$E_c \in \mathcal{B}(Y; U_c), \quad E \in \mathcal{B}(Y_c; U).$$

Now, interconnect the subsystems via the relation

$$\mathfrak{B}x = -E(C_c x_c + D_c E_c \mathfrak{C}x).$$

Remark 4.1. To keep the presentation as simple as possible, in this exposition we will always assume that $Y = U_c$ and $U = Y_c$ as well as $E_c = I$ and $E = I$ are the identity maps.

We may then define the following operator $\widehat{A}$ on $\widehat{X}$

$$\widehat{A}\,\widehat{x} := (\mathfrak{A}x, A_c x_c + B_c \mathfrak{C}x),$$
$$D\big(\widehat{A}\big) := \big\{\widehat{x} = (x, x_c) \in D(\mathfrak{A}) \times X_c : \mathfrak{B}x = -(C_c x_c + D_c \mathfrak{C}x)\big\}.$$

Example 4.2. Let us consider two particular special cases:

1. If $m_c = 0$, i.e. $X_c = U_c = Y_c = \{0\}$, we can identify $\widehat{X} = X$, $\widehat{Y} = Y$, $\widehat{U} = U$ and $C_c x_c + D_c \mathfrak{C}x = D_c \mathfrak{C}x$, so that

$$\widehat{A}\,\widehat{x} = \mathfrak{A}x, \quad D\big(\widehat{A}\big) = \big\{x \in D(\mathfrak{A}) : \mathfrak{B}x = -D_c \mathfrak{C}x\big\}.$$

In this case, no finite dimensional control system is present and this just describes the interconnection of port-Hamiltonian systems $\mathfrak{S}_i$ by boundary feedback, with the special case $m = 1$ being the case of a port-Hamiltonian system closed by linear boundary feedback.

2. If $m = m_c = 1$ and $U = Y_c = U_c = Y$, the operator $\widehat{A}$ reads as

$$\widehat{A}\,\widehat{x} = (\mathfrak{A}x, A_c x_c + B_c \mathfrak{C}x),$$
$$D(\widehat{A}) = \{\widehat{x} \in D(\mathfrak{A}) \times X_c : \mathfrak{B}x = -C_c x_c - D_c \mathfrak{C}x\}$$

so that we are in the case of dynamic boundary feedback with a finite dimensional control system interconnected by standard feedback interconnection with the port-Hamiltonian system.

Remark 4.3. 1. Note that the abstract Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{x}(t) = \widehat{A}\,\widehat{x}(t) \ (t \geq 0), \quad \widehat{x}(0) = (x_0, x_{c,0}) \in X \times X_c$$

is equivalent to the system of PDE and ODE

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= \mathfrak{A}x(t), \\
\frac{\mathrm{d}}{\mathrm{d}t}x_c(t) &= A_c x_c(t) + B_c u_c(t), \\
\mathfrak{B}x(t) &= -(C_c x_c + D_c u_c), \\
u_c(t) &= \mathfrak{C}x(t), \quad t \geq 0.
\end{aligned}$$

2. The definition of $\widehat{A}$ does, at first glance, not allow complex systems where the input into one finite dimensional control system depends upon the output from another finite dimensional controller. However, in most cases it should be possible, to merge such two finite control systems into a larger control system, by plugging in the equations of one of these systems into the other.

Our proof of the general generation result, Theorem 4.6, below uses the special case where $X_c = \{0\}$, i.e. no finite dimensional control systems are present within the network. We, therefore, begin by considering the generation result for this particular special case.

Proposition 4.4. *Assume that $X_c = \{0\}$. Then $\widehat{A}$ generates a contractive C_0-semigroup on $\widehat{X} \cong X$ if and only if $\widehat{A}$ is dissipative.*

Proof. Since $\mathcal{H} = \operatorname{diag}_{j\in\mathcal{J}} \mathcal{H}^j$ is a strictly coercive (matrix) multiplication operator on X, by Lemma 7.2.3 in [11] we can restrict ourselves to the case $\mathcal{H} = I \in \mathcal{B}(X)$. Further, let us for the moment assume that all $P_0^j = 0$ (or a constant matrix independent of $\zeta \in (0,1)$ with negative semi-definite symmetric part). Since $\mathfrak{B}x = \mathfrak{C}x = 0$ for all $x \in \prod_{j\in\mathcal{J}} C_c^\infty(0,1;\mathbb{K}^{d_j})$ and this set is dense in X, the operator $\widehat{A}$ is densely defined, so that by the Lumer–Phillips Theorem, see e.g. Theorem II.3.15 in [10], it remains to prove that $\lambda I - \widehat{A}$ is surjective for some $\lambda > 0$ whenever $\widehat{A}$ is dissipative. Here, we choose $\lambda = 1$. Take $f = \big(f_j\big)_{j\in\mathcal{J}} \in \prod_{j\in\mathcal{J}} X^j = X$. Then we have to find $x \in D(\mathfrak{A})$ such that

$$(\mathfrak{A} - I)x = f, \quad \mathfrak{B}x = -D_c\mathfrak{C}x =: K\mathfrak{C}x.$$

We can identify the operator $\widehat{A} : D\big(\widehat{A}\big) \subseteq X \times \{0\} \to X \times \{0\}$ with the operator $A = \mathfrak{A}\big|_{\ker(\mathfrak{B}-K\mathfrak{C}x)} : D(A) = \ker(\mathfrak{B} - K\mathfrak{C}) \subseteq X \to X$. For every $j \in \mathcal{J}$ we now write

$$h^j = \big(x^j, (x^j)', \ldots, (x^j)^{(N_j-1)}\big), \quad g^j = \big(0, \ldots, 0, \big(P^j_{N_j}\big)^{-1} f^j\big), \quad j \in \mathcal{J}.$$

Then

$$\begin{aligned}
&(\mathfrak{A} - I)x = f\\
\iff\quad & (\mathfrak{A}^j - I)x^j = f^j, \quad j \in \mathcal{J}\\
\iff\quad & \sum_{k=0}^{N_j} P^j_k (x^j)^{(k)}(\zeta) - x^j(\zeta) = f^j(\zeta), \quad \text{a.e. } \zeta \in (0,1),\ j \in \mathcal{J}\\
\iff\quad & (x^j)^{(N_j)}(\zeta) = (P^j_{N_j})^{-1}\Bigg(x^j(\zeta) - \sum_{k=0}^{N_j-1} P^j_k (x^j)^{(k)}(\zeta) + f^j(\zeta)\Bigg),\\
& \quad \text{a.e. } \zeta \in (0,1),\ j \in \mathcal{J}\\
\iff\quad & (h^j)'(\zeta) = L^j h^j(\zeta) + g^j(\zeta), \quad \text{a.e. } \zeta \in (0,1),\ j \in \mathcal{J}\\
\iff\quad & h^j(\zeta) = \mathrm{e}^{\zeta L^j} h^j(0) + \int_0^\zeta \mathrm{e}^{(\zeta-s)L^j} g^j(s)\,\mathrm{d}s, \quad \text{a.e. } \zeta \in (0,1),\ j \in \mathcal{J}
\end{aligned}$$

where

$$L^j = \begin{bmatrix} 0 & 1 & & & 0\\ \vdots & & & \ddots & \vdots\\ 0 & & & 1 & 0\\ \big(P^j_{N_j}\big)^{-1} - \big(P^j_{N_j}\big)^{-1} P^j_0 & -\big(P^j_{N_j}\big)^{-1} P^j_1 & \ldots & \ldots & -\big(P^j_{N_j}\big)^{-1} P^j_{N_j-1} \end{bmatrix}$$

belongs to $\mathbb{K}^{N^j d^j \times N^j d^j}$. In that case, we have, writing $D_c = (D_c^{ij})_{i,j\in\mathcal{J}}$ for $D_c^{ij} \in \mathcal{B}(Y^j; U^i)$, that $x \in D(A)$ if and only if

$$\begin{aligned}
&W^j_B \begin{bmatrix} \mathrm{e}^{L^j}\\ I \end{bmatrix} h^j(0) + W^j_B \begin{bmatrix} \int_0^1 \mathrm{e}^{(1-s)L^j} g^j(s)\,\mathrm{d}s\\ 0 \end{bmatrix}\\
&= W^j_B \tau^j(\mathcal{H}^j x^j) = -\sum_{i\in\mathcal{J}} D_c^{ji} W^i_C \tau^i(\mathcal{H}^i x^i)\\
&= -\sum_{i\in\mathcal{J}} D_c^{ji}\left(W^i_C \begin{bmatrix} \mathrm{e}^{L^i}\\ I \end{bmatrix} h^i(0) + W^i_C \begin{bmatrix} \int_0^1 \mathrm{e}^{(1-s)L^i} g^i(s)\,\mathrm{d}s\\ 0 \end{bmatrix}\right), \quad j \in \mathcal{J}.
\end{aligned}$$

Letting

$$\widehat{\xi} := \left(\widehat{\xi^j}\right)_{j\in\mathcal{J}}, \quad \widehat{\xi^j} := \begin{bmatrix} \int_0^1 \mathrm{e}^{(1-s)L^j} g^j(s)\,\mathrm{d}s \\ 0 \end{bmatrix},$$
$$\widehat{h} := \left(\widehat{h^j}\right)_{j\in\mathcal{J}}, \quad \widehat{h^j} := \begin{bmatrix} \mathrm{e}^{L^j} \\ I \end{bmatrix} h^j(0), \quad j \in \mathcal{J}$$

and defining

$$W_B := \operatorname{diag}\left(W_B^j\right)_{j\in\mathcal{J}}, \quad W_C := \operatorname{diag}\left(W_C^j\right)_{j\in\mathcal{J}} \in \mathcal{B}(U^2;U),$$
$$T := \operatorname{diag}\left(\begin{bmatrix} \mathrm{e}^{M_j} \\ I \end{bmatrix}\right)_{j\in\mathcal{J}} \in \mathcal{B}(U;U^2),$$

this equation reads as

$$(W_B + D_c W_C)\big(T\widehat{h} - \widehat{\xi}\big) = 0$$

where ξ is determined by f. We are done after showing that $(W_B + D_c W_C)T \in \mathcal{B}(U)$ is invertible. Namely, then the unique solution $x \in D\big(\widehat{A}\big)$ is given by

$$x^j = h_1^j, \quad h^j(0) = \widehat{h}^j, \quad \widehat{h} = \big((W_B - KW_C)T\big)^{-1}(W_B + D_c W_C)\widehat{\xi}.$$

So, let us show that $(W_B + D_c W_C)T \in \mathcal{B}(U)$ is invertible. Since U is finite dimensional it suffices to show that $(W_B + D_c W_C)T$ is injective. *Assume* there were $\widehat{h} \in U \setminus \{0\}$ such that

$$(W_B + D_c W_C)T\widehat{h} = 0$$

Then $h^j(0) = \widehat{h}^j$, $j \in \mathcal{J}$, are well-defined and for $f = 0$ the problem $(I - \mathfrak{A})x = 0$ has a solution $x = (x^j)_{j\in\mathcal{J}} := \left(h^j\right)_{j\in\mathcal{J}} \in D(\mathfrak{A})$ for which we also have

$$\mathfrak{B}x - K\mathfrak{C}x = (W_B + D_c W_C)T\widehat{h} = 0,$$

i.e. $x \in D(A)$ with $Ax = x$, a contradiction to A being dissipative, so $1 \notin \sigma(A)$. This concludes the proof for the case $P_0 = 0$.

For the case of general $P_0 \neq 0$, note that $P_0\mathcal{H}$ is a bounded perturbation of $\widehat{A} - P_0\mathcal{H}$, hence, $\widehat{A} - P_0\mathcal{H}$ generates a C_0-semigroup if and only if $\widehat{A}$ generates a C_0-semigroup. The proof is then completed by the following small observation. □

Lemma 4.5. *The operator $\widehat{A}$ is dissipative on $\widehat{X}$ if and only if the operator $\widehat{A}'$ where the P_0^j are replaced by constant zero matrices is dissipative and additionally for all $j \in \mathcal{J}$ one has*

$$\operatorname{Re}\left(P_0^j(\zeta)\xi^j \mid \xi^j\right)_{\mathbb{K}^{d^j}} \leq 0, \quad a.e.\ \zeta \in (0,1),\ all\ \xi^j \in \mathbb{K}^{d^j},\ j \in \mathcal{J},$$

i.e. the symmetric parts $\operatorname{Sym} P_0^j(\zeta) = \frac{P_0^j(\zeta) + P_0^j(\zeta)^*}{2} \leq 0$ *are negative semi-definite for a.e. $\zeta \in (0,1)$.*

Proof. Use the same strategy as in the proof of Theorem 2.3 and Lemma 2.4 in [5] for every $j \in \mathcal{J}$. □

Having the generation result for static feedback interconnection at hand, we are able to prove the generation result for dynamic feedback interconnection via a finite dimensional linear control system as well.

Theorem 4.6. *$\widehat{A}$ generates a contractive C_0-semigroup on $\widehat{X}$ if and only if $\widehat{A}$ is dissipative. In that case, $\widehat{A}$ has compact resolvent.*

Proof. Clearly, by the Lumer–Phillips Theorem, $\widehat{A}$ is necessarily dissipative if it generates a strongly continuous contraction semigroup $\big(T(t)\big)_{t\geq 0}$. Therefore, we only have to show that this condition is (just as for single port-Hamiltonian systems with static or dynamic boundary feedback) even sufficient. Let us further note that we can restrict ourselves to the case $\mathcal{H}^j = I$ for all $j \in \mathcal{J}$, see e.g. Lemma 7.2.3 in [11], and $P_0 = 0$, cf. Lemma 4.5. By the Lumer–Phillips Theorem, we have to show that ran $\big(\lambda - \widehat{A}\big) = \widehat{X}$ for some $\lambda > 0$ and that $\widehat{A}$ is densely defined. First, we show that $\widehat{A}$ is densely defined. Take any $(x, x_c) \in \widehat{X} = X \times X_c$ and $\varepsilon > 0$. Then, the condition

$$\mathfrak{B}x = -(C_c x_c + D_c \mathfrak{C}x)$$

is equivalent to the condition

$$\mathfrak{B}x + D_c\mathfrak{C}x = -C_c x_c =: w \in U.$$

The left-hand side can be written as

$$\mathfrak{B}x + D_c\mathfrak{C}x = [I\ D_c]\begin{bmatrix} W_B \\ W_C \end{bmatrix}\tau(\mathcal{H}x)$$

where we used the notation

$$W_B = \operatorname{diag}_{j\in\mathcal{J}}\{W_B^j\} \in \mathcal{B}(U\times Y; U), \quad W_C = \operatorname{diag}_{j\in\mathcal{J}}\{W_C^j\} \in \mathcal{B}(U\times Y; Y)$$

and $\tau(\mathcal{H}x) = \big(\tau^j(\mathcal{H}_j x^j)\big)_{j\in\mathcal{J}} \in U^2 = U \times Y = Y^2$. By the definition of a port-Hamiltonian system, the matrix $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]$ is invertible as it is similar to the block-diagonal matrix diag $\left(\left[\begin{smallmatrix} W_B^j \\ W_C^j \end{smallmatrix}\right]\right)_{j\in\mathcal{J}}$. Moreover, the matrix $[I\ D_c] \in \mathcal{B}(U \times Y; U)$ has full rank, in particular $[I\ D_c]\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]$ is surjective, i.e. there is $v \in U \times Y$ such that

$$[I\ D_c]\begin{bmatrix} W_B \\ W_C \end{bmatrix} v = w.$$

One then finds $x_0 \in D(\mathfrak{A})$ such that $\tau(\mathcal{H}x_0) = v$, hence, $(x_0, x_c) \in D\big(\widehat{A}\big)$. Since $\prod_{j\in\mathcal{J}} C_c^\infty\big(0,1;\mathbb{K}^{d_j}\big)$ is dense in X, there is $x_1 \in \prod_{j\in\mathcal{J}} C_c^\infty\big(0,1;\mathbb{K}^{d_j}\big)$ such that $\|x_1 - (x - x_0)\|_X = \|(x_0 + x_1) - x\|_X \leq \varepsilon$, i.e. we find that $\widehat{x}_2 := (x_0 + x_1, x_c) \in D\big(\widehat{A}\big)$ with $\|(x, x_c) - \widehat{x}_2\|_{\widehat{X}} \leq \varepsilon$, i.e. $D\big(\widehat{A}\big)$ is dense in $\widehat{X}$.

It remains to show that ran $\big(\lambda - \widehat{A}\big) = \widehat{X}$ for some $\lambda > 0$. Here, we take $\lambda > 0$ large enough such that $\lambda \in \rho(A_c)$, i.e. $\big(\lambda - A_c\big)^{-1} \in \mathcal{B}(X_c)$ exists. (Note that X_c is finite dimensional, hence, such a choice is always possible.) Take $(f, f_c) \in \widehat{X}$.

We need to find $(x, x_c) \in D(\widehat{A})$ such that $(\lambda I - \widehat{A})(x, x_c) = (f, f_c)$, i.e. $(x, x_c) \in D(\mathfrak{A}) \times X_c$ such that $(\lambda I - \mathfrak{A})x = f$, $(\lambda - A_c)x_c - B_c \mathfrak{C} x = f_c$ and

$$\mathfrak{B}x = -(C_c x_c + D_c \mathfrak{C} x).$$

Since $\lambda \in \rho(A_c)$, this means that in particular $x_c \in X_c$ is given by

$$x_c = (\lambda - A_c)^{-1}(f_c + B_c \mathfrak{C} x)$$

and the interconnection condition then reads

$$\begin{aligned} &\mathfrak{B}x = -C_c(\lambda - A_c)^{-1} f_c - \big(C_c(\lambda - A_c)^{-1} B_c + D_c\big)\mathfrak{C}x \\ \iff \quad &\mathfrak{B}_{\mathrm{cl}} x := \mathfrak{B}x + (C_c(\lambda - A_c)^{-1} B_c + D_c)\mathfrak{C}x = -C_c(\lambda - A_c)^{-1} f_c =: \widetilde{f_c}. \end{aligned}$$

Just as in the single port-Hamiltonian system case, the boundary operator $\mathfrak{B}_{\mathrm{cl}} \in \mathcal{B}(D(\mathfrak{A}); U)$ has a right-inverse $B_{\mathrm{cl}} \in \mathcal{B}(U; D(\mathfrak{A}))$, so we may set

$$x_{\mathrm{new}} := x - B_{\mathrm{cl}} \widetilde{f_c}$$

which, therefore, has to be a solution to the problem

$$\begin{aligned} (\lambda I - \mathfrak{A}) x_{\mathrm{new}} &= f - (\lambda I - \mathfrak{A}) B_{\mathrm{cl}} \widetilde{f_c} =: \widetilde{f}, \\ \mathfrak{B}_{\mathrm{cl}} x_{\mathrm{new}} &= \mathfrak{B}_{\mathrm{cl}} x - \mathfrak{B}_{\mathrm{cl}} B_{\mathrm{cl}} \widetilde{f_c} = 0. \end{aligned}$$

To show that this problem has a (unique) solution, we show that the operator $A_{\mathrm{cl}} := \mathfrak{A}_{\mathrm{cl}}|_{\ker \mathfrak{B}_{\mathrm{cl}}}$ is dissipative and, hence, generates a strongly continuous contraction semigroup on $\widehat{X}$, in particular, $x_{\mathrm{new}} = (\lambda - A_{\mathrm{cl}})^{-1} \widetilde{f}$. In fact, for any $x \in D(A_{\mathrm{cl}})$, set $x_c := (\lambda - A_c)^{-1} B_c \mathfrak{C} x \in X_c$. Then

$$\mathfrak{B}x + C_c x_c + D_c \mathfrak{C} x = \mathfrak{B} x + \big(C_c(\lambda - A_c)^{-1} B_c + D_c\big)\mathfrak{C}x = \mathfrak{B}_{\mathrm{cl}} x = 0$$

so that $(x, x_c) \in D(\widehat{A})$ and, hence,

$$\begin{aligned} &\operatorname{Re}\,(A_{\mathrm{cl}} x \mid x)_X \\ &= \operatorname{Re}\,(\mathfrak{A}x \mid x)_X = \operatorname{Re}\,\Big(\widehat{A}(x, x_c) \mid (x, x_c)\Big)_{\widehat{X}} - \operatorname{Re}\,(A_c x_c + B_c \mathfrak{C} x \mid x_c)_{X_c} \\ &\le - \operatorname{Re}\,(A_c x_c + B_c \mathfrak{C} x \mid x_c)_{X_c} \\ &= - \operatorname{Re}\,\Big(A_c(\lambda - A_c)^{-1} B_c \mathfrak{C} x + (\lambda - A_c)(\lambda - A_c)^{-1} B_c \mathfrak{C} x \mid (\lambda - A_c)^{-1} B_c \mathfrak{C} x\Big)_{X_c} \\ &= -\lambda \big\| (\lambda - A_c)^{-1} B_c \mathfrak{C} x \big\|_{X_c}^2 \le 0. \end{aligned}$$

This shows that A_{cl} is dissipative and by Proposition 4.4 above A_{cl} generates a strongly continuous contraction semigroup on X, in particular $(0, \infty) \subseteq \rho(A_{\mathrm{cl}})$ and, hence, $x_{\mathrm{new}} = (\lambda - A_{\mathrm{cl}})^{-1} \widetilde{f}$. Putting everything together, we obtain the

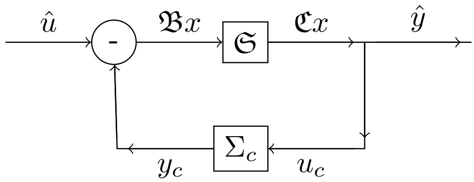

FIGURE 1. A system of port-Hamiltonian type $\mathfrak{S}$ coupled with a finite-dimensional controller Σ_c and external input $\widehat{u}$ and output $\widehat{y}$.

desired $(x, x_c) \in D\big(\widehat{A}\big)$ by solving the problem $\big(\lambda - \widehat{A}\big)(x, x_c) = (f, f_c)$ as

$$\begin{aligned} x &= x_{\text{new}} + B_{\text{cl}}\widetilde{f_c} = \big(\lambda - A_{\text{cl}}\big)^{-1}\widetilde{f} + B_{\text{cl}}\widetilde{f_c} \\ &= \big(\lambda - A_{\text{cl}}\big)^{-1}(f - (\lambda - \mathfrak{A})B_{\text{cl}}\widetilde{f_c}) + B_{\text{cl}}\widetilde{f_c}, \\ x_c &= \big(\lambda - A_c\big)^{-1}(f_c + B_c\mathfrak{C}x) \\ &= \big(\lambda - A_c\big)^{-1}(f_c + B_c\mathfrak{C}\big(\big(\lambda - A_{\text{cl}}\big)^{-1}(f - (\lambda - \mathfrak{A})B_{\text{cl}}\widetilde{f_c}) + B_{\text{cl}}\widetilde{f_c})\big). \end{aligned}$$

The operator $\lambda - \widehat{A}$, therefore, is surjective and the Lumer–Phillips Theorem provides the characterisation of the generator property. The compactness of the resolvent follows for generators $\widehat{A}$ since $D\big(\widehat{A}\big) \subset D(\mathfrak{A}) \times X_c$, where $D(\mathfrak{A}) = \prod_{j\in\mathcal{J}} D(\mathfrak{A}_j)$ is relatively compact as a product of relatively compact (in X_j) spaces $D(\mathfrak{A}_j)$ (by the Rellich–Kondrachov theorem, see e.g. Theorem 8.9 in [16]; all spaces shall be equipped with their respective graph norms), and X_c is finite dimensional, so compactly embedded into itself. □

Similar to the case of Dirac structures, where an interconnection of Dirac structures is a Dirac structure again, the interconnection of port-Hamiltonian systems in boundary control and observation form defines a boundary control and observation system.

Definition 4.7. *For a system as above consisting of a family of port-Hamiltonian systems $\mathfrak{S}^j$ and finite dimensional control system Σ_c^j, we may also introduce external inputs and outputs by setting*

$$\widehat{u} = \widehat{\mathfrak{B}}\widehat{x} = \mathfrak{B}x + C_c x_c + D_c\mathfrak{C}x \in U, \quad \widehat{y} = \widehat{\mathfrak{C}}\,\widehat{x} = \mathfrak{C}x \in Y, \quad (x, x_c) \in D(\mathfrak{A}) \times X_c,$$

see Fig. 1. Moreover, we define the triple $\widehat{\mathfrak{S}} = \big(\widehat{\mathfrak{A}}, \widehat{\mathfrak{B}}, \widehat{\mathfrak{C}}\big)$ with

$$\begin{aligned} \widehat{\mathfrak{A}}(x, x_c) &= \begin{bmatrix} \mathfrak{A} & 0 \\ B_c\mathfrak{C} & A_c \end{bmatrix}(x, x_c), \\ D(\widehat{\mathfrak{A}}) &= D(\mathfrak{A}) \times X_c. \end{aligned}$$

In the following, we call $\widehat{\mathfrak{S}}$ *an (open-loop)* hybrid port-Hamiltonian system*. Note that* $\widehat{A} = \widehat{\mathfrak{A}}|_{\ker \widehat{\mathfrak{B}}}$*. More generally, we also call* $\widehat{\mathfrak{S}} = \big(\widehat{\mathfrak{A}}, \widehat{\mathfrak{B}}, \widehat{\mathfrak{C}}\big)$ *an (open-loop)* hybrid port-Hamiltonian system*, if*

$$\begin{pmatrix} \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} \end{pmatrix} \widehat{x} = \widehat{W} \begin{pmatrix} \mathfrak{B}x + C_c x_c + D_c \mathfrak{C}x \\ \mathfrak{C}x \end{pmatrix}, \quad \widehat{x} \in D(\widehat{\mathfrak{A}})$$

for some invertible matrix $\widehat{W} \in \mathcal{B}(U \times Y)$*.*

These input and output maps $\widehat{\mathfrak{B}}$ and $\widehat{\mathfrak{C}}$ may then be used to interconnect several of such hybrid PDE–ODE systems $\mathfrak{S}$ with each other. As each of such systems consists of infinite-dimensional port-Hamiltonian systems on an interval and finite dimensional control systems, the interconnection of such hybrid systems then again generates a contractive C_0-semigroup if and only if the interconnection makes the total system dissipative. Therefore, with respect to well-posedness such a point of view does not give more information than just considering the system of these hybrid PH systems as one large hybrid PH system. In the next section, however, we exploit structural conditions on the arrangement of such a system to deduce better stability results, i.e. stability under less restrictive conditions.

Remark 4.8. If one chooses $\widehat{W} = I$ in the above definition of an open-loop hybrid PH–ODE system, and additionally all port-Hamiltonian systems $\mathfrak{S}^j = \big(\mathfrak{A}^j, \mathfrak{B}^j, \mathfrak{C}^j\big)$ and the linear controller $\Sigma_c = (A_c, B_c, C_c, D_c)$ are impedance passive, then the triple $\mathfrak{S} = \big(\widehat{\mathfrak{A}}, \widehat{\mathfrak{B}}, \widehat{\mathfrak{C}}\big)$ is impedance passive as well, since for all $\widehat{x} \in D\big(\widehat{\mathfrak{A}}\big)$ one has

$$\begin{aligned} \operatorname{Re}\big(\widehat{\mathfrak{A}}\widehat{x} \mid \widehat{x}\big)_{\widehat{X}} &= \operatorname{Re}\big(\mathfrak{A}x \mid x\big)_X + \operatorname{Re}\big(A_c x_c + B_c \mathfrak{C}x \mid x_c\big)_{X_c} \\ &\le \operatorname{Re}\big(\mathfrak{B}x \mid \mathfrak{C}x\big)_U + \operatorname{Re}\big(C_c x_c + D_c \mathfrak{C}x \mid \mathfrak{C}x\big)_{U_c} \\ &= \operatorname{Re}\big((\mathfrak{B} + D_c\mathfrak{C})x + C_c x_c \mid \mathfrak{C}x\big)_U = \operatorname{Re}\big(\widehat{\mathfrak{B}}\widehat{x} \mid \widehat{\mathfrak{C}}\widehat{x}\big)_U . \end{aligned}$$

5. Stability Properties of Hybrid Multi-PHS-control systems

Let us take the operator $\widehat{A}$ from the previous section, i.e.

$$\begin{aligned} \widehat{A}\,\widehat{x} &= (\mathfrak{A}x, A_c x_c + B_c \mathfrak{C}x) = \big(\big(\mathfrak{A}^j x^j\big)_{j\in\mathcal{J}}, \big(A_c^j x_c^j + B_c^j \mathfrak{C}x\big)_{j\in\mathcal{J}_c}\big), \\ D(\widehat{A}) &= \big\{\widehat{x} = (x, x_c) \in D(\mathfrak{A}) \times X_c : \mathfrak{B}x = -(C_c x_c + D_c \mathfrak{C}x)\big\}, \end{aligned}$$

in particular, we assume $U_c = Y_c = U = Y$ and $E_c = I$, $E = I$. Stability, as for single port-Hamiltonian operators, is much more involved than the generation property.

Proposition 5.1. *Let $\widehat{A}$ be as in Theorem 4.6 with port-Hamiltonian order $N^j = 1$ for all $j \in \mathcal{J}$ and assume that the Hamiltonian density matrix functions $\mathcal{H}^j : [0,1] \to \mathbb{K}^{d^j \times d^j}$ are Lipschitz continuous for all $j \in \mathcal{J}$. If*

$$\operatorname{Re}\left(\widehat{A}\widehat{x} \mid \widehat{x}\right)_{\widehat{X}} \lesssim -\sum_{j=1}^{m} \left|(\mathcal{H}^j x^j)(0)\right|^2, \quad \widehat{x} \in D(\widehat{A})$$

and $\sigma_p(A_c) \subseteq \mathbb{C}_0^-$, then the C_0-semigroup $\left(\widehat{T}(t)\right)_{t\geq 0}$ generated by $\widehat{A}$ is uniformly exponentially stable.

Proof. This result already follows from Corollary 3.10 in [5]. □

Note that the condition imposed in Proposition 5.1 on the interconnection is by far too restrictive for complex systems consisting of several subsystems of infinite-dimensional port-Hamiltonian type and finite-dimensional control systems: All port-Hamiltonian subsystems have to be interconnected in a way that they dissipate energy at the boundary, and all control systems have to be internally stable. The result does in no way require any special structure for the interconnection of the port-Hamiltonian systems, whereas for systems which interconnection structure forms a special class of graphs much less restrictive condition on the dissipative terms can be expected.

In the following, we restrict ourselves to impedance passive port-Hamiltonian systems and strictly input passive control systems as follows.

Assumption 5.2. *We assume that the following hold:*

1. $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *is* impedance passive, *i.e.*

$$\operatorname{Re}\left(\mathfrak{A}x \mid x\right)_X \leq \operatorname{Re}\left(\mathfrak{B}x \mid \mathfrak{C}x\right)_U - \|\mathfrak{R}x\|_Z^2, \quad x \in D(\mathfrak{A})$$

   for some linear operator $\mathfrak{R} : D(\mathfrak{R}) = D(\mathfrak{A}) \subseteq X \to Z$ and some Hilbert space Z,
2. $\Sigma_c = (A_c, B_c, C_c, D_c)$ *is* strictly input passive, *more precisely, there is an orthogonal projection $\Pi : U_c \to U_c$ such that*

$$\ker \Pi = \ker D_c \subseteq \ker B_c$$

   and for some $\kappa > 0$, and all $x_c \in X_c$, $u_c \in U_c$,

$$\operatorname{Re}\left(A_c x_c + B_c u_c \mid x_c\right)_{X_c} \leq \operatorname{Re}\left(C_c x_c + D_c u_c \mid u_c\right)_{U_c} - \kappa \left|\Pi u_c\right|_{U_c}^2,$$

3. $\sigma_p(A_c) \subseteq \mathbb{C}_0^-$, *i.e.* $\left(\mathrm{e}^{tA_c}\right)_{t\geq 0}$ *is uniformly exponentially stable on the finite dimensional space X_c, and*
4. *there are linear operators $\mathfrak{R}^j : D(\mathfrak{R}^j) = D(\mathfrak{A}^j) \subseteq X^j \to Z^j$ (for some Hilbert spaces Z^j), $j \in \mathcal{J}$, such that*

$$\|\mathfrak{R}x\|_Z^2 + |\Pi\mathfrak{C}x|_U^2 + |\mathfrak{B}x|_Y^2 \geq \sum_{j\in\mathcal{J}} \left\|\mathfrak{R}^j x^j\right\|_{Z^j}^2, \quad x \in D(\mathfrak{A}).$$

Remark 5.3. Note that, as a consequence of Assumption 5.2,

$$\operatorname{Re}\left(\widehat{A}\,\widehat{x} \mid \widehat{x}\right)_{\widehat{X}} \leq -\left\|\mathfrak{R}x\right\|_Z^2 - \kappa\left|\Pi\mathfrak{C}x\right|_Y^2, \quad \widehat{x} \in D(\widehat{A}).$$

Moreover, $\ker D_c \subseteq \ker C_c^*$.

Proof. The first assertion directly follows from impedance passivity and standard feedback interconnection. Let us show that $\ker D_c \subseteq \ker C_c^*$. Take $u_c \in \ker D_c \subseteq \ker B_c$. Then, from the impedance passivity of Σ_c, we have for all $x_c \in X_c$ that

$$\begin{aligned}\operatorname{Re}\left(A_c x_c \mid x_c\right)_{X_c} &= \left(A_c x_c + B_c u_c \mid x_c\right)_{X_c} \\ &\leq \operatorname{Re}\left(C_c x_c + D_c u_c \mid u_c\right)_{U_c} = \operatorname{Re}\left(x_c \mid C_c^* u_c\right)_{U_c}.\end{aligned}$$

Since this inequality holds for all $x_c \in X_c$, we deduce that $C_c^* u_c \in X_c^{\perp} = \{0\}$. □

To relate stability properties of the interconnected system, that is, the C_0-semigroup $\left(\widehat{T}(t)\right)_{t\geq 0}$, with structural and damping properties of the involved port-Hamiltonian subsystems, let us introduce the following notions: properties ASP and AIEP (which have already been used in the research article [5]), as well as property AIEP_S (which is a slight modification of property AIEP).

Definition 5.4. *Let $B : D(B) \subseteq H_1 \to H_1$ be a closed linear operator and $R \in \mathcal{B}(D(B); H_2)$, $S \in \mathcal{B}(D(B); H_3)$ for Hilbert spaces H_1, H_2 and H_3, and where $D(B)$ is equipped with its graph norm. We then say that the pair (B, R) has property*

1. *ASP, if $\ker(\mathrm{i}\beta - B) \cap \ker R = \{0\}$ for all $\beta \in \mathbb{R}$, i.e.*

$$\mathrm{i}\beta x = Bx \text{ and } Rx = 0 \implies x = 0.$$

2. *AIEP, if for all sequences $(x_n, \beta_n)_{n\geq 1} \subseteq D(B) \times \mathbb{R}$ with $\sup_{n\geq 1}\|x_n\| < \infty$ and $|\beta_n| \to \infty$,*

$$\mathrm{i}\beta_n x_n - Bx_n \to 0 \text{ and } Rx_n \to 0 \implies x_n \to 0 \text{ in } H_1.$$

3. AIEP_S, *if for all sequences $(x_n, \beta_n)_{n\geq 1} \subseteq D(B) \times \mathbb{R}$ with $\sup_{n\geq 1}\|x_n\| < \infty$ and $|\beta_n| \to \infty$,*

$$\mathrm{i}\beta_n x_n - Bx_n \to 0 \text{ and } Rx_n \to 0 \quad\implies\quad x_n \to 0 \text{ in } H_1 \text{ and } Sx_n \to 0 \text{ in } H_3.$$

With these abstract notions at hand, we can formulate the following stability results.

Theorem 5.5 (Stability properties). *Assume that $\widehat{A}$ satisfies Assumption* 5.2.

1. *If all pairs $(\mathfrak{A}^j, \mathfrak{R}^j)$, $j \in \mathcal{J}$, have property ASP, then the C_0-semigroup $\left(\widehat{T}(t)\right)_{t\geq 0}$ generated by $\widehat{A}$ is (asymptotically) strongly stable.*
2. *If $\left(\widehat{T}(t)\right)_{t\geq 0}$ is asymptotically stable and all pairs $(\mathfrak{A}^j, \mathfrak{R}^j)$ have property AIEP, then $\left(\widehat{T}(t)\right)_{t\geq 0}$ is uniformly exponentially stable.*

3. *If all pairs* $(\mathfrak{A}^j, \mathfrak{R}^j)$ *have property* $\mathrm{AIEP}_{\tau^j \circ \mathcal{H}^j}$, *then the pair*

$$\left(\begin{bmatrix} \mathfrak{A} & 0 \\ B_c \mathfrak{C} & A_c \end{bmatrix}, \mathfrak{B}x + D_c \mathfrak{C}x + C_c x_c \right)$$

has property $\mathrm{AIEP}_{\tau \circ \mathcal{H}}$ *as well, where* $\tau(\mathcal{H}x) = \left(\tau^j\left(\mathcal{H}_j x^j\right)\right)_{j \in \mathcal{J}}$.

Proof. 1. We show strong stability by demonstrating that $\sigma_p\big(\widehat{A}\big) \subseteq \mathbb{C}_0^-$, which by the Arendt–Batty–Lyubich–Vũ Theorem is enough for strong stability as $\widehat{A}$ has compact resolvent. Clearly, since $\widehat{A}$ is dissipative, we have $\sigma\big(\widehat{A}\big) \subseteq \overline{\mathbb{C}_0^-}$, i.e. we only need to check that no $\mathrm{i}\beta \in \mathrm{i}\mathbb{R}$ is an eigenvalue of $\widehat{A}$. Thus, let $\widehat{x} = (x, x_c) \in D\big(\widehat{A}\big)$ be such that $\widehat{A}\,\widehat{x} = \mathrm{i}\beta\,\widehat{x}$ for some $\beta \in \mathbb{R}$. Then, in particular

$$\begin{pmatrix} \mathfrak{A}x \\ A_c x_c + B_c \mathfrak{C}x \end{pmatrix} = \begin{pmatrix} \mathrm{i}\beta x \\ \mathrm{i}\beta x_c \end{pmatrix} \implies \begin{pmatrix} (\mathfrak{A} - \mathrm{i}\beta)x \\ x_c \end{pmatrix} = \begin{pmatrix} 0 \\ (\mathrm{i}\beta - A_c)^{-1} B_c \mathfrak{C}x \end{pmatrix}$$

(note that $\mathrm{i}\mathbb{R} \subseteq \rho(A_c)$ by Assumption 5.2). Since $\widehat{x} \in D\big(\widehat{A}\big)$, we then have

$$\mathfrak{B}x = -(C_c x_c + D_c \mathfrak{C}x) = -\left[C_c(\mathrm{i}\beta - A_c)^{-1} B_c + D_c\right] \mathfrak{C}x$$

and from impedance passivity of $\mathfrak{S}$ and Σ_c, we obtain

$$\begin{aligned} 0 &= \mathrm{Re}\, \left(\mathrm{i}\beta x \mid x\right)_X = \mathrm{Re}\, \left(\mathfrak{A}x \mid x\right)_X \\ &\le \mathrm{Re}\, \left(\mathfrak{B}x \mid \mathfrak{C}x\right)_U = -\,\mathrm{Re}\, \left(\left(C_c(\mathrm{i}\beta - A_c)^{-1} B_c + D_c\right)\mathfrak{C}x \mid \mathfrak{C}x\right)_Y \\ &\le -\,\mathrm{Re}\, \left(A_c(\mathrm{i}\beta - A_c)^{-1} B_c \mathfrak{C}x + B_c \mathfrak{C}x \mid (\mathrm{i}\beta - A_c)^{-1} B_c \mathfrak{C}x\right)_{X_c} - \kappa\, |\Pi \mathfrak{C}x|^2 \\ &= -\,\mathrm{Re}\, \left(\mathrm{i}\beta(\mathrm{i}\beta - A_c)^{-1} B_c \mathfrak{C}x \mid (\mathrm{i}\beta - A_c)^{-1} B_c \mathfrak{C}x\right)_{X_c} - \kappa\, |\Pi \mathfrak{C}x|^2 \\ &= -\kappa\, |\Pi \mathfrak{C}x|^2 \le 0. \end{aligned}$$

This chain of inequalities shows that $\Pi \mathfrak{C}x = 0$, hence, $B_c \mathfrak{C}x = 0$ due to $\ker D_c \subseteq \ker B_c$, and then $x_c = (\mathrm{i}\beta - A_c)^{-1} B_c \mathfrak{C}x = 0$ so that

$$\mathfrak{B}x = -\left[C_c(\mathrm{i}\beta - A_c)^{-1} B_c + D_c\right] \mathfrak{C}x = 0.$$

Moreover,

$$0 = \mathrm{Re}\, \left(\mathrm{i}\beta\,\widehat{x} \mid \widehat{x}\right)_{\widehat{X}} = \mathrm{Re}\, \left(\widehat{A}\,\widehat{x} \mid \widehat{x}\right)_{\widehat{X}} \le -\, \|\mathfrak{R}x\|_Z^2 - \kappa\, |\Pi \mathfrak{C}x|^2 \le 0,$$

so that $\mathfrak{R}x = 0$, $\mathfrak{B}x = 0$ and $\Pi \mathfrak{C}x = 0$, in particular $\mathfrak{R}^j x^j = 0$ for all $j \in \mathcal{J}$, and by property ASP of the pairs $(\mathfrak{A}^j, \mathfrak{R}^j)$ this implies that $x^j = 0$ for all $j \in \mathcal{J}$, but then $\mathfrak{C}x = 0$ as well as $x_c = 0$, i.e. $\widehat{x} = 0$ and $\sigma_p\big(\widehat{A}\big) \cap \mathrm{i}\mathbb{R} = \emptyset$. Strong stability follows.

2. For uniform exponential stability, we use the Gearhart–Prüss–Huang Theorem, i.e. we show that $\sup_{\beta \in \mathbb{R}} \left\| \left(\mathrm{i}\beta - \widehat{A}\right)^{-1} \right\|_{\mathcal{B}(\widehat{X})} < \infty$. By Remark 2.6, this property is equivalent to showing that for every sequence $(\widehat{x}_n, \beta_n)_{n \ge 1} \subseteq D\big(\widehat{A}\big) \times \mathbb{R}$ with

$\sup_{n\in\mathbb{N}} \|\widehat{x}_n\|_{\widehat{X}} < \infty$ and $|\beta_n| \to \infty$ and $\widehat{A}\,\widehat{x}_n - \mathrm{i}\beta_n\widehat{x}_n$, we have $\widehat{x}_n \to 0$ in $\widehat{X}$. In view of the third assertion, we even show a little bit more, namely

$$\left.\begin{array}{l} (\widehat{x}_n)_{n\geq 1} \subseteq D(\mathfrak{A}) \times X_c,\ \sup_{n\in\mathbb{N}} \|\widehat{x}_n\|_{\widehat{X}} < \infty \\ (\beta_n)_{n\geq 1} \subseteq \mathbb{R},\ |\beta_n| \to \infty \\ (\mathrm{i}\beta_n - \mathfrak{A})x_n \to 0 \ \text{ in } X \\ (\mathrm{i}\beta_n - A_c)x_{c,n} - B_c\mathfrak{C}x_n \to 0 \ \text{ in } X_c \\ \mathfrak{B}x_n + C_c x_{c,n} + D_c\mathfrak{C}x_n \to 0 \ \text{ in } \operatorname{ran}\begin{bmatrix} C_c & D_c \end{bmatrix} \subseteq U \end{array}\right\} \implies \widehat{x}_n \to 0 \ \text{ in } \widehat{X}. \quad (*)$$

Let $(\widehat{x}_n, \beta_n)_{n\geq 1}$ be a sequence as on the left-hand side. Using Assumption 5.2, we obtain that

$$0 \leftarrow \operatorname{Re} \big((\mathfrak{A} - \mathrm{i}\beta_n)x_n \;\big|\; x_n\big)_X = \operatorname{Re} \big(\mathfrak{A}x_n \;\big|\; x_n\big)_X \leq \operatorname{Re} \big(\mathfrak{B}x_n \;\big|\; \mathfrak{C}x_n\big)_U - \|\mathfrak{R}x_n\|_Z^2,$$

$$\begin{aligned} 0 &\leftarrow \operatorname{Re} \big((A_c - \mathrm{i}\beta_n)x_{c,n} + B_c\mathfrak{C}x_n \;\big|\; x_{c,n}\big)_{X_c} \\ &\leq \operatorname{Re} \big(C_c x_{c,n} + D_c\mathfrak{C}x_n \;\big|\; \mathfrak{C}x_n\big) - \kappa\,|\Pi\mathfrak{C}x_n|^2 \end{aligned}$$

and adding up these two inequalities we derive

$$\liminf_{n\to\infty} \operatorname{Re} \big((\mathfrak{B} + D_c\mathfrak{C})x_n + C_c x_{c,n} \;\big|\; \mathfrak{C}x_n\big) - \|\mathfrak{R}x_n\|_Z^2 - \kappa\,|\Pi\mathfrak{C}x_n|^2 \geq 0.$$

Now, since $\ker D_c \subseteq \ker B_c \cap \ker C_c^*$, and $(\mathfrak{B} + D_c\mathfrak{C})x_n + C_c x_{c,n}$ by choice of the sequence, cf. $(*)$, lies in $\operatorname{ran}\begin{bmatrix} C_c & D_c \end{bmatrix}$, this inequality is equivalent to the statement

$$\liminf_{n\to\infty} \operatorname{Re} \big((\mathfrak{B} + D_c\mathfrak{C})x_n + C_c x_{c,n} \;\big|\; \Pi\mathfrak{C}x_n\big) - \|\mathfrak{R}x_n\|_Z^2 - \kappa\,|\Pi\mathfrak{C}x_n|^2 \geq 0.$$

Namely, for every $C_c\eta$, $D_c\mu$ one has

$$\begin{aligned} \big(C_c\eta \;\big|\; (I - \Pi)\mathfrak{C}x\big) &= \big(\eta \;\big|\; C_c^*(I - \Pi)\mathfrak{C}x\big) = 0, \\ \big(D_c\mu \;\big|\; (I - \Pi)\mathfrak{C}x\big) &= \big((I - \Pi)D_c\mu \;\big|\; \mathfrak{C}x\big) = 0 \end{aligned}$$

as $(I - \Pi)$ projects onto $\ker D_c$. Since $(\mathfrak{B} + D_c\mathfrak{C})x_n + C_c x_{c,n} \to 0$ by $(*)$, we then deduce that $\Pi\mathfrak{C}x_n \to 0$ and $\mathfrak{R}x_n \to 0$: *Assume* $\limsup_{n\to\infty} |\Pi\mathfrak{C}x_n| > 0$. Dividing by $|\Pi\mathfrak{C}x_n|$ for a suitable subsequence then gives

$$\liminf_{n\to\infty} -\frac{\|\mathfrak{R}x_n\|_Z^2}{\|\Pi\mathfrak{C}x_n\|} - \kappa\,|\Pi\mathfrak{C}x_n| \geq 0$$

and $\limsup_{n\to\infty} |\Pi\mathfrak{C}x_n| = 0$, a contradiction. Hence, $\lim_{n\to\infty} |\mathfrak{C}x_n| = 0$ and then

$$\liminf_{n\to\infty} -\|\mathfrak{R}x_n\|_Z^2 = 0$$

gives $\lim_{n\to\infty} \mathfrak{R}x_n = 0$ as well. Since $\ker \Pi \subseteq \ker B_c \cap \ker D_c$, this also implies that

$$B_c\mathfrak{C}x_n, D_c\mathfrak{C}x_n \to 0 \ \text{ in } Y.$$

Therefore,

$$x_{c,n} = \big(\mathrm{i}\beta_n - A_c\big)^{-1}\big[B_c\mathfrak{C}x_n - (B_c\mathfrak{C}x_n + A_c x_{c,n} - \mathrm{i}\beta_n x_{c,n})\big] \to 0 \ \text{ in } X_c,$$

using that $\sup_{\beta\in\mathbb{R}} \left\|\big(\mathrm{i}\beta - A_c\big)^{-1}\right\|_{\mathcal{B}(X_c)} < \infty$ and both $B_c\mathfrak{C}x_n$ and $B_c\mathfrak{C}x_n + A_c x_{c,n} - \mathrm{i}\beta_n x_{c,n}$ tend to zero. As a consequence, also

$$\mathfrak{B}x_n = (\mathfrak{B}x_n + D_c\mathfrak{C}x_n + C_c x_{c,n}) - D_c\mathfrak{C}x_n - C_c x_{c,n} \to 0 \ \text{ in } U$$

as all three summands converge to zero. Then

$$\sum_{j\in\mathcal{J}} \left\|\mathfrak{R}^j x_n^j\right\|_{Z_j}^2 \leq |\mathfrak{B}x_n|_U^2 + |\Pi\mathfrak{C}x_n|_Y^2 + \|\mathfrak{R}x_n\|_Z^2 \to 0 \implies \mathfrak{R}^j x_n^j \to 0,\ j\in\mathcal{J}.$$

Now, for every $j \in \mathcal{J}$, we have $(\mathfrak{A}^j - \mathrm{i}\beta_n)x_n^j \to 0$ and $\mathfrak{R}^j x_n^j \to 0$, so that by property AIEP we obtain $x_n^j \to 0$ in X^j for all $j \in \mathcal{J}$, i.e. $x_n \to 0$ in X as well, i.e. $\widehat{x}_n \to 0$ in $\widehat{X}$.

Next, let us show the assertion on uniform exponential stability. By the Gearhart–Prüss–Huang Theorem, we need to show that

$$\begin{cases} (\widehat{x}_n)_{n\geq 1} \subseteq D\big(\widehat{A}\big),\ \sup_{n\in\mathbb{N}} \|\widehat{x}_n\|_{\widehat{X}} < \infty \\ (\widehat{\beta}_n)_{n\geq 1} \subseteq \mathbb{R},\ |\beta_n| \to \infty \\ \big(\widehat{A} - \mathrm{i}\beta_n\big)x_n \to 0 \text{ in } \widehat{X} \end{cases} \implies \widehat{x}_n \to 0 \text{ in } \widehat{X}.$$

So let $(\widehat{x}_n, \beta_n)_{n\geq 1} \subseteq D\big(\widehat{A}\big) \times \mathbb{R}$ be such a sequence. Then, by dissipativity of $\widehat{A}$ we have

$$0 \leftarrow \operatorname{Re}\left(\big(\widehat{A} - \mathrm{i}\beta_n\big)\widehat{x}_n \mid \widehat{x}_n\right)_{\widehat{X}} = \operatorname{Re}\left(\widehat{A}\,\widehat{x}_n \mid \widehat{x}_n\right)_{\widehat{X}} \leq -\|\mathfrak{R}x_n\|_Z^2 - \kappa\,|\Pi\mathfrak{C}x_n|_Y^2 \leq 0$$

and, therefore, $\mathfrak{R}x_n \to 0$ and $\Pi\mathfrak{C}x_n \to 0$. Moreover, $(\mathfrak{B} + D_c\mathfrak{C})x_n + C_c x_{c,n} = 0$ by definition of $D\big(\widehat{A}\big)$ and $\big(\widehat{A} - \mathrm{i}\beta_n\big)\widehat{x}_n \to 0$ means that in particular

$$(\mathfrak{A} - \mathrm{i}\beta_n)x_n \to 0, \quad (A_c - \mathrm{i}\beta_n)x_{c,n} + B_c\mathfrak{C}x_n \to 0.$$

By property $(*)$, this means that $\widehat{x}_n \to 0$ in $\widehat{X}$ and uniform exponential stability follows.

3. If for all $j \in \mathcal{J}$, we even have property $\mathrm{AIEP}_{\tau^j \circ \mathcal{H}^j}$, then for the sequence $(\widehat{x}_n, \beta_n)_{n\geq 1}$ as in $(*)$ of the previous case we do not only have $x_n^j \to 0$, but also $\tau^j\big(\mathcal{H}^j x_n^j\big) \to 0$ for all $j \in \mathcal{J}$, so that the last assertion follows as well. □

6. Networks of Hybrid PH-ODE Systems

Next, we want to exploit possible *structural conditions* on the hybrid interconnected port-Hamiltonian-control system to have uniform exponential stability under more restrictive structural assumptions, but weaker assumptions on the dissipativity of the subsystems. Instead of viewing the system as a family of port-Hamiltonian systems $\mathfrak{S}^j$ which are coupled via boundary feedback and control with a finite-dimensional control system, we cluster the port-Hamiltonian systems and parts of the finite-dimensional control system into hybrid PH-ODE systems $\widehat{\mathfrak{S}}^j$ $(j \in \widehat{\mathcal{J}})$ as in Definition 4.7 and assume that the resulting evolutionary system can be written in an equivalent *serially connected* (or, maybe more precisely, *rooted graph*) form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\widehat{x}^j = \widehat{\mathfrak{A}}^j\widehat{x}^j := \widehat{\mathfrak{A}}^j\widehat{x}^j, \\ \widehat{\mathfrak{B}}^j\widehat{x}^j = \sum_{i\in\widehat{\mathcal{J}}} \widehat{K}^{ij}\widehat{\mathfrak{C}}^i\widehat{x}^i, \quad j \in \widehat{\mathcal{J}} \end{cases}$$

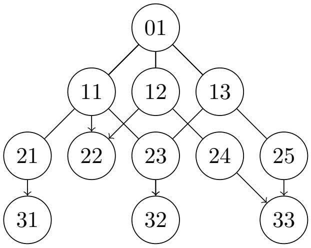

FIGURE 2. Example of a rooted graph.

where $D(\widehat{\mathfrak{A}}^j) = D(\widehat{\mathfrak{C}}^j) = D(\widehat{\mathfrak{B}}^j) = D(\widehat{\mathfrak{A}}^j)$ and

$$\widehat{\mathfrak{B}}^j : D(\widehat{\mathfrak{B}}^j) \subseteq \widehat{X} \to \widehat{U}^j, \quad \widehat{\mathfrak{C}}^j : D(\widehat{\mathfrak{C}}^j) \subseteq \widehat{X} \to \widehat{Y}^j$$

and where the Hilbert spaces $\widehat{U}^j$ and $\widehat{Y}^j$ may be distinct, but $\widehat{U}^j \times \widehat{Y}^j \cong \widehat{U} \times \widehat{Y}$. Moreover, for this interconnection to be *serial* (or, in *rooted graph form*) we demand the following.

Assumption 6.1. *Assume that* $\widehat{K} = (\widehat{K}^{ij})_{i,j\in\widehat{\mathcal{J}}}$ *is strictly lower-block triangular, i.e.* $\widehat{K}^{ij} = 0$ *for* $i, j \in \widehat{\mathcal{J}}$ *with* $\mathrm{i} \leq j$.

Under this assumption one can hope for better (i.e. less restrictive) conditions for asymptotic or uniform exponential stability, similar to the interconnection of a PHS with a finite dimensional control system.

Assumption 6.2. *There are linear maps* $\widehat{\mathfrak{R}}^j : D(\widehat{\mathfrak{R}}^j) = D(\widehat{\mathfrak{A}}^j) \to \widehat{\mathcal{Z}}^j$ $(j \in \widehat{\mathcal{J}})$ *such that*

$$\operatorname{Re}\left(\widehat{A}\,\widehat{x} \mid \widehat{x}\right)_{\widehat{X}} \leq -\sum_{j\in\widehat{\mathcal{J}}} \left\|\widehat{\mathfrak{R}}^j \widehat{x}^j\right\|^2_{\widehat{\mathcal{Z}}^j}, \quad \widehat{x} \in D(\widehat{A}).$$

Under these two assumptions we can formulate the following

Theorem 6.3 (Asymptotic stability). *Let Assumptions* 6.1 *and* 6.2 *hold true. Assume that* $\sigma(A_c) \subseteq \mathbb{C}_0^-$, *i.e.* $\left(\mathrm{e}^{tA_c}\right)_{t\geq 0}$ *is an exponentially stable semigroup on* X_c, *and that for all* $j \in \widehat{\mathcal{J}}$ *the pairs* $\left(\widehat{\mathfrak{A}}^j, \left(\widehat{\mathfrak{B}}^j, \widehat{\mathfrak{R}}^j\right)\right)$ *have property ASP, i.e.*

$$\left.\begin{array}{l} \widehat{x}^j \in D(\widehat{\mathfrak{A}}^j) \\ \beta \in \mathbb{R} \\ \widehat{\mathfrak{A}}^j \widehat{x}^j = \mathrm{i}\beta\, \widehat{x}^j \\ \left(\widehat{\mathfrak{B}}^j \widehat{x}^j, \widehat{\mathfrak{R}}^j \widehat{x}^j\right) = 0 \end{array}\right\} \quad \Longrightarrow \quad \widehat{x}^j = 0. \tag{ASP}$$

Then $\widehat{A}$ *generates an (asymptotically) strongly stable* C_0*-semigroup* $(\widehat{T}(t))_{t\geq 0}$ *on* $\widehat{X}$.

Proof. We use the Arendt–Batty–Lyubich–Vũ Theorem again. Since $\widehat{A}$ generates a contractive C_0-semigroup and has compact resolvent by Theorem 4.6, we need to show that $\sigma_p(\widehat{A}) \cap \mathrm{i}\mathbb{R} = \emptyset$. Let $\widehat{x} \in D(\widehat{A})$ such that $\widehat{A}\,\widehat{x} = \mathrm{i}\beta\,\widehat{x}$ for some $\beta \in \mathbb{R}$. Then, in particular

$$0 = \mathrm{Re}\,\left(\mathrm{i}\beta\,\widehat{x} \mid \widehat{x}\right)_{\widehat{X}} = \mathrm{Re}\,\left(\widehat{A}\,\widehat{x} \mid \widehat{x}\right)_{\widehat{X}} \leq - \sum_{j \in \widehat{\mathcal{J}}} \left\| \widehat{\mathfrak{R}}^j \widehat{x}^j \right\|_{\mathcal{Z}^j}^2 \leq 0$$

and, therefore, $\widehat{\mathfrak{R}}^j \widehat{x}^j = 0$ for all $j \in \widehat{\mathcal{J}}$. Moreover, by definition of $\widehat{A}$ and Assumption 6.1, we have

$$\widehat{\mathfrak{B}}^j \widehat{x}^j = \sum_{i=1}^{j-1} \widehat{K}^{ij} \widehat{\mathfrak{C}}^i \widehat{x}^i, \quad j \in \widehat{\mathcal{J}}.$$

Hence, whenever we know that $\widehat{x}^i = 0$ for all $i < j$, then $\left(\widehat{\mathfrak{R}}^j \widehat{x}^j, \widehat{\mathfrak{B}}^j \widehat{x}^j\right) = 0$ and since also $\widehat{\mathfrak{A}}^j \widehat{x}^j = \mathrm{i}\beta\,\widehat{x}^j$, property ASP implies that then $\widehat{x}^j = 0$ as well. Since this is certainly true for $j = 1$, it follows iteratively that $\widehat{x}^j = 0$ for all $j \in \widehat{\mathcal{J}}$, i.e. $\widehat{x} = 0$ and, therefore, $\sigma_p(\widehat{A}) \cap \mathrm{i}\mathbb{R} = \emptyset$. The Arendt–Batty–Lyubich–Vũ Theorem gives us strong stability of the semigroup $\left(\widehat{T}(t)\right)_{t \geq 0}$. □

Similarly, for uniform exponential stability the following result relies on property AIEP_τ.

Theorem 6.4 (Uniform exponential stability). *Assume that Assumption* 6.1 *and* 6.2 *hold true. Further assume that* $\widehat{A}$ *generates an* (*asymptotically*) *strongly stable contraction semigroup* $\left(\widehat{T}(t)\right)_{t \geq 0}$ *on* $\widehat{X}$*, and that for all* $j \in \mathcal{J}$ *the pairs* $\left(\widehat{\mathfrak{A}}^j, \left(\widehat{\mathfrak{B}}^j, \widehat{\mathfrak{R}}^j\right)\right)$ *have property* $\mathrm{AIEP}_{\widehat{\mathfrak{C}}^j}$*, i.e.*

$$\left.\begin{array}{l} \left(\widehat{x}_n^j\right)_{n \geq 1} \subseteq D\left(\widehat{\mathfrak{A}}^j\right) \\ \sup_{n \in \mathbb{N}} \|\widehat{x}_n^j\|_{\widehat{X}^j} < \infty \\ (\beta_n)_{n \geq 1} \subseteq \mathbb{R} \\ |\beta_n| \to \infty \\ \left(\widehat{\mathfrak{A}}^j - \mathrm{i}\beta_n\right)\widehat{x}_n \to 0 \\ \left(\widehat{\mathfrak{B}}^j \widehat{x}_n^j, \widehat{\mathfrak{R}}^j \widehat{x}_n^j\right) \to 0 \end{array}\right\} \implies \left\{\begin{array}{l} \widehat{x}_n^j \to 0 \quad \text{in } \widehat{X}^j, \\ \widehat{\mathfrak{C}}^j \widehat{x}_n^j \to 0 \quad \text{in } \widehat{Y}^j. \end{array}\right. \qquad (\mathrm{AIEP}_{\widehat{\mathfrak{C}}^j})$$

Then the C_0*-semigroup* $\left(\widehat{T}(t)\right)_{t \geq 0}$ *is uniformly exponentially stable.*

Remark 6.5. The assumption that in $(\mathrm{AIEP}_{\widehat{\mathfrak{C}}^j})$ one has $\widehat{\mathfrak{C}}^j \widehat{x}_n^j \to 0$ in $\widehat{Y}^j$ could be weakened to $\Pi^j \widehat{\mathfrak{C}}^j \widehat{x}_n^j \to 0$ in $\widehat{Y}^j$ where $\Pi^j : \widehat{Y}^j \to \widehat{Y}^j$ is the orthogonal projection onto $\left(\bigcap_{i > j} \ker \widehat{K}^{ij}\right)^\perp$, however, in concrete examples this does not make any difference. If necessary, one could extend the system by an artificial additional hybrid system $\mathfrak{S}$ to ensure the structure of Theorem 6.4.

Proof of Theorem 6.4. Since $\widehat{A}$ generates an asymptotically stable semigroup and has compact resolvent, $\sigma(\widehat{A}) = \sigma_p(\widehat{A}) \subseteq \mathbb{C}_0^-$ and we thus only have to prove that

$\sup_{\beta\in\mathbb{R}} \left\| (\mathrm{i}\beta - \widehat{A})^{-1} \right\| < \infty$. Therefore, take any sequence $(\widehat{x}_n, \beta_n)_{n\geq 1} \subseteq D(\widehat{A}) \times \mathbb{R}$ such that $\sup_{n\in\mathbb{N}} \|\widehat{x}_n\|_{\widehat{X}} < \infty$, $|\beta_n| \to \infty$ and $(\mathrm{i}\beta_n - \widehat{A})\widehat{x}_n \to 0$ in $\widehat{X}$. Then, by Assumption 6.2 we obtain

$$0 \leftarrow \mathrm{Re}\left((\widehat{A} - \mathrm{i}\beta_n)\widehat{x}_n \mid \widehat{x}_n \right)_{\widehat{X}} = \mathrm{Re}\left(\widehat{A}\widehat{x}_n \mid \widehat{x}_n \right)_{\widehat{X}} \leq - \sum_{j\in\widehat{\mathcal{J}}} \left\| \widehat{\mathfrak{R}}^j \widehat{x}_n^j \right\|^2_{\widehat{\mathcal{Z}}^j} \leq 0$$

and, therefore, $\widehat{\mathfrak{R}}^j \widehat{x}^j \to 0$ for all $j \in \widehat{\mathcal{J}}$. Moreover, by Assumption 6.1, we have

$$\widehat{\mathfrak{B}}^j \widehat{x}_n^j = \sum_{i=1}^{j-1} \widehat{K}^{ji} \widehat{\mathfrak{C}}^i \widehat{x}_n^i, \quad j \in \widehat{\mathcal{J}}$$

and property $\mathrm{AIEP}_{\widehat{\mathfrak{C}}^j}$ now implies that $\widehat{x}_n^j \to 0$ and $\widehat{\mathfrak{C}}^j \widehat{x}_n^j \to 0$ whenever $\widehat{\mathfrak{C}}^i \widehat{x}_n^i \to 0$ for all $i < j$. Again, this is true for $j = 1$ and by induction it follows that $\widehat{x}^j \to 0$ and $\widehat{\mathfrak{C}}^j \widehat{x}_n^j \to 0$ for all $j \in \widehat{\mathcal{J}}$. In particular, $\widehat{x}_n \to 0$ in $\widehat{X}$ and, therefore, by the Gearhart–Prüss–Huang Theorem the semigroup $(\widehat{T}(t))_{t\geq 0}$ is uniformly exponentially stable. □

7. Applications

We now discuss the properties ASP and $\mathrm{AIEP}_{\widehat{\mathfrak{C}}^j}$ for some particular classes of PDE which are of port-Hamiltonian type. We aim to give several types of interconnection structures, thus motivating the abstract results of the previous sections. We begin with

Proposition 7.1. *Assume that $N_j = 1$ for all $\mathcal{J} = \widehat{\mathcal{J}}$ (i.e. every hybrid PH-ODE systems consists of exactly one port-Hamiltonian system $\mathfrak{S}^j$ and a controller Σ_c^j), all Hamiltonian matrix density functions $\mathcal{H}^j$ $(j \in \mathcal{J})$ are Lipschitz continuous on $[0,1]$, $\sigma_p(A_c^j) \subseteq \mathbb{C}_0^-$ for all $j \in \mathcal{J}$, there are $\mathfrak{R}^j : D(\mathfrak{A}^j) \to \mathcal{Z}^j$ such that*

$$\mathrm{Re}\left(\widehat{A}x \mid x \right)_{\widehat{X}} \leq - \sum_{j\in\mathcal{J}} \left| \mathfrak{R}^j x^j \right|^2$$

and

$$\left| (\mathcal{H}^j x^j)(0) \right| \lesssim \left| \mathfrak{R}^j \widehat{x}^j \right| + \left| \mathfrak{B}^j x^j \right|, \quad x^j \in D(\mathfrak{A}^j), \quad j \in \mathcal{J}.$$

Then the C_0-semigroup generated by $\widehat{A}$ is uniformly exponentially stable.

Proof. This proposition follows from the Theorems 6.3 and 6.4 above, and with the following lemma on port-Hamiltonian systems of order $N = 1$ and Theorem 5.5 (the latter traducing properties ASP and $\mathrm{AIEP}_{\tau\circ\mathcal{H}}$ from the systems $\mathfrak{S}^j$ $(j \in \mathcal{J})$ to $\mathfrak{A}^j$ $(j \in \mathcal{J})$). □

Lemma 7.2. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a port-Hamiltonian system of order $N = 1$ and $\mathcal{H} : [0,1] \to \mathbb{K}^{d\times d}$ be Lipschitz continuous. Then the following assertions hold true:*

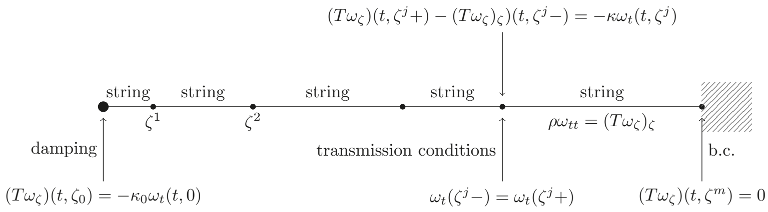

FIGURE 3. A Chain of Serially Interconnected Strings.

1. *If* $x \in D(\mathfrak{A})$ *with* $\mathfrak{A}x = \mathrm{i}\beta x$ *for some* $\beta \in \mathbb{R}$*, and additionally* $(\mathcal{H}x)(0) = 0$*, then* $x = 0$*.*
2. *If* $(x_n, \beta_n)_{n\geq 1} \subseteq D(\mathfrak{A}) \times \mathbb{R}$ *with* $\sup_{n\in\mathbb{N}} \|x_n\|_X < \infty$*,* $|\beta_n| \to \infty$ *and* $(\mathfrak{A} - \mathrm{i}\beta_n)x_n \to 0$ *in* X*,* $(\mathcal{H}x_n)(0) \to 0$ *in* $\mathbb{K}^d$*, then* $x_n \to 0$ *in* X *and* $(\mathcal{H}x_n)(1) \to 0$ *in* $\mathbb{K}^d$*.*

Proof. 1. See the proof of Proposition 2.11 in [5].
2. For the property that $x_n \to 0$ in X, see the proof of Proposition 2.12 in [5]. Repeating the proof presented there for $q = 1$ shows that that

$$\frac{1}{2}\|x_n\|_X^2 + \frac{1}{2}\Big[\big(x_n(\zeta) \;\big|\; \mathcal{H}(\zeta)x_n(\zeta)\big)_{\mathbb{K}^d}\Big]_0^1 \to 0,$$

and since $x_n \to 0$ in X, $(\mathcal{H}x_n)(0) = \mathcal{H}(0)x_n(0) \to 0$ and $\mathcal{H}(1)$ is symmetric positive definite, this implies that $(\mathcal{H}x_n)(1) \to 0$ as well. □

Example 7.3 (Serially Connected Strings). As an example where the structure of the interconnection can be employed to ensure uniform exponential stability, consider the following chain of serially connected strings, see Fig. 3, which are modelled by the *inhomogeneous one-dimensional wave equation*:

$$\rho(\zeta)\omega_{tt}(t,\zeta) - (T(\zeta)\omega_\zeta)_\zeta(t,\zeta) = 0, \quad \zeta \in (\zeta^{j-1},\zeta^j),\ t \geq 0, \quad j = 1,\dots,m$$

where $0 =: \zeta^0 < \zeta^1 < \cdots < \zeta^m := L$ and $0 < \varepsilon \leq \rho^j := \rho|_{(\zeta^{j-1},\zeta^j)}, T^j := T|_{(\zeta^{j-1},\zeta^j)} \in \mathrm{Lip}(\zeta^{j-1},\zeta^j;\mathbb{R})$. The chain of strings is damped at the left end, free at the right end, and interconnected in a dissipative or conservative way:

$$\begin{aligned}
(T\omega_\zeta)(t,\zeta_0) &= -\kappa^0\omega_t(t,\zeta_0), \quad t \geq 0 \quad (\text{for some } \kappa^0 > 0),\\
(T\omega_\zeta)(t,\zeta^m) &= 0, \quad t \geq 0,\\
\omega_t(t,\zeta^j-) &= \omega_t(t,\zeta^j+), \quad t \geq 0,\ j = 1,\dots,m-1,\\
(T\omega_\zeta)(t,\zeta^j-) - (T\omega_\zeta)(t,\zeta^j+) &= -\kappa^j\omega_t(t,\zeta^j),\\
&\qquad t \geq 0,\ j = 1,\dots,m-1 \quad (\text{for some } \kappa^j \geq 0).
\end{aligned} \tag{1}$$

We show that this example can be written as a network of port-Hamiltonian systems of order $N = 1$, and the theory developed in this section can be applied to deduce stability properties for this system. Using a scaling argument we may and reduce the general case to the special case $\zeta^j = j$. We may then identify $x^j(t,\zeta) := (\rho(j+\zeta)\omega_t(t,j+\zeta), -\omega_\zeta(t,j+\zeta))$ and $\mathcal{H}^j(\zeta) := \operatorname{diag}(1/\rho(j+\zeta), T(j+\zeta))$ and obtain for $P_1^j = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], P_0^j = 0 \in \mathbb{K}^{2\times 2}$, $j = 2, \dots, m$, the first order port-Hamiltonian systems

$$\begin{aligned} \mathfrak{A}^j x^j &= \left[P_1 \frac{\partial}{\partial \zeta} + P_0\right] (\mathcal{H}^j x^j)(\zeta), \\ x^j &\in D(\mathfrak{A}^j) = \{x^j \in L^2(0,1;\mathbb{K}^2) : \, (\mathcal{H}^j x^j) \in H^1(0,1;\mathbb{K}^2)\} \end{aligned}$$

with boundary input and output maps

$$\begin{aligned} \mathfrak{B}^j x^j &= \begin{pmatrix} -(\mathcal{H}_2^j x_2^j)(0) \\ (\mathcal{H}_1^j x_1^j)(1) \end{pmatrix}, \quad \mathfrak{C}^j x^j = \begin{pmatrix} (\mathcal{H}_1^j x_1^j)(0) \\ (\mathcal{H}_2^j x_2^j)(1) \end{pmatrix}, \\ D(\mathfrak{B}^j) &= D(\mathfrak{C}^j) = D(\mathfrak{A}^j), \quad j \in \mathcal{J} \setminus \{m\} \end{aligned}$$

and

$$\begin{aligned} \mathfrak{B}^m x^m &= \begin{pmatrix} -\big(\mathcal{H}_2^m x_2^m\big)(0) \\ \big(\mathcal{H}_2^m x_2^m\big)(1) \end{pmatrix}, \quad \mathfrak{C}^m x^m = \begin{pmatrix} \big(\mathcal{H}_1^m x_1^m\big)(0) \\ \big(\mathcal{H}_1^m x_1^m\big)(1) \end{pmatrix}, \\ D(\mathfrak{B}^m) &= D(\mathfrak{C}^m) = D(\mathfrak{A}^m). \end{aligned}$$

For this choice of the boundary input and output maps, the port-Hamiltonian systems $\mathfrak{S}^j = (\mathfrak{A}^j, \mathfrak{B}^j, \mathfrak{C}^j)$ become impedance passive with energy state spaces $X^j = (L_2(0,1;\mathbb{K}^2), \|\cdot\|_{\mathcal{H}^j})$ and input and output spaces $U^j = Y^j = \mathbb{K}^2$. This property corresponds to the formal *power balance equation*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} H_{\mathrm{WE}}(t) &:= \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_0^1 \rho(\zeta) \left|\omega_t(t,\zeta)\right|^2 + T(\zeta) \left|\omega_\zeta(t,\zeta)\right|^2 \, \mathrm{d}\zeta \\ &= \mathrm{Re} \, \left[\big(\omega_t(t,\zeta) \mid T(\zeta)\omega_\zeta(t,\zeta)\big)_{\mathbb{K}}\right]_0^1 \end{aligned}$$

for the wave equation $\rho(\zeta)\omega_{tt}(t,\zeta) = (T(\cdot)\omega_\zeta)_\zeta(t,\zeta)$. The interconnection structure (1) can then be written in the boundary feedback form

$$\mathfrak{B}x = \begin{bmatrix} -\kappa_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -\kappa_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -\kappa_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\kappa_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathfrak{C}x =: K\mathfrak{C}x.$$

Clearly, the symmetric part of K,

$$\text{Sym } K = \begin{bmatrix} -\kappa_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\kappa_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\kappa_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\kappa_{m-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is negative semi-definite, thus the operator

$$Ax = \left(\mathfrak{A}^j x^j\right)_{j \in \mathcal{J}}, \quad D(A) = \left\{ x \in D(\mathfrak{A}) = \prod_{j \in \mathcal{J}} D(\mathfrak{A}^j) : \mathfrak{B}x = K\mathfrak{C}x \right\}$$

is dissipative on the product Hilbert space $X = \prod_{j \in \mathcal{J}} X^j$ and thus generates a contractive C_0-semigroup $\left(T(t)\right)_{t \geq 0}$ on $\widehat{X}$ by Theorem 4.6 (or, by Theorem 4.1 in [14]). We employ Theorem 6.3 and Theorem 6.4 to deduce uniform exponential stability, as long as the parameter functions ρ^j and T are Lipschitz continuous on $\left(\zeta^{j-1}, \zeta^j\right)$, $j = 1, \ldots, m$. For this end, we reformulate the boundary conditions in a form more suitable for the setting of these theorems, and set

$$\begin{aligned} \mathfrak{B}^1 x^1 &= \left(\left(\mathcal{H}_2^1 x_2^1\right)(0) + \kappa^0 \left(\mathcal{H}_1^1 x_1^1\right)(0)\right) \in \mathbb{K}, \\ \mathfrak{C}^1 x^1 &= \left(\left(\mathcal{H}_1^1 x_1^1\right)(0), \left(\mathcal{H}^1 x^1\right)(1)\right) \in \mathbb{K}^3, \\ \mathfrak{B}^j x^j &= \left(\mathcal{H}^j x^j\right)(0) \in \mathbb{K}^2, \\ \mathfrak{C}^j x^j &= \left(\mathcal{H}^j x^j\right)(1) \in \mathbb{K}^2, \qquad j = 2, \ldots, m-1, \\ \mathfrak{B}^m x^m &= \left(\left(\mathcal{H}^m x^m\right)(0), \left(\mathcal{H}_2^m x_2^m\right)(1)\right) \in \mathbb{K}^3, \\ \mathfrak{C}^m x^m &= \left(\mathcal{H}_1^m x_1^m\right)(0) \in \mathbb{K}. \end{aligned}$$

(In this situation, we simply have $X_c = \{0\}$.) Then, the boundary conditions can be rewritten in the form

$$\mathfrak{B}^j x^j = \sum_{i=1}^{j-1} K^{ij} \mathfrak{C}^i x^i, \quad j \in \widehat{\mathcal{J}} = \{1, \ldots, m\}$$

for appropriate matrices K^{ij}, $i, j \in \mathcal{J}$, and such that $K = \left(K^{ij}\right)_{i,j \in \mathcal{J}}$ is strictly lower-block triangular.

Corollary 7.4. *In the situation of Example* 7.3*, assume that* $\rho^j, T^j : (\zeta^{j-1}, \zeta^j) \to (0, \infty)$ *are Lipschitz continuous for each string* $j \in \mathcal{J}$ *of the serially connected chain, and assume that* $\kappa^0 > 0$ *whereas* $\kappa^j \geq 0$ *for* $j \in \mathcal{J}$*. Then the problem is* *well-posed, i.e. for every initial datum*

$$(\omega(0,\cdot),\omega_t(0,\cdot)) = (\omega_0,\omega_1) \in \prod_{j\in\mathcal{J}} H^1(\zeta^{j-1},\zeta^j) \times \prod_{j\in\mathcal{J}} L_2(\zeta^{j-1},\zeta^j)$$

there is a unique strong solution $\omega : \mathbb{R} \to \prod_{j\in\mathcal{J}} H^1(\zeta^{j-1},\zeta^j)$ *such that*

$$\omega \in C\Big(\mathbb{R}_+; \prod_{j\in\mathcal{J}} H^1(\zeta^{j-1},\zeta^j)\Big), \quad \omega_t \in C\Big(\mathbb{R}_+; \prod_{j\in\mathcal{J}} L_2\big(\zeta^{j-1},\zeta^j\big)\Big)$$

with non-increasing energy

$$H_{\mathrm{WE}}(t) = \frac{1}{2}\sum_{j=1}^{m}\int_{\zeta_{j-1}}^{\zeta_j} \rho_j|\omega_t|^2 + T_j|\omega_\zeta|^2\,\mathrm{d}\zeta$$

and there are constants $M \geq 1$ *and* $\eta < 0$ *such that*

$$H_{\mathrm{WE}}(t) \leq M\mathrm{e}^{\eta t} H_{\mathrm{WE}}(0), \quad t \geq 0$$

holds uniformly for all initial data. Moreover, if additionally

$$\big((T\omega_0)_\zeta,\omega_1\big) \in \prod_{j\in\mathcal{J}} H^1(\zeta^{j-1},\zeta^j) \times \prod_{j\in\mathcal{J}} H^1(\zeta^{j-1},\zeta^j)$$

and satisfy the compatibility conditions for (1), *i.e.*

$$\begin{cases} \left(T(\omega_0)_\zeta\right)(\zeta_0) = -\kappa^0\omega_1^1(\zeta_0), \\ \left(T(\omega_0^m)_\zeta\right)(\zeta^m) = 0, \\ \omega_1(\zeta^j-) = \omega_1(\zeta^j+), \\ \left(T(\omega_0)_\zeta\right)(\zeta^j-) - \left(T(\omega_0)_\zeta\right)(\zeta^j+) = -\kappa^j\omega_1(\zeta^j), \quad j = 1,\dots,m-1, \end{cases}$$

the solution is classical, i.e.

$$\omega \in C^1\Big(\mathbb{R}_+; \prod_{j\in\mathcal{J}} H^1\big(\zeta^{j-1},\zeta^j\big)\Big), \quad \omega_t \in C\Big(\mathbb{R}_+; \prod_{j\in\mathcal{J}} H^1\big(\zeta^{j-1},\zeta^j\big)\Big),$$
$$T\omega_\zeta \in C\Big(\mathbb{R}_+; \prod_{j\in\mathcal{J}} H^1\big(\zeta^{j-1},\zeta^j\big)\Big).$$

Proof. By the port-Hamiltonian formulation, we can see that the impedance passivity of the systems $\mathfrak{S}^j = (\mathfrak{A}^j,\mathfrak{B}^j,\mathfrak{C}^j)$ and the structure of the interconnection by the static feedback matrix K imply that

$$\begin{aligned} \mathrm{Re}\,\big(Ax \mid x\big)_X &\leq -\sum_{j=1}^{m}\kappa_{j-1}\left|\mathfrak{B}_1^j x^j\right|^2 = -\sum_{j=1}^{m}\kappa_{j-1}\left|(\mathcal{H}_2^j x_2^j)(0)\right|^2 \\ &\leq -\kappa_0\left|(\mathcal{H}_2^1 x_2^1)(0)\right|^2 = -\frac{1}{\kappa_0}\left|(\mathcal{H}_1^1 x_1^1)(0)\right|^2 \\ &\leq -\frac{1}{2}\min\{\kappa_0,\kappa_0^{-1}\}\left|(\mathcal{H}^1 x^1)(0)\right|^2, \quad x \in D(A). \end{aligned}$$

This already implies well-posedness. Moreover, for each $j \geq 2$ we have

$$\left|\mathfrak{B}^j x^j\right| \leq \left|(\mathcal{H}^j x^j)(0)\right|.$$

Since all the pairs $(\mathfrak{A}^j, (\mathcal{H}^j x^j)(0))$ have property ASP by Lemma 7.2, as long as the parameter functions ρ^j, T^j are Lipschitz continuous, it follows asymptotic stability from Theorem 6.3, and then, since by Lemma 7.2 the pairs $\left(\mathfrak{A}^j, (\mathcal{H}^j x^j)(0)\right)$ also have property $\mathrm{AIEP}_{\tau^j \circ \mathcal{H}^j}$ as well, uniform exponential stability follows by Theorem 6.4. □

Remark 7.5. It would be nice if one could apply Theorems 6.3 and 6.4 to the case of a chain of Euler–Bernoulli beam models, cf. [7], as well. Unfortunately, as it turns out a dissipativity condition like

$$\operatorname{Re}\left(\widehat{A}x \mid x\right)_X \leq -\kappa\left(\left|(\mathcal{H}x)(0)\right|^2 + \left|(\mathcal{H}x)'(0)\right|^2\right), \quad x \in D(\widehat{A}) \tag{2}$$

is not sufficient for uniform exponential stability of (closed-loop) port-Hamiltonian systems of order $N = 2$, and also for the special case of an Euler–Bernoulli beam such a property is not known. In particular, though clearly $\left(\mathfrak{A}, \left((\mathcal{H}x)(0), (\mathcal{H}x)'(0)\right)\right)$ has property ASP for port-Hamiltonian operators of order $N = 2$ with Lipschitz-continuous $\mathcal{H} : [0,1] \to \mathbb{K}^{d \times d}$, it is not known whether there are classes, e.g. Euler–Bernoulli beam type systems, for which properties AIEP and AIEP_τ hold for the pair $\left(\mathfrak{A}, \left((\mathcal{H}x)(0), (\mathcal{H}x)'(0)\right)\right)$. Even more, dissipation of the form (2) is not what can be ensured by the most usual damping conditions for the Euler–Bernoulli beam, namely only dissipation in three of the four components (or, the component being zero by the boundary conditions imposed on the system) of $\left((\mathcal{H}x)(0), (\mathcal{H}x)'(0)\right)$ for the Euler–Bernoulli beam (where $d = 2$) is a realistic assumption. However, it is already known for 30 years [7], that serially interconnected, homogeneous (i.e. constant parameters along each beam) Euler–Bernoulli beams can be uniformly exponentially stabilised at one end by suitable (realistic) boundary conditions, if one additionally assumes that the parameters are ordered in a monotone way. The same result for inhomogeneous beams, where the parameter functions on each beam are allowed to have Lipschitz continuous dependence on the spatial parameter ζ, but still satisfy monotonicity conditions at the joints ζ^j, will be shown in a forthcoming paper [4].

Example 7.6 (The Euler–Bernoulli Beam). The Euler–Bernoulli beam equation

$$\rho(\zeta)\omega_{tt}(t,\zeta) + \frac{\partial^2}{\partial\zeta^2}\left(EI(\zeta)\omega_{\zeta\zeta}(t,\zeta)\right), \quad t \geq 0,\, \zeta \in (a,b)$$

can be written in port-Hamiltonian form for $N = 2$ and the identification

$$x(t,\zeta) = \begin{pmatrix} \rho(\zeta)\omega_t(t,\zeta) \\ \omega_{\zeta\zeta}(t,\zeta) \end{pmatrix}, \quad \mathcal{H}(\zeta) = \begin{bmatrix} \rho(\zeta)^{-1} & 0 \\ 0 & EI(\zeta) \end{bmatrix}.$$

Choosing $P_2 = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], P_1 = P_0 = 0 \in \mathbb{K}^{2\times 2}$, we arrive at the first order in time, second order in space evolution equation

$$\frac{\partial}{\partial t} x(t,\zeta) = \mathfrak{A} x(t,\zeta) := \left[P_2 \frac{\partial^2}{\partial \zeta^2} + P_1 \frac{\partial}{\partial \zeta} + P_0\right] (\mathcal{H}(\zeta) x(t,\zeta)), \quad t \geq 0,\, \zeta \in (a,b).$$

After appropriate scaling, w.l.o.g. we may and will assume that $a = 0$ and $b = 1$ in the following. There are several possible choices for conservative boundary conditions (e.g. at the right end), such as

1. $\omega(t,1) = (EI\omega_{\zeta\zeta})(t,1)$ (*simply supported* or *pinned* right end),
2. $\omega_{\zeta\zeta}(t,1) = (EI\omega_{\zeta\zeta})_\zeta(t,1) = 0$ (*free* right end),
3. $\omega_\zeta(t,1) = (EI\omega_{\zeta\zeta})_\zeta(t,0) = 0$ (*shear hinge* right end),
4. $\omega_t(t,1) = \omega_\zeta(t,1)$ (*clamped* left end),
5. $\omega_t(t,1) = (EI\omega_{\zeta\zeta})(t,1) = 0$,
6. $\omega_{t\zeta}(t,1) = (EI\omega_{\zeta\zeta})_\zeta(t,1) = 0$.

Here, the first and third case are just special cases of the fifth (there, $\omega(t,1) = c \in \mathbb{K}$) and sixth case (there, $\omega_\zeta(t,1) = c \in \mathbb{K}$), so the most important conservative boundary conditions in energy state space formulation read as

1. $\big(\mathcal{H}^m x^m\big)(1) = 0$,
2. $\big(\mathcal{H}_2^m x_2^m\big)(1) = \big(\mathcal{H}_2^m x_2^m\big)'(1) = 0$,
3. $\big(\mathcal{H}^m x^m\big)'(1) = 0$,
4. $\big(\mathcal{H}_1^m x_1^m\big)(1) = \big(\mathcal{H}_1^m x_1^m\big)'(1) = 0$.

At the other end we want to impose dissipative boundary conditions to obtain uniform exponential energy decay for the solution of the Euler–Bernoulli beam model closed in this linear way, the most popular being (cf. [7])

$$\begin{pmatrix} (EI\omega_{\zeta\zeta})(0) \\ -(EI\omega_{\zeta\zeta})_\zeta(0) \end{pmatrix} = -K_0 \begin{pmatrix} \omega_{t\zeta}(t,\zeta) \\ \omega_t(t,\zeta) \end{pmatrix} \tag{3}$$

for some matrix $K_0 \in \mathbb{K}^{2\times 2}$ such that

$$\text{either} \quad K_0 = \begin{bmatrix} k_0^{11} & 0 \\ 0 & 0 \end{bmatrix} \text{ for some } k_0^{11} > 0, \text{ or } \operatorname{Sym}(K_0) > 0 \text{ is positive definite.}$$

For the first of these options, conservative boundary conditions at the right end of type *clamped end* or *shear hinge right end* ensure well-posedness and uniform exponential energy, whereas in the second case any of the conservative boundary conditions listed above, i.e. also *free right end* or *pinned right end* boundary conditions are allowed, lead to well-posedness with uniform exponential decay of the energy functional.

Lemma 7.7. *For the Euler–Bernoulli beam of Example* 7.6 *assume that* $\rho, EI : [0,1] \to \mathbb{R}$ *are uniformly positive and Lipschitz continuous. Then, for* $\mathfrak{A}$ *and the following choices of* $\mathfrak{R} : D(\mathfrak{R}) = D(\mathfrak{A}) \to \mathbb{K}^4$*, the pair* $(\mathfrak{A}, \mathfrak{R})$ *has property ASP:*

$$\mathfrak{R}x = \begin{pmatrix} (\mathcal{H}_1 x_1)(0) \\ (\mathcal{H}_1 x_1)'(0) \\ (\mathcal{H}_2 x_2)(0) \\ (\mathcal{H}_2 x_2)'(1) \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} (\mathcal{H}_1 x_1)(0) \\ (\mathcal{H}_1 x_1)'(0) \\ (\mathcal{H}_2 x_2)'(0) \\ (\mathcal{H}_2 x_2)(1) \end{pmatrix}$$
$$\textit{or} \quad \begin{pmatrix} (\mathcal{H}_1 x_1)(0) \\ (\mathcal{H}_2 x_2)(0) \\ (\mathcal{H}_2 x_2)'(0) \\ (\mathcal{H}_1 x_1)'(1) \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} (\mathcal{H}_1 x_1)'(0) \\ (\mathcal{H}_2 x_2)(0) \\ (\mathcal{H}_2 x_2)'(0) \\ (\mathcal{H}_1 x_1)(1) \end{pmatrix}.$$

Moreover, for the following choices of $\mathfrak{R}' : D(\mathfrak{R}') = D(\mathfrak{A}) \to \mathbb{K}^5$*, the pair* $(\mathfrak{A}, \mathfrak{R}')$ *has property* AIEP_τ

$$\mathfrak{R}'x = \begin{pmatrix} (\mathcal{H}x)(0) \\ (\mathcal{H}_1 x_1)'(0) \ \textit{or}\ (\mathcal{H}_2 x_2)'(0) \\ (\mathcal{H}_1 x_1)(1) \ \textit{or}\ (\mathcal{H}_2 x_2)'(1) \\ (\mathcal{H}_1 x_1)'(1) \ \textit{or}\ (\mathcal{H}_2 x_2)(1) \end{pmatrix}, \quad \textit{or}$$
$$\mathfrak{R}'x = \begin{pmatrix} (\mathcal{H}x)(0) \\ (\mathcal{H}x)(1) \\ (\mathcal{H}_{j_0} x_{j_0})'(\zeta_0) \end{pmatrix} \quad \textit{for some } j_0 \in \{1,2\}, \zeta_0 \in \{0,1\}$$

In particular, for the following choices of $\mathfrak{R}'$*, the pair* $(\mathfrak{A}, \mathfrak{R}')$ *has both properties ASP and* AIEP_τ*:*

$$\mathfrak{R}'x = \begin{pmatrix} (\mathcal{H}x)(0) \\ (\mathcal{H}_1 x_1)'(0) \\ (\mathcal{H}_1 x_1)(1) \,\textit{or}\, (\mathcal{H}_2 x_2)'(1) \\ (\mathcal{H}_2 x_2)(1) \end{pmatrix}, \quad \textit{or}$$
$$\mathfrak{R}'x = \begin{pmatrix} (\mathcal{H}x)(0) \\ (\mathcal{H}_2 x_2)'(0) \\ (\mathcal{H}_1 x_1)(1) \,\textit{or}\, (\mathcal{H}_2 x_2)'(1) \\ (\mathcal{H}_1 x_1)'(1) \end{pmatrix}, \quad \textit{or}$$
$$\mathfrak{R}'x = \begin{pmatrix} (\mathcal{H}x)(0) \\ (\mathcal{H}x)(1) \\ \left(\mathcal{H}_{j_0} x_{j_0}\right)'(\zeta_0) \end{pmatrix} \quad \textit{for some } j_0 \in \{1,2\},\ \zeta_0 \in \{0,1\}.$$

Proof. Partly, this is part of Proposition 2.9 in [5]. For the full proof of properties ASP and AIEP considered here, except for the latter case, and even in the more general setting of a chain of Euler–Bernoulli beams, see the upcoming article [4]. In these cases it remains to prove property AIEP_τ. This follows from property AIEP and Lemma 9.2 in the appendix. Let us prove the statement for the choice $\mathfrak{R}'x = \big((\mathcal{H}x)(0), (\mathcal{H}x)(1), (\mathcal{H}_1 x_1)'(0)\big)$, then it is clear how the remaining other choices for the fifth component can be handled. First of all, the pair $(\mathfrak{A}, \mathfrak{R}')$

has property ASP which can be seen by using e.g. [2, Lemma 4.2.9] and in fact is a special case of [2, Corollary 4.2.10]. Then, by [2, (4.28) on p. 108] in the proof of [2, Proposition 4.3.19], for every sequence $(x_n, \beta_n)_{n\geq 1} \subseteq D(\mathfrak{A}) \times \mathbb{R}$ with $\sup_{n\in\mathbb{N}} \|x_n\|_X < \infty$ and $|\beta_n| \to \infty$ such that $\mathfrak{A}x_n - \mathrm{i}\beta_n x_n \to 0$ in X, and for every $q \in C^2([0,1];\mathbb{R})$ one has the equality

$$\begin{aligned}
&\operatorname{Re}\left(x_n \mid (2q'\mathcal{H} - q\mathcal{H}')x_n\right)_{L_2} \\
&= \Big[-2\operatorname{Re}\left((\mathcal{H}_2 x_{n,2})'(\zeta) \mid \tfrac{iq(\zeta)}{\beta_n}(\mathcal{H}_1 x_{n,1})'(\zeta)\right)_{\mathbb{K}} - \left(x_n(\zeta) \mid (q\mathcal{H})(\zeta)x_n(\zeta)\right)_{\mathbb{K}} \\
&\quad - \operatorname{Re}\left((\mathcal{H}_2 x_{n,2})'(\zeta) \mid \tfrac{iq'(\zeta)}{\beta_n}(\mathcal{H}_1 x_{n,1})(\zeta)\right)_{\mathbb{K}} \\
&\quad + \operatorname{Re}\left((\mathcal{H}_2 x_{n,2})'(\zeta) \mid \tfrac{iq'(\zeta)}{\beta_n}(\mathcal{H}_1 x_{n,1})(\zeta)\right)_{\mathbb{K}} \Big]_0^1 + o(1)
\end{aligned}$$

where $o(1)$ denotes terms that vanish as $n \to \infty$. Also $\frac{\mathcal{H}x_n}{\beta_n} \to 0$ in $C^1([0,1];\mathbb{K}^2)$ has been shown there. Therefore, if we additionally assume that $\mathfrak{R}'x_n \to 0$ and take $q \in C^2([0,1];\mathbb{R})$ such that

$$2q'\mathcal{H} - q\mathcal{H} \geq \varepsilon I, \quad \text{a.e. } \zeta \in (0,1),$$

which is possible by the coercivity of $\mathcal{H}$ and the uniform boundedness of $\mathcal{H}'$, we obtain that

$$\varepsilon \|x_n\|_{L_2}^2 \leq \left(x_n \mid (2q'\mathcal{H} - q\mathcal{H})x_n\right)_{L_2} = o(1)$$

and thus $x_n \to 0$ in X. This shows property AIEP. By Lemma 9.2 in the Appendix, it follows that $\tau(\mathcal{H}x_n) \to 0$ as well, so that $(\mathfrak{A}, \mathfrak{R}')$ has properties ASP and AIEP$_\tau$. □

Example 7.8. Consider the system of Fig. 4 consisting of a string which is damped at the left end, and is interconnected at the right end with an Euler–Bernoulli beam. We denote by $\omega(t,\zeta)$ and $\widetilde{\omega}(t,\zeta)$ the transversal position of the string and the Euler–Bernoulli beam at time time $t \geq 0$ and position $\zeta \in (0,1)$, respectively. (Here, w.l.o.g. we may and assume that both the string and the beam have unit length.) Moreover, we denote by $\rho(\zeta)$ and $\widetilde{\rho}(\zeta)$ the mass density times transversal area at position $\zeta \in (0,1)$ for the string and the Euler–Bernoulli beam, respectively, by $T(\zeta)$ Young's modulus of the string and by $\widetilde{EI}(\zeta)$ the elasticity times moment of inertia per area element of the Euler–Bernoulli beam. We assume that $\rho, \widetilde{\rho}, T, \widetilde{EI}$ are all Lipschitz-continuous and uniformly positive on $[0,1]$. Then the dynamics of the system are described by the evolution equations for the string and the beam,

$$\begin{aligned}
\rho(\zeta)\omega_{tt}(t,\zeta) &= +(T\omega_\zeta)_\zeta(t,\zeta), \\
\widetilde{\rho}(\zeta)\widetilde{\omega}_{tt}(t,\zeta) &= -(\widetilde{EI}\widetilde{\omega}_{\zeta\zeta})_{\zeta\zeta}(t,\zeta), \qquad t \geq 0,\ \zeta \in (0,1),
\end{aligned}$$

the damping by feedback boundary condition for the string at the left end

$$(T\omega_\zeta)(t,0) = -\kappa\omega_t(t,0), \quad t \geq 0$$

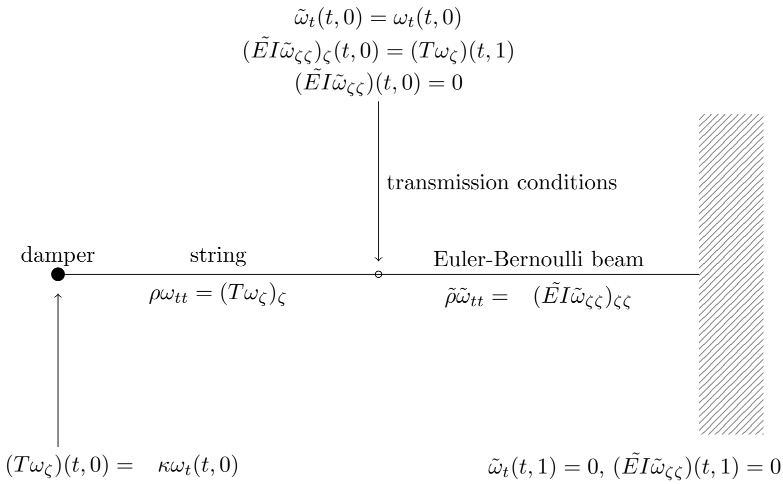

FIGURE 4. A damper–string–beam system.

for some constant $\kappa > 0$, the transmission conditions

$$\begin{aligned} \widetilde{\omega}_t(t,0) &= \omega_t(t,1), \\ \left(\widetilde{EI}\widetilde{\omega}_{\zeta\zeta}\right)_\zeta(t,0) &= -(T\omega_\zeta)(t,1), \\ (\widetilde{EI}\widetilde{\omega}_{\zeta\zeta})(t,0) &= 0, \qquad t \geq 0 \end{aligned}$$

and the conservative *pinned end* boundary conditions of the Euler–Bernoulli beam at the right end

$$\begin{aligned} \widetilde{\omega}_t(t,1) &= 0, \\ (\widetilde{EI}\widetilde{\omega}(t,\zeta))(t,1) &= 0, \qquad t \geq 0. \end{aligned}$$

The total energy of this system consists of the string part and the beam part of the energy

$$\begin{aligned} H_{\mathrm{tot}}(t) &= H_{\mathrm{WE}}(t) + H_{\mathrm{EB}}(t) \\ &= \frac{1}{2}\Big[\int_0^1 \rho(\zeta) \left|\omega_t(t,\zeta)\right|^2 + T(\zeta) \left|\omega_\zeta\right|^2 \,\mathrm{d}\zeta \\ &\quad + \int_0^1 \widetilde{\rho}(\zeta) \left|\widetilde{\omega}_t(t,\zeta)\right|^2 + \widetilde{EI}(\zeta) \left|\widetilde{\omega}_{\zeta\zeta}(t,\zeta)\right|^2 \,\mathrm{d}\zeta \Big] \end{aligned}$$

and along solutions of the systems which are sufficiently regular, one readily computes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H_{\mathrm{tot}}(t) &= \operatorname{Re}\Big[\big((T\omega_\zeta)(t,\zeta)\mid\omega_t(t,\zeta)\big)_{\mathbb{K}}\\&\qquad+\Big(-(\widetilde{EI}\widetilde{\omega}_{\zeta\zeta})_\zeta(t,\zeta)\mid\widetilde{\omega}_t(t,\zeta)\Big)_{\mathbb{K}}+\Big((\widetilde{EI}\widetilde{\omega}_{\zeta\zeta})(t,\zeta)\mid\widetilde{\omega}_{t\zeta}\Big)_{\mathbb{K}}\Big]_0^1\\&= -\kappa\left|\omega_t(t,0)\right|^2\le 0,\quad t\ge 0.\end{aligned}$$

So the system is dissipative, and the corresponding operator of port-Hamiltonian type $\widehat{A}$ below generates a contractive C_0-semigroup. Since the subsystems are a string modelled by the one-dimensional wave equation and an Euler–Bernoulli beam, the port-Hamiltonian formulation reads as follows:

$$\begin{aligned}X^1&=L_2(0,1;\mathbb{K}^2)\quad\text{with}\quad\mathcal{H}^1=\operatorname{diag}\left(\frac{1}{\rho},T\right),\quad x^1(t,\zeta)=\begin{pmatrix}(\rho\omega_t)(t,\zeta)\\\omega_\zeta(t,\zeta)\end{pmatrix},\\X^2&=L_2(0,1;\mathbb{K}^2)\quad\text{with}\quad\mathcal{H}^2=\operatorname{diag}\left(\frac{1}{\widetilde{\rho}},\widetilde{EI}\right),\quad x^2(t,\zeta)=\begin{pmatrix}(\widetilde{\rho}\widetilde{\omega}_t)(t,\zeta)\\\omega_{\zeta\zeta}(t,\zeta)\end{pmatrix},\end{aligned}$$

there is no dynamic controller (i.e. $X_c^j=\{0\}$) and the differential operators are given by

$$\begin{aligned}\mathfrak{A}^1x^1&=\begin{bmatrix}0&1\\1&0\end{bmatrix}\frac{\partial}{\partial\zeta}(\mathcal{H}^1x^1),\\\mathfrak{A}^2x^2&=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\frac{\partial^2}{\partial\zeta^2}(\mathcal{H}^2x^2),\end{aligned}$$

and we get

$$\begin{aligned}Ax&=\begin{pmatrix}\mathfrak{A}^1x^1\\\mathfrak{A}^2x^2\end{pmatrix},\\D(A)&=\big\{(x^1,x^2)\in X^1\times X^2:\ \big(\mathcal{H}^1x^1\big)\in H^1(0,1;\mathbb{K}^2),\ \big(\mathcal{H}^2x^2\big)\in H^2(0,1;\mathbb{K}^2),\\&\qquad\big(\mathcal{H}_2^1x_2^1\big)(0)=-\kappa\big(\mathcal{H}_1^1x_1^1\big)(0),\big(\mathcal{H}_1^2x_1^2\big)(0)=\big(\mathcal{H}_1^1x_1^1\big)(1),\\&\qquad\big(\mathcal{H}_2^2x_2^2\big)'(0)=\big(\mathcal{H}_2^1x_2^1\big)(0),\ \big(\mathcal{H}_2^2x_2^2\big)(0),\big(\mathcal{H}^2x^2\big)(1)=0\big\}.\end{aligned}$$

For this operator one has

$$\begin{aligned}\operatorname{Re}\big(Ax\mid x\big)_X&\le-\kappa\left|(\mathcal{H}_1^1x_1^1)(0)\right|^2=-\frac{1}{\kappa}\left|(\mathcal{H}_2^1x_2^1)(0)\right|^2\\&=-\frac{1}{2}\left[\kappa\left|(\mathcal{H}_1^1x_1^1)(0)\right|^2+\frac{1}{\kappa}\left|(\mathcal{H}_2^1x_2^1)(0)\right|^2\right],\quad x\in D\big(\widehat{A}\big).\end{aligned}$$

Let us give its formulation as a serial interconnection of port-Hamiltonian systems:

$$\begin{aligned}\widehat{\mathfrak{A}}^j&=\mathfrak{A}^j\big|_{D(\widehat{\mathfrak{A}}^j)},\\D\big(\widehat{\mathfrak{A}}^1\big)&=\big\{x^1\in D(\mathfrak{A}^1):\ \big(\mathcal{H}_2^1x_2^1\big)(0)=-\kappa\big(\mathcal{H}_1^1x_1^1\big)(0)\big\},\\D\big(\widehat{\mathfrak{A}}^2\big)&=\big\{x^1\in D(\mathfrak{A}^1):\ \big(\mathcal{H}_2^2x_2^2\big)(0)=0,\ \big(\mathcal{H}^2x^2\big)(0)=0\big\},\end{aligned}$$

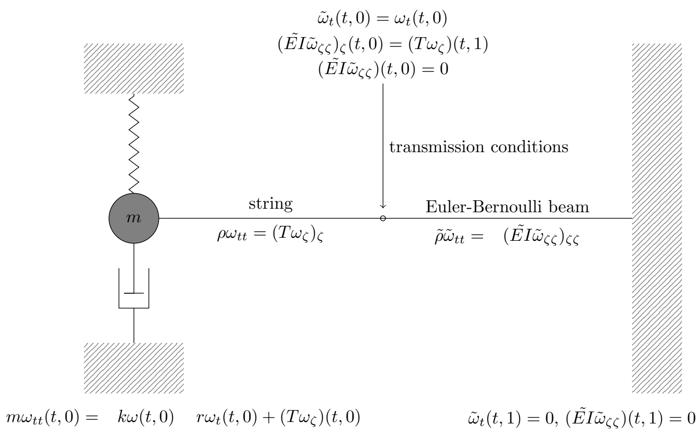

Figure 5. An Euler–Bernoulli Beam–Spring–Mass–Damper–String System.

and the conditions of the stability theorems are satisfied for

$$\mathfrak{R}^1 x^1 = \big(\sqrt{\kappa}\big(\mathcal{H}_1^1 x_1^1\big)(0), \sqrt{\kappa}^{-1}\big(\mathcal{H}_2^1 x_2^1\big)(0)\big),$$
$$\mathfrak{B}^2 x^2 = \big(\big(\mathcal{H}^2 x^2\big)(0), \big(\mathcal{H}^2 x^2\big)(1), \big(\mathcal{H}_2^2 x_2^2\big)'(0)\big).$$

Now, the pairs $\big(\widehat{\mathfrak{A}}^1, \mathfrak{R}^1\big)$ and $\big(\widehat{\mathfrak{A}}^2, \mathfrak{B}^2\big)$ have properties ASP and AIEP$_{\tau^j \circ \mathcal{H}^j}$ since both the pairs $\big(\mathfrak{A}^1, \big(\mathcal{H}^1 x^1\big)(0)\big)$ and $\big(\mathfrak{A}^2, \big(\big(\mathcal{H}^2 x^2\big)(0), \big(\mathcal{H}^2 x^2\big)(1), \big(\mathcal{H}_1^2 x_1^2\big)(0)\big)\big)$ have properties ASP and AIEP$_{\tau^j \circ \mathcal{H}^j}$. Therefore, by Theorems 6.3 and 6.4 the operator $\widehat{A}$ generates a uniformly exponentially stable contraction semigroup on $X = X^1 \times X^2$, i.e. there are constants $M \geq 1$ and $\eta < 0$ such that uniformly for all finite energy initial data the energy decays uniformly exponentially,

$$H_{\mathrm{tot}}(t) \leq M \mathrm{e}^{\eta t} H_{\mathrm{tot}}(0), \quad t \geq 0.$$ □

Example 7.9. Consider the following interconnection of a string modelled by a wave equation, damped at the left end by a spring–mass damper and attached to an Euler–Bernoulli beam at the right, and where the latter is pinned at the right end, see Fig. 5. For the interconnection, the transmission conditions

$$\omega(t,1) = \widetilde{\omega}_t(t,0),\ (T\omega_\zeta)(t,1) + \big(\widetilde{EI}\widetilde{\omega}_{\zeta\zeta}\big)_\zeta(t,0) = 0,\ \widetilde{\omega}_\zeta(t,0) = 0, \quad t \geq 0.$$

are assumed, i.e. in particular the transversal position of the string and the beam continuous is at the joint and no force is acting on the joint. The spring–mass damper is modelled by the ODE

$$m\omega_{tt}(t,0) = -k\omega(t,0) - r\omega_t(t,0) + (T\omega_\zeta)(t,0),$$

i.e. the tip of mass $m > 0$ moves under the influence of forces from a spring with spring constant $k > 0$ and a damper with damping constant $r > 0$, as well as the stress $(T\omega_\zeta)(t, 0)$ of the string at the left end. The pinned end boundary conditions of the Euler–Bernoulli beam are modelled by

$$\widetilde{\omega}(t,1) = 0, \ \big(\widetilde{EI}\widetilde{\omega}_{\zeta\zeta}\big)(t,1) = 0.$$

The total energy of this system is given by the potential and kinetic energies of the spring, the string and the beam

$$\begin{aligned} H_{\mathrm{tot}}(t) &= H_{\mathrm{WE}}(t) + H_{\mathrm{EB}}(t) + H_{m,k}(t) \\ &= \frac{1}{2}\left[\int_0^1 \rho(\zeta)\left|\omega_t(t,\zeta)\right|^2 \,\mathrm{d}\zeta + T(\zeta)\left|\omega_\zeta(t,\zeta)\right| \,\mathrm{d}\zeta\right] \\ &\quad + \frac{1}{2}\left[\int_0^1 \widetilde{\rho}(\zeta)\left|\widetilde{\omega}_t(t,\zeta)\right|^2 + \widetilde{EI}(\zeta)\left|\widetilde{\omega}_{\zeta\zeta}(t,\zeta)\right|^2 \,\mathrm{d}\zeta\right] \\ &\quad + \frac{1}{2}\left[m\left|\omega_t(t,0)\right|^2 + k\left|\omega(t,0)\right|^2\right]. \end{aligned}$$

Then, the formal energy balance along sufficiently regular solutions shows that

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{\mathrm{tot}}(t) = -r\left|\omega_t(t,0)\right|^2 \le 0, \quad t \ge 0.$$

Therefore, the system is *dissipative* and after reformulation as network of port-Hamiltonian type, it is clear that well-posedness in the sense of unique solutions with non-increasing energy holds for all sufficiently regular initial data. For this end, we take

$$X^1 = L_2(0,1;\mathbb{K}^2), \quad X^2 = L_2(0,1;\mathbb{K}^2) \quad \text{and} \quad X_c^2 = \{0\}$$

as in the previous example, but this time

$$X_c^1 = \mathbb{K}^2, \quad x_c^1 = \begin{pmatrix} \omega(t,0) \\ \omega_t(t,0) \end{pmatrix}, \quad \|x_c\|_{X_c^1}^2 = k\left|x_{c,1}^1\right|^2 + m\left|x_{c,2}^1\right|^2.$$

Also the operators $\mathfrak{A}^1$ and $\mathfrak{A}^2$ are defined as before, but now we additionally have the control system given by the operators

$$A_c^1 = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{r}{m} \end{bmatrix}, \ B_c^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C_c^1 = (B_c^1)^* = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ D_c^1 = 0$$

for $U_c^1 = \mathbb{K}$. The resulting operator $\widehat{A} : D\big(\widehat{A}\big) \subseteq \widehat{X} = X \times X_c = X^1 \times X^2 \times X_c^1 \to \widehat{X}$ is, therefore, given by

$$\begin{aligned} \widehat{A}\begin{pmatrix} x^1 \\ x^2 \\ x_c^1 \end{pmatrix} &= \begin{pmatrix} \mathfrak{A}^1 x^1 \\ \mathfrak{A}^2 x^2 \\ A_c^1 x_c^1 + B_c^1\big(\mathcal{H}_2^1 x_2^1\big)(0) \end{pmatrix} \\ D(\widehat{A}) &= \big\{(x^1, x^2, x_c^1) \in D(\mathfrak{A}^1) \times D(\mathfrak{A}^2) \times X_c^1 : \big(\mathcal{H}_1^1 x_1^1\big)(0) = -C_c^1 x_c^1, \\ &\qquad \big(\mathcal{H}_1^2 x_1^2\big)(0) = \big(\mathcal{H}_1^1 x_1^1\big)(1), \big(\mathcal{H}_2^2 x_2^2\big)'(0) = -\big(\mathcal{H}_2^1 x_2^1\big)(1), \\ &\qquad \big(\mathcal{H}_2^2 x_2^2\big)(0) = 0, \big(\mathcal{H}^2 x^2\big)(1) = 0\big\} \end{aligned}$$

and it is dissipative with

$$\operatorname{Re}\left(\widehat{A}\widehat{x} \mid \widehat{x}\right)_{\widehat{X}} = -r\left|x^1_{c,2}\right|^2 = -r\left|(\mathcal{H}^1_1 x^1_1)(0)\right|^2, \quad \widehat{x} = (x^1, x^2, x^1_c) \in D\big(\widehat{A}\big).$$

As a result, by Theorem 4.6 the operator $\widehat{A}$ generates a strongly continuous contraction semigroup on $\widehat{X}$. Let us investigate stability properties next. For this end, we write

$$\begin{aligned}
\widehat{\mathfrak{A}}^1\widehat{x}^1 &= \begin{pmatrix} \mathfrak{A}^1 x^1 \\ A^1_c x^1_c + B^1_c\big(\mathcal{H}^1_2 x^1_2\big)(0) \end{pmatrix},\\
D\big(\widehat{\mathfrak{A}}^1\big) &= \big\{\widehat{x}^1 = (x^1, x^1_c) \in D(\mathfrak{A}^1) \times X^1_c : \big(\mathcal{H}^1_1 x^1_1\big)(0) = -C^1_c x^1_c\big\}\\
\widehat{\mathfrak{A}}^2\widehat{x}^2 &= \mathfrak{A}^1 x^1\\
D\big(\widehat{\mathfrak{A}}^2\big) &= \big\{\widehat{x}^2 = x^2 \in D\big(\mathfrak{A}^2\big) : \big(\mathcal{H}^2_2 x^2_2\big)(0) = 0,\ \big(\mathcal{H}^2 x^2\big)(1) = 0\big\}.
\end{aligned}$$

Then $\widehat{A} \cong \operatorname{diag}\big(\widehat{\mathfrak{A}}^1, \widehat{\mathfrak{A}}^2\big)\big|_{D(\widehat{A})}$ for

$$D\big(\widehat{A}\big) = \left\{\widehat{x} = \big(\widehat{x}^1, \widehat{x}^2\big) \in D\big(\widehat{\mathfrak{A}}^1\big) \times D\big(\widehat{\mathfrak{A}}^2\big) : \begin{pmatrix} \big(\mathcal{H}^1_1 x^1_1\big)(1) \\ \big(\mathcal{H}^1_2 x^1_2\big)(1) \end{pmatrix} = \begin{pmatrix} \big(\mathcal{H}^2_1 x^2_1\big)(0) \\ -\big(\mathcal{H}^2_2 x^2_2\big)'(0) \end{pmatrix}\right\}.$$

To show uniform exponential energy decay, by Theorems 6.3 and 6.4 it suffices to prove that the pairs $\big(\widehat{\mathfrak{A}}^1, \big(\mathcal{H}^1_1 x^1_1\big)(0)\big)$ and $\big(\widehat{\mathfrak{A}}^2, \big(\big(\mathcal{H}^2_1 x^2_1\big)(0), \big(\mathcal{H}^2_2 x^2_2\big)'(0)\big)$ have properties ASP and AIEP$_\tau$. The latter, we have already seen in the previous example, as long as $\widetilde{\rho}, \widetilde{EI} \in \operatorname{Lip}(0,1;\mathbb{R})$. It remains to prove these properties for the pair $\big(\widehat{\mathfrak{A}}^1, \big(\mathcal{H}^1_1 x^1_1\big)(0)\big)$. We assume that $\rho, T \in \operatorname{Lip}(0,1;\mathbb{R})$ are Lipschitz continuous as well. For the matrix A^1_c we can calculate the eigenvalues as $\lambda_{1,2} = -\frac{r \pm \sqrt{r^2 - 4km}}{2m} \in \mathbb{C}^-_0$, thus A^1_c is a Hurwitz matrix and $\big(\mathrm{e}^{tA^1_c}\big)_{t \geq 0}$ uniformly exponentially stable on X^1_c. Since the pair $\big(\mathfrak{A}^1, \big(\mathcal{H}^1 x^1\big)(0)\big)$ has properties ASP and AIEP$_\tau$, this implies that also the pair $\big(\widehat{\mathfrak{A}}^1, \big(\mathcal{H}^1_1 x^1_1\big)(0)\big)$ has properties ASP and AIEP. Uniform exponential stability of the semigroup $\big(\widehat{T}(t)\big)_{t \geq 0}$ on $\widehat{X}$ thus follows by Theorems 6.3 and 6.4. □

8. Conclusion and Open Problems

In this paper, we have considered dissipative systems resulting from conservative or dissipative interconnection of several infinite-dimensional port-Hamiltonian systems $\mathfrak{S}^j = \big(\mathfrak{A}^j, \mathfrak{B}^j, \mathfrak{C}^j\big)$ of arbitrary, possibly distinct orders N^j via boundary control and observation and static or dynamic feedback via a finite-dimensional linear control system $\Sigma_c = \big(A_c, B_c, C_c, D_c\big)$ such that the total, interconnected system on the product energy Hilbert space $\widehat{X}$ becomes dissipative. The generation theorems from single infinite-dimensional port-Hamiltonian systems (or, port-Hamiltonian systems of the same differential order $N^j = N$ for all $j \in \mathcal{J}$) with static or dynamic boundary feedback have been shown to directly extend to systems of mixed-order port-Hamiltonian systems: The existence of a contractive C_0-semigroup $\left(\widehat{T}(t)\right)_{t\geq 0}$ acting as the (unique) solution operator for the abstract Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{x}(t) = \widehat{A}\,\widehat{x}(t)\ (t \geq 0), \quad \widehat{x}(0) = \widehat{x}_0 \in \widehat{X}$$

is equivalent to the operator $\widehat{A}$ simply being dissipative (w.r.t. the energy inner product $\left(\cdot \,\middle|\, \cdot\right)_{\widehat{X}}$). Therefore, whenever beam and wave equations are interconnected with each other and finite dimensional control systems via boundary control and observation, it is enough to choose the boundary and interconnection conditions such that the energy does not increase along classical solutions.

For multi-component systems consisting of subsystems of finite dimensional or infinite-dimensional port-Hamiltonian type on an interval, we presented a scheme to ensure asymptotic and uniform exponential stability from the structure of the interconnection and dissipative elements. Especially, we applied the results to a chain of strings modelled by the wave equation and hinted at possible arrangements of beam-string-controller-dissipation structures leading to uniform stabilisation of the total interconnected system.

All results presented here are based on linear semigroup theory, especially the Arendt–Batty–Lyubich–Vũ Theorem and the Gearhart–Prüss–Huang Theorem on stability properties for one-parameter semigroups of linear operators. Therefore, the techniques used are not accessible for nonlinear problems, e.g. nonlinear boundary feedback or nonlinear control systems which may be encountered in practice a lot. Whereas for the generation theorem the Komura–Kato Theorem is a nonlinear analogue to the Lumer–Phillips Theorem for the generation of strongly continuous contraction semigroups by m-dissipative operators, handling stability properties for nonlinear systems is much more involved, see [3] for some efforts in this direction.

9. Appendix: Some technical results on the Euler–Bernoulli Beam

Within this section we consider a port-Hamiltonian system operator $\mathfrak{A}$ of Euler–Bernoulli type, i.e. we assume that

$$\mathfrak{A}x = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial^2}{\partial\zeta^2} + P_0(\zeta) \right) \begin{bmatrix} \mathcal{H}_1(\zeta) & 0 \\ 0 & \mathcal{H}_2(\zeta) \end{bmatrix} \begin{pmatrix} x_1(\zeta) \\ x_2(\zeta) \end{pmatrix}$$

where additionally $x_1(\zeta), x_2(\zeta) \in \mathbb{K}$ are assumed scalars and the Hamiltonian densities $\mathcal{H}_i \in \mathrm{Lip}(0,1)$ as well as the bounded perturbation $P_0 \in \mathrm{Lip}\left(0,1;\mathbb{K}^{2\times 2}\right)$ are Lipschitz continuous. We consider the situation that we have sequences

$$(x_n)_{n\geq 1} \subseteq D\big(\widehat{A}\big) = \left\{x \in L_2\big(0,1;\mathbb{K}^2\big) : \mathcal{H}x \in H^2\big(0,1;\mathbb{K}^2\big)\right\}$$

and $(\beta_n)_{n\geq 1} \subseteq \mathbb{R}$ such that the following hold (with convergence in $L_2(0,1;\mathbb{K}^d)$)

$$x_n \to 0, \quad |\beta_n| \to 0, \quad \text{and} \quad \mathfrak{A}x_n - \mathrm{i}\beta_n x_n \to 0. \tag{4}$$

We first note that then also $(\mathfrak{A} - P_0\mathcal{H})x_n - \mathrm{i}\beta_n x_n \to 0$. Therefore, in the following we can ourselves often essentially restrict to the case $P_0 = 0$. We investigate what can be said about the sequence of traces $\tau(\mathcal{H}x_n)$, if we additionally assume that parts of the trace, e.g. $(\mathcal{H}x_n)(0)$, are already known to converge to zero.

The first important observation is the following.

Lemma 9.1. *Let $j \in \{1,2\}$. Assume additionally that for both boundary points $\zeta \in \{0,1\}$, either $\big(\mathcal{H}_j x_{n,j}\big)(\zeta) \to 0$ or $\big(\mathcal{H}_j x_{n,j}\big)'(\zeta) \to 0$ is known. Then*

$$\frac{1}{\sqrt{|\beta_n|}} \big\|(\mathcal{H}_j x_{n,j})'\big\|_{L_2(0,1)} \to 0, \quad n \to \infty.$$

Proof. As $x_n \to 0$ and $(\mathfrak{A} - P_0\mathcal{H})x_n - \mathrm{i}\beta_n \to 0$ in $L_2(0,1;\mathbb{K}^2)$, we also have that

$$\frac{1}{\beta_n}\big(\mathcal{H}_j x_{n,j}\big)'' \to 0 \quad \text{in } L_2(0,1),$$

and also (already without any of the extra conditions)

$$\frac{1}{\beta_n}\big(\mathcal{H}_j x_{n,j}\big)(\zeta), \frac{1}{\beta_n}\big(\mathcal{H}_j x_{n,j}\big)'(\zeta) \to 0 \quad \text{in } \mathbb{K} \quad \text{for } \zeta = 0, 1.$$

Therefore,

$$\begin{aligned}
\frac{1}{|\beta_n|}\|(\mathcal{H}_j x_{n,j})'\|^2_{L_2(0,1)} &= \frac{1}{|\beta_n|}\int_0^1 \big((\mathcal{H}_j x_{n,j})'(\zeta) \mid (\mathcal{H}_j x_{n,j})'(\zeta)\big)_{\mathbb{K}}\, \mathrm{d}\zeta \\
&= -\frac{1}{|\beta_n|}\int_0^1 \big((\mathcal{H}_j x_{n,j})(\zeta) \mid (\mathcal{H}_j x_{n,j})''(\zeta)\big)_{\mathbb{K}}\, \mathrm{d}\zeta \\
&\quad + \frac{1}{|\beta_n|}\Big[\big((\mathcal{H}_j x_{n,j})(\zeta) \mid (\mathcal{H}_j x_{n,j})'(\zeta)\big)_{\mathbb{K}}\Big]_0^1 \\
&\le \|\mathcal{H}_j x_{n,j}\|_{L_2(0,1)} \frac{1}{|\beta_n|}\|(\mathcal{H}_j x_{n,j})'\|_{L_2(0,1)} \\
&\quad + \sum_{\zeta=0}^{1} |(\mathcal{H}_j x_{n,j})(\zeta)| \frac{1}{|\beta_n|} |(\mathcal{H}_j x_{n,j})'(\zeta)| \to 0, \quad n \to \infty
\end{aligned}$$

where in the last step we used the extra condition on the trace at boundary points $\zeta = 0, 1$. □

Lemma 9.2. *Let $(\beta_n)_{n\ge 1} \subseteq \mathbb{R}$ and $(x_n)_{n\ge1} \subseteq D(\mathfrak{A})$ be as above, i.e. $|\beta_n| \to \infty$, $\sup_{n\in\mathbb{N}} \|x_n\|_X < \infty$, and $\mathfrak{A}x_n - \mathrm{i}\beta_n x_n \to 0$ in $L_2(0,1;\mathbb{K}^2)$ and P_0 and $\mathcal{H}$ are Lipschitz-continuous.*

1. *Assuming $(\mathcal{H}x_n)'(1) \to 0$, and $(\mathcal{H}^1 x_{n,1})(0) \to 0$ or $(\mathcal{H}_1 x_{n,1})'(0) \to 0$, and $(\mathcal{H}_2 x_{n,2})(0) \to 0$ or $(\mathcal{H}_2 x_{n,2})'(0) \to 0$ are known, $(\mathcal{H}x_n)(1) \to 0$.*
2. *Assuming $(\mathcal{H}x_n)'(0) \to 0$, and $(\mathcal{H}_1 x_{n,1})(1) \to 0$ or $(\mathcal{H}_1 x_{n,1})'(1) \to 0$, and $(\mathcal{H}_2 x_{n,2})(1) \to 0$ or $(\mathcal{H}_2 x_{n,2})'(1) \to 0$ are known, $(\mathcal{H}x_n)(0) \to 0$.*
3. *Assuming $(\mathcal{H}x_n)'(0) \to 0$ and $(\mathcal{H}x_n)'(1) \to 0$ are known, $\tau(\mathcal{H}x_n) \to 0$.*

4. *Assuming* $(\mathcal{H}x_n)(0) \to 0$, *and* $(\mathcal{H}_1 x_{n,1})'(0) \to 0$ *or* $(\mathcal{H}_2 x_{n,2})'(0) \to 0$, *as well as* $|(\mathcal{H}_1 x_{n,1})(1)| + |(\mathcal{H}_1 x_{n,1})'(1)| \to 0$ *or* $|(\mathcal{H}_2 x_{n,2})(1)| + |(\mathcal{H}_2 x_{n,2})'(1)| \to 0$, *are known,* $\tau(\mathcal{H}x_n) \to 0$.
5. *Assuming* $(\mathcal{H}x_n)(1) \to 0$, *and* $(\mathcal{H}_1 x_{n,1})'(1) \to 0$ *or* $(\mathcal{H}_2 x_{n,2})'(1) \to 0$, *as well as* $|(\mathcal{H}_1 x_{n,1})(0)| + |(\mathcal{H}_1 x_{n,1})'(0)| \to 0$ *or* $|(\mathcal{H}_2 x_{n,2})(0)| + |(\mathcal{H}_2 x_{n,2})'(0)| \to 0$, *are known,* $\tau(\mathcal{H}x_n) \to 0$.
6. *Assuming* $(\mathcal{H}x_n)(1) \to 0$, *and* $(\mathcal{H}_1 x_{n,1})(0) \to 0$ *or* $(\mathcal{H}_1 x_{n,1})'(0) \to 0$, *and* $(\mathcal{H}_2 x_{n,2})(0) \to 0$ *or* $(\mathcal{H}_2 x_{n,2})'(0) \to 0$ *are known,* $(\mathcal{H}x_n)'(0) \to 0$.
7. *Assuming* $(\mathcal{H}x_n)(0) \to 0$, *and* $(\mathcal{H}_1 x_{n,1})(1) \to 0$ *or* $(\mathcal{H}_1 x_{n,1})'(1) \to 0$, *and* $(\mathcal{H}_2 x_{n,2})(1) \to 0$ *or* $(\mathcal{H}_2 x_{n,2})'(1) \to 0$ *are known,* $(\mathcal{H}x_n)'(1) \to 0$.
8. *Assuming* $(\mathcal{H}x_n)(0) \to 0$ *and* $(\mathcal{H}x_n)(1) \to 0$ *are known,* $\tau(\mathcal{H}x_n) \to 0$.

Proof. The first five cases 1 to 5 are based on the following multiplier argument. As in any case the sequence

$$\left(\frac{iq}{\beta_n} (\mathcal{H}x_n)' \right)_{n \geq 1} \subseteq L_2(0,1;\mathbb{K}^2)$$

is bounded, for any fixed $q \in C^1([0,1];\mathbb{R})$, we obtain from

$$(\mathfrak{A} - P_0 \mathcal{H})x_n - \mathrm{i}\beta_n x_n \to 0 \quad \text{in } L_2(0,1;\mathbb{K}^2)$$

that

$$\begin{aligned}
0 \leftarrow & \left((\mathfrak{A} - P_0\mathcal{H})x_n - \mathrm{i}\beta_n x_n \mid \frac{iq}{\beta_n}(\mathcal{H}x_n)' \right)_{L_2} \\
&= - \left((\mathcal{H}_2 x_{n,2})'' \mid \frac{iq}{\beta_n} (\mathcal{H}_1 x_{n,1})' \right)_{L_2} \\
&\quad + \left((\mathcal{H}_1 x_{n,1})'' \mid \frac{iq}{\beta_n} (\mathcal{H}_2 x_n)' \right)_{L_2} - \left(x_n \mid q(\mathcal{H}x_n)' \right)_{L_2} \\
&= \left((\mathcal{H}_2 x_{n,2})' \mid \frac{iq}{\beta_n} (\mathcal{H}_1 x_{n,1})'' \right)_{L_2} + \left((\mathcal{H}_1 x_{n,1})'' \mid \frac{iq}{\beta_n} (\mathcal{H}_2 x_{n,2})' \right)_{L_2} \\
&\quad + \left((\mathcal{H}_2 x_{n,2})' \mid \frac{iq'}{\beta_n} (\mathcal{H}_1 x_{n,1})' \right)_{L_2} \\
&\quad - \left[\left((\mathcal{H}_2 x_{n,2})'(\zeta) \mid \frac{iq(\zeta)}{\beta_n} (\mathcal{H}_1 x_{n,1})'(\zeta) \right)_{\mathbb{K}} \right]_0^1 \\
&\quad - \left((\mathcal{H}x_n) \mid (q\mathcal{H}^{-1})'(\mathcal{H}x_n) \right)_{L_2} + \frac{1}{2} \left[\left((\mathcal{H}x_n)(\zeta) \mid (q\mathcal{H}^{-1})(\zeta)(\mathcal{H}x_n)(\zeta) \right)_{\mathbb{K}} \right]_0^1 \\
&= 2i \operatorname{Im} \left((\mathcal{H}_2 x_{n,2})' \mid \frac{iq}{\beta_n} (\mathcal{H}_1 x_{n,1})'' \right)_{L_2} + \left((\mathcal{H}_2 x_{n,2})' \mid \frac{iq'}{\beta_n} (\mathcal{H}_1 x_{n,1})' \right)_{L_2} \\
&\quad - \left[\left((\mathcal{H}_2 x_{n,2})'(\zeta) \mid \frac{iq}{\beta_n} (\mathcal{H}_1 x_{n,1})'(\zeta) \right)_{\mathbb{K}} \right]_0^1 \\
&\quad + \frac{1}{2} \left[\left((\mathcal{H}x_n)(\zeta) \mid (q(-q\mathcal{H}^{-1})(\zeta)(\mathcal{H}x_n)(\zeta) \right)_{\mathbb{K}} \right]_0^1 + o(1)
\end{aligned}$$

where we denote by $o(1)$ any terms that vanish as $n \to 0$. Taking the real part, this equality gives us that

$$\operatorname{Re}\left(\left((\mathcal{H}_2 x_{n,2})' \mid \frac{iq'}{\beta_n} (\mathcal{H}_1 x_{n,1})' \right)_{L_2} - \left[\left((\mathcal{H}_2 x_{n,2})'(\zeta) \mid \frac{iq(\zeta)}{\beta_n} (\mathcal{H}_1 x_{n,1})'(\zeta) \right)_{\mathbb{K}} \right]_0^1 \right) + \frac{1}{2} \left[\left((\mathcal{H} x_n)(\zeta) \mid (q\mathcal{H}^{-1})(\zeta)(\mathcal{H} x_n)(\zeta) \right)_{\mathbb{K}} \right]_0^1 = o(1). \tag{5}$$

The first of these terms can be estimated by

$$\frac{c}{|\beta_n|} \left\| (\mathcal{H}_2 x_{n,2})' \right\|_{L_2} \left\| (\mathcal{H}_1 x_{n,1})' \right\|_{L_2}$$

which by Lemma 9.1 and under the constraints of the first or second case tends to zero as $n \to \infty$. The assertion for the first five cases then follow, namely

1. In the first case choose q such that $q(0) = 0$ and $q(1) > 0$, then

$$\frac{1}{2} \left((\mathcal{H} x_n)(1) \mid (q\mathcal{H}^{-1})(1)(\mathcal{H} x_n)(1) \right)_{\mathbb{K}} \leq o(1),$$

   so $(\mathcal{H} x_n)(1) \to 0$ by positive definiteness of $\mathcal{H}(1)$.
2. As before, this time choosing $q(1) = 0$ and $q(0) > 0$.
3. Follows by combining the previous two cases 1 and 2 iteratively.
4. In this case we do not have Lemma 9.1 at hand, but we may choose q to be a constant $c \neq 0$. From equation (5) and the assumption on the boundary trace we then obtain that

$$\left((\mathcal{H} x_n)(1) \mid \mathcal{H}^{-1}(1)(\mathcal{H} x_n)(1) \right)_{\mathbb{K}} = o(1)$$

   so that $(\mathcal{H} x_n)(1) \to 0$ and $(\mathcal{H} x_n)(1) \to 0$. The assertion then follows from cases 6 and 7 below.
5. For this case, repeat the argument of case 4.

We proceed by showing the assertion for the cases 6 and 7 by a similar multiplier argument, but this time using the multiplier

$$\frac{q}{\beta_n} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x_n'.$$

Since $\mathcal{H}$ is Lipschitz continuous, this is a bounded sequence in $L_2(0,1;\mathbb{K}^2)$ as well, so we find that

$$
\begin{aligned}
0 \leftarrow & \left((\mathfrak{A} - P_0\mathcal{H})x_n - \mathrm{i}\beta_n x_n \;\middle|\; \frac{q}{\beta_n}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} x_n' \right)_{L_2} \\
&= \left((\mathcal{H}_1 x_{n,1})'' \;\middle|\; \frac{q}{\beta_n} x_{n,1}' \right)_{L_2} + \left((\mathcal{H}_2 x_{n,2})'' \;\middle|\; \frac{q}{\beta_n} x_{n,2}' \right)_{L_2} \\
&\quad - \left(i x_{n,1} \;\middle|\; q x_{n,2}' \right)_{L_2} + \left(i x_{n,2} \;\middle|\; q x_{n,1}' \right)_{L_2} \\
&= \left((\mathcal{H}_1 x_{n,1})'' \;\middle|\; \frac{q(\mathcal{H}_1)^{-1}}{\beta_n} (\mathcal{H}_1 x_{n,1})' \right)_{L_2} - \left((\mathcal{H}_1 x_{n,1})'' \;\middle|\; \frac{q(\mathcal{H}_1)^{-1}}{\beta_n} (\mathcal{H}_1)' x_{n,1} \right)_{L_2} \\
&\quad + \left((\mathcal{H}_2 x_{n,2})'' \;\middle|\; \frac{q(\mathcal{H}_2)^{-1}}{\beta_n} (\mathcal{H}_2 x_{n,2})' \right)_{L_2} - \left((\mathcal{H}_2 x_{n,2})'' \;\middle|\; \frac{q(\mathcal{H}_2)^{-1}}{\beta_n} (\mathcal{H}_2)' x_{n,2} \right)_{L_2} \\
&\quad + \left(i x_{n,1}' \;\middle|\; q x_{n,2} \right)_{L_2} - \left(i x_{n,2} \;\middle|\; q x_{n,1}' \right)_{L_2} \\
&\quad - \left(i x_{n,2} \;\middle|\; q' x_{n,1} \right)_{L_2} + \Big[\left(i x_{n,1}(\zeta) \;\middle|\; q(\zeta) x_{n,2}(\zeta) \right)_{\mathbb{K}} \Big]_0^1 \\
&= -\frac{1}{2}\Bigg(\left((\mathcal{H}_1 x_{n,1})' \;\middle|\; \frac{(q(\mathcal{H}_1)^{-1})'}{\beta_n} (\mathcal{H}_1 x_{n,1})' \right)_{L_2} \\
&\quad + \left((\mathcal{H}_2 x_{n,2})' \;\middle|\; \frac{(q(\mathcal{H}_2)^{-1})'}{\beta_n} (\mathcal{H}_2 x_{n,2})' \right)_{L_2} \Bigg) \\
&\quad + \frac{1}{2}\Bigg[\left((\mathcal{H}_1 x_{n,1})'(\zeta) \;\middle|\; \frac{(q(\mathcal{H}_1)^{-1})(\zeta)}{\beta_n} (\mathcal{H}_1 x_{n,1})'(\zeta) \right)_{\mathbb{K}} \\
&\quad + \left((\mathcal{H}_2 x_{n,2})'(\zeta) \;\middle|\; \frac{(q(\mathcal{H}_2)^{-1})(\zeta)}{\beta_n} (\mathcal{H}_2 x_{n,2})'(\zeta) \right)_{\mathbb{K}} \Bigg]_0^1 \\
&\quad + 2i\,\mathrm{Im}\, \left(i x_{n_1}' \;\middle|\; q x_{n,2} \right)_{L_2} + \Big[\left(i x_{n,1}(\zeta) \;\middle|\; q(\zeta) x_{n,2}(\zeta) \right)_{\mathbb{K}} \Big]_0^1 + o(1).
\end{aligned}
$$

Thus, taking the real part, we arrive at the equation

$$
\begin{aligned}
&\frac{1}{2}\Bigg[\left((\mathcal{H}_1 x_{n,1})'(\zeta) \;\middle|\; \frac{(q(\mathcal{H}_1)^{-1})(\zeta)}{\beta_n} (\mathcal{H}_1 x_{n,1})'(\zeta) \right)_{\mathbb{K}} \\
&\quad + \left((\mathcal{H}_2 x_{n,2})'(\zeta) \;\middle|\; \frac{(q(\mathcal{H}_2)^{-1})(\zeta)}{\beta_n} (\mathcal{H}_2 x_{n,2})'(\zeta) \right)_{\mathbb{K}} \Bigg]_0^1 \\
&\quad + \mathrm{Re}\, \Big[\left(i x_{n,1}(\zeta) \;\middle|\; q(\zeta) x_{n,2}(\zeta) \right)_{\mathbb{K}} \Big]_0^1 \\
&= \frac{1}{2}\Bigg(\left((\mathcal{H}_1 x_{n,1})' \;\middle|\; \frac{(q(\mathcal{H}_1)^{-1})'}{\beta_n} (\mathcal{H}_1 x_{n,1})' \right)_{L_2} \\
&\quad + \left((\mathcal{H}_2 x_{n,2})' \;\middle|\; \frac{(q(\mathcal{H}_2)^{-1})'}{\beta_n} (\mathcal{H}_2 x_{n,2})' \right)_{L_2} \Bigg) + o(1).
\end{aligned}
$$

Also for cases 6 and 7, Lemma 9.1 gives us that

$$\frac{1}{\sqrt{|\beta_n|}} \left\| (\mathcal{H}x_n)' \right\|_{L_2} = o(1),$$

so that we obtain the result by choosing $q(0) = 0$ and $q(1) > 0$ or $q(0) > 0$ and $q(1) = 0$, respectively. Finally, case 8 follows by combining the results of cases 6 and 7. □

Remark 9.3. Note that all the assertions of Lemmas 9.1 and 9.2 also hold for $\mathfrak{A}$ of the form

$$\mathfrak{A}x = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial^2}{\partial \zeta^2} + P_1 \frac{\partial}{\partial \zeta} + P_0(\zeta) \right) \begin{bmatrix} \mathcal{H}_1(\zeta) & 0 \\ 0 & \mathcal{H}_2(\zeta) \end{bmatrix} \begin{pmatrix} x_1(\zeta) \\ x_2(\zeta) \end{pmatrix}$$

where $P_1 \in \mathbb{K}^{2\times 2}$ is any symmetric matrix. Namely, in cases 1 to 5 of Lemma 9.2 one may use that

$$\operatorname{Re} \left(P_1 (\mathcal{H}x_n)' \mid \frac{iq}{\beta_n} (\mathcal{H}x_n)' \right) = 0,$$

as iqP_1 is skew-symmetric. For the latter three cases 6 to 8 of Lemma 9.2 one may always use Lemma 9.1 to deduce that $\frac{1}{\sqrt{|\beta_n|}} \left\| (\mathcal{H}x_n)' \right\|_{L_2} = o(1)$, but then also

$$\left| \left(P_1 (\mathcal{H}x_n)' \mid \frac{q}{\beta_n} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x_n' \right) \right| \le C \frac{1}{|\beta_n|} \left\| \mathcal{H}x_n \right\|_{H^1}^2 = o(1).$$

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A distance between operators acting in different Hilbert spaces and operator convergence

Olaf Post and Jan Simmer

1. Introduction

The aim of the present article is to give an introduction to the concept of quasi-unitary equivalence and to define several (pseudo-)metrics on the space of self-adjoint operators acting possibly in different Hilbert spaces. As some of the "metrics" do not fulfil all properties of a metric (e.g. some lack the triangle inequality or the definiteness), we call them "distances" here. To the best of our knowledge, such distances are treated for the first time here. The present article shall serve as a starting point of further research.

1.1. Operator convergence in varying Hilbert spaces

A main motivation for the definition of a distance for operators acting in different Hilbert spaces is apparent: In many applications, operators such as a Laplacian $\Delta_\varepsilon \geq 0$ act on a Hilbert space $\mathscr{H}_\varepsilon$ that changes with respect to a parameter ε, and one is interested in some sort of convergence. Our concept allows to define a *generalised norm convergence* for the resolvents $R_\varepsilon = (\Delta_\varepsilon + 1)^{-1}$ acting on $\mathscr{H}_\varepsilon$ towards a resolvent $R_0 = (\Delta_0 + 1)^{-1}$ acting on $\mathscr{H}_0$ using identification operators $J_\varepsilon \colon \mathscr{H}_0 \to \mathscr{H}_\varepsilon$. One can first assume that J_ε is unitary and that

$$\|J_\varepsilon R_0 - R_\varepsilon J_\varepsilon\| \to 0 \tag{1.1a}$$

as $\varepsilon \to 0$. In applications (as the one presented in Section 3 on shrinking manifolds) it is more convenient to use maps J_ε that are unitary only in an asymptotic sense, i.e., where

$$\|(\mathrm{id}_{\mathscr{H}_0} - J_\varepsilon^* J_\varepsilon) R_0\| \to 0 \quad \text{and} \quad \|(\mathrm{id}_{\mathscr{H}_\varepsilon} - J_\varepsilon J_\varepsilon^*) R_\varepsilon\| \to 0. \tag{1.1b}$$

We call such operators J_ε *quasi-unitary*, see Subsection 2.2. For example, the second estimate of (1.1b) means that if $(u_\varepsilon)_\varepsilon$ is a family with $\|(\Delta_\varepsilon + 1)u_\varepsilon\|_{\mathscr{H}_\varepsilon} = 1$, then $\|u_\varepsilon - J_\varepsilon J_\varepsilon^* u_\varepsilon\| \to 0$ as $\varepsilon \to 0$. In our example, we even have $J_\varepsilon^* J_\varepsilon = \mathrm{id}_{\mathscr{H}_0}$, and

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functions in the range of $\mathrm{id}_{\mathscr{H}_\varepsilon} - J_\varepsilon J_\varepsilon^*$ do not concentrate at "negligible" regions and at high (transversal) modes, see (3.8a)–(3.8b).

We illustrate in Section 3 the abstract theory on generalised norm resolvent convergence: Consider a family of thin Riemannian manifolds X_ε that shrink towards a metric graph X_0 (i.e., a topological graph where the edges are metrically identified with compact intervals). We show that the Laplacians on X_ε converge in generalised norm resolvent sense to the so-called Kirchhoff Laplacian on X_0. The example of thin branched manifolds shrinking towards a metric graph has already been treated in [12] (see also [4, 13, 5] and references therein). In this note we use a sightly different proof as we directly compare the resolvent difference and we do not make use of the corresponding quadratic forms as in [12, 13]. Other topological perturbations of manifolds such as removing many small balls are treated in a similar way in [9, 1], see also the references therein. The concept of generalised norm resolvent convergence also applies to approximations of metric spaces with a Laplace-like operator by finite dimensional operators such as graph approximations of fractals, see [14, 15, 16] for details.

1.2. Metrics on sets of operators acting in different Hilbert spaces

When defining a distance between unbounded operators such as Laplacians, it is convenient to work with the resolvent $R = (\Delta+1)^{-1}$ where Δ is an unbounded, self-adjoint and non-negative operator in a Hilbert space $\mathscr{H}$. In particular, we consider the space of all self-adjoint, injective and bounded operators R with spectrum in $[0,1]$ as space of operators. In all our examples, the distance will not change when passing from an operator R to a unitarily equivalent operator URU^* for a unitary map $U\colon \mathscr{H} \to \widetilde{\mathscr{H}}$. The simplest distance we define is

$$d_{\mathrm{uni}}(R, \widetilde{R}) := \inf\bigl\{ \|\widetilde{R} - URU^*\| \bigm| U\colon \mathscr{H} \to \widetilde{\mathscr{H}} \text{ unitary} \bigr\} \tag{1.2}$$

for operators R on $\mathscr{H}$ and $\widetilde{R}$ on $\widetilde{\mathscr{H}}$ as above. If (1.1a) is fulfilled for some unitary map J_ε, then $d_{\mathrm{uni}}(R_\varepsilon, R_0) \to 0$.

From an abstract point of view we could also work with operators in a *fixed* Hilbert space $\mathscr{H}$ using an abstract unitary map U, but the identification is in general not natural. For example, if R and $\widetilde{R}$ are both compact, then one can define a unitary map via $U\psi_k = \widetilde{\psi}_k$, where $(\psi_k)_k$ resp. $(\widetilde{\psi}_k)_k$ are orthonormal bases of eigenfunctions of R resp. $\widetilde{R}$. Then $\|\widetilde{R} - URU^*\| = \sup_k |\widetilde{\mu}_k - \mu_k|$ where μ_k resp. $\widetilde{\mu}_k$ denote the corresponding eigenvalues. This observation is not very useful in examples, as one needs at least information on one of the eigenfunction or eigenvalue families.

Later on, we want to use more general maps $J\colon \mathscr{H} \to \widetilde{\mathscr{H}}$ instead of unitary ones, and allow J to be unitary only "up to a small error", measured e.g. by quantities such as $\|(\mathrm{id}_{\mathscr{H}} - J^*J)R\|$ and $\|(\mathrm{id}_{\widetilde{\mathscr{H}}} - JJ^*)\widetilde{R}\|$ (see Subsection 2.2 for details).

If R is a compact operator, then more can be said. Basically, the different distances defined later on (such as d_{uni}) depend only on the spectrum, i.e., the sequence of eigenvalues $(\mu_k)_k$ (ordered non-increasingly and repeated according to multiplicity). In particular, for compact R and $\widetilde{R}$, we have $d_{\mathrm{uni}}(R, \widetilde{R}) = 0$ if and only if R and $\widetilde{R}$ are unitarily equivalent (this is no longer true for general R and $\widetilde{R}$, see Remark 2.3).

In this article, we only treat *operators*: there is a more elaborated version of the concept of quasi-unitary equivalence for closed quadratic forms using not only identification operators $J\colon \mathscr{H} \to \widetilde{\mathscr{H}}$ on the level of the Hilbert spaces, but also identification operators $J^1\colon \mathscr{H}^1 \to \widetilde{\mathscr{H}}^1$ and $J'^1\colon \widetilde{\mathscr{H}}^1 \to \mathscr{H}^1$ on the level of the form domains $\mathscr{H}^1 = \operatorname{dom} \Delta^{1/2}$ and $\widetilde{\mathscr{H}}^1 = \operatorname{dom} \widetilde{\Delta}^{1/2}$ (see e.g. [12, 13]).

1.3. Related works

There are of course a lot of classical results on operator convergence (and resolvent convergence) for operators acting in a *fixed* Hilbert space, see e.g. [8, Sect. IV.2] or [17, Chap. VIII.7]. The concept of *generalised norm resolvent convergence* has already been introduced by [18, Sect. 9.3] and is closely related to ours: a sequence of self-adjoint operators $\Delta_n \geq 0$ converges in generalised norm resolvent sense to Δ_∞ if and only if $R_n = (\Delta_n + 1)^{-1}$ and $R_\infty = (\Delta_\infty + 1)^{-1}$ are δ_n-quasi-unitarily equivalent (see Definition 2.10) with $\delta_n \to 0$ as $n \to \infty$, or, equivalently, if $d_{\text{q-uni}}(R_n, R_\infty) \to 0$, see Definition 2.14. Bögli recently proved spectral convergence of spectra and pseudospectra in [2, 3], we refer also to the extensive list of references therein.

We compare the different concepts of Kuwae and Shioya [10, Sect. 2] (and related concepts such as Mosco and Γ-convergence) generalising *strong* resolvent convergence, the concept of the generalised norm (and strong) resolvent convergence in the sense of Weidmann and the results of Bögli in a subsequent paper. Here, we focus on the definition of some metrics on a set of operators defined on different Hilbert spaces. The aim is to express (operator norm) convergence in metric terms. Note that our concept easily allows to define a *convergence* speed, which in many other works is not treated.

2. Distances between operators acting in different Hilbert spaces

In this section, we introduce a generalisation of a distance of two operators acting in different Hilbert spaces.

2.1. A spectral distance for operators acting in different Hilbert spaces

For a Hilbert space $\mathscr{H}$, denote by

$$\mathcal{B}_{(0,1]}(\mathscr{H}) := \left\{ R\colon \mathscr{H} \to \mathscr{H} \,\middle|\, R = R^*,\ \ker R = \{0\},\ \operatorname{spec}(R) \subset [0,1] \right\} \tag{2.1}$$

the set of all self-adjoint and injective operators with spectrum in $[0,1]$, i.e., the set of non-negative, self-adjoint and injective operators with operator norm bounded by 1.[1] Moreover, let HS be a set of separable Hilbert spaces of infinite dimension[2] and let

$$\mathcal{B}_{(0,1]} := \bigcup_{\mathscr{H} \in \mathsf{HS}} \mathcal{B}_{(0,1]}(\mathscr{H}). \tag{2.2}$$

We first define the following distance function:

Definition 2.1. For $R, \widetilde{R} \in \mathcal{B}_{(0,1]}$ we define the *unitary distance* of R and $\widetilde{R}$ as in (1.2), i.e.,

$$d_{\mathrm{uni}}(R, \widetilde{R}) := \inf\{\, \|\widetilde{R} - URU^*\| \,\big|\, U \text{ unitary} \,\}.$$

Proposition 2.2. *The function d_{uni} is a pseudometric on $\mathcal{B}_{(0,1]}$ (i.e., it is a metric execpt for the positive definiteness). Moreover, $d_{\mathrm{uni}}(R, \widetilde{R}) = 0$ is equivalent with the fact that there is a sequence of unitary operators $U_n \colon \mathscr{H} \to \widetilde{\mathscr{H}}$ such that*

$$\|\widetilde{R} - U_n R U_n^*\| \to 0 \qquad \text{as } n \to \infty. \tag{2.3}$$

Finally, if R and $\widetilde{R}$ are unitarily equivalent, then $d_{\mathrm{uni}}(R, \widetilde{R}) = 0$.

Proof. We have $d_{\mathrm{uni}}(R, \widetilde{R}) \ge 0$, $d_{\mathrm{uni}}(R, R) = 0$, $d_{\mathrm{uni}}(R, \widetilde{R}) = d_{\mathrm{uni}}(\widetilde{R}, R)$ and the triangle inequality

$$d_{\mathrm{uni}}(R_1, R_3) \le d_{\mathrm{uni}}(R_1, R_2) + d_{\mathrm{uni}}(R_2, R_3).$$

follows from

$$\begin{aligned} \|R_3 - U_{13} R_1 U_{13}^*\| &\le \|R_3 - U_{23} R_2 U_{23}^*\| + \|U_{23}(R_2 - U_{12} R_1 U_{12}^*) U_{23}^*\| \\ &= \|R_3 - U_{23} R_2 U_{23}^*\| + \|R_2 - U_{12} R_1 U_{12}^*\| \end{aligned}$$

using $U_{ij} \colon \mathscr{H}_i \to \mathscr{H}_j$ as unitary operators with $U_{13} = U_{23} U_{12}$. Taking the infimum over all unitary operators U_{12} and U_{23} we obtain the desired inequality. Note that all unitary operators $U_{13} \colon \mathscr{H}_1 \to \mathscr{H}_3$ can be written as $U_{23} U_{12}$, e.g., with $U_{23} = U_{13} U_{12}^*$ for some fixed U_{12}. The remaining claims are easily seen. □

Remark 2.3. The condition of the two operators R and $\widetilde{R}$ in (2.3) is closely related to the notion *approximate unitary equivalence* defined for the C*-algebras generated by R and $\widetilde{R}$, cf. [11] and references therein. Note that the unitary orbit of R in $\mathcal{B}(\widetilde{\mathscr{H}})$, i.e., the set $\{\, URU^* \,|\, U \colon \mathscr{H} \to \widetilde{\mathscr{H}} \text{ unitary} \,\}$ is *not* closed in the operator topology; in particular, $\widetilde{R}$ and R are not (necessarily) unitarily equivalent if $d_{\mathrm{uni}}(R, \widetilde{R}) = 0$. It follows from the next result (see also Proposition 2.15) that such operators must have the same spectrum.

[1] We could use any other positive number $c > 0$ as norm bound, but 1 makes the following estimates simpler and 1 is also the norm bound in our main application where $R = (\Delta + 1)^{-1}$ for some non-negative, self-adjoint and possibly unbounded operator Δ.

[2] We need to define a fixed set of Hilbert spaces to avoid some set-theoretic problems related to self-referencing definitions such as "the set of all sets ... ". Typically, HS is s family of Hilbert spaces such as $\mathsf{HS} = \{\, \mathscr{H}_m \,|\, m \in \mathbb{N} \cup \{\infty\} \,\}$ or $\mathsf{HS} = \{\, \mathscr{H}_\varepsilon \,|\, \varepsilon \in [0,1] \,\}$. Moreover, we assume here for simplicity that all Hilbert spaces have infinite dimension.

Proposition 2.4 ([7, Lemma A.1])**.** *We have*

$$d_{\mathrm{H}}(\operatorname{spec}(R), \operatorname{spec}(\widetilde{R})) \leq d_{\mathrm{uni}}(R, \widetilde{R}), \tag{2.4}$$

where d_{H} *denotes the Hausdorff distance.*

Proof. For a unitary map $U\colon \mathscr{H} \to \widetilde{\mathscr{H}}$, we have $\operatorname{spec}(URU^*) = \operatorname{spec}(R)$. In Lemma A.1 of [7] it is proved that

$$d_{\mathrm{H}}(\operatorname{spec}(R), \operatorname{spec}(\widetilde{R})) \leq \|URU^* - \widetilde{R}\|.$$

As U is arbitrary, the assertion follows. □

We restrict now our space of operators to certain compact operators. Denote by

$$\mathcal{K}_{(0,1]}(\mathscr{H}) := \{\, R \in \mathcal{B}_{(0,1]}(\mathscr{H}) \mid R \text{ compact} \,\}$$

the set of compact and injective operators such that $\operatorname{spec}(R) \subset [0, 1]$. Moreover, set

$$\mathcal{K}_{(0,1]} := \bigcup_{\mathscr{H} \in \mathsf{HS}} \mathcal{K}_{(0,1]}(\mathscr{H}). \tag{2.5}$$

For $R \in \mathcal{K}_{(0,1]}(\mathscr{H})$, denote by $(\mu_k)_k$ its (discrete spectrum), ordered in non-increasing order, repeated according to multiplicity. Note that $\mu_k \to 0$ as $k \to \infty$. Denote by Σ the space of all such sequences, i.e.,

$$\Sigma := \Big\{\, \mu = (\mu_k)_{k \in \mathbb{N}} \;\Big|\; \lim_{k \to \infty} \mu_k = 0,\ \forall k \in \mathbb{N}\colon 0 < \mu_{k+1} \leq \mu_k \leq 1 \Big\}. \tag{2.6}$$

Denote by $(\psi_k)_k$ a corresponding sequence of orthonormal eigenfunctions. As R is injective, $(\psi_k)_k$ is an orthonormal basis. Similarly, let $(\widetilde{\mu}_k)_k \in \Sigma$ and $(\widetilde{\psi}_k)_k$ be the ordered eigenvalue sequence with corresponding orthonormal basis of eigenvectors for $\widetilde{R} \in \mathcal{K}_{(0,1]}(\widetilde{\mathscr{H}})$.

We set

$$\mathsf{K}_{(0,1]} := \mathcal{K}_{(0,1]}/{\sim} \tag{2.7}$$

where $R \sim \widetilde{R}$ if and only if R and $\widetilde{R}$ are unitarily equivalent.

As the class of operators unitarily equivalent with R is actually determined by the sequence of eigenvalues $(\mu_k)_k$ we have the following result:

Lemma 2.5. *The map* $\sigma\colon \mathcal{K}_{(0,1]} \to \Sigma$ *associating to* R *its ordered sequence of eigenvalues* $(\mu_k)_k$ *descends to a bijective map onto the quotient, i.e.,* $\widetilde{\sigma}\colon \mathsf{K}_{(0,1]} \to \Sigma$, $[R] \mapsto \sigma(R)$, *is well-defined and bijective.*

The main reason why we restrict to the space of compact operators is that operators with d_{uni}-distance 0 are now actually unitarily equivalent:

Proposition 2.6. *We have* $d_{\mathrm{uni}}(R, \widetilde{R}) = 0$ *if and only if* R *and* $\widetilde{R}$ *are unitarily equivalent. In particular,* d_{uni} *induces a metric on* $\mathsf{K}_{(0,1]}$.

Proof. If $d_{\mathrm{uni}}(R, \widetilde{R}) = 0$, then R and $\widetilde{R}$ are unitarily equivalent by (2.11) and Proposition 2.15 (c), the arguments used there are independent of what we have used so far. If the eigenvalues of R and $\widetilde{R}$ are simple, we could also use Proposition 2.4 to conclude $d_{\mathrm{H}}(\operatorname{spec}(R), \operatorname{spec}(\widetilde{R})) = 0$ and hence $\operatorname{spec}(R) = \operatorname{spec}(\widetilde{R})$. The simplicity of the spectra implies that R and $\widetilde{R}$ are actually unitarily equivalent. □

Let us now define a spectral distance respecting also the multiplicity of the eigenvalues:

Definition 2.7. For $R, \widetilde{R} \in \mathcal{K}_{(0,1]}$ denote by

$$d_{\mathrm{spec}}(R, \widetilde{R}) := \sup_{k \in \mathbb{N}} |\mu_k - \widetilde{\mu}_k|$$

the (*multiplicity respecting*) *distance of the spectra.*

We have some simple consequences:

Lemma 2.8. (a) *The supremum in Definition* 2.7 *is actually a maximum.*

(b) d_{spec} *defines a pseudometric on* $\mathcal{K}_{(0,1]}$.

(c) *We have* $d_{\mathrm{spec}}(R, \widetilde{R}) = 0$ *if and only if* R *and* $\widetilde{R}$ *are unitarily equivalent. In particular,* d_{spec} *induces a metric on* $\mathsf{K}_{(0,1]}$.

Proof. (a) This is clear as the sequences are monotone decreasing and converge to 0. (b) As the right hand side in the definition of d_{spec} is the supremum norm, the claim is standard. (c) If $d_{\mathrm{spec}}(R, \widetilde{R}) = 0$ then $\mu_k = \widetilde{\mu}_k$ for all indices k. Define a unitary map by $U\psi_k = \widetilde{\psi}_k$, then $\widetilde{R} = URU^*$. □

Proposition 2.9. *For* $R, \widetilde{R} \in \mathcal{K}_{(0,1]}$, *we have*

$$d_{\mathrm{spec}}(R, \widetilde{R}) \geq d_{\mathrm{uni}}(R, \widetilde{R}).$$

Proof. Let $U \colon \mathscr{H} \to \widetilde{\mathscr{H}}$ be the unitary map given by $U\psi_k = \widetilde{\psi}_k$, then it is easily seen that

$$d_{\mathrm{spec}}(R, \widetilde{R}) = |\widetilde{\mu}_{k_0} - \mu_{k_0}| = \|(\widetilde{R} - URU^*)\widetilde{\psi}_{k_0}\| = \|\widetilde{R} - URU^*\|,$$

where the maximum is achieved at k_0. As U is unitary, the inequality follows by the definition of $d_{\mathrm{uni}}(R, \widetilde{R})$ via an infimum over all unitary maps $U \colon \mathscr{H} \to \widetilde{\mathscr{H}}$. □

Passing to the sequence space Σ, we define

$$d_{\mathrm{H}}(\mu, \widetilde{\mu}) := d_{\mathrm{H}}\big(\{\, \mu_k \,|\, k \in \mathbb{N} \,\}, \{\, \widetilde{\mu}_k \,|\, k \in \mathbb{N} \,\}\big).$$

This is actually only a pseudometric as the multiple appearance of a value in a *sequence* $\mu = (\mu_k)_k$ is not detected in the *set* $\{\, \mu_k \,|\, k \in \mathbb{N} \,\}$. Using the symbols d_{uni} and d_{spec} also for the induced metrics on Σ (see (2.6)), we have

$$d_{\mathrm{H}}(\mu, \widetilde{\mu}) \leq d_{\mathrm{uni}}(\mu, \widetilde{\mu}) \leq d_{\mathrm{spec}}(\mu, \widetilde{\mu}) \tag{2.8}$$

combining Propositions 2.4 and 2.9.

Note that the metric space $(\Sigma, d_{\mathrm{spec}})$ is not complete, choose e.g. the sequence $(\mu^{(n)})_n$ with $\mu^{(n)} = (1/(kn))_{k \in \mathbb{N}}$, then $d_{\mathrm{spec}}(\mu^{(n)}, \mu^{(m)}) = |1/n - 1/m| \to 0$ as

$m, n \to \infty$, i.e., $(\mu^{(n)})_n$ is a Cauchy sequence but the limit $0 = (0)_k$ is not in Σ. It can be seen similarly that $(\Sigma, d_{\mathrm{uni}})$ and (Σ, d_{H}) are not complete.

2.2. Quasi-unitary equivalence

We now want to weaken the condition that U is unitary in Definition 2.1 and use a slightly more general concept. We define the correspondent distance in Subsection 2.4.

Definition 2.10. Let $\delta \ge 0$. Moreover, let $R \in \mathcal{B}_{(0,1]}(\mathscr{H})$ and $\widetilde{R} \in \mathcal{B}_{(0,1]}(\widetilde{\mathscr{H}})$. We say that, R and $\widetilde{R}$ are *δ-quasi-unitarily equivalent*, if there are bounded operators $J \colon \mathscr{H} \to \widetilde{\mathscr{H}}$ and $J' \colon \widetilde{\mathscr{H}} \to \mathscr{H}$ such that

$$\|J\| \le 1 + \delta, \qquad \|J'\| \le 1 + \delta, \tag{2.9a}$$

$$\|J' - J^*\| \le \delta, \tag{2.9b}$$

$$\|(\mathrm{id}_{\mathscr{H}} - J'J)R\| \le \delta, \qquad \|(\mathrm{id}_{\widetilde{\mathscr{H}}} - JJ')\widetilde{R}\| \le \delta, \tag{2.9c}$$

$$\|JR - \widetilde{R}J\| \le \delta, \qquad \|J'\widetilde{R} - RJ'\| \le \delta. \tag{2.9d}$$

We call J and J' *identification operators* and δ the *error*.

Actually, some conditions follow from others with possibly different δ, see e.g. the next lemma; we have included all of them in the above definition to make them *symmetric* with respect to R and $\widetilde{R}$.

Obviously, if $\delta = 0$ in (2.9a)–(2.9c) then J is unitary, and (2.9d) is equivalent to the norm estimate $\|\widetilde{R} - J^*RJ\| \le \delta$. In particular, 0-quasi-unitary equivalence is just unitary equivalence.

For example, we have the following simple facts:

Lemma 2.11. (a) *If $\|J\| \le 1 + \delta$ and* (2.9b) *hold, then $\|J'\| \le 1 + 2\delta$.*

(b) *If $\|JR - \widetilde{R}J\| \le \delta$ and* (2.9b) *hold, then $\|J'\widetilde{R} - RJ'\| \le 3\delta$.*

(c) *If $J' = J^*$ then $\|J\| = \|J'\|$ and $\|JR - \widetilde{R}J\| = \|J'\widetilde{R} - RJ'\|$, i.e., only one of the estimates in* (2.9a) *and* (2.9d) *is enough to ensure δ-quasi-unitary equivalence.*

(d) *If J is unitary, then R and $\widetilde{R}$ are δ-quasi unitarily equivalent* (*with unitary J*) *if and only if $\|\widetilde{R} - JRJ^*\| \le \delta$.*

(e) *If R_1 and R_2 are δ_{12}-quasi-unitarily equivalent and R_2 and R_3 are δ_{23}-quasi-unitarily equivalent, then R_1 and R_3 are δ_{13}-quasi-unitarily equivalent with $\delta_{13} = \Phi(\delta_{12}, \delta_{23}) = \mathrm{O}(\delta_{12}) + \mathrm{O}(\delta_{23})$, where Φ is defined in* (2.10)*.*

(f) *If R and $\widetilde{R}$ are δ-quasi-unitarily equivalent with $\delta \in [0, 1]$ and with identification operators J and J' then R and $\widetilde{R}$ are 3δ-quasi-unitarily equivalent with identification operators J and J^*. In particular, we can assume, without loss of generality, that $J' = J^*$ in Definition* 2.10*.*

Proof. The first four assertions are obvious. (e) The transitivity of quasi-unitary equivalence can be seen as follows (see also [13, Theorem 4.2.5] and [16]): Denote by $J_{12} \colon \mathscr{H}_1 \to \mathscr{H}_2$ and $J_{21} \colon \mathscr{H}_2 \to \mathscr{H}_1$ the identification operators for $R_1 \in \mathcal{B}_{(0,1]}(\mathscr{H}_1)$

and $R_2 \in \mathcal{B}_{(0,1]}(\mathscr{H}_2)$, and similarly, denote by $J_{23}\colon \mathscr{H}_2 \to \mathscr{H}_3$ and $J_{32}\colon \mathscr{H}_3 \to \mathscr{H}_2$ the identification operators for $R_2 \in \mathcal{B}_{(0,1]}(\mathscr{H}_2)$ and $R_3 \in \mathcal{B}_{(0,1]}(\mathscr{H}_3)$.

We define the identification operators for R_1 and R_3 by $J_{13} := J_{23}J_{12}$ and $J_{31} := J_{21}J_{32}$. Then

$$\|J_{13}\| = \|J_{23}J_{12}\| \le (1+\delta_{23})(1+\delta_{12}) = 1 + (\delta_{12}+\delta_{23}) + \delta_{12}\delta_{23}$$

and similarly for $\|J_{31}\|$. Inequality (2.9b) follows from

$$\begin{aligned}\|J_{13}^* - J_{31}\| &\le \|J_{12}^*(J_{23}^* - J_{32})\| + \|(J_{12}^* - J_{21})J_{32}\| \\ &\le (1+\delta_{12})\delta_{23} + \delta_{12}(1+\delta_{23}) = \delta_{12} + \delta_{23} + 2\delta_{12}\delta_{23}.\end{aligned}$$

The first inequality in (2.9c) is also satisfied because

$$\begin{aligned}&\|(\mathrm{id}_{\mathscr{H}_1} - J_{31}J_{13})R_1\| \\ &\quad\le \|(\mathrm{id}_{\mathscr{H}_1} - J_{21}J_{12})R_1\| + \|J_{21}(J_{12}R_1 - R_2J_{12})\| \\ &\qquad + \|J_{21}(\mathrm{id}_{\mathscr{H}_2} - J_{32}J_{23})R_2J_{12}\| + \|J_{21}J_{32}J_{23}(R_2J_{12} - J_{12}R_1)\| \\ &\quad\le \delta_{12} + (1+\delta_{12})\delta_{12} + (1+\delta_{12})^2\delta_{23} + (1+\delta_{12})(1+\delta_{23})^2\delta_{12} \\ &\quad= 3\delta_{12} + \delta_{23} + 4\delta_{12}\delta_{23} + \delta_{12}^2 + \delta_{12}^3 + 3\delta_{12}^2\delta_{23} + \delta_{12}\delta_{23}^2 + \delta_{12}^2\delta_{23}^2\end{aligned}$$

and similarly we have

$$\begin{aligned}\|(\mathrm{id}_{\mathscr{H}_3} - J_{13}J_{31})R_3\| \le 3\delta_{23} + \delta_{12} + 4\delta_{12}\delta_{23} + \delta_{23}^2 \\ + 3\delta_{12}\delta_{23}^2 + \delta_{12}^2\delta_{23} + \delta_{23}^3 + \delta_{12}^2\delta_{23}^2.\end{aligned}$$

For the first inequality of (2.9d), we estimate

$$\begin{aligned}\|J_{13}R_1 - R_3J_{13}\| &\le \|J_{23}(J_{12}R_1 - R_2J_{12})\| + \|(J_{23}R_2 - R_3J_{23})J_{12}\| \\ &\le (1+\delta_{23})\delta_{12} + \delta_{23}(1+\delta_{12}) = \delta_{12} + \delta_{23} + 2\delta_{12}\delta_{23},\end{aligned}$$

and similarly for the second inequality of (2.9d). In particular, the desired estimate holds if we define $\delta_{13} := \Phi(\delta_{12}, \delta_{23})$ with

$$\Phi(a,b) := 3(a+b) + (a+b)^2 + 2ab + (a+b)^3 + a^2b^2. \tag{2.10}$$

Note that $\Phi(a,b) \le 12(a+b)$ if $a, b \in [0,1]$.

(f) We have

$$\|(\mathrm{id}_{\mathscr{H}} - J^*J)R\| \le \|(\mathrm{id}_{\mathscr{H}} - J'J)R\| + \|J' - J^*\|\,\|J\|\,\|R\| \le \delta(2+\delta) \le 3\delta$$

provided $\delta \in [0,1]$, and similarly for the second inequality of (2.9c). For the second inequality of (2.9d) we have

$$\|J^*\widetilde{R} - RJ^*\| \le \|(J^* - J')R\| + \|J'\widetilde{R} - RJ'\| + \|R(J' - J^*)\| \le 3\delta. \qquad \square$$

2.3. Consequences of quasi-unitary equivalence

Let us cite here some consequences of quasi-unitary equivalence; for details we refer to [13, Chap. 4] and [16]. Note that there, we applied quasi-unitary equivalence to the resolvents $R = (\Delta + 1)^{-1}$ and $\widetilde{R} = (\widetilde{\Delta} + 1)^{-1}$ of two non-negative and self-adjoint operators Δ and $\widetilde{\Delta}$.

Theorem 2.12. (a) (***Convergence of operator functions***) *Let $\varphi\colon [0,1] \to \mathbb{C}$ be a continuous function then there are functions $\eta_\varphi(\delta), \eta'_\varphi(\delta) \to 0$ as $\delta \to 0$ depending only on φ such that*

$$\|\varphi(\widetilde{R})J - J\varphi(R)\| \le \eta_\varphi(\delta) \quad \textit{and} \quad \|\varphi(\widetilde{R}) - J\varphi(R)J'\| \le \eta'_\varphi(\delta)$$

for all operators R and $\widetilde{R}$ being δ-quasi-unitarily equivalent (with identification operators J and J').

If φ is holomorphic in a neighbourhood of $[0,1]$ then we can choose $\eta_\varphi(\delta) = C_\varphi\delta$ and similarly for η'_φ.

(b) (***Convergence of spectra***) *Let $R \in \mathcal{B}_{(0,1]}(\mathscr{H})$, then there is a function η with $\eta(\delta) \to 0$ as $\delta \to 0$ depending only on R such that*

$$d_{\mathrm{H}}(\mathrm{spec}(R), \mathrm{spec}(\widetilde{R})) \le \eta(\delta)$$

for all $\widetilde{R}$ being δ-quasi-unitarily equivalent with R. Here, d_{H} denotes the Hausdorff distance of the two spectra. A similar assertion holds for the essential *spectra.*

(c) (***Convergence of discrete spectra***) *Let μ be an eigenvalue of R with multiplicity m, then there is $\delta_0 > 0$ and there exist m eigenvalues $\widetilde{\mu}_j$ of $\widetilde{R}$ ($j = 1, \dots, m$, not all necessarily distinct) such that $|\mu - \widetilde{\mu}_j| \le C\delta$ if $\delta \in [0, \delta_0]$, where C is a universal constant depending only on μ and its distance from the remaining spectrum of R.*

(d) (***Convergence of eigenfunctions***) *Let μ be a simple*[3] *eigenvalue with normalised eigenfunction ψ, and denote by $\widetilde{\psi}$ the normalised eigenfunction associated with $\widetilde{\mu}$ as in* (c) *, then*

$$\|\widetilde{\psi} - J\psi\|_{\widetilde{\mathscr{H}}} \le C'\delta.$$

Here, C' is again universal constants depending only on μ and its distance from the remaining spectrum of R.

The proof of the assertions and more details can be found in [13, Chap. 4].

Remark 2.13. We say that Δ and $\widetilde{\Delta}$ are *δ-quasi-unitarily equivalent* if R and $\widetilde{R}$ are.

(a) As an example in Theorem 2.12 (a) one can choose $\varphi_t((\lambda+1)^{-1}) = \mathrm{e}^{-t\lambda}$ and one obtains operator estimates for the *heat* or *evolution operator* of Δ and

[3] This assumption is for simplicity only.

$\widetilde{\Delta}$. In this case, one can use the *holomorphic functional calculus* and give a precise estimate on $\eta'_{\varphi_t}(\delta)$, (namely $\eta'_{\varphi_t}(\delta) = (16/t + 5)\delta$, and we have

$$\|\mathrm{e}^{-t\widetilde{\Delta}} - J'\mathrm{e}^{-t\Delta} J\| \le \Big(\frac{16}{t} + 5\Big)\delta$$

if Δ and $\widetilde{\Delta}$ are δ-quasi-unitarily equivalent (or equivalently, $d_{\text{q-uni}}(R, \widetilde{R}) < \delta$), see [16] for details. Such an estimate might be of interest in control theory.

(b) It suffices in (a) of this remark that φ is only continuous on $[0,1] \setminus \operatorname{spec}(R)$. One can then show norm estimates also for *spectral projections*.

2.4. A distance arising from quasi-unitary equivalence

We now use the concept of quasi-unitary equivalence to define another distance function:

Definition 2.14. For $R, \widetilde{R} \in \mathcal{B}_{(0,1]}$ we define the *quasi-unitary distance* of R and $\widetilde{R}$ by

$$d_{\text{q-uni}}(R, \widetilde{R}) := \inf\big\{\, \delta \ge 0 \,\big|\, R \text{ and } \widetilde{R} \text{ are } \delta\text{-quasi-unitarily equivalent} \,\big\}.$$

Clearly, we have

$$\begin{aligned} d_{\text{q-uni}}(R, \widetilde{R}) = \inf \Big\{ &\max\big\{ \|J\| - 1, \|J'\| - 1, \|J^* - J'\|, \|(\mathrm{id}_{\mathscr{H}} - J'J)R\|, \\ &\qquad \|(\mathrm{id}_{\widetilde{\mathscr{H}}} - JJ')\widetilde{R}\|, \|JR - \widetilde{R}J\|, \|J'\widetilde{R} - RJ'\| \big\} \\ &\qquad \big| J \colon \mathscr{H} \to \widetilde{\mathscr{H}},\ J' \colon \widetilde{\mathscr{H}} \to \mathscr{H} \text{ bounded} \Big\}. \end{aligned}$$

Obviously, we have (using also Proposition 2.9)

$$d_{\text{q-uni}}(R, \widetilde{R}) \le d_{\text{uni}}(R, \widetilde{R}) \le d_{\text{spec}}(R, \widetilde{R}) \tag{2.11}$$

as in the definition of d_{uni}, we only use unitary maps J instead of general ones in the definition of $d_{\text{q-uni}}$.

The function $d_{\text{q-uni}}$ has the following properties:

Proposition 2.15. *Let $R, \widetilde{R} \in \mathcal{B}_{(0,1]}$.*

(a) *We have $d_{\text{q-uni}}(R, \widetilde{R}) \ge 0$, $d_{\text{q-uni}}(R, R) = 0$, $d_{\text{q-uni}}(R, \widetilde{R}) = d_{\text{q-uni}}(\widetilde{R}, R)$ and*

$$d_{\text{q-uni}}(R_1, R_3) \le \Phi\big(d_{\text{q-uni}}(R_1, R_2), d_{\text{q-uni}}(R_2, R_3)\big)$$

where $\Phi(a, b) = 3(a + b) + \mathrm{o}(a) + \mathrm{o}(b)$ is defined in (2.10).

(b) *If $d_{\text{q-uni}}(R, \widetilde{R}) = 0$ then the essential spectra of R and $\widetilde{R}$ agree. Also the discrete spectra agree and have the same multiplicity.*

(c) *If R and $\widetilde{R}$ are compact, i.e., $R, \widetilde{R} \in \mathcal{K}_{(0,1]}$, then $d_{\text{q-uni}}(R, \widetilde{R}) = 0$ if and only if R and $\widetilde{R}$ are unitarily equivalent. In particular, $d_{\text{q-uni}}$ induces a metric on $\mathsf{K}_{(0,1]}$ and hence on Σ (see* (2.6)*).*

Remark. Function $d_{\text{q-uni}}$ in (a) is sometimes referred to as a *semi(pseudo)metric* with a relaxed triangle equation, and Φ is referred to as a *triangle function.* As $\Phi(a,b) \le 12(a+b)$ if $a,b \in [0,1]$, and since $d_{\text{q-uni}}(R,\widetilde{R}) \le d_{\text{spec}}(R,\widetilde{R}) \le 1$, $d_{\text{q-uni}}$ is a semi(pseudo)metric with 12-relaxed triangle inequality.

Proof. (a) For the first assertion, only the relaxed triangle inequality is non-trivial; but it follows easily from the transitivity in Lemma 2.11 (e).

(b) For the second assertion, note that bounded operators $J_n \colon \mathscr{H} \to \widetilde{\mathscr{H}}$ and $J_n' \colon \widetilde{\mathscr{H}} \to \mathscr{H}$ exist such that (2.9a)–(2.9d) are fulfilled for some sequence $\delta_n \to 0$. In particular, R and $\widetilde{R}$ are δ_n-quasi-unitarily equivalent. It follows from Theorem 2.12 that $d_{\mathrm{H}}(\operatorname{spec}(R), \operatorname{spec}(\widetilde{R})) \le \eta(\delta_n)$ for any n, hence the Hausdorff distance is 0. The same is true for the essential spectra, and also for the discrete spectrum (including multiplicity). In particular, operators with $d_{\text{q-uni}}(R,\widetilde{R}) = 0$ have the same essential and discrete spectrum (the latter even with the same multiplicity).

(c) If R and $\widetilde{R}$ are compact, then the essential spectrum of both is $\{0\}$ (which is not an eigenvalue) and there are orthonormal bases $(\psi_n)_n$ and $(\widetilde{\psi}_n)_n$ of eigenfunctions of R and $\widetilde{R}$, respectively. Then $U\psi_n = \widetilde{\psi}_n$ defines a unitary map from $\mathscr{H}$ to $\widetilde{\mathscr{H}}$ such that $\widetilde{R} = URU^*$. □

Due to Proposition 2.15 (c), we can again define a (semi)metric on Σ via $d_{\text{q-uni}}(\mu,\widetilde{\mu}) = d_{\text{q-uni}}(R,\widetilde{R})$ if $\mu = (\mu_k)_k$ is the sequence of eigenvalues of R and similarly for $\widetilde{\mu}$ and $\widetilde{R}$. Moreover, we have

$$d_{\text{q-uni}}(\mu,\widetilde{\mu}) \le d_{\text{uni}}(\mu,\widetilde{\mu}) \le d_{\text{spec}}(\mu,\widetilde{\mu}) \tag{2.12}$$

(see (2.8)). We will investigate the structure of Σ with respect to these metrics and related questions in a forthcoming publication. It is in particular of interest to express $d_{\text{uni}}(\mu,\widetilde{\mu})$ and $d_{\text{q-uni}}(\mu,\widetilde{\mu})$ directly in terms of the sequences μ and $\widetilde{\mu}$.

3. Laplacians on thin branched manifolds shrinking towards a metric graph

Let us present our main example here, the Laplacian on a manifold that shrinks to a metric graph. Our result holds also for non-compact Riemannian manifolds and metric graphs under some uniform conditions. We would like to stress that this example has already been treated in [12] (see also [4, 13, 5] and references therein). However, here we present a sightly different proof as we directly compare the resolvent difference and we do not make use of the corresponding quadratic forms as in [12, 13].

3.1. Metric graphs

Let (V,E,∂) be a discrete oriented graph, i.e., V and E are at most countable sets, and $\partial \colon E \to V \times V$, $e \mapsto (\partial_- e, \partial_+ e)$ is a map that associates to an edge its *initial* $(\partial_- e)$ and *terminal* $(\partial_+ e)$ vertex. We set $E_v^\pm := \{\, e \in E \,|\, \partial_\pm e = v \,\}$ and

$E_v := E_v^+ \mathbin{\dot\cup} E_v^-$ (the set of incoming/outgoing resp. adjacent edges at v). A *metric* graph is given by (V, E, ∂) together with a map $\ell \colon E \to (0, \infty)$, $e \mapsto \ell_e > 0$, where we interpret ℓ_e as the *length* of the edge e. In particular, we set

$$M_e := [0, \ell_e] \quad \text{and} \quad M := \dot{\bigcup_{e \in E}} M_e / \Psi,$$

where Ψ identifies the end points of the intervals M_e according to the graph structure, i.e.,

$$\Psi \colon \dot{\bigcup_{e \in E}} \partial M_e \to V, \qquad \begin{cases} 0 \in \partial M_e \mapsto \partial_- e \in V, \\ \ell_e \in \partial M_e \mapsto \partial_+ e \in V. \end{cases}$$

Any point in M not being a vertex after the identification is uniquely determined by $e \in E$ and $s_e \in M_e$. In the sequel, we often omit the subscript and write $s \in M_e$.

To avoid some technical complications, we assume that

$$\ell_0 := \inf_{e \in E} \ell_e > 0. \tag{3.1}$$

Then M becomes a metric space by defining the distance $d(x, y)$ of two points $x, y \in M$ as the length of the shortest path in M (the path may not be unique). We also have a natural measure on M denoted by $\mathrm{d}s$, given by the sum of the Lebesgue measures $\mathrm{d}s_e$ on M_e (up to the boundary points, a null set).

As Laplacian on M, we define $\Delta_M f$ via $(\Delta_M f)_e = -f_e''$ with f in

$$\operatorname{dom} \Delta_M = \Big\{ f \in \bigoplus_{e \in E} \mathsf{H}^2(M_e) \,\Big|\, f \text{ continuous}, \forall v \in V \colon \sum_{e \in E_v} f_e'(v) = 0 \Big\}. \tag{3.2}$$

Here, $f_e'(v) = f_e'(\partial_+ e)$ if $v = \partial_+ e$ and $f_e'(v) = -f_e'(0)$ if $v = \partial_- e$ denotes the derivative of f along the edge e towards the vertex. It can be shown that this operator is self-adjoint, see [13] and references therein for details. Because of the sum condition on the derivatives, this operator is also called *Kirchhoff Laplacian* on M. We later on write $X_0 = M$ and Δ_0 for the Kirchhoff Laplacian Δ_M.

3.2. Thin branched manifolds ("fat graphs")

Let $X_0 = M$ be a metric graph. Let us now describe a family of manifolds X_ε shrinking to X_0 as $\varepsilon \to 0$. We will show that a suitable Laplacian on X_ε (actually, the Neumann Laplacian, if $\partial X_\varepsilon \neq \emptyset$) converges to the Kirchhoff Laplacian Δ_0 on X_0

According to the metric graph X_0 we associate a family $(X_\varepsilon)_\varepsilon$ of smooth Riemannian manifolds of dimension $m + 1 \geq 2$ for small $\varepsilon > 0$. We call X_ε a *thin branched manifold*, if the following holds:

- We have a decomposition

$$X_\varepsilon = \bigcup_{e \in E} X_{\varepsilon, e} \cup \bigcup_{v \in V} X_{\varepsilon, v}, \tag{3.3}$$

where $X_{\varepsilon,e}$ and $X_{\varepsilon,v}$ are compact Riemannian manifolds with boundary, $(X_{\varepsilon,e})_{e\in E}$ and $(X_{\varepsilon,v})_{v\in V}$ are pairwise disjoint and

$$X_{\varepsilon,e} \cap X_{\varepsilon,v} = \begin{cases} \emptyset, & e \notin E_v, \\ Y_{\varepsilon,e}, & e \in E_v, \end{cases}$$

where $Y_{\varepsilon,e}$ is a compact Riemannian manifold of dimension m, isometric to a Riemannian manifold $(Y_e, \varepsilon^2 h_e)$.

- The *edge neighbourhood* $X_{\varepsilon,e}$ is isometric to a cylinder

$$X_{\varepsilon,e} \cong M_e \times Y_{\varepsilon,e},$$

i.e., $X_{\varepsilon,e} = M_e \times Y_e$ as manifold with metric $g_{\varepsilon,e} = \mathrm{d}s^2 + \varepsilon^2 h_e$. Recall that $M_e = [0, \ell_e]$.

- The *vertex neighbourhood* $X_{\varepsilon,v}$ is isometric to

$$X_{\varepsilon,v} \cong \varepsilon X_v,$$

i.e., $X_{\varepsilon,v} = X_v$ as manifold with metric $g_{\varepsilon,v} = \varepsilon^2 g_v$, where (X_e, g_v) is a compact Riemannian manifold. In other words, $X_{\varepsilon,v}$ is ε-homothetic with a fixed Riemannian manifold (X_v, g_v).

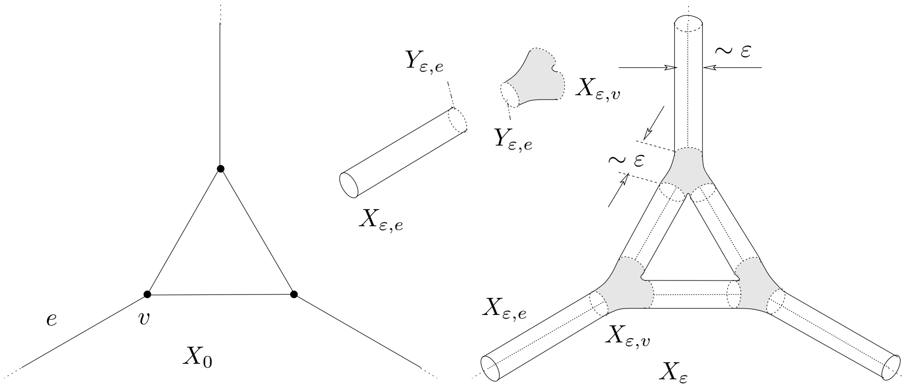

FIGURE 1. A part of a metric graph, the (scaled) building blocks and the corresponding (part of a) thin branched manifold (here, $Y_e = \mathbb{S}^1$, and X_ε is the surface of the pipeline network). The vertex neighbourhoods $X_{\varepsilon,v}$ are drawn in gray.

Remark 3.1.

(a) Note that we write $X_{\varepsilon,v}$, etc., for a *Riemannian* manifold. More precisely, we should write $(X_v, g_{\varepsilon,v})$ for this manifold, since the underlying space can be chosen to be ε-independent, the ε-dependence only enters via the metric.

(b) The manifold X_ε may have boundary or not. If X_ε has boundary, then also (some of) the transversal manifolds Y_e have boundary.

(c) We define the space X_ε in an abstract manner, although we have concrete examples in mind. If we consider a graph M embedded in, say, $\mathbb{R}^d$, and if $\widetilde{X}_\varepsilon$ denotes its ε-neighbourhood, then we can define a similar decomposition as in (3.3), but the building blocks $\widetilde{X}_{\varepsilon,e}$ and $\widetilde{X}_{\varepsilon,v}$ are only *approximatively* isometric with $M_e \times \varepsilon Y_e$ and εX_v for some fixed Riemannian manifolds (Y_e, h_e) and (X_v, g_v). This may have two reasons:
- We need a little space for the vertex neighbourhoods (of order ε), so that we need to replace the interval M_e by a slightly smaller one of length $\ell_e - \mathrm{O}(\varepsilon)$.
- The edges may be embedded as non-straight *curves* in $\mathbb{R}^2$. This leads to a slight deviation from the product metric.

All these cases can be treated as a *perturbation* of the abstract situation above, see e.g. [13, Sects. 5.4 and 6.7]).

Laplacians on thin branched manifolds. The underlying Hilbert space is $\mathsf{L}_2(X_\varepsilon)$ (with the natural volume measure denoted by $\mathrm{d}X_\varepsilon$ induced by the ε-dependent Riemannian metric). In particular, we have

$$\begin{aligned}\|u\|^2_{\mathsf{L}_2(X_\varepsilon)} &= \int_{X_\varepsilon} |u(x)|^2 \,\mathrm{d}X_\varepsilon(x)\\ &= \varepsilon^m \sum_{e\in E} \int_0^{\ell_e} \int_{Y_e} |u_e(s_e,y)|^2 \,\mathrm{d}Y_e(y)\,\mathrm{d}s_e + \varepsilon^{m+1} \sum_{v\in V} \int_{X_v} |u_v(x)|^2 \,\mathrm{d}X_v(x)\end{aligned}$$

using the decomposition (3.3) and suitable identifications. In particular, u_e resp. u_v denote the restriction onto $X_{\varepsilon,e}$ and $X_{\varepsilon,v}$, respectively.

As operator on X_ε, we consider the Laplacian $\Delta_\varepsilon \geq 0$ (with Neumann boundary conditions if $\partial X_\varepsilon \neq \emptyset$). This operator is given as

$$(\Delta_\varepsilon u)_v = \frac{1}{\varepsilon^2} \Delta_{X_v} u_v \quad \text{and} \quad (\Delta_\varepsilon u)_e = -u_e'' + (\mathrm{id} \otimes \Delta_{Y_{\varepsilon,e}}) u_e,$$

where $\Delta_{Y_{\varepsilon,e}} \varphi = \varepsilon^{-2} \Delta_{Y_e} \varphi$ for a smooth function φ on Y_e, and Δ_{Y_e} is the (Neumann) Laplacian on Y_e. Moreover, $(\cdot)'_e$ denotes the derivative with respect to the longitudinal variable $s \in M_e$.

3.3. Convergence of the Laplacian on thin branched manifolds

Let us first define a suitable identification operator

$$J_\varepsilon \colon \mathsf{L}_2(X_0) \to \mathsf{L}_2(X_\varepsilon).$$

For simplicity, we assume here that $\mathrm{vol}_m(Y_e, h_e) = 1$. As identification operator we choose

$$(J_\varepsilon f)_e = f_e \otimes \mathbf{1}_{\varepsilon,e} \quad \text{and} \quad (J_\varepsilon f)_v = 0$$

where $(J_\varepsilon f)_e$ is the contribution on the edge neighbourhood $X_{\varepsilon,e}$ and $(J_\varepsilon f)_v$ is the contribution on the vertex neighbourhood, according to the decomposition (3.3). Moreover, $\mathbf{1}_{\varepsilon,e}$ is the constant function on $Y_{\varepsilon,e}$ with value $\varepsilon^{-m/2}$ (the first normalised eigenfunction of $Y_{\varepsilon,e}$).

Remark 3.2. The setting $(J_\varepsilon f)_v = 0$ seems at first sight a bit rough, but we cannot set something like $(J_\varepsilon f)_v = \varepsilon^{-m/2} f(v)$, since on $\mathsf{L}_2(X_0)$, the value of f at v is not defined. There is a finer version of identification operators on the level of the quadratic form domains, again see [13, Chap. 4] for details.

Let us now calculate the resolvent difference $R_\varepsilon J_\varepsilon - J_\varepsilon R_0$: For $g \in \mathsf{L}_2(X_0)$ and $w \in \mathsf{L}_2(X_\varepsilon)$, we have

$$\begin{aligned}\langle (R_\varepsilon J_\varepsilon - J_\varepsilon R_0)g, w\rangle_{\mathsf{L}_2(X_\varepsilon)} &= \langle J_\varepsilon g, R_\varepsilon w\rangle_{\mathsf{L}_2(X_\varepsilon)} - \langle J_\varepsilon R_0 g, w\rangle_{\mathsf{L}_2(X_\varepsilon)}\\ &= \langle J_\varepsilon \Delta_0 f, u\rangle_{\mathsf{L}_2(X_\varepsilon)} - \langle J_\varepsilon f, \Delta_\varepsilon u\rangle_{\mathsf{L}_2(X_\varepsilon)},\end{aligned}$$

where $u = R_\varepsilon w \in \operatorname{dom} \Delta_\varepsilon$ and $f = R_0 g \in \operatorname{dom} \Delta_0$. Moreover, using the definition of $J_\varepsilon f$, we obtain

$$\begin{aligned}&\langle (R_\varepsilon J_\varepsilon - J_\varepsilon R_0)g, w\rangle_{\mathsf{L}_2(X_\varepsilon)}\\ &\quad= \sum_{e\in E}\Big(\langle(-f_e''\otimes \mathbf{1}_{\varepsilon,e}, u_e\rangle_{\mathsf{L}_2(X_{\varepsilon,e})} - \langle f_e \otimes \mathbf{1}_{\varepsilon,e}, -u_e''\rangle_{\mathsf{L}_2(X_{\varepsilon,e})}\\ &\qquad - \langle f_e\otimes \mathbf{1}_{\varepsilon,e}, (\mathrm{id}\otimes\Delta_{Y_{\varepsilon,e}})u_e\rangle_{\mathsf{L}_2(X_{\varepsilon,e})}\Big)\\ &\quad= \sum_{e\in E}\Big(\langle(-f_e''\otimes \mathbf{1}_{\varepsilon,e}, u_e\rangle_{\mathsf{L}_2(X_{\varepsilon,e})} - \langle f_e \otimes \mathbf{1}_{\varepsilon,e}, -u_e''\rangle_{\mathsf{L}_2(X_{\varepsilon,e})}\Big)\end{aligned}$$

since we can bring $(\mathrm{id}\otimes\Delta_{Y_{\varepsilon,e}})$ on the other side of the inner product (the operator is self-adjoint!) and $\Delta_{Y_{\varepsilon,e}}\mathbf{1}_{\varepsilon,e} = 0$. Using $\mathrm{d}X_{\varepsilon,e} = \varepsilon^m \,\mathrm{d}Y_e \,\mathrm{d}s$ and performing a partial integration (Green's first formula), we obtain

$$\langle (R_\varepsilon J_\varepsilon - J_\varepsilon R_0)g, w\rangle_{\mathsf{L}_2(X_\varepsilon)} = \sum_{e\in E}\varepsilon^{m/2}\Big[\int_{Y_e}(-f_e'\overline{u}_e + f_e\overline{u}_e')\,\mathrm{d}Y_e\Big]_{\partial M_e}.$$

Using the conventions $f_e(v) = f_e(0)$ resp. $f_e(v) = f_e(\ell_e)$, $u_e(v) := u_e(0,\cdot)$ resp. $u_e(v) = u_e(\ell_e,\cdot)$ and $f_e'(v) = -f_e'(0)$ resp. $f_e'(v) = f_e'(\ell_e)$ if $v = \partial_- e$ resp. $v = \partial_+ e$, and after reordering, we obtain

$$\begin{aligned}\langle (R_\varepsilon J_\varepsilon - J_\varepsilon R_0)g, w\rangle_{\mathsf{L}_2(X_\varepsilon)} &= \sum_{e\in E}\sum_{v=\partial_\pm e}\varepsilon^{m/2}\int_{Y_e}\big(-f_e'(v)\overline{u}_e(v) + f_e(v)\overline{u}_e'(v)\big)\,\mathrm{d}Y_e\\ &= \sum_{v\in V}\sum_{e\in E_v}\varepsilon^{m/2}\int_{Y_e}\big(-\underbrace{f_e'(v)\overline{u}_e(v)}_{=:I_1} + \underbrace{f_e(v)\overline{u}_e'(v)}_{=:I_2}\big)\,\mathrm{d}Y_e.\end{aligned}$$

Consider now

$$\textstyle{⨍}_v u_v := \frac{1}{\operatorname{vol} X_v}\int_{X_v} u_v\,\mathrm{d}X_v \quad\text{and}\quad {⨍}_e u_e(v) := \frac{1}{\operatorname{vol} Y_e}\int_{Y_e} u_e(v)\,\mathrm{d}Y_e,$$

then we can express the sums over I_1 as

$$\begin{aligned}\sum_{e\in E_v} \varepsilon^{m/2} \int_{Y_e} f_e'(v)\overline{u}_e(v) &= \sum_{e\in E_v} \varepsilon^{m/2} f_e'(v)\big(\textstyle\fint_e \overline{u}_e(v) - \fint_v \overline{u}_v\big)\\ &\quad + \Big(\sum_{e\in E_v} \varepsilon^{m/2} f_e'(v)\Big)\textstyle\fint_v \overline{u}_v\\ &= \sum_{e\in E_v} \varepsilon^{m/2} f_e'(v)\big(\textstyle\fint_e \overline{u}_e(v) - \fint_v \overline{u}_v\big).\end{aligned}$$

The last sum in the first line vanishes since $f \in \operatorname{dom}\Delta_0$ fulfils the so-called Kirchhoff condition $\sum_{e\in E_v} f_e'(v) = 0$. For the second summand I_2, we use the fact that $f_e(v) = f(v)$ is independent of $e \in E_v$, hence

$$\begin{aligned}\sum_{e\in E_v} \varepsilon^{m/2} \int_{Y_e} f_e(v)\overline{u}_e'(v)\big)\,\mathrm{d}Y_e &= \varepsilon^{m/2} f(v) \int_{\partial X_v} \partial_{\mathrm{n}} \overline{u}_v \,\mathrm{d}\partial X_v\\ &= \varepsilon^{m/2} f(v) \int_{X_v} \Delta_{X_v} \overline{u}_v \,\mathrm{d}X_v,\end{aligned}$$

performing again a partial integration (Green's first formula, writing u_v as $1\cdot u_v$). Summing up the contributions, we have

$$\begin{aligned}&\langle (R_\varepsilon J_\varepsilon - J_\varepsilon R_0)g, w\rangle_{\mathsf{L}_2(X_\varepsilon)}\\ &\quad= \sum_{v\in V} \varepsilon^{m/2}\big(-\sum_{e\in E_v} f_e'(v)\big(\textstyle\fint_e \overline{u}_e(v) - \fint_v \overline{u}_v\big) + f(v)\displaystyle\int_{X_v} \Delta_{X_v}\overline{u}_v \,\mathrm{d}X_v\big)\\ &\quad=: -\langle B_0 g, A_\varepsilon w\rangle_{\mathcal{G}^{\max}} + \langle A_0 g, B_\varepsilon w\rangle_{\mathcal{G}},\end{aligned}$$

where $\mathcal{G} := \ell_2(V,\deg)$ (with norm $\|\varphi\|^2_{\ell_2(V,\deg)} := \sum_{v\in V} |\varphi(v)|^2 \deg v < \infty$), $\mathcal{G}^{\max} := \bigoplus_{v\in V} \mathbb{C}^{E_v}$ and

$$\begin{aligned}&B_0\colon \mathsf{L}_2(X_0) \to \mathcal{G}^{\max}, && (B_0 g)_v = \big((R_0 g)_e'(v)\big)_{e\in E_v},\\ &A_\varepsilon\colon \mathsf{L}_2(X_\varepsilon) \to \mathcal{G}^{\max}, && (A_\varepsilon w)_v = \varepsilon^{m/2}\big(\textstyle\fint_e (R_\varepsilon w)_e(v) - \fint_v (R_\varepsilon w)_v\big)_{e\in E_v},\\ &B_\varepsilon\colon \mathsf{L}_2(X_\varepsilon) \to \mathcal{G}, && (B_\varepsilon w)(v) = \frac{\varepsilon^{m/2}}{\deg v}\int_{X_v} \Delta_{X_v}(R_\varepsilon w)\,\mathrm{d}X_v,\\ &A_0\colon \mathsf{L}_2(X_0) \to \mathcal{G}, && (A_0 g)(v) = (R_0 g)(v).\end{aligned}$$

In particular, we have shown

Proposition 3.3. *We can express the resolvent differences of Δ_ε and Δ_0, sandwiched with the identification operator J_ε, as*

$$R_\varepsilon J_\varepsilon - J_\varepsilon R_0 = -A_\varepsilon^* B_0 + B_\varepsilon^* A_0 \colon \mathsf{L}_2(X_0) \to \mathsf{L}_2(X_\varepsilon).$$

We will now show that the ε-dependent operators are actually small if $\varepsilon \to 0$. In order to do so, we need two important estimates:

Sobolev trace estimate. We have

$$\|u(0,\cdot)\|^2_{\mathsf{L}_2(Y_e)} \le C(\ell_0)\|u\|^2_{\mathsf{H}^1(X_{v,e})}\big(\|u\|^2_{\mathsf{L}_2(X_{v,e})} + \|\nabla u\|^2_{\mathsf{L}_2(X_{v,e})}\big) \tag{3.4}$$

for all $u\colon X_{v,e} \to \mathbb{C}$ smooth enough, where $X_{v,e} = [0,\ell_0] \times Y_e$ is a collar neighbourhood of the boundary component of X_v touching the edge neighbourhood X_e. The optimal constant is actually $C(\ell_0) = \coth(\ell_0/2)$. The proof of (3.4) is just a vector-valued version of the Sobolev trace estimate

$$|f(0)|^2 \le C(\ell_0)\big(\|f\|^2_{\mathsf{L}_2([0,\ell_0])} + \|f'\|^2_{\mathsf{L}_2([0,\ell_0])}\big). \tag{3.5}$$

A min–max estimate. We have

$$\|u - ⨍u\|^2_{\mathsf{L}_2(X_v)} \le \frac{1}{\lambda_2(X_v)}\|\nabla u\|^2_{\mathsf{L}_2(X_v)} \tag{3.6}$$

for all u smooth enough, where $\lambda_2(X_v)$ is the first (non-vanishing) Neumann eigenvalue of X_v. Note that $u - ⨍u$ is the projection onto the space orthogonal to the first (constant) eigenfunction on X_v.

As a consequence, we obtain

Lemma 3.4. *We have*

$$\varepsilon^m \sum_{e\in E_v} \big|⨍_e u_e(v) - ⨍_v u\big|^2 \le \varepsilon C(\ell_0)\Big(\frac{1}{\lambda_2(X_v)} + 1\Big)\|\nabla u\|^2_{\mathsf{L}_2(X_{\varepsilon,v})}.$$

Proof. We have (denoting by $\ell_0 > 0$ a lower bound on the edge lengths)

$$\begin{aligned}
\varepsilon^m \sum_{e\in E_v} \big|⨍_e u_e(v) - ⨍_v u\big|^2 &= \varepsilon^m \sum_{e\in E_v} \big|⨍_e (u - ⨍_v u)\big|^2 \\
&\le \varepsilon^m \sum_{e\in E_v} \int_{Y_e} \big|u - ⨍_v u\big|^2 \,\mathrm{d}Y_e \\
&\le \varepsilon^m C(\ell_0) \sum_{e\in E_v} \Big(\|u - ⨍_v u\|^2_{\mathsf{L}_2(X_{v,e})} + \|\nabla u\|^2_{\mathsf{L}_2(X_{v,e})}\Big) \\
&\le \varepsilon^m C(\ell_0)\Big(\|u - ⨍_v u\|^2_{\mathsf{L}_2(X_v)} + \|\nabla u\|^2_{\mathsf{L}_2(X_v)}\Big) \\
&\le \varepsilon C(\ell_0)\Big(\frac{1}{\lambda_2(X_v)} + 1\Big)\|\nabla u\|^2_{\mathsf{L}_2(X_{\varepsilon,v})}
\end{aligned}$$

using Cauchy–Schwarz in the first inequality, (3.4) and the fact that $\nabla ⨍_v u = 0$ in the second estimate, the fact that $\bigcup_{e\in E_v} X_{v,e} \subset X_v$ in the third estimate and (3.6) and the scaling behaviour $\varepsilon^{m-1}\|\nabla u\|^2_{\mathsf{L}_2(X_v)} = \|\nabla u\|^2_{\mathsf{L}_2(X_{\varepsilon,v})}$ in the fourth estimate. □

The following result is not hard to see using the Sobolev trace estimate (3.5):

Proposition 3.5. *Assume that $0 < \ell_0 \le \ell_e$ for all $e \in E$, then the operators A_0 and B_0 are bounded by a constant depending only on ℓ_0.*

Proposition 3.6. *Assume that*

$$0 < \ell_0 \le \ell_e \ \forall e \in E, \quad 0 < \lambda_2 \le \lambda_2(X_v) \quad \textit{and} \quad \frac{\operatorname{vol} X_v}{\deg v} \le c_{\mathrm{vol}} < \infty \ \forall v \in V \qquad (3.7)$$

holds, then $\|A_\varepsilon\| = \mathrm{O}(\varepsilon^{1/2})$ *and* $\|B_\varepsilon\| = \mathrm{O}(\varepsilon^{3/2})$, *and the errors depend only on* ℓ_0, λ_0 *and* c_{vol}.

Proof. For A_ε, we have

$$\begin{aligned}\|A_\varepsilon w\|^2_{\mathcal{G}^{\max}} &= \varepsilon^m \sum_{v \in V} \sum_{e \in E_v} \big| ⨍_e u_e(v) - ⨍_v u_v \big|^2 \\ &\le \varepsilon C(\ell_0)\Big(\frac{1}{\lambda_2}+1\Big) \sum_{v\in V} \|\nabla u\|^2_{\mathsf{L}_2(X_{\varepsilon,v})} \le \varepsilon C(\ell_0)\Big(\frac{1}{\lambda_2}+1\Big)\|\nabla u\|^2_{\mathsf{L}_2(X_\varepsilon)}\end{aligned}$$

using Lemma 3.4, where $u = R_\varepsilon w$. Now, since $u \in \operatorname{dom} \Delta_{X_\varepsilon}$, and since Δ_{X_ε} is the operator associated with the quadratic form, we have

$$\begin{aligned}\|\nabla u\|^2_{\mathsf{L}_2(X_\varepsilon)} &= \langle \Delta_{X_\varepsilon} u, u\rangle_{\mathsf{L}_2(X_\varepsilon)} \\ &= \langle \Delta_{X_\varepsilon}(\Delta_{X_\varepsilon}+1)^{-1} w, (\Delta_{X_\varepsilon}+1)^{-1} w\rangle_{\mathsf{L}_2(X_\varepsilon)} \le \|w\|^2_{\mathsf{L}_2(X_\varepsilon)}\end{aligned}$$

and the inequality is true by the spectral calculus.

For B_ε, we have

$$\begin{aligned}\|B_\varepsilon g\|^2_{\mathcal{G}} = \varepsilon^m \sum_{v\in V} \frac{1}{\deg v} \Big| \int_{X_v} \Delta_{X_v} u \Big|^2 &\le \varepsilon^m \sum_{v \in V} \frac{\operatorname{vol} X_v}{\deg v} \|\Delta_{X_v} u\|^2_{\mathsf{L}_2(X_v)} \\ &= \varepsilon^3 \sum_{v\in V} \frac{\operatorname{vol} X_v}{\deg v} \|\Delta_{X_{\varepsilon,v}} u\|^2_{\mathsf{L}_2(X_{\varepsilon,v})} \\ &\le \varepsilon^3 c_{\mathrm{vol}} \|\Delta_{X_\varepsilon}(\Delta_{X_\varepsilon}+1)^{-1} w)\|^2_{\mathsf{L}_2(X_\varepsilon)} \\ &\le \varepsilon^3 c_{\mathrm{vol}} \|w\|^2_{\mathsf{L}_2(X_\varepsilon)}\end{aligned}$$

using the scaling behaviour $\Delta_{X_{\varepsilon,v}} = \varepsilon^{-2}\Delta_{X_v}$ and $\|w\|^2_{\mathsf{L}_2(X_{\varepsilon,v})} = \varepsilon^{m+1}\|w\|^2_{\mathsf{L}_2(X_v)}$, where again $u = R_\varepsilon w$. □

Combining the previous results (Propositions 3.3, 3.5 and 3.6), we have shown the following:

Theorem 3.7. *Assume that* (3.7) *holds, then*

$$\|R_\varepsilon J_\varepsilon - J_\varepsilon R_0\|_{\mathsf{L}_2(X_0)\to\mathsf{L}_2(X_\varepsilon)} = \mathrm{O}(\varepsilon^{1/2}),$$

where the error depends only on ℓ_0, λ_0 *and* c_{vol}.

Theorem 3.8. *Assume that* (3.7) *holds, then the* (*Neumann*) *Laplacian* Δ_{X_ε} *converges to the standard* (*Kirchhoff*) *Laplacian* Δ_{X_0} *in the generalised norm resolvent sense.*

In particular, the results of Theorem 2.12 *apply, i.e., we have convergence of the spectrum* (*discrete or essential*) *and can approximate* $\varphi(\Delta_{X_\varepsilon})$ *by* $J_\varepsilon\varphi(\Delta_{X_0})J_\varepsilon^*$ *in operator norm up to an error of order* $\mathrm{O}(\varepsilon^{1/2})$.

Idea of proof. We have to show that J_ε is δ_ε-quasi unitary. It is not hard to see that

$$(J_\varepsilon^* u)_e(s) = \varepsilon^{m/2} \int_{Y_e} u_e(s,\cdot)\,\mathrm{d}Y_e,$$

and that

$$J_\varepsilon^* J_\varepsilon f = f \tag{3.8a}$$

for all $f \in \mathsf{L}_2(X_0)$ (i.e., going from the metric graph to the manifold and back, we do not loose information).

Hence we only have to show that

$$\begin{aligned}\|u - J_\varepsilon J_\varepsilon^* u\|^2 &= \sum_{v\in V} \|u_v\|^2_{\mathsf{L}_2(X_{\varepsilon,v})} + \sum_{e\in E} \int_{M_e} \|u_e(s,\cdot) - \textstyle\fint_e u_e(s,\cdot)\|^2_{\mathsf{L}_2(Y_{\varepsilon,e})}\,\mathrm{d}s \\ &\le \delta_\varepsilon^2 \|(\Delta_\varepsilon + 1)u\|^2\end{aligned} \tag{3.8b}$$

for some $\delta_\varepsilon \to 0$. Actually, this can be done using similar ideas as before. For details, we refer again to [13, Sect. 6.3], and one can show that $\delta_\varepsilon = \mathrm{O}(\varepsilon^{1/2})$ under the additional assumption that $0 < \lambda_2' \le \lambda_2(Y_e)$ (the first non-zero eigenvalue of Δ_{Y_e} on Y_e). □

Remark 3.9. Note that Grieser showed in [6] that the kth eigenvalue $\lambda_k(\Delta_\varepsilon)$ of the (Neumann) Laplacian converges to the kth eigenvalue of the metric graph (Kirchhoff) Laplacian $\lambda_k(\Delta_0)$, i.e., $\lambda_k(\Delta_\varepsilon) - \lambda_k(\Delta_0) = \mathrm{O}(\varepsilon)$, for compact metric graphs and a corresponding family of compact Riemannian manifolds using asymptotic expansions. From our analysis, we only obtain the error $\mathrm{O}(\varepsilon^{1/2})$ as we use less elaborated methods. From Grieser's result, it follows that $d_{\text{spec}}(R_\varepsilon, R_0) = \mathrm{O}(\varepsilon)$ where $R_\varepsilon = (\Delta_\varepsilon + 1)^{-1}$. We conclude from (2.11) that $d_{\text{q-uni}}(R_\varepsilon, R_0) = \mathrm{O}(\varepsilon)$. Our identification operator J_ε only shows the estimate $d_{\text{q-uni}}(R_\varepsilon, R_0) \le \mathrm{O}(\varepsilon^{1/2})$. Knowing already Grieser's result, we can directly define a unitary map sending eigenfunctions of the metric graph to eigenfunctions of the manifold.

Our identification operator J_ε just imitates the eigenfunctions up to an error. It would be interesting to see whether one can also obtain the optimal error estimate $\mathrm{O}(\varepsilon)$ using other identification operators J_ε respecting in more detail the domains and also the local structure of the spaces.

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Infinite-time admissibility under compact perturbations

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Abstract. We investigate the behavior of infinite-time admissibility under compact perturbations. We show, by means of two completely different examples, that infinite-time admissibility is not preserved under compact perturbations Q of the underlying semigroup generator A, even if A and $A+Q$ both generate strongly stable semigroups.

Keywords. Infinite-time admissibility, compact perturbations, stabilization of collocated linear systems.

1. Introduction

In this note, we investigate the behavior of infinite-time admissibility under compact perturbations of the underlying semigroup generator. So, we consider semigroup generators $A : D(A) \subset X \to X$ (with X a Hilbert space) and possibly unbounded control operators B (defined on another Hilbert space U) and we ask how the property of infinite-time admissibility of B behaves under compact perturbations of the generator A. Infinite-time admissibility of B for A means that for every control input $u \in L^2([0,\infty), U)$ the mild solution of the initial value problem

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{and} \quad x(0) = 0 \tag{1.1}$$

is a bounded function from $[0,\infty)$ with values in X. (A priori, the mild solution has values only in the extrapolation space X_{-1} of A and, a fortiori, need not be bounded in the norm of X, of course.)

It is well-known that (finite-time) admissibility is preserved under very general perturbations Q of the generator A, in particular, under bounded perturbations. It is also clear that infinite-time admissibility, by contrast, is not preserved under bounded perturbations. Just think of a generator A of an exponentially stable semigroup and a bounded perturbation Q (for example, a sufficiently large

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multiple of the identity operator) such that the operator $A+Q$ has spectral points in the right half-plane.

In this note, we will show by way of two completely different kinds of examples that infinite-time is also not preserved under compact perturbations Q which are such that both A and $A+Q$ generate strongly stable (but not exponentially stable) semigroups. So, in other words, we show that there exist semigroup generators A and $A+Q$ with Q being compact and a control operator B such that

- the semigroups $\mathrm{e}^{A\cdot}$ and $\mathrm{e}^{(A+Q)\cdot}$ are strongly stable but not exponentially stable,
- B is infinite-time admissible for A but not infinite-time admissible for $A+Q$.

In our first – more elementary – example, we will use an old and well-known result from the 1970s, namely a stabilization result for collocated linear systems. In that example, the compact perturbation Q will be of rank 1 and the control operator B will be bounded. In particular, none of the technicalities coming along with unbounded control operators will bother us there. In our second – less elementary – example, we will use a more advanced result from the 1990s, namely a characterization of infinite-time admissibility for diagonal semigroup generators. In that example, the control operator B will be unbounded and the compact perturbation Q will be of rank ∞.

In the entire note, we will use the following notation.

$$\mathbb{R}_0^+ := [0,\infty), \quad \mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}, \quad \mathbb{C}^- := \{z \in \mathbb{C} : \operatorname{Re} z < 0\}.$$

As usual, $L(X,Y)$ denotes the Banach space of bounded linear operators between two Banach spaces X and Y and $\|\cdot\|_{X,Y}$ stands for the operator norm on $L(X,Y)$. Also, $\|u\|_2$ denotes the norm of a square-integrable function $u \in L^2(\mathbb{R}_0^+, U)$ with values in the Banach space U. When speaking of a semigroup, we will always mean a strongly continuous semigroup of bounded linear operators and we refer to [3], [4] or [9] for basic definitions and facts from semigroup theory. And finally, for a semigroup generator A and bounded operators B, C between appropriate spaces, the symbol $\mathfrak{S}(A,B,C)$ will stand for the state-linear system [3]

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{with } y(t) = Cx(t).$$

2. Some basic facts about admissibility and infinite-time admissibility

In this section, we briefly recall the definition of and some basic facts about admissibility and infinite-time admissibility. If $A : D(A) \subset X \to X$ is a semigroup generator on the Hilbert space X and X_{-1} is the corresponding extrapolation space, then an operator $B \in L(U, X_{-1})$ (with U another Hilbert space) is called *control operator for* A. Also, B is called a *bounded control operator* iff $B \in L(U,X)$ and an *unbounded control operator* iff $B \in L(U,X_{-1}) \setminus L(U,X)$. See [9] (Sect. 2.10) or [4] (Sect. II.5) for basic facts about extrapolation spaces.

Definition 2.1. Suppose $A : D(A) \subset X \to X$ is a semigroup generator on X and $B \in L(U, X_{-1})$, where X, U are both Hilbert spaces. Then B is called *admissible for A* iff for every $u \in L^2(\mathbb{R}_0^+, U)$

$$(0, \infty) \ni t \mapsto \Phi_t(u) := \int_0^t \mathrm{e}^{A_{-1}s} Bu(s)\,\mathrm{d}s \tag{2.1}$$

is a function with values in X, where A_{-1} is the generator of the strongly continuous extension of the semigroup $\mathrm{e}^{A\cdot}$ to X_{-1}.

Clearly, for a given semigroup generator A every bounded control operator $B \in L(U, X)$ is admissible (because $\mathrm{e}^{A_{-1}s}\big|_X = \mathrm{e}^{As}$ for $s \in \mathbb{R}_0^+$). It should also be noted that if $B \in L(U, X_{-1})$ is admissible for A, then for every $t \in (0, \infty)$ the linear operator $L^2(\mathbb{R}_0^+, U) \ni u \mapsto \Phi_t(u) \in X$ defined in (2.1) is closed and thus continuous by the closed graph theorem. Consequently, $B \in L(U, X_{-1})$ is admissible for A if and only if

$$\Phi_t \in L\big(L^2(\mathbb{R}_0^+, U), X\big) \quad (t \in (0, \infty)). \tag{2.2}$$

Definition 2.2. Suppose $A : D(A) \subset X \to X$ is a semigroup generator on X and $B \in L(U, X_{-1})$, where X, U are both Hilbert spaces. Then B is called *infinite-time admissible for A* iff for every $u \in L^2(\mathbb{R}_0^+, U)$

$$(0, \infty) \ni t \mapsto \Phi_t(u) := \int_0^t \mathrm{e}^{A_{-1}s} Bu(s)\,\mathrm{d}s \tag{2.3}$$

is a function with values in X that is bounded (in the norm of X), where A_{-1} is the generator of the strongly continuous extension of the semigroup $\mathrm{e}^{A\cdot}$ to X_{-1}.

Clearly, if $B \in L(U, X_{-1})$ is infinite-time admissible for a given semigroup generator A, then it is also admissible for A. It should also be noted that, by the uniform boundedness principle, $B \in L(U, X_{-1})$ is infinite-time admissible for A if and only if

$$\Phi_t \in L\big(L^2(\mathbb{R}_0^+, U), X\big) \quad (t \in (0, \infty)) \quad \text{and} \quad \sup_{t \in (0,\infty)} \big\|\Phi_t\big\|_{L^2(\mathbb{R}_0^+, U), X} < \infty. \tag{2.4}$$

Some authors [8, 2, 10] use the term input-stability for the system $\mathfrak{S}(A, B)$ instead of infinite-time admissibility.

It is well-known that admissibility is preserved under bounded perturbations.

Proposition 2.3. *Suppose $A : D(A) \subset X \to X$ is a semigroup generator on X and $B \in L(U, X_{-1})$, where X, U are both Hilbert spaces. Also, let $Q \in L(X)$. Then B is admissible for A if and only if B is admissible for $A + Q$.*

In fact, the conclusion of this proposition remains true for much more general perturbations Q, namely for perturbations of the (feedback) form $Q = B_0 C_0$, where $B_0 \in L(U_0, X_{-1})$ is an admissible control operator for A and $C_0 \in L(X, U_0)$ with U_0 an arbitrary Hilbert space. See Corollary 5.5.1 from [9], for instance.

Proposition 2.4. *Suppose $A : D(A) \subset X \to X$ is the generator of an exponentially stable semigroup on X and $B \in L(U, X_{-1})$ is admissible for A. Then B is even infinite-time admissible for A.*

See Proposition 4.4.5 in [9], for instance, and notice that for bounded control operators B the above proposition is trivial. In view of that proposition, it is clear that infinite-time admissibility – unlike admissibility – is not preserved under bounded perturbations. Choose, for example, a bounded generator A of an exponentially stable semigroup and let $Q := -A \in L(X)$ and $B := I \in L(X, X)$ (identity operator on X).

3. An example using a stabilization result for collocated linear systems

3.1. Stabilization of collocated linear systems

We will use the following well-known stabilization result for collocated systems, that is, systems of the form $\mathfrak{S}(A, B, B^*)$ with a bounded control operator B. It essentially goes back to [1] (Corollary 3.1) and, in the form below, can be found in [8] (Lemma 2.2.6), for instance. (Actually, for the more general version with the countability assumption on $\sigma(A_0) \cap \mathrm{i}\mathbb{R}$ we have to refer to [10], but this more general version will not be used in the sequel.) See [10] or the upcoming second edition of [3] for the definition of approximate controllability and approximate observability in infinite time (in [3] the additional clarifying qualifier “in infinite time” is not used).

Theorem 3.1. *Suppose A_0 is a contraction semigroup generator on a Hilbert space X with compact resolvent (or, more generally, with $\sigma(A_0) \cap \mathrm{i}\mathbb{R}$ being countable). Suppose furthermore that $B \in L(U, X)$ with another Hilbert space U and that $\mathfrak{S}(A_0, B, B^*)$ is approximately controllable or observable in infinite time. Then*

(i) *B is infinite-time admissible for $A_0 - BB^*$, more precisely,*

$$\left\| \int_0^t \mathrm{e}^{(A_0 - BB^*)s} Bu(s) \, \mathrm{d}s \right\|_X^2 \leq \frac{1}{2} \|u\|_2^2 \quad \big(u \in L^2(\mathbb{R}_0^+, U), t \in \mathbb{R}_0^+\big).$$

(ii) *$\mathrm{e}^{(A_0 - BB^*)\cdot}$ is a strongly stable contraction semigroup on X.*

A far-reaching generalization of this result to the case of unbounded control operators was obtained by Curtain and Weiss [10]. See Theorem 5.1 and 5.2 in conjunction with Proposition 1.5 from [10]. We also refer to [2] for a parallel result on exponential stabilization.

3.2. Infinite-time admissibility under compact perturbations

Example 3.2. Set $X := \ell^2(\mathbb{N}, \mathbb{C})$ and let $A_0 : D(A_0) \subset X \to X$ be defined by

$$A_0 x := \big(\lambda_{0k} x_k\big)_{k \in \mathbb{N}} \quad (x \in D(A_0)),$$

where $D(A_0) := \{(x_k) \in X : (\lambda_{0k} x_k) \in X\}$ and $\lambda_{0k} := -\alpha_k + \mathrm{i}\beta_k$ with

$$\operatorname{Re}\lambda_{0k} = -\alpha_k := -1/k \quad (k \in \mathbb{N}) \quad \text{and} \quad \operatorname{Im}\lambda_{0k} = \beta_k \longrightarrow \infty \quad (k \to \infty).$$

Set $U := \mathbb{C}$ and let $B : U \to \mathbb{C}^{\mathbb{N}}$ be defined by

$$Bu := (ub_k)_{k\in\mathbb{N}} \quad (u \in U),$$

where

$$b_k := 1/k^{3/8} \quad (k \in I_1) \quad \text{and} \quad b_k := 1/k \quad (k \in I_2),$$
$$I_1 := \{l^2 : l \in \mathbb{N}\} \quad \text{and} \quad I_2 := \mathbb{N} \setminus I_1.$$

Clearly, $(b_k) \in X$ and therefore $B \in L(U, X)$ and $B^* \in L(X, U)$ with

$$B^* x = \sum_{k\in\mathbb{N}} \overline{b}_k x_k = \langle b, x\rangle_X \quad (x \in X), \tag{3.1}$$

where $b := (b_k)$, of course. We now define

$$A := A_0 - BB^* \quad \text{and} \quad A' := A_0$$

and show, in various steps, that A and A' are generators of strongly but not exponentially stable contraction semigroups on X, that $A' = A + Q$ for a compact perturbation Q of rank one, and that B is infinite-time admissible for A but not infinite-time admissible for A'.

As a first step, we observe that $A' = A + Q$ with $Q := BB^*$ and that Q has rank one (because the same is true for B), whence Q is compact.

As a second step, we observe from

$$\lambda_{0k} \in \mathbb{C}^- \quad (k \in \mathbb{N}) \quad \text{and} \quad \sup\{\operatorname{Re}\lambda_{0k} : k \in \mathbb{N}\} = 0 \tag{3.2}$$

that A' is the generator of a strongly stable but not exponentially stable contraction semigroup on X. Indeed, by (3.2.a) one directly verifies that $\|\mathrm{e}^{A_0 t}x\|_X^2 \to 0$ as $t \to \infty$ for every $x \in X$ and by (3.2.b) the spectral bound and hence the growth bound of $\mathrm{e}^{A_0 \cdot}$ is at least 0.

As a third step, we show that B is not infinite-time admissible for A'. In view of (2.4), we have to show that

$$\sup_{\|u\|_2=1} \sup_{t\in(0,\infty)} \left\| \int_0^t \mathrm{e}^{A_0 s} Bu(s)\,\mathrm{d}s \right\|_X = \infty. \tag{3.3}$$

We first observe by Fatou's lemma that

$$\begin{aligned}\liminf_{t\to\infty} \left\| \int_0^t \mathrm{e}^{A_0 s} Bu(s)\,\mathrm{d}s \right\|_X^2 &\geq \sum_{k\in\mathbb{N}} \left| \int_0^\infty u(s)\mathrm{e}^{\lambda_{0k} s}\,\mathrm{d}s \right|^2 |b_k|^2 \\ &\geq \left| \int_0^\infty u(s)\mathrm{e}^{\lambda_{0n} s}\,\mathrm{d}s \right|^2 |b_n|^2 \end{aligned} \tag{3.4}$$

for every $u \in L^2(\mathbb{R}_0^+, U)$ and $n \in \mathbb{N}$. Setting $u_n(s) := n^{-1/2}\chi_{[0,n]}(s) \cdot \mathrm{e}^{-\mathrm{i}\beta_n s}$ for $s \in \mathbb{R}_0^+$ and $n \in \mathbb{N}$, we see that

$$\|u_n\|_2 = 1, \tag{3.5}$$

$$\left| \int_0^\infty u_n(s)\mathrm{e}^{\lambda_{0n} s}\,\mathrm{d}s \right|^2 = \frac{1}{n}\left| \int_0^n \mathrm{e}^{-\alpha_n s}\,\mathrm{d}s \right|^2 = \frac{1}{\alpha_n^2 n}\Big(1 - \mathrm{e}^{-\alpha_n n}\Big)^2$$
$$= n(1-\mathrm{e}^{-1})^2 \tag{3.6}$$

for every $n \in \mathbb{N}$. Combining now (3.4), (3.5) and (3.6) we get

$$\sup_{\|u\|_2 = 1} \sup_{t \in (0,\infty)} \left\| \int_0^t \mathrm{e}^{A_0 s} Bu(s)\,\mathrm{d}s \right\|_X^2 \geq \sup_{n\in\mathbb{N}} \left(\liminf_{t\to\infty} \left\| \int_0^t \mathrm{e}^{A_0 s} Bu_n(s)\,\mathrm{d}s \right\|_X^2 \right)$$
$$\geq (1-\mathrm{e}^{-1})^2 \sup_{n\in\mathbb{N}} \big(n|b_n|^2\big).$$

Since $\sup_{n\in\mathbb{N}} \big(n|b_n|^2\big) \geq \sup_{n\in I_1} \big(n|b_n|^2\big) = \infty$, the desired relation (3.3) follows.

As a fourth step, we show that B is infinite-time admissible for A and that A is the generator of a strongly stable contraction semigroup on X. In order to do so, we apply the stabilization theorem above (Theorem 3.1). Since

$$\operatorname{Re}\lambda_{0k} \leq 0 \quad (k \in \mathbb{N}) \quad \text{and} \quad |\lambda_{0k}| \to \infty \quad (k \to \infty),$$

we see that A_0 is a contraction semigroup generator on X with compact resolvent, and since the eigenvalues λ_{0k} of A_0 are pairwise distinct and $b_k \neq 0$ for every $k \in \mathbb{N}$, we see by Theorem 4.2.3 of [3] that the collocated linear system $\mathfrak{S}(A_0, B, B^*)$ is approximately controllable and approximately observable in infinite time. So, by the stabilization theorem above (Theorem 3.1), B is infinite-time admissible for $A_0 - BB^* = A$ and $\mathrm{e}^{A\cdot}$ is a strongly stable contraction semigroup on X.

As a fifth and last step, we convince ourselves that the semigroup generated by A is not exponentially stable. Assume the contrary. Then there exist $M \geq 1$ and $\omega < 0$ such that $\{z \in \mathbb{C} : \operatorname{Re} z > \omega\} \subset \rho(A)$ and

$$\big\|(A-z)^{-1}\big\|_{X,X} \leq \frac{M}{\operatorname{Re} z - \omega} \quad (\operatorname{Re} z > \omega).$$

So, since $\operatorname{Re}\lambda_{0n} \to 0$ as $n \to \infty$, we conclude that

$$\limsup_{n\to\infty} \big\|(A-\lambda_{0n})^{-1}\big\|_{X,X} \leq \limsup_{n\to\infty} \frac{M}{\operatorname{Re}\lambda_{0n} - \omega} = \frac{M}{|\omega|}. \tag{3.7}$$

We now observe from (3.1) that

$$(A - \lambda_{0n})e_n = -BB^* e_n = -b_n \cdot b \to 0 \quad (n \to \infty). \tag{3.8}$$

Combining (3.7) and (3.8), we arrive at

$$1 = \limsup_{n\to\infty} \|e_n\|_X = \limsup_{n\to\infty} \big\|(A-\lambda_{0n})^{-1} b_n \cdot b\big\|_X \leq \frac{M}{|\omega|} \limsup_{n\to\infty} \|b_n \cdot b\|_X = 0.$$

Contradiction! (Alternatively, we could prove the fifth step as follows. Since $e^{(A+Q)\cdot}$ is strongly stable by the second step and since Q is compact, we have

$$\left\| e^{(A+Q)t} Q \right\|_{X,X} \to 0 \quad (t \to \infty). \tag{3.9}$$

Assuming that $e^{A\cdot}$ is exponentially stable and using (3.9), it is not difficult to conclude from

$$e^{(A+Q)t}x = e^{At}x + \int_0^t e^{(A+Q)s} Q \cdot e^{A(t-s)} x \, ds \quad (x \in X)$$

that $\|e^{(A+Q)t}\|_{X,X} \to 0$ as $t \to \infty$. And from this, in turn, it follows by Proposition IV.2.2 of [4] that the semigroup $e^{(A+Q)\cdot}$ is exponentially stable, contradicting the second step!) □

4. An example using an admissibility result for diagonal linear systems

4.1. Characterization of infinite-time admissibility

We will use the following well-known characterization of infinite-time admissibility for diagonal semigroup generators A_0. It essentially goes back to [5] (Proposition 2.2) and can also be found in [9] (Theorem 5.3.9 in conjunction with Remark 4.6.5), for instance.

Theorem 4.1. *Suppose $X = \ell^2(I, \mathbb{C})$ with a countable infinite index set I and let $A_0 : D(A_0) \subset X \to X$ be the diagonal operator given by*

$$A_0 x := \left(\lambda_{0k} x_k \right)_{k \in I} \quad (x \in D(A_0)),$$

where $D(A_0) := \left\{ (x_k) \in X : (\lambda_{0k} x_k) \in X \right\}$ and $\lambda_{0k} \in \mathbb{C}^-$ for every $k \in I$. Suppose further that $B \in L(U, X_{-1})$ with $U := \mathbb{C}$, that is,

$$Bu = \left(u b_k \right)_{k \in I} \qquad (u \in U)$$

for a unique sequence $(b_k) \in X_{-1} = \left\{ (c_k) \in \mathbb{C}^I : \sum_{k \in I} |c_k|^2 / (1 + |\lambda_k|^2) < \infty \right\}$. Then the following statements are equivalent:

(i) *B is infinite-time admissible for A_0.*
(ii) *There exists a constant $M \in \mathbb{R}_0^+$ such that*

$$\sum_{k \in I} \frac{|b_k|^2}{|z - \lambda_{0k}|^2} \leq \frac{M}{\operatorname{Re} z} \quad (z \in \mathbb{C}^+).$$

Clearly, in the situation of the above theorem the condition (ii) is equivalent to the existence of a constant $M \in \mathbb{R}_0^+$ such that

$$\left\| (z - A)^{-1} B \right\|_{U,X} \leq \frac{M}{\sqrt{\operatorname{Re} z}} \quad (z \in \mathbb{C}^+). \tag{4.1}$$

A far-reaching generalization of the above theorem to the case of general contraction semigroup generators A_0 on a separable Hilbert space X was obtained by

Jacob and Partington [6]. See Theorem 1.3 from [6]. It states that for a contraction semigroup generator A_0 on a separable Hilbert space X a control operator $B \in L(U, X_{-1})$ with $U := \mathbb{C}$ is infinite-time admissible if and only if there is a constant $M \in \mathbb{R}_0^+$ such that the resolvent estimate (4.1) is satisfied. We also refer to [7] and [9] (Section 5.6) for an overview of many more admissibility results, for example, for infinite-dimensional input-value spaces U.

4.2. Infinite-time admissibility under compact perturbations

Example 4.2. Set $X := \ell^2(\mathbb{Z}, \mathbb{C})$ and let $A : D(A) \subset X \to X$ and $A' : D(A') \subset X \to X$ be defined by

$$Ax := \big(\lambda_k x_k\big)_{k\in\mathbb{Z}} \quad (x \in D(A)) \quad \text{and} \quad A'x := \big(\lambda'_k x_k\big)_{k\in\mathbb{Z}} \quad (x \in D(A')),$$

where $D(A) := \big\{(x_k) \in X : (\lambda_k x_k) \in X\big\}$ and $D(A') := \big\{(x_k) \in X : (\lambda'_k x_k) \in X\big\}$ with

$$\lambda_k := \begin{cases} -1/k^{1/2} + \mathrm{i}k, & k \in \mathbb{N}, \\ -(|k|+1)^{1/2}, & k \in -\mathbb{N}_0, \end{cases} \quad \text{and} \quad \lambda'_k := \begin{cases} -\mathrm{e}^{-k} + \mathrm{i}k, & k \in \mathbb{N}, \\ -(|k|+1)^{1/2}, & k \in -\mathbb{N}_0. \end{cases}$$

Set $U := \mathbb{C}$ and let $B : U \to \mathbb{C}^{\mathbb{Z}}$ be defined by

$$Bu := \big(ub_k\big)_{k\in\mathbb{Z}} \quad (u \in U),$$

where

$$b_k := 1/k \quad (k \in \mathbb{N}), \quad b_0 := 0, \quad b_k := 1/|k|^{1/2} \quad (k \in -\mathbb{N}).$$

Clearly, $\sum_{k\in\mathbb{Z}} |b_k|^2/(1+|\lambda_k|^2) < \infty$ and $\sum_{k\in\mathbb{Z}} |b_k|^2 = \infty$ whence $(b_k) \in X_{-1} \setminus X$. And therefore

$$B \in L(U, X_{-1}) \setminus L(U, X).$$

We now show, in various steps, that A and A' are generators of strongly but not exponentially stable contraction semigroups on X, that $A' = A + Q$ for a compact perturbation Q of infinite rank, and that B is infinite-time admissible for A but not infinite-time admissible for A'.

As a first step, we observe from

$$\begin{gathered} \lambda_k, \lambda'_k \in \mathbb{C}^- \quad (k \in \mathbb{Z}), \\ \sup\big\{\operatorname{Re}\lambda_k : k \in \mathbb{Z}\big\},\ \sup\big\{\operatorname{Re}\lambda'_k : k \in \mathbb{Z}\big\} = 0 \end{gathered} \tag{4.2}$$

that A and A' are generators of strongly stable but not exponentially stable contraction semigroups on X. Indeed, this follows in exactly the same way as the second step of Example 3.2.

As a second step, we observe that $A' = A + Q$ for a compact operator Q of infinite rank. Indeed, the operator $Q : X \to X$ defined by

$$Qx := \big((\lambda'_k - \lambda_k)x_k\big)_{k\in\mathbb{Z}} \quad (x \in X)$$

is a bounded operator on X because $(\lambda_k' - \lambda_k)_{k\in\mathbb{Z}}$ is a bounded sequence. Also, Q is the limit in norm operator topology of the finite-rank operators $Q_N : X \to X$ defined by

$$Q_N x := \big(\dots, 0, 0, (\lambda_1' - \lambda_1)x_1, \dots, (\lambda_N' - \lambda_N)x_N, 0, 0, \dots\big) \quad (x \in X)$$

and therefore Q is compact, as desired.

As a third step, we show that B is infinite-time admissible for A. We have that

$$\sum_{k\in\mathbb{Z}} \frac{|b_k|^2}{|z-\lambda_k|^2} \le \sum_{k\in\mathbb{Z}} \frac{|b_k|^2}{(\operatorname{Re} z + |\operatorname{Re}\lambda_k|)^2} \le \frac{1}{2\operatorname{Re} z} \sum_{k\in\mathbb{Z}} \frac{|b_k|^2}{|\operatorname{Re}\lambda_k|} \tag{4.3}$$

for every $z \in \mathbb{C}^+$ and that

$$M := \sum_{k\in\mathbb{Z}} \frac{|b_k|^2}{|\operatorname{Re}\lambda_k|} < \infty. \tag{4.4}$$

So, by the admissibility theorem above (Theorem 4.1), the claimed infinite-time admissibility of B for A follows from (4.3) and (4.4).

As a fourth and last step, we show that B is not infinite-time admissible for A'. We have that

$$\sum_{k\in\mathbb{Z}} \frac{|b_k|^2}{|z-\lambda_k'|^2} \ge \frac{|b_n|^2}{|z-\lambda_n'|^2} = \frac{1}{(\operatorname{Re} z + \mathrm{e}^{-n})^2 + (\operatorname{Im} z - n)^2} \frac{1}{n^2} \tag{4.5}$$

for every $z \in \mathbb{C}^+$ and $n \in \mathbb{N}$. Choosing $z_n := \mathrm{e}^{-n} + \mathrm{i}n \in \mathbb{C}^+$ for $n \in \mathbb{N}$, we see from (4.5) that

$$\begin{aligned} \sup_{z\in\mathbb{C}^+} \Big(\operatorname{Re} z \sum_{k\in\mathbb{Z}} \frac{|b_k|^2}{|z-\lambda_k'|^2} \Big) &\ge \sup_{n\in\mathbb{N}} \Big(\frac{\operatorname{Re} z_n}{(\operatorname{Re} z_n + \mathrm{e}^{-n})^2 + (\operatorname{Im} z_n - n)^2} \frac{1}{n^2} \Big) \\ &= \sup_{n\in\mathbb{N}} \frac{\mathrm{e}^n}{4n^2} = \infty. \end{aligned} \tag{4.6}$$

So, by the admissibility theorem above (Theorem 4.1), B is not infinite-time admissible for A', as desired. □

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Input-to-state stability for parabolic boundary control: linear and semilinear systems

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Abstract. Input-to-state stability (ISS) for systems described by partial differential equations has seen intensified research activity recently, and in particular the class of boundary control systems, for which truly infinite-dimensional effects enter the situation. This note reviews input-to-state stability for parabolic equations with respect to general L^p-input-norms in the linear case and includes extensions of recent results on semilinear equations.

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1. Introduction

In the study of control of partial differential equations two main types of inputs can be distinguished: *distributed* and *boundary* inputs (or disturbances or controls). The latter emerge e.g. by the following reason: Although a system is described by an infinite-dimensional state space, the ability to influence the system may only be possible through an "infinitesimal small number" of states. As a simple motivating example consider a metal rod of length 1 whose temperature flux at both boundary point is subject to control. Neglecting the width of the rod and normalizing parameters, the heat distribution may be governed by the following

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equations:

$$\begin{cases} \dfrac{\partial x}{\partial t}(\xi,t) = \dfrac{\partial^2 x}{\partial \xi^2}(\xi,t) - ax(\xi,t), & (\xi,t) \in [0,1] \times (0,\infty), \\ \dfrac{\partial x}{\partial \xi}(0,t) = \dfrac{\partial x}{\partial \xi}(1,t) = u(t), & t \in (0,\infty), \\ \quad x(\xi,0) = x_0(\xi), & \text{where } a > 0. \end{cases} \tag{1.1}$$

In this setting input-to-state stability (ISS) can be understood as follows: The given "data" of the model is the initial temperature distribution x_0 and the *input function* $u : \mathbb{R} \to U = \mathbb{R}$, representing the temperature flux at the boundary points. Since it is (physically) clear that the system is causal, that is, the solution x at time t does not depend on the values of the function u at later values, we may ask for an estimate on the (norm of the) *state* x at time t depending on (the norms of) $u\big|_{[0,t]}$ and x_0. In particular, if we choose for the *state space* $X = L^2(0,1)$, we could aim for the following type of time-space estimate for solutions to the differential equation:

$$\|x(t)\|_X \lesssim \mathrm{e}^{-t\omega_0} \|x_0\|_X + \|u\|_{L^q(0,t)}, \tag{1.2}$$

for some fixed $q \in [1,\infty]$, $\omega_0 > 0$ and all $x_0 \in L^2(0,1)$, $u \in L^q(0,t)$ and $t > 0$. In this (linear) case, we call the system *L^q-input-to-state stable* (ISS), and in fact, the above system is L^q-ISS for the parameters $q \in (4/3,\infty]$ and $\omega_0 \in (0, a\pi^2]$, see Example 2.14. In the literature, the most commonly studied ISS property is with respect to L^∞-functions. Clearly, the notion of "solution" is ambiguous here, and we shall, for simplicity, confine ourselves in this introduction to classical solutions of the PDE with sufficiently smooth input functions u.

Since the above example is a linear system, estimate (1.2) clearly superposes the uniform global asymptotic stability of the *internal system*, that is the dependence of $x(t)$ on x_0 in the case that $u \equiv 0$, and the *external stability*, that is, the stability of $u \mapsto x(t)$ when $x_0 = 0$. This combination of stability notions lies at the heart of ISS and has proved very useful particularly for nonlinear ODE systems where this superposition principle does not hold. For a detailed overview on why this concept has become a practical tool in systems and control theory we refer to [2]. Note that the linear PDE case is more subtle compared to the rather trivial linear finite-dimensional situation: Imagine, for instance, a simple space-discretization of the above heat equation which leads to a system of the form

$$\dot{\widetilde{x}}(t) = A\widetilde{x}(t) + Bu(t), \quad \widetilde{x}(0) = \widetilde{x}_0, \tag{1.3}$$

where $\widetilde{x}$ is vector-valued and A, B are matrices of appropriate dimensions. By the variation-of-constants formula the spatially-discrete system is L^q-ISS for any $q \in [1,\infty]$ if and only if A is Hurwitz. If more generally (1.3) describes a system with A being the generator of a strongly continuous semigroup on the (possibly infinite-dimensional) state space X and $B : U \to X$ being a bounded linear operator, the corresponding assertion, that the semigroup is exponentially stable if and only if the system is L^q-ISS, remains valid. Comparing this to the afore-mentioned range of $q \in (4/3, \infty]$ for which the heat equation with Neumann control at the boundary is L^q-ISS, Example 2.14, reveals fundamental differences between systems of the form (1.3) (with bounded input operator B) and the ones with boundary control. So *what goes wrong?*[1] Apparently, (1.1) does not fit into the framework of (1.3) with *bounded* B. Instead, (1.1) is of the abstract form

$$\begin{cases} \dot{x}(t) = \mathfrak{A}x(t), & t > 0, \\ \mathfrak{B}x(t) = u(t), & t > 0, \\ x(0) = x_0, \end{cases} \tag{1.4}$$

where both $\mathfrak{A}$ and $\mathfrak{B}$ are unbounded operators — we will elaborate on the precise assumptions in Section 2. Such systems have become known as *boundary control systems*. Whereas it is formally clear that our example fits into the setting of (1.4) rather than into (1.3), it is a little less clear how ISS estimates can be assessed in this case (or how to discuss existence of solutions, to begin with). However, there is a way of interpreting a boundary control system as a variant of (1.3). Although this is rather well-known to the operator theorists in systems theory, the explicit argument will be recalled in Section 2, also revealing the natural connection to weak formulations from PDEs. This is also done in order to place approaches and results that were recently obtained for ISS together with more classic — but sometimes a bit folklore — results known in the literature.

Whereas these different view-points for linear systems are often rather subject to taste or one's background — however, the amount of effort for obtaining ISS results may differ greatly, not only because solution concepts are intimately linked with the approach — they (can) become crucial when considering systems governed by nonlinear PDEs. In line with the introductory 1D-heat equation, one may be interested in the following semilinear system

$$\begin{cases} \dfrac{\partial x}{\partial t}(\xi, t) = \dfrac{\partial^2 x}{\partial \xi^2}(\xi, t) + f(x(t, \xi)), & (\xi, t) \in [0, 1] \times (0, \infty), \\ \dfrac{\partial x}{\partial \xi}(0, t) = \dfrac{\partial x}{\partial \xi}(1, t) = u(t) & t \in (0, \infty), \\ x(\xi, 0) = 0. \end{cases} \tag{1.5}$$

where f is e.g. of the form $f(x) = -x - x^3$. In general, to account for nonlinearities, the aimed ISS estimate has to be adapted to an inequality of the more general form

$$\|x(t)\|_X \lesssim \beta(\|x_0\|_X, t) + \gamma(\|u\|_{L^q(0,t)}), \tag{1.6}$$

[1] [34], A. Mironchenko and F. Wirth. *Restatements of input-to-state stability in infinite dimensions: what goes wrong?* In: *Proc. of the 22th International Symposium on Mathematical Theory of Networks and Systems*, pages 667–674, 2016.

where $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ being classical comparison functions from Lyapunov theory,

$$\begin{aligned}
\mathcal{K} &= \big\{\mu : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid \mu(0) = 0,\ \mu \text{ continuous, strictly increasing}\big\},\\
\mathcal{K}_\infty &= \big\{\gamma \in \mathcal{K} \mid \lim_{x\to\infty} \gamma(x) = \infty\big\},\\
\mathcal{L} &= \big\{\theta : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid \gamma \text{ continuous, strictly decreasing, } \lim_{t\to\infty} \theta(t) = 0\big\},\\
\mathcal{KL} &= \big\{\beta : (\mathbb{R}_0^+)^2 \to \mathbb{R}_0^+ \mid \beta(\cdot, t) \in \mathcal{K}\ \forall t \text{ and } \beta(s, \cdot) \in \mathcal{L}\ \forall s\big\}.
\end{aligned}$$

Of course, even in the uncontrolled setting $u(t) \equiv 0$, equation (1.5) is more delicate to deal with than a linear equation, both in terms of existence of solutions as well as asymptotic behaviour, but well-known [38, 16]. In particular, the "sign" of f may be crucial for the existence of global solutions, which is necessary for ISS. Regarding ISS, we now have typical nonlinear effects (for which ISS was originally studied for ODEs [42]) blended with infinite-dimensional effects (through both the heat diffusion and the boundary control).

Recently, several steps have been made to address ISS for semilinear systems, for both distributed and boundary control, e.g. [13, 39, 33, 48, 49] and the references in Section 1.1. The employed methods are diverse — see the section paragraph — and it seems that a unified approach for more general systems is missing and open problems remain. In the following we try to offer yet another approach to the ISS for parabolic semilinear equations from a mere functional-analytic point of view. This, though linked to the spirit with [48], generalizes to more general equations of the form

$$\begin{cases} \dot{x}(t) = \mathfrak{A}x(t) + f(t, x(t)), & t > 0,\\ \mathfrak{B}x(t) = u(t), & t > 0,\\ x(0) = x_0. \end{cases} \tag{1.7}$$

Before we summarize on the state-of-the-art in the literature, let us identify the crucial tasks in identifying ISS for a parabolic system of the form (1.7):

(I) global existence (and uniqueness) of solutions to (1.7) for u in the considered function class;

(II) uniform global asymptotic stability of the undisturbed system, $u \equiv 0$;

(III) the L^q-ISS estimate, (1.6).

The first task is classical in the study of (parabolic) PDEs and is typically approached by local fix-point arguments and iteratively extending the solutions to a maximal interval and a-posteriori regularity investigations. The second step, sometimes phrased by the "geometric properties" of an evolution equation in the PDE literature, is dealt with differently than in (I); with methods, such as Lyapunov functions, carefully adjusted from the finite-dimensional theory. The final step (III) is closely connected to (II) and, at least in the situations studied in the literature so far, can often be accessed by weaker arguments than the ones in (II). In particular, a local (in time) version of estimate (1.6) does in general not suffice to guarantee global solutions. However, after having settled global existence, in

Section 3 we shall see relatively simple Lyapunov arguments which are sufficient for ISS.

This note has two goals: First and foremost we would like to survey on recent developments that fall under the concept of ISS for boundary-controlled (parabolic) evolution equations: This is done with particular care at those instances where the literature has seen results in similar spirit, but emerging from different approaches. An example of such an instance is the use of the notion of *admissible operators* which is classic in infinite-dimensional systems theory, but comes along with quite an operator-theoretic "flavour" compared to (direct) PDE arguments. We will avoid the notion of "admissibility" throughout this manuscript as it is, in case of uniformly globally asymptotically stable linear systems, equivalent to ISS, [19]. Thus admissibility in the context of ISS is rather "another name" than an additional property, which, for linear equations, can be used interchangeably. By this, we hope to contribute to clarify on some things that may be folklore knowledge in one community, while possibly unknown in others. The author strongly believes that the fact that ISS for PDEs is currently studied by view-points from different fields, such as operator theory, systems theory and control of PDEs, has and has had a very positive effect on the topic. Apart from this survey-character, the article slightly extends recent findings around ISS for semilinear equations, in particular the ones in [48]. This includes the goal to unify some of the approaches from the literature and or to reveal common features and difficulties. We emphasize that in contrast to the introductory example and several results in the literature, we will not restrict ourselves to spatially one-dimensional systems in the following. Thereby we hope to set the ground for coming efforts in the study of ISS estimates for PDE systems, which even in the semilinear parabolic case are by far not completed.

What this note does not cover is the link to a profound application of ISS. Instead, we confine ourselves to some of the — as we believe — mathematical essentials and refer to the literature for important topics such as ISS feedback redesign and ISS small-gain theorems, which have had great success in finite-dimensional theory. Furthermore, ISS Lyapunov functions — interesting from both the application and the general theory — for which even the linear case is not completely understood yet, see [17] for an interesting partial result, will not be discussed here in detail.[2]

Altogether we hope to address with this article both experts in ISS for infinite-dimensional systems as well as researchers new to the field. This intention has also led to the style of the presentation which is chosen in a way that, the author hopes, is more intuitive than a plain arrangement of definitions and results. Like in the introduction, we will try to stick closely to some tutorial examples and develop/recap the ISS theory around them. This also means that some of the results of Section 3 should rather be seen as a first step (or better second step after what has already been done in the literature) far from being settled conclusively.

[2]At least not explicitly.

We will point out such incomplete situations and comment on difficulties. For example, one of these seems to be L^q-ISS for semilinear parabolic equations with Dirichlet boundary control, where, to the best of the authors knowledge, so far only the case $q = \infty$ has partially been resolved [33, 47, 49].

1.1. ISS for parabolic semilinear systems — what is known

As mentioned before the notion of ISS in the context of PDEs has only been studied in the last ten years. However, particularly for linear systems, several results had previously been known — at least implicitly — by other notions arising in the control of PDEs or boundary value problems. For example, for linear systems L^q-ISS is equivalent to uniform global asymptotic stability together with L^q-admissibility — the latter property being particularly satisfied if distributed controls are considered, see [9, 19, 35]. Therefore, classical results for L^2-admissibility, e.g. [44] and L^q-admissibility, $q \in [1, \infty)$ e.g. [14, 43, 46], can be applied to derive ISS for linear systems. Recall that $q \in \{1, \infty\}$ are special choices for linear systems: Whereas $q = 1$ can practically only arise for distributed controls [46], the case $q = \infty$ is implied by any other L^p-ISS estimate with $p < \infty$. By now there are several results for general linear, not necessarily parabolic, systems for distributed and boundary control, see e.g. [5, 8, 9, 26, 19, 35, 36] and the references therein.

In the following we concentrate on works that focus on parabolic equations. The assessment for particular parabolic equations, both linear and semilinear, has been studied by several authors. In [8, 9, 32, 39] (coercive) ISS-Lyapunov functions are constructed for semilinear parabolic equations with distributed control. In these references, spatially one-dimensional equations are considered with the diffusion term being the Laplacian and primarily L^∞-ISS is shown with input functions being continuous or piecewise continuous. Boundary control (or mixed boundary and distributed control) for parabolic equations has been studied in [19, 21, 24, 25, 33, 29, 30, 48, 50, 49]: More precisely, in [24, 25] L^∞-ISS estimates for classical solutions were proved for spatially one-dimensional linear parabolic equations where $\mathfrak{A}$ referred to a regular Sturm–Liouville differential operator and with controls acting through general Robin boundary conditions.[3] The proof technique rested on a careful analysis of the solutions represented via the spectral decomposition, available in this case. In [19, Sect. 4] general Riesz-spectral operators were considered and more general ISS estimates. Recently, another abstract extension of [24, 25] to Riesz-spectral boundary control systems has been given in [30], also for generalized solutions and more generally, continuous inputs. The assumptions used in these works, which particularly include that the differential operators have discrete spectra, are not required in [21], where a very general class of linear parabolic equations and inputs in L^∞ are considered, see Theorem 2.18 below. Note that all these references require finite-dimensional input spaces.

[3]Here, "Robin boundary conditions" includes Dirichlet and Neumann boundary conditions.

Semilinear diffusion equations (with constant diffusion coefficients) in one spatial coordinate have appeared in [48, 49, 50] with different scenarios of boundary control. In particular, it is shown in [48] that Robin boundary control which is not Dirichlet control allows for L^q-ISS estimates, $q \in [2, \infty]$, under sufficient assumptions on f in order to guarantee global existence of classical solutions. We will revisit these results in the present paper and show how they generalize to more general differential operators on higher-dimensional spatial domains. In [33] maximum principles and their compatibility with monotonicity are used to assess L^∞-ISS for a broad class of semilinear parabolic equations with Dirichlet boundary control and infinite-dimensional input spaces. Dirichlet control has also appeared in [49, 50] for a viscous Burger's type equation, however with a technical assumptions on the L^∞-norm of the input functions. We also mention a recent result in [17, Proposition 4.1] which establishes L^∞-ISS Lyapunov functions for parabolic boundary control problems (and even a bit more general settings). Furthermore, we remark that also linear control systems with nonlinear (closed-loop) feedback law can be interpreted as semilinear control systems, e.g. [43]. In particular, we mention the extensive results for Lur'e systems in [13] and the prior work [22].

1.2. Notation

In the following let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers respectively and $\mathbb{R}_+ = [0, \infty)$. The letters X and U will always refer to complex Banach spaces with norms $\|\cdot\|_X, \|\cdot\|_U$ where we omit the reference to the space whenever it is clear from the context. Let $I \subset [0, \infty)$ be a bounded interval. By $L^p(I; X)$, $p \in [1, \infty)$ we refer to the X-valued Lebesgue spaces of measurable, p-integrable functions $f : I \to X$, where the Bochner integral is used to define the vector-valued integrals. The space $W^{k,p}(I; X) \subset L^p(I; X)$ refers to the vector-valued Sobolev functions of order k. The space of essentially bounded X-functions is denoted by $L^\infty(I; X)$, the space of X-valued regulated functions by $\mathrm{Reg}(I; X)$, which is the closure of the step functions in $L^\infty(I; X)$, and the space of continuous functions by $C(I; X)$; all equipped with their natural (essential) supremum norms. Furthermore, $C^k(I; X)$ refers to the space of k-times continuously differentiable functions $f : I \to X$. By $C_c^\infty(I; X)$ we refer to the functions which are k-times differentiable for any $k > 0$ and compactly supported in I. If $\mathcal{Z}(I; X)$ refers to one of the defined function spaces, then $\mathcal{Z}_{\mathrm{loc}}(\mathbb{R}_+; X)$ denotes the space of functions $f : \mathbb{R}_+ \to X$ such that the restriction $f\big|_I : I \to X$ lies in $\mathcal{Z}(I; X)$ for all compact subintervals $I \subset \mathbb{R}_+$. We will also identify a function $f : I \to X$with its zero extension to $\mathbb{R}$ or $\mathbb{R}_+$. For a Banach space Y let $\mathcal{L}(X, Y)$ denote the space of bounded linear operators from X to Y. We assume that the reader is familiar with basics from strongly continuous semigroups (or "C_0-semigroups") for which we refer to the textbooks [7, 38, 43, 44]. Typically we will denote a semigroup by T and its generator by A. The growth bound of T will be denoted by ω_A. For a Hilbert space X the scalar product will be denoted by $\langle \cdot, \cdot \rangle$ and for a densely defined, closed operator A on X, let A^* denote the Hilbert space adjoint.

The notation "$F(x) \lesssim G(x)$" means that there exists a constant $C > 0$, which is independent of the involved variable x, such that $F(x) \leq CG(x)$.

2. A recap on ISS for linear boundary control systems

Intuitively, and in particular if one has a certain class of systems in mind, it is rather straight-forward how input-to-state stability for PDEs should be defined in order to generalize the finite-dimensional theory. However, as various solution concepts such as weak, mild and strong solutions for infinite-dimensional systems exist, the following abstract definition in the language of dynamical systems seems to be natural for what we need in the following, see [9, 23] and the references therein, for similar notions in the context of ISS which have motivated the following.

Definition 2.1 (Dynamical control systems). Let X and U be a Banach spaces. Let $D \subset X \times U^{\mathbb{R}_+}$ and let $\Phi : \mathbb{R}_+ \times D \to X$ be a function satisfying the following properties for any $t, h \in \mathbb{R}_+$, $(x, u), (x, u') \in D$:

(i) $\Phi(0, x, u) = x$;
(ii) $(\Phi(t, x, u), u(t + \cdot)) \in D$ and $\Phi(t + h, x, u) = \Phi(h, \Phi(t, x, u), u(t + \cdot))$;
(iii) $\left(x, u\big|_{[0,t]}\right) \in D$ and $u|_{[0,t]} = u'|_{[0,t]}$ implies that $\Phi(t, x, u) = \Phi(t, x, u')$.

The mapping Φ is called *semiflow* and

- X the *state space*
- U the *input space*
- $D(\Phi) := D$ the space of *input data*
- $D_X(\Phi) = \big\{x \in X : \exists u \text{ such that } (x, u) \in D(\Phi)\big\}$ the *initial values*
- $D_U(\Phi) = \{u \in U^{\mathbb{R}_+} : \exists x \text{ such that } (x, u) \in D(\Phi)\}$ the *input functions*

The triple (X, U, Φ) is called a *dynamical control system.*

Note that for linear systems it is often possible to "separate" $D(\Phi)$ in the sense that $D(\Phi) = D_X(\Phi) \times D_U(\Phi)$. However, in the case of Φ referring to the semiflow arising from the classical solutions of a boundary control system — even in the linear case — this is not true.

Remark 2.2. It is debatable whether the definition of a dynamic control system (as we decided to call it here) should include any continuity assumptions on the flow. For example, as a "minimal" property, one could require that $t \mapsto \Phi(t, x, u)$ is continuous for any $(x, u) \in D(\Phi)$, as suggested e.g. in [33]. This condition sounds reasonable in most concrete situations involving the solution concept of the PDE. However, we remark that checking this property may not be trivially satisfied even in the context of linear ISS with respect to inputs from L^∞, see [19, 21]. As mentioned, several abstract settings have been introduced in the literature and the assumptions vary from one to the other. We do not claim that our definition is more suitable than others, but it seems to be reasonable for our needs.

Definition 2.3 (ISS of dynamical control system). Let (X, U, Φ) be a dynamical control system and let $q \in [1, \infty]$. We say that the dynamical control system is *L^q-input-to-state stable, L^q-ISS,* if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$\left\|\Phi(t, x, u)\right\|_X \leq \beta\big(\left\|x_0\right\|_X, t\big) + \gamma\big(\left\|u|_{[0,t]}\right\|_{L^q(0,t;U)}\big) \tag{2.1}$$

for all $t > 0$, $(x_0, u) \in D(\Phi) \cap \big(X \times L^q_{\mathrm{loc}}(0, \infty; U)\big)$.

More "exotic" norms other than L^q can be considered in the study of ISS. For instance, Orlicz spaces, a generalization of L^p-spaces, appear naturally when studying *integral input-to-state stability*, a variant of ISS [19, 37, 21]. We remark that in the above definition one could more generally refrain from the completeness of the spaces X and U. It is also important to keep in mind that the definition of an input-to-state stable dynamical control systems requires the global existence of solutions in time, known as "forward-completeness" of the function Φ. Infinite-dimensional examples of dynamical control systems that are ISS can readily be given by means of linear PDE systems with distributed control.

Example 2.4. Let A be the generator of a strongly continuous semigroup on X and $B : U \to X$ be a bounded operator. It is well-known, see e.g. [44, Proposition 4.2.10], that for any $x_0 \in D(A)$ and $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+; U)$ there exists a classical solution $x : [0, \infty) \to X$ to the abstract linear equation

$$\dot{x} = Ax(t) + Bu(t), \quad t > 0 \tag{2.2}$$

$$x(0) = x_0 \tag{2.3}$$

and by the (abstract) variation-of-constants formula,

$$x(t) = T(t)x_0 + \int_0^t T(t - s)Bu(s)\,\mathrm{d}s \tag{2.4}$$

one sees that (X, U, Φ) with $\Phi(t, x_0, u) = x(t)$ and $D(\Phi) = D(A) \times W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+; U)$, where x denotes the classical solution for $x_0 \in D(A)$, is a dynamical control system which is L^p-ISS for any $p \in [1, \infty]$ if and only if A generates an exponentially stable semigroup, see e.g. [19, Proposition 2.10]. On the other hand, if we 'define' a solution only by (2.4), which is possible for any $x_0 \in X$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+; U)$, we have that (X, U, Ψ) is an L^p-ISS dynamical control system, $p \in [1, \infty]$, with semiflow $\Psi(t, x_0, u)$ defined as the left-hand-side of (2.4) and $D(\Phi) = X \times L^1_{\mathrm{loc}}(\mathbb{R}_+; U)$, if and only if A generates an exponentially stable semigroup.

For instance, this can be applied to show that that the following system is L^p-ISS for any $p \in [1, \infty]$ with $X = U = L^2(\Omega)$ and $a > 0$,

$$\begin{aligned} \dot{x}(\xi, t) &= \Delta x(\xi, t) - ax(\xi, t) + u(\xi, t), && (\xi, t) \in \Omega \times (0, \infty), \\ \frac{\partial x}{\partial \nu}(\xi, t) &= 0 && (\xi, t) \in \partial\Omega \times (0, \infty), \\ x(\xi, 0) &= x_0(\xi), && \xi \in \Omega, \end{aligned}$$

where Δ denotes the Laplace operator on a bounded domain $\Omega \in \mathbb{R}^n$ with smooth boundary.

In Example 2.4 we have seen that for a linear system with distributed control the space of initial values $D_X(\Phi)$ can be chosen identical to X provided that Φ was extended to a more general solution concept. In fact, the ISS estimate was only assessed from the variation-of-constants formula which is a hint that this integrated version of the PDE is a more natural object to study ISS estimates (of course not only ISS estimates). However, as indicated in the introduction, system (1.1) does not fit into the framework of Example 2.4. Before we present a work-around to this issue, let us formalize the type of system that (1.1) is representing.

Definition 2.5 (Linear boundary control system). Let X and U be Banach spaces and $\mathfrak{A} : D(\mathfrak{A}) \subset X \to X$ and $\mathfrak{B} : D(\mathfrak{A}) \to U$ be closed operators such that

1. $\mathfrak{A}|_{\ker \mathfrak{B}}$ generates a C_0-semigroup on X, and
2. $\mathfrak{B}$ is right-invertible, i.e. there exists $B_0 \in \mathcal{L}(U, D(\mathfrak{A}))$ with $\mathfrak{B}B_0 = id_U$.

Here and in the following, we equip $D(\mathfrak{A})$ with the graph norm

$$\|\cdot\|_{\mathfrak{A}} := \|\cdot\|_X + \|\mathfrak{A}\cdot\|_X.$$

Then we call both the pair $(\mathfrak{A}, \mathfrak{B})$ and the formally associated set of equations

$$\begin{cases} \dot{x}(t) = \mathfrak{A}x(t), & t > 0, \\ \mathfrak{B}x(t) = u(t), & t > 0, \\ x(0) = x_0 \in X, \end{cases} \tag{2.5}$$

a (linear) *boundary control system*. Given a continuous function $u : [0,\infty) \to U$ and $x_0 \in X$, a function $x : [0,\infty) \to X$ is called a *classical solution* of the boundary control system if $x \in C^1([0,\infty); X) \cap C([0,\infty); D(\mathfrak{A}))$ and x satisfies (2.5) pointwise.

Note that the definition of a classical solution implies that $\mathfrak{B}x_0 = u(0)$. Let us now provide an argument for the L^q-ISS estimate (1.2) for the linear heat equation, (1.1), stated in the introduction. Suppose $x : [0,\infty) \to X$ is a classical solution to the PDE satisfying the boundary condition for some continuous function $u : [0,\infty) \to U$. Integration by parts then readily yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{L^2(0,1)} \\
&\quad = \mathrm{Re}\langle x(t), \dot{x}(t)\rangle \\
&\quad = \mathrm{Re}\left\langle x(t), \frac{\partial^2}{\partial\xi^2}x(t) - ax(t)\right\rangle \\
&\quad = -\left\|\frac{\partial}{\partial\xi}x(t)\right\|^2_{L^2(0,1)} - a\|x(t)\|^2_{L^2(0,1)} + \mathrm{Re}\left(x(\xi,t)\overline{u(t)}\Big|_{\xi=0}^{\xi=1}\right) \\
&\quad \le -\left\|\frac{\partial}{\partial\xi}x(t)\right\|^2_{L^2(0,1)} - a\|x(t)\|^2_{L^2(0,1)} + \varepsilon\|x(t)\|^2_{H^1(0,1)} + \frac{C}{\varepsilon}|u(t)|^2,
\end{aligned}$$

where in the last step we used the fact that the boundary trace is a continuous linear operator from the Sobolev space $H^1(0,1)$ to $\mathbb{C}^2$ and where $C>0$ is some absolute constant. Therefore, by Gronwall's lemma, we conclude that for any $\omega < a$ there exists $\widetilde{C} > 0$ such that

$$\left\|x(t)\right\|^2_{L^2(0,1)} \leq \mathrm{e}^{-\omega t} \left\|x_0\right\|^2_{L^2(0,1)} + \widetilde{C} \int_0^t \mathrm{e}^{-a(t-s)} \left|u(s)\right|^2 \mathrm{d}s$$

and thus, by Hölder's inequality, L^q-ISS, (1.2), for any $q \in [2,\infty]$ follows. Note that an argument in this spirit has been applied in [48] to assess ISS even for a class of semilinear one-dimensional heat equations, provided that "the nonlinearity behaves well" in the above estimates — we will be more explicit on that in Section 3. Let us make a few remarks on this proof: Although eventually L^q-ISS is derived for $q \in [2,\infty]$, it is essential for the argument to bound the term involving $u(t)$ such that the resulting $x(t)$ is bounded in the H^1-norm squared and consequently derive an implicit inequality in $\left\|x(t)\right\|_{L^2(0,1)}$. However, the result is not sharp. In fact, the considered controlled heat equation (1.1) is L^q-ISS for all $q \in (4/3,\infty]$. To see this, we will rewrite the boundary control system such that an explicit solution representation of the form (2.4) as in the distributed case can be used. Here the defining properties of a boundary control system are essential. This transformation is a well-known technique for operator theorists in systems theory [44, 43], but appears to be a type of folklore result that is hard to find explicitly in the literature. What can be found more easily, e.g. in [7], is the so-called Fattorini trick which rewrites the boundary control system into a linear system of the form (2.2) with bounded operator B at the price that the new input is the derivative of the initial u. As we are interested in L^q-estimates of the input u, this is undesirable. This can be overcome by an additional step: To show that this is a natural view-point, we briefly lay-out the "general Fattorini trick" in the following. Recall that the assumptions made in the definition of a boundary control system are intimately linked with semigroups and thus with (2.4).

Let $(\mathfrak{A},\mathfrak{B})$ be a boundary control system. Denote by T the semigroup generated by $A := \mathfrak{A}\big|_{\ker\mathfrak{B}}$ and let $B_0 : U \to D(\mathfrak{A})$ be a right-inverse of $\mathfrak{B}$. A simple calculation shows that for continuously differentiable $u : [0,\infty) \to U$ and a classical solution x to (2.5), the function $z = x - B_0 u$ solves the following differential equation

$$\dot{z}(t) = Az(t) + \mathfrak{A}B_0u(t) - B_0\dot{u}(t), \quad z(0) = x_0 - B_0u(0), \tag{2.6}$$

in the classical sense. Note in particular that by the defining properties of B_0 we have that $x - B_0u \in D(A)$ if and only if $x \in D(\mathfrak{A})$ and $\mathfrak{B}x = u$. This simple reformulation, however, paves the way to derive an equation that again only depends on u and not on $\dot{u}$. For that consider the representation of the solution to the inhomogeneous equation (2.6),

$$z(t) = T(t)(x_0 - B_0u_0) + \int_0^t T(t-s)\mathfrak{A}B_0u(s)\,\mathrm{d}s - \int_0^t T(t-s)B_0\dot{u}(s)\,\mathrm{d}s. \tag{2.7}$$

Note that $\mathfrak{A}B_0 \in \mathcal{L}(U,X)$ so that the second term is well-defined even for any $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+;U)$. The third term is also well-defined, even for functions $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+;U)$. In order to get rid of the term $\dot{u}$ we want to (formally) integrate the second term by parts. To do so, an extension of the semigroup to a larger space X_{-1} is considered. This is done to make sure that $t \mapsto T(t)x$ is differentiable for $x \in X$. For some $\lambda \in \mathbb{C}$ in the resolvent set of the generator A, X_{-1} is defined to be the completion of the space X with respect to the norm $\|(\lambda I - A)^{-1} \cdot \|$ which is independent of λ. The semigroup uniquely extends to a strongly continuous semigroup $T_{-1}(t)$ on X_{-1} with the generator A_{-1} being an extension of A with $D(A_{-1}) = X$. For this standard procedure to define X_{-1}, we refer to [38, 43, 44]. Thus, (2.7) and particular the integrals can be viewed in the larger space X_{-1}. Therefore, integration by parts yields

$$\int_0^t T(t-s)B_0\dot{u}(s)\,\mathrm{d}s = \int_0^t T_{-1}(t-s)A_{-1}B_0u(s)\,\mathrm{d}s + B_0u(t) - T(t)B_0u(0).$$

Inserting this in (2.7) and transforming back to x gives

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)\left[\mathfrak{A}B_0 - A_{-1}B_0\right]u(s)\mathrm{d}s. \tag{2.8}$$

We emphasize that the integral will in general only exist as a limit in X_{-1} whereas its value happens to be an element of X for any $t > 0$ by our assumption that x is a classical solution to the boundary control system. Also note that $A_{-1}B_0 \in \mathcal{L}(U, X_{-1})$ and that $x - B_0 \in D(A)$ is in turn equivalent to $A_{-1}x + [\mathfrak{A}B_0 - A_{-1}B_0]u \in X$. All this leads to the definition of mild solutions.

Definition 2.6 (Mild solutions of boundary control systems). Let $(\mathfrak{A}, \mathfrak{B})$ be a boundary control system with state space X and input space U. Let T denote the semigroup generated by $A := \mathfrak{A}|_{\ker \mathfrak{B}}$ and B_0 be a right-inverse of $\mathfrak{B}$. Let $x_0 \in X$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+;U)$. If the function $x : [0,\infty) \to X_{-1}$ defined in (2.8) takes values only in X, i.e., $x(t) \in X$ for all $t > 0$, and x is continuous from $[0,\infty)$ to X, then x is called a (*continuous*) *mild solution* of (2.5).

Remark 2.7. We want to point that in the literature the notion of a mild solution may be defined in a more general way. E.g. in [19] an arbitrary function $x : [0,\infty) \to X_{-1}$ defined by (2.8) is called a mild solution, without any assumption on the range of x and its continuity. Since $B \in \mathcal{L}(U, X_{-1})$, any such function will however be continuous in the weaker norm of X_{-1}. The assumption that a mild solution should be X-valued is rather natural — not least as one models a differential equation by choosing for a norm/space initially — and so is the continuity (in X). While the first one is necessary for ISS, the second (continuity) could be dropped, if we would be interested in minimal a-priori requirements for ISS estimates. However, we will see shortly that for linear systems the continuity is implicit if the system is L^q-ISS for $q < \infty$, and also for $q = \infty$, if only continuous input functions are considered, see below and [19, 30, 44].

The following properties of mild solutions corresponding to boundary control systems are well-known and can for instance be found in [44, Chap. 11] (in the case of Hilbert spaces). The proofs extend to the general Banach space setting in a straightforward way, see also [43]. Note that in the literature there exists slightly different versions of the definition of abstract boundary control systems, e.g. in [12].

Proposition 2.8. *Let* $(\mathfrak{A}, \mathfrak{B})$ *be a boundary control system with associated operators* A *and* B_0. *Let* $x_0 \in X$ *and* $u \in L^1_{\text{loc}}(0, \infty; U)$. *Then the following assertions hold:*

1. *Any continuous mild solution* x *as in* (2.8) *solves the equation*

$$x(t) - x(0) = \int_0^t A_{-1}x(s) + [\mathfrak{A}B_0 - A_{-1}B_0]\, u(s)\, \mathrm{d}s \quad t > 0, \tag{2.9}$$

   where equality is understood in X_{-1}. *Conversely, any* $x \in C([0, \infty); X)$ *satisfying* (2.9) *in* X_{-1} *is of the form* (2.8) *with* $x_0 = x(0)$.
2. *The operator* $B = \mathfrak{A}B_0 - A_{-1}B_0$ *is uniquely determined by the boundary control system and does not depend on the chosen right-inverse* B_0 *of* $\mathfrak{B}$.
3. *If* $x_0 \in D(\mathfrak{A})$ *and* $u \in W^{2,1}(\mathbb{R}_+; U)$ *such that* $\mathfrak{B}x_0 = u(0)$, *then there exists a unique classical solution to* (2.5) *given by* (2.8).
4. $\mathfrak{A} = A_{-1}|_{D(\mathfrak{A})} + B\mathfrak{B}$ *where* $B = \mathfrak{A}B_0 - A_{-1}B_0 \in \mathcal{L}(U; X_{-1})$.

Proof. For the first item we refer to [43, Theorem 3.8.2]. The rest can be found in [44, Chap. 11] upon the straight-forward adaption of proofs to general Banach spaces. ☐

In the case of Hilbert spaces (in fact, reflexive spaces suffice), we have several alternatives to characterize the operator B as well as the mild solutions to a boundary control system. Note that with this one could in principle avoid the space X_{-1}.

Proposition 2.9. *Let the assumptions of Proposition* 2.8 *hold and additionally assume that* X *and* U *are Hilbert spaces. Then the following assertions hold:*

1. *If* X *and* U *are Hilbert spaces, then*

$$\langle \mathfrak{A}x, \psi \rangle - \langle x, A^*\psi \rangle = \langle \mathfrak{B}x, B^*\psi \rangle \quad \forall x \in D(\mathfrak{A}), \psi \in D(A^*).$$

2. *A continuous function* $x : [0, \infty) \to X$ *is a mild solution of the form* (2.8) *if and only if it is a* (*weak/strong*) *solution in one of the following senses:*
   (i) *For all* $v \in D(A^*)$ *it holds that* $\langle v, x(\cdot) \rangle$ *is absolutely continuous and*

$$\frac{d}{dt}\langle v, x(t) \rangle = \langle x(t), A^*v \rangle + \langle v, \mathfrak{A}B_0u(t) \rangle - \langle A^*v, B_0u(t) \rangle$$

   holds for almost every $t \geq 0$ *and* $x(0) = x_0$.

(ii) *For all $T>0$ and all $z\in C([0,T];D(A^*))\cap C^1([0,T];X)$ with $z(T)=0$ it holds that*

$$\begin{aligned}\langle z(0),x_0\rangle &- \int_0^T \langle \dot z(t),x(t)\rangle \mathrm{d}t\\ &= \int_0^T \langle A^*z(t),x(t)\rangle + \langle z(t),\mathfrak{A}B_0u(t)\rangle - \langle A^*z(t),B_0u(t)\rangle \mathrm{d}t.\end{aligned}$$

(iii) $x\in W^{1,1}_{\mathrm{loc}}([0,\infty);X_{-1})$, $x(0)=x_0$ *and*

$$\dot x(t) = A_{-1}x(t) + [\mathfrak{A}B_0 - A_{-1}B_0]\,u(t)$$

holds in X_{-1} for almost every $t\geq 0$.

Proof. Assertion 1. follows directly from Proposition 2.8, see also [44, Remark 10.1.6]. That the solution concept (i) is equivalent to the one of a mild solution readily follows from (2.9) in Proposition 2.8 and the fundamental theorem of calculus for the Lebesgue integral, see also [44, Remark 4.1.2]. Also recall the duality of X_{-1} and $D(A^*)$ (see e.g. [44, Proposition 2.10.2]).

Similarly, (2.9) shows the equivalence with (ii) by the fundamental theorem of calculus for vector-valued functions (see e.g. [3] and note that X possesses the Radon–Nikodym property) and again using the duality of X_{-1} and $D(A^*)$.

To see that (iii) implies (ii) note first that the function $t\mapsto \langle z(t),x(t)\rangle$ is differentiable for a.e. t and

$$\begin{aligned}\frac{\partial}{\partial t}\langle z(t),x(t)\rangle =&\langle \dot z(t),x(t)\rangle + \langle z(t),A_{-1}x(t)+[\mathfrak{A}B_0-A_{-1}B_0]\,u(t)\rangle_{D(A^*)\times X_{-1}}\\ =&\ \langle \dot z(t),x(t)\rangle + \langle A^*z(t),x(t)-B_0u(t)\rangle + \langle z(t),\mathfrak{A}B_0u(t)\rangle\end{aligned}$$

and thus, by integrating, x satisfies the identity in (ii) for all $z\in C([0,T];D(A^*))\cap C^1([0,T];X)$ with $z(T)=0$. Conversely, assume that x satisfies the condition in (ii) and consider $z(t)=v\widetilde{z}(t)$, with $v\in D(A^*)$, $\widetilde{z}\in C^1([0,T];\mathbb{C})$ and $\widetilde{z}(T)=0$. It readily follows by the definition of the scalar-valued weak derivative and the characterization of scalar-valued Sobolev functions $W^{1,1}$ that

$$\langle x(T),v\rangle - \langle x_0,v\rangle = \int_0^T \langle x(t)-B_0u(t),A^*v\rangle + \langle v,\mathfrak{A}B_0u(t)\rangle \mathrm{d}t$$

holds. Thus, $\langle x(0),v\rangle = \langle x(\cdot),v\rangle(0) = \langle x_0,v\rangle$ for all $v\in D(A^*)$. Thus, by density, $x(0)=x_0$ and hence, (i) holds. For a similar proof showing that mild solutions are weak solutions in the sense of (ii) see e.g. [7, pp. 631–632] (there, however, only bounded B's are considered). □

Remark 2.10. 1. In [30] ISS estimates for boundary control systems are shown for continuous weak solutions in the sense of (ii) of Proposition 2.9. There it is also shown that for smooth inputs, this definition of weak solutions coincides with solutions of the form (2.7). In fact, as Proposition 2.9 shows, the notions of a mild solution as introduced in Definition 2.6, weak solutions of the form (i), (ii) and a "strong solution" (iii) are all equivalent provided we assume continuity. Note that the definitions of weak solutions have the advantage that they do not refer to the space X_{-1}.

2. It is easy to see that the definition of classical and mild solutions can be adapted to more general boundary control systems of the form

$$\begin{cases} \dot{x}(t) = \mathfrak{A}x(t) + \widetilde{B}u_1(t), & t > 0, \\ \mathfrak{B}x(t) = u_2(t), & t > 0, \\ x(0) = x_0, \end{cases} \tag{2.10}$$

where U_1 is a Banach space, $\widetilde{B} \in \mathcal{L}(U_1, X)$ and $u_1 : \mathbb{R}_+ \to U_1$ account for some distributed control.

3. Comparing the form of a mild solution (2.8) with the usual variation-of-constants formula suggests to view a boundary control system as a special case of a system of the form

$$\dot{x}(t) = A_{-1}x(t) + Bu(t), \quad t > 0,\ x(0) = x_0 \in X, \tag{2.11}$$

where the differential equation is understood in the larger space X_{-1} for $B \in \mathcal{L}(U, X_{-1})$. Clearly, for any x_0 and $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+; U)$ this equation has a unique "mild" solution $x : [0, \infty) \to X_{-1}$. Definition 2.6 of a mild solution for a boundary system now additionally requires that such x maps indeed to X. Also note that this setting as the advantage that systems of the form (2.10) are automatically encoded in that form. Conversely, if we are given a system of the form (2.11) with a semigroup generator A and $B \in \mathcal{L}(U, X_{-1})$, it is always possible to find operators $\mathfrak{A} : D(\mathfrak{A}) \to X$, $\mathfrak{B} : D(\mathfrak{A}) \to U$ and $\widetilde{B} : U \to X$ so that we have a boundary control system as in (2.10) with $A = \mathfrak{A}|_{\ker \mathfrak{B}}$ and $B = (\mathfrak{A} - A)B_0$ for some (all) right-inverses B_0 of $\mathfrak{B}$. This result, in the case that B is injective, can be found in [40]. The non-injective case can be seen upon considering the quotient space $\widetilde{U} = U/\ker B$. In conclusion, the study of boundary control systems rather than systems (2.11) is not a restriction.

So far we have encountered — having in mind the equivalence of Proposition 2.9 — two types of solutions for boundary control systems: *classical* and, more generally, *mild* solutions. The use of the latter is also motivated by the fact that the objects in the ISS estimate naturally only require initial values to be in X and input functions in L^q (or the respective functions space). However, for linear systems, this choice is less "conceptual" than rather a technicality, as the following results shows. Note that the case of ISS with respect to continuous functions has already appeared in [30] (where weak solutions haven been considered instead of mild solutions). In the view of systems (A, B) of the form (2.11), the following result is a simple consequence of the linearity and the density of the involved functions spaces.

Proposition 2.11 (ISS w.r.t. different solution concepts). *Let $(\mathfrak{A}, \mathfrak{B})$ be a boundary control system on a Banach space X with associated operators B_0,* *A and B. Let $q \in [1,\infty)$ and let Φ_{classic} and Φ_{mild} refer to the semiflow defined by the classical and mild solutions, respectively. Then the following assertions are equivalent:*

1. $(X, U, \Phi_{\text{classic}})$ *with* $D(\Phi_{\text{classic}}) = D(A) \times C_c^\infty(0,\infty;U)$ *is* L^q*-ISS.*
2. $(X, U, \Phi_{\text{mild}})$ *with* $D(\Phi_{\text{mild}}) = X \times L^q_{\text{loc}}([0,\infty);U)$ *is* L^q*-ISS.*
3. $(X, U, \Phi_{\text{classic}})$ *with*
$$D(\Phi_{\text{classic}}) = \left\{(x,u) \in D(\mathfrak{A}) \times W^{2,1}_{\text{loc}}(0,\infty;U) \colon \mathfrak{B}x = u(0)\right\}$$
is L^q*-ISS.*

If $q = \infty$, then the following assertions are equivalent:

1. $(X, U, \Phi_{\text{classic}})$ *with*
$$D(\Phi_{\text{classic}}) = \left\{(x,u) \in D(\mathfrak{A}) \times W^{2,1}_{\text{loc}}(0,\infty;U) \colon \mathfrak{B}x = u(0)\right\}$$
is L^∞*-ISS.*
2. $(X, U, \Phi_{\text{mild}})$ *with* $D(\Phi_{\text{mild}}) = X \times C([0,\infty);U)$ *is* L^∞*-ISS.*

Note that the statements above particularly include that the considered dynamical control systems are well-defined.

Proof. Since classical solutions are mild solutions the implication (2) to (3) for $q < \infty$ and (2) to (1) when $q = \infty$ are clear. Moreover, the implication (3) to (1) is trivial in the case $q < \infty$. It remains to show (1) $\implies$ (2) in both regimes. Let $t > 0$ be fixed and consider the operator

$$L_t : D(L_t) \subset X \times L^1([0,t];U) \to C([0,t];X), \begin{bmatrix} x \\ u \end{bmatrix} \mapsto \Phi_{\text{classic}}(\cdot, x, u)\big|_{[0,t]}$$

with $D(L_t) = \left\{\left(x, u|_{[0,t]}\right) \colon (x,u) \in D(\Phi_{\text{classic}})\right\}$. By Proposition 2.8, L_t is well-defined and the assumed ISS estimate together with linearity implies that L_t is continuous with respect to the sum norm $\|x\|_X + \|u\|_{L^q(0,t;U)}$. Since classical solutions are mild solutions, Proposition 2.8, L_t extends to an operator, again denoted by L_t, continuous from $D := \left\{\left(x, u|_{[0,t]}\right) \colon (x,u) \in D(\Phi_{\text{mild}})\right\}$ to $C([0,t];X_{-1})$. Consider now $q < \infty$. Thus L_t is continuous even from D to $C([0,t];X)$ since $D(A) \times C_c^\infty(0,\infty;U)$ lies dense in D and since X is continuously embedded in X_{-1}. For the case $q = \infty$, it may not be immediate why $D(L_t)$ is dense in D. To see this, let $x \in X$ and $u \in C([0,t];U)$. Since $D(A)$ is dense in X, we find a sequence $(\widetilde{x}_n)_{n\geq 0}$ in $D(A) = \ker \mathfrak{B}$ such that $\widetilde{x}_n \to x - B_0 u(0)$ for $n \to \infty$. Let $x_n = \widetilde{x}_n + B_0 u(0)$, $n \in \mathbb{N}$. Then $(x_n, u) \in D(\Phi_{\text{classic}})$ and $x_n \to x$ for $n \to \infty$. Now choose a sequence of smooth functions u_n which satisfy $u_n = u(0)$ for all $n \in \mathbb{N}$ and approximate u on $[0,t]$ in the supremum norm. It follows that $(x_n, u_n) \in D(\Phi_{\text{classic}})$. Therefore, L_t is continuous from D to $C([0,t];X)$. From the representation (2.8), it follows that $\Phi_{\text{mild}}(s,x,u) = (L_t(x,u))(s)$ for any s,t such that $s \leq t$ and $(x,u) \in D(\Phi_{\text{mild}})$.

Hence, in both cases, the continuity of the norms and the $\mathcal{KL}$, $\mathcal{K}$ functions directly gives the ISS estimates for $(X, U, \Phi_{\text{mild}})$. □

Remark 2.12. 1. The result of Proposition 2.11 remains true if one replaces L^q, $q < \infty$ by the Orlicz space E_Φ as defined in [19], since $W^{2,1}(0,t;U)$ is dense in $E_\Phi(0,t;U)$.

2. The proof of Proposition 2.11 can also be easily given by viewing the boundary control system as a linear system of the form (2.11). This completely reduces to the fact that an operator is bounded if and only if it is bounded (with an explicit estimate) on a dense subspace.
3. In the view of Proposition 2.11, one could also completely avoid the space X_{-1} in the above considerations and define generalized solutions for the case that L^q estimates are known for the classical solutions. Then Φ can be defined as abstract extension of Φ_{classic} on the space $X \times L^q_{\text{loc}}(0,\infty;U)$ or $X \times C(0,\infty;U)$ respectively, in a similar way as followed in the proof. Such solutions concepts (which coincide in this case) are known as "generalized solutions" in the literature, see e.g. [41], [45, Definition 4.2].
4. The proof of Proposition 2.11 also shows the following: Let $q < \infty$ (for the case $q = \infty$ see below) and (X, U, Φ) be a dynamical control system for a boundary control system with $D(\Phi) \subseteq X \times L^q_{\text{loc}}(0,t;U)$ with the property that it extends the dynamical control system given by the classical solutions $(X, U, \Phi_{\text{classic}})$ in the following sense:
   - $\Phi(\cdot, x, u) \in C(0;\infty;X)$ for any $(x,u) \in D(\Phi)$,
   - $(x,u) \mapsto \Phi(t,x,u)$ is linear for any $t \in \mathbb{R}_+$,
   - $D(\Phi_{\text{classic}}) \subseteq D(\Phi)$,
   - $\Phi_{\text{classic}}(t,x,u) = \Phi(t,x,u)$ for all $(t,x,u) \in \mathbb{R}_+ \times D(\Phi_{\text{classic}})$.

   Then, it holds that
   $$\Phi = (\Phi_{\text{mild}})\big|_{\mathbb{R}_+ \times D(\Phi)}$$
   if $(X, U, \Phi_{\text{classic}})$ is L^q-ISS. The same assertion holds for $q = \infty$ with the modification as in Proposition 2.11. As a side effect, this provides another proof that the weak solutions considered in [30] coincide with the mild definitions defined here, at least if the dynamical control system is L^q-ISS. Note that, by Proposition 2.9, this holds true even without any assumption on ISS.

   All of this shows that ISS estimates for *continuous* input functions and linear systems do ultimately not rely on the "solution concept", but essentially only on the classical solutions, see [30] for a similar conclusion.
5. In contrast to the previous comment in this remark, we want to point out that if one aims to study L^∞-ISS for input functions in $L^\infty_{\text{loc}}(0,\infty;U)$ or the regulated functions $\text{Reg}_{\text{loc}}(0,\infty;U)$, then L^∞-ISS estimates for the classical solutions are not sufficient. This issue is crucial as one may want to allow for non-continuous input-functions.

Above we have seen that the regularity of the boundary trace was the key to derive the L^2-ISS estimate in the case of the toy example heat equation with Neumann boundary control. In fact, this conclusion follows from the upcoming Proposition 2.13, which will also show that a better L^q-ISS estimate can be obtained. Before let us recap a few essentials about parabolic equations in the view of semigroup theory. Recall that a semigroup T is called *analytic* if T can be extended analytically to an open sector $S_\phi = \{z \in \mathbb{C}\setminus\{0\} \colon |\arg z| < \phi\}$ and *bounded analytic* if T is bounded on S_ϕ. An important characteristic of analytic semigroups is that $\operatorname{ran} T(t) \subset D(A)$ for all $t > 0$ and

$$\sup_{t>0} t\,\mathrm{e}^{-t\omega}\,\|AT(t)\| < \infty \tag{2.12}$$

for $\omega > \omega_A$. We will now introduce interpolation spaces X_α for analytic semigroups. Note that there are several approaches to do so and we only touch the topic very briefly here. Let us without loss of generality assume that $\omega_A < 0$. If A generates an analytic semigroup, one can define the fractional power $(-A)^{-\alpha} : X \to X$ for any $\alpha \in (0,1)$ by the contour integral

$$(-A)^{-\alpha} = \int_{\partial S_{\phi'}} z^{-\alpha}(zI + A)^{-1}\mathrm{d}z,$$

where $\partial S_{\phi'}$ is the boundary of a sufficiently large sector $S_{\phi'}$ which particularly contains the spectrum of A. Since $(-A)^{-\alpha}$ is a bounded injective operator on X, one can further define $(-A)^{\alpha} = ((-A)^{-\alpha})^{-1} : \operatorname{ran}(-A)^{-\alpha} \to X$. The domain of $(-A)^\alpha$ equipped with the graph norm is denoted by X_α. Analogously to the space X_{-1}, we can define $X_{-\alpha}$ as the completion of X with respect to the norm $\|(-A)^{-\alpha}\cdot\|$. The operator $(-A)^{-\alpha}$ extends uniquely to an isometric isomorphism from $X_{-\alpha}$ to X which we denote again by $(-A)^{-\alpha}$. Its inverse is the unique bounded extension of $(-A)^\alpha$ from X to $X_{-\alpha}$. For reflexive spaces there is an equivalent view-point of the space $X_{-\alpha}$ as the dual space of the space X_α^* where X_α^* denotes the corresponding fractional space for the dual semigroup T^* with generator A^* and where duality is understood in the sense of the underlying pivot space X, see [44, Chap. 3] and [46]. One of the many basic properties of these spaces are the following (continuous) inclusions,

$$X_{-1} \supset X_{-\alpha} \supset X_{-\beta} \supset X \supset X_\beta \supset X_\alpha \supset X_1,$$

where $0 < \beta < \alpha < 1$. If the growth bound of the semigroup satisfies $\omega_A \geq 0$, the above construction can be performed for a suitably rescaled semigroup $\mathrm{e}^{-t\omega}\,T(t)$ and it can be shown that X_α does not depend on the chosen $\omega > \omega_A$. For specific examples (for example when A is the Laplacian with Dirichlet boundary conditions) these abstract spaces indeed reduce to well-known fractional Sobolev spaces, which is why X_α is sometimes called an "abstract Sobolev space".

In the spirit of (2.12), the fractional powers $(-A)^\alpha$ of a generator of an exponentially stable analytic semigroup satisfy $\operatorname{ran} T(t) \subset D(-A^\alpha)$ for all $t > 0$ and

$$\sup_{t>0} t^\alpha\,\mathrm{e}^{-t\omega}\,\|(-A)^\alpha T(t)\| < \infty,$$

for any $\omega > \omega_A$. Moreover, it holds that $\operatorname{ran} T_{-1}(t) \subset D(A)$ for all $t > 0$. For details on interpolation spaces for analytic semigroup generators we refer e.g. to [10, 15, 38].

With these preparatory comments on analytic semigroups, we can prove the following sufficient condition for ISS.

Proposition 2.13 (L^q-ISS for analytic semigroups). *Let $(\mathfrak{A}, \mathfrak{B})$ be a boundary control system on a Banach space X with associated operators A and B_0 and $B = \mathfrak{A}B_0 - A_{-1}B_0$. Furthermore, assume that A generates an exponentially stable analytic semigroup T and that one of the following properties is satisfied for some $\alpha \in (0, 1]$:*

(i) $B_0 \in \mathcal{L}(U, X_\alpha)$.
(ii) $B \in \mathcal{L}(U, X_{-1+\alpha})$.
(iii) $B^* \in \mathcal{L}(X^*_{1-\alpha}, U^*)$ *and X is reflexive.*[4]

Then $(\mathfrak{A}, \mathfrak{B})$ is L^q-ISS for $q \in (\alpha^{-1}, \infty]$. More precisely, the dynamical control system $(X, U, \Phi_{\text{mild}})$ is L^q-ISS for $D(\Phi_{\text{mild}}) = X \times L^q_{\text{loc}}(0, \infty; U)$, where $\Phi_{\text{mild}}(t, x_0, u)$ refers to the mild solution $x(t)$ defined in (2.8).

Proof. Either of the assumptions on B imply that $(-A)^{-1+\alpha}B \in \mathcal{L}(U, X)$ and hence,

$$\begin{aligned}
\left\|T_{-1}(t)B\right\|_{\mathcal{L}(U,X)} &= \left\|T_{-1}(t)(-A)^{1-\alpha}(-A)^{-1+\alpha}B\right\|_{\mathcal{L}(U,X)} \\
&\le \left\|T_{-1}(t)(-A)^{1-\alpha}\right\|_{\mathcal{L}(X)}\left\|(-A)^{-1+\alpha}B\right\|_{\mathcal{L}(U,X)} \\
&\lesssim t^{-1+\alpha}\,\mathrm{e}^{t\omega}\left\|B\right\|_{\mathcal{L}(U,X_{-1+\alpha})}.
\end{aligned}$$

Thus, for the Hölder conjugate p of $q > (1 - 1 + \alpha)^{-1} = \alpha^{-1}$,

$$\int_0^t \|T_{-1}(t-s)Bu(s)\|_X \,\mathrm{d}s \lesssim \left\|B\right\|_{\mathcal{L}(U,X_{-1+\alpha})} C_{q,\omega} \left\|u\right\|_{L^q(0,t;U)},$$

where we used that $(-1 + \alpha)p \in (-1, 0)$ and $C_{q,\omega} = \left\| \mathrm{e}^{(t-\cdot)\omega}(t-\cdot)^{(-1+\alpha)}\right\|_{L^p(0,t)}$. Therefore the integral $\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s$ converges in X for any $u \in L^q(0, t; U)$ and the assertion follows. □

Recalling that the operator B_0 is not uniquely determined by the boundary control system in general, it is, however, easily seen that Condition (i) in the above proposition holds for *all* right inverses of $\mathfrak{B}$ if and only if it holds for some B_0. Looking at the proof, Proposition 2.13 may seem rather elementary. However, it is widely applicable to settle ISS for linear parabolic boundary control problems as the assumption can often be checked by known properties of boundary trace operators. We now come back to the discussion of the heat equation mentioned in the introduction for general n-dimensional spatial domains.

Example 2.14 (Heat equation with Neumann boundary control). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary $\partial\Omega$. Consider the Neumann

[4] Here X^*_β denotes the dual space of $X_{-\beta}$ with respect to the pivot space X.

boundary controlled heat equation with additional distributed control d, i.e.

$$\begin{aligned} \dot{x}(\xi,t) &= \Delta x(\xi,t) - ax(\xi,t) + d(\xi,t), && (\xi,t)\in\Omega\times(0,\infty),\\ \frac{\partial x}{\partial\nu}(\xi,t) &= u(\xi,t) && (\xi,t)\in\partial\Omega\times(0,\infty),\\ x(\xi,0) &= x_0(\xi), && \xi\in\Omega. \end{aligned}$$

We can formulate this as a boundary control system of the form (2.10) with

$$X = L^2(\Omega),\quad \mathfrak{A} = \Delta - aI_X,\quad \mathfrak{B} = \frac{\partial}{\partial\nu},\quad \widetilde{B} = I_X,\quad U = L^2(\partial\Omega)$$

with $A = \Delta - aI_X$ and $D(A) = \left\{x \in H^1(\Omega) : \Delta x \in L^2(\Omega), \frac{\partial}{\partial\nu}x = 0\right\}$. Integrating by parts twice gives for $x \in H^2(\Omega)$, $\psi \in D(A)$,

$$\langle \mathfrak{A}x, \psi\rangle = \left\langle \mathfrak{B}x, \psi|_{\partial\Omega}\right\rangle_{L^2(\partial\Omega)} + \langle x, A\psi\rangle,$$

Since A is self-adjoint, we conclude by Propositions 2.9 that B^* equals the boundary trace operator γ_0. It is known that $\gamma_0 \in \mathcal{L}\big(H^\beta(\Omega), L^2(\partial\Omega)\big)$ for any $\beta > \frac{1}{2}$, where $H^\beta(\Omega)$ refers to the classical fractional Sobolev space (note, however, that γ_0 is bounded from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$, see [44, 13.6.1]). In terms of the abstract Sobolev spaces X_α this means that $B^* \in \mathcal{L}(X_\beta, U^*)$ for any $\beta > \frac{1}{4}$, see e.g. [28]. Also recall that the Neumann Laplacian on $L^2(\Omega)$ has spectrum in $(-\infty, 0]$ which implies that A generates an exponentially stable analytic semigroup as $a > 0$. Thus, we can infer from Proposition 2.13 that the system is L^q-ISS with respect for any $q > (1-\frac{1}{4})^{-1} = \frac{4}{3}$. Because $\widetilde{B}$ is bounded from X to X, we obtain the ISS estimates for any $q > \frac{3}{4}$ and $\widetilde{q} \geq 1$.

$$\|x(t)\|_{L^2(\Omega)} \lesssim \mathrm{e}^{-at}\,\|x_0\|_{L^2(\Omega)} + \|u\|_{L^q(0,t;L^2(\partial\Omega))} + \|d\|_{L^{\widetilde{q}}(0,t;X)}$$

for all $t > 0$, $d \in L^{\widetilde{q}}(0,t;X)$ and $u \in L^q\big(0,t;L^2(\partial\Omega)\big)$.

Similarly, we can consider the situation where the control does only act on a part of the boundary $\partial\Omega$, and adapt the argumentation in [6, p. 351].

Remark 2.15.
1. The author is not aware of way to sharpen the "Lyapunov argument" for ISS from the introduction on the Neumann controlled heat equation in order to derive the same (sharp) result $p > 4/3$ as in Example 2.14. It seems that such Lyapunov arguments heavily rely on the fact that the space of input functions is L^2 (in time).
2. It is straight-forward to generalize Example 2.14 to a Neumann boundary problem for a general uniformly elliptic second-order differential operator with smooth coefficients.

Another, and in the view of the Lyapunov arguments mentioned in the introduction, more interesting example is the Dirichlet-boundary controlled heat equation.

Example 2.16 (Dirichlet controlled heat equation). Let $\Omega \subset \mathbb{R}^n$ be a domain with C^2-boundary $\partial\Omega$. The Dirichlet boundary controlled heat equation

$$\begin{aligned} \dot{x}(\xi,t) &= \Delta x(\xi,t), && (\xi,t) \in \Omega \times (0,\infty), \\ x(\xi,t) &= u(\xi,t) && (\xi,t) \in \partial\Omega \times (0,\infty), \\ x(\xi,0) &= x_0(\xi), && \xi \in \Omega \end{aligned}$$

can be formulated as a boundary control system with

$$X = L^2(\Omega), \quad \mathfrak{A} = \Delta, \quad \mathfrak{B}x = x\big|_{\partial\Omega}, \quad U = L^2(\partial\Omega)$$

with $A = \Delta$ and $D(A) = \{x \in H^1(\Omega) : \Delta x \in L^2(\Omega), \gamma_0 x = 0\}$, where γ_0 denotes the boundary trace. Integrating by parts twice gives for $x \in D(A)$, $\psi \in C^\infty(\Omega)$,

$$\langle \mathfrak{A}x, \psi\rangle = \left\langle \mathfrak{B}x, \frac{\partial \psi}{\partial \nu}\right\rangle_{L^2(\partial\Omega)} + \langle x, A\psi\rangle,$$

Since A is self-adjoint, we conclude by Proposition 2.9 that B^* equals the Neumann boundary trace operator γ_1 for which $\gamma_1 \in \mathcal{L}(H^\beta(\Omega), L^2(\partial\Omega))$ for any $\beta > \frac{3}{2}$, see e.g. [44, Appendix]. In terms of the abstract Sobolev spaces X_α this means that $B^* \in \mathcal{L}(X_\beta, U^*)$ for any $\beta > \frac{3}{4}$, see e.g. [28]. Since the Dirichlet Laplacian on $L^2(\Omega)$ generates an exponentially stable analytic semigroup, by Proposition 2.13 the system is L^q-ISS with respect for any $q > (1 - \frac{3}{4})^{-1} = 4$. Thus,

$$\|x(t)\|_{L^2(\Omega)} \lesssim \mathrm{e}^{-\lambda_0 t}\,\|x_0\|_{L^2(\Omega)} + \|u\|_{L^q(0,t;L^2(\partial\Omega))}$$

for all $t > 0$, some $\lambda_0 < 0$ and all $u \in L^q(0,t;L^2(\partial\Omega))$.

As seen above, Proposition 2.13 L^q-ISS for parabolic equation provided sufficient properties of the boundary operator can be shown. In concrete situations this typically reduces to knowledge of boundary traces. Let us briefly elaborate on what can be said in situations where this information is not accessible. Furthermore, one may also ask the question whether at all boundary systems exist which are not L^q-ISS for any finite q. Let us first answer this positively with a, admittedly pathologic, example.

Example 2.17. Let $X = \ell^2(\mathbb{N})$ be the space of complex-valued, square-summable sequences and let $(e_n)_{n\geq 1}$ denote the canonical orthonormal basis. Define $\mathfrak{A} : D(\mathfrak{A}) \to X$ and $\mathfrak{B} : D(\mathfrak{A}) \to \mathbb{C}$ by

$$\begin{aligned} D(\mathfrak{A}) &= \left\{ x \in \ell^2(\mathbb{N})\colon \exists c_x \in \mathbb{C} \text{ such that } \sum_{n=1}^\infty \left| -2^n \langle x, e_n\rangle + \frac{c_x 2^n}{n}\right|^2 < \infty \right\}, \\ \mathfrak{A}x &= \sum_{n=1}^\infty \left(-2^n \langle x, e_n\rangle e_n + \frac{c_x 2^n}{n}\right), \\ \mathfrak{B}x &= c_x. \end{aligned}$$

To see that $\mathfrak{A}$ and $\mathfrak{B}$ are well-defined, suppose that $x \in X$ and $c_x, \widetilde{c}_x \in \mathbb{C}$. Then it holds that

$$\sum_{n=1}^{\infty} \left| -2^n \langle x, e_n \rangle + \frac{c_x 2^n}{n} \right|^2 < \infty, \quad \sum_{n=1}^{\infty} \left| -2^n \langle x, e_n \rangle + \frac{\widetilde{c}_x 2^n}{n} \right|^2 < \infty.$$

By triangle inequality (in $\ell^2(\mathbb{N})$), it follows that $|c_x - \widetilde{c}_x|^2 \sum_{n=1}^{\infty} \frac{2^{2n}}{n^2} < \infty$, and hence $c_x = \widetilde{c}_x$. Similarly, it follows that both operators are linear. Since $(\frac{1}{n})_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, it is clear that $\mathfrak{B}$ possesses a right-inverse, e.g. given by $B_0 c = c \sum_{n=1}^{\infty} \frac{1}{n} e_n$, $c \in \mathbb{C}$. The operator $A = \mathfrak{A}|_{\ker \mathfrak{B}}$ is given through $Ae_n = -2^n e_n$, $n \in \mathbb{N}$ on its maximal domain. This operator generates an exponentially stable, analytic semigroup T determined by $T(t)e_n = \mathrm{e}^{-2^n t}$, $n \in \mathbb{N}$. Thus, $(\mathfrak{A}, \mathfrak{B})$ constitute a boundary control system. The operator $B = \mathfrak{A}B_0 - A_{-1}B_0$ thus becomes $Bc = -c\sum_{n=1}^{\infty} \frac{2^n}{n} e_n$, which has to be interpreted as an operator from $\mathbb{C}$ to

$$X_{-1} = \left\{ \sum_{n=1}^{\infty} x_n e_n \colon \left(\frac{x_n}{2^n} \right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \right\}.$$

In [19, Example 5.2], which in turn was based on a result from [20], it was shown that the system $\Sigma(A, B)$ of the form (2.11) is not L^q-ISS for any $q < \infty$. In particular, this implies that for any $q < \infty$ there exists a time t_0 and a sequence of continuously differentiable functions $u_m : [0, \infty) \to \mathbb{C}$ such that

- $\sup_{m\in\mathbb{N}} \left\| u_m \right\|_{L^q(0,t_0)} < \infty$ and
- the classical solution $x_m : [0, \infty) \to X$ to the boundary control system $(\mathfrak{A}, \mathfrak{B})$ with initial value $x_0 = 0$ and input function u_m satisfy

$$\lim_{m\to\infty} \left\| x_m(t_0) \right\|_X \to \infty.$$

However, the boundary control system is L^∞-ISS, by the upcoming Theorem 2.18.

Theorem 2.18. *Let $(\mathfrak{A}, \mathfrak{B})$ be a boundary control system on a Hilbert space with associated operator A. If the following assumptions are satisfied:*

- *A generates an exponentially stable, analytic semigroup, and*
- *there exists an equivalent scalar product $\langle \cdot, \cdot \rangle_{new}$ on X such that A is dissipative, i.e. $\mathrm{Re}\langle Ax, x\rangle_{new} \leq 0$,*
- *the range of $\mathfrak{B}$ is finite-dimensional,*

then $(\mathfrak{A}, \mathfrak{B})$ is L^∞-ISS and the (mild) solutions are continuous for all

$$(x_0, u) \in X \times L^\infty_{\mathrm{loc}}(\mathbb{R}_+; U).$$

Moreover, there exist positive constants $C_1, C_2, \omega,$ and ϵ, as well as a strictly increasing, smooth, convex function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0, \lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$ such that

$$\left\| x(t) \right\|_X \leq C_1 \, \mathrm{e}^{-\omega t} \left\| x_0 \right\|_X + C_2 \, \mathrm{e}^{-\epsilon t} \inf \left\{ k \geq 0 \colon \int_0^t \Phi\left(\frac{\mathrm{e}^{s\epsilon} \, \|u(s)\|_U}{k} \right) \mathrm{d}s \leq 1 \right\} \tag{2.13}$$

for any mild solution x, $t > 0$, $u \in L^\infty_{\mathrm{loc}}(0, \infty)$ and $x_0 \in X$.

Proof. This is a direct consequence of the results in [21] where similar results were stated for systems of the form (2.11). It remains to observe the following. Because of the assumed dissipativity, the semigroup T is similar to a contraction semigroup. Since $\mathfrak{B}$ has a right-inverse, it follows that $\dim U = \dim \operatorname{ran} \mathfrak{B} < \infty$. Hence, $B = (\mathfrak{A} - A)B_0$ is an operator from a finite-dimensional space to X_{-1}. In order to derive Estimate (2.13), we use a rescaling argument: Let $\epsilon > 0$ such that $\widetilde{T} = \mathrm{e}^{\epsilon \cdot} T$ is exponentially stable and consider the boundary control system $(\mathfrak{A} + \epsilon I, \mathfrak{B})$. Note that the spaces X_{-1} and the corresponding one for $A + \epsilon I$, the generator of $\widetilde{T}$, coincide and also $B = (\mathfrak{A} - A_{-1})B_0 = (\mathfrak{A} + \epsilon I - A_{-1} - \epsilon I)B_0$. Corollary 21 and Theorem 19 from [21] show that there exist positive constants $\widetilde{C}_1, \widetilde{C}_2$ and ω and a function Φ with the properties described in the statement of the theorem such that

$$\left\|\widetilde{x}(t)\right\|_X \leq \widetilde{C}_1 \,\mathrm{e}^{-\omega t} \left\|x_0\right\|_X + \widetilde{C}_2 \inf \left\{ k \geq 0\colon \int_0^t \Phi\left(\frac{\left\|\widetilde{u}(s)\right\|_U}{k}\right) \mathrm{d}s \leq 1 \right\} \tag{2.14}$$

for any mild solution $\widetilde{x}$ of $(\mathfrak{A} + \epsilon I, \mathfrak{B})$ and $t > 0$, $\widetilde{u} \in L^\infty_{\mathrm{loc}}(0, \infty)$ and $x_0 \in X$. On the other hand it follows from the representation (2.8), that any mild solution x to $(\mathfrak{A}, \mathfrak{B})$ with input function u, the function $\widetilde{x}(t) = \mathrm{e}^{\epsilon t}\, x(t)$ defines a mild solution of the boundary control problem $(\mathfrak{A} + \epsilon I, \mathfrak{B})$ with input function $\widetilde{u} = \mathrm{e}^{\cdot \epsilon}\, u$. Combining this with (2.14) shows (2.13).

To see that (2.13) implies that $(\mathfrak{A}, \mathfrak{B})$ is L^∞-ISS, we show that there exists a constant C_3 such that for all $t > 0$,

$$\int_0^t \Phi\left(\frac{\mathrm{e}^{s\epsilon} \left\|u(s)\right\|_U}{C_3 \,\mathrm{e}^{\epsilon t} \left\|u\right\|_{L^\infty(0,t;U)}}\right) \mathrm{d}s \leq 1.$$

Since Φ is strictly increasing it thus suffices to show that

$$\sup_{t>0} \int_0^t \Phi\left(\mathrm{e}^{-s\epsilon}\, C_3^{-1}\right) \mathrm{d}s \leq 1,$$

which follows easily by the property that $\lim_{x \to 0} \frac{\Phi(x)}{x} = 0$. □

Remark 2.19. The following remarks can be made about Theorem 2.18:

- Let us point out that (2.13) is indeed stronger than the corresponding estimate with $\epsilon = 0$: By monotonicity of Φ,

$$\mathrm{e}^{-\epsilon t} \inf \left\{ k \geq 0\colon \int_0^t \Phi\left(\frac{\mathrm{e}^{s\epsilon} \left\|u(s)\right\|_U}{k}\right) \mathrm{d}s \leq 1 \right\}$$
$$\leq \inf \left\{ k \geq 0\colon \int_0^t \Phi\left(\frac{\left\|u(s)\right\|_U}{k}\right) \mathrm{d}s \leq 1 \right\}.$$

  Furthermore, in case that Φ can be chosen as $\Phi(x) = x^q$, $x \in [0, \infty)$ for $q \in (1, \infty)$, the estimate reduces to an L^q-ISS estimate.

- The BCS in Example 2.17 satisfies the assumptions of Theorem 2.18 as can be checked by the explicit expression for the semigroup. However, the function Φ cannot be taken of the form $\Phi(x) = x^q$ for any $q < \infty$, [19, Example 5.2].
- The assumption that there exists an equivalent scalar product such that A is dissipative is rather weak from a practical point of view: Most known practically-relevant examples of differential operators satisfy this condition, [27] which can be rephrased as the property that the semigroup is similar to a contraction semigroup. However, it is not difficult to construct counterexamples assuring that not every analytic semigroup on a Hilbert space is similar to a contractive one. This can be done by diagonal operators with respect to a (Schauder) basis which is not a Riesz basis, [4, 15].

Example 2.20. Let $(\mathfrak{A}, \mathfrak{B})$ be a boundary control system with $\dim U < \infty$ and A being a Riesz-spectral operator, i.e. $A = S^{-1}\Lambda S$ for a bijective operator $S \in \mathcal{L}(X)$ and a densely defined closed operator $\Lambda : D(\Lambda) \subset X \to X$ with discrete spectrum $\sigma(\Lambda)$ contained in a left-half-plane of the complex plane and such that the eigenvectors establish an orthonormal basis of X, where we also assume that the eigenvalues are pairwise distinct. By Parseval's identity, it follows that A is dissipative with respect to the scalar product $\langle S\cdot, S\cdot\rangle$. If, moreover, it is assumed that $\sigma(A) = \sigma(\Lambda)$ is contained in a sector $S_\theta := \{z \in \mathbb{C}\colon \arg(z) \leq \theta\}$ with $\theta < \frac{\pi}{2}$, then A generates an analytic semigroup, which is exponentially stable if and only if $\sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(A)\} < 0$. For details on Riesz-spectral operator we refer for instance to [7]. Therefore, the conditions of Theorem 2.18 are satisfied and $(\mathfrak{A}, \mathfrak{B})$ is L^∞-ISS for input data $(x_0, u) \in X \times L^\infty_{\mathrm{loc}}(\mathbb{R}_+; U)$. See also [30] and [18, 19] for different proofs of this fact. In particular in the latter, more generally q-Riesz-spectral operators are considered.

3. A primer on semilinear boundary control systems

In the following we extend the linear systems considered in Section 2 to semilinear ones. As motivating example serves (1.5). The abstract theory of semilinear PDEs (without controls/disturbances) with our without using semigroups is comparably old and can be found e.g. in the textbooks [16, 38]. There is a particularly rich theory for parabolic equations as smoothing effect of the linear part through the analytic semigroups allows for rather general nonlinearities. In the following we are interested in ISS estimates similar to the ones we derived for linear systems: This includes the property that the undisturbed system is uniformly asymptotically stable which requires already restrictive conditions on the nonlinearity, particularly, if we aim for abstract results covering whole classes of examples. The simplest condition guaranteeing this global stability is a global Lipschitz condition with sufficiently small Lipschitz constant, as we shall see in Theorem 3.3. There it is shown that the usual proof technique to assess uniform global asymptotic stability for uncontrolled systems also goes through for boundary control systems using the results we discussed in Section II. The final result of this section is Theorem 3.4, which provides a generalization of the findings in [48].

Definition 3.1 (Semilinear boundary control system). Let $(\mathfrak{A}, \mathfrak{B})$ be a linear boundary control system with state space X and input space U. Denote by A the associated semigroup generator and by B_0 a right-inverse of $\mathfrak{B}$. Further let

- $\alpha \in [0, 1)$ if A generates an analytic semigroup, or
- $\alpha = 0$ else (in which case we set $X_0 = X$).

Let $f : \mathbb{R}_+ \times X_\alpha \to X$ be a function continuous in the first variable and locally Lipschitz in the second variable with respect to the norm X_α. Then the triple $(\mathfrak{A}, \mathfrak{B}, f)$ formally representing the equations

$$\begin{cases} \dot{x}(t) = \mathfrak{A}x(t) + f(t, x(t)), \\ \mathfrak{B}x(t) = u(t), \\ x(0) = x_0, \end{cases} \tag{3.1}$$

$t > 0$, is called a *semilinear boundary control system.*

Let $x_0 \in D(\mathfrak{A})$, $T > 0$ and $u \in C([0, T]; U)$. A function

$$x \in C([0, T]; D(\mathfrak{A})) \cap C^1([0, T]; X)$$

is called a *classical solution* to the nonlinear BCS (3.1) on $[0, T]$ if $x(t) \in X_\alpha$ for all $t > 0$ and the equations (3.1) are satisfied pointwise for $t \in (0, T]$. A function $x : [0, \infty) \to X$ is called *(global) classical solution* to the BCS, if $x|_{[0,T]}$ is a classical solution on $[0, T]$ for every $T > 0$. If $x \in C([0, T]; D(\mathfrak{A})) \cap C^1((0, T]; X)$ and $x(t) \in X_\alpha$ for all $t > 0$ and the equations (3.1) are satisfied pointwise for $t \in (0, T]$, then we say that x is a classical solution on $(0, T]$.

Similar as in the previous section, we can define mild solutions.

Definition 3.2 (Mild solutions of semilinear boundary control systems). Suppose $(\mathfrak{A}, \mathfrak{B}, f)$ is a semilinear boundary control system with associated $A, B_0, \alpha \in [0, 1)$. Let $x_0 \in X$, $T > 0$ and $u \in L^1_{\mathrm{loc}}([0, T]; U)$. A continuous function $x : [0, T] \to X$ is called *mild solution* to the BCS (3.1) on $[0, T]$ if $x(t) \in X_\alpha$ for all $t > 0$ and x solves

$$x(t) = T(t)x_0 + \int_0^t T(t - s)\,[f(s, x(s)) + Bu(s)]\,\mathrm{d}s, \tag{3.2}$$

for all $t \in [0, T]$ and where $B = \mathfrak{A}B_0 - A_{-1}B_0$. A function $x : [0, \infty) \to X$ is called a *global mild solution* if $x\big|_{[0,T]}$ is a mild solution on $[0, T]$ for all $T > 0$.

It is not hard to see that the definition 3.2 coincides with the one for linear BCS in case that $f(t, x) = Cx$ for any bounded operator $C : X \to X$, or, more generally, when C is unbounded and $A + C$ generate a strongly continuous semigroup. Moreover, any (global) classical solution is a (global) mild solution.

The following result is not very surprising as it shows that a semilinear system is ISS if the linear subsystem is ISS and the nonlinearity is globally Lipschitz.

Theorem 3.3. *Let* $(\mathfrak{A}, \mathfrak{B})$ *be a boundary control system which is assumed to be* $\mathcal{Z}$*-ISS where* $\mathcal{Z}$ *refers to either* L^q *with* $q < \infty$*,* C *or* Reg*. Let* $M \geq 1$ *and* $\omega < 0$ *be such that for the associated semigroup* T *it holds that* $\|T(t)\| \leq M\,\mathrm{e}^{\omega t}$ *for all* $t > 0$*. Furthermore, let* $f : \mathbb{R}_+ \times X \to X$ *be continuous in the first and uniformly Lipschitz continuous in the second variable with Lipschitz constant* $L_f > 0$ *and* $f(t, 0) = 0$ *for all* $t \geq 0$*. If*

$$\omega + ML_f < 0,$$

then the semilinear boundary control system $\Sigma(\mathfrak{A}, \mathfrak{B}, f)$ *is* $\mathcal{Z}$*-ISS. More precisely, for any* $x_0 \in X$ *and* $u \in \mathcal{Z}(\mathbb{R}_+; U)$ *System* (3.1) *has a unique global mild solution* $x \in C([0, \infty); X)$*. Furthermore, there exist* $\beta \in \mathcal{KL}$ *and a constant* $\sigma > 0$ *such that for all* $t > 0$*,* $x_0 \in X$ *and* $u \in \mathcal{Z}([0, t]; U)$*,*

$$\big\|x(t)\big\|_X \leq \beta(\|x_0\|, t) + \sigma\big\|u\big\|_{\mathcal{Z}([0,t];U)}, \tag{3.3}$$

thus $\Sigma(A, B, f)$ *is* $\mathcal{Z}$*-ISS.*

Proof. The proof follows the lines of a standard technique for semilinear equations with (global) Lipschitz continuous nonlinearity. For fixed $u \in \mathcal{Z}_{\mathrm{loc}}(0, \infty; U)$ and $x_0 \in X$, it follows from the assumed ISS that the mapping

$$t \mapsto g(t) := T(t)x_0 + \int_0^t T(t - s)Bu(s)\mathrm{d}s$$

is continuous from $[0, \infty)$ to X. Indeed, the continuity follows by [43, Theorem 4.3.2] (noting that ISS implies $\mathcal{Z}$-admissibility/$\mathcal{Z}$-well-posedness). The existence of a mild solution to $\Sigma(A, B, f)$ is equivalent to the existence of a fixed-point $x \in C([0, \infty); X)$ of

$$x(t) = g(t) + \int_0^t T(t - s)f(s, x(s))\mathrm{d}s, \quad t \in [0, \infty).$$

The latter follows from [38, Corollary 6.1.3] by the assumptions on f and the continuity of g. The ISS property can now be shown by a Gronwall-type argument: Since the linear boundary control system is ISS, there exists $\sigma > 0$ such that

$$\|g(t)\| \leq M\,\mathrm{e}^{\omega t}\,\|x_0\| + \sigma\big\|u\big\|_{\mathcal{Z}([0,t];U)}, \quad t > 0,$$

where M and ω are chosen as in the statement of the theorem. By the definition of the mild solution,

$$\begin{aligned}\|x(t)\| &\leq \|g(t)\| + \int_0^t M\,\mathrm{e}^{(t-s)\omega}\,\|f(s, x(s)\|\mathrm{d}s \\ &\leq \|g(t)\| + ML_f\,\mathrm{e}^{t\omega}\int_0^t \mathrm{e}^{-\omega s}\,\|x(s)\|\mathrm{d}s.\end{aligned}$$

Now Gronwall's inequality implies that

$$\begin{aligned}
\|x(t)\| &\le \|g(t)\| + \mathrm{e}^{t\omega} \int_0^t \mathrm{e}^{-\omega s} ML_f \|g(s)\| \,\mathrm{e}^{ML_f(t-s)} \,\mathrm{d}s \\
&\le \|g(t)\| + ML_f \,\mathrm{e}^{t(\omega+ML_f)} \left(\int_0^t M \,\mathrm{e}^{-ML_f s} \,\mathrm{d}s \|x_0\| + \right. \\
&\quad \left. + \sigma \|u\|_{\mathcal{Z}([0,t];U)} \int_0^t \mathrm{e}^{-\omega s - ML_f s} \,\mathrm{d}s \right) \\
&= M \,\mathrm{e}^{t(\omega+ML_f)} \|x_0\| + \left[ML_f \frac{\mathrm{e}^{t\omega+ML_f t} - 1}{\omega + ML_f} + 1 \right] \sigma \|u\|_{\mathcal{Z}([0,t];U)}.
\end{aligned}$$

Since the coefficient of the second term on the right hand side is bounded in t, the assertion follows. □

It is trivially seen that the condition on the Lipschitz constant is in general sharp as the finite-dimensional example

$$\dot{x} = -x + 2x + u,$$

with $f(x) = 2x$, shows. On the other hand, the slight adaption $X = \mathbb{R}$, $A = 1$, $f(x) = -2x$, $B = 0$ shows, that the result is not optimal in the sense that the "sign" of the nonlinearity is crucial for asymptotic stability.

Theorem 3.4. *Let $(\mathfrak{A}, \mathfrak{B}, f)$ be a semilinear boundary control system with associated operators A and B_0. Let the following be satisfied for the linear system $(\mathfrak{A}, \mathfrak{B})$:*

- (i) *the operator $A = \mathfrak{A}\big|_{\ker \mathfrak{B}}$ is self-adjoint and bounded from above by $\omega_A \in \mathbb{R}$, i.e. $\langle Ax, x\rangle \le \omega_A$ for all $x \in D(A)$,*
- (ii) *$B \in \mathcal{L}\big(U, X_{-\frac{1}{2}}\big)$, where $B := (\mathfrak{A} - A)B_0$.*

Furthermore, the function $f : [0,\infty) \times X_{\frac{1}{2}} \to X$ satisfies the following properties:

- (1) *f is locally Hölder continuous in the first and Lipschitz in the second variable, i.e. for any $(t,x) \in \mathbb{R}_+ \times X_{\frac{1}{2}}$ there exists $L > 0$, $\theta \in (0,1)$, $\rho > 0$ such that*

$$\|f(t,x) - f(s,y)\| \le L\big(|t-s|^\theta + \|x-y\|_{\frac{1}{2}}\big)$$

  for all (s,y) in the ball $B_\rho(t,x)$ in $\mathbb{R}_+ \times X$ with radius ρ and centre (t,x);
- (2) *there exists a continuous, nondecreasing function $k : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|f(t,x)\| \le k(t)\big(1 + \|x\|_{\frac{1}{2}}\big), \quad \forall (t,x) \in \mathbb{R}_+ \times X;$$

- (3) *there exist constants $m_1, m_2 \in \mathbb{R}$ such that for any $(t,x) \in \mathbb{R}_+ \times X_{\frac{1}{2}}$ it holds that $\langle f(t,x), x\rangle \in \mathbb{R}$ and*

$$\langle f(t,x), x\rangle \le -m_1 \langle Ax, x\rangle + m_2 \|x\|^2;$$

- (4) *above constants satisfy the inequality*

$$1 - m_1 > 0 \quad \text{and} \quad (1 - m_1)\omega_A + m_2 < 0.$$

Then, for any $x_0 \in X_{\frac{1}{2}}$ *and* $u \in W^{2,1}(\mathbb{R}_+;U)$ *with* $A_{-1}x_0 + Bu(0) \in X$*, the semilinear boundary control system* (3.1) *has a unique mild solution* x*, which is classical on* $(0,\infty)$*, and* $(\mathfrak{A},\mathfrak{B},f)$ *is* L^q*-ISS for any* $q \geq 2$*. More precisely, for any* $q \geq 2$ *there exist constants* $C_1, C_2, \omega > 0$ *such that for all* $(t,x_0,u) \in \mathbb{R}_+ \times X_\alpha \times W^{2,1}(\mathbb{R}_+;U)$ *with* $A_{-1}x_0 + Bu(0) \in X$ *the solution* x *satisfies*

$$\|x(t)\|_X \leq C_1 \, \mathrm{e}^{-\omega t} \, \|x_0\|_X + C_2 \|u\|_{L^q(0,t;U)}.$$

Proof. First note that — upon considering $\widetilde{\mathfrak{A}} = \mathfrak{A} - \omega_A - \epsilon$ and $\widetilde{f}(s,x) = f(s,x) + (\omega_A + \epsilon)x$ we can without loss of generality assume that $\omega_A < 0$ and thus that the semigroup is exponentially stable.
In order to show existence and uniqueness of the solutions, we closely follow the proof of the classical result in [38, Theorem 6.3.1 and Theorem 6.3.3] which has to be adapted to allow for boundary inputs u. Under the made assumptions on A and f, it follows by [38, Theorem 6.3.1], that the uncontrolled system, $u \equiv 0$, has a unique local classical solution for any $x_0 \in X_{\frac{1}{2}}$, which, by the assumption (2) and [38, Theorem 6.3.3], extends to a global solution. The key argument for local existence [38, Theorem 6.3.1] is to consider the unique solution y of

$$y(t) = T(t)(-A)^{\frac{1}{2}}x_0 + \int_0^t (-A)^{\frac{1}{2}} T(t-s) f(s, (-A)^{-\frac{1}{2}} y(s)) \mathrm{d}s \tag{3.4}$$

for $t \in [0,\tau]$, where $\tau > 0$ and to show that $t \mapsto y(t)$ is Hölder continuous on $(0,\tau)$, so that the sought solution is given by the solution of

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(t, (-A)^{-\frac{1}{2}} y(t)), \\ x(0) &= x_0. \end{aligned} \tag{3.5}$$

To apply an analogous reasoning in the controlled case, $u \neq 0$, it remains to adapt (3.4) and (3.5) by adding the terms $\int_0^t (-A)^{\frac{1}{2}} T(t-s) Bu(s) \mathrm{d}s$ and $Bu(t)$ to the right-hand sides, respectively. Since $B \in \mathcal{L}(U; X_{-\frac{1}{2}})$, we have that $\widetilde{B} := (-A)^{-\frac{1}{2}} B \in \mathcal{L}(U;X)$ and thus

$$\begin{aligned} t \mapsto \int_0^t (-A)^{\frac{1}{2}} T(t-s) Bu(s) \mathrm{d}s &= -\int_0^t AT(t-s)\widetilde{B}u(s)\mathrm{d}s \\ &= -\int_0^t T(t-s)\widetilde{B}\dot{u}(s)\mathrm{d}s + T(t)\widetilde{B}u(0) - \widetilde{B}u(t) \end{aligned}$$

is a continuous function on $[0,\infty)$ and, by the analyticity of the semigroup, even Hölder continuous on $(0,\infty)$. Therefore, analogously to the proof of [38, Theorem 6.3.1], we conclude that the equation

$$y(t) = T(t)(-A)^{\frac{1}{2}}x_0 + \int_0^t (-A)^{\frac{1}{2}} T(t-s) \left[f(s, (-A)^{-\frac{1}{2}} y(s)) + Bu(s) \right] \mathrm{d}s, \tag{3.6}$$

allows for a unique continuous solution $y : [0,\tau] \to X$ for some $\tau > 0$ such that $t \mapsto f(t, (-A)^{-\frac{1}{2}} y(t))$ is Hölder continuous on $(0,\tau)$. Therefore, and since

$u \in W^{2,1}(\mathbb{R}_+;U)$ with $A_{-1}x_0 + Bu(0) \in X$, the mild solution $x \in C([0,\tau];X)$ of

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f\big(t, (-A)^{-\frac{1}{2}}y(t)\big) + Bu(t), \\ x(0) &= x_0 \end{aligned} \tag{3.7}$$

is in fact a classical solution on $(0,\tau)$, [38, Corollary 4.3.3] and [44, Proposition 4.2.10]. From the representation of the mild solution of (3.7),

$$x(t) = T(t)x_0 + \int_0^t T(t-s)\left[f\big(s, (-A)^{-\frac{1}{2}}y(s)\big) + Bu(s)\right] \mathrm{d}s,$$

it moreover follows that $x(t) = (-A)^{-\frac{1}{2}}y(t)$ and thus, x is a mild solution of the original boundary control problem (3.1) on $[0,\tau]$ and even a classical solution on $(0,\tau)$. From assumption (4), it follows that x remains bounded in the $\|\cdot\|_{\frac{1}{2}}$-norm on $[0,\tau)$, so that, by iterating the argument, x can be extended to a global solution, see [38, Theorem 6.3.3].

We now show the L^q-ISS estimate. Let x be the mild solution to an initial value $x_0 \in X_{\frac{1}{2}}$. Since x is a classical solution on $(0,\infty)$, we have for any $t > 0$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 = \langle Ax(t), x(t)\rangle + \langle f(t,x(t)), x(t)\rangle + \mathrm{Re}\langle u(t), B^*x(t)\rangle.$$

Therefore, by Assumption (3) and noting that

$$\|x\|_{\frac{1}{2}}^2 = \left\langle (-A)^{\frac{1}{2}}x, (-A)^{\frac{1}{2}}x\right\rangle = -\langle Ax, x\rangle \geq -\omega_A\|x\|^2, \tag{3.8}$$

it follows that for any $t > 0$ and sufficiently small $\epsilon > 0$

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 &\leq (1-m_1)\langle Ax(t), x(t)\rangle + m_2\|x(t)\|^2 + \left|\left\langle u(t), B^*x(t)\right\rangle_{U\times U}\right| \\ &\leq (1-m_1-\epsilon)\langle Ax(t), x(t)\rangle + m_2\|x(t)\|^2 + \frac{1}{4\epsilon}\left\|B^*\right\|^2_{\mathcal{L}(X_{\frac{1}{2}},U)}\|u(t)\|^2 \\ &\leq ((1-m_1-\epsilon)\omega_A + m_2)\|x(t)\|^2 + \frac{1}{4\epsilon}\left\|B^*\right\|^2_{\mathcal{L}(X_{\frac{1}{2}},U)}\|u(t)\|^2, \end{aligned} \tag{3.9}$$

where we used (3.8) and Assumption (4) in the last inequality. Gronwall's inequality now yields the assertion for $q = 2$ and an additional application of Hölder's inequality the one for $q > 2$. □

Remark 3.5. Theorem 3.4 is a generalization of the result in [48] where only the Laplacian with Robin/Neumann boundary control (excluding Dirichlet control) in one spatial variable was considered and the assumptions on f were tuned to guarantee the existence of classical solutions. We decided to give a full proof (or at least a sketch of the necessary adaptations from [38]) of the existence of solutions for the convenience of the reader, but also since the classical literature on semilinear PDEs does not cover the presence of the inputs.[5] The assumption that the inputs should lie $W^{2,1}(\mathbb{R})$ with the additional property that $A_{-1}x_0 + Bu(0) \in X$ is clearly tuned in order to guarantee for classical solutions (in $(0,\infty)$), cf. [44,

[5]At least the author is not aware of any explicit reference in this operator-theoretic framework.

Proposition 4.2.10]. This, however, can be weakened with a more careful analysis on the regularity of the solutions and by deriving (3.9) only for almost every $t > 0$. Although our proof follows standard arguments in the semigroup approach to semilinear equations instead, the derivation of the ISS estimate can be seen as abstraction of the procedure in [48]. Recall that it is well-known that the corresponding boundary operator B in the situation of Neumann or Robin control in [48] satisfies the condition $B \in \mathcal{L}(U, X_{-\frac{1}{2}})$, see also Example 2.14.

Example 3.6 (Semilinear parabolic equation with cubic nonlinearity). Let $\Omega \subset \mathbb{R}^n$ with $n \in \{1,2,3\}$. Under the setting of Example 2.14 consider

$$\begin{aligned} \dot{x}(\xi,t) &= \Delta x(\xi,t) - ax(\xi,t) - x(\xi,t)^3 + d(\xi,t), && (\xi,t) \in \Omega \times (0,\infty), \\ \frac{\partial x}{\partial \nu}(\xi,t) &= u(\xi,t) && (\xi,t) \in \partial\Omega \times (0,\infty), \\ x(\xi,0) &= x_0(\xi), && \xi \in \Omega, \end{aligned}$$

which establishes a semilinear BCS $(\mathfrak{A}, \mathfrak{B}, f)$ with $f(x) = -x^3$ and the same operators $\mathfrak{A}, \mathfrak{B}$, A, B as in Example 2.14. As seen in the previous example, $(\mathfrak{A}, \mathfrak{B})$ is a linear boundary control system for $d = 0$ and, in the generalized sense of Remark 2.10, for $d \neq 0$. The conditions (i) and (ii) of Theorem 3.4 are satisfied with $\omega_A = -a$. Conditions (1) and (2) both follow from the Sobolev embedding $W^{1,2}(\Omega) \subseteq L^6(\Omega)$ valid for $n \in \{1,2,3\}$, see e.g. [1], and the fact that $X_{\frac{1}{2}} = W^{1,2}(\Omega)$, see e.g. [28].

4. Concluding remarks and outlook

In the situation of Dirichlet boundary control and the choice $X = L^2(\Omega)$ for the state space, it is well-known that an L^2-ISS-estimate (in time) cannot be expected. More precisely, even for a linear heat equation the input operator represented by Dirichlet boundary control is not L^2-admissible if the state space is $L^2(\Omega)$, see [31, p. 217] for a counterexample. Instead, as we have seen in Example 2.16, we only have L^p-ISS for $p > 4$ in general, see also [11, Proposition 5.1] for another proof in the case that $p = \infty$. Therefore, the results of Section 3 cannot be applied and the situation becomes more involved. The question is if Lyapunov arguments such as used in Theorem 3.4 can at all be used to assess ISS in situations which are not L^2-ISS. A work-around — typical in the theory of linear L^2-well-posed systems [44]— is as follows: If in the setting of Example 3.6 one considers Dirichlet boundary control instead of Neumann boundary control, we could change the considered state space X to be the Sobolev space $H^{-1}(\Omega)$ in order to obtain L^2-ISS, i.e.

$$\|x(t)\|_{H^{-1}(\Omega)} \lesssim \mathrm{e}^{-at} \|x_0\|_{H^{-1}(\Omega)} + \|u\|_{L^2(0,t;L^2(\partial\Omega))}.$$

On the other hand, if we aim for L^∞-ISS estimates only, other techniques may be more suitable; such as the maximum principle methods in [33]. These methods, however, seem to be practical only for L^∞-ISS estimates.

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Null-controllability and control cost estimates for the heat equation on unbounded and large bounded domains

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Abstract. We survey recent results on the control problem for the heat equation on unbounded and large bounded domains. First we formulate new uncertainty relations, respectively spectral inequalities. Then we present an abstract control cost estimate which improves upon earlier results. The latter is particularly interesting when combined with the earlier mentioned spectral inequalities since it yields sharp control cost bounds in several asymptotic regimes. We also show that control problems on unbounded domains can be approximated by corresponding problems on a sequence of bounded domains forming an exhaustion. Our results apply also for the generalized heat equation associated with a Schrödinger semigroup.

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1. Introduction

We survey several new results on the control problem of the heat equation on unbounded and large bounded domains. The study of heat control on bounded domains has a long history, while unbounded domains became a focus of interest only quite recently. In order to compare and interpolate these two geometric situations it is natural to study the control problem on large bounded domains including a quantitative and explicit analysis of the influence of the underlying geometry. Here the term 'large domain' may be made precise in at least two ways. For instance, it could mean that we study the control problem on a sequence of domains which form an exhaustion of the whole Euclidean space. Alternatively, it could

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mean that the considered domain is large compared to some characteristic length scale of the system, e.g. determined by the properties of the control/observability set. Not surprisingly, the results on unbounded and large bounded domains which we present draw on concepts and methods which have been developed initially for control problems on generic bounded domains. While these previous results focused on giving precise criteria for (null-)controllability to hold, only a partial analysis of the influence of the underlying geometry on the control cost has been carried out. Merely the dependence on the time interval length in which the control is allowed to take place has been studied thoroughly. However, recently there has been an increased interest in the role of geometry for the control cost. We survey a number of recent results which perform a systematic analysis of the dependence of control cost estimates on characteristic length scales of the control problem. As a side benefit we obtain new qualitative results, most prominently a sharp, i.e. sufficient and necessary, condition on the control/observability set which ensures the null-controllability of the classical heat equation on the whole of $\mathbb{R}^d$.

The results on null-controllability, in accordance with previous proofs, are obtained in two steps. The first consists in some hard analysis and depends on the specific partial differential equation at hand whereas the second one can be formulated in an abstract operator theoretic language. Let us discuss these two ingredients separately.

The mentioned hard analysis component of the proof consists in a variant of the *uncertainty relation* or *uncertainty principle*. These terms stem from quantum physics and encode the phenomenon that the position and the momentum of a particle cannot be measured simultaneously with arbitrary precision. Note that the momentum representation of an observable is obtained from the position representation via the Fourier transform. Hence the fact that a non-trivial function and its Fourier transform cannot be simultaneously compactly supported is a particular manifestation of the uncertainty principle. This qualitative theorem can be given a quantitative form in various ways, e.g. by the Paley–Wiener Theorem or the Logvinenko–Sereda Theorem which we discuss in Section 2. If the property that a function has compactly supported Fourier transform is replaced by some similar restriction, for instance that it is an element of a spectral subspace of a self-adjoint Hamiltonian describing the total energy of the system, other variants of the uncertainty relation are obtained. In the particular case that the Hamiltonian is represented by a second order elliptic partial differential operator with sufficiently regular coefficients a particular instance of an uncertainty principle is embodied in (a quantitative version of) the *unique continuation principle*. The latter states that an eigenfunction (or, more generally a finite linear combination of eigenfunctions or elements from spectral subspaces associated to bounded energy intervals) cannot vanish in the neighborhood of a point faster than a specified rate. Such a quantitative unique continuation estimate in turn implies what is called a *spectral inequality* in the context of control theory. This term was first coined for evolutions determined by the Laplace operator but is now used also for abstract systems. Thus, it is hardly distinguishable from the notion of an uncertainty relation. Note however that the term spectral inequality is used in other areas of mathematics with a different meaning, e.g. in Banach algebras or matrix analysis.

The second mentioned step uses operator theoretic methods and ODEs in Hilbert space to deduce observability and controllability results from the hard analysis bound obtained in the first step. There are several related but distinct approaches to implement this. One of them we present in full for pedagogical reasons. The other ones are not developed in this paper, but we discuss the resulting quantitative bounds on the control cost. In fact, these seem to be better than what can be obtained by the mentioned pedagogical approach.

Let us point out several special features of this survey (and the underlying original research articles): The uncertainty principles or spectral inequalities, and consequently the implied control cost estimates, which we develop, are scale-free. This means that the same bound holds uniformly over a sequence of bounded domains which exhaust all of $\mathbb{R}^d$.

The control cost estimates which we present are optimal in several asymptotic regimes. More precisely, the estimate becomes optimal for the large time $T \to \infty$ and small time $T \to 0$ limit, as well as for the homogenization limit. The latter corresponds to a sequence of observability sets in $\mathbb{R}^d$ which have a common positive density but get evenly distributed on finer and finer scales. Effectively this leads to a control problem with control set equal to the whole domain but with a weight factor.

Last but not least, we point out two fields of analysis where related or complementary results to spectral inequalities in control theory have been developed. One of them is the theory of random Schrödinger operators. There, uncertainty principles play a crucial role for the study of the integrated density of states and proofs of Anderson localization. The other is the use of uncertainty principles developed with the help of complex or harmonic analysis to study semi-norms on L^p-spaces.

2. Scale-free spectral inequalities based on complex analysis

In this section we give an overview of scale-free spectral inequalities obtained through complex analytical methods, in contrast to the ones obtained through Carleman estimates, discussed in a subsequent section. The term *scale-free* stands for the independence of the estimates on the size of the underlying domain. In particular, only a dependence on the dimension, on the geometry of the observability set, and on the class of functions considered is present.

These inequalities deal with the class of L^p-functions on $\mathbb{R}^d$ with compactly supported Fourier transform or with L^p-functions on the d-dimensional torus with sides of length $2\pi L$, $L > 0$, with active Fourier frequencies contained in a parallelepiped of $\mathbb{R}^d$, and with observability sets which are measurable and well-distributed in $\mathbb{R}^d$ in the following sense:

Definition 2.1. Let S be a subset of $\mathbb{R}^d$, $d \in \mathbb{N}$. We say that S is a *thick set* if it is measurable and there exist $\gamma \in (0,1]$ and $a = (a_1, \dots, a_d) \in \mathbb{R}^d_+$ such that

$$|S \cap (x + [0,a_1] \times \dots \times [0,a_d])| \geq \gamma \prod_{j=1}^{d} a_j, \quad \forall\, x \in \mathbb{R}^d.$$

Here $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^d$. We will call thick sets (γ, a)-thick to emphasise the parameters.

This geometric condition relates the volume of cubes to the volume of the part of these cubes inside S. It can equivalently also be formulated with respect to balls in $\mathbb{R}^d$, in which case $a \in \mathbb{R}^d_+$ is replaced by a radius $r > 0$. The latter is considered in the proof of Lemma 2.3 below.

Before presenting the most current results, we discuss how these spectral inequalities and the above geometric condition were identified originally.

2.1. Earlier literature and historical development: Equivalent norms on subspaces

Let $d \in \mathbb{N}$, $p \in [1,\infty]$, $\Omega \subset \mathbb{R}^d$, and $S \subset \mathbb{R}^d$ be measurable subsets. We define

$$F(\Omega, p) := \bigl\{ f \in L^p(\mathbb{R}^d) \colon \operatorname{supp} \widehat{f} \subset \Omega \bigr\},$$

where $\widehat{f}$ is the Fourier transform of f. If Ω is bounded, we ask for which sets S there exists a constant $C = C(S, \Omega) > 0$ such that

$$\|f\|_{L^p(S)} \geq C \|f\|_{L^p(\mathbb{R}^d)}, \quad \forall\, f \in F(\Omega, p). \tag{2.1}$$

Since $\|\cdot\|_{L^p(S)}$ defines a semi-norm on $F(\Omega, p)$ and $\|\cdot\|_{L^p(S)} \leq \|\cdot\|_{L^p(\mathbb{R}^d)}$, we are actually asking for which sets S this semi-norm defines a norm equivalent to the L^p-norm on $\mathbb{R}^d$.

This question was (at the best of our knowledge) first considered by Panejah in [43]. The author treated the case $p = 2$ and characterized the class of sets S satisfying (2.1) through a property of their complement. Indeed, our initial question is equivalent to the problem for which sets S there exists a constant $\widetilde{C} = \widetilde{C}(S^c, \Omega) \in (0,1)$, S^c being the complement of S in $\mathbb{R}^d$, such that

$$\|f\|_{L^p(S^c)} \leq \widetilde{C} \|f\|_{L^p(\mathbb{R}^d)}, \quad \forall\, f \in F(\Omega, p). \tag{2.2}$$

If we set

$$\rho(S^c, p) := \sup \bigl\{ \|f\|_{L^p(S^c)} \colon f \in F(\Omega, p), \|f\|_{L^p(\mathbb{R}^d)} = 1 \bigr\},$$

then (2.2) is satisfied for a $\widetilde{C} < 1$ if and only if $\rho(S^c, p) < 1$. The main result in [43] is a necessary condition for $\rho(S^c, 2) < 1$.

Theorem 2.2 ([43]). *Let $d \in \mathbb{N}$. Let $S \subset \mathbb{R}^d$ be a measurable set and S^c its complement in $\mathbb{R}^d$. Let $B(x,r)$ be the ball in $\mathbb{R}^d$ centered at x of radius $r > 0$. If*

$$\beta(S^c) := \lim_{r \to +\infty} \sup_{x \in \mathbb{R}^d} \frac{|S^c \cap B(x,r)|}{|B(x,r)|} = 1, \tag{2.3}$$

then $\rho(S^c, 2) = 1$*.*

Let us observe that Eq. (2.3) in Theorem 2.2 is just a different characterization for S not being a thick set. Indeed, we have the following:

Lemma 2.3. *Let* $d \in \mathbb{N}$*. Let* $S \subset \mathbb{R}^d$ *be a measurable set with complement* S^c*. Then* S *is thick if and only if* $\beta(S^c) < 1$*.*

Proof. Let us assume that S is not thick. Then for all $\gamma, r > 0$ there exists a ball $B(x_{\gamma,r}, r)$ centered at some point $x_{\gamma,r} \in \mathbb{R}^d$ dependent on γ and r, such that $|S \cap B(x_{\gamma,r}, r)| < \gamma |B(x_{\gamma,r}, r)|$. Let now $r > 0$ and choose $\gamma = 1/r$, then

$$\inf_{x\in\mathbb{R}^d} \frac{|S \cap B(x,r)|}{|B(x,r)|} \leq \frac{|S \cap B(x_{1/r,r}, r)|}{|B(x_{1/r,r}, r)|} < \frac{1}{r},$$

which implies

$$\lim_{r\to+\infty} \inf_{x\in\mathbb{R}^d} \frac{|S \cap B(x,r)|}{|B(x,r)|} = 0.$$

Since $|S^c \cap B(x,r)| = |B(x,r)| - |S \cap B(x,r)|$, we obtain

$$\lim_{r\to+\infty} \sup_{x\in\mathbb{R}^d} \frac{|S^c \cap B(x,r)|}{|B(x,r)|} = 1 - \lim_{r\to+\infty} \inf_{x\in\mathbb{R}^d} \frac{|S \cap B(x,r)|}{|B(x,r)|} = 1,$$

that is, $\beta(S^c) = 1$.

Conversely, if S is a thick set, we find some positive γ and r such that

$$\inf_{x\in\mathbb{R}^d} \frac{|S \cap B(x,r)|}{|B(x,r)|} \geq \gamma$$

and hence

$$\lim_{r\to+\infty} \inf_{x\in\mathbb{R}^d} \frac{|S \cap B(x,r)|}{|B(x,r)|} \geq \gamma.$$

Arguing as above, we see that $\beta(S^c) \leq 1 - \gamma < 1$, which completes the proof. ☐

Summarizing we have collected the following implications (at least for $p = 2$):

$$\begin{aligned}&(2.1) \text{ holds for some } C(S,\Omega) > 0 \Longleftrightarrow (2.2) \text{ holds for some } \widetilde{C}(S^c,\Omega) \in (0,1)\\ &\qquad\Longleftrightarrow \rho(S^c,2) < 1 \Longrightarrow \beta(S^c) < 1 \Longleftrightarrow S \text{ is thick}\end{aligned}$$

So, this leaves open the (hard) question whether thickness of S is a sufficient criterion to ensure the equivalence of norms in (2.1).

In the subsequent paper [44] Panejah shows that in dimension one the condition $\beta(S^c) < 1$ is also sufficient for $\rho(S^c, 2) < 1$, while in higher dimensions he provides a sufficient condition unrelated to the necessary one. In both papers, the methods used rely essentially on L^2-properties of the Fourier transform.

A different approach was taken by Logvinenko & Sereda [34] and Kacnel'son [23]. Using the theory of harmonic functions they considered the case $p \in (0, \infty)$ and, almost simultaneously, proved the following theorem.

Theorem 2.4 ([34, 23])**.** *Let* $d \in \mathbb{N}$*,* $\sigma > 0$*,* $p \in [1, \infty)$*, and* $S \subset \mathbb{R}^d$ *be a measurable set. Then the following statements are equivalent:*

(i) *S is a thick set;*
(ii) *there exists a constant $C = C(S, \sigma) > 0$ such that for all entire functions $f : \mathbb{C}^d \to \mathbb{C}$ satisfying $f\big|_{\mathbb{R}^d} \in L^p(\mathbb{R}^d)$ and*

$$\limsup_{|z_1|+\cdots+|z_d|\to\infty} \left(\sum_{i=1}^{d} |z_i|\right)^{-1} \ln f(z) \leq \sigma,$$

we have

$$\|f\|_{L^p(S)} \geq C\|f\|_{L^p(\mathbb{R}^d)}. \tag{2.4}$$

In addition, they exhibit the dependence on σ establishing the relation

$$C = c_1 e^{\sigma c_2},$$

where c_1 and c_2 depend only on the thickness parameters of S and dimension d.

We observe that a function f satisfying the assumption in part (ii) is called entire L^p-functions of exponential type σ. Equivalently, the space of such functions is the space of functions with Fourier transform supported in ball of radius σ (see for example [45, Theorem IX.11] or [2]). Hence, Theorem 2.4 may be regarded as the first quantitative statement related to the problem formulated in (2.1).

2.2. Current state-of-the-art

A quantitatively improved version of Theorem 2.4 was given in early 2000s by Kovrijkine (see [29] for the one dimensional case and [28] for the higher dimensional case). Using complex analytical techniques, he shows that the constant $C(S, \Omega)$ in (2.4) depends polynomially on the thickness parameters of the set S. Moreover, he analyzes the case when the support of the Fourier transform is contained in a finite union of parallelepipeds, which may or may not be disjoint. His approach is inspired by work of Nazarov [42], which studies topics related to the classical Turan Lemma [56].

More precisely, Kovrijkine proved the following statement.

Theorem 2.5 ([28])**.** *Let $d \in \mathbb{N}$, $p \in [1, \infty]$, and let S be a (γ, a)-thick set in $\mathbb{R}^d$.*

(i) *Let J be a parallelepiped with sides of length $b_1, \ldots, b_d$ parallel to the coordinate axes and let $f \in F(J, p)$. Set $b = (b_1, \ldots, b_d)$, then*

$$\|f\|_{L^p(S)} \geq \left(\frac{\gamma}{K_1^d}\right)^{K_1(a\cdot b+d)} \|f\|_{L^p(\mathbb{R}^d)}, \tag{2.5}$$

where $K_1 > 0$ is a universal constant.

(ii) *Let $n \in \mathbb{N}$ and let $J_1, \ldots, J_n$ be parallelepipeds with sides parallel to the coordinate axes and of length $b_1, \ldots, b_d$. Let $f \in F(J_1 \cup \cdots \cup J_n, p)$ and set $b = (b_1, \ldots, b_d)$. Then*

$$\|f\|_{L^p(S)} \geq \left(\frac{\gamma}{K_2^d}\right)^{\left(\frac{K_2^d}{\gamma}\right)^n a\cdot b+n-\frac{p-1}{p}} \|f\|_{L^p(\mathbb{R}^d)}, \tag{2.6}$$

for $K_2 > 0$ a universal constant.

Here $a \cdot b$ denotes the Euclidean inner product in $\mathbb{R}^d$.

The different nature of the constants in (2.5) and (2.6) originates in the different approaches used in the proofs. While the bound in (2.6) allows for more general situations, it is substantially weaker than (2.5) in the case $n = 1$. The bound in (2.5), however, is essentially optimal, which is exhibited in the following example (see also [28]).

Example 2.6. Let $d \in \mathbb{N}$, $p \geq 1$, $a_1 = \cdots = a_d = 1$, and $\gamma \in (0,1)$. We choose $b > 0$ such that $\mathbb{N} \ni \alpha := b/(4\pi) \geq 3$. We consider the 1-periodic set A in $\mathbb{R}$ such that $A \cap \left[-\frac{1}{2}, \frac{1}{2}\right] = \left[-\frac{1}{2}, -\frac{1}{2} + \frac{\gamma}{2}\right] \cup \left[\frac{1}{2} - \frac{\gamma}{2}, \frac{1}{2}\right]$, and define the set $S := A \times \mathbb{R}^{d-1}$. Clearly, S is a $(\gamma, 1)$-thick set in $\mathbb{R}^d$. Let now $g \colon \mathbb{R}^{d-1} \to \mathbb{C}$ be an L^p-function such that $\operatorname{supp} \widehat{g} \subset B(0,r) \subset \mathbb{R}^{d-1}$ for some $r < b/4$, and let $f \colon \mathbb{R} \to \mathbb{R}$ defined as $f(x_1) := \left(\frac{\sin(2\pi x_1)}{x_1}\right)^{\alpha}$. Since $\operatorname{supp} \widehat{f} \subset \left[-\frac{b}{2}, \frac{b}{2}\right]$, the function

$$\varphi \colon \mathbb{R}^d \to \mathbb{C}, \quad \varphi(x) = f(x_1) g(x_2, \ldots, x_d)$$

has Fourier Transform supported in a cylinder inside the cube $\left[-\frac{b}{2}, \frac{b}{2}\right]^d$. Theorem 2.5(i) says

$$\|\varphi\|_{L^p(S)} \geq \left(\frac{\gamma}{K_1^d}\right)^{K_1(db+d)} \|\varphi\|_{L^p(\mathbb{R}^d)},$$

for a constant $K_1 > 0$. We now show that the L^p-norm of φ on S can also be bounded from above by a constant of type γ^b. In order to do so, it is enough to bound the L^p-norm of f on A from above.

We first observe that $\|f\|_{L^p(\mathbb{R})} \geq 1$. Then, taking into account the inequality $\sin(2\pi t)/t \leq 6\pi(1/2 - t)$ for all $t \in [0, 1/2]$, we calculate

$$\begin{aligned}
\frac{\|f\|_{L^p(A)}}{\|f\|_{L^p(\mathbb{R})}} &\leq \left(\int_A \left|\frac{\sin(2\pi x_1)}{x_1}\right|^{p\alpha - 2} \left|\frac{\sin(2\pi x_1)}{x_1}\right|^2 \mathrm{d}x_1\right)^{1/p} \\
&\leq \left(\sup_{x_1 \in A} \left|\frac{\sin(2\pi x_1)}{x_1}\right|^{p\alpha - 2} \int_A \left|\frac{\sin(2\pi x_1)}{x_1}\right|^2 \mathrm{d}x_1\right)^{1/p} \\
&\leq \left(\sup_{x_1 \in A} \left|\frac{\sin(2\pi x_1)}{x_1}\right|\right)^{\alpha - 2/p} (2\pi^2)^{1/p} \\
&= \left(\frac{\sin(2\pi(1/2 - \gamma/2))}{1/2 - \gamma/2}\right)^{\alpha - 2/p} (2\pi^2)^{1/p} \\
&\leq (2\pi^2)^{\alpha - 2/p} \left(\frac{\gamma}{2} 6\pi\right)^{\alpha - 2/p} = \left(\frac{\gamma}{1/(6\pi^3)}\right)^{\alpha - 2/p}.
\end{aligned}$$

Using $\alpha - 2/p = \frac{b}{4\pi} - 2/p \geq \frac{b}{4\pi} - 2 \geq 1$, we obtain for $\gamma < 1/(6\pi^3)$,

$$\|f\|_{L^p(A)} \leq \left(\frac{\gamma}{1/(6\pi^3)}\right)^{\frac{b}{4\pi} - 2} \|f\|_{L^p(\mathbb{R})}.$$

Hence, by separation of variables, we conclude

$$\begin{aligned}\|\varphi\|_{L^p(S)} &= \|f\|_{L^p(A)}\|g\|_{L^p(\mathbb{R}^{d-1})} \\ &\leq \left(\frac{\gamma}{1/(6\pi^3)}\right)^{\frac{b}{4\pi}-2} \|f\|_{L^p(\mathbb{R})}\|g\|_{L^p(\mathbb{R}^{d-1})} \\ &= \left(\frac{\gamma}{1/(6\pi^3)}\right)^{\frac{b}{4\pi}-2} \|\varphi\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

which shows the optimality of the γ^b term.

For $p = 2$, the statement of Theorem 2.5(i) can be easily turned into a spectral inequality. Let $E > 0$ and let $\chi_{(-\infty,E]}(-\Delta_{\mathbb{R}^d})$ be the spectral projector of $-\Delta_{\mathbb{R}^d}$ up to energy E, $\Delta_{\mathbb{R}^d}$ being the Laplacian on $\mathbb{R}^d$. Then

$$\chi_{(-\infty,E]}(-\Delta_{\mathbb{R}^d})\colon L^2(\mathbb{R}^d) \to \Big\{f \in L^2(\mathbb{R}^d) :\ \operatorname{supp}\widehat{f} \subset B\big(0,\sqrt{E}\,\big)\Big\},$$

where $B(0,\sqrt{E})$ is the Euclidean ball with center 0 and radius $\sqrt{E}$. Clearly,

$$\operatorname{Ran}\big(\chi_{(-\infty,E]}(-\Delta_{\mathbb{R}^d})\big) \subset \Big\{f \in L^2(\mathbb{R}^d) :\ \operatorname{supp}\widehat{f} \subset \big[-\sqrt{E},\sqrt{E}\,\big]^d\Big\}.$$

Therefore, as explained in [14, §5], Theorem 2.5(i) implies:

Corollary 2.7. *Let $d \in \mathbb{N}$. There exists a constant $K_1 > 0$ such that for all $E > 0$, all (γ, a)-thick sets S, and all $f \in \operatorname{Ran}\big(\chi_{(-\infty,E]}\big(-\Delta_{\mathbb{R}^d}\big)\big)$ we have*

$$\|f\|_{L^2(S)} \geq \left(\frac{\gamma}{K_1^d}\right)^{K_1(2\sqrt{E}\|a\|_1+d)} \|f\|_{L^2(\mathbb{R}^d)},$$

where $\|a\|_1 = a_1 + \cdots + a_d$.

Using similar techniques as in [28], Logvinenko–Sereda-type estimates have been recently established also on the torus $\mathbb{T}_L^d = [0, 2\pi L]^d$ with sides of length $2\pi L$, $L > 0$, $d \in \mathbb{N}$, for $L^p(\mathbb{T}_L^d)$-functions with active Fourier frequencies contained in a parallelepiped of arbitrary size, see [13]. This leads to a spectral inequality for linear combinations of eigenfunctions of the Laplacian on $\mathbb{T}_L^d$ with suitable boundary conditions, see [14, §5].

For $f \in L^p(\mathbb{T}_L^d)$ we adopt the convention:

$$\widehat{f}\colon \left(\frac{1}{L}\mathbb{Z}\right)^d \to \mathbb{R}^d, \qquad \widehat{f}\left(\frac{k_1}{L},\dots,\frac{k_d}{L}\right) = \frac{1}{(2\pi L)^d}\int_{\mathbb{T}_L^d} f(x)e^{-i\frac{1}{L}x\cdot k}\mathrm{d}x.$$

In particular, $\operatorname{supp}\widehat{f} \subset \big(\frac{1}{L}\mathbb{Z}\big)^d \subset \mathbb{R}^d$.

Theorem 2.8 ([13])**.** *Let $p \in [1,\infty]$ and $L > 0$. Let $\mathbb{T}_L^d = [0, 2\pi L]^d$, $f \in L^p(\mathbb{T}_L^d)$, and S be a (γ, a)-thick set with $a = (a_1, \dots, a_d)$ such that $0 < a_j \leq 2\pi L$ for all $j = 1, \dots, d$.*

(i) *Assume that* $\operatorname{supp}\widehat{f} \subset J$*, where* J *is a parallelepiped in* $\mathbb{R}^d$ *with sides of length* $b_1, \dots, b_d$ *and parallel to coordinate axes. Set* $b = (b_1, \dots, b_d)$*, then*

$$\|f\|_{L^p(S\cap\mathbb{T}_L^d)} \geq \left(\frac{\gamma}{K_3^d}\right)^{K_3 a\cdot b + \frac{6d+1}{p}} \|f\|_{L^p(\mathbb{T}_L^d)}, \tag{2.7}$$

where $K_3 > 0$ *is a universal constant.*

(ii) *Let* $n \in \mathbb{N}$ *and assume that* $\operatorname{supp}\widehat{f} \subset \bigcup_{l=1}^n J_l$*, where each* J_l *is a parallelepiped in* $\mathbb{R}^d$ *with sides of length* $b_1, \dots, b_d$ *and parallel to coordinate axes. Set* $b = (b_1, \dots, b_d)$*, then*

$$\|f\|_{L^p(S\cap\mathbb{T}_L^d)} \geq \left(\frac{\gamma}{K_4^d}\right)^{\left(\frac{K_4^d}{\gamma}\right)^n a\cdot b + n - \frac{(p-1)}{p}} \|f\|_{L^p(\mathbb{T}_L^d)},$$

for $K_4 > 0$ *a universal constant.*

Here $a \cdot b$ *denotes the Euclidean inner product in* $\mathbb{R}^d$*.*

We stress that these estimates are uniform for all $L \geq (2\pi)^{-1} \max_{j=1,\dots,d} a_j$ and are independent of the position of the parallelepipeds J_l. Note that for growing L the number of possible Fourier frequencies in the set $\bigcup_{l=1}^n J_l$ grows unboundedly.

Let us also note that in [54, Corollary 3.3], related techniques from complex analysis, in particular a version of the Turan Lemma, are used to establish an estimate similar to the one in Theorem 2.8. However, there the control set S is assumed to contain a parallelepiped and the constant comparing $\|\cdot\|_{L^2(S)}$ and $\|\cdot\|_{L^2(\mathbb{T}_L^d)}$ depends on its volume.

Comparing (i) and (ii) of the above theorem, we again see that, although (ii) allows for more general situations, the corresponding constant is worse that the one in (i) in the case $n = 1$. Example 2.9 below, inspired by Example 2.6, shows that for general L^p-functions on $\mathbb{T}_L^d$ estimate (2.7) is optimal up to the unspecified constant K_3. However, this bound may be improved once special classes of functions are considered, for example Fourier series with few, but spread out Fourier coefficients, as discussed in Example 2.10.

Example 2.9. Let $a_1 = \dots = a_d = 1$, $p \geq 1$, $b \geq 8\pi$, and $\varepsilon \in (0,1)$. We consider the set

$$S = A_1 \times \dots \times A_d \subset \mathbb{R}^d$$

such that each A_j is 1-periodic and $A_j \cap [0,1] = \left[\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2}\right]$. Then, S is $(\gamma, 1)$-thick in $\mathbb{R}^d$ with $\gamma = \varepsilon^d$.

Let now $\mathbb{N} \ni \alpha := \lfloor \frac{b}{4\pi} \rfloor$ and $L = 1/(2\pi)$. On the torus $\mathbb{T}_L^1 = [0, 2\pi L] = [0,1]$ and on its d-dimensional counterpart $\mathbb{T}_L^d = [0,1]^d$ we consider the functions

$$f\colon [0,1] \to \mathbb{R}, \quad f(x) := (\sin(2\pi x))^\alpha$$

$$g\colon [0,1]^d \to \mathbb{R}, \quad g(x) := \prod_{j=1}^d f(x_j) = \prod_{j=1}^d \sin(2\pi x_j)^\alpha.$$

Clearly, $\operatorname{supp}\widehat{f} \subset [-2\pi\alpha, 2\pi\alpha] \subset \left[-\frac{b}{2}, \frac{b}{2}\right]$ and $\operatorname{supp}\widehat{g} \subset \left[-\frac{b}{2}, \frac{b}{2}\right]^d$, and the Fourier coefficients are uniformly spaced.

Consequently, by Theorem 2.8(i) we know

$$\|g\|_{L^p(S\cap[0,1]^d)} \geq \left(\frac{\varepsilon^d}{K_3^d}\right)^{K_3 db + \frac{6d+1}{p}} \|g\|_{L^p([0,1]^d)}.$$

We now show that the prefactor cannot be improved qualitatively. To obtain an upper bound on $\|g\|_{L^p(S\cap[0,1]^d)}$ we proceed as follows. By separation of variables, $\|g\|_{L^p(S\cap[0,1]^d)} = \prod_{j=1}^d \|f\|_{L^p(A_j\cap[0,1])}$ and similarly for $\|g\|_{L^p([0,1]^d)}$. It is therefore enough to analyze the L^p-norm of f on $A_1 \cap [0,1]$.

By Jensen's inequality, we have

$$\|f\|_{L^p([0,1])}^p = \int_0^1 |\sin(2\pi x)|^{p\alpha} \mathrm{d}x \geq \left(\int_0^1 |\sin(2\pi x)| \mathrm{d}x\right)^{p\alpha} = \left(\frac{2}{\pi}\right)^{p\alpha}.$$

By symmetry of the sine function, $\sin x \leq x$, the choice of α, and the change of variable $y = 2\pi x$, we estimate

$$\begin{aligned}
\frac{\|f\|_{L^p(A_1\cap[0,1])}}{\|f\|_{L^p([0,1])}} &\leq \left(\frac{\pi}{2}\right)^\alpha \left(\int_{A_1\cap[0,1]} |\sin(2\pi x)|^{p\alpha} \mathrm{d}x\right)^{1/p} \\
&= \left(\frac{\pi}{2}\right)^\alpha \left(\frac{1}{\pi}\int_0^{\pi\varepsilon} \sin^{p\alpha}(y) \mathrm{d}y\right)^{1/p} \\
&\leq \left(\frac{\pi}{2}\right)^\alpha \left(\frac{1}{\pi}\int_0^{\pi\varepsilon} y^{p\alpha} \mathrm{d}y\right)^{1/p} \\
&= \left(\frac{\pi}{2}\right)^\alpha \left(\frac{1}{\pi}\frac{(\pi\varepsilon)^{1+p\alpha}}{1+p\alpha}\right)^{1/p} \\
&\leq \left(\frac{\varepsilon}{(2/\pi^2)}\right)^{\alpha+1/p}.
\end{aligned}$$

Using $\alpha + 1/p = \lfloor \frac{b}{4\pi} \rfloor + 1/p \geq \frac{b}{4\pi} - 1 \geq 1$, we obtain for $\varepsilon < 2/\pi^2$,

$$\|f\|_{L^p(A_1\cap[0,1])} \leq \left(\frac{\varepsilon}{(2/\pi^2)}\right)^{\frac{b}{4\pi}-1} \|f\|_{L^p([0,1])},$$

which holds also for $\varepsilon \geq 2/\pi^2$ trivially. Consequently,

$$\|g\|_{L^p(S\cap[0,1]^d)} \leq \left(\frac{\gamma}{(2/\pi^2)^d}\right)^{\frac{b}{4\pi}-1} \|g\|_{L^p([0,1]^d)}.$$

This shows that in general we cannot obtain a Logvinenko–Sereda constant which is qualitatively better than $(\gamma/c^d)^{c(b+d)}$, for some $c > 0$.

Example 2.10. Let $b \in \mathbb{N}$, $\gamma \in (0,1)$, S be the 1-periodic set such that $S \cap [0,1] = [0,\gamma]$, and $f\colon [0,1] \to \mathbb{R}$ be defined as $f(x) := \sin(2b\pi x)$. This function has two non-zero Fourier coefficients at $-2b\pi$ and $2b\pi$, growing apart as b increases. For the L^1-norm of f on $[0,1]$ and $[0,\gamma]$ we calculate

$$\frac{\|f\|_{L^1([0,\gamma])}}{\|f\|_{L^1([0,1])}} \le \frac{\pi}{2}\int_0^\gamma 2b\pi x \,\mathrm{d}x = \frac{\pi^2}{2}b\gamma^2,$$

suggesting a behavior of type $b\gamma^2$ instead of γ^b as in Theorem 2.8(i).

As anticipated, the case $p = 2$ in Theorem 2.8(i) is of particular interest, since it can be interpreted as a statement for functions in the range of the spectral projector of $-\Delta_{\mathbb{T}_L^d}$ with periodic, Dirichlet, or Neumann boundary conditions. Let $\Delta^P_{\mathbb{T}_L^d}, \Delta^D_{\mathbb{T}_L^d}, \Delta^N_{\mathbb{T}_L^d}$ be the Laplacian on $\mathbb{T}_L^d$ with periodic, Dirichlet, and Neumann boundary conditions, respectively. To shorten the notation we set $\bullet \in \{P, D, N\}$. Let $\chi_{(-\infty,E]}(-\Delta^\bullet_{\mathbb{T}_L^d})$ be the spectral projector of $-\Delta^\bullet_{\mathbb{T}_L^d}$ up to energy $E > 0$. Namely, let $\lambda^\bullet$ and $\phi^\bullet_{\lambda^\bullet}$ be the eigenvalues and corresponding eigenfunctions of $-\Delta^\bullet_{\mathbb{T}_L^d}$, then

$$\chi_{(-\infty,E]}(-\Delta^\bullet_{\mathbb{T}_L^d}) : L^p(\mathbb{T}_L^d) \to \left\{ \sum_{\lambda^\bullet \le E} \alpha_{\lambda^\bullet}\phi^\bullet_{\lambda^\bullet}(x) \mid \alpha_{\lambda^\bullet} \in \mathbb{C} \right\}.$$

Similarly as before, Theorem 2.8(i) implies by simple arguments performed in [14, §5]:

Corollary 2.11. *Let $d \in \mathbb{N}$, and let $\mathbb{T}_L^d = [0, 2\pi L]^d$, $L > 0$. There exists a universal constant $K_5 > 0$ such that for all $L > 0$, all (γ, a)-thick sets $S \subset \mathbb{R}^d$ with $a = (a_1, \ldots, a_d)$ such that $0 < a_j \le 2\pi L$ for all $j = 1, \ldots, d$, all $E > 0$, and all $f \in \operatorname{Ran}\left(\chi_{(-\infty,E]}(-\Delta^\bullet_{\mathbb{T}_L^d})\right)$ we have*

$$\|f\|_{L^2(S\cap\mathbb{T}_L^d)} \ge \left(\frac{\gamma}{K_5^d}\right)^{K_5\sqrt{E}\|a\|_1 + \frac{6d+1}{2}} \|f\|_{L^2(\mathbb{T}_L^d)}, \tag{2.8}$$

where $\|a\|_1 = a_1 + \cdots + a_d$.

In the case of periodic boundary conditions, Corollary 2.11 is a direct consequence of Theorem 2.8(i): Since the eigenfunctions of $-\Delta^P_{\mathbb{T}_L^d}$ are $e^{i(k/L)\cdot x}$ (up to a normalization factor), corresponding to eigenvalues $\|k\|_2^2/L^2$, $k \in \mathbb{Z}^d$, the Fourier frequencies of any $f \in \operatorname{Ran}\left(\chi_{(-\infty,E]}(-\Delta^P_{\mathbb{T}_L^d})\right)$ are contained in $[-\sqrt{E}, \sqrt{E}]^d$, and the statement follows immediately.

In contrast, when Dirichlet or Neumann boundary conditions are considered, the respective eigenfunctions do not have Fourier frequencies contained in a compact set. However, once these functions are extended to functions on $\mathbb{T}_{2L}^d$ in a suitable way depending on the boundary conditions, the Fourier frequencies of the extensions are concentrated in $[-\sqrt{E}, \sqrt{E}]^d$. Correspondingly, one can construct a new thick set with controllable thickness parameters by first extending $S \cap \mathbb{T}_L^d$ to $\mathbb{T}_{2L}^d$ using reflections with respect to the boundary of $\mathbb{T}_L^d$, and then taking the union of translates of this set with respect to the group $(4\pi L\mathbb{Z})^d$. Finally, Theorem 2.8(i) applied to the extensions and the new thick set yields Corollary 2.11. For more details we refer the reader to [14, §5].

Remark 2.12. Recently, a Logvinenko–Sereda-type estimate has also been obtained for L^2-functions on the infinite strip $\Omega_L := \mathbb{T}_L^{d-1} \times \mathbb{R}$, $d \geq 2$ and $L > 0$, having finite Fourier series as functions on $\mathbb{T}_L^{d-1}$ and compactly supported Fourier transform as functions on $\mathbb{R}$. In this case, the set $S \subset \mathbb{R}^d$ is assumed to be thick with parameters $a = (a_1, \ldots, a_d) \in \mathbb{R}_+^d$ such that $a_j \leq 2\pi L$ for $j \in \{1, \ldots, d-1\}$, and $\gamma \in (0, 1]$, see [12, Theorem 9]. With similar arguments as in [14, §5], we obtain, as a consequence, a corresponding variant of Corollary 2.11 on the strip, that is, a spectral inequality analogous to (2.8) for functions in the range of the spectral projector $\chi_{(-\infty,E]}(-\Delta_{\Omega_L})$, where $-\Delta_{\Omega_L}$ is the Laplacian on Ω_L with either Dirichlet or Neumann boundary conditions.

3. Scale-free spectral inequalities based on Carleman estimates

Most of the results which we present here have originated in works devoted to the spectral theory and asymptotic analysis of evolution of solutions of random Schrödinger equations. The interested reader may consult for instance the monographs [50, 58, 1] for an exposition of this research area. In this theory one is (among others) interested in lifting estimates for eigenvalues. The particular task we want to discuss here can be formulated in operator theoretic language in the following way: Given a self-adjoint and lower semi-bounded operator H with purely discrete spectrum $\lambda_1(H) \leq \cdots \leq \lambda_k(H) \leq \cdots$, a parameter interval $I \subset \mathbb{R}$, a cut-off energy $E \in \mathbb{R}$, and a positive semi-definite perturbation B, find a positive constant C such that

$$\frac{\mathrm{d}}{\mathrm{dt}} \lambda_k(H + tB) \geq C$$

for all indices $k \in \mathbb{N}$ for which the associated eigenvalue curve $I \ni t \mapsto \lambda_k(H + tB)$ stays below the level E for all $t \in I$. Depending on the properties of H and B, this exercise may be trivial, demanding, or impossible, so we should say a bit more about the structure of the operators of interest.

The self-adjoint Hamiltonian H models a condensed matter system, and studying it will require investigating it on several scales, on the one hand the macroscopic scale of the solid and on the other the microscopic scale of atoms. Let us explain this in more detail: If we choose a coordinate system such that the typical distance between atomic nuclei is equal to one, the size L of the macroscopic solid may be very large – of the order of magnitude of 10^{23} or so. Hence the Hamiltonian of the system H will be defined on the Hilbert space $L^2(\Lambda_L)$ where $\Lambda_L = (-L/2, L/2)^d$ and $L \gg 1$. Since often the only possibility to understand the full system is to consider first smaller sub-systems and subsequently analyze how they interact, one is also interested in intermediate scales. Thus in the discussion which follows, the scale L will always be larger than one, but will range over many orders of magnitude.

The Hamiltonian H will be a Schrödinger operator of the form $H = -\Delta + V$ in $L^2(\Lambda_L)$. The electric potential V, which mainly models the force of the atomic nuclei in the solid on an electron wave packet, will have a characteristic length scale corresponding to the typical distance between atoms (which as above we set equal to one). This characteristic scale could manifest itself in different ways, for instance, V may be the restriction $\chi_{\Lambda_L} V_{\mathrm{per}}$ of a $\mathbb{Z}^d$-periodic potential $V_{\mathrm{per}} \colon \mathbb{R}^d \to \mathbb{R}$. It could also have a structure which is not exactly periodic but incorporates some deviations from periodicity. Furthermore, it can happen that the exact shape of V is not known. In this case, V is modeled by a random field allowing for local fluctuations. The values of the field at two different points with a distance of order one may be correlated, but the field will exhibit a mixing behavior on large scales. Since we study the system in $L^2(\Lambda_L)$ for many scales $L \geq 1$, the ratio between the scale L of the whole system and the scale one can grow unboundedly. In view of this challenge, for a comprehensive understanding of the system it is required to derive so-called *scale-free* results, i.e. results which hold for all scales $L \geq 1$, or at least for an unbounded sequence of length scales.

In the light of this multi-scale structure, the problem formulated above takes now a more specific form: We are given a bounded potential V as well as the corresponding Schrödinger operator $H = -\Delta + V$ on Λ_L with self-adjoint, say Dirichlet, boundary conditions, a bounded non-negative perturbation potential $W \colon \Lambda_L \to [0, \infty)$, which is the restriction to Λ_L of a (more or less) periodic potential $W_{\mathrm{per}} \colon \mathbb{R}^d \to [0, \infty)$, an interval $I \subset \mathbb{R}$, a cut-off energy $E \in \mathbb{R}$ and aim to find a positive constant C, independent of $L \geq 1$, such that

$$\frac{\mathrm{d}}{\mathrm{dt}} \lambda_k(H + tW) \geq C \tag{3.1}$$

for all $k \in \mathbb{N}$ such that the associated eigenvalue parametrization $I \ni t \mapsto \lambda_k(H + tB)$ stays below E for all $t \in I$. The real challenge of the problem is to obtain a bound C which is scale-independent. Furthermore, C should only depend on some rough features of V and W such as their sup-norms but not on minute details of their shape. This is required since, as explained above, the potentials V and W might by modeled as realizations of a random field where it is possible to control certain global properties, but not the detailed shape of the realization.

The eigenvalue lifting bound in (3.1) can be derived from an *uncertainty relation for spectral projectors* of the type

$$\chi_{(-\infty,E]}(H) W \chi_{(-\infty,E]}(H) \geq C \chi_{(-\infty,E]}(H),$$

where the inequality is understood in quadratic form sense. In fact, since the operators considered so far have purely discrete spectrum, this inequality can be rewritten in terms of linear combinations of eigenfunctions, so that the conclusion (3.1) is almost immediate. If H is the Dirichlet-Laplacian on a bounded domain the above uncertainty relation is called *spectral inequality* in the literature on control theory. It is also sometimes called a *quantitative unique continuation estimate* (for spectral projectors), because its proof uses a refined version of the proof of the classical qualitative unique continuation principle for solutions of second order elliptic operators, based on Carleman estimates. In the particular case where, as explained above, the constant C in the estimate is independent of the length scale $L \geq 1$, the inequality is called *scale-free unique continuation estimate*. Let us present a summary of such results derived in the context of random Schrödinger operators.

3.1. Development of scale-free unique continuation estimates applicable to Schrödinger operators with random potential

We will not be able to review all publications dealing with the topic, in particular older ones, but have to be selective due to limitations of space. As the starting point we choose an important and intuitive result of [6] (which was fully exploited only in [7]).

Theorem 3.1 ([6, 7])**.** *Let $E \in \mathbb{R}$, $V_{\mathrm{per}} \colon \mathbb{R}^d \to \mathbb{R}$ be a measurable, bounded and $\mathbb{Z}^d$-periodic potential and H_L^{per} the restriction of $-\Delta + V_{\mathrm{per}}$ to the cube Λ_L with periodic boundary conditions. Denote by $\chi_{(-\infty,E]}(H_L^{\mathrm{per}})$ the spectral projector of H_L^{per} associated to the energy interval $(\infty, E]$. If $\mathcal{O} \neq \emptyset$ is an open $\mathbb{Z}^d$-periodic subset of $\mathbb{R}^d$ and $W \colon \mathbb{R}^d \to [0,\infty)$ a measurable, bounded and $\mathbb{Z}^d$-periodic potential such that $W \geq \chi_{\mathcal{O}}$ then there exists a constant $C > 0$ depending on E, V_{per} and W, but not on $L \in \mathbb{N}$, such that*

$$\chi_{(-\infty,E]}(H_L^{\mathrm{per}})\ W\chi_{\Lambda_L}\ \chi_{(-\infty,E]}(H_L^{\mathrm{per}}) \geq C\chi_{(-\infty,E]}(H_L^{\mathrm{per}})$$

for all scales $L \in \mathbb{N}$.

The theorem gives no estimate on the constant $C > 0$, since its proof invokes a compactness argument. Moreover, it is based on the Floquet–Bloch decomposition and thus cannot be extended to a situation without periodicity. This explains also the restriction to integer valued scales $L \in \mathbb{N}$.

An improvement of the above theorem with an explicit lower bound on C was given in [19]. The method which allowed to derive this quantitative estimate was a *Carleman estimate*. It was the seminal paper [5] which introduced Carleman estimates to the realm of random Schrödinger operators and stimulated the further development. With this tool at hand it was possible to circumvent the use of Floquet–Bloch theory in the proof of scale-free unique continuation estimates. Consequently, it was possible to remove the periodicity assumptions on the potential function V and the set $\mathcal{O}$. They can be be replaced by a geometric condition which we define next.

Definition 3.2. Given $G, \delta > 0$, we say that a sequence $Z = (z_j)_{j \in (G\mathbb{Z})^d} \subset \mathbb{R}^d$ is (G,δ)-*equidistributed*, if

$$\forall j \in (G\mathbb{Z})^d \colon \quad B(z_j, \delta) \subset \Lambda_G + j.$$

Corresponding to a (G, δ)-equidistributed sequence Z we define the set

$$S_{\delta,Z} = \bigcup_{j \in (G\mathbb{Z})^d} B(z_j, \delta),$$

see Fig. 1 for an illustration. Note that the set $S_{\delta,Z}$ depends on G and the choice of the (G, δ)-equidistributed sequence Z.

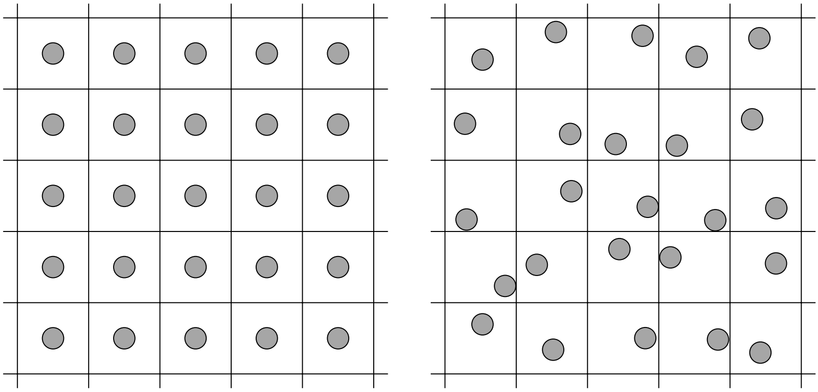

FIGURE 1. Illustration of $S_{\delta,Z} \subset \mathbb{R}^2$ for periodically (left) and non-periodically (right) arranged balls.

For $L > 0$ we denote by $\mathcal{D}(\Delta_L^{\mathrm{per}})$ and $\mathcal{D}(\Delta_L^{\mathrm{Dir}})$ the domain of the Laplacian on $L^2(\Lambda_L)$ subject to periodic or Dirichlet boundary conditions. With this notion at hand we formulate the following result:

Theorem 3.3 ([46])**.** *Let $E \in \mathbb{R}$. There exists a constant $K \in (0, \infty)$ depending merely on the dimension d, such that for any $E \in \mathbb{R}$, any $G > 0$, any $\delta \in (0, G/2]$, any (G, δ)-equidistributed sequence Z, any measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, any $L \in G\mathbb{N}$, and any real-valued $\psi \in \mathcal{D}(\Delta_L^{\mathrm{per}}) \cup \mathcal{D}(\Delta_L^{\mathrm{Dir}})$ satisfying $|\Delta\psi| \leq |(V - E)\psi|$ almost everywhere on Λ_L we have*

$$\|\psi\|_{L^2(\Lambda_L)} \geq \|\psi\|_{L^2(S_{\delta,Z} \cap \Lambda_L)} \geq \left(\frac{\delta}{G}\right)^{K\left(1+G^{4/3}\|V-E\|_\infty^{2/3}\right)} \|\psi\|_{L^2(\Lambda_L)}. \tag{3.2}$$

The last inequality implies by first order perturbation theory the lifting estimate (3.1) with

$$C = \left(\frac{\delta}{G}\right)^{K\left(1+G^{4/3}\|V-E\|_\infty^{2/3}\right)}. \tag{3.3}$$

The theorem has been extended to $\mathbb{R}^d$ in [53]. Lower bounds like (3.2) (with less explicit constants) have previously been known for

(1) Schrödinger operators in one dimension (see [57, 26] where periodicity was assumed, and [20], where the additional periodicity assumption was eliminated),
(2) energies E sufficiently close to $\min\sigma(H) = \min\sigma(-\Delta+V)$ (see [24, 5] under a periodicity assumption and [18] without it), and similarly for
(3) energies E sufficiently close to a spectral band edge of a periodic Schrödinger operator $-\Delta+V_{\mathrm{per}}$. (This has been implemented in [25] for periodic potentials using Floquet theory.)

In the two latter cases one uses perturbative arguments, whereas in the one-dimensional situation one has methods from ordinary differential equations at disposal. The result of [46] unifies and generalizes this set of earlier results.

Remark 3.4 (Dependence of the constant on parameters). Apart form being scale independent the constant C from (3.3) is also explicit with respect to the model parameters. Only the sup-norm $\|V\|_\infty$ of the potential enters, no knowledge of V beyond this is used, in particular no regularity properties. This is essential since in applications V is chosen from an infinite ensemble of potentials with possibly quite different local features. The constant is polynomial in δ and (almost) exponential in $\|V\|_\infty$.

For $L>0$ and $V\in L^\infty(\mathbb{R}^d)$, we define the operator

$$H_L = -\Delta + V \quad \text{in } L^2(\Lambda_L)$$

with Dirichlet, Neumann or periodic boundary conditions. In view of the known scale-free uncertainty relation for periodic spectral projectors of [6], see Theorem 3.1, the authors of [46] asked whether (3.2) holds also for linear combinations of eigenfunctions, i.e. for $\psi\in\operatorname{Ran}\chi_{(-\infty,E]}(H_L)$. This is equivalent to

$$\chi_{(-\infty,E]}(H_L)\,\chi_{S_{\delta,Z}\cap\Lambda_L}\,\chi_{(-\infty,E]}(H_L) \geq C\chi_{(-\infty,E]}(H_L), \tag{3.4}$$

with an explicit dependence of C on the parameters G, δ, E and $\|V\|_\infty$ as in (3.3). Here $\chi_I(H_L)$ denotes the spectral projector of H_L associated to the interval I. If $\chi_{S_{\delta,Z}\cap\Lambda_L}$ is periodic and a lower bound for the potential W, we recover an estimate as in Theorem 3.1. A partial answer was given in [27].

Theorem 3.5 ([27]). *There exists $K=K(d)$ such that for all $E,G>0$, all $\delta\in(0,G/2)$, all (G,δ)-equidistributed sequences Z, all measurable and bounded $V\colon\mathbb{R}^d\to\mathbb{R}$, all $L\in\mathbb{N}$, all intervals $I\subset(-\infty,E]$ with*

$$|I|\leq 2\gamma \quad \textit{where} \quad \gamma^2 = \frac{1}{2G^4}\left(\frac{\delta}{G}\right)^{K\left(1+G^{4/3}(2\|V\|_\infty+E)^{2/3}\right)},$$

and all $\psi\in\operatorname{Ran}\chi_I(H_L)$ we have

$$\|\psi\|_{L^2(S_{\delta,Z}\cap\Lambda_L)} \geq G^4\gamma^2\|\psi\|_{L^2(\Lambda_L)}.$$

Again the scale-free unique continuation principle of [27] on the finite cube Λ_L was adapted to functions on $\mathbb{R}^d$ in [53]. Theorem 3.5 left open what happens if the energy interval I has length larger than 2γ, which is quite small for typical choices of G, δ, E, V. In particular, Theorem 3.5 is not sufficient for applications in control theory which we discuss in Section 4. The full answer to the above question, confirming (3.4), has been given in [40, 41].

Theorem 3.6 ([40, 41]). *There is $K = K(d) > 0$ such that for all $G > 0$, all $\delta \in (0, G/2)$, all (G,δ)-equidistributed sequences Z, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $L \in G\mathbb{N}$, all $E \geq 0$, and all $\psi \in \mathrm{Ran}\chi_{(-\infty,E]}(H_L)$ we have*

$$\|\psi\|^2_{L^2(S_{\delta,Z}\cap\Lambda_L)} \geq C_{\mathrm{uc}}\|\psi\|^2_{L^2(\Lambda_L)}$$

where

$$C_{\mathrm{uc}} = C_{\mathrm{uc}}(d, G, \delta, E, \|V\|_\infty) := \left(\frac{\delta}{G}\right)^{K\left(1+G^{4/3}\|V\|_\infty^{2/3}+G\sqrt{E}\right)}.$$

Note that since Λ_L is bounded, H_L has compact resolvent, thus any $\psi \in \mathrm{Ran}\chi_{(-\infty,E]}(H_L)$ is a finite linear combination of eigenfunctions. In [52] this assumption has been relaxed to allow certain infinite linear combinations of eigenfunctions where the coefficients decay sufficiently fast.

3.2. Current state-of-the-art

Let $d \in \mathbb{N}$. For $G > 0$ we say that a set $\Gamma \subset \mathbb{R}^d$ is *G-admissible*, if there exist $\alpha_i, \beta_i \in \mathbb{R} \cup \{\pm\infty\}$ with $\beta_i - \alpha_i \geq G$ for $i \in \{1, \ldots, d\}$, such that

$$\Gamma = \bigtimes_{i=1}^{d} (\alpha_i, \beta_i) \quad \text{and} \quad (-G/2, G/2)^d \subset \Gamma. \tag{3.5}$$

If instead of the second condition in (3.5) one can only find a $\xi \in \mathbb{R}^d$ such that the cube $(-G/2, G/2)^d + \xi$ is contained in Γ, then our assumption $(-G/2, G/2)^d \subset \Gamma$ can be achieved by a global shift of the coordinate system. For a G-admissible set Γ and a real-valued $V \in L^\infty(\Gamma)$, we define the self-adjoint operator H_Γ on $L^2(\Gamma)$ as

$$H_\Gamma = -\Delta + V$$

with Dirichlet or Neumann boundary conditions.

Theorem 3.7 ([39]). *There is $K = K(d) > 0$ depending only on the dimension, such that for all $G > 0$, all G-admissible $\Gamma \subset \mathbb{R}^d$, all $\delta \in (0, G/2)$, all (G,δ)-equidistributed sequences Z, all real-valued $V \in L^\infty(\Gamma)$, all $E \in \mathbb{R}$, and all $\psi \in$ Ran $\chi_{(-\infty,E]}(H_\Gamma)$ we have*

$$\|\psi\|^2_{L^2(S_{\delta,Z}\cap\Gamma)} \geq C^{(G)}_{\mathrm{uc}}\|\psi\|^2_{L^2(\Gamma)},$$

where

$$C^{(G)}_{\mathrm{uc}} = \sup_{\lambda\in\mathbb{R}} \left(\frac{\delta}{G}\right)^{K\left(1+G^{4/3}\|V-\lambda\|_\infty^{2/3}+G\sqrt{(E-\lambda)_+}\right)},$$

and $t_+ := \max\{0, t\}$ for $t \in \mathbb{R}$.

Remark 3.8. If $\Gamma = \Lambda_L$ for some $L \geq G$, then H_Γ has compact resolvent, and hence the spectrum of H_Γ consists of a non-decreasing sequence of eigenvalues whose only accumulation point is at infinity. As a consequence, functions $\psi \in \operatorname{Ran} \chi_{(-\infty,E]}(H_\Gamma)$ considered in Theorem 3.7 are finite linear combinations of eigenfunctions corresponding to eigenvalues smaller than or equal to E. On the contrary, if Γ is an unbounded set like $\mathbb{R}^d$ or an infinite strip, the bulk of the spectrum of H_Γ will in general consist of essential spectrum, and eigenfunctions, if any exist, might span only a subspace. Hence, the subspace $\operatorname{Ran} \chi_{(-\infty,E]}$ might be infinite dimensional – a challenge.

The proofs of Theorems 3.3 and 3.5 are heavily based on the fact that the function ψ satisfies the pointwise differential inequality $|\Delta\psi| \leq |V\psi|$ almost everywhere on Λ_L, or are perturbative arguments thereof. Functions from a spectral subspace as considered in Theorem 3.7 do in general not have this property. In what follows, we explain one main idea how to bypass this difficulty. It is inspired by a technique developed for operators with compact resolvent in the context of control theory for the heat equation, see e.g. [31, 32, 22, 30].

We denote by $\{P_{H_\Gamma}(\lambda) = \chi_{(-\infty,\lambda]}(H_\Gamma) \colon \lambda \in \mathbb{R}\}$ the resolution of identity of H_Γ, and define the family of self-adjoint operators $(\mathcal{F}_t)_{t\in\mathbb{R}}$ on $L^2(\Gamma)$ by

$$\mathcal{F}_t = \int_{-\infty}^{\infty} s_t(\lambda) \mathrm{d}P_{H_\Gamma}(\lambda) \quad \text{where } s_t(\lambda) = \begin{cases} \sinh(\sqrt{\lambda}t)/\sqrt{\lambda}, & \lambda > 0, \\ t, & \lambda = 0, \\ \sin(\sqrt{-\lambda}t)/\sqrt{-\lambda}, & \lambda < 0. \end{cases}$$

The operators $\mathcal{F}_t$ are self-adjoint, lower semi-bounded, and satisfy $\operatorname{Ran} P_{H_\Gamma}(E) \subset \mathcal{D}(\mathcal{F}_t)$ for $E \in \mathbb{R}$, where $\mathcal{D}(\mathcal{F}_t)$ denotes the domain of $\mathcal{F}_t$. For $\psi \in \operatorname{Ran} P_{H_\Gamma}(E)$ and $T > 0$ we define the function $\Psi \colon \Gamma \times (-T,T) \to \mathbb{C}$ as

$$\Psi(x,t) = (\mathcal{F}_t\psi)(x).$$

Note that $\Psi(\cdot,t) \in L^2(\Gamma)$ for all $t \in (-T,T)$. Moreover, we define the (non-self-adjoint) operator $\widehat{H}_\Gamma$ on $L^2(\Gamma \times (-T,T)) \cong L^2((-T,T), L^2(\Gamma))$ on

$$\begin{aligned} &\mathcal{D}(\widehat{H}_\Gamma) \\ &= \Big\{\Phi \in L^2\big((-T,T), L^2(\Gamma)\big) \colon t \mapsto H_\Gamma(\Phi(t)) - \Big(\frac{\partial^2}{\partial t^2}\Phi\Big)(t) \in L^2\big((-T,T), L^2(\Gamma)\big)\Big\} \end{aligned}$$

by $\widehat{H}_\Gamma = -\Delta + \widehat{V}$, where $\widehat{V}(x,t) = V(x)$. Here, Δ denotes the $(d+1)$-dimensional Laplacian. We formulate a special case of Lemma 2.5 in [39].

Lemma 3.9. *For all $T > 0$, $E \in \mathbb{R}$ and all $\psi \in \operatorname{Ran} P_{H_\Gamma}(E)$ we have:*

(i) *The map $(-T,T) \ni t \mapsto \Psi(\cdot,t) \in L^2(\Gamma)$ is infinitely L^2-differentiable with*

$$\Big(\frac{\partial}{\partial t}\Psi\Big)(\cdot,0) = \psi.$$

(ii) $\Psi \in \mathcal{D}(\widehat{H}_\Gamma)$ *and* $\widehat{H}_\Gamma\Psi = 0$.

From Lemma 3.9(ii) we infer that Ψ is an eigenfunction of $\widehat{H}_\Gamma$. This allows us to apply similar techniques to the function Ψ as used in the proofs of the results presented in Subsection 3.1. In order to recover properties of ψ from properties of Ψ one combines a second Carleman estimate with boundary terms already used in [31, 22] with part (i) of Lemma 3.9.

4. From uncertainty to control

We introduce the notion of (null-)controllability in an abstract setting. Let $\mathcal{H}$ and $\mathcal{U}$ be Hilbert spaces, A a lower semi-bounded, self-adjoint operator in $\mathcal{H}$ and B a bounded operator from $\mathcal{U}$ to $\mathcal{H}$. Given $T > 0$, we consider the abstract, inhomogeneous Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u(t) + Au(t) = Bf(t), & t \in (0,T], \\ u(0) = u_0 \in \mathcal{H}, \end{cases} \tag{4.1}$$

where $u \in L^2((0,T),\mathcal{H})$ and $f \in L^2((0,T),\mathcal{U})$. The function f is also called *control function* or simply *control* and the operator B is called *control operator*. The *mild solution* to (4.1) is given by the Duhamel formula

$$u(t) = \mathrm{e}^{-tA}u_0 + \int_0^t \mathrm{e}^{-(t-s)A}Bf(s)\mathrm{d}s, \quad t \in [0,T]. \tag{4.2}$$

One central question in control theory is whether, given an input state u_0, a time $T > 0$ and a target state u_T, it is possible to find a control f, such that $u(T) = u_T$.

Definition 4.1. Let $T > 0$. The system (4.1) is *null-controllable in time* T if for every $u_0 \in \mathcal{H}$ there exists a control $f = f_{u_0} \in L^2((0,T),\mathcal{U})$ such that the solution of (4.1) satisfies $u(T) = 0$. In this case the function f_{u_0} will be called a *null-control* in time T for the initial state u_0.

The *controllability map* or *input map* is the mapping $\mathcal{B}^T : L^2((0,T),\mathcal{U}) \to \mathcal{H}$ given by

$$\mathcal{B}^T f = \int_0^T \mathrm{e}^{-(T-s)A}Bf(s)\mathrm{d}s.$$

Taking into account (4.2), clearly a function f is a null-control for (4.1) if and only if $\mathrm{e}^{-TA}u_0 + \mathcal{B}^T f = 0$. Thus, the system (4.1) is null-controllable in time $T > 0$ if and only if one has the relation $\operatorname{Ran} \mathcal{B}^T \supset \operatorname{Ran} \mathrm{e}^{-TA}$, which gives an alternative definition of null-controllability in terms of the controllability map.

Remark 4.2. Note that if the system (4.1) is null-controllable in time $T > 0$, then, by linearity of e^{-TA}, it is also controllable on the range of e^{-TA}. This means that for every $u_0 \in \mathcal{H}$ and every $u_T \in \operatorname{Ran} \mathrm{e}^{-TA}$ there is a control $f \in L^2((0,T),\mathcal{U})$ such that the solution of (4.1) satisfies $u(T) = u_T$.

In the context of the heat equation on a compact, connected and smooth manifold with control operator $B = \chi_S$, null-controllability was proved for all $T > 0$ in [31, Theorem 1] and independently in [17]:

Theorem 4.3. *Let $\mathcal{H} = \mathcal{U} = L^2(\Omega)$ for a compact and connected C^∞ manifold Ω, $A = -\Delta$ and $B = \chi_S$ for some non-empty, open $S \subset \Omega$, and $T > 0$. Then, the system* (4.1) *is null-controllable in time T.*

In fact, the statement in [31] is stronger since it allows for the control set S to change in time and it states that the null-control can be chosen smooth and with compact support.

The concept of null-controllability is closely related to a second one, the so-called *final-state-observability*: For $T > 0$ we consider the homogeneous system

$$\begin{cases} \frac{\partial}{\partial t} u(t) + Au(t) = 0, & t \in (0, T], \\ u(0) = u_0 \in \mathcal{H} \end{cases} \tag{4.3}$$

with solution given by $u(t) = \mathrm{e}^{-At} u_0$ for $t \in [0, T]$.

Definition 4.4. The system (4.3) is called *final-state-observable* in time $T > 0$ if there is a constant $C_{\mathrm{obs}} > 0$ such that for all $u_0 \in \mathcal{H}$ we have

$$\left\| \mathrm{e}^{-AT} u_0 \right\|_{\mathcal{H}}^2 \leq C_{\mathrm{obs}}^2 \int_0^T \left\| B^* \mathrm{e}^{-At} u_0 \right\|_{\mathcal{U}}^2 \mathrm{d}t \tag{4.4}$$

with B from (4.1). Inequality (4.4) is called *observability inequality.*

In [31, Corollary 2], it is noted that null-controllability of the system (4.1) leads to final-state-observability of (4.3). In fact, it is known that the notions of null-controllability and final-state-observability are equivalent:

Theorem 4.5 ([47]**, see also** [61, Chap. IV.2.]**).** *Let $T > 0$. The system* (4.1) *is null-controllable in time T if and only if the system* (4.3) *is final-state-observable in time T.*

Theorem 4.5 is, in fact, a direct consequence of the following lemma, which is a well-known result going back to [11], see also [10, 8, 33]. The proof given here is inspired by the corresponding proofs in [61] and [55, Proposition 12.1.2].

Lemma 4.6. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be Hilbert spaces, and let $X\colon \mathcal{H}_1 \to \mathcal{H}_3$, $Y\colon \mathcal{H}_2 \to \mathcal{H}_3$ be bounded operators. Then, the following are equivalent:*

(a) $\operatorname{Ran} X \subset \operatorname{Ran} Y$*;*
(b) *There is $c > 0$ such that $\|X^* z\| \leq c\|Y^* z\|$ for all $z \in \mathcal{H}_3$.*
(c) *There is a bounded operator $Z\colon \mathcal{H}_1 \to \mathcal{H}_2$ satisfying $X = YZ$.*

Moreover, in this case, one has

$$\inf \{c\colon c \text{ as in (b)}\} = \inf\{\|Z\|\colon Z \text{ as in (c)}\}, \tag{4.5}$$

and both infima are actually minima.

Proof. (a)⇒(b). First, suppose that $\operatorname{Ker} Y = \{0\}$. Let $\widetilde{\mathcal{H}}_3 = \overline{\operatorname{Ran} Y}$ be the Hilbert space with the same scalar product as in $\mathcal{H}_3$. Then, we can regard X, Y as operators with the codomains $\widetilde{\mathcal{H}}_3$ and $Y^{-1}\colon \widetilde{\mathcal{H}}_3 \to \mathcal{H}_2$ exists and is densely defined. The operator $Y^{-1}X$ is an everywhere defined closed operator, hence bounded by the closed graph theorem. In turn, also $(Y^{-1}X)^*$ is bounded. From [48, Proposition 1.7] it follows that $X^{(*)}Y^{-(*)} \subset (Y^{-1}X)^*$, where $(*)$ denotes the adjoint with respect to $\widetilde{\mathcal{H}}_3$. Hence there exists $c \geq 0$ such that $\|X^{(*)}z\| \leq c\|Y^{(*)}z\|$ for all $z \in \widetilde{\mathcal{H}}_3$. But it is easy to see that $X^{(*)}z = X^*z$ and $Y^{(*)}z = Y^*z$ for all $z \in \overline{\operatorname{Ran} Y}$. Finally note that if $z \in \big(\overline{\operatorname{Ran} Y}\big)^\perp$ then $Y^*z = 0 = X^*z$ since $\operatorname{Ker} Y^* = (\operatorname{Ran} Y)^\perp \subset (\operatorname{Ran} X)^\perp = \operatorname{Ker} X^*$ by hypothesis. Hence, in this case, (b) is proved.

If $\operatorname{Ker} Y$ is not trivial, instead of Y we take $\widehat{Y}$, the restriction of Y to the space $(\operatorname{Ker} Y)^\perp$. Since $\operatorname{Ran} \widehat{Y} = \operatorname{Ran} Y$, we can apply the first part of the proof to show $\|X^*z\| \leq c\|\widehat{Y}^*z\|$ for all $z \in \mathcal{H}_3$. Since $\widehat{Y}^*z = Y^*z$ for all $z \in \mathcal{H}_3$, the claim follows.

(b)⇒(c). We define the operator $K\colon \operatorname{Ran} Y^* \to \operatorname{Ran} X^*$ by $K(Y^*z) = X^*z$ for all $z \in \mathcal{H}_3$. The hypothesis implies that K is well defined and bounded with norm less or equal to c. We continuously extend K to $\overline{\operatorname{Ran} Y^*}$ and by zero to a bounded operator on $\mathcal{H}_2$. Then still $\|K\|_{\mathcal{H}_2\to\mathcal{H}_1} \leq c$. We obviously have $KY^* = X^*$ by construction and hence also $X = YK^*$, which implies the claim with $Z = K^*$. Since $\|Z\| = \|K\| \leq c$, this also shows that the right-hand side of (4.5) does not exceed the left-hand side.

(c)⇒(b). We clearly have

$$\|X^*z\| = \|Z^*Y^*z\| \leq \|Z^*\| \cdot \|Y^*z\|$$

for all $z \in \mathcal{H}_3$, which proves the claim with $c = \|Z^*\| = \|Z\|$. This also shows that the left-hand side of (4.5) does not exceed the right-hand side.

(c)⇒(a). This is obvious.

This concludes the proof of the equivalence of (a)–(c) and also of the identity (4.5). It remains to show that both minima in (4.5) are actually minima. This is clear for the infimum on the left-hand side. In turn, it then follows from (4.5) and the proof of (b)⇒(c) that also the infimum on the right-hand side is a minimum, which completes the proof. □

Proof of Theorem 4.5. Observe that

$$\int_0^T \big\|B^*\mathrm{e}^{-At}u_0\big\|_{\mathcal{U}}^2 \mathrm{d}t = \big\|(\mathcal{B}^T)^*u_0\big\|_{L^2((0,T),\mathcal{U})}^2.$$

The claim therefore follows from the equivalence between (a) and (b) in Lemma 4.6 by taking $X = \mathrm{e}^{-AT}\colon \mathcal{H} \to \mathcal{H}$ and $Y = \mathcal{B}^T\colon L^2((0,T),\mathcal{U}) \to \mathcal{H}$. □

Lemma 4.6 actually gives much more information: If the system (4.1) is null-controllable, corresponding to case (a) in the lemma, there exists according to case (c) a bounded operator $\mathcal{F}\colon \mathcal{H} \to L^2((0,T),\mathcal{U})$ such that

$$\|\mathcal{F}\| = \min\left\{c > 0 \mid \forall z \in \mathcal{H} : \left\|\mathrm{e}^{-TA} z\right\|_{\mathcal{H}} \leq c\left\|(\mathcal{B}^T)^* z\right\|_{L^2((0,T),\mathcal{U})} = \int_0^T \left\|B^* \mathrm{e}^{-At} z\right\|_{\mathcal{U}}^2 \mathrm{d}t\right\} \tag{4.6}$$

and $\mathrm{e}^{-TA} + \mathcal{B}^T \mathcal{F} = 0$. In particular, $\mathcal{F} u_0$ is a null-control in time T for an initial state $u_0 \in \mathcal{H}$. Moreover, if we fix an initial datum u_0 and a time $T > 0$, and are given one particular null-control f_0, the set of all null-controls is given by the closed affine space

$$f_0 + \operatorname{Ker} \mathcal{B}^T.$$

If P denotes the orthogonal projection onto $\operatorname{Ker} \mathcal{B}^T$ we have $-\mathrm{e}^{-AT} = \mathcal{B}^T(I - P)\mathcal{F}$ and the operator $\mathcal{F}^T := (I - P)\mathcal{F}$ does not depend on the choice of $\mathcal{F}$. It follows that for every $u_0 \in \mathcal{H}$, the function $\mathcal{F}^T u_0 \in L^2((0,T),\mathcal{U})$ is the unique control with minimal norm associated to the initial datum u_0.

Together with the identity (4.6), this justifies the following definition.

Definition 4.7. If the system (4.1) is null-controllable, then the norm of the above defined optimal operator $\mathcal{F}^T\colon \mathcal{H} \to L^2((0,T),\mathcal{U})$ is called *control cost in time T*. It satisfies

$$\begin{aligned} C_T := \|\mathcal{F}^T\| &= \sup_{\|u_0\|_{\mathcal{H}}=1} \min\left\{\|f\|_{L^2((0,T),\mathcal{U})} \colon \mathrm{e}^{-TA} u_0 + \mathcal{B}^T f = 0\right\} \\ &= \min\left\{C_{\mathrm{obs}} \colon C_{\mathrm{obs}} \text{ satisfies (4.4)}\right\}. \end{aligned}$$

The equivalence between final-state-observability and null-controllability can be seen as a way to reduce the study of properties of the inhomogeneous system (null-controllability) to properties of the homogeneous system (final-state-observability).

A crucial ingredient for proving observability estimates are *uncertainty relations*. An uncertainty relation is an estimate of the form

$$\forall E \in \mathbb{R}, u \in \mathcal{H}\colon \quad \left\|\chi_{(-\infty,E]}(A)u\right\|_{\mathcal{H}}^2 \leq C_{\mathrm{ur}}(E)\left\|B^* \chi_{(-\infty,E]}(A)u\right\|_{\mathcal{U}}^2 \tag{4.7}$$

for some function $C_{\mathrm{ur}}\colon \mathbb{R} \to [0,\infty)$. As we will see below, in the context of interest to us, it is possible to prove estimates of this type with

$$C_{\mathrm{ur}}(E) = d_0 \mathrm{e}^{d_1 E_+^s} \tag{4.8}$$

for some $s \in (0,1)$ and constants $d_0, d_1 > 0$. Recall that $t_+ = \max\{0,t\}$ for $t \in \mathbb{R}$.

In the case of the pure Laplacian, such estimates can be deduced from the Logvinenko–Sereda theorem, cf. Corollaries 2.7 and 2.11. In the case of Schrödinger operators, they can be proved by means of Carleman estimates as discussed in Section 3.

Remark 4.8 (Terminology). In the case where A is an elliptic second order differential operator (on a subset of $\mathbb{R}^d$ or on a manifold) and B is the indicator function of a non-empty, open subset, inequality (4.7) is also referred to as a *quantitative unique continuation principle.* In the context of control theory, it is also called *spectral inequality.*

In [30], a very transparent interplay between null-controllability, final-state-observability and spectral inequalities is used to iteratively construct a null-control and thus establish null-controllability. Since this approach is very instructive in nature, we are going to present their strategy in detail here. Even though in [30] the special case of the heat equation on bounded domains Ω with $B = \chi_S$ for some open $S \subset \Omega$ has been considered, we formulate their proof here in an abstract setting. In particular, it does not require the operator A to have purely discrete spectrum and thus can also be applied for the heat equation on unbounded domains, provided that a corresponding spectral inequality has been established.

Theorem 4.9. *Assume that $A \geq 0$ is a self-adjoint operator and that the spectral inequality* (4.7) *holds for $E \geq 0$ with $C_{\mathrm{ur}}(E) = C\mathrm{e}^{C\sqrt{E}}$ for some $C \geq 1$. Then, for every $T > 0$ the system* (4.1) *is null-controllable.*

The main idea in the proof of Theorem 4.9 in [30] are so-called *active and passive phases.* For that purpose, the time interval is decomposed as $[0, T] = \bigcup_{j\in\mathbb{N}_0}[a_j, a_{j+1}]$ where $a_0 = 0$, $a_{j+1} = a_j + 2T_j$ for $T_j > 0$ to be specified in the proof, and with $\lim_{j\to\infty} a_j = T$. The subintervals $[a_j, a_j + T_j]$ are called *active phases* and the subintervals $[a_j + T_j, a_{j+1}]$ *passive phases.* The idea is now to choose a sequence $(E_j)_{j\in\mathbb{N}_0}$, tending to infinity and to split for every $j \in \mathbb{N}_0$ the system according to $\mathcal{H} = \operatorname{Ran}\chi_{(-\infty,E_j]}(A) \oplus \operatorname{Ran}\chi_{(E_j,\infty)}(A)$ into a *low energy* and a *high energy part.* In every active phase $[a_j, a_j + T_j]$, one then deduces final-state-observability of the low energy part $\operatorname{Ran}\chi_{(-\infty,E_j]}(A)$ and thus finds a control in this time interval such that at time $a_j + T_j$, the solution will be in $\operatorname{Ran}\chi_{(E_j,\infty)}(A)$, i.e. it will be in the high energy part of the state space. Then, in the passive phase, no control will be applied and by contractivity of the semigroup e^{-At}, the solution will decay proportional to $\mathrm{e}^{-T_jE_j}$. Repeating this procedure, we will see that with appropriate choices of the T_j and the E_j, the solution tends to zero as $j \to \infty$, i.e. as $t \to T$.

In order to make these ideas more precise, the following energy-truncated control system is introduced:

$$\begin{cases} \frac{\partial}{\partial t}v(t) + Av(t) = \chi_{(-\infty,E]}(A)Bf(t), \\ v(0) = v_0 \in \operatorname{Ran}\chi_{(-\infty,E]}(A). \end{cases} \tag{4.9}$$

Lemma 4.10. *Let $\mathcal{T} > 0$ and assume that the spectral inequality* (4.7) *holds for all $E \geq 0$. Then for every $E \geq 0$, the system* (4.9) *is null-controllable in time $\mathcal{T}$ with cost $C_{\mathcal{T}}$ satisfying $C_{\mathcal{T}}^2 = C_{\mathrm{ur}}(E)/\mathcal{T}$.*

Proof. It suffices to see that the system

$$\begin{cases} \frac{\partial}{\partial t}v(t) + Av(t) = 0, \quad t > 0, \\ v(0) = v_0 \in \operatorname{Ran}\chi_{(-\infty,E]}(A), \end{cases}$$

as a system on the Hilbert space $\operatorname{Ran}\chi_{(-\infty,E]}(A)$, is final-state-observable in time $\mathcal{T}$. For that purpose, we calculate, using spectral calculus and in particular the fact that e^{At} leaves $\operatorname{Ran}\chi_{(-\infty,E]}(A)$ invariant, and (4.7) that

$$\mathcal{T}\|\mathrm{e}^{-A\mathcal{T}}v_0\|_{\mathcal{H}}^2 \leq \int_0^{\mathcal{T}} \|\mathrm{e}^{-At}v_0\|_{\mathcal{H}}^2\mathrm{d}t \leq C_{\mathrm{ur}}(E)\int_0^{\mathcal{T}} \|B^*\mathrm{e}^{-At}v_0\|_{\mathcal{U}}^2\mathrm{d}t. \qquad \square$$

Proof of Theorem 4.9. Following [30, Sect. 6.2], we split the time interval $[0,T] = \bigcup_{j\in\mathbb{N}_0}[a_j,a_{j+1}]$ with $a_0=0$, $a_{j+1}=a_j+2T_j$, and $T_j = K2^{-j/2}$ for a constant K defined by the relation $2\sum_{j=0}^{\infty}T_j = T$. Furthermore, we choose $E_j = 2^{2j}$.

Our aim is to choose in every active phase $[a_j,a_j+T_j]$ an appropriate null-control $f_j \in L^2([a_j,a_j+T_j],\mathcal{U})$ such that $u(a_j+T_j)\in\operatorname{Ran}\chi_{(E_j,\infty)}(A)$.

Therefore, let $j\in\mathbb{N}_0$ and $u(a_j)\in\mathcal{H}$ be given. In the active phase $[a_j,a_j+T_j]$, we apply Lemma 4.10 with $v_0 = \chi_{(-\infty,E_j]}(A)u(a_j)$, and $\mathcal{T}=T_j$. This yields a function $f_j\in L^2([a_j,a_j+T_j],\mathcal{U})$ with

$$\int_{a_j}^{a_j+T_j} \|f_j(t)\|_{\mathcal{U}}^2\mathrm{d}t \leq \frac{C\mathrm{e}^{C2^j}}{T_j}\|u(a_j)\|_{\mathcal{H}}^2,$$

such that the solution of the system

$$\begin{cases} \frac{\partial}{\partial t}v(t) + Av(t) = \chi_{(-\infty,E_j]}(A)Bf_j(t), & t\in(a_j,a_j+T_j],\\ v(a_j) = \chi_{(-\infty,E_j]}(A)u(a_j)\in\operatorname{Ran}\chi_{(-\infty,E_j]}(A), \end{cases}$$

satisfies $v(a_j+T_j)=0$. Since the spectral projectors of A commute with e^{-tA}, with this control function f_j in $(a_j,a_j+T_j]$ we then have $\chi_{(-\infty,E_j]}(A)u(a_j+T_j)=0$ and

$$\begin{aligned} &u(a_j+T_j)\\ &= \chi_{(E_j,\infty)}(A)u(a_j+T_j)\\ &= \mathrm{e}^{-T_jA}\chi_{(E_j,\infty)}(A)u(a_j) + \int_{a_j}^{a_j+T_j}\mathrm{e}^{-(a_j+T_j-t)A}\chi_{(E_j,\infty)}(A)Bf_j(t)\mathrm{d}t. \end{aligned} \tag{4.10}$$

We use the notation $F(t) := \mathrm{e}^{-(a_j+T_j-t)A}\chi_{(E_j,\infty)}(A)Bf_j(t)$ and estimate

$$\begin{aligned} &\left\|\int_{a_j}^{a_j+T_j}\mathrm{e}^{-(a_j+T_j-t)A}\chi_{(E_j,\infty)}(A)Bf_j(t)\mathrm{d}t\right\|_{\mathcal{H}}^2\\ &\qquad\leq \int_{a_j}^{a_j+T_j}\int_{a_j}^{a_j+T_j}\|F(t)\|_{\mathcal{H}}\cdot\|F(s)\|_{\mathcal{H}}\mathrm{d}t\mathrm{d}s\\ &\qquad\leq \frac{1}{2}\int_{a_j}^{a_j+T_j}\int_{a_j}^{a_j+T_j}\|F(t)\|_{\mathcal{H}}^2\mathrm{d}t\mathrm{d}s + \frac{1}{2}\int_{a_j}^{a_j+T_j}\int_{a_j}^{a_j+T_j}\|F(s)\|_{\mathcal{H}}^2\mathrm{d}t\mathrm{d}s\\ &\qquad= \int_{a_j}^{a_j+T_j}\int_{a_j}^{a_j+T_j}\|F(t)\|_{\mathcal{H}}^2\mathrm{d}t\mathrm{d}s \leq T_j\|B\|^2\frac{C\mathrm{e}^{C2^j}}{T_j}\|u(a_j)\|_{\mathcal{H}}^2. \end{aligned}$$

Hence, we obtain from (4.10) and using that $C \geq 1$

$$\begin{aligned}\|u(a_j+T_j)\|_{\mathcal{H}} &\leq \left(1+\|B\|\sqrt{C}\mathrm{e}^{(C/2)2^j}\right)\|u(a_j)\|_{\mathcal{H}}\\ &\leq \mathrm{e}^{(2+\|B\|)C2^j}\|u(a_j)\|_{\mathcal{H}} =: \mathrm{e}^{D2^j}\|u(a_j)\|_{\mathcal{H}}.\end{aligned}$$

Now, using $u(a_j+T_j) \in \operatorname{Ran}\chi_{(E_j,\infty)}(A) = \operatorname{Ran}\chi_{(2^{2j},\infty)}(A)$ and recalling that $T_j = K2^{-j/2}$, we find

$$\|u(a_{j+1})\|_{\mathcal{H}} \leq \mathrm{e}^{-2^{2j}T_j}\|u(a_j+T_j)\|_{\mathcal{H}} \leq \mathrm{e}^{D2^j - K2^{3j/2}}\|u(a_j)\|_{\mathcal{H}}.$$

Inductively, this yields

$$\|u(a_{j+1})\|_{\mathcal{H}} \leq \exp\left(\sum_{k=0}^{j} D2^k - K2^{3k/2}\right)\|u(0)\|_{\mathcal{H}}.$$

Thus, $\lim_{j\to\infty}\|u(a_j)\|_{\mathcal{H}}^2 = 0$. It remains to show that the function $f\colon [0,T] \to \mathcal{U}$, defined by

$$f(t) := \begin{cases} f_j(t) & \text{if } t \in [a_j, a_j+T_j],\\ 0 & \text{else}\end{cases}$$

is in $L^2((0,T),\mathcal{U})$. For that purpose, we calculate

$$\begin{aligned}&\|f\|_{L^2((0,T),\mathcal{U})}^2\\ &= \sum_{j=0}^{\infty}\int_{a_j}^{a_j+T_j}\|f_j(t)\|_{\mathcal{H}}^2\mathrm{d}t \leq \sum_{j=0}^{\infty}\frac{C\mathrm{e}^{C2^j}}{T_j}\|u(a_j)\|_{\mathcal{H}}^2\\ &\leq \left(\frac{C\mathrm{e}^{2C}}{T_0} + \sum_{j=1}^{\infty}\frac{C\mathrm{e}^{C2^j}}{T_j}\exp\left(\sum_{k=0}^{j-1}2D2^k - 2K2^{3k/2}\right)\right)\|u(0)\|_{\mathcal{H}}^2 \qquad (4.11)\\ &= \left(\frac{C\mathrm{e}^{2C}}{T_0} + \sum_{j=1}^{\infty}\frac{C}{K}\exp\left(C2^j + \frac{\ln(2)j}{2} + \sum_{k=0}^{j-1}2D2^k - 2K2^{3k/2}\right)\right)\|u(0)\|_{\mathcal{H}}^2,\end{aligned}$$

and since there are $\widetilde{C}_1, \widetilde{C}_2 > 0$ such that

$$\begin{aligned}C2^j &+ \frac{\ln(2)j}{2} + \sum_{k=0}^{j-1}2D2^k - 2K2^{3k/2}\\ &= C2^j + \frac{\ln(2)j}{2} + 2D\frac{2^j-1}{2-1} - 2K\frac{2^{3j/2}-1}{2^{3/2}-1}\\ &\leq \left(C + \frac{\ln(2)}{2} + 2D + \frac{2K}{2^{3/2}-1}\right)2^j - \left(\frac{2K}{2^{3/2}}\right)2^{3j/2}\\ &\leq \widetilde{C}_1 - \widetilde{C}_2 2^j \quad \text{for all } j \in \mathbb{N},\end{aligned}$$

the series in (4.11) converges. This concludes the proof. □

We have now seen how a spectral inequality leads to null-controllability. While being very constructive in nature, the above method makes it challenging to keep track of the estimate on the control cost, that is, on the norm of the null-control f, in terms of model parameters. Even trying to understand its T-dependence is difficult. This becomes even more involved if we endow the spectral inequality with more constants, e.g. by choosing $C_{\mathrm{ur}}(E) = d_0 \mathrm{e}^{d_1 \sqrt{E}}$, and attempt to also understand the dependence of the control cost in terms of d_0 and d_1.

However, there exist other works which have derived more explicit upper bounds on the control cost. There, usually an observability estimate for the whole system is proved without going through the active-passive-phases construction. The first work we cite here is [37], where ideas of [31] have been streamlined and generalized to a more abstract situation. In fact, Miller considered a situation where the operator A is no longer self-adjoint, but merely the generator of a strongly continuous semigroup. Due to the lack of spectral calculus, an additional assumption on contractivity of the semigroup on certain invariant subspaces (4.12) is required and serves as a replacement for the strict contractivity of the semigroup on high energy spectral subspaces. Furthermore, the situation is treated where the spectral inequality holds for an additional reference operator B_0 which is in some relation to the actual control operator B (actually, it will be the identity operator in our applications below).

Theorem 4.11 **(**[37, Theorem 2.2]**).** *Let a (not necessarily self-adjoint) operator $-A$ in $\mathcal{H}$ be the generator of a strongly continuous semigroup $\{\mathrm{e}^{-tA}\colon t \geq 0\}$. Assume that there is a family $\mathcal{H}_\lambda \subset \mathcal{H}$, $\lambda > 0$, of semigroup invariant subspaces such that for some $\nu \in (0,1)$, $m \geq 0$, $m_0 \geq 0$, and $T_0 > 0$ we have*

$$\forall \lambda > 0, x \in \mathcal{H}_\lambda^\perp, t \in (0, T_0), \quad \left\| \mathrm{e}^{-tA} x \right\|_{\mathcal{H}} \leq m_0 \mathrm{e}^{m\lambda^\nu} \mathrm{e}^{-\lambda t} \left\| x \right\|_{\mathcal{H}}. \tag{4.12}$$

Let B_0 be an operator, mapping from $\mathcal{D}(A)$ to $\mathcal{U}$, satisfying

$$\forall x \in \mathcal{H}_\lambda, \lambda > 0, \quad \left\| B_0 x \right\|_{\mathcal{H}}^2 \leq a_0 \mathrm{e}^{2a\lambda^\alpha} \left\| Bx \right\|_{\mathcal{H}}^2 \tag{4.13}$$

for some $a_0, a, \alpha > 0$. Assume that there are b_0, β, $b > 0$ such that

$$\forall x \in \mathcal{D}(A), T \in (0, T_0), \quad \left\| \mathrm{e}^{-TA} x \right\|_{\mathcal{H}}^2 \leq b_0 \mathrm{e}^{\frac{2b}{T^\beta}} \int_0^T \left\| B_0 \mathrm{e}^{-tA} x \right\|_{\mathcal{H}}^2 \mathrm{d}t. \tag{4.14}$$

Assume that we can choose $\beta = \alpha/(1-\alpha) = \nu/(1-\nu)$.

Then, for all $T > 0$, we have the observability estimate

$$\left\| \mathrm{e}^{-TA} x \right\|_{\mathcal{H}}^2 \leq \kappa_T \int_0^T \left\| B \mathrm{e}^{-tA} x \right\|_{\mathcal{H}}^2 \mathrm{d}t, \quad \forall x \in \mathcal{D}(A)$$

where κ_T satisfies $2c = \limsup_{T \to 0} T^\beta \ln \kappa_T < \infty$ with the constant c satisfying

$$c \leq c_* = \left(\frac{(\beta+1)b}{a+m} \right)^{\frac{\beta+1}{\beta}} \frac{\beta^\beta}{s^{\frac{(\beta+1)^2}{\beta}}}$$

with

$$s(s+\beta+1)^{\beta} = (\beta+1)\beta^{\frac{\beta^2}{\beta+1}} \frac{b^{\frac{1}{\beta+1}}}{a+m}.$$

Moreover, if we have

$$\forall x \in \mathcal{D}(A), T > 0, \quad \int_0^T \|B\mathrm{e}^{-tA}\|_{\mathcal{H}}^2 \mathrm{d}t \le \mathrm{Adm}_T \|x\|_{\mathcal{H}}^2$$

with a constant Adm_T *satisfying* $\lim_{T\to 0} \mathrm{Adm}_T = 0$*, then there exists* $T' > 0$ *such that for all* $T \in (0, T']$*, we have*

$$\kappa_T \le 4a_0 b_0 \exp\left(\frac{2c_*}{T^{\beta}}\right).$$

In particular, the control cost κ_T is estimated only for sufficiently small times.

One can apply Theorem 4.11 in various ways. For instance, it is possible to choose $B_0 = I$, in which case (4.14) is obviously satisfied for small times and (4.13) becomes a spectral inequality. Depending on the system, the latter can be challenging to establish or not. Alternatively, one might be able to prove (4.14) for a convenient operator B_0 for which (4.13) is easier to establish.

In the case of the system (4.1), Theorem 4.11 simplifies to the following result.

Corollary 4.12. *Let* $A \ge 0$ *be a self-adjoint operator in a Hilbert space* $\mathcal{H}$ *and* $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$*. Then,* (4.12) *holds with* $m_0 = 1$ *and* $m = 0$*. Let* (4.7) *be satisfied with* $C_{\mathrm{ur}}(\lambda) = a_0 \mathrm{e}^{2a\lambda^{\alpha}}$ (*i.e.* (4.13) *is valid for* $B_0 = I$ *and* $\mathcal{H}_\lambda$ *being spectral subspaces of* A *corresponding to the interval* $(-\infty, \lambda]$)*. Then* (4.14) *is satisfied for any choice of* $b, b_0 > 0$*, provided* T_0 *is small enough. Consequently the conclusions of Theorem* 4.11 *hold true.*

In the particular case where the spectral inequality (4.7) with $C_{\mathrm{ur}}(E)$ as in (4.8) and $s = 1/2$ holds, the result of [37] implies that the system (4.3) is final-state-observable in sufficiently small time T. Thus the system (4.1) is null-controllable in time T with cost satisfying

$$C_T \le d_0 \exp\left(\frac{c_*}{T}\right), \quad 0 < T \le T'$$

for some $T', c_* > 0$, depending in an implicit manner on d_0 and d_1. We emphasize that this result provides estimates on the control cost only for small times $0 < T \le T'$, where T' also depends in an implicit way on the model parameters.

In [4, Theorem 2.1], Beauchard, Pravda-Starov and Miller removed this restriction to small times in the specific situation where $\mathcal{H} = L^2(\Omega)$ and $B = \chi_S$ for $S \subset \Omega \subset \mathbb{R}^d$.

Theorem 4.13 ([4, Theorem 2.1])**.** *Let* Ω *be an open subset of* $\mathbb{R}^d$*,* S *be an open subset of* Ω*,* $\{\pi_k : k \in \mathbb{N}\}$ *be a family of orthogonal projections on* $L^2(\Omega)$*,* $\{\mathrm{e}^{-tA} : t \ge 0\}$ *be a contraction semigroup on* $L^2(\Omega)$*,* c_1*,* c_2*,* a*,* b*,* t_0*,* $m > 0$ *be positive constants with* $a < b$*. If the spectral inequality*

$$\forall g \in L^2(\Omega), \forall k \ge 1, \quad \|\pi_k g\|_{L^2(\Omega)} \le \mathrm{e}^{c_1 k^a} \|\pi_k g\|_{L^2(S)},$$

and the dissipation estimate

$$\forall g \in L^2(\Omega), \forall k \geq 1, \forall 0 < t < t_0\colon \left\|(1-\pi_k)(\mathrm{e}^{-TA} g)\right\|_{L^2(\Omega)} \leq \frac{1}{c_2}\mathrm{e}^{-c_2 t^m k^b} \left\|g\right\|_{L^2(\Omega)}$$

hold, then there exists a positive constant $C > 1$ *such that the following observability estimate holds*

$$\forall T > 0, \forall g \in L^2(\Omega), \quad \left\|\mathrm{e}^{-TA}\right\|^2_{L^2(\Omega)} \leq C \exp\left(\frac{C}{T^{\frac{am}{b-a}}}\right) \int_0^T \left\|\mathrm{e}^{-tA} g\right\|^2_{L^2(S)} \mathrm{d}t.$$

Let us remark that the proof of [4, Theorem 2.1] does not require S to be open, but merely to have positive measure as observed in [14].

In the applications we discuss below, the projectors π_k will be spectral projectors corresponding to the operator A and the dissipation estimate will hold automatically. Thus, the verification of the conditions of the theorem is again reduced to the verification of a spectral inequality.

In Theorem 4.13, the estimate on the control cost is again given in the form

$$C_T = \widetilde{C} \exp\left(\frac{\widetilde{C}}{T}\right), \quad T > 0 \tag{4.15}$$

for a non-explicit constant $\widetilde{C}$. Note that this constant C_T does not converge to zero as T tends to ∞. In some situations, however, the constant can be strengthened to show this asymptotic behavior at large times. A step in this direction is [54, Theorem 1.2]. We note that there, more general control operators B are considered, while A is assumed to be a non-negative self-adjoint operator with purely discrete spectrum.

Theorem 4.14 ([54, Theorem 1.2])**.** *Let A be a non-negative operator in $\mathcal{H}$ and let $B \in \mathcal{L}(\mathcal{U}, \mathcal{H}_\beta)$ for some $\beta \leq 0$, where $\mathcal{H}_\beta$ is the completion of $\mathcal{H}$ with respect to the scalar product*

$$\langle x, y\rangle_{\mathcal{H}_\beta} = \left\langle (\mathrm{Id} + A^2)^{\beta/2} x, (\mathrm{Id} + A^2)^{\beta/2} x\right\rangle_{\mathcal{H}}.$$

Assume that A is diagonalizable, that $\{\phi_k \colon k \in \mathbb{N}\}$ is an orthonormal basis of eigenvectors with corresponding non-decreasing sequence of eigenvalues $\{\lambda_k \colon k \in \mathbb{N}\}$ such that $\lim_{k\to\infty} \lambda_k = \infty$. Assume furthermore that there exists $s \in (0,1)$ such that for some $d_0, d_1 > 0$, we have

$$\forall \{a_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C}), \mu > 1, \quad \left(\sum_{\lambda_k^s \leq \mu} \left\|\alpha_k\right\|^2\right)^{1/2} \leq d_0 \mathrm{e}^{d_1 \mu} \left\| \sum_{\lambda_k^s \leq \mu} a_k B^* \phi_k \right\|_{\mathcal{U}}.$$

Then, the system

$$\dot{w} = -Aw + Bu, \quad w(0) = z \tag{4.16}$$

is null-controllable in any time $T > 0$. Moreover, given $c > h^{gh} g^{-g^2} d_1^h$, where $g = s/(1-s)$, $h = g + 1 = 1/(1-s)$, the control cost satisfies $C_T \leq \widetilde{C} T^{-1/2} \mathrm{e}^{c/T^g}$ for a constant $\widetilde{C}$ depending only on d_0, d_1, c, β, s, and $\left\|B\right\|_{\mathcal{L}(\mathcal{U},\mathcal{H}_\beta)}$.

Note that in Theorem 4.14 as well as in Theorem 4.15 below, the Duhamel formula (4.2), defining the mild solution of the system (4.16), now describes a function in the Hilbert space $\mathcal{H}_\beta$ whence also the semigroup $\mathrm{e}^{-A\cdot}$ needs to be appropriately extended from $\mathcal{H}$ to $\mathcal{H}_\beta$, see e.g. [15, II.5.a] for details.

Theorem 4.14 shows that if A has compact resolvent, then a spectral inequality with $C_{\mathrm{ur}} = d_0 \mathrm{e}^{d_1 \lambda^s}$ for all $\lambda \geq 0$ and some $s \in (0,1)$ implies null-controllability in all times $T > 0$ with cost satisfying

$$C_T \leq \frac{C_1}{\sqrt{T}} \exp\left(\frac{C_2}{T^{\frac{s}{1-s}}}\right). \tag{4.17}$$

The upper bound in (4.17) decays proportional to $\sqrt{T}^{-1}$ as T tends to infinity and thus improves upon the upper bound in (4.15). Furthermore, [54] provides an estimate on C_2 in terms of s and d_1. However, it remains unclear whether and how C_1 depends on s, C_2, d_0, d_1, and on the operator B.

While the results in [37] and in [4] are both inspired by [31] and thus the structure of the proofs is rather similar, the proof in [54] has a different structure which makes it easier to keep track of the dependence of the constant C_T in terms of the model parameters, even though this analysis has not been thoroughly performed in [54].

In the recent paper [38], the result of [54] is generalized to non-negative self-adjoint operators (regardless of the spectral type) with explicit dependence on the model parameter. This unifies advantages of all the control cost bounds mentioned above, at least for heat flow control problems.

Theorem 4.15 ([38])**.** *Let A be a non-negative, self-adjoint operator in a Hilbert space $\mathcal{H}$ and $B \in \mathcal{L}(\mathcal{U}, \mathcal{H}_\beta)$ for some $\beta \leq 0$, where $\mathcal{U}$ is a Hilbert space and $\mathcal{H}_\beta$ is defined as in Theorem* 4.14*. Assume that there are $d_0 > 0$, $d_1 \geq 0$ and $s \in (0,1)$ such that for all $\lambda > 0$ we have the spectral inequality* (4.7) *with $C_{\mathrm{ur}}(\lambda) = d_0 \mathrm{e}^{d_1 \lambda^s}$. Then for all $T > 0$, we have*

$$\left\|\mathrm{e}^{-AT} u_0\right\|_{\mathcal{H}}^2 \leq C_{\mathrm{obs}}^2 \int_0^T \left\|B^* \mathrm{e}^{-At} u_0\right\|_{\mathcal{U}}^2 \mathrm{d}t$$

where

$$C_{\mathrm{obs}}^2 = \frac{C_1 d_0}{T} K^{C_2} \exp\left(C_3 \left(\frac{d_1 + (-\beta)^{C_4}}{T^s}\right)^{\frac{1}{1-s}}\right)$$

with $K = 2 d_0 \mathrm{e}^{-\beta} \|B\|_{\mathcal{L}(\mathcal{U}, \mathcal{H}_\beta)} + 1$. Here C_1, C_2, C_3, and C_4 are constants which depend only on s.

5. Null-controllability of the heat and Schrödinger semigroups

In the previous Section 4, we have seen how uncertainty relations, respectively spectral inequalities, lead to null-controllability of abstract systems. In particular, Theorem 4.15 provides a very explicit estimate on the resulting control cost. We now combine this abstract result with the results of Sections 2 and 3 to deduce null-controllability of the heat equation on cubes and on $\mathbb{R}^d$ with so-called interior control and provide explicit estimates on the control cost. In particular, the cost will be explicitly given in terms of parameters which describe the geometry of the control set.

We start by examining the classical heat equation. Recall from Section 3 that $\Lambda_L = (-L/2, L/2)^d \subset \mathbb{R}^d$ for $L > 0$. Let $\Omega \in \{\Lambda_L, \mathbb{R}^d\}$. If $\Omega = \mathbb{R}^d$, then Δ denotes the self-adjoint Laplacian in $L^2(\mathbb{R}^d)$. If $\Omega = \Lambda_L$, then Δ denotes the self-adjoint Laplacian in $L^2(\Lambda_L)$ with Dirichlet, Neumann or periodic boundary conditions. Given a measurable $S \subset \mathbb{R}^d$, the *controlled heat equation* in time $[0, T]$ with control operator $B = \chi_{S\cap\Omega}$ (this choice is also called *interior control*) is

$$\frac{\partial}{\partial t}u - \Delta u = \chi_{S\cap\Omega} f, \quad u, f \in L^2((0,T)\times\Omega), \quad u(0,\cdot) = u_0 \in L^2(\Omega). \tag{5.1}$$

Note that by the above convention, the boundary conditions are fixed by the choice of the self-adjoint Laplacian. If $\Omega = \mathbb{R}^d$, the system (5.1) is null-controllable if and only if S is a thick set, see [14, 59]. If $\Omega = \Lambda_L$, the system (5.1) is null-controllable if and only if $\|\Lambda_L \cap S\| > 0$, see [3]. Furthermore, in [14], combining the spectral inequalities from Corollaries 2.7 and 2.11 with the technique by [4], cf. Theorem 4.13, the following estimate on the control cost is provided:

Theorem 5.1. *Let $L > 0$, $\Omega \in \{\Lambda_L, \mathbb{R}^d\}$, $S \subset \mathbb{R}^d$ a (γ, a)-thick set with $a = (a_1, \dots, a_d)$ and $\gamma > 0$. If $\Omega = \Lambda_L$, we assume that $0 < a_j \leq L$ for all $j = 1, \dots, d$. Then, for every $T > 0$, the system* (5.1) *is null-controllable in time T with cost satisfying*

$$C_T \leq C_1^{1/2} \exp\left(\frac{C_1}{2T}\right), \quad \text{where} \quad C_1 = \left(\frac{K^d}{\gamma}\right)^{K(d+\|a\|_1)}, \tag{5.2}$$

where K is a universal constant and $\|a\|_1 = \sum_{j=1}^d a_j$.

As discussed in Section 2, the spectral inequalities used in the proof of Theorem 5.1 have recently been extended in [12] to strips, see Remark 2.12. This has led in an analogous way to the following result which, to the best of our knowledge, is the first result of this kind dealing with an unbounded domain Ω that is not the whole of $\mathbb{R}^d$.

Theorem 5.2 ([12])**.** *Let $L > 0$, $\Omega = (-L/2, L/2)^{d-1} \times \mathbb{R}$, $S \subset \mathbb{R}^d$ a (γ, a)-thick set with $\gamma > 0$, and $0 < a_j \leq L$ for all $j = 1, \dots, d-1$. Then, for every $T > 0$, the system* (5.1) *with Dirichlet or Neumann boundary conditions is null-controllable in time T with cost satisfying the bound* (5.2).

Here, thickness of S is again a necessary requirement for null-controllability (where obviously S can be arbitrarily modified outside Ω). We refer to [12] for more details.

In light of the discussion made in the previous section, the bound in Theorem 5.1 (and, of course, Theorem 5.2) can be strengthened if Theorem 4.13 in the last step of the proof is replaced by Theorem 4.15. For $\Omega \in \{\Lambda_L, \mathbb{R}^d\}$, this has been performed in [38, 51]:

Theorem 5.3. *Let $L > 0$, $\Omega \in \{\Lambda_L, \mathbb{R}^d\}$, $S \subset \mathbb{R}^d$ a (γ, a)-thick set with $a = (a_1, \dots, a_d)$ and $\gamma > 0$. If $\Omega = \Lambda_L$, we assume that $0 < a_j \leq L$ for all $j = 1, \dots, d$. Then, for every $T > 0$, the system* (5.1) *is null-controllable in time T with cost satisfying*

$$C_T \leq \frac{D_1}{\gamma^{D_2}\sqrt{T}} \exp\left(\frac{D_3 \|a\|_1^2 \ln^2(D_4\gamma)}{T}\right). \tag{5.3}$$

where D_1 to D_4 are constants which depend only on the dimension.

Proof. By Corollaries 2.7 and 2.11 we have the spectral inequality

$$\forall E \geq 0, u \in \operatorname{Ran} \chi(-\infty, E](-\Delta): \quad \|u\|_{L^2(\Omega)}^2 \leq d_0 \mathrm{e}^{d_1\sqrt{E}} \|\chi_{S\cap\Omega} u\|_{L^2(\Omega)}^2$$

with

$$d_0 = \left(\frac{N_1}{\gamma}\right)^{N_2} \quad \text{and} \quad d_1 = N_3 \|a\|_1 \ln\left(\frac{N_1}{\gamma}\right),$$

where N_1, N_2, and N_3 are constants, depending only on the dimension. Theorem 4.15 together with the equivalence between null-controllability and final-state-observability, and the absorbtion of all universal constants into D_1 to D_4 yields the result. ☐

Remark 5.4. In order to discuss the bound (5.3), let us first compare it to *lower bounds* on the control cost. For the controlled heat equation (5.1) with open S it is known that the control cost grows at least proportional to $\exp(C/T)$ as T tends to zero unless $S = \Omega$, see e.g. [16, 36]. Thus, the T-dependence (5.3) is optimal in the small time regime.

On the other hand, the $T^{-1/2}$ term will dominate for large T. This is also optimal. One way to see this is to study the ODE system

$$\begin{cases} y'(t) = Cf(t), \quad y, f \in L^2((0,T), \mathbb{C}), \\ y(0) = y_0 \in \mathbb{C}, \end{cases}$$

the control cost of which can be explicitly computed and is $C/\sqrt{T}$ in time T for every $T > 0$. This also shows that the minimal possible lower bound on the control cost in time T of abstract controlled systems as in (4.1) is of order $T^{-1/2}$. This argument can be slightly generalized to show that this lower bound holds in fact for *all* systems of the form (4.1), see [38] for details. We conclude that the control cost in time T is lower bounded by $C/\sqrt{T}$ for all $T > 0$ for some constant C.

An interesting limit is the *homogenization limit* of the control set where the parameter a tends to zero while the parameter γ remains constant. This corresponds to requiring an equidistribution on finer and finer scales a while keeping the overall density γ constant. We see that the exponential term, which is characteristic for the heat equation with control operator $B = \chi_{S\cap\Omega}$ where $S \subset \Omega, S \neq \Omega$, is annihilated. On the other hand the $1/\sqrt{T}$ factor, which is universal in the class of abstract linear control systems, remains unaffected. This limit can be interpreted as the control cost of the system with *weighted full control*, i.e. where χ_S has been replaced by $c_\gamma \chi_\Omega$ with a γ-dependent constant $c_\gamma \in (0, 1]$.

Now we study the heat equation with *non-negative* potential or homogeneous source term. Instead of considering thick control sets $S \subset \mathbb{R}^d$, we will restrict our attention to a special geometric setting, namely to equidistributed unions of δ-balls. Recall the notation from Section 3: If $G > 0$, $\delta \in (0, G/2)$, and Z is a (G, δ)-equidistributed sequence then

$$S_{\delta,Z} = \bigcup_{j\in(G\mathbb{Z})^d} B(z_j, \delta).$$

Let $L \geq G$ and $\Omega \in \{\Lambda_L, \mathbb{R}^d\}$. For a non-negative $V \in L^\infty(\Omega)$ the *controlled heat equation with potential* V in time $[0, T]$ with interior control in $S_{\delta,Z} \cap \Omega$ is

$$\frac{\partial}{\partial t}u - \Delta u + Vu = \chi_{S_{\delta,Z}\cap\Omega} f, \quad u, f \in L^2((0,T)\times\Omega), \quad u(0,\cdot) = u_0 \in L^2(\Omega). \tag{5.4}$$

In [41], Theorem 3.6 and Miller's Theorem 4.11 were combined to prove:

Theorem 5.5. *There exists $T' > 0$, depending on G, δ, and $\|V\|_\infty$ such that for all $T \leq T'$, the system* (5.4) *is null-controllable in time T with cost C_T satisfying*

$$C_T \leq 2\left(\frac{G}{\delta}\right)^{K\left(1+G^{4/3}\|V\|_\infty^{2/3}\right)} \exp\left(\|V\|_\infty + \frac{\ln^2(\delta/G)\,(KG + 4/\ln(2))^2}{T}\right)$$

with a dimension-dependent K.

Again we can improve this bound by replacing Theorem 4.11 with a more suitable estimate. Furthermore, certain unbounded domains can be treated as well. More precisely, combining Theorems 3.7 and 4.15, we obtain analogously to Theorem 5.3 the following result.

Theorem 5.6. *Let $G > 0$, $0 < \delta < G/2$, Z a (G, δ)-equidistributed sequence, $L \geq G$, and $\Omega \in \{\Lambda_L, \mathbb{R}^d\}$. Then, for every $T > 0$, the system* (5.4) *is null-controllable in time T with cost satisfying*

$$C_T \leq \frac{D_1}{\sqrt{T}}\left(\frac{G}{\delta}\right)^{D_2\left(1+G^{4/3}\|V\|_\infty^{2/3}\right)} \exp\left(\frac{D_3 G^2 \ln^2(\delta/G)}{T}\right). \tag{5.5}$$

where D_1, D_2, D_3 are constants which depend only on the dimension.

Theorem 5.6 improves upon Theorem 5.5 since it allows for all times $T > 0$ and since the argument of the exponential term is now of order G^2 as $G \to 0$, which is optimal.

The difference between Theorems 5.3 and 5.6 is that Theorem 5.3 allows for more general control sets, while Theorem 5.6 treats Schrödinger operators with non-negative potential instead of the pure Laplacian.

Remark 5.7. By the same arguments as in Remark 5.4, we see that the asymptotic T-dependence in Theorem 5.6 is optimal. Homogenization of the control set now corresponds to $G, \delta \to 0$ with $\delta/G = \rho$ for some $\rho \in (0, 1/2)$. In the limit, the upper bound in (5.5) tends to

$$\frac{D_1}{\sqrt{T}} \rho^{D_2}.$$

We see that homogenization not only annihilates the term $\exp(C/T)$ which is characteristic for the heat equation, but also the influence of a non-negative potential V on the control cost estimate disappears.

Furthermore, the dependence of the exponential term on the parameter G in (5.5) is optimal. This can best be seen in the special case $V = 0$ by comparing it to a lower bound on the control cost in terms of the geometry deduced in [36]. In fact, for the heat equation on smooth, connected manifolds Ω with control operator $B = \chi_S$ for an open $S \subset \Omega$ it is proved in [36] that the control cost C_T in time T satisfies

$$\sup_{\overline{B_\rho} \subset \Omega \setminus \overline{S}} \rho^2/4 \leq \liminf_{T \to 0} T \ln C_T. \tag{5.6}$$

Inequality (5.5), on the other hand, implies

$$\limsup_{T \to 0} T \ln C_T \leq D_3 G^2 \ln^2(\delta/G). \tag{5.7}$$

Thus, we complement the lower bound in (5.6) by an upper bound. More precisely, for a (G, δ)-equidistributed sequence Z, it is clear that the complement of $\overline{S_{\delta,Z}}$ (in Λ_L or $\mathbb{R}^d$, respectively) always contains a ball of radius

$$\rho = \frac{1}{2}\left(\frac{G}{2} - \delta\right) = G\frac{1 - 2\delta/G}{4} \quad \text{whence} \quad G\frac{1 - 2\delta/G}{4} \leq \sup_{B_\rho \subset \Omega \setminus S_{\delta,Z}} \rho.$$

Combining this with (5.6) and (5.7), we find

$$\begin{aligned} G^2 \frac{(1 - 2\delta/G)^2}{64} \leq \sup_{B_\rho \subset \Omega \setminus S_{\delta,Z}} \rho^2/4 &\leq \liminf_{T \to 0} T \ln C_T \\ &\leq \limsup_{T \to 0} T \ln C_T \leq D_3 G^2 \ln^2(\delta/G). \end{aligned}$$

If we perform the limit $G \to 0$ or $G \to \infty$, respectively, while keeping δ/G constant, this reasoning shows that the factor G^2 in the exponential term in (5.5) is optimal.

Remark 5.8. So far, we only used the fact that $V \geq 0$. If, however, we have $V \geq \kappa > 0$, then the control cost should decay proportional to $\exp(-\kappa T)$ at large times. This can be seen by modifying the construction of the null-control, see [51, 38]

Conversely, if we only have $V \in L^\infty$, but $\inf V < 0$, then the situation might become even more interesting. In fact, the relevant quantity is $\min \sigma(-\Delta + V)$. If

$\min\sigma(-\Delta+V)<0$, then the semigroup $\exp((\Delta-V)t)$ will be non-contractive and the control cost will be bounded away from zero uniformly for all times $T>0$. This situation can also be studied by an appropriate generalization of the above arguments, see [51, 38].

Remark 5.9. One can also study the *fractional heat equation* for $\theta\in(1/2,\infty)$:

$$\frac{\partial}{\partial t}u+(-\Delta)^\theta u=\chi_{S\cap\Omega}f,\quad u,f\in L^2((0,T)\times\Omega),\quad u(0,\cdot)=u_0\in L^2(\Omega)\tag{5.8}$$

and deduce an estimate on the control cost. Here, again $\Omega\in\{\Lambda_L,\mathbb{R}^d\}$, and S is a (γ,a)-thick set such that $0<a_j\le L$ for all $j=1,\dots,d$ in case $\Omega=\Lambda_L$. It is known that the fractional heat equation on one-dimensional intervals is null-controllable if and only if $\theta>1/2$, see [35]. In order to deduce a control cost estimate, it suffices to deduce an uncertainty relation for the operator $(-\Delta)^\theta$. For that purpose, we estimate using the transformation formula for spectral measures, cf. [48, Proposition 4.24], and the uncertainty relation for the pure Laplacian in Corollaries 2.7 and 2.11

$$\begin{aligned}\|\chi_{(-\infty,\lambda]}((-\Delta)^\theta)u\|^2_{L^2(\Omega)}&=\|\chi_{(-\infty,\lambda^{1/\theta}]}(-\Delta)u\|^2_{L^2(\Omega)}\\&\le d_0\mathrm{e}^{d_1\lambda^{1/(2\theta)}}\|\chi_S\cdot\chi_{(-\infty,\lambda^{1/\theta}]}(-\Delta)u\|^2_{L^2(\Omega)}\\&=d_0\mathrm{e}^{d_1\lambda^{1/(2\theta)}}\|\chi_S\cdot\chi_{(-\infty,\lambda]}((-\Delta)^\theta)u\|^2_{L^2(\Omega)}\end{aligned}\tag{5.9}$$

for all $\lambda\ge0$ and all $u\in L^2(\Omega)$ where

$$d_0=\left(\frac{N_1}{\gamma}\right)^{N_2}\quad\text{and}\quad d_1=N_3\|a\|_1\ln\left(\frac{N_1}{\gamma}\right),$$

with constants N_1, N_2, and N_3, depending only on the dimension. Combining (5.9) and Theorem 4.15, we obtain the following result.

Corollary 5.10. *Let $\theta\in(1/2,\infty)$, $\Omega\in\{\Lambda_L,\mathbb{R}^d\}$, and S be a (γ,a)-thick set such that, in case $\Omega=\Lambda_L$, $0<a_j\le L$ for all $j=1,\dots,d$. Then the system* (5.8) *is null-controllable in any time $T>0$ with cost satisfying*

$$C_T\le\frac{D_1}{\gamma^{D_2}\sqrt{T}}\exp\left(\frac{D_3\left(\|a\|_1\ln(D_4/\gamma)\right)^{\frac{2\theta}{2\theta-1}}}{T^{\frac{1}{2\theta-1}}}\right)$$

for constants $D_1,\dots,D_4$, depending only on $\theta>1/2$ and on the dimension.

6. Convergence of solutions along exhausting cubes

In this section we review certain approximation results which have been indicated in [14] and spelled out with proofs in [49]. They describe how controllability problems on unbounded domains can be approximated by corresponding problems on a sequence of bounded domains. Since these results apply to a larger class of Schrödinger operators than discussed so far, we will introduce them first.

Let $\Omega \subset \mathbb{R}^d$ be an open set and $\Lambda_L := (-L/2, L/2)^d$ with $L > 0$ as before. Let $V\colon \mathbb{R}^d \to \mathbb{R}$ be a potential such that $V_+ := \max(V, 0) \in L^1_{\mathrm{loc}}(\Omega)$ and $V_- := \max(-V, 0)$ is in the Kato class; see, e.g., [9, Sect. 1.2] for a discussion of the Kato class in $\mathbb{R}^d$. Under these hypotheses, one can define the Dirichlet Schrödinger operators H_Ω and $H_L = H_{\Omega\cap\Lambda_L}$ as lower semi-bounded self-adjoint operators on $L^2(\Omega)$ and $L^2(\Omega \cap \Lambda_L)$, respectively, associated with the differential expression $-\Delta + V$ via their quadratic forms, with form core $C_c^\infty(\Omega)$ and $C_c^\infty(\Omega \cap \Lambda_L)$, respectively. For details of this construction we refer to [9, Sect. 1.2], [21, Sect. 2], and the references therein. In fact, our arguments apply to Schrödinger operators incorporating a magnetic vector potential as well, see [49] for details.

Since we want to compare operators defined on two different Hilbert spaces, namely $L^2(\Omega)$ and $L^2(\Omega \cap \Lambda_L)$, we need a notion of extension. Corresponding to the orthogonal decomposition $L^2(\Omega) = L^2(\Omega \cap \Lambda_L) \oplus L^2(\Omega \setminus \Lambda_L)$, we identify H_L with the direct sum $H_L \oplus 0$ on $L^2(\Omega)$. Consequently, the subspace $L^2(\Omega \cap \Lambda_L) \subset L^2(\Omega)$ is a reducing subspace for the self-adjoint operator H_L on $L^2(\Omega)$. Hence, the exponential $\mathrm{e}^{-tH_L} = \mathrm{e}^{-tH_L} \oplus I$ for all $t \geq 0$ decomposes as well; see, e.g., [48, Definition 1.8] and [60, Satz 8.23]. In particular, e^{-tH_L} is a bounded self-adjoint operator on $L^2(\Omega)$, and $\mathrm{e}^{-tH_L} f = 0$ on $\Omega \setminus \Lambda_L$ for all $f \in L^2(\Omega \cap \Lambda_L)$.

6.1. Approximation of semigroups based on an exhaustion of the domain

An important tool in what follows is an approximation result for Schrödinger semigroups. It applies to a sequence of semigroups, all of the same type, but defined on different domains.

Lemma 6.1 **(**[49]**).** *Let $R > 0$, $u_0 \in L^2(\Omega \cap \Lambda_R) \subset L^2(\Omega)$, and $t > 0$. Then, there exists a constant $C = C(t, d, V_-) > 0$ such that for every $L \geq 2R$ one has*

$$\big\| \big(\mathrm{e}^{-tH_\Omega} - \mathrm{e}^{-tH_L}\big) u_0 \big\|^2_{L^2(\Omega)} \leq C \exp\big(-\frac{L^2}{32t}\big) \|u_0\|^2_{L^2(\Omega)}.$$

The lemma implies that for every $t > 0$ the exponential e^{-tH_L} converges strongly to e^{-tH_Ω} as $L \to \infty$. Moreover, it exhibits a very explicit error bound if the support of the function u_0 is located inside some cube. However, for what we present here the qualitative statement on strong convergence will be all what we will use.

6.2. Continuous dependence on inhomogeneity

In the applications we have in mind, the above approximation estimate for a sequence of semigroups needs to be complemented by an approximation result with respect to change of the right-hand side of the partial differential equation and truncation of the initial datum. This is presented next in a more general framework.

Let $\mathcal{H}$ and $\mathcal{U}$ be Hilbert spaces, and let $T > 0$. Recall (cf. Sect. 4 above) that given a lower semi-bounded self-adjoint operator A on $\mathcal{H}$, a bounded operator

$B\colon \mathcal{U} \to \mathcal{H}$, $u_0 \in \mathcal{H}$, and $f \in L^2((0,T),\mathcal{U})$, the continuous function $u\colon [0,T] \to \mathcal{H}$ with

$$u(t) = \mathrm{e}^{-tA}u_0 + \int_0^t \mathrm{e}^{-(t-s)A}Bf(s)\mathrm{d}s$$

is called the *mild solution* to the abstract Cauchy problem

$$\frac{\partial}{\partial t}u(t) + Au(t) = Bf(t) \quad \text{for} \quad 0 < t < T, \quad u(0) = u_0.$$

Lemma 6.2 ([49]). *Let A, A_n, $n \in \mathbb{N}$, be lower semi-bounded self-adjoint operators on the Hilbert space $\mathcal{H}$ with a common lower bound $a \in \mathbb{R}$. Assume that $(\mathrm{e}^{-tA_n})_n$ converges strongly to e^{-tA} for all $t > 0$. Let B, B_n, $n \in \mathbb{N}$, be bounded operators from $\mathcal{U}$ to $\mathcal{H}$ such that $(B_n)_n$ and $(B_n^*)_n$ converge strongly to B and B^*, respectively. Moreover, let $(u_{0,n})_n$ be a sequence in $\mathcal{H}$ converging in norm to some $u_0 \in \mathcal{H}$. Let $f, f_n \in L^2((0,T),\mathcal{U})$, $n \in \mathbb{N}$. Denote by u and u_n, $n \in \mathbb{N}$, the mild solutions to the abstract Cauchy problems*

$$\frac{\partial}{\partial t}u(t) + Au(t) = Bf(t) \quad \textit{for } 0 < t < T, \quad u(0) = u_0,$$

and

$$\frac{\partial}{\partial t}u_n(t) + A_n u_n(t) = B_n f_n(t) \quad \textit{for } 0 < t < T, \quad u_n(0) = u_{0,n},$$

respectively.

(a) *If $(f_n)_n$ converges to f in $L^2((0,T),\mathcal{U})$, then $(u_n(t))_n$ converges to $u(t)$ in $\mathcal{H}$ for all $t \in (0,T]$. Moreover, $(u_n)_n$ converges to u in $L^2((0,T),\mathcal{U})$.*

(b) *If $(f_n)_n$ converges to f weakly in $L^2((0,T),\mathcal{U})$, then $(u_n(t))_n$ converges to $u(t)$ weakly in $\mathcal{H}$ for all $t \in (0,T]$. Moreover, the sequence $(u_n)_n$ converges to u weakly in $L^2((0,T),\mathcal{H})$.*

If the sequence $(f_n)_n$ consists of null-controls as in Definition 4.1, then this property is inherited by the limit f, more precisely:

Corollary 6.3. *If in the situation of Lemma 6.2 the sequence $(f_n)_n$ converges to f weakly in $L^2((0,T),\mathcal{U})$ and for every n one has $u_n(T) = 0$, then also $u(T) = 0$.*

6.3. Construction of controls via exhaustion of the domain

In certain situations it may be easier to infer (or is already known) that a certain variant of the heat equation exhibits a null-control provided the domain of the problem is bounded. With the operators H_L and H_Ω introduced above we present a criterion how one can infer the existence of a null-control of the corresponding problem on an unbounded domain.

Theorem 6.4 ([49]). *Let $S \subset \mathbb{R}^d$ be measurable, $\widetilde{u} \in L^2(\Omega)$, and $(L_n)_n$ a sequence in $(0,\infty)$ with $L_n \nearrow \infty$ as $n \to \infty$. Let $f_n \in L^2((0,T), L^2(\Omega \cap \Lambda_{L_n} \cap S))$ for each $n \in \mathbb{N}$ be a null-control for the initial value problem*

$$\frac{\partial}{\partial t}u(t) + H_{L_n}u(t) = \chi_{\Omega\cap\Lambda_{L_n}\cap S}f_n(t) \quad \textit{for } 0 < t < T, \quad u(0) = \chi_{\Omega\cap\Lambda_{L_n}}\widetilde{u}, \quad (6.1)$$

and u_n the corresponding mild solution.

Suppose that $(f_n)_n$ *converges weakly in* $L^2((0,T),L^2(\Omega))$ *to some function* f*. Then,* f *is a null-control for*

$$\frac{\partial}{\partial t}u(t) + H_\Omega u(t) = \chi_{\Omega\cap S}f(t) \quad \text{for } 0<t<T, \quad u(0)=\widetilde{u}, \tag{6.2}$$

and the corresponding mild solution is the weak limit of $(u_n)_n$ *in* $L^2((0,T),L^2(\Omega))$*.*

The above theorem is based on Lemma 6.2 in the situation $\mathcal{U} = \mathcal{H} = L^2(\Omega)$ with $A = H_\Omega$, $A_n = H_{L_n}$, $B = \chi_{\Omega\cap S}$, and $B_n = \chi_{\Omega\cap\Lambda_{L_n}\cap S}$. In this case, due to the discussion before Definition 4.7, the null-controls for (6.1) and (6.2) can indeed be assumed to be supported in $\Omega\cap S$ and $\Omega\cap\Lambda_{L_n}\cap S$, respectively.

Note that if the null-controls f_n in Theorem 6.4 are uniformly bounded, that is,

$$\|f_n\|_{L^2((0,T),L^2(\Omega\cap\Lambda_{L_n}\cap S))} \leq c \quad \text{for all } \; n\in\mathbb{N} \tag{6.3}$$

for some constant $c>0$, then $(f_n)_n$ has a weakly convergent subsequence with limit in $L^2((0,T),L^2(\Omega\cap S))$. Theorem 6.4 can then be applied to every such weakly convergent subsequence, and the corresponding weak limit f of the subsequence of $(f_n)_n$ automatically satisfies the bound

$$\|f\|_{L^2((0,T),L^2(\Omega\cap S))} \leq c. \tag{6.4}$$

This leads to the following corollary to Theorem 6.4.

Corollary 6.5. *Let* $S\subset\mathbb{R}^d$ *be measurable,* $\widetilde{u}\in L^2(\Omega)$*, and* $(L_n)_n$ *a sequence in* $(0,\infty)$ *with* $L_n \nearrow \infty$ *as* $n\to\infty$*. Let* $f_n \in L^2((0,T),L^2(\Omega\cap\Lambda_{L_n}\cap S))$ *for each* $n\in\mathbb{N}$ *be a null-control for the initial value problem* (6.1)*, and let* u_n *be the corresponding mild solution.*

Assume that there is a constant $c\in\mathbb{R}$ *such that* (6.3) *holds. Then there exists a subsequence of* $(f_n)_n$ *which converges weakly to a null-control*

$$f\in L^2((0,T),L^2(\Omega\cap S))$$

for (6.2)*, satisfying* (6.4) *as well. The mild solution* u *associated to (any such weak accumulation point)* f *is the weak limit of the corresponding subsequence of* $(u_n)_n$ *in* $L^2((0,T),L^2(\Omega))$*.*

As discussed in previous sections, the control cost estimate (6.3) can be inferred by a final state observability estimate. Consequently, a scale-free uncertainty principle or spectral inequality, as formulated in Theorem 2.8 or Theorem 3.6, leads not only to control cost estimates on a sequence of bounded cubes Λ_L but also to the limiting domain $\Omega=\mathbb{R}^d$. This means that results like Theorem 5.1 or Theorem 5.6 (for $\Omega=\mathbb{R}^d$) could be obtained by a (partially) alternative method, where one performs hard analysis for partial differential equations only on bounded domains and then invokes operator theoretic methods to lift the results to unbounded domains. For details see [49].

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Dichotomous Hamiltonians and Riccati equations for systems with unbounded control and observation operators

Christian Wyss

Abstract. The control algebraic Riccati equation is studied for a class of systems with unbounded control and observation operators. Using a dichotomy property of the associated Hamiltonian operator matrix, two invariant graph subspaces are constructed which yield a nonnegative and a nonpositive solution of the Riccati equation. The boundedness of the nonnegative solution and the exponential stability of the associated feedback system is proved for the case that the generator of the system has a compact resolvent.

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1. Introduction

In systems theory, the algebraic Riccati equation

$$A^*X + XA - XBB^*X + C^*C = 0 \tag{1}$$

plays an important role in many areas. One example is the problem of linear quadratic optimal control where a selfadjoint nonnegative solution is of particular interest. For infinite-dimensional systems such a solution is often constructed in parallel to a solution of the optimal control problem. This has been done for different kinds of linear systems, e.g. in [6, 15, 16, 17, 20].

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On the other hand, the Riccati equation is closely connected to the so-called Hamiltonian operator matrix

$$T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}. \tag{2}$$

An operator X is a solution of (1) if and only if its associated graph $\mathcal{R}\left(\begin{smallmatrix} I \\ X \end{smallmatrix}\right)$ is an invariant subspace of the Hamiltonian. In the finite-dimensional case, this connection has lead to a complete characterisation of all solutions of the Riccati equation, see e.g. [3, 13] and the references therein. For infinite-dimensional linear systems, this "Hamiltonian approach" to the Riccati equation has been studied under different boundedness assumptions on the control and observation operators B, C and for different classes of Hamiltonians concerning their spectral properties. For the case that B, C are bounded and have finite rank, a characterisation of all nonnegative solutions of (1) has been obtained in [5]. In [12] the class of Hamiltonians possessing a Riesz basis of eigenvectors was considered for systems with bounded B and C, and characterisations of solutions and their properties were obtained. In [22, 23] this was extended to unbounded B, C and to more general kinds of Riesz bases. The Riesz basis setting typically leads to the existence of an infinite number of solutions of (1).

However, the existence of a Riesz basis of eigenvectors of T is a strong assumption and might be to restrictive. An often weaker condition is that T is *dichotomous*. This means that the spectrum of T does not contain points in a strip around the imaginary axis and that there exist invariant subspaces corresponding to the parts of the spectrum in the left and right half-plane, respectively. Dichotomous Hamiltonians with bounded B and C were considered in [4, 14] and the existence of a nonnegative and a nonpositive solution of (1) was shown. This result was extended in [18] to a setting where BB^* and C^*C are unbounded closed operators acting on the state space. This however excludes PDE systems with control or observation on the boundary. In this article we will construct a nonnegative and a nonpositive solution of (1) for a class of dichotomous Hamiltonians which allows for systems with boundary control and observation.

In the infinite-dimensional setting the Hamiltonian approach typically leads to unbounded solutions of the Riccati equation in the first instance, see [14, 18, 22, 23]. This means that the boundedness of solutions is an additional question now. Moreover, due to the unboundedness of the operators in (1), additional care has to be taken to exactly determine the domain on which the Riccati equation actually holds.

Our setting is as follows: Let H, U, Y be Hilbert spaces. Let A be a *quasi-sectorial* operator on H, i.e., $A - \mu$ is sectorial for some $\mu \geq 0$. This means that A may have spectrum on and to the right of the imaginary axis up to the line $\operatorname{Re} z = \mu$ and that A generates an analytic semigroup. The operator A determines two scales of Hilbert spaces $\{H_s\}$ and $\{H_s^{(*)}\}$,

$$H_s \subset H \subset H_{-s}, \quad H_s^{(*)} \subset H \subset H_{-s}^{(*)}, \quad s > 0,$$

whose norms are given by $\|x\|_s = \left\|\left(I+AA^*\right)^{\frac{s}{2}}x\right\|$ and $\|x\|_s^{(*)} = \left\|\left(I+A^*A\right)^{\frac{s}{2}}x\right\|$. If A is a normal operator, then both scales coincide with the usual fractional power spaces, $H_s = H_s^{(*)} = \mathcal{D}(|A|^s)$. In general, however, the two scales are different and must be distinguished. Our assumption on the control and observation operators is now

$$B \in L(U, H_{-r}), \quad C \in L(H_s^{(*)}, Y)$$

where $r, s \geq 0$ and $r + s < 1$. Examples of systems with boundary control and observation which fit into this setting may be found e.g. in [19, 23]. The adjoints of B and C are defined using a duality relation in each of the scales of Hilbert spaces, which is induced by the inner product $\left(\,\cdot\,|\,\cdot\,\right)$ on H: the mapping $y \mapsto \left(\,\cdot\,|y\right)$, $y \in H$, extends by continuity to isometric isomorphisms $H_{-r} \to (H_r)'$ and $H_{-s}^{(*)} \to (H_s^{(*)})'$. This is also referred to as duality with respect to the pivot space H. With this duality we obtain

$$BB^* : H_r \to H_{-r}, \quad C^*C : H_s^{(*)} \to H_{-s}^{(*)}.$$

The Hamiltonian is now considered as an unbounded operator

$$T_0 = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}$$

acting on $V_0 = H_{-r} \times H_{-s}^{(*)}$, with appropriate extensions of the operators A and A^*. We prove that if

(a) $\sigma(A) \cap i\mathbb{R} = \varnothing$, or
(b) A has a compact resolvent and

$$\ker(A - it) \cap \ker C = \ker(A^* + it) \cap \ker B^* = \{0\}, \quad t \in \mathbb{R},$$

then T_0 is dichotomous and hence there is a decomposition $V_0 = V_{0+} \oplus V_{0-}$ into T_0-invariant subspaces such that $\sigma(T_0|_{V_{0\pm}}) \subset \mathbb{C}_\pm$, i.e., V_{0-} corresponds to the spectrum in the open left half-plane $\mathbb{C}_-$ and V_{0+} to the one in the open right half-plane $\mathbb{C}_+$. For the rest of this introduction we assume that (a) or (b) is satisfied.

We derive that $V_{0\pm}$ are graph subspaces in two different situations. In the first we assume that

$$\bigcap_{\lambda \in i\mathbb{R}\cap\varrho(A^*)} \ker B^*(A^* - \lambda)^{-1} = \{0\}. \tag{3}$$

Then $V_{0\pm}$ are graphs, $V_{0\pm} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{0\pm} \end{smallmatrix}\right)$, of closed, possibly unbounded operators $X_{0\pm} : \mathcal{D}(X_{0\pm}) \subset H_{-r} \to H_{-s}^{(*)}$. If in addition

$$\bigcap_{\lambda \in i\mathbb{R}\cap\varrho(A)} \ker C(A - \lambda)^{-1} = \{0\}, \tag{4}$$

then $X_{0\pm}$ are also injective and hence $V_{0\pm} = \mathcal{R}\left(\begin{smallmatrix} Y_{0\pm} \\ I \end{smallmatrix}\right)$ with $Y_{0\pm} = X_{0\pm}^{-1}$. The conditions (3) and (4) were also used in [14, 18, 22, 23], sometimes in different but equivalent forms; (3) amounts to the approximate controllability, (4) to the approximate observability of the system (A, B, C), see [14, 23]. In the second situation, we assume that $\sigma(A) \subset \mathbb{C}_-$. Hence the semigroup generated by A is exponentially stable. In this case we obtain $V_{0-} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{0-} \end{smallmatrix}\right)$ and $V_{0+} = \mathcal{R}\left(\begin{smallmatrix} Y_{0+} \\ I \end{smallmatrix}\right)$ where, again, X_{0-} and Y_{0+} are closed and possibly unbounded, but not necessarily injective.

Under the additional assumption that A has a compact resolvent, we can show that X_{0-} and Y_{0+} are bounded. More precisely, if A has a compact resolvent and either (3) and (4) or $\sigma(A) \subset \mathbb{C}_-$ hold, then $X_{0-} \in L\big(H_{-r}, H_{-s}^{(*)}\big)$, $Y_{0+} \in L\big(H_{-s}^{(*)}, H_{-r}\big)$. In this case we also obtain that X_{0-} is a solution of the Riccati equation on the domain $H_{1-r}^{(*)}$ and that the operator $A - BB^*X_{0-}$ associated with the closed loop system generates an exponentially stable semigroup on H_{-r}.

In [14, 18] the two solutions of the Riccati equation are selfadjoint operators on H, one being nonnegative, the other nonpositive. Here the situation is more involved. While $X_{0\pm}$ can be restricted to symmetric operators on H that are nonnegative and nonpositive, respectively, selfadjoint restrictions need not exist in general. More specifically, $X_{0\pm}$ admit restrictions to closed operators $X_{1\pm}$ from $H_s^{(*)}$ to H_r such that

$$X_{1\pm} \subset X_{1\pm}^* = X_{0\pm},$$

where the adjoint is computed with respect to the duality in the scales $\{H_s\}$ and $\{H_s^{(*)}\}$. In particular, $X_{1\pm}$ is symmetric when considered as an operator on H. If $X_{M\pm}$ is the closure of $X_{1\pm}$ as an operator on H and $X_\pm$ is the part of $X_{0\pm}$ in H, then

$$X_{1\pm} \subset X_{M\pm} \subset X_{M\pm}^* = X_\pm \subset X_{0\pm},$$

X_{M-} is symmetric and nonnegative, X_{M+} is symmetric and nonpositive. We can also consider the restriction of the Hamiltonian T_0 to an operator T on $V = H \times H$. Then T has invariant subspaces $V_\pm$ corresponding to the spectrum in $\mathbb{C}_\pm$ and $V_\pm$ is in fact the graph of $X_\pm$. Note here that T will in general not be dichotomous since $V_+ \oplus V_-$ will only be dense in V. Also note that the above statements hold for X_{0-} and its restrictions provided that $V_{0-} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{0-} \end{smallmatrix}\right)$, i.e., if (3) or $\sigma(A) \subset \mathbb{C}_-$ holds. Likewise the statements for the restrictions of X_{0+} hold if $V_{0+} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{0+} \end{smallmatrix}\right)$, i.e., if (3) is true.

Finally assume that $\max\{r, s\} < \frac{1}{2}$. In this case T is in fact dichotomous and we obtain $X_{M\pm} = X_\pm$. Hence X_- is selfadjoint nonnegative, X_+ is selfadjoint nonpositive. If in addition A has a compact resolvent, then X_- is also bounded and a restriction of $A - BB^*X_{0-}$ generates an exponentially stable semigroup on H.

This article is organised as follows: In Section 2 we collect some general operator theoretic statements, in particular about dichotomous, sectorial and bisectorial operators. The scales of Hilbert spaces are defined in Section 3 and their basic properties are recalled, in particular concerning interpolation. Section 4 contains the definition of the Hamiltonian and basic facts about its spectrum. Moreover, we describe the symmetry of the Hamiltonian with respect to two indefinite inner products, which will be essential in Sections 6 and 7. In Section 5 we prove the bisectoriality and dichotomy of T_0 and T using interpolation in the Hilbert scales. The graph subspace properties of $V_{0\pm}$ and $V_\pm$ are derived in Section 6 as well as the boundedness of X_{0-} and Y_{0+}. The symmetry relations between $X_{0\pm}$ and its restrictions are the subject of Section 7, while the Riccati equation and the closed loop operator are studied in Section 8.

A few remarks on the notation: We denote the domain of a linear operator T by $\mathcal{D}(T)$, its range by $\mathcal{R}(T)$, the spectrum by $\sigma(T)$ and the resolvent set by $\varrho(T)$. The space of all bounded linear operators mapping a Banach space V to another Banach space W is denoted by $L(V,W)$. For the operator norm of $T \in L(V,W)$ we occasionally write $\|T\|_{V\to W}$ to make the dependence on the spaces V and W explicit.

2. Preliminaries

In this section, we summarise some concepts and results for linear operators on Banach spaces. Unless stated explicitly, linear operators are not assumed to be densely defined.

Lemma 2.1. *Let T be a linear operator on a Banach space V. Let W be another Banach space such that $\mathcal{D}(T) \subset W \subset V$ and such that the imbedding $W \hookrightarrow V$ is continuous. Let $\lambda \in \varrho(T)$.*

(a) *The resolvent $(T-\lambda)^{-1}$ yields a bounded operator from V into W, i.e., $(T-\lambda)^{-1} \in L(V,W)$.*
(b) *If the imbedding $W \hookrightarrow V$ is compact, then the resolvent is compact as an operator from V into V, i.e., $(T-\lambda)^{-1} : V \to V$ is compact.*

Proof. (a) The assumption $\mathcal{D}(T) \subset W$ implies that $(T-\lambda)^{-1}$ maps V into W. The operator $(T-\lambda)^{-1} : V \to W$ is thus well defined, and by the closed graph theorem it suffices to show that it is closed. Let $x_n \in V$ with $x_n \to x$ in V and $(T-\lambda)^{-1}x_n \to y$ in W as $n \to \infty$. Then $(T-\lambda)^{-1}x_n \to y$ in V by the continuity of the imbedding $W \hookrightarrow V$, and also $(T-\lambda)^{-1}x_n \to (T-\lambda)^{-1}x$ in V since the resolvent is a bounded operator on V. Consequently $(T-\lambda)^{-1}x = y$ and hence $(T-\lambda)^{-1} : V \to W$ is closed.

(b) This follows immediately from (a) by composing the bounded operator $(T-\lambda)^{-1} : V \to W$ with the compact imbedding $W \hookrightarrow V$. □

Lemma 2.2. *Let T_0 be a linear operator on a Banach space V_0. Let V be another Banach space satisfying $\mathcal{D}(T_0) \subset V \subset V_0$ with continuous imbedding $V \hookrightarrow V_0$. Let T be the part of T_0 in V, i.e., T is the restriction of T_0 to the domain*

$$\mathcal{D}(T) = \{x \in \mathcal{D}(T_0) \,|\, T_0 x \in V\},$$

considered as an operator $T : \mathcal{D}(T) \subset V \to V$. Then

(a) *$\sigma_p(T) = \sigma_p(T_0)$,*
(b) *$\varrho(T_0) \subset \varrho(T)$ and $(T-\lambda)^{-1} = (T_0-\lambda)^{-1}|_V$ for all $\lambda \in \varrho(T_0)$,*

(c) *if* $\mathcal{D}(T_0)$ *is dense in* V, V *is dense in* V_0 *and* $\varrho(T_0) \neq \varnothing$, *then* T *is densely defined.*

Proof. (a) This is clear, since $\mathcal{D}(T_0) \subset V$ implies that all eigenvectors of T_0 belong to V.

(b) Let $\lambda \in \varrho(T_0)$. Then $T - \lambda : \mathcal{D}(T) \to V$ is injective as a restriction of $T_0 - \lambda$. Let $y \in V$ and set $x = (T_0 - \lambda)^{-1}y$. Then $x \in \mathcal{D}(T_0)$, which implies $x \in V$ and $T_0 x = \lambda x + y \in V$. Therefore $x \in \mathcal{D}(T)$ and $(T - \lambda)x = y$. Hence $T - \lambda : \mathcal{D}(T) \to V$ is bijective with inverse $(T - \lambda)^{-1} = (T_0 - \lambda)^{-1}\big|_V$. Since $(T_0 - \lambda)^{-1} \in L(V_0, V)$ by Lemma 2.1 and since $V \hookrightarrow V_0$ is continuous, we obtain $(T - \lambda)^{-1} \in L(V)$ and thus $\lambda \in \varrho(T)$.

(c) Let $\lambda \in \varrho(T_0)$. Since $(T_0 - \lambda)^{-1} \in L(V_0, V)$ and since $V \subset V_0$ is dense, we get that $\mathcal{D}(T) = (T_0 - \lambda)^{-1}(V)$ is dense in $\mathcal{D}(T_0) = (T_0 - \lambda)^{-1}(V_0)$ with respect to the norm in V. As $\mathcal{D}(T_0) \subset V$ is dense, we conclude that $\mathcal{D}(T) \subset V$ is dense. □

Let us recall the definitions and basic properties of sectorial, bisectorial and dichotomous operators. For more details we refer the reader to [7, 8, 21]. We denote by

$$\Sigma_{\frac{\pi}{2}+\theta} = \left\{\lambda \in \mathbb{C} \setminus \{0\} \,\middle|\, \arg\lambda \in \left[-\frac{\pi}{2} - \theta, \frac{\pi}{2} + \theta\right]\right\} \tag{5}$$

the sector containing the positive real axis with semi-angle $\frac{\pi}{2} + \theta$. We also consider the corresponding bisector around the imaginary axis

$$\Omega_\theta = \Sigma_{\frac{\pi}{2}+\theta} \cap \left(-\Sigma_{\frac{\pi}{2}+\theta}\right) = \left\{\lambda \in \mathbb{C} \setminus \{0\} \,\middle|\, |\arg\lambda| \in \left[\frac{\pi}{2} - \theta, \frac{\pi}{2} + \theta\right]\right\}. \tag{6}$$

For sectorial operators we adopt the convention that the spectrum is contained in a sector in the left half-plane:

Definition 2.3. A linear operator S on a Banach space V is called *sectorial* if there exist $\theta \geq 0$ and $M > 0$ such that $\Sigma_{\frac{\pi}{2}+\theta} \subset \varrho(S)$ and

$$\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|} \qquad \text{for all } \lambda \in \Sigma_{\frac{\pi}{2}+\theta}. \tag{7}$$

S is called *quasi-sectorial* if $S - \mu$ is sectorial for some $\mu \in \mathbb{R}$.

If (7) holds for some θ, then it also holds for some $\theta' > \theta$ (with a typically larger constant M). We may therefore always assume that $\theta > 0$. S is quasi-sectorial if and only if there exist $\theta, M, \rho > 0$ such that[1] $\Sigma_{\frac{\pi}{2}+\theta} \setminus B_\rho(0) \subset \varrho(S)$ and

$$\|(S - \lambda)^{-1}\| \leq \frac{M}{|\lambda|} \qquad \text{for all } \lambda \in \Sigma_{\frac{\pi}{2}+\theta},\ |\lambda| \geq \rho. \tag{8}$$

An operator is sectorial and densely defined if and only if it is the generator of a bounded analytic semigroup. On reflexive Banach spaces every sectorial and quasi-sectorial operator is densely defined. If S is a (quasi-) sectorial operator on a

[1] $B_r(z) \subset \mathbb{C}$ denotes the open disc with radius r centred at z.

Hilbert space, then its adjoint S^* is also (quasi-) sectorial with the same constants θ, M (and μ, ρ).

Definition 2.4. A linear operator S on V is called *bisectorial* if $i\mathbb{R} \setminus \{0\} \subset \varrho(S)$ and

$$\|(S-\lambda)^{-1}\| \leq \frac{M}{|\lambda|} \quad \text{for all } \ \lambda \in i\mathbb{R} \setminus \{0\} \tag{9}$$

with some constant $M > 0$. S is *almost bisectorial* if $i\mathbb{R} \setminus \{0\} \subset \varrho(S)$ and there exist $0 < \beta < 1$, $M > 0$ such that

$$\|(S-\lambda)^{-1}\| \leq \frac{M}{|\lambda|^\beta} \quad \text{for all } \ \lambda \in i\mathbb{R} \setminus \{0\}. \tag{10}$$

If S is bisectorial, then for some $\theta > 0$ the bisector Ω_θ is contained in the resolvent set $\varrho(S)$, and an estimate (9) holds for all $\lambda \in \Omega_\theta$. Similarly, for an almost bisectorial operator a parabola shaped region around the imaginary axis belongs to $\varrho(S)$. If S is bisectorial and $0 \in \varrho(S)$, then S is almost bisectorial too, for any $0 < \beta < 1$. Note that an almost bisectorial operator always satisfies $0 \in \varrho(S)$, while for a bisectorial operator $0 \in \sigma(S)$ is possible. Bisectorial operators on reflexive spaces are always densely defined; for almost bisectorial operators this need not be the case.

Definition 2.5. A linear operator S on a Banach space V is called *dichotomous* if $i\mathbb{R} \subset \varrho(S)$ and there exist closed S-invariant subspaces $V_\pm$ of V such that $V = V_+ \oplus V_-$ and

$$\sigma\big(S\big|_{V_+}\big) \subset \mathbb{C}_+, \quad \sigma\big(S\big|_{V_-}\big) \subset \mathbb{C}_-.$$

S is *strictly dichotomous* if in addition $\big\|\big(S\big|_{V_\pm} - \lambda\big)^{-1}\big\|$ is bounded on $\mathbb{C}_\mp$.

A dichotomous operator is block diagonal with respect to the decomposition $V = V_+ \oplus V_-$, see [18, Remark 2.3 and Lemma 2.4]. In particular, $\sigma(S) = \sigma\big(S\big|_{V_+}\big) \cup \sigma\big(S\big|_{V_-}\big)$ and the subspaces $V_\pm$ are also $(S-\lambda)^{-1}$-invariant for all $\lambda \in \varrho(S)$. The additional condition of strict dichotomy ensures that the invariant subspaces $V_\pm$ are uniquely determined by the operator.

One of the main results from [21] is that if the resolvent of an operator S is uniformly bounded along the imaginary axis, then S possesses invariant subspaces $V_\pm$ having the same properties as in Definition 2.5, with the exception that $V_+ \oplus V_-$ might be a proper subspace of V, i.e., S need not necessarily be dichotomous. In this case, the corresponding projections are unbounded. We summarise the results for the almost bisectorial situation here.

Let S be an almost bisectorial operator. Then there exists $h > 0$ such that $\{\lambda \in \mathbb{C} \,|\, |\operatorname{Re}\lambda| \leq h\} \subset \varrho(S)$ and the integrals

$$L_\pm = \frac{\pm 1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda}(S-\lambda)^{-1}\, d\lambda \tag{11}$$

define bounded operators $L_\pm \in L(V)$ which satisfy

$$L_+ L_- = L_- L_+ = 0, \quad L_+ + L_- = S^{-1}, \tag{12}$$

see [21, §5].

Theorem 2.6. *Let S be almost bisectorial on the Banach space V. Then $P_\pm = SL_\pm$ are closed complementary projections, the subspaces $V_\pm = \mathcal{R}(P_\pm)$ are closed, S- and $(S-\lambda)^{-1}$-invariant for all $\lambda \in \varrho(S)$, and*

(a) $\sigma(S) = \sigma\big(S|_{V_+}\big) \cup \sigma\big(S|_{V_-}\big)$ *with* $\sigma\big(S|_{V_\pm}\big) \subset \mathbb{C}_\pm$,
(b) $\big\|\big(S|_{V_\pm} - \lambda\big)^{-1}\big\|$ *is bounded on* $\mathbb{C}_\mp$,
(c) $\mathcal{D}(S) \subset \mathcal{D}(P_\pm) = V_+ \oplus V_- \subset V$,
(d) $I = P_+ + P_-$ *on* $\mathcal{D}(P_\pm)$.

The projections satisfy the identity

$$P_+x - P_-x = \frac{1}{\pi i}\int_{-i\infty}^{i\infty\,\prime} (S-\lambda)^{-1}x\,d\lambda, \quad x \in \mathcal{D}(S), \tag{13}$$

where the prime denotes the Cauchy principal value at infinity. Moreover, S is strictly dichotomous if and only if $P_\pm \in L(V)$.

Proof. All assertions follow from Theorem 4.1 and 5.6 as well as Corollary 4.2 and 5.9 in [21]. □

Note that $P_\pm$ are closed complementary projections in the sense that they are closed operators on V and satisfy $\mathcal{R}(P_\pm) \subset \mathcal{D}(P_\pm)$, $P_\pm^2 = P_\pm$, $\mathcal{D}(P_+) = \mathcal{D}(P_-)$ and $I = P_+ + P_-$ on $\mathcal{D}(P_\pm)$. In other words, $P_\pm$ are complementary projections in the algebraic sense acting on the space $\mathcal{D}(P_+) = \mathcal{D}(P_-)$. Since S is invertible, we obtain

$$V_\pm = \mathcal{R}(P_\pm) = \ker P_\mp = \ker L_\mp. \tag{14}$$

The case that $P_\pm$ are unbounded may occur even for bisectorial and almost bisectorial S, see Examples 5.8 and 8.2 in [21].

For use in later sections, we collect some properties of the spaces $\mathcal{R}(L_\pm)$:

Lemma 2.7. *Let S be an almost bisectorial operator. Then the inclusions*

$$\mathcal{D}(S) \cap V_\pm \subset \mathcal{R}(L_\pm) \subset V_\pm \tag{15}$$

hold, in particular $\overline{\mathcal{R}(L_\pm)} \subset V_\pm$. In addition,

(a) *if S is also densely defined, then $\overline{\mathcal{D}(S) \cap V_\pm} = \overline{\mathcal{R}(L_\pm)}$;*
(b) *if S is densely defined and strictly dichotomous, then $\mathcal{D}(S) \cap V_\pm = \mathcal{R}(L_\pm)$ and $\overline{\mathcal{R}(L_\pm)} = V_\pm$.*

Proof. From (12) and the invariance properties of $V_\pm$ we get

$$\mathcal{D}(S) \cap V_\pm = S^{-1}(V_\pm) = L_\pm(V_\pm) \subset \mathcal{R}(L_\pm) \subset \ker L_\mp = V_\pm.$$

Since $V_\pm$ are closed, $\overline{\mathcal{R}(L_\pm)} \subset V_\pm$ follows. If S is densely defined, then part (c) of the previous theorem yields $\overline{V_+ \oplus V_-} = V$. Therefore

$$\mathcal{R}(L_\pm) = L_\pm(\overline{V_+ \oplus V_-}) \subset \overline{L_\pm(V_+ \oplus V_-)} = \overline{L_\pm(V_\pm)} = \overline{\mathcal{D}(S) \cap V_\pm},$$

and hence the inclusion "$\supset$" in (a) holds. The other inclusion is clear by (15). If now S is also strictly dichotomous, then $P_\pm$ are bounded. In particular $\mathcal{R}(L_\pm) \subset \mathcal{D}(S)$ and hence $\mathcal{R}(L_\pm) = \mathcal{D}(S) \cap V_\pm$. Using that S and $L_\pm$ commute, we obtain

$$V_\pm = \mathcal{R}(P_\pm) = P_\pm(\overline{\mathcal{D}(S)}) \subset \overline{P_\pm(\mathcal{D}(S))} = \overline{L_\pm S(\mathcal{D}(S))} = \overline{\mathcal{R}(L_\pm)}$$

and hence $\overline{\mathcal{R}(L_\pm)} = V_\pm$ by (15). □

We remark that the inclusion $\overline{\mathcal{R}(L_\pm)} \subset V_\pm$ is strict in general, see [21, §6] and Examples 8.3 and 8.5 in [21].

3. Two scales of Hilbert spaces associated with a closed operator

In this section we construct two scales of Hilbert spaces $\{H_s\}$ and $\{H_s^{(*)}\}$ associated with a closed, densely defined operator A. Although the results are well known, the presentations found in the literature often cover only parts of the full theory or are restricted to certain special cases: The construction of the spaces $H_{\pm 1}$ and $H_{\pm 1}^{(*)}$ for general A can be found e.g. in [9, 19]. The intermediate spaces for $s = \pm\frac{1}{2}$ are defined in [9] for general, and in [19] for selfadjoint positive A. The spaces H_s with arbitrary s are constructed in [10] for selfadjoint A, while a general theory of scales of Hilbert spaces including interpolation results is contained in [2]. Note that in [19] a different naming convention and different but equivalent definitions of the spaces are used. Our presentation follows [2, 9].

Let A be a closed, densely defined linear operator on a separable Hilbert space H. We denote by $\|\cdot\|$ the norm on H and consider the positive selfadjoint operator $\Lambda = \bigl(I + AA^*\bigr)^{\frac{1}{2}}$. For $s > 0$ let $H_s = \mathcal{D}(\Lambda^s)$ be equipped with the norm $\|x\|_s = \|\Lambda^s x\|$, and let H_{-s} be the completion of H with respect to the norm $\|x\|_{-s} = \|\Lambda^{-s} x\|$. Then H_s and H_{-s} are Hilbert spaces,

$$H_s \subset H \subset H_{-s},$$

and the imbeddings are continuous and dense. The family of spaces $\{H_s\}$ is called a *scale of Hilbert spaces*. In particular we obtain $H_1 = \mathcal{D}(A^*)$ and

$$\|x\|_1 = \bigl(\|x\|^2 + \|A^*x\|^2\bigr)^{\frac{1}{2}}, \quad x \in H_1.$$

For any $s > 0$, the spaces H_s and H_{-s} are dual to each other with respect to the inner product $\bigl(\cdot \,|\, \cdot\bigr)$ of H. More precisely, the norm on H_s satisfies

$$\|y\|_{-s} = \sup\bigl\{|(x|y)| \bigm| x \in H_s,\ \|x\|_s = 1\bigr\}, \quad y \in H,$$

which implies that the inner product of H extends by continuity to a bounded sesquilinear form on $H_s \times H_{-s}$, which we denote by $\bigl(\cdot \,|\, \cdot\bigr)_{s,-s}$. In fact,

$$\bigl(x|y\bigr)_{s,-s} = \bigl(\Lambda^s x|\Lambda^{-s} y\bigr), \quad x \in H_s,\ y \in H.$$

The space H_{-s} can now be identified with the dual space of H_s by means of the isometric isomorphism $H_{-s} \to (H_s)'$, $y \mapsto \big(\cdot \, | y \big)_{s,-s}$. For convenience, we also define a sesquilinear form on $H_{-s} \times H_s$ by

$$\big(y|x\big)_{-s,s} = \overline{\big(x|y\big)_{s,-s}}, \quad x \in H_s,\, y \in H_{-s}.$$

With respect to the duality in the scale $\{H_s\}$, we obtain the following notion of adjoint operators:

Definition 3.1. Let W be a Hilbert space and $C \in L(H_s, W)$. Then the operator $C^* \in L(W, H_{-s})$ satisfying

$$\big(Cx|w\big)_W = \big(x|C^*w\big)_{s,-s}, \quad x \in H_s,\, w \in W, \tag{16}$$

where $\big(\cdot \, | \, \cdot \big)_W$ denotes the inner product of W, is called the *adjoint of* C *with respect to the scale* $\{H_s\}$. Similarly the adjoint of $B \in L(W, H_{-s})$ with respect to $\{H_s\}$ is the operator $B^* \in L(H_s, W)$ such that

$$\big(x|Bw\big)_{s,-s} = \big(B^*x|w\big)_W, \quad x \in H_s,\, w \in W. \tag{17}$$

The adjoints exist, are uniquely determined and satisfy $B = B^{**}$, $C = C^{**}$, $\|B\| = \|B^*\|$ and $\|C\| = \|C^*\|$. The adjoints of $\widetilde{C} \in L(W, H_s)$ and $\widetilde{B} \in L(H_{-s}, W)$ are defined in a similar way. If $C \in L(H_s, W)$ is an isomorphism, then C^* is an isomorphism too and $(C^*)^{-1} = (C^{-1})^*$.

Remark 3.2. The notion of adjoints with respect to the scale $\{H_s\}$ generalises the usual definition of adjoints of unbounded operators on Hilbert spaces: Let $C \in L(H_s, W)$. Then C can be regarded as a densely defined unbounded operator $C_1 : \mathcal{D}(C_1) \subset H \to W$ with domain $\mathcal{D}(C_1) = H_s$. The adjoint of C_1 in the usual sense of unbounded operators is an operator $C_1^* : \mathcal{D}(C_1^*) \subset W \to H$. Observe that C_1 and C_1^* satisfy (16) provided that $w \in \mathcal{D}(C_1^*)$. Consequently, C_1^* is a restriction of $C^* : W \to H_{-s}$. In fact,

$$\mathcal{D}(C_1^*) = \{w \in W \,|\, C^*w \in H\}\,.$$

Note here that $C \in L(H_s, W)$ does not imply that C_1 is closable. Hence C_1^* need not be densely defined and even $\mathcal{D}(C_1^*) = \{0\}$ is possible.

Since $H_1 = \mathcal{D}(A^*)$ and since $\|\cdot\|_1$ is equal to the graph norm of A^*, we can consider A^* as a bounded operator $A^* : H_1 \to H$. The adjoint with respect to $\{H_s\}$ is a bounded operator $A^{**} : H \to H_{-1}$ and in view of the last remark A^{**} is an extension of the original operator A. We will denote this extension by A again,

$$A : H \to H_{-1}.$$

Now for any $\lambda \in \varrho(A)$, the operator $A^* - \overline{\lambda} : H_1 \to H$ is an isomorphism. Hence its adjoint $A - \lambda : H \to H_{-1}$ is an isomorphism too. In particular $\|(A-\lambda)^{-1} \cdot \|$ is an equivalent norm on H_{-1}.

Consider now the positive selfadjoint operator $\Lambda_* = \big(I + A^*A\big)^{\frac{1}{2}}$, and let $\{H_s^{(*)}\}$ be the scale of Hilbert spaces associated with it. In other words, we repeat the above construction with the roles of A and A^* interchanged. We denote the respective norms and the extension of the inner product by $\|\cdot\|_s^{(*)}$, $\|\cdot\|_{-s}^{(*)}$ and $(\cdot\,|\,\cdot)_{s,-s}^{(*)}$. Moreover, $H_1^{(*)} = \mathcal{D}(A)$, the norm on $H_1^{(*)}$ is equal to the graph norm of A, the norm on $H_{-1}^{(*)}$ is equivalent to $\|(A^* - \lambda)^{-1} \cdot\|$ for $\lambda \in \varrho(A^*)$, and we get bounded operators

$$A : H_1^{(*)} \to H, \qquad A^* : H \to H_{-1}^{(*)}.$$

Lemma 3.3. *If A has a compact resolvent, then the imbeddings $H_s \hookrightarrow H$ and $H_s^{(*)} \hookrightarrow H$ are compact for all $s > 0$.*

Proof. Let $\lambda \in \varrho(A)$. So $(A-\lambda)^{-1}$ and $(A^* - \overline{\lambda})^{-1}$ are compact operators in $L(H)$. The imbedding $H_1 \hookrightarrow H$ can be written as the composition

$$H_1 \xrightarrow{A^*-\overline{\lambda}} H \xrightarrow{(A^*-\overline{\lambda})^{-1}} H.$$

Since $A^* - \overline{\lambda} : H_1 \to H$ is bounded, it follows that $H_1 \hookrightarrow H$ is compact. Since $\Lambda^{-1} : H \to H_1$ is bounded, the sequence

$$H \xrightarrow{\Lambda^{-1}} H_1 \hookrightarrow H$$

implies that the operator $\Lambda^{-1} : H \to H$ is compact. Consequently $\Lambda^{-s} : H \to H$ is also compact for all $s > 0$. Decomposing $H_s \hookrightarrow H$ as

$$H_s \xrightarrow{\Lambda^s} H \xrightarrow{\Lambda^{-s}} H$$

where $\Lambda^s : H_s \to H$ is bounded, we conclude that $H_s \hookrightarrow H$ is compact. The proof for $H_s^{(*)} \hookrightarrow H$ is analogous. □

For operators acting between two scales of Hilbert spaces, there is the following interpolation result, which is also known as Heinz' inequality, see [11, Theorem I.7.1]. Let H and G be Hilbert spaces. Consider the scales of Hilbert spaces $\{H_s\}$ and $\{G_r\}$ with corresponding positive selfadjoint operators Λ and Δ on H and G, respectively.

Theorem 3.4 ([2, Theorem III.6.10])**.** *Let $r_1 < r_2$, $s_1 < s_2$ and let $B : G_{r_1} \to H_{s_1}$ be a bounded linear operator which restricts to a bounded operator $B : G_{r_2} \to H_{s_2}$. Let $0 < \lambda < 1$ and*

$$r = \lambda r_1 + (1-\lambda) r_2, \quad s = \lambda s_1 + (1-\lambda) s_2.$$

Then B also restricts to a bounded operator $B : G_r \to H_s$ and

$$\|B\|_{G_r \to H_s} \le \|B\|_{G_{r_1} \to H_{s_1}}^{\lambda} \|B\|_{G_{r_2} \to H_{s_2}}^{1-\lambda}.$$

We remark that if B restricts to an operator $B : G_{r_2} \to H_{s_2}$, i.e., if B maps G_{r_2} into H_{s_2}, then the boundedness of the restriction already follows from the closed graph theorem.

Applying interpolation to $A : H_1^{(*)} \to H$ and its extension $A : H \to H_{-1}$, we obtain that A also acts as a bounded operator

$$A : H_{1-s}^{(*)} \to H_{-s}, \quad s \in [0,1].$$

Similarly,

$$A^* : H_{1-s} \to H_{-s}^{(*)}, \quad s \in [0,1].$$

Moreover, if $\lambda \in \varrho(A)$ then $A - \lambda : H_{1-s}^{(*)} \to H_{-s}$ and $A^* - \overline{\lambda} : H_{1-s} \to H_{-s}^{(*)}$ are both isomorphisms. Here surjectivity follows from the fact that for example the resolvent $(A-\lambda)^{-1}$ is an operator in $L\big(H, H_1^{(*)}\big)$ and $L(H_{-1}, H)$ and hence by interpolation also in $L\big(H_{-s}, H_{1-s}^{(*)}\big)$.

The extensions of A and A^* satisfy the identity

$$\big(Ax|y\big)_{-s,s} = \big(x|A^*y\big)_{1-s,s-1}^{(*)}, \quad x \in H_{1-s}^{(*)},\, y \in H_s. \tag{18}$$

This follows from an extension by continuity of the relation $\big(Ax|y\big) = \big(x|A^*y\big)$, $x \in \mathcal{D}(A)$, $y \in \mathcal{D}(A^*)$.

In view of the above, using appropriate restrictions and extensions, the resolvent $(A-\lambda)^{-1}$ belongs to $L(H)$ as well as $L(H_{-1})$ and $L(H_1^{(*)})$. Similarly, $(A^* - \overline{\lambda})^{-1}$ belongs to $L(H)$, $L(H_{-1}^{(*)})$ and $L(H_1)$. The corresponding operator norms can be estimated as follows:

Lemma 3.5. *For any $\lambda \in \varrho(A)$ the estimates*

$$\big\|(A-\lambda)^{-1}\big\|_{L(H_{-1})} \le \big\|(A^*-\overline{\lambda})^{-1}\big\|_{L(H_1)} \le \big\|(A-\lambda)^{-1}\big\|_{L(H)}$$

and

$$\big\|(A^*-\overline{\lambda})^{-1}\big\|_{L(H_{-1}^{(*)})} \le \big\|(A-\lambda)^{-1}\big\|_{L(H_1^{(*)})} \le \big\|(A^*-\overline{\lambda})^{-1}\big\|_{L(H)}$$

hold.

Proof. From

$$\begin{aligned}\big\|(A^*-\overline{\lambda})^{-1}x\big\|_1^2 &= \|(A^*-\overline{\lambda})^{-1}x\|^2 + \|A^*(A^*-\overline{\lambda})^{-1}x\|^2 \\ &= \|(A^*-\overline{\lambda})^{-1}x\|^2 + \|(A^*-\overline{\lambda})^{-1}A^*x\|^2 \\ &\le \big\|(A^*-\overline{\lambda})^{-1}\big\|_{L(H)}^2 \|x\|_1^2\end{aligned}$$

for $x \in H_1$ we obtain

$$\big\|(A^*-\overline{\lambda})^{-1}\big\|_{L(H_1)} \le \big\|(A^*-\overline{\lambda})^{-1}\big\|_{L(H)} = \big\|(A-\lambda)^{-1}\big\|_{L(H)}.$$

Moreover, for $x \in H_1$, $y \in H_{-1}$,

$$\big|\big(x|(A-\lambda)^{-1}y\big)\big| = \big|\big((A^*-\overline{\lambda})^{-1}x|y\big)_{1,-1}\big| \le \big\|(A^*-\overline{\lambda})^{-1}\big\|_{L(H_1)}\|x\|_1\|y\|_{-1},$$

which implies $\|(A-\lambda)^{-1}y\|_{-1} \le \big\|(A^*-\overline{\lambda})^{-1}\big\|_{L(H_1)}\|y\|_{-1}$ and hence

$$\big\|(A-\lambda)^{-1}\big\|_{L(H_{-1})} \le \big\|(A^*-\overline{\lambda})^{-1}\big\|_{L(H_1)}.$$

The other estimates are analogous. □

Interpolation now yields the following:

Corollary 3.6. *For* $\lambda \in \varrho(A)$*,* $s \in [0,1]$*,*

$$\big\|(A-\lambda)^{-1}\big\|_{L(H_{-s})} \le \big\|(A-\lambda)^{-1}\big\|_{L(H)},$$
$$\big\|(A^*-\overline{\lambda})^{-1}\big\|_{L(H_{-s}^{(*)})} \le \big\|(A^*-\overline{\lambda})^{-1}\big\|_{L(H)}.$$

4. The Hamiltonian

Let A be a closed, densely defined operator on a Hilbert space H and let $\{H_s\}$ and $\{H_s^{(*)}\}$ be the associated scales of Hilbert spaces defined in Section 3. Let

$$B \in L(U, H_{-r}), \quad C \in L\big(H_s^{(*)}, Y\big)$$

where U, Y are additional Hilbert spaces and $r, s \in [0,1]$ satisfy $r + s \le 1$. The adjoints of B and C with respect to the scales of Hilbert spaces are

$$B^* \in L(H_r, U), \quad C^* \in L\big(Y, H_{-s}^{(*)}\big).$$

We define the *Hamiltonian* as the operator matrix

$$T_0 = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}.$$

Then T_0 is a well-defined linear operator from $\mathcal{D}(T_0) = H_{1-r}^{(*)} \times H_{1-s}$ to the product Hilbert space

$$V_0 = H_{-r} \times H_{-s}^{(*)}.$$

Indeed we have

$$A : H_{1-r}^{(*)} \to H_{-r}, \quad BB^* : H_r \to H_{-r},$$
$$C^*C : H_s^{(*)} \to H_{-s}^{(*)}, \qquad A^* : H_{1-s} \to H_{-s}^{(*)},$$

and the assumption $r + s \le 1$ implies

$$H_{1-r}^{(*)} \subset H_s^{(*)}, \quad H_{1-s} \subset H_r.$$

We consider T_0 as an unbounded operator on V_0 with domain $\mathcal{D}(T_0)$ as above. In particular, T_0 is densely defined.

Alongside V_0 we will also consider the two product Hilbert spaces

$$V_1 = H_s^{(*)} \times H_r \quad \text{and} \quad V = H \times H.$$

Thus

$$\mathcal{D}(T_0) \subset V_1 \subset V \subset V_0.$$

Let T be the part of T_0 in V. Then $\sigma_p(T) = \sigma_p(T_0)$. Moreover, T will be densely defined as soon as $\varrho(T_0) \neq \varnothing$. This follows from Lemma 2.2 since both inclusions $\mathcal{D}(T_0) \subset V$ and $V \subset V_0$ are dense.

Lemma 4.1. *The Hamiltonian satisfies*

$$\sigma_p(T_0) \cap i\mathbb{R} = \varnothing$$

if and only if

$$\ker(A - it) \cap \ker C = \ker(A^* + it) \cap \ker B^* = \{0\} \quad \text{for all } t \in \mathbb{R}. \tag{19}$$

Proof. Suppose first that (19) holds and that

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}(T_0), \quad T_0 \begin{pmatrix} x \\ y \end{pmatrix} = it \begin{pmatrix} x \\ y \end{pmatrix}, \quad t \in \mathbb{R}.$$

Then

$$(A - it)x - BB^*y = 0, \quad -C^*Cx - (A^* + it)y = 0$$

where $x \in H_{1-r}^{(*)} \subset H_s^{(*)}$, $y \in H_{1-s} \subset H_r$. Using the extended inner products of the scales $\{H_s\}$ and $\{H_s^{(*)}\}$, we find

$$\begin{aligned} 0 &= \big((A - it)x - BB^*y|y\big) = \big((A - it)x|y\big)_{-r,r} - \big(BB^*y|y\big)_{-r,r}, \\ 0 &= \big(-C^*Cx - (A^* + it)y|x\big) = -\big(C^*Cx|x\big)_{-s,s}^{(*)} - \big((A^* + it)y|x\big)_{-s,s}^{(*)}. \end{aligned} \tag{20}$$

From (18) we see that

$$\big(Ax|y\big)_{-r,r} = \big(x|A^*y\big)_{s,-s}^{(*)}, \quad x \in H_{1-r}^{(*)}, \; y \in H_{1-s}.$$

Adding the two equations in (20) and taking the real part, we thus obtain

$$0 = -\big(BB^*y|y\big)_{-r,r} - \big(C^*Cx|x\big)_{-s,s}^{(*)} = -\|B^*y\|_U^2 - \|Cx\|_Y^2.$$

Consequently, $B^*y = Cx = 0$ and hence also $(A - it)x = (A^* + it)y = 0$. Now (19) implies $x = y = 0$ and so $it \notin \sigma_p(T_0)$. For the reverse implication note that if for example $x \in \ker(A - it) \cap \ker C$ and $x \neq 0$, then $(x, 0)$ is an eigenvector of T_0 with eigenvalue it. □

Lemma 4.2. *The Hamiltonian satisfies*

$$\sigma_{\mathrm{app}}(T_0) \cap i\mathbb{R} \subset \sigma(A). \tag{21}$$

Proof. Let $t \in \mathbb{R}$, $it \in \sigma_{\mathrm{app}}(T_0)$. Then there exist $v_n \in \mathcal{D}(T_0)$ such that $\|v_n\|_{V_0} = 1$ and

$$\lim_{n\to\infty} (T_0 - it)v_n = 0 \quad \text{in } V_0.$$

By the continuity of the imbedding $V_1 \hookrightarrow V_0$ there is a constant $c > 0$ such that

$$1 = \|v_n\|_{V_0} \le c\|v_n\|_{V_1}.$$

Thus also

$$\lim_{n\to\infty} (T_0 - it)\frac{v_n}{\|v_n\|_{V_1}} = 0 \quad \text{in } V_0.$$

Setting $(x_n, y_n) = v_n/\|v_n\|_{V_1}$ we obtain $\|x_n\|_s^{(*)2} + \|y_n\|_r^2 = 1$ and

$$\lim_{n\to\infty} (T_0 - it)\begin{pmatrix} x_n \\ y_n \end{pmatrix} = 0 \quad \text{in } V_0,$$

or

$$\begin{aligned} (A-it)x_n - BB^*y_n &\to 0 \quad \text{in } H_{-r}, \\ -C^*Cx_n - (A^*+it)y_n &\to 0 \quad \text{in } H_{-s}^{(*)} \end{aligned} \tag{22}$$

as $n \to \infty$. Since the sequences (x_n) and (y_n) are bounded in $H_s^{(*)}$ and H_r, respectively, this implies that

$$\big((A-it)x_n - BB^*y_n|y_n\big)_{-r,r} \to 0,$$
$$\big(-C^*Cx_n - (A^*+it)y_n|x_n\big)_{-s,s}^{(*)} \to 0.$$

Similarly to the previous proof, we add these identities and take the real part to obtain

$$-\big(BB^*y_n|y_n\big)_{-r,r} - \big(C^*Cx_n|x_n\big)_{-s,s}^{(*)} = -\|B^*y_n\|_U^2 - \|Cx_n\|_Y^2 \to 0.$$

Consequently, $B^*y_n \to 0$ and $Cx_n \to 0$.

Now suppose in addition that $it \in \varrho(A)$. Then $A-it$ is an isomorphism from $H_{1-r}^{(*)}$ to H_{-r}, see Section 3. Therefore $(A-it)^{-1} \in L(H_{-r}, H_s^{(*)})$ and analogously $(A^*+it)^{-1} \in L(H_{-s}^{(*)}, H_r)$. It follows that

$$(A-it)^{-1}BB^*y_n \to 0 \quad \text{in } H_s^{(*)}, \quad (A^*+it)^{-1}C^*Cx_n \to 0 \quad \text{in } H_r.$$

On the other hand, we infer from (22) that

$$x_n - (A-it)^{-1}BB^*y_n \to 0 \quad \text{in } H_s^{(*)},$$
$$-(A^*+it)^{-1}C^*Cx_n - y_n \to 0 \quad \text{in } H_r.$$

Thus $x_n \to 0$ in $H_s^{(*)}$ and $y_n \to 0$ in H_r, which contradicts $\|x_n\|_s^{(*)2} + \|y_n\|_r^2 = 1$. □

Lemma 4.3. *If A has a compact resolvent, $r+s<1$ and $\varrho(T_0) \neq \varnothing$, then both T and T_0 have a compact resolvent, too.*

Proof. First we have $\varrho(T) \neq \varnothing$ by Lemma 2.2. Lemma 3.3 shows that the imbeddings $H_{1-r}^{(*)} \times H_{1-s} \hookrightarrow V$ and $H_{1-r}^{(*)} \times H_{1-s} \hookrightarrow V_0$ are compact. Since $\mathcal{D}(T) \subset \mathcal{D}(T_0) = H_{1-r}^{(*)} \times H_{1-s}$, Lemma 2.1 implies that the resolvents of T and T_0 are compact. □

On $V = H \times H$ we consider the two indefinite inner products

$$[v|w] = \big(Jv|w\big), \quad [v|w]_\sim = \big(\widetilde{J}v|w\big), \quad v,w \in H \times H, \tag{23}$$

with fundamental symmetries

$$J = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad \widetilde{J} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

For $v = (x,y), w = (\widetilde{x}, \widetilde{y})$ this yields

$$[v|w] = i\big(x|\widetilde{y}\big) - i\big(y|\widetilde{x}\big), \quad [v|w]_\sim = \big(x|\widetilde{y}\big) + \big(y|\widetilde{x}\big).$$

For the first inner product, we also consider its extension to $v \in V_1 = H_s^{(*)} \times H_r$ and $w \in V_0 = H_{-r} \times H_{-s}^{(*)}$ which we denote again by $[\cdot|\cdot]$ and which is given by

$$\begin{aligned}[v|w] &= i\big(x|\widetilde{y}\big)^{(*)}_{s,-s} - i\big(y|\widetilde{x}\big)_{r,-r},\\ [w|v] &= i\big(\widetilde{x}|y\big)_{-r,r} - i\big(\widetilde{y}|x\big)^{(*)}_{-s,s} = \overline{[v|w]}.\end{aligned} \tag{24}$$

Note that the extended inner product is non-degenerate in the sense that if $w \in V_0$ is such that $[v|w] = 0$ for all $v \in V_1$, then $w = 0$. Analogously $v \in V_1$ with $[v|w] = 0$ for all $w \in V_0$ implies $v = 0$.

The Hamiltonian has the following properties with respect to the inner products defined above:

Lemma 4.4.

$$\begin{aligned}[T_0 v|w] &= -[v|T_0 w], \quad v, w \in \mathcal{D}(T_0),\\ \operatorname{Re}[Tv|v]_\sim &\le 0, \quad v \in \mathcal{D}(T).\end{aligned}$$

Proof. Let $v, w \in \mathcal{D}(T_0) = H_{1-r}^{(*)} \times H_{1-s}$ and $v = (x, y), w = (\widetilde{x}, \widetilde{y})$. Then

$$x, \widetilde{x} \in H_{1-r}^{(*)} \subset H_s^{(*)}, \quad y, \widetilde{y} \in H_{1-s} \subset H_r, \quad T_0 v, T_0 w \in V_0 = H_{-r} \times H_{-s}^{(*)}.$$

We obtain

$$\begin{aligned}[T_0 v|w] &= i\big(Ax - BB^*y|\widetilde{y}\big)_{-r,r} - i\big(-C^*Cx - A^*y|\widetilde{x}\big)^{(*)}_{-s,s}\\ &= i\big(Ax|\widetilde{y}\big)_{-r,r} - i\big(BB^*y|\widetilde{y}\big)_{-r,r} + i\big(C^*Cx|\widetilde{x}\big)^{(*)}_{-s,s} + i\big(A^*y|\widetilde{x}\big)^{(*)}_{-s,s}\\ &= i\big(x|A^*\widetilde{y}\big)^{(*)}_{s,-s} - i\big(y|BB^*\widetilde{y}\big)_{r,-r} + i\big(x|C^*C\widetilde{x}\big)^{(*)}_{s,-s} + i\big(y|A\widetilde{x}\big)_{r,-r}\\ &= i\big(x|C^*C\widetilde{x} + A^*\widetilde{y}\big)^{(*)}_{s,-s} - i\big(y| - A\widetilde{x} + BB^*\widetilde{y}\big)_{r,-r}\\ &= [v| - T_0 w].\end{aligned}$$

Let now $v = (x, y) \in \mathcal{D}(T)$. Then

$$\begin{aligned}[Tv|v]_\sim &= \big(Ax - BB^*y|y\big) + \big(-C^*Cx - A^*y|x\big)\\ &= \big(Ax|y\big)_{-r,r} - \big(BB^*y|y\big)_{-r,r} - \big(C^*Cx|x\big)^{(*)}_{-s,s} - \big(A^*y|x\big)^{(*)}_{-s,s}\\ &= \big(Ax|y\big)_{-r,r} - \|B^*y\|_U^2 - \|Cx\|_Y^2 - \big(y|Ax\big)_{r,-r}\end{aligned}$$

and hence

$$\operatorname{Re}[Tv|v]_\sim = -\|B^*y\|_U^2 - \|Cx\|_Y^2 \le 0. \qquad \square$$

Corollary 4.5. (a) *If there exists $\lambda \in \mathbb{C}$ such that $\lambda, -\overline{\lambda} \in \varrho(T_0)$, then T is J-skew-selfadjoint and $\sigma(T)$ is symmetric with respect to the imaginary axis.*

(b) *If both T and T_0 have a compact resolvent, then $\sigma(T_0)$ is symmetric with respect to the imaginary axis.*

Proof. The previous lemma yields $[Tv|w] = -[v|Tw]$ for $v, w \in V$. Also recall that T is densely defined since $\varrho(T_0) \neq \varnothing$. Lemma 2.2 implies $\varrho(T_0) \subset \varrho(T)$ and hence $\lambda, -\overline{\lambda} \in \varrho(T)$. By the theory of operators in Krein spaces, we conclude that T is skew-selfadjoint with respect to the J-inner product, which in turn implies the symmetry of the spectrum. If now both resolvents are compact, then $\sigma(T) = \sigma_p(T) = \sigma_p(T_0) = \sigma(T_0)$ and the symmetry of the spectrum follows from part (a). □

Remark 4.6. The symmetries of the Hamiltonian with respect to the two indefinite inner products on $H \times H$ have been used already in [14, 18, 22, 23]. The use of the Hamiltonian T_0 on the extended space V_0 as well as the extended indefinite inner product is new here and is motivated by the better properties of T_0 compared to T.

5. Bisectorial Hamiltonians

Starting from this section we consider Hamiltonians whose operator A is quasi-sectorial, see Definition 2.3. Recall from Section 4 that

$$V_1 = H_s^{(*)} \times H_r, \quad V_0 = H_{-r} \times H_{-s}^{(*)}$$

and

$$BB^* \in L(H_r, H_{-r}), \quad C^*C \in L(H_s^{(*)}, H_{-s}^{(*)}),$$

We consider the following decomposition of T_0 on V_0:

$$T_0 = S_0 + R, \qquad S_0 = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -BB^* \\ -C^*C & 0 \end{pmatrix}. \tag{25}$$

Here S_0, like T_0, is an unbounded operator on V_0 with domain $\mathcal{D}(S_0) = \mathcal{D}(T_0) = H_{1-r}^{(*)} \times H_{1-s}$. On the other hand, R is a bounded operator $R \in L(V_1, V_0)$.

By Corollary 3.6 the extensions of A and A^* to unbounded operators on H_{-r} and $H_{-s}^{(*)}$, respectively, are quasi-sectorial and satisfy

$$\left\|(A-\lambda)^{-1}\right\|_{L(H_{-r})} \leq \frac{M}{|\lambda|}, \quad \left\|(A^*-\lambda)^{-1}\right\|_{L(H_{-s}^{(*)})} \leq \frac{M}{|\lambda|}$$

for all $\lambda \in \Sigma_{\frac{\pi}{2}+\theta}$, $|\lambda| \geq \rho$ where θ, M, ρ are the constants from (8). Consequently,

$$\left\|(S_0-\lambda)^{-1}\right\|_{L(V_0)} \leq \frac{M}{|\lambda|}, \qquad \lambda \in \Omega_\theta, |\lambda| \geq \rho, \tag{26}$$

with Ω_θ the bisector from (6).

We derive a few estimates for the resolvents of A and A^* with respect to the scales of Hilbert spaces $\{H_s\}$ and $\{H_s^{(*)}\}$.

Lemma 5.1. *Let A be quasi-sectorial and let $\theta, M, \rho > 0$ be the corresponding constants from* (8)*. Then for all $\lambda \in \Sigma_{\frac{\pi}{2}+\theta}$ with $|\lambda| \geq \rho$ the estimates*

$$\left\|(A-\lambda)^{-1}\right\|_{H\to H_1^{(*)}} \leq M_1, \quad \left\|(A-\lambda)^{-1}\right\|_{H_{-1}\to H} \leq M_1,$$
$$\left\|(A^*-\lambda)^{-1}\right\|_{H\to H_1} \leq M_1, \quad \left\|(A^*-\lambda)^{-1}\right\|_{H_{-1}^{(*)}\to H} \leq M_1$$

hold where $M_1 = M\left(\frac{1}{\rho}+1\right)+1$.

Proof. For $x \in H$ we have

$$\begin{aligned}\left\|(A-\lambda)^{-1}x\right\|_1^{(*)} &\leq \|(A-\lambda)^{-1}x\| + \|A(A-\lambda)^{-1}x\| \\ &\leq \|(A-\lambda)^{-1}x\| + \|x\| + |\lambda|\|(A-\lambda)^{-1}x\| \\ &\leq \left(\frac{M}{|\lambda|}+1+M\right)\|x\| \leq \left(\frac{M}{\rho}+1+M\right)\|x\|\end{aligned}$$

and hence $\left\|(A-\lambda)^{-1}\right\|_{H\to H_1^{(*)}} \leq M_1$. Since the adjoint of $(A-\overline{\lambda})^{-1} : H \to H_1^{(*)}$ with respect to the scale $\{H_s^{(*)}\}$ is $(A^*-\lambda)^{-1} : H_{-1}^{(*)} \to H$, see Section 3, we also get

$$\left\|(A^*-\lambda)^{-1}\right\|_{H_{-1}^{(*)}\to H} = \left\|(A-\overline{\lambda})^{-1}\right\|_{H\to H_1^{(*)}} \leq M_1.$$

Note here that if λ belongs to $\Sigma_{\frac{\pi}{2}+\theta}$ then so does $\overline{\lambda}$. The other estimates follow by interchanging the roles of A and A^*. □

Corollary 5.2. *Let A be quasi-sectorial, θ, M, ρ as above. Let $r, s \geq 0$ with $r+s \leq 1$. Then for $\lambda \in \Sigma_{\frac{\pi}{2}+\theta}$, $|\lambda| \geq \rho$:*

$$\left\|(A-\lambda)^{-1}\right\|_{H_{-r}\to H_s^{(*)}} \leq \frac{M_2}{|\lambda|^{1-r-s}}, \quad \left\|(A^*-\lambda)^{-1}\right\|_{H_{-s}^{(*)}\to H_r} \leq \frac{M_2}{|\lambda|^{1-r-s}}.$$

The constant M_2 depends on M, ρ, r, s only.

Proof. We apply interpolation to the results of Lemma 5.1. As a first step we get

$$\begin{aligned}\left\|(A-\lambda)^{-1}\right\|_{H\to H_{r+s}^{(*)}} &\leq \left\|(A-\lambda)^{-1}\right\|_{H\to H_1^{(*)}}^{r+s}\left\|(A-\lambda)^{-1}\right\|_{H\to H}^{1-r-s} \\ &\leq M_1^{r+s}\left(\frac{M}{|\lambda|}\right)^{1-r-s} = \frac{M_2}{|\lambda|^{1-r-s}}\end{aligned}$$

with $M_2 = M_1^{r+s}M^{1-r-s}$ and similarly

$$\left\|(A-\lambda)^{-1}\right\|_{H_{-r-s}\to H} \leq \frac{M_2}{|\lambda|^{1-r-s}}.$$

From this we obtain with $\tau = \frac{r}{r+s}$,

$$\left\|(A-\lambda)^{-1}\right\|_{H_{-r}\to H_s^{(*)}} \leq \left\|(A-\lambda)^{-1}\right\|_{H_{-r-s}\to H}^{\tau}\left\|(A-\lambda)^{-1}\right\|_{H\to H_{r+s}^{(*)}}^{1-\tau} \leq \frac{M_2}{|\lambda|^{1-r-s}}.$$

The estimates for $\left\|(A^*-\lambda)^{-1}\right\|_{H_{-s}^{(*)}\to H_r}$ are again analogous. □

Lemma 5.3. *Let A be quasi-sectorial, let θ, ρ be the constants from* (8)*. Suppose that $r+s<1$. Then there exists $\rho_1 \geq \rho$ and $c_0, c_1 > 0$ such that $\Omega_\theta \setminus B_{\rho_1}(0) \subset \varrho(T_0)$ and*

$$\|(T_0-\lambda)^{-1}\|_{L(V_0)} \leq \frac{c_0}{|\lambda|}, \tag{27}$$

$$\|(T_0-\lambda)^{-1} - (S_0-\lambda)^{-1}\|_{L(V_0)} \leq \frac{c_1}{|\lambda|^{2-r-s}} \tag{28}$$

for all $\lambda \in \Omega_\theta$, $|\lambda| \geq \rho_1$.

Proof. This is a standard perturbation argument for $T_0 = S_0 + R$ on V_0: For $\lambda \in \varrho(S_0)$, the identity

$$T_0 - \lambda = \big(I - R(S_0-\lambda)^{-1}\big)(S_0-\lambda)$$

holds. Corollary 5.2 implies that

$$\|(S_0-\lambda)^{-1}\|_{L(V_0,V_1)} \leq \frac{M_2}{|\lambda|^{1-r-s}}, \quad \lambda \in \Omega_\theta,\ |\lambda| \geq \rho.$$

Since $\|R(S_0-\lambda)^{-1}\|_{L(V_0)} \leq \|R\|\|(S_0-\lambda)^{-1}\|_{L(V_0,V_1)}$ and $1-r-s>0$, it follows that there exists $\rho_1 \geq \rho$ such that

$$\|R(S_0-\lambda)^{-1}\|_{L(V_0)} \leq \frac{1}{2} \quad \text{for all } \lambda \in \Omega_\theta,\ |\lambda| \geq \rho_1.$$

Hence $I - R(S_0-\lambda)^{-1}$ is an isomorphism on V_0 and thus $\lambda \in \varrho(T_0)$ with

$$(T_0-\lambda)^{-1} = (S_0-\lambda)^{-1}\big(I - R(S_0-\lambda)^{-1}\big)^{-1} \tag{29}$$

and

$$\big\|(T_0-\lambda)^{-1}\big\|_{L(V_0)} \leq \big\|(S_0-\lambda)^{-1}\big\|_{L(V_0)} \big\|\big(I - R(S_0-\lambda)^{-1}\big)^{-1}\big\|_{L(V_0)} \leq \frac{2M}{|\lambda|}$$

for $\lambda \in \Omega_\theta$, $|\lambda| \geq \rho_1$. Moreover,

$$(S_0-\lambda)^{-1} - (T_0-\lambda)^{-1} = (T_0-\lambda)^{-1}R(S_0-\lambda)^{-1}, \tag{30}$$

which implies

$$\begin{aligned}\big\|(S_0-\lambda)^{-1} - (T_0-\lambda)^{-1}\big\|_{L(V_0)} &\leq \big\|(T_0-\lambda)^{-1}\big\|_{L(V_0)}\|R\|\big\|(S_0-\lambda)^{-1}\big\|_{L(V_0,V_1)} \\ &\leq \frac{2M\|R\|M_2}{|\lambda|^{2-r-s}}.\end{aligned}$$

□

Lemma 5.4. *Let A be quasi-sectorial and let $Q_{0\pm} \in L(V_0)$ be the projections*

$$Q_{0-} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_{0+} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{31}$$

Consider the integration contours $\gamma_1(t) = it$, $t \in \,]-\infty,-\rho] \cup [\rho,\infty[$ as well as $\gamma_{0+}(t) = \rho e^{it}$, $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\gamma_{0-}(t) = \rho e^{-it}$, $t \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ where ρ is the constant from (8) *for A. Then*

$$Q_{0+}v - Q_{0-}v = \frac{1}{\pi i}\int_{\gamma_1}' (S_0-\lambda)^{-1}v\,d\lambda + Kv, \quad v \in V_0,$$

where the prime denotes the Cauchy principal value at infinity and $K \in L(V_0)$ *is given by* $K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ *with*

$$K_1 = \frac{1}{\pi i} \int_{\gamma_{0+}} (A - \lambda)^{-1} \, d\lambda, \quad K_2 = \frac{1}{\pi i} \int_{\gamma_{0-}} (-A^* - \lambda)^{-1} \, d\lambda.$$

Proof. We consider A as an operator on H_{-r}. Since $A - \rho$ is sectorial and $0 \in \varrho(A - \rho)$,

$$\frac{1}{\pi i} \int_{-i\infty}^{i\infty}{}' (A - \rho - \lambda)^{-1} x \, d\lambda = -x, \quad x \in H_{-r},$$

holds by [14, Lemma 6.1]. Using Cauchy's theorem in conjunction with the resolvent decay of A to alter the integration contour, we obtain

$$\begin{aligned} -x &= \frac{1}{\pi i} \int_{\rho - i\infty}^{\rho + i\infty}{}' (A - \lambda)^{-1} x \, d\lambda \\ &= \frac{1}{\pi i} \int_{\gamma_1}{}' (A - \lambda)^{-1} x \, d\lambda + \frac{1}{\pi i} \int_{\gamma_{0+}} (A - \lambda)^{-1} x \, d\lambda, \quad x \in H_{-r}. \end{aligned}$$

Looking at $-A^*$, we get

$$\frac{1}{\pi i} \int_{-i\infty}^{i\infty}{}' (-A^* + \rho - \lambda)^{-1} y \, d\lambda = y, \quad y \in H_{-s}^{(*)},$$

and hence

$$y = \frac{1}{\pi i} \int_{\gamma_1}{}' (-A^* - \lambda)^{-1} y \, d\lambda + \frac{1}{\pi i} \int_{\gamma_{0-}} (-A^* - \lambda)^{-1} y \, d\lambda, \quad y \in H_{-s}^{(*)}.$$

Combining both identities and noting that $Q_{0+}v - Q_{0-}v = (-x, y)$ for $v = (x, y)$, we obtain the claim. □

Theorem 5.5. *Let* A *be quasi-sectorial and let* $r + s < 1$*. If* $\sigma(A) \cap i\mathbb{R} = \varnothing$ *or if* A *has a compact resolvent and*

$$\ker(A - it) \cap \ker C = \ker(A^* + it) \cap \ker B^* = \{0\} \quad \text{for all } t \in \mathbb{R}, \qquad (32)$$

then the Hamiltonian T_0 *is bisectorial and strictly dichotomous.*

Proof. We first show that $i\mathbb{R} \subset \varrho(T_0)$. If $\sigma(A) \cap i\mathbb{R} = \varnothing$, then Lemma 4.2 implies $\sigma_{\mathrm{app}}(T_0) \cap i\mathbb{R} = \varnothing$. Since $\partial\sigma(T_0) \subset \sigma_{\mathrm{app}}(T_0)$ and $i\mathbb{R} \cap \varrho(T_0) \neq \varnothing$, by Lemma 5.3 it follows that $i\mathbb{R} \subset \varrho(T_0)$. Suppose, on the other hand, that A has a compact resolvent and that (32) holds. By Lemma 4.3, T_0 has a compact resolvent too and therefore $\sigma(T_0) = \sigma_p(T_0)$. Lemma 4.1 then implies $\sigma(T_0) \cap i\mathbb{R} = \varnothing$.

From $i\mathbb{R} \subset \varrho(T_0)$ and the estimate (27) we obtain that T_0 is bisectorial. In particular Theorem 2.6 can be applied to T_0 and yields corresponding closed projections on V_0, which we denote by $P_{0\pm}$. By Lemma 5.4 the mapping

$$v \mapsto \frac{1}{\pi i} \int_{\gamma_1}{}' (S_0 - \lambda)^{-1} v \, d\lambda, \quad v \in V_0,$$

defines a bounded operator in $L(V_0)$. In view of (28) the integral

$$\int_{\gamma_1} (T_0 - \lambda)^{-1} - (S_0 - \lambda)^{-1}\, d\lambda$$

converges in $L(V_0)$. Consequently, $v \mapsto \frac{1}{\pi i}\int_{\gamma_1}' (T_0 - \lambda)^{-1} v\, d\lambda$ and hence also

$$v \mapsto \frac{1}{\pi i}\int_{-i\infty}^{i\infty\,\prime} (T_0 - \lambda)^{-1} v\, d\lambda, \quad v \in V_0,$$

defines a bounded operator in $L(V_0)$. By (13) this last operator coincides with $P_{0+} - P_{0-}$ on $\mathcal{D}(T_0)$. Since $P_{0+} - P_{0-}$ is closed and $\mathcal{D}(T_0)$ is dense in V_0, we conclude that $\mathcal{D}(P_{0\pm}) = V_0$ and hence $P_{0\pm} \in L(V_0)$ by the closed graph theorem. Therefore T_0 is strictly dichotomous. □

Remark 5.6. Combining the results from Lemma 5.3 with the dichotomy of T_0 from Theorem 5.5 we find that, in fact,

$$\big(\Omega_\theta \setminus B_{\rho_1}(0)\big) \cup \big\{\lambda \in \mathbb{C} \,\big|\, |\lambda| \le h\big\} \subset \varrho(T_0)$$

where $\rho_1 \ge \rho$, $h > 0$, and θ, ρ are the constants from (8) corresponding to the quasi-sectoriality of A. Also note that the last proof shows that T_0 is bisectorial and strictly dichotomous whenever $r + s < 1$ and $i\mathbb{R} \subset \varrho(T_0)$.

We close this section by investigating the dichotomy properties of the Hamiltonian on $V = H \times H$, i.e., of the operator T. Let

$$S = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$$

with domain $\mathcal{D}(S) = H_1^{(*)} \times H_1$, considered as an unbounded operator on V, i.e., S is the part of S_0 in V. Note that a decomposition similar to (25) does not hold for the operators T and S since R maps out of V into the larger space V_0. In particular we have $\mathcal{D}(T) \ne \mathcal{D}(S)$ in general.

Lemma 5.7. *Let A be quasi-sectorial with constants θ, ρ as in* (8)*. Let $r + s < 1$. Then there exist $\rho_1 \ge \rho$ and $c_0, c_1 > 0$ such that $\Omega_\theta \setminus B_{\rho_1}(0) \subset \varrho(T)$ and*

$$\big\|(T-\lambda)^{-1}\big\|_{L(V)} \le \frac{c_0}{|\lambda|^\beta}, \tag{33}$$

$$\big\|(T-\lambda)^{-1} - (S-\lambda)^{-1}\big\|_{L(V)} \le \frac{c_1}{|\lambda|^{2(1-\max\{r,s\})}}, \tag{34}$$

for all $\lambda \in \Omega_\theta$, $|\lambda| \ge \rho_1$ where

$$\beta = \begin{cases} 1, & \max\{r,s\} \le \frac{1}{2}, \\ 2(1-\max\{r,s\}), & \max\{r,s\} > \frac{1}{2}. \end{cases}$$

Proof. By Corollary 5.2, there exist $M_2, M_2' > 0$ with

$$\big\|(A-\lambda)^{-1}\big\|_{L(H_{-r},H)} \le \frac{M_2}{|\lambda|^{1-r}}, \qquad \big\|(-A^*-\lambda)^{-1}\big\|_{L(H_{-s}^{(*)},H)} \le \frac{M_2'}{|\lambda|^{1-s}}$$

for all $\lambda \in \Omega_\theta$, $|\lambda| \ge \rho$. Since $\rho > 0$ we can thus find $c > 0$ such that

$$\big\|(S_0-\lambda)^{-1}\big\|_{L(V_0,V)} \le \frac{c}{|\lambda|^{1-\max\{r,s\}}} \quad \text{for } \lambda \in \Omega_\theta,\ |\lambda| \ge \rho.$$

Similarly there exists $c' > 0$ with

$$\big\|(S-\lambda)^{-1}\big\|_{L(V,V_1)} \le \frac{c'}{|\lambda|^{1-\max\{r,s\}}} \quad \text{for } \lambda \in \Omega_\theta,\ |\lambda| \ge \rho.$$

Let now $\rho_1 \ge \rho$ be chosen as in Lemma 5.3 and let $\lambda \in \Omega_\theta$, $|\lambda| \ge \rho_1$. Then $\lambda \in \varrho(T_0)$ and we obtain from (29) that

$$\begin{aligned} \big\|(T_0-\lambda)^{-1}\big\|_{L(V_0,V)} &\le \big\|(S_0-\lambda)^{-1}\big\|_{L(V_0,V)} \big\|\big(I - R(S_0-\lambda)^{-1}\big)^{-1}\big\|_{L(V_0)} \\ &\le \frac{2c}{|\lambda|^{1-\max\{r,s\}}} \end{aligned} \tag{35}$$

and, consequently,

$$\begin{aligned} \big\|(T_0-\lambda)^{-1}R(S-\lambda)^{-1}\big\|_{L(V)} &\le \big\|(T_0-\lambda)^{-1}\big\|_{L(V_0,V)} \|R\| \big\|(S-\lambda)^{-1}\big\|_{L(V,V_1)} \\ &\le \frac{2cc'\|R\|}{|\lambda|^{2(1-\max\{r,s\})}}. \end{aligned} \tag{36}$$

Lemma 2.2 implies that $\lambda \in \varrho(T)$ and $(T-\lambda)^{-1} = (T_0-\lambda)^{-1}\big|_V$. Restricting (30) to the space V, we get

$$(S-\lambda)^{-1} - (T-\lambda)^{-1} = (T_0-\lambda)^{-1}R(S-\lambda)^{-1}. \tag{37}$$

Combining this with (36) and $\big\|(S-\lambda)^{-1}\big\|_{L(V)} \le M/|\lambda|$, we obtain the desired estimates. □

Remark 5.8. The statement of Lemma 5.4 remains true if all involved operators are restricted to V. This means that V_0, S_0 and $Q_{0\pm}$ are replaced by V, S and $Q_\pm$, respectively, where $Q_\pm$ are the restrictions of $Q_{0\pm}$ to V. The proof remains unchanged except for an adaption of the spaces.

Theorem 5.9. *Let A be quasi-sectorial and let $r+s<1$. If $\sigma(A) \cap i\mathbb{R} = \varnothing$ or if A has a compact resolvent and*

$$\ker(A-it) \cap \ker C = \ker(A^*+it) \cap \ker B^* = \{0\} \quad \textit{for all} \quad t \in \mathbb{R},$$

then T is almost bisectorial; in particular, there exist closed, T- and $(T-\lambda)^{-1}$-invariant subspaces $V_\pm \subset V$ such that $\sigma\big(T\big|_{V_\pm}\big) \subset \mathbb{C}_\pm$. If in addition $\max\{r,s\} < \frac{1}{2}$, then T is even bisectorial and strictly dichotomous.

Proof. From Theorem 5.5 we know that $i\mathbb{R} \subset \varrho(T_0)$. Hence also $i\mathbb{R} \subset \varrho(T)$ by Lemma 2.2. From (33) in Lemma 5.7 we thus conclude that T is almost bisectorial with $0 < \beta < 1$ if $\max\{r,s\} > \frac{1}{2}$ and bisectorial if $\max\{r,s\} \le \frac{1}{2}$. Note that bisectoriality implies almost bisectoriality here since $0 \in \varrho(T)$. The existence of $V_\pm$ follows by Theorem 2.6. If now $\max\{r,s\} < \frac{1}{2}$ then (34) yields

$$\|(T-\lambda)^{-1} - (S-\lambda)^{-1}\| \le \frac{c_1}{|\lambda|^{1+\varepsilon}}, \quad \lambda \in \Omega_\theta,\ |\lambda| \ge \rho_1,$$

with some $\varepsilon > 0$. In view of Remark 5.8 we can then derive in the same way as in the proof of Theorem 5.5 that T is dichotomous. □

6. Graph and angular subspaces

In this section we consider a Hamiltonian with quasi-sectorial A, $r + s < 1$, and $i\mathbb{R} \subset \varrho(T_0)$. From the last section we know that then T_0 is bisectorial and strictly dichotomous and T is almost bisectorial. We denote by $V_{0\pm}$ and $V_\pm$ the corresponding invariant subspaces of T_0 and T, respectively, and by $P_{0\pm}$ and $P_\pm$ the associated projections; see Theorem 2.6. In particular $P_{0\pm} \in L(V_0)$ while $P_\pm$ are closed operators on V. The projections $P_{0\pm}$ are given by $P_{0\pm} = TL_{0\pm}$ where $L_{0\pm} \in L(V_0)$,

$$L_{0\pm} = \frac{\pm 1}{2\pi i} \int_{\pm h - i\infty}^{\pm h + i\infty} \frac{1}{\lambda} (T_0 - \lambda)^{-1} \, d\lambda. \tag{38}$$

Recall from (24) the extended indefinite inner product $[\cdot|\cdot]$ defined on $V_1 \times V_0$ as well as $V_0 \times V_1$.

Lemma 6.1. *The operators $L_{0\pm}$ satisfy $L_{0\pm} \in L(V_0, V_1)$ and*

$$[L_{0+}v|w] = -[v|L_{0-}w] \quad \text{for all } v, w \in V_0.$$

Proof. In the proof of Lemma 5.3 we have seen that there exists $\rho_1 > 0$ such that

$$(T_0 - \lambda)^{-1} = (S_0 - \lambda)^{-1}\big(I - R(S_0 - \lambda)^{-1}\big)^{-1}$$

for $\lambda \in \Omega_\theta$, $|\lambda| > \rho_1$, and the estimates

$$\big\|(S_0 - \lambda)^{-1}\big\|_{L(V_0,V_1)} \le \frac{M_2}{|\lambda|^{1-r-s}}, \quad \big\|R(S_0 - \lambda)^{-1}\big\|_{L(V_0)} \le \frac{1}{2}$$

hold. It follows that

$$\|(T_0 - \lambda)^{-1}\|_{L(V_0,V_1)} \le \frac{2M_2}{|\lambda|^{1-r-s}}. \tag{39}$$

Since $1 - r - s > 0$ this implies that the integral in (38) converges in $L(V_0, V_1)$; in particular, $L_{0\pm} \in L(V_0, V_1)$. For $v, w \in V_0$ we can now derive, using Lemma 4.4,

$$\begin{aligned}
[L_{0+}v|w] &= \Big[\frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda}(T_0 - \lambda)^{-1} v \, d\lambda \Big| w\Big] \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big[\frac{1}{h + it}(T_0 - h - it)^{-1} v \Big| w\Big] \, dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big[v \Big| \frac{1}{h - it}(-T_0 - h + it)^{-1} w\Big] \, dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big[v \Big| \frac{1}{-h + it}(T_0 + h - it)^{-1} w\Big] \, dt \\
&= \Big[v \Big| \frac{1}{2\pi i} \int_{-h-i\infty}^{-h+i\infty} \frac{1}{\lambda}(T_0 - \lambda)^{-1} w \, d\lambda\Big] = -[v|L_{0-}w].
\end{aligned}$$

□

Corollary 6.2.

$$[v|w] = 0 \qquad \textit{for all } v \in V_{0\pm},\ w \in \mathcal{R}(L_{0\pm}).$$

Proof. This is immediate since $V_{0\pm} = \ker L_{0\mp}$. □

We can now establish conditions for the subspaces $V_{0\pm}$ to be graphs of operators. We say that a subspace $U \subset V_0 = H_{-r} \times H_{-s}^{(*)}$ is the graph of a (possibly unbounded) operator $X : \mathcal{D}(X) \subset H_{-r} \to H_{-s}^{(*)}$ if

$$U = \left\{ \begin{pmatrix} x \\ Xx \end{pmatrix} \,\middle|\, x \in \mathcal{D}(X) \right\} = \mathcal{R}\begin{pmatrix} I \\ X \end{pmatrix}.$$

We also consider the inverse situation where $U \subset H_{-r} \times H_{-s}^{(*)}$ is the graph of an operator $Y : \mathcal{D}(Y) \subset H_{-s}^{(*)} \to H_{-r}$, i.e.,

$$U = \left\{ \begin{pmatrix} Yy \\ y \end{pmatrix} \,\middle|\, y \in \mathcal{D}(Y) \right\} = \mathcal{R}\begin{pmatrix} Y \\ I \end{pmatrix}.$$

Proposition 6.3. *If*

$$\bigcap_{\lambda \in i\mathbb{R} \cap \varrho(A^*)} \ker B^*(A^* - \lambda)^{-1} = \{0\} \quad \textit{on } H_{-s}^{(*)}, \tag{40}$$

then $V_{0\pm} = \mathcal{R}\begin{pmatrix} I \\ X_{0\pm} \end{pmatrix}$ *with closed operators* $X_{0\pm} : \mathcal{D}(X_{0\pm}) \subset H_{-r} \to H_{-s}^{(*)}$. *If*

$$\bigcap_{\lambda \in i\mathbb{R} \cap \varrho(A)} \ker C(A - \lambda)^{-1} = \{0\} \quad \textit{on } H_{-r}, \tag{41}$$

then $V_{0\pm} = \mathcal{R}\begin{pmatrix} Y_{0\pm} \\ I \end{pmatrix}$ *with closed operators* $Y_{0\pm} : \mathcal{D}(Y_{0\pm}) \subset H_{-s}^{(*)} \to H_{-r}$. *If both* (40) *and* (41) *hold then* $X_{0\pm}$ *are injective and* $X_{0\pm}^{-1} = Y_{0\pm}$.

Proof. For the first assertion, since $V_{0\pm}$ are closed linear subspaces of V_0, it suffices to show that $(0, w) \in V_{0\pm}$ implies $w = 0$. Let $(0, w) \in V_{0\pm}$ and $t \in \mathbb{R}$ such that $-it \in \varrho(A^*)$. Set

$$\begin{pmatrix} x \\ y \end{pmatrix} = (T_0 - it)^{-1} \begin{pmatrix} 0 \\ w \end{pmatrix}.$$

Then $(x, y) \in \mathcal{D}(T_0) \cap V_{0\pm}$ by the invariance of $V_{0\pm}$. By Lemma 2.7 it follows that $(x, y) \in \mathcal{R}(L_{0\pm})$. Using Corollary 6.2, we get

$$0 = \left[\begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{pmatrix} 0 \\ w \end{pmatrix} \right] = i\big(x|w\big)_{s,-s}^{(*)}.$$

From

$$\begin{pmatrix} 0 \\ w \end{pmatrix} = (T_0 - it) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (A - it)x - BB^*y \\ -C^*Cx - (A^* + it)y \end{pmatrix}$$

we thus obtain

$$\begin{aligned}0 &= \big(x|w\big)^{(*)}_{s,-s} = -\big(x|C^*Cx\big)^{(*)}_{s,-s} - \big(x|(A^*+it)y\big)^{(*)}_{s,-s}\\ &= -\|Cx\|^2 - \big((A-it)x|y\big)_{-r,r} = -\|Cx\|^2 - \big(BB^*y|y\big)_{-r,r}\\ &= -\|Cx\|^2 - \|B^*y\|^2\end{aligned}$$

and therefore $Cx = B^*y = 0$. This implies $w = -(A^* + it)y$ and hence $-B^*y = B^*(A^*+it)^{-1}w = 0$. Since $t \in \mathbb{R}$ with $-it \in \varrho(A^*)$ was arbitrary, (40) implies that $w = 0$. For the second assertion, we show in an analogous way that $(w, 0) \in V_{0\pm}$ implies $w = 0$ provided that (41) holds. The final statement is then clear. □

Proposition 6.4. *Suppose that A is sectorial with $0 \in \varrho(A)$. Then*

$$V_{0-} = \mathcal{R}\begin{pmatrix} I \\ X_{0-}\end{pmatrix}, \quad V_{0+} = \mathcal{R}\begin{pmatrix} Y_{0+} \\ I\end{pmatrix}$$

with closed operators $X_{0-} : \mathcal{D}(X_{0-}) \subset H_{-r} \to H^{(*)}_{-s}$ and $Y_{0+} : \mathcal{D}(Y_{0+}) \subset H^{(*)}_{-s} \to H_{-r}$.

Proof. Let $(0, w) \in V_{0-}$ and $t \in \mathbb{R}$. Proceeding as in the previous proof, we set

$$\begin{pmatrix} x \\ y\end{pmatrix} = (T_0 - it)^{-1}\begin{pmatrix} 0 \\ w\end{pmatrix}$$

and obtain $Cx = B^*y = 0$ and hence $(A - it)x = 0$ and $w = -(A^* + it)y$. Since $i\mathbb{R} \subset \varrho(A)$ it follows that

$$(T_0 - it)^{-1}\begin{pmatrix} 0 \\ w\end{pmatrix} = \begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} 0 \\ (-A^* - it)^{-1}w\end{pmatrix}.$$

We consider now the two functions

$$\varphi(\lambda) = (T_0 - \lambda)^{-1}\begin{pmatrix} 0 \\ w\end{pmatrix}, \quad \psi(\lambda) = \begin{pmatrix} 0 \\ (-A^* - \lambda)^{-1}w\end{pmatrix}.$$

φ is analytic on a strip $\{\lambda \in \mathbb{C} \,|\, |\operatorname{Re}\lambda| < \varepsilon\}$ while ψ is analytic on a half-plane $\{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda < \varepsilon\}$ where $\varepsilon > 0$ is sufficiently small. The above derivation shows that φ and ψ coincide on $i\mathbb{R}$. Hence they coincide for $|\operatorname{Re}\lambda| < \varepsilon$ by the identity theorem. Moreover, ψ is bounded on $\overline{\mathbb{C}_-}$ since A is sectorial with $0 \in \varrho(A)$. On the other hand φ extends to a bounded analytic function on $\overline{\mathbb{C}_+}$ since $(0, w) \in V_{0-}$, see Theorem 2.6. Therefore φ extends to a bounded entire function and is thus constant by Liouville's theorem. This implies $w = 0$.

Similarly for $(w, 0) \in V_{0+}$, $t \in \mathbb{R}$ and

$$\begin{pmatrix} x \\ y\end{pmatrix} = (T_0 - it)^{-1}\begin{pmatrix} w \\ 0\end{pmatrix},$$

we derive $Cx = B^*y = 0$, $w = (A - it)x$ and $(A^* + it)y = 0$; hence

$$(T_0 - it)^{-1}\begin{pmatrix} w \\ 0\end{pmatrix} = \begin{pmatrix} (A - it)^{-1}w \\ 0\end{pmatrix}.$$

In this case the analytic functions

$$\varphi(\lambda) = (T_0 - \lambda)^{-1}\begin{pmatrix} w \\ 0 \end{pmatrix}, \qquad \psi(\lambda) = \begin{pmatrix} (A-\lambda)^{-1}w \\ 0 \end{pmatrix}$$

coincide on $i\mathbb{R}$, φ is bounded on $\overline{\mathbb{C}_-}$ since $(w,0) \in V_{0+}$, and ψ is bounded on $\overline{\mathbb{C}_+}$. Therefore φ is again constant and hence $w = 0$. □

We turn to the question of the boundedness of the operators $X_{0\pm}$, $Y_{0\pm}$. To this end we use the concept of angular subspaces, see [1, §5.1], [23, Lemma 7.1]. Consider again the projections from Lemma 5.4,

$$Q_{0-} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_{0+} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

acting on $V_0 = H_{-r} \times H_{-s}^{(*)}$.

Lemma 6.5. *Let U be a closed subspace of V_0. Then:*

(a) $U = \mathcal{R}\left(\begin{smallmatrix} I \\ X \end{smallmatrix}\right)$ *with a closed operator* $X : \mathcal{D}(X) \subset H_{-r} \to H_{-s}^{(*)}$ *if and only if*

$$U \cap \ker Q_{0-} = \{0\}.$$

$U = \mathcal{R}\left(\begin{smallmatrix} I \\ X \end{smallmatrix}\right)$ *with a bounded operator* $X \in L(H_{-r}, H_{-s}^{(*)})$ *if and only if*

$$V_0 = U \oplus \ker Q_{0-}. \tag{42}$$

(b) $U = \mathcal{R}\left(\begin{smallmatrix} Y \\ I \end{smallmatrix}\right)$ *with a closed operator* $Y : \mathcal{D}(Y) \subset H_{-s}^{(*)} \to H_{-r}$ *if and only if*

$$U \cap \ker Q_{0+} = \{0\}.$$

$U = \mathcal{R}\left(\begin{smallmatrix} Y \\ I \end{smallmatrix}\right)$ *with* $Y \in L(H_{-s}^{(*)}, H_{-r})$ *if and only if*

$$V_0 = U \oplus \ker Q_{0+}. \tag{43}$$

Proof. Observe that $\ker Q_{0-} = \{0\} \times H_{-s}^{(*)}$. Since U is the graph of some closed operator $X : \mathcal{D}(X) \subset H_{-r} \to H_{-s}^{(*)}$ if and only if $(0,y) \in U$ implies $y = 0$, the first assertion of (a) follows. By [1, Proposition 5.1], (42) holds if and only if $U = \{Xx + x \,|\, x \in \mathcal{R}(Q_{0-})\}$ with $X \in L(\mathcal{R}(Q_{0-}), \ker Q_{0-})$. Identifying $\mathcal{R}(Q_{0-}) \cong H_{-r}$ and $\ker Q_{0-} \cong H_{-s}^{(*)}$, we obtain the second assertion of (a). The proof of (b) is analogous; here $\mathcal{R}(Q_{0+}) \cong H_{-s}^{(*)}$, $\ker Q_{0+} \cong H_{-r}$. □

If (42) holds then U is called *angular* with respect to Q_{0-} and X is the *angular operator* for U. Similarly in case of (43), U is called angular with respect to Q_{0+} and angular operator Y.

The next lemma is the key step in proving that $V_{0\pm}$ are angular subspaces. The idea for its proof goes back to [4, Theorem 2.3] where instead of F_1 and F_2 the operator $Q_{0-}P + Q_{0+}\widetilde{P}$ was used, see also [1, §6.4].

Lemma 6.6. *Suppose $V_0 = U \oplus \widetilde{U}$ with closed subspaces $U, \widetilde{U} \subset V_0$. Let $P, \widetilde{P} \in L(V_0)$ be the associated complementary projections, $U = \mathcal{R}(P)$, $\widetilde{U} = \mathcal{R}(\widetilde{P})$, $I = P + \widetilde{P}$. Let $F_1 = I - Q_{0-} + P$ and $F_2 = I - P + Q_{0-}$.*

(a) *If*
$$U=\mathcal{R}\begin{pmatrix} I \\ X \end{pmatrix}, \quad \widetilde{U}=\mathcal{R}\begin{pmatrix} Y \\ I \end{pmatrix} \tag{44}$$
with some $X:\mathcal{D}(X)\subset H_{-r}\to H_{-s}^{(*)}$ *and* $Y:\mathcal{D}(Y)\subset H_{-s}^{(*)}\to H_{-r}$, *then* F_1 *and* F_2 *are injective.*

(b) *If* F_1 *and* F_2 *are bijective, then* (44) *holds with bounded operators* $X\in L(H_{-r},H_{-s}^{(*)})$, $Y\in L(H_{-s}^{(*)},H_{-r})$.

Proof. (a) By the previous lemma, identity (44) implies that $U\cap\ker Q_{0-}=\widetilde{U}\cap\ker Q_{0+}=\{0\}$. Let $F_1v=0$. Then $(I-Q_{0-})v=-Pv\in U\cap\ker Q_{0-}$, which implies $(I-Q_{0-})v=Pv=0$. It follows that $v\in\mathcal{R}(Q_{0-})\cap\ker P=\ker Q_{0+}\cap\widetilde{U}$ and hence $v=0$. The injectivity of F_2 is analogous.

(b) Let $v\in U\cap\ker Q_{0-}$. Then $(I-P)v=Q_{0-}v=0$, which yields $F_2v=0$ and thus $v=0$. On the other hand we can write $w\in V_0$ as $w=F_1v=(I-Q_{0-})v+Pv$ and so $w\in U+\ker Q_{0-}$. This shows that $V_0=U\oplus\ker Q_{0-}$, i.e., U is angular with respect to Q_{0-}. Since $F_1=I-\widetilde{P}+Q_{0+}$ and $F_2=I-Q_{0+}+\widetilde{P}$, we get by symmetry that $\widetilde{U}$ is angular to Q_{0+}. The assertion follows by the previous lemma. □

Corollary 6.7. *Suppose that* $P_{0-}-Q_{0-}$ *is compact. If*
$$V_{0-}=\mathcal{R}\begin{pmatrix} I \\ X_{0-} \end{pmatrix}, \quad V_{0+}=\mathcal{R}\begin{pmatrix} Y_{0+} \\ I \end{pmatrix},$$
with some operators X_{0-},Y_{0+}, *then these operators are in fact bounded,* $X_{0-}\in L(H_{-r},H_{-s}^{(*)})$, $Y_{0+}\in L(H_{-s}^{(*)},H_{-r})$.

Proof. We use the previous lemma with $U=V_{0-}$, $\widetilde{U}=V_{0+}$, $P=P_{0-}$, $\widetilde{P}=P_{0+}$. Then $F_1=I+(P_{0-}-Q_{0-})$ and $F_2=I-(P_{0-}-Q_{0-})$, and the assertion follows from Fredholm's alternative. □

Theorem 6.8. *Suppose that* A *has a compact resolvent. If*
$$\bigcap_{\lambda\in i\mathbb{R}\cap\varrho(A^*)}\ker B^*(A^*-\lambda)^{-1}=\{0\} \quad \text{on } H_{-s}^{(*)}, \tag{45}$$
and
$$\bigcap_{\lambda\in i\mathbb{R}\cap\varrho(A)}\ker C(A-\lambda)^{-1}=\{0\} \quad \text{on } H_{-r}, \tag{46}$$
then $V_{0\pm}=\mathcal{R}\begin{pmatrix} I \\ X_{0\pm} \end{pmatrix}$ *where the operators* X_{0-} *and* X_{0+} *are injective,* $X_{0-}\in L(H_{-r},H_{-s}^{(*)})$ *and* $X_{0+}^{-1}\in L(H_{-s}^{(*)},H_{-r})$.

Proof. If A has a compact resolvent, then the same is true for S_0 and T_0, compare Lemma 4.3. From Theorem 2.6 and Lemma 5.4 we know that
$$P_{0+}v-P_{0-}v=\frac{1}{\pi i}\int_{-i\infty}^{i\infty\,\prime}(T_0-\lambda)^{-1}v\,d\lambda, \quad v\in\mathcal{D}(T_0),$$

$$Q_{0+}v - Q_{0-}v = \frac{1}{\pi i}\int'_{\gamma_1}(S_0-\lambda)^{-1}v\,d\lambda + Kv, \quad v \in V_0,$$

where $K \in L(V_0)$. Since

$$Q_{0+} - Q_{0-} - (P_{0+} - P_{0-}) = I - 2Q_{0-} - (I - 2P_{0-}) = 2(P_{0-} - Q_{0-}),$$

we find

$$2(P_{0-} - Q_{0-})v = \frac{1}{\pi i}\int_{\gamma_1}(S_0-\lambda)^{-1} - (T_0-\lambda)^{-1}\,d\lambda\, v$$
$$- \frac{1}{\pi i}\int_{-i\rho}^{i\rho}(T_0-\lambda)^{-1}\,d\lambda\, v + Kv$$

for $v \in \mathcal{D}(T_0)$. Note here that because of (28) the first integral converges in the operator norm topology of $L(V_0)$. In particular, both integrals on the right-hand side define bounded operators in $L(V_0)$ and hence the above identity holds for all $v \in V_0$. Since $(T_0-\lambda)^{-1}$ and $(S_0-\lambda)^{-1}$ are compact, both integrals yield in fact compact operators. The expression for K in Lemma 5.4 implies that K is compact too. Consequently $P_{0-} - Q_{0-}$ is compact. The assertion is now a consequence of Proposition 6.3 and Corollary 6.7. □

Theorem 6.9. *Suppose that A has a compact resolvent, is sectorial and $0 \in \varrho(A)$. Then*

$$V_{0-} = \mathcal{R}\begin{pmatrix} I \\ X_{0-}\end{pmatrix}, \quad V_{0+} = \mathcal{R}\begin{pmatrix} Y_{0+} \\ I\end{pmatrix}$$

with $X_{0-} \in L(H_{-r}, H_{-s}^{(*)})$, $Y_{0+} \in L(H_{-s}^{(*)}, H_{-r})$.

Proof. As in the previous theorem we obtain that $P_{0-} - Q_{0-}$ is compact. Hence Proposition 6.4 and Corollary 6.7 complete the proof. □

Next we investigate the graph properties of the invariant subspaces $V_\pm$ of T. We know that $V_\pm = \mathcal{R}(P_\pm)$ where $P_\pm$ are the closed projections on V given by $P_\pm = TL_\pm$ with $L_\pm \in L(V)$,

$$L_\pm = \frac{\pm 1}{2\pi i}\int_{\pm h - i\infty}^{\pm h + i\infty}\frac{1}{\lambda}(T-\lambda)^{-1}\,d\lambda.$$

In particular, $L_\pm$ are the restrictions of $L_{0\pm}$ to V. Since $V_\pm = \ker L_\mp$ and $\ker L_\mp = \ker L_{0\mp} \cap V$ it follows that

$$V_\pm = V_{0\pm} \cap V. \tag{47}$$

This implies that graph subspace structures of $V_{0\pm}$ are inherited by the spaces $V_\pm$:

Lemma 6.10. *If*

$$V_{0+} = \mathcal{R}\begin{pmatrix} I \\ X_{0+}\end{pmatrix}$$

with a closed operator $X_{0+} : \mathcal{D}(X_{0+}) \subset H_{-r} \to H_{-s}^{(*)}$, then also

$$V_{+} = \mathcal{R}\begin{pmatrix} I \\ X_{+}\end{pmatrix}$$

where $X_+ : \mathcal{D}(X_+) \subset H \to H$ *is closed and is the part of* X_{0+} *in* H*, i.e.,* $\mathcal{D}(X_+) = \{x \in \mathcal{D}(X_{0+}) \cap H \,|\, X_{0+}x \in H\}$*. Similarly, if*

$$V_{0+} = \mathcal{R}\begin{pmatrix} Y_{0+} \\ I \end{pmatrix}$$

with a closed operator $Y_{0+} : \mathcal{D}(Y_{0+}) \subset H^{(*)}_{-s} \to H_{-r}$*, then*

$$V_+ = \mathcal{R}\begin{pmatrix} Y_+ \\ I \end{pmatrix}$$

where $Y_+ : \mathcal{D}(Y_+) \subset H \to H$ *is closed and is the part of* Y_{0+} *in* H*. The corresponding statements hold for* V_{0-} *and* V_-*.*

Proof. This is immediate from (47) and the fact that $V_\pm$ are closed subspaces of $V = H \times H$. □

Remark 6.11. A result analogous to Corollary 6.7 holds for the subspaces $V_\pm$ of V in the case that T is strictly dichotomous, i.e., if $P_\pm \in L(V)$. In particular, if $P_- - Q_-$ is compact where $Q_- = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right) \in L(V)$ and

$$V_- = \mathcal{R}\begin{pmatrix} I \\ X_- \end{pmatrix}, \quad V_+ = \mathcal{R}\begin{pmatrix} Y_+ \\ I \end{pmatrix},$$

then $X_-, Y_+ \in L(H)$.

Theorem 6.12. *Suppose that* A *has a compact resolvent and that* $\max\{r, s\} < \frac{1}{2}$*.*

(a) *If* (45) *and* (46) *hold, then* $V_\pm = \mathcal{R}\left(\begin{smallmatrix} I \\ X_\pm \end{smallmatrix}\right)$ *where* $X_\pm$ *are the parts of* $X_{0\pm}$ *in* H*. The operators* $X_\pm$ *are injective and satisfy* $X_-, X_+^{-1} \in L(H)$*.*
(b) *If* A *is sectorial and* $0 \in \varrho(A)$*, then* $V_- = \mathcal{R}\left(\begin{smallmatrix} I \\ X_- \end{smallmatrix}\right)$*,* $V_+ = \mathcal{R}\left(\begin{smallmatrix} Y_+ \\ I \end{smallmatrix}\right)$ *where* X_- *and* Y_+ *are the parts of* X_{0-} *and* Y_{0+} *in* H*, respectively, and* $X_-, Y_+ \in L(H)$*.*

Proof. The proof is analogous to the ones of Theorem 6.8 and 6.9, where it is shown that $V_{0\pm}$ are angular subspaces. First note that S and T have a compact resolvent, see Lemma 4.3. Second, since $\max\{r, s\} < \frac{1}{2}$ and since $i\mathbb{R} \subset \varrho(T)$ by our general assumption in this section, Theorem 5.9 in conjunction with Lemma 4.1 implies that T is strictly dichotomous. Consequently the projections $P_\pm$ are bounded and satisfy

$$P_+v - P_-v = \frac{1}{\pi i}\int_{-i\infty}^{i\infty}{}' (T - \lambda)^{-1} v \, d\lambda, \quad v \in \mathcal{D}(T).$$

On the other hand, for $Q_\pm \in L(V)$ given by $Q_- = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $Q_+ = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$ the identity

$$Q_+v - Q_-v = \frac{1}{\pi i}\int_{\gamma_1}{}' (S - \lambda)^{-1} v \, d\lambda + Kv, \quad v \in V,$$

holds with some $K \in L(V)$, see Lemma 5.4 and Remark 5.8. Consequently,

$$2(P_- - Q_-)v = \frac{1}{\pi i}\int_{\gamma_1} (S-\lambda)^{-1} - (T-\lambda)^{-1}\, d\lambda\, v - \frac{1}{\pi i}\int_{-i\rho}^{i\rho} (T-\lambda)^{-1}\, d\lambda\, v + Kv$$

for $v \in V$, where we have used that in view of $\max\{r,s\} < \frac{1}{2}$ and (34) all terms on the right-hand side yield bounded operators from $L(V)$. Since the resolvents of S and T are compact, we conclude that $P_- - Q_-$ is compact too. The assertion now follows from Theorems 6.8 and 6.9, Lemma 6.10 and Remark 6.11. □

7. Symmetries of the angular operators

The aim of this section is to derive symmetry properties for the operators $X_{0\pm}$ and $X_\pm$. We keep our general assumptions on the Hamiltonian: A is quasi-sectorial, $r + s < 1$ and $i\mathbb{R} \subset \varrho(T_0)$. Hence T_0 is bisectorial, strictly dichotomous and the invariant subspaces are given by

$$V_{0\pm} = \mathcal{R}(P_{0\pm}) = \ker L_{0\mp}$$

where $P_{0\pm} = TL_{0\pm}$, $L_{0\pm} \in L(V_0, V_1)$ and

$$[L_{0+}v|w] = -[v|L_{0-}w], \quad v, w \in V_0, \tag{48}$$

with the extended indefinite inner product defined in (24), see Lemma 6.1.

For a subspace $U \subset V_1$, we consider its orthogonal complement $U^{[\perp]} \subset V_0$ with respect to the extended inner product:

$$U^{[\perp]} = \{w \in V_0 \mid [v|w] = 0 \text{ for all } v \in V_1\}.$$

For $\widetilde{U} \subset V_0$ the orthogonal complement $\widetilde{U}^{[\perp]} \subset V_1$ is defined analogously. Then, as in the usual Hilbert or Krein space setting, orthogonal complements are closed and $U^{[\perp][\perp]} = \overline{U}$. Let $V_{1\pm}$ be the closure of $\mathcal{R}(L_{0\pm})$ in V_1,

$$V_{1\pm} = \overline{\mathcal{R}(L_{0\pm})}^{V_1}. \tag{49}$$

Lemma 7.1. *The following identities hold:*

(a) $V_{1\pm}^{[\perp]} = V_{0\pm}$,
(b) $V_{1\pm} = V_{0\pm} \cap V_1$.

Proof. (a) From (48) we get

$$V_{0\pm} = \ker L_{0\mp} \subset \mathcal{R}(L_{0\pm})^{[\perp]} = V_{1\pm}^{[\perp]}.$$

If on the other hand $w \in V_{1\pm}^{[\perp]}$, then $[v|L_{0\mp}w] = -[L_{0\pm}v|w] = 0$ for all $v \in V_0$. Since the inner product is non-degenerate, this implies $L_{0\mp}w = 0$ and thus $w \in V_{0\pm}$.

(b) By Lemma 2.7 we have $\mathcal{R}(L_{0\pm}) \subset V_{0\pm}$. By the continuity of the imbedding $V_1 \hookrightarrow V_0$, the subspace $V_{0\pm} \cap V_1$ is closed in V_1, and hence the inclusion from left to right follows. For the reverse inclusion let $v \in V_{0\pm} \cap V_1$. Then

$$[w|v] = 0 \qquad \text{for all } w \in V_{1\pm}$$

by (a). Since T_0 is densely defined and strictly dichotomous, Lemma 2.7 implies $\overline{\mathcal{R}(L_{0\pm})}^{V_0} = V_{0\pm}$. Hence $\overline{V_{1\pm}}^{V_0} = V_{0\pm}$ and therefore

$$[w|v] = 0 \quad \text{for all } w \in V_{0\pm}.$$

Consequently, $v \in V_{0\pm}^{[\perp]} = V_{1\pm}^{[\perp][\perp]} = V_{1\pm}$. □

Let $X_1 : \mathcal{D}(X_1) \subset H_s^{(*)} \to H_r$ be a densely defined operator. We define its adjoint with respect to the scales of Hilbert spaces $\{H_r\}$ and $\{H_s^{(*)}\}$ as the operator $X_1^* : \mathcal{D}(X_1^*) \subset H_{-r} \to H_{-s}^{(*)}$ with maximal domain such that

$$\big(X_1 x|y\big)_{r,-r} = \big(x|X_1^* y\big)_{s,-s}^{(*)}, \qquad x \in \mathcal{D}(X_1),\, y \in \mathcal{D}(X_1^*). \tag{50}$$

Then X_1^* is uniquely determined and closed.

Lemma 7.2. *If $V_{0-} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{0-} \end{smallmatrix}\right)$ with a closed operator*

$$X_{0-} : \mathcal{D}(X_{0-}) \subset H_{-r} \to H_{-s}^{(*)},$$

then also $V_{1-} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{1-} \end{smallmatrix}\right)$ with a closed operator

$$X_{1-} : \mathcal{D}(X_{1-}) \subset H_s^{(*)} \to H_r.$$

In this case:

(a) $\mathcal{D}(X_{1-}) = \left\{x \in \mathcal{D}(X_{0-}) \cap H_s^{(*)} \,\middle|\, X_{0-}x \in H_r\right\}$*, i.e., X_{1-} is the part of X_{0-} in the space of operators from $H_s^{(*)}$ to H_r;*

(b) *X_{1-} and X_{0-} are densely defined and $X_{1-}^* = X_{0-}$;*

(c) *the set $\left\{x \in \mathcal{D}(X_{0-}) \cap H_{1-r}^{(*)} \,\middle|\, X_{0-}x \in H_{1-s}\right\}$ is a core for X_{1-} and X_{0-}.*

Analogous statements hold for the spaces V_{0+}, V_{1+} and the operators X_{0+}, X_{1+}.

Proof. The inclusion $V_{1-} \subset V_{0-}$ implies that if V_{0-} is a graph, then so is V_{1-} and that X_{1-} is a restriction of X_{0-}. X_{1-} is closed since V_{1-} is closed in $V_1 = H_s^{(*)} \times H_r$. (a) is now immediate from $V_{1-} = V_{0-} \cap V_1$.

To show (b), suppose $x \in H_{-r}$, $y \in H_{-s}^{(*)}$ are such that

$$\big(X_{1-}u|x\big)_{r,-r} = \big(u|y\big)_{s,-s}^{(*)} \quad \text{for all } u \in \mathcal{D}(X_{1-}). \tag{51}$$

Then

$$\left[\begin{pmatrix} u \\ X_{1-}u \end{pmatrix} \,\middle|\, \begin{pmatrix} x \\ y \end{pmatrix}\right] = 0, \quad u \in \mathcal{D}(X_{1-}),$$

i.e., $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in V_{1-}^{[\perp]} = V_{0-}$ and thus $x \in \mathcal{D}(X_{0-})$, $X_{0-}x = y$. This implies that $\mathcal{D}(X_{1-})$ is dense in $H_s^{(*)}$. Indeed if $y \in H_{-s}^{(*)}$ with $\big(u|y\big)_{s,-s} = 0$ for all $u \in \mathcal{D}(X_{1-})$, then

(51) holds with $x = 0$ and it follows that $y = 0$. On the other hand $V_{1-}^{[\perp]} = V_{0-}$ implies

$$i\big(u|X_{0-}x\big)^{(*)}_{s,-s} - i\big(X_{1-}u|x\big)_{r,-r} = \left[\begin{pmatrix} u \\ X_{1-}u \end{pmatrix} \,\Big|\, \begin{pmatrix} x \\ X_{0-}x \end{pmatrix}\right] = 0$$

for all $u \in \mathcal{D}(X_{1-})$, $x \in \mathcal{D}(X_{0-})$ and therefore $X_{0-} \subset X_{1-}^*$. Moreover, if $x \in \mathcal{D}(X_{1-}^*)$ and $y = X_{1-}^* x$, then x, y satisfy (51) and we obtain $x \in \mathcal{D}(X_{0-})$. Consequently $X_{0-} = X_{1-}^*$. Finally X_{0-} is densely defined since $\mathcal{D}(X_{1-})$ is dense in $H_s^{(*)}$ and the imbedding $H_s^{(*)} \hookrightarrow H_{-r}$ is continuous and dense.

Finally (c) follows from the equivalence

$$u \in \mathcal{D}(X_{0-}) \cap H_{1-r}^{(*)} \;\wedge\; X_{0-}u \in H_{1-s} \quad\Longleftrightarrow\quad \begin{pmatrix} u \\ X_{0-}u \end{pmatrix} \in V_{0-} \cap \mathcal{D}(T_0)$$

in conjunction with $\mathcal{R}(L_{0-}) = V_{0-} \cap \mathcal{D}(T_0)$, $V_{0-} = \overline{\mathcal{R}(L_{0-})}^{V_0}$, see Lemma 2.7, and $V_{1-} = \overline{\mathcal{R}(L_{0-})}^{V_1}$. □

Remark 7.3. The previous lemma implies $X_{1\pm} \subset X_{0\pm} = X_{1\pm}^*$. From this identity and (50) we obtain

$$\big(X_{1\pm}x|y\big) = \big(x|X_{1\pm}y\big), \qquad x, y \in \mathcal{D}(X_{1\pm}).$$

Consequently, if we consider $X_{1\pm}$ as an unbounded operator on H, then it is densely defined and symmetric and hence closable. The corresponding closure will be determined in Lemma 7.5.

Now we turn to the symmetry properties of the operators $X_\pm$. To this end, we look at the subspaces

$$M_\pm = \overline{\mathcal{R}(L_\pm)}^{V} \tag{52}$$

of V. By Lemma 2.7, we have $M_\pm \subset V_\pm$ and this inclusion may be strict. The next lemma shows that $M_\pm^{[\perp]}$ coincides with $V_\pm$. Note here that since $M_\pm \subset V$, $M_\pm^{[\perp]}$ is the orthogonal complement with respect to the inner product $[\cdot|\cdot]$ *in* V, i.e., $M_\pm^{[\perp]} \subset V$ in the usual Krein space sense.

Lemma 7.4. *The following identities hold:*

(a) $V_{1\pm} \subset M_\pm$ *and* $\overline{V_{1\pm}}^{V} = M_\pm$*;*
(b) $M_\pm^{[\perp]} = V_\pm$.

Proof. (a) Since $\mathcal{D}(T_0)$ is dense in V_0 and $L_{0\pm} \in L(V_0, V_1)$, we have

$$V_{1\pm} = \overline{\mathcal{R}(L_{0\pm})}^{V_1} \subset \overline{L_{0\pm}(\mathcal{D}(T_0))}^{V_1} \subset \overline{L_{0\pm}(\mathcal{D}(T_0))}^{V} \subset \overline{L_{0\pm}(V)}^{V} = M_\pm.$$

On the other hand $\mathcal{R}(L_\pm) \subset \mathcal{R}(L_{0\pm}) \subset V_{1\pm}$, which implies $M_\pm \subset \overline{V_{1\pm}}^{V}$ and thus equality.

(b) Lemma 6.1 implies $[L_+v|w] = -[v|L_-w]$ for all $v, w \in V$. Using this and the definitions of $V_\pm$ and $M_\pm$, the proof is completely analogous to Lemma 7.1(a). □

Lemma 7.5. *Suppose V_{0-} is a graph subspace $V_{0-} = \mathcal{R}\binom{I}{X_{0-}}$. Then $V_- = \mathcal{R}\binom{I}{X_-}$ and $M_- = \mathcal{R}\binom{I}{X_{M-}}$ where X_-, X_{M-} are closed operators on H. Moreover,*

(a) $X_{M-} \subset X_-$,
(b) X_- *is the part of* X_{0-} *in* H,
(c) X_{M-} *is the closure of* X_{1-} *when considered as an operator on* H,
(d) $\{x \in \mathcal{D}(X_{0-}) \cap H_{1-r}^{(*)} \mid X_{0-}x \in H_{1-s}\}$ *is a core for* X_{M-},
(e) X_{M-} *and* X_- *are densely defined and* $X_{M-}^* = X_-$. *In particular* X_{M-} *is symmetric.*

Again, analogous statements hold for V_{0+}, V_+ and M_+ and the respective operators.

Proof. The first assertions up to (c) follow readily from $M_- \subset V_- \subset V_{0-}$, $V_- = V_{0-} \cap V$, $\overline{V_{1-}}^V = M_-$ and the closedness of M_- and V_- in V. (d) is a consequence of (c) and Lemma 7.2(c), and (e) follows from $M_-^{[\perp]} = V_-$ in an analogous way to the proof of Lemma 7.2(b). □

Lemma 7.6. *The symmetric operators X_{M-} and X_{M+} are nonnegative and nonpositive, respectively.*

Proof. Here we employ the indefinite inner product $[\cdot|\cdot]_\sim$ defined in (23). Observe that X_{M-} is nonnegative, i.e., $(X_{M-}x|x) \geq 0$ for all $x \in \mathcal{D}(X_{M-})$, if and only if $[v|v]_\sim \geq 0$ for all $v \in M_-$. Likewise $(X_{M+}x|x) \leq 0$ for all $x \in \mathcal{D}(X_{M+})$ if and only if $[v|v]_\sim \leq 0$ for all $v \in M_+$. Consider first $v \in \mathcal{D}(T)$. Using (13) and Lemma 4.4, we calculate

$$\begin{aligned}\mathrm{Re}[P_+v - P_-v|v]_\sim &= \frac{1}{\pi}\int_{-\infty}^{\infty\,\prime} \mathrm{Re}[(T-it)^{-1}v|v]_\sim\,dt\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty\,\prime} \mathrm{Re}[(T-it)^{-1}v|(T-it)(T-it)^{-1}v]_\sim\,dt\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty\,\prime} \mathrm{Re}[T(T-it)^{-1}v|(T-it)^{-1}v]_\sim\,dt \leq 0.\end{aligned}$$

If now $v \in \mathcal{D}(T) \cap V_-$ then $P_+v - P_-v = -v$ and hence $[v|v]_\sim \geq 0$. Since $\mathcal{D}(T) \cap V_-$ is dense in M_- by Lemma 2.7, we conclude that $[v|v]_\sim \geq 0$ for $v \in M_-$. Similarly for $v \in \mathcal{D}(T) \cap V_+$ we obtain $P_+v - P_-v = v$ and thus $[v|v]_\sim \leq 0$ for all $v \in M_+$. □

Corollary 7.7. *If $\max\{r, s\} < \frac{1}{2}$, then $X_{M\pm} = X_\pm$. The operator X_- is selfadjoint and nonnegative, X_+ is selfadjoint and nonpositive.*

Proof. The assumption implies that T is strictly dichotomous. Then $M_\pm = V_\pm$ by Lemma 2.7 and hence $X_{M\pm} = X_\pm$. □

8. The Riccati equation

We keep the general assumptions of the previous section.

Lemma 8.1. *Suppose $X_0 \in L(H_{-r}, H_{-s}^{(*)})$ is such that its graph subspace $U = \mathcal{R}\left(\begin{smallmatrix} I \\ X_0 \end{smallmatrix}\right)$ is T_0- and $(T_0-\lambda)^{-1}$-invariant. Consider the isomorphism $\varphi : H_{-r} \to U$, $x \mapsto \left(\begin{smallmatrix} x \\ X_0 x \end{smallmatrix}\right)$. Then*

(a) $X_0(H_{1-r}^{(*)}) \subset H_{1-s}$;

(b) $(A - BB^*X_0)x = \varphi^{-1}T_0\big|_U \varphi x$ *for all* $x \in H_{1-r}^{(*)}$;

(c) $A^*X_0x + X_0Ax - X_0BB^*X_0x + C^*Cx = 0$ *for all* $x \in H_{1-r}^{(*)}$.

Proof. First note that φ is indeed an isomorphism between H_{-r} and U since X_0 is bounded. The inverse is $\varphi^{-1} = \mathrm{pr}_1\big|_U$ where

$$\mathrm{pr}_1 : V_0 = H_{-r} \times H_{-s}^{(*)} \to H_{-r}$$

denotes the projection onto the first component. Recall the decomposition $T_0 = S_0 + R$ from (25) and consider the two operators $F = \varphi^{-1}T_0\big|_U\varphi$ and $A_0 = \mathrm{pr}_1 S_0 \varphi$, both understood as unbounded operators on H_{-r}. Since $\mathcal{D}(T_0) = \mathcal{D}(S_0) = H_{1-r}^{(*)} \times H_{1-s}$, their domains are

$$\mathcal{D}(F) = \mathcal{D}(A_0) = \left\{ x \in H_{1-r}^{(*)} \,\middle|\, X_0 x \in H_{1-s} \right\}.$$

Moreover,

$$A_0 x = Ax \quad \text{for } x \in \mathcal{D}(A_0),$$

i.e., A_0 is a restriction of A when A is considered as an operator on H_{-r} with $\mathcal{D}(A) = H_{1-r}^{(*)}$. Since φ is an isomorphism we get $\varrho(F) = \varrho\big(T_0\big|_U\big)$. Also $\varrho(T_0) \subset \varrho\big(T_0\big|_U\big)$ by the invariance of U. Therefore $i\mathbb{R} \subset \varrho(F)$. For $t \in \mathbb{R}$ we compute

$$\begin{aligned}(A_0 - F)(F - it)^{-1} &= (\mathrm{pr}_1 S_0\varphi - \varphi^{-1}T_0\varphi)(\varphi^{-1}T_0\varphi - it)^{-1} \\ &= \mathrm{pr}_1(S_0 - T_0)\varphi\varphi^{-1}(T_0 - it)^{-1}\varphi = -\mathrm{pr}_1 R(T_0 - it)^{-1}\varphi.\end{aligned}$$

From (39) in the proof of Lemma 6.1 we know that $\left\|(T_0 - it)^{-1}\right\|_{L(V_0,V_1)} \to 0$ as $t \to \infty$, and we conclude that $\|(A_0 - F)(F - it)^{-1}\| < 1$ for $t > 0$ sufficiently large. Now

$$A_0 - it = F - it + A_0 - F = \left(I + (A_0 - F)(F - it)^{-1}\right)(F - it),$$

which implies that $it \in \varrho(A_0)$. Since also $it \in \varrho(A)$ for large t and $A_0 \subset A$, it follows that in fact

$$\mathcal{D}(A_0) = \mathcal{D}(A) = H_{1-r}^{(*)}.$$

Consequently, $X_0\big(H_{1-r}^{(*)}\big) \subset H_{1-s}$. Since $Fx = Ax - BB^*X_0x$ for $x \in \mathcal{D}(F) = \mathcal{D}(A_0)$, (b) is now clear. To show (c), let $x \in H_{1-r}^{(*)}$. Then $X_0x \in H_{1-s}$ and $\varphi x \in \mathcal{D}(T_0)$. By the invariance of U, there exists $y \in H_{1-r}^{(*)}$ such that $T_0\varphi x = \varphi y$, i.e.,

$$\begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}\begin{pmatrix} x \\ X_0x \end{pmatrix} = \begin{pmatrix} y \\ X_0y \end{pmatrix}$$

and thus

$$X_0Ax - X_0BB^*X_0x = X_0(Ax - BB^*X_0x) = X_0y = -C^*Cx - A^*X_0x. \qquad \square$$

Corollary 8.2. *If* $V_{0-} = \mathcal{R}\left(\begin{smallmatrix} I \\ X_{0-} \end{smallmatrix}\right)$ *with a bounded operator* $X_{0-} \in L(H_{-r}, H_{-s}^{(*)})$*, then* $X_{0-}(H_{1-r}^{(*)}) \subset H_{1-s}$*, the Riccati equation*

$$A^*X_{0-}x + X_{0-}Ax - X_{0-}BB^*X_{0-}x + C^*Cx = 0, \quad x \in H_{1-r}^{(*)},$$

holds, and $A - BB^*X_{0-}$ *considered as an unbounded operator on* H_{-r} *is sectorial with spectrum* $\sigma(A - BB^*X_{0-}) \subset \mathbb{C}_-$*. In particular, it generates an exponentially stable analytic semigroup on* H_{-r}*.*

Proof. $A - BB^*X_{0-}$ is similar to $T_0|_{V_{0-}}$ via the isomorphism φ from the previous lemma, $\sigma(T_0|_{V_{0-}}) \subset \mathbb{C}_-$, and $T_0|_{V_{0-}}$ is sectorial by [21, Theorem 5.6]. □

Remark 8.3. If $X_{0-} \in L(H_{-r}, H_{-s}^{(*)})$ and hence $X_{0-}(H_{1-r}^{(*)}) \subset H_{1-s}$, Lemmas 7.2 and 7.5 imply that $H_{1-r}^{(*)} \subset \mathcal{D}(X_{1-}) \subset \mathcal{D}(X_-)$. Since the operator $A - BB^*X_{0-}$ considered on H_{-r} has domain $H_{1-r}^{(*)}$ we find that

$$A - BB^*X_{0-} = A - BB^*X_- = A - BB^*X_{1-}.$$

Hence the Riccati equation can be written as

$$A^*X_{1-}x + X_{0-}Ax - X_{0-}BB^*X_{1-}x + C^*Cx = 0, \quad x \in H_{1-r}^{(*)},$$

or in weak form, using $X_{0-} = X_{1-}^*$, as

$$\begin{aligned}\big(X_{1-}x|Ay\big)_{r,-r} + \big(Ax|X_{1-}y\big)_{-r,r} - \big(B^*X_{1-}x|B^*X_{1-}y\big)_U \\ + \big(Cx|Cy\big)_Y = 0, \quad x, y \in H_{1-r}^{(*)}.\end{aligned}$$

Of course, in both Riccati equations X_{1-} may be replaced by one of its extensions X_{M-} and X_-.

Remark 8.4. For $X_{0-} \in L\big(H_{-r}, H_{-s}^{(*)}\big)$, Corollary 8.2 yields that $A - BB^*X_-$ is sectorial *when considered as an operator in* H_{-r}*.* On the other hand, we can consider the part of $A - BB^*X_-$ in H, which we denote by $(A - BB^*X_-)\big|_H$. Then $(A - BB^*X_-)\big|_H$ is almost sectorial: First note that

$$\sigma\big((A - BB^*X_-)\big|_H\big) \subset \sigma(A - BB^*X_-).$$

From $A - BB^*X_- = \varphi^{-1}T_0\big|_{V_{0-}}\varphi$ we obtain

$$\|(A - BB^*X_- - \lambda)^{-1}\|_{L(H_{-r},H)} \le \big\|\big(T_0\big|_{V_{0-}} - \lambda\big)^{-1}\big\|_{L(V_{0-},V)}\|\varphi\|,$$

and (35) in conjunction with $i\mathbb{R} \subset \varrho(A - BB^*X_-)$ implies

$$\big\|\big(A - BB^*X_- - \lambda\big)^{-1}\big\|_{L(H_{-r},H)} \le \frac{c_0}{|\lambda|^{1-\max\{r,s\}}} \quad \text{for } \lambda \in i\mathbb{R} \setminus \{0\},$$

with some constant $c_0 > 0$. Moreover, since $\big\|\big(T_0\big|_{V_{0-}} - \lambda\big)^{-1}\big\|_{L(V_0)}$ is bounded on $\mathbb{C}_+$, $\big\|\big(T_0\big|_{V_{0-}} - \lambda\big)^{-1}\big\|_{L(V_0,\mathcal{D}(T_0))}$ does not grow faster than $|\lambda|$ on $\mathbb{C}_+$, where $\mathcal{D}(T_0)$ is equipped with the graph norm. As the imbedding $\mathcal{D}(T_0) \hookrightarrow V$ is continuous,

$\|(A - BB^*X_- - \lambda)^{-1}\|_{L(H_{-r},H)}$ does not grow faster than $|\lambda|$ on $\mathbb{C}_+$ too. The Phragmén–Lindelöf theorem then implies that

$$\|(A - BB^*X_- - \lambda)^{-1}\|_{L(H_{-r},H)} \leq \frac{c_0}{|\lambda|^{1-\max\{r,s\}}} \quad \text{for } \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}$$

and hence $(A - BB^*X_-)|_H$ is almost sectorial, see [21, §5].

Now suppose in addition that $\max\{r, s\} < \frac{1}{2}$ and that $X_- \in L(H)$, e.g. as a consequence of Theorem 6.12. Then

$$(A - BB^*X_-)\big|_H = \varphi\big|_H^{-1} T\big|_{V_-} \varphi\big|_H$$

where $\varphi\big|_H : H \to V_-,\ x \mapsto \binom{x}{X_- x}$ is an isomorphism. Since T is bisectorial by Theorem 5.9, $T\big|_{V_-}$ is sectorial by [21, Theorem 5.6], and hence $(A - BB^*X_-)\big|_H$ is sectorial, too.

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