The QuaSEFE Problem

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Abstract. We initiate the study of Simultaneous Graph Embedding with Fixed Edges in the beyond planarity framework. In the QuaSEFE problem, we allow edge crossings, as long as each graph individually is drawn quasiplanar, that is, no three edges pairwise cross. We show that a triple consisting of two planar graphs and a tree admit a QuaSEFE. This result also implies that a pair consisting of a 1-planar graph and a planar graph admits a QuaSEFE. We show several other positive results for triples of planar graphs, in which certain structural properties for their common subgraphs are fulfilled. For the case in which simplicity is also required, we give a triple consisting of two quasiplanar graphs and a star that does not admit a QuaSEFE. Moreover, in contrast to the planar SEFE problem, we show that it is not always possible to obtain a QuaSEFE for two matchings if the quasiplanar drawing of one matching is fixed.

Keywords: Quasiplanar · SEFE · Simultaneous graph drawing

1 Introduction

Simultaneous Graph Embedding is a family of problems where one is given a set of graphs $G_1, \ldots, G_k$ with shared vertex set V and is required to produce drawings $\Gamma_1, \ldots, \Gamma_k$ of them, each satisfying certain readability properties, so that each vertex has the same position in every Γ_i. The readability property that is usually pursued is the planarity of the drawing, and a large body of research has been devoted to establish the complexity of the corresponding decision problem, or to determine whether such embeddings always exist, given the number and the types of the graphs; for a survey refer to [9].

Work started at Dagstuhl Seminar 19092, "Beyond-Planar Graphs: Combinatorics, Models and Algorithms". Research supported by MIUR Project "MODE" under PRIN 20157EFM5C, by MIUR Project "AHeAD" under PRIN 20174LF3T8, by Roma Tre University Azione 4 Project "GeoView", by DFG grant Ka812/17-1, by NSF under grants CCF-1740858 and CCF-1712119, and by SNSF Project 200021E-171681.

D. Archambault and C. D. Tóth (Eds.): GD 2019, LNCS 11904, pp. 268–275, 2019.
https://doi.org/10.1007/978-3-030-35802-0_21

These problems have been studied both from a geometric (*Geometric Simultaneous Embedding* - GSE) [6,16] and from a topological point of view (*Simultaneous Embedding with Fixed Edges* - SEFE) [10,12,19]. In particular, in GSE the edges are straight-line segments, while in SEFE they are topological curves, but the edges shared between two graphs G_i and G_j have to be drawn in the same way in Γ_i and Γ_j. Unless otherwise specified, we focus on the topological setting.

We study a relaxation of the SEFE problem, where the graphs can be drawn with edge crossings. However, we prohibit certain crossing configurations in the drawings $\Gamma_1, \dots, \Gamma_k$, to guarantee their readability, i.e., we require that they satisfy the conditions of a graph class in the area of *beyond-planarity*; see [15] for a survey on this topic. We initiate this study with the class of *quasiplanar* graphs [2,3,18], by requiring that no Γ_i contains three mutually crossing edges.

Definition 1 (QuaSEFE). *Given a set of graphs $G_1 = (V, E_1), \dots, G_k = (V, E_k)$ with shared vertex set V, we say that $\langle G_1, \dots, G_k \rangle$ admits a QuaSEFE if there exist quasiplanar drawings $\Gamma_1, \dots, \Gamma_k$ of $G_1, \dots, G_k$, respectively, so that each vertex of V has the same position in every Γ_i and each edge shared between two graphs G_i and G_j is drawn in the same way in Γ_i and Γ_j. Further, the QuaSEFE problem asks whether an instance $\langle G_1, \dots, G_k \rangle$ admits a QuaSEFE.*

It may be worth mentioning that the problem of computing quasiplanar simultaneous embeddings of graph pairs has been studied in the geometric setting [13,14]. Also, simultaneous embeddings have been considered in relation to another beyond-planarity geometric graph class, namely *RAC graphs* [7,8,17,20].

We prove in Sect. 2 that any triple of two planar graphs and a tree admits a QuaSEFE, which also implies that any pair consisting of a 1-planar graph[1] and a planar graph admits a QuaSEFE. Recall that, for the original SEFE problem, there exist even negative instances composed of two outerplanar graphs [19]. Further, we investigate triples of planar graphs in which the common subgraphs have specific structural properties. Finally, we show negative results in more specialized settings in Sect. 3 and conclude with open problems in Sect. 4.

2 Sufficient Conditions for QuaSEFEs

In this section, we provide several sufficient conditions for the existence of a QuaSEFE, mainly focusing on instances composed of three planar graphs G_1, G_2, and G_3. We start with a theorem relating the existence of a SEFE of two of the input graphs to the existence of a QuaSEFE of the three input graphs.

Theorem 1. *Let $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, and $G_3 = (V, E_3)$ be planar graphs with shared vertex set V. If $\langle G_1 \setminus G_3, G_2 \setminus G_3 \rangle$ admits a SEFE, then $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE, in which the drawing of G_3 is planar.*

[1] A graph is k-planar if it admits a drawing where each edge has at most k crossings.

Proof. First construct a SEFE of $\langle G_1 \setminus G_3, G_2 \setminus G_3\rangle$, and then construct a planar drawing of G_3, whose vertices have already been placed, but whose edges have not been drawn yet, using the algorithm by Pach and Wenger [23].

The drawing of G_3 is planar, by construction. The drawing of G_1 is quasiplanar, as it is partitioned into two subgraphs, $G_1 \setminus G_3$ and $G_1 \cap G_3$, each of which is drawn planar. Analogously, the drawing of G_2 is quasiplanar. □

Since every pair composed of a planar graph and a tree admits a SEFE [19], we derive from Theorem 1 the following positive result for the QuaSEFE problem.

Corollary 1. *Let $G_1 = (V, E_1)$ and $G_3 = (V, E_3)$ be planar graphs and $T_2 = (V, E_2)$ be a tree with shared vertex set V. Then $\langle G_1, T_2, G_3\rangle$ admits a QuaSEFE, in which the drawing of G_3 is planar.*

Corollary 1 already shows that allowing quasiplanarity significantly enlarges the set of positive instances. We further strengthen this result, by additionally guaranteeing that even the tree is drawn planar. For this, we use a result on the *partially embedded planarity* [5] problem (PEP): Given a planar graph G, a subgraph H of G, and a planar embedding $\mathcal{H}$ of H, is there a planar embedding of G whose restriction to H coincides with $\mathcal{H}$? In particular, we will exploit the following characterization, which is the core of a linear-time algorithm for PEP.

Lemma 1 **([5]).** *Let $(G, H, \mathcal{H})$ be an instance of PEP. A planar embedding $\mathcal{G}$ of G is a solution for $(G, H, \mathcal{H})$ if and only if the following conditions hold:* **(C.1)** *for every vertex $v \in V$, the edges incident to v in H appear in the same cyclic order in the rotation schemes of v in $\mathcal{H}$ and in $\mathcal{G}$; and* **(C.2)** *for every cycle C of H, and for every vertex v of $H \setminus C$, we have that v lies in the interior of C in $\mathcal{G}$ if and only if it lies in the interior of C in $\mathcal{H}$.*

Theorem 2. *Let $G_1 = (V, E_1)$ and $G_3 = (V, E_3)$ be planar graphs and $T_2 = (V, E_2)$ be a tree with shared vertex set V. Then $\langle G_1, T_2, G_3\rangle$ admits a QuaSEFE, in which the drawings of G_1 and T_2 are planar.*

Proof. Consider planar embeddings $\mathcal{G}_1$ and $\mathcal{G}_3^*$ of G_1 and $G_3 \setminus G_1$, respectively. We draw G_1 according to $\mathcal{G}_1$. This fixes the embedding of the subgraph $T_2 \cap G_1$ of T_2, thus resulting in an instance of the PEP problem. Since T_2 is acyclic, Condition C.1 of Lemma 1 is trivially fulfilled. Also, since every rotation scheme of T_2 is planar, we choose for the edges of $(T_2 \cap G_3) \setminus G_1$ an order compatible with $\mathcal{G}_3^*$, still satisfying Condition C.1. Finally, we draw the remaining edges of G_3 by considering the instance of PEP defined by its embedded subgraph $(T_2 \cap G_3) \setminus G_1$. Condition C.1 is trivially satisfied, and Condition C.1 is satisfied by construction, if we add the edges of G_3 according to $\mathcal{G}_3^*$. Since crossings edges of the same graph belong to $G_3 \setminus G_1$ and $G_3 \cap G_1$, the drawing of G_3 is quasiplanar. □

The additional property guaranteed by Theorem 2 is crucial to infer the first result in the simultaneous embedding setting for a class of beyond-planar graphs.

Theorem 3. *Let $G_1 = (V, E_1)$ be a 1-planar graph and $G_2 = (V, E_2)$ be a planar graph. Then $\langle G_1, G_2\rangle$ admits a QuaSEFE.*

Proof. As G_1 is 1-planar, it is the union of a planar graph G'_1 and a forest F_1 [1]. We augment F_1 to a tree T_1. By Theorem 2, there is a QuaSEFE of $\langle G'_1, T_1, G_2 \rangle$ where G'_1 and T_1 are drawn planar. Thus, G_1 is drawn quasiplanar. □

We now study properties of the subgraphs induced by the edges that belong to one, to two, or to all the input graphs. We denote by H_i the subgraph induced by the edges only in G_i; by $H_{i,j}$ the subgraph induced by the edges only in G_i and G_j; and by H the subgraph induced by the edges in all graphs; see Fig. 1a.

The following two corollaries of Theorem 1 list sufficient conditions for $G_1 \setminus G_3$ and $G_2 \setminus G_3$ to have a SEFE. In the first case, $H_{1,2}$ has a unique embedding, which fulfills the conditions of Lemma 1 with respect to any planar embedding of G_1 and of G_2. In the second case, this is because $G_1 \setminus G_3$ is a subgraph of $G_2 \setminus G_3$.

Corollary 2. *Let $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, $G_3 = (V, E_3)$ be planar graphs with shared vertex set V. If $H_{1,2}$ is acyclic and has maximum degree 2, then $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE.*

Corollary 3. *Let $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, $G_3 = (V, E_3)$ be planar graphs with shared vertex set V. If $H_1 = \emptyset$, then $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE.*

Contrary to the previous corollaries, Theorem 1 has no implication for the graph H, as there are instances with $H = \emptyset$ where no pair of graphs has a SEFE. However, we show that a simple structure of H is still sufficient for a QuaSEFE.

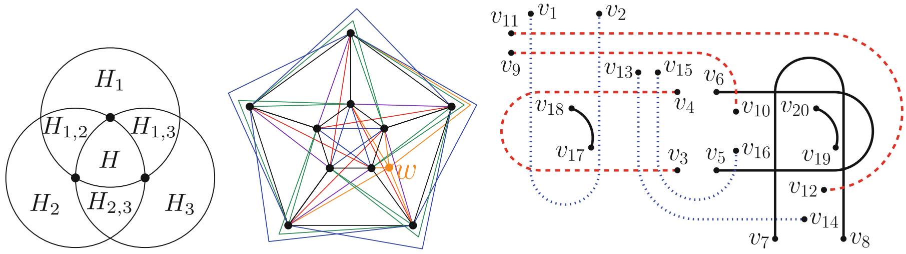
Theorem 4. *Let $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, $G_3 = (V, E_3)$ be planar graphs with shared vertex set V. If H has a planar embedding that can be extended to a planar embedding $\mathcal{G}_i$ of each graph G_i, then $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE.*

Proof. We draw the graph $G_1 \setminus H_{1,3} = H_1 \cup H_{1,2} \cup H$ with embedding $\mathcal{G}_1$, the graph $G_2 \setminus H_{1,2} = H_2 \cup H_{2,3} \cup H$ with embedding $\mathcal{G}_2$, and the graph $G_3 \setminus H_{2,3} = H_3 \cup H_{1,3} \cup H$ with embedding $\mathcal{G}_3$. Then, the edges of G_1 are partitioned into two sets, one belonging to $G_1 \setminus H_{1,3}$ and one to $G_3 \setminus H_{2,3}$, each of which is drawn planar. As the same holds for the edges of G_2 and G_3, the statement follows. □

Corollary 4. *Let $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, $G_3 = (V, E_3)$ be planar graphs with shared vertex set V. If H is acyclic and has maximum degree 2, then $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE.*

The above discussion shows that, if one of the seven subgraphs in Fig. 1a is empty, or has a sufficiently simple structure, $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE. Most notably, this is always the case in the *sunflower* setting [4,21,24], in which every edge belongs either to a single graph or to all graphs, i.e., $H_{1,2} = H_{1,3} = H_{2,3} = \emptyset$. We extend this result to any set of planar graphs. We remark that SEFE is NP-complete in the sunflower setting for three planar graphs [4,24].

Theorem 5. *Let $G_1 = (V, E_1), \ldots, G_k = (V, E_k)$ be planar graphs with shared vertex set V in the sunflower setting. Then $\langle G_1, \ldots, G_k \rangle$ admits a QuaSEFE.*



(a) (b) (c)

Fig. 1. (a) Subgraphs induced by the edges in one, two, or three graphs. (b) A simple quasiplanar drawing of Q_1 in Theorem 6, obtained by adding w to the drawing of K_{10} by Brandenburg [11]. (c) Theorem 7: Edge (v_{18}, v_{20}) crosses either all dotted blue or all dashed red edges, making (v_5, v_6) and (v_7, v_8) uncrossable (Color figure online).

Proof. Let H be the graph induced by the edges belonging to all graphs. We independently draw planar the graph H and every subgraph $G_i \setminus H$, for $i = 1, \ldots, k$. This guarantees that each G_i is drawn quasiplanar. □

We remark that all our proofs are constructive. Moreover, the corresponding algorithms run in linear time, as they exploit linear-time algorithms for constructing planar embeddings of graphs [22], for extending their partial embeddings [5], and for partitioning 1-planar graphs into planar graphs and forests [1].

3 Counterexamples for `QuaSEFE`

In this section we complement our positive results, by providing negative instances of the `QuaSEFE` problem in two specific settings. We start with a negative result about the existence of a *simple* `QuaSEFE` for two quasiplanar graphs and one star. Here *simple* means that a pair of independent edges in the same graph is allowed to cross at most once and a pair of adjacent edges in the same graph is not allowed to cross. Note that our algorithms in Sect. 2 may produce non-simple drawings. Also, the maximum number of edges in a quasiplanar graph with n vertices depends on whether simplicity is required or not [2].

Theorem 6. *There exist two quasiplanar graphs $Q_1 = (V, E_1)$, $Q_2 = (V, E_2)$ and a star $S_3 = (V, E_3)$ with shared vertex set V such that $\langle Q_1, Q_2, S_3 \rangle$ does not admit a simple `QuaSEFE`.*

Proof. Let $V = \{v_1, \ldots, v_{10}, w\}$ and let E_{10} be the edges of the complete graph on $V \setminus \{w\}$. Further, let $E_1 = E_{10} \cup \{(w, v_1), \ldots, (w, v_6)\}$, let $E_2 = E_{10} \cup \{(w, v_7)\}$, and let $E_3 = \{(w, v_1), \ldots, (w, v_{10}\}$. By construction, S_3 is the star on all eleven vertices with center w, while Fig. 1b shows that there is a simple quasiplanar drawing of Q_1 (and of Q_2, which is a subgraph of Q_1, up to vertex relabeling).

Suppose that $\langle Q_1, Q_2, S_3 \rangle$ has a simple QuaSEFE, and let $\Gamma_{1,2}$ be the drawing of the union of Q_1 and Q_2 that is part of it. Since the union of Q_1 and Q_2 has 52 edges, which exceeds the upper bound of $6.5n - 20$ edges in a simple quasiplanar graph [2], $\Gamma_{1,2}$ is not simple or not quasiplanar. Since (w, v_7) is the only edge in $\Gamma_{1,2}$ that is not in Q_1, edge (w, v_7) is involved in every crossing violating simplicity or quasiplanarity. Analogously, one of $(w, v_1), \dots, (w, v_6)$, say (w, v_1), is involved in every crossing violating simplicity or quasiplanarity; in particular, (w, v_1) crosses (w, v_7). Since both (w, v_1) and (w, v_7) belong to S_3, the drawing of S_3 that is part of the simple QuaSEFE is not simple, a contradiction. □

The second special setting is the one in which one of the input graphs is already drawn in a quasiplanar way, and the goal is to draw the other input graphs so that the resulting simultaneous drawing is a QuaSEFE. This setting is motivated by the natural approach, for an instance $\langle G_1, \dots, G_k \rangle$, of first constructing a solution for $\langle G_1, \dots, G_{k-1} \rangle$ and then adding the remaining edges of G_k. Note that, since the drawing of the first graph partially fixes a drawing of the second graph, this can be seen as a version of the PEP problem for quasiplanarity.

For the original SEFE problem, this setting always has a solution when the graph that is already drawn (in a planar way) is a general planar graph, and the other graph is a tree [19]. In a surprising contrast, we construct negative instances for the QuaSEFE problem that are composed of two matchings only.

Theorem 7. *Let $M_1 = (V, E_1)$ and $M_2 = (V, E_2)$ be two matchings on the same vertex set V and let Γ_1 be a quasiplanar drawing of M_1. Instance $\langle M_1, M_2 \rangle$ does not always admit a QuaSEFE in which the drawing of M_1 is Γ_1.*

Proof. First recall that the edges in $E_1 \cap E_2$ have to be drawn in the quasiplanar drawing Γ_2 of G_2 as they are in Γ_1. Consider the quasiplanar drawing Γ_1 of the matching (v_{2i-1}, v_{2i}), with $i = 1, \dots, 10$, in Fig. 1c, and let E_2 contain the edges (v_{17}, v_{19}) and (v_{18}, v_{20}). Since v_{17} is enclosed in a region bounded by the crossing edges (v_1, v_2) and (v_3, v_4), in any quasiplanar drawing of M_2 edge (v_{17}, v_{19}) crosses exactly one of (v_1, v_2) and (v_3, v_4). In the first case, (v_{17}, v_{19}) crosses also (v_{13}, v_{14}) and (v_{15}, v_{16}) (dotted blue). In the second case, (v_{17}, v_{19}) crosses also (v_9, v_{10}) and (v_{11}, v_{12}) (dashed red). In both cases, (v_5, v_6) and (v_7, v_8) cannot be crossed, and thus (v_{17}, v_{19}) cannot be drawn so that Γ_2 is quasiplanar. □

4 Conclusions and Open Problems

We initiated the study of simultaneous embeddability in the beyond planar setting, which is a fertile and almost unexplored research direction that promises to significantly enlarge the families of representable graphs when compared with the planar setting. We conclude the paper by listing a few open problems.

– A natural question is whether two 1-planar graphs, a quasiplanar graph and a matching, three outerplanar graphs, or four paths admit a QuaSEFE. All our algorithms construct drawings with a stronger property than quasiplanarity, namely that they are composed of two sets of planar edges. Exploiting quasiplanarity in full generality may lead to further positive results.

- Motivated by Theorem 6, we ask whether some of the constructions presented in Sect. 2 can be modified to guarantee the simplicity of the drawings.
- Another intriguing direction is to determine the computational complexity of the `QuaSEFE` problem, both in its general version and in the two restrictions studied in Sect. 3. In particular, the setting in which one of the graphs is already drawn can be considered as a quasiplanar version of the `PEP` problem, which is known to be linear-time solvable in the planar case [5].
- Extend the study to other beyond-planarity classes. For example, do any two planar graphs admit a k-planar `SEFE` for some constant k?

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