Chapter 30
Multibody Modelling

A vehicle on elastic suspensions may be modelled as a system made by a certain number of rigid bodies connected with each other by mechanisms of various kinds and by a set of massless springs and dampers simulating the suspensions. A vehicle with four wheels can be modelled as a system with 10 degrees of freedom, six for the body and one for each wheel. This holds for any type of suspension, if the motion of the wheels due to the compliance of the system constraining the motion of the suspensions (longitudinal and transversal compliance of the suspensions) is neglected. The wheels of each axle may be suspended separately (independent suspensions) or together (solid axle suspensions), but the total number of degrees of freedom is the same (Fig. 30.1). Additional degrees of freedom, such as the rotation of the wheels about their axis or about the kingpin, can be inserted into the model to allow the longitudinal slip or the compliance of the steering system to be taken into account.

The multibody approach can be pushed much further, by modelling, for instance, each of the links of the suspensions as a rigid body. To model a short-long arms (SLA) suspension it is possible to resort to three rigid bodies, simulating the lower and upper triangles and the strut, plus a further rigid body simulating the steering bar. While modelling the system in greater detail, the number of rigid bodies included in the model increases. However, if the compliance of the various elements is neglected (i.e. if these bodies are rigid bodies), the number of degrees of freedom does not increase along with the number of bodies: an SLA suspension always has a single degree of freedom, even if it is made up of a number of rigid bodies simulating its various elements.

The mathematical model of a multibody system is thus made up of the equations of motion of the various elements, which in tri-dimensional space are $6n$, if n is the number of the rigid bodies, plus a suitable number of constraint equations.

Consider for instance the articulated truck of Fig. 30.1d. The rigid bodies are 8 (tractor, trailer and 6 rigid axles) and thus the equations of motion are 48 second order differential equations. The constraint equations are 27: 3 equations for the constraint between tractor and trailer (these state that the coordinates of the center of the hitch, assumed to be a spherical hinge, are the same if this point is seen as belonging to the tractor or to the trailer) and 4 equations for each axle, leaving to each

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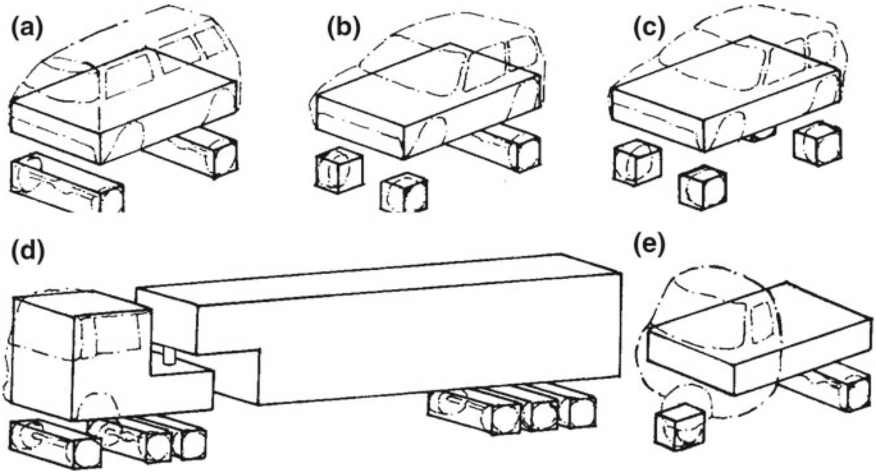

Fig. 30.1 Example of models for the dynamic study of road vehicles. **a**–**c** Vehicle with two axles, 10 d.o.f.; **d** Articulated truck with 6 axles, 21 d.o.f.; **e** Vehicle with 3 wheels; 9 d.o.f

one of them just two, out of its 6 degrees of freedom. The 27 constraint equations are algebraic equations containing only the generalized coordinates but not the velocities (holonomic constraints).

By using the 27 constraint equations to eliminate 27 of the generalized coordinates, a set of 21 equations in the 21 independent generalized coordinates is obtained. It must be emphasized that in the 48 equations of motion originally written, the forces the various bodies exchange at the constraints are included; these forces are then eliminated when the constraint equations are introduced. The 27 equations so eliminated can be used to compute the constraint forces.

This approach is the broadest, and is usually implemented in *general purpose* multibody computer codes.

It is possible to resort to a simpler approach in the case of the multibody models used in motor vehicle dynamics, because the internal constraints are holonomic and the system is branched. One of the bodies may be chosen as *main body*, to which a number of secondary, *first level* bodies are attached. Other secondary bodies, considered as *second level* bodies, are then attached, and so on. Secondary bodies have only the degrees of freedom allowed by the constraints. In this way, the minimum number of equations needed for the study are directly obtained. Such equations are all differential equations, usually of the second order. The forces exchanged at the constraints between the bodies do not appear explicitly in the equations and need not be computed in the dynamic study of the system as a whole. The model of the articulated truck with 6 axles shown in Fig. 30.1d obtained in this way is sketched in Fig. 30.2: the tractor is considered as the main body, the axles of the tractor and the trailer are first level secondary bodies, and the axles of the trailer are second level secondary bodies.

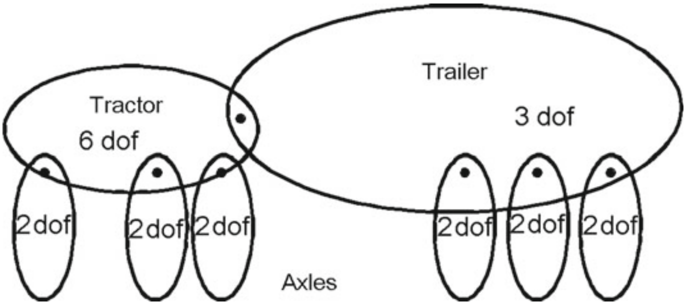

Fig. 30.2 Model of the articulated truck with 6 axles shown as a branched model: the tractor is the main body, the axles of the tractor and the trailer are first level secondary bodies and the axles of the trailer are second level secondary bodies. The constraint between tractor and trailer is a spherical hinge constraining 3 degrees of freedom, while those constraining the axles to the sprung masses lock 4 degrees of freedom each

30.1 Isolated Vehicle

The model for an isolated vehicle can thus be easily built through the following steps:

- choice of the generalized coordinates;
- computation of the expressions for the kinetic and potential energies, the dissipation function and the virtual work of external forces (road-wheels forces and aerodynamic forces);
- writing the equations of motion through Lagrange equations.

The basic degrees of freedom of the model are:

- six degrees of freedom for the sprung mass (usually three components of the displacement define the position of the vehicle and three rotations define its orientation in space);
- two degrees of freedom for each rigid axle;
- one degree of freedom for each independent suspension.

The total number of degrees of freedom is then $6 + 2m$, where m is the number of axles.

To these basic degrees of freedom, it is possible to add the following:

- The rotation χ of each wheel, or better its angular velocity $\dot{\chi}$, if the angle of rotation of each wheel is considered as an independent variable. It is then possible to compute longitudinal forces at the road-wheel contact from the longitudinal slip.

  As an alternative, it is possible to neglect the longitudinal slip and to compute the wheel rotations from the space covered by the vehicle or, better, to compute their velocity from the velocity of the vehicle, but in this case the wheel rotations are not degrees of freedom of the system.

- The steering angle δ of the steering wheels. It may be considered as:
  - a given quantity, or better, an input, if the motion is studied with locked controls and the compliance of the steering system is neglected;
  - a known function of a single variable, the angle at the steering wheel δ_v, if the motion is studied with free controls but the compliance of the steering system is neglected;
  - a variable, linked by a an equation expressing the compliance of the steering system, to the angle at the steering wheel δ_v that is a given quantity, or better, an input, if the motion is studied with locked controls and the compliance of the steering system is accounted for;
  - an independent variable, to which a further independent variable, the angle at the steering wheel δ_v is added, if the motion is studied with free controls and the compliance of the steering system is accounted for.

A motor vehicle with four wheels can then be described by a model with 10 degrees of freedom (Fig. 30.1) in the simplest case (locked controls, neglecting longitudinal slip and the compliance of the steering system), that become 14 if the slip of the wheels is considered, 15 if the study is performed with free controls and rigid steering, or 16, 17 or 19 depending on how the steering system is modelled and, in the latter case, on how many wheels steer.

Once the kinematics of the suspensions has been defined, it is possible to write the equations of motion. An approach also used in commercial codes is to introduce the elasto-kinematic characteristics of the suspensions directly (often the kinematic characteristics alone, because the suspensions are assumed to be made of rigid bodies) to define the kinematics of the system. The characteristics of the tires, including the cornering forces, the aligning torques, and the relationships linking the longitudinal slip with the longitudinal forces (in case the longitudinal slip is not neglected) can be expressed using the magic formula.

The ten (or more, depending on the model used) equations can thus be written. They are quite complicated nonlinear equations, difficult to write in explicit form. They will not be shown here.

The solution of such a set of equations can be undertaken only by numerically integrating the equations in time starting from a given set of initial conditions and specifying the time history of the various inputs (steering angle, if the manoeuvre with locked controls, or the torque acting on the steering wheel for the motion with free controls). As an alternative, it is possible to use a model of the driver to simulate the behavior of the vehicle-driver system. Suspensions are modelled by introducing their elasto-kinematic characteristics directly.

Several commercial computer codes operating in this way exist. One of the most common is Carsim®, based on a model with 14 degrees of freedom.

A more complex alternative is the use of one of the standard multibody general purpose codes to simulate the suspensions in detail, taking into account the exact kinematics of the system and again simulating the tires using the magic formula. An example of a commercial code operating along these lines is ADAMS-Car®, based on the general purpose code ADAMS®.

Codes of the latter type draw upon a much larger number of equations, because they do not use just the minimum number of generalized coordinates, but are based on the explicit equations of motion of the various parts and on the relevant constraint equations.

The two approaches are equivalent to the user, because in both cases it is necessary to resort to the numerical integration of the equations of motion, simulating the dynamic behavior of the vehicle. The only actual difference for the user is that in the first case the behavior of the suspensions is introduced in synthetic form, computing their kinematic characteristics separately or measuring experimentally and then introducing them into the computations, while in the second case the geometry of the suspensions is directly introduced in analytic form.

30.2 Linearized Model for the Isolated Vehicle

30.2.1 Basic Assumptions

While in the case of the nonlinear model the exact geometry or the elasto-kinematic characteristics of the suspensions must be introduced, in the case of linearization it is possible to write a model that is fairly precise and quite general, while allowing analytical solutions to be obtained. In this case it is worthwhile to write the equations of motion in explicit form, so that it is possible to obtain general results and closed form solutions. In particular, it will be possible to obtain solutions in the frequency domain and to perform stability studies.

The model is based on the following additional assumptions:

- Reference is made to a certain configuration of the vehicle. It may be the static equilibrium condition with the vehicle at standstill or travelling at a constant speed. If the shape of the vehicle is such that it produces little aerodynamic lift and pitching moment, the first choice may be the best, because it allows the motion at constant speed to be studied. However, if aerodynamic forces are important, as in racing cars, the configuration of the suspensions may change considerably at varying speed, so much so that linearization is not possible, and reference must then be made to the equilibrium configuration at the given speed. The linearization of the model allows us to resort to the superimposition of effects and then to neglect static forces (weight, aerodynamic lift in the reference conditions, etc.) in the dynamic study.

- The kinematics of suspensions is linearized around the reference position.
- Pitch and roll angles are small enough to linearize their trigonometric functions. Also, the displacements in Z direction and all linear and angular velocities, with the exception of the forward speed and the rotation speed of wheels, are considered as small quantities.

Two other assumptions, needed to further simplify the equations of motion, are then added:

- The vertical plane xz through the center of mass is a symmetry plane for the vehicle and its parts.
- Sideslip angles of the wheels are small enough to linearize the cornering forces and the aligning torque.

30.2.2 Sprung Mass

Let the reference frame of the sprung mass be $x_s y_s z_s$, with axis x_s parallel to the roll axis and axis z_s lying in the plane of symmetry. Axis x^* coincides with the projection of the roll axis on the ground. A plane perpendicular to the road and to axis x^* containing the centre of mass G of the vehicle in the reference position is defined (Sect. B-B in Fig. 30.3). The roll axis intersects such a plane in H; O is the point on the ground vertically under H (it may be located above H, as the roll axis may lie below the ground in particular cases). Two further reference frames are defined. They are:
— xyz, fixed to the sprung mass, with origin in point H, x-axis coinciding with the roll axis, z axis laying in the symmetry plane of the sprung mass;
— $x^*y^*z^*$ with origin in point O; x^*-axis coincides with the projection on the ground of the roll axis and z^* axis is perpendicular to the road.

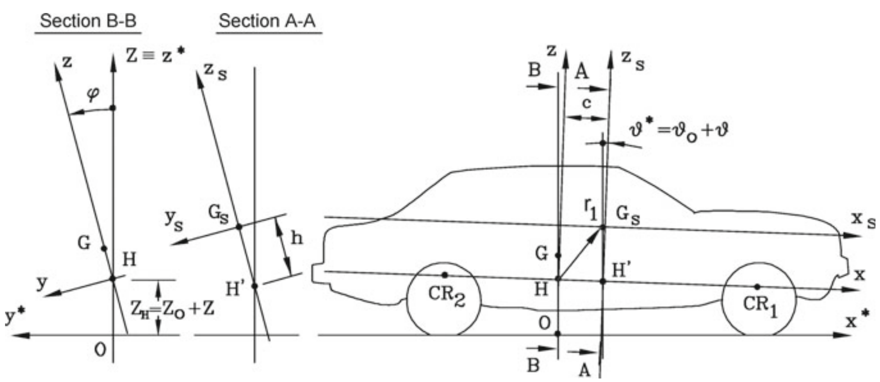

Fig. 30.3 Reference frames for the sprung mass and definition of points H and O

Instead of using the coordinates of the centre of mass G_s of the sprung mass to define the generalized coordinates for the translational degrees of freedom, the coordinates X_H, Y_H and Z_H of point H in the inertial frame $OX_iY_iZ_i$ will be used. In the following, to simplify the notation it will be $X = X_H$ and $Y = Y_H$. Operating in this way, if the roll and pitch motions are locked, the frame $x^*y^*z^*$ coincides with frame xyz defined in Fig. 25.15 and the model reduces to that of a rigid vehicle.

Coordinate Z_H can be considered as the sum of a constant value Z_0 corresponding to a reference position and a displacement Z:

$$Z_H = Z_0 + Z \ . \tag{30.1}$$

The generalized coordinates for translations of the sprung mass are then X, Y and Z, with Z considered as a small displacement with respect to the reference position.

The generalized coordinates for rotations are three Tait-Bryan angles (Appendix A, Fig. A.5): the yaw angle ψ, the pitch angle, here considered as the sum of a constant value θ_0 related to the reference position and a pitch generalized coordinate θ, and the roll angle ϕ. Angle θ_0 is the inclination on the horizontal direction of the roll axis in the reference position and will be considered as a small angle. All generalized coordinates and velocities, except v_x, are then small quantities.

Velocity v_x may be confused with the velocity V along the vehicle path. This is made possible by the smallness of the pitch angle $\theta_0 + \theta$ and of the sideslip angle β.

The rotation matrix allowing us to pass from the frame $Gxyz$ fixed to the body to the inertial frame $X_iY_iZ_i$ is:

$$\mathbf{R} = \mathbf{R}_1\mathbf{R}_2\mathbf{R}_3 \ , \tag{30.2}$$

where:

$$\mathbf{R}_1 = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{R}_2 = \begin{bmatrix} \cos(\theta_0+\theta) & 0 & \sin(\theta_0+\theta) \\ 0 & 1 & 0 \\ -\sin(\theta_0+\theta) & 0 & \cos(\theta_0+\theta) \end{bmatrix},$$

$$\mathbf{R}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}.$$

Its explicit expression is reported in Appendix A (Equation (A.106)), in which $\theta_0 + \theta$ is substituted for θ).

If the pitch and roll angles are small, i.e. their cosine is approximately equal to 1 and their sine is equal to the angle, product $\mathbf{R}_2\mathbf{R}_3$ is, approximately:

$$\mathbf{R}_2\mathbf{R}_3 \approx \begin{bmatrix} 1 & 0 & \theta_0+\theta \\ 0 & 1 & -\phi \\ -\theta_0-\theta & \phi & 1 \end{bmatrix}. \tag{30.3}$$

Because the generalized forces are written in the body-fixed frame, it is expedient to write the kinetic energy in term of the components v_x, v_y and v_z of the velocity written in the $x^*y^*z^*$ frame and the components Ω_x, Ω_y and Ω_z of the angular velocity in frame Gxyz.

The components of the velocity and angular velocity so obtained are not the derivatives of true coordinates, but are linked with the derivatives of the coordinates by the six kinematic equations

$$\mathbf{V} = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = \mathbf{R}_1^T \begin{Bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{Bmatrix} , \tag{30.4}$$

$$\begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -\sin(\theta) \\ 0 & \cos(\phi) & \sin(\phi)\cos(\theta) \\ 0 & -\sin(\phi) & \cos(\theta)\cos(\phi) \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} . \tag{30.5}$$

The third Eq. (30.4) is justified because Z differs from Z_H by a constant.

The vector of the generalized coordinates is then

$$\mathbf{q} = \begin{bmatrix} X & Y & Z & \phi & \theta & \psi \end{bmatrix}^T . \tag{30.6}$$

Let the generalized velocities for translational degrees of freedom be the components of the velocity in the $x^*y^*z^*$ frame. For the rotational degrees of freedom, on the other hand, the derivatives $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ of coordinates ϕ, θ and ψ, which in the following will be indicated as v_ϕ, v_θ and v_ψ, will be used instead of the components Ω_x, Ω_y and Ω_z of the angular velocity, as shown in Appendix A. This choice is due to the fact that the yawing moments are more easily expressed when considering an axis perpendicular to the road, and also because the linearization of the model allows us to proceed in this way without difficulties.

The generalized velocities are then

$$\mathbf{w} = \begin{bmatrix} v_x & v_y & v_z & v_\phi & v_\theta & v_\psi \end{bmatrix}^T . \tag{30.7}$$

The relationship linking the generalized velocities to the derivatives of the generalized coordinates may then be written as

$$\mathbf{w} = \mathbf{A}^T \dot{\mathbf{q}} , \tag{30.8}$$

where matrix **A**[1] is:

$$\mathbf{A} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} \end{bmatrix} . \tag{30.9}$$

[1]Matrix **A** here defined has nothing to do with the dynamic matrix of the system, which is also indicated as **A**.

The inverse transformation is Eq. (A.85):

$$\dot{\mathbf{q}} = \mathbf{B}\mathbf{w} \,,$$

where[2] $\mathbf{B} = \mathbf{A}^{-\mathbf{T}}$ is the inverse of the transpose of **A**. In this case, **A** is a rotation matrix, and then

$$\mathbf{A}^{-1} = \mathbf{A}^{T} \;;\mathbf{B} = \mathbf{A}. \tag{30.10}$$

If $\mathbf{r}_1$ is the vector defining the position of the center of mass of the sprung mass $\mathrm{G_S}$ with respect to point H, the position of the former in the inertial frame is

$$(\overline{\mathrm{G_S{-}O'}}) = (\overline{\mathrm{H} - \mathrm{O'}}) + \mathbf{R}\mathbf{r}_1. \tag{30.11}$$

Assume that the vehicle body has a symmetry plane and that this plane coincides with plane xz in Fig. 30.3. In the reference position, points G and $\mathrm{G_S}$ then belong to the symmetry plane and the second component of vector $\mathbf{r}_1$ vanishes. The coordinates of the center of the sprung mass are c, 0 and h in the xyz frame, and thus the expression of vector $\mathbf{r}_1$ is:

$$\mathbf{r}_1 = \left[\, c \;0\; h \,\right]^T \,. \tag{30.12}$$

Because $\mathbf{r}_1$ is constant, the velocity of point $\mathrm{G_S}$ is:

$$\mathbf{V}_{\mathrm{G_S}} = \left[\, \dot{X} \;\dot{Y}\; \dot{Z} \,\right]^T + \dot{\mathbf{R}}\mathbf{r}_1 \,. \tag{30.13}$$

i.e.,

$$\mathbf{V}_{\mathrm{G_S}} = \mathbf{R}_1\mathbf{V} + \dot{\mathbf{R}}\mathbf{r}_1 \,. \tag{30.14}$$

and then the translational kinetic energy of the sprung mass is

$$\mathcal{T}_t = \frac{1}{2}m\left(\mathbf{V}^T\mathbf{V} + \mathbf{r}_1^T\dot{\mathbf{R}}^T\dot{\mathbf{R}}\mathbf{r}_1 + 2\mathbf{V}^T\mathbf{R}_1^T\dot{\mathbf{R}}\mathbf{r}_1\right) \,. \tag{30.15}$$

Because xz is a symmetry plane for the sprung mass, its inertia tensor is

$$\mathbf{J}_s = \begin{bmatrix} J_{x_s} & 0 & -J_{xz_s} \\ 0 & J_{y_s} & 0 \\ -J_{xz_s} & 0 & J_{z_s} \end{bmatrix}. \tag{30.16}$$

The rotational kinetic energy of the sprung mass is then

$$\mathcal{T}_w = \frac{1}{2}\boldsymbol{\Omega}^T\mathbf{J}_s\boldsymbol{\Omega}. \tag{30.17}$$

[2]Matrix **B** here used must not be confused with the input gain matrix, which is also usually indicated as **B**.

Performing the relevant computations, expressing the components of the angular velocity as functions of the derivatives of the coordinates and neglecting the terms containing powers higher than 2 of small quantities, it follows that

$$\begin{array}{c}\mathcal{T}_s = \frac{1}{2}m_s\left({v_x}^2 + {v_y}^2 + {v_z}^2\right) + \frac{1}{2}\left(m_s h^2 + J_{x_s}\right)^2 \dot{\phi}^2 + \\ +\frac{1}{2}\left[m_s\left(h^2 + c^2\right) + J_{y_s}\right]\dot{\theta}^2 + \frac{1}{2}\left(m_s c^2 + J_{z_s}\right)\dot{\psi}^2 - \left(m_s ch + J_{xz_s}\right)\dot{\psi}\,\dot{\phi} + \\ -m_s v_x\left[\left(c\theta_0 - h\right)\dot{\theta} - h\dot{\psi}\,\phi + c\dot{\theta}\theta\right] - m_s v_y\left(h\dot{\phi} - c\dot{\psi}\right) - m_s c v_z\dot{\theta}.\end{array} \tag{30.18}$$

The height of the center of mass of the sprung mass on the road is

$$Z_G = Z_0 + Z + \mathbf{e}_3^T\mathbf{R}\mathbf{r}_1\ , \tag{30.19}$$

where:

$$\mathbf{e}_3 = \left[\,0\;0\;1\,\right]^T$$

is the unit vector of axis Z.

The potential energy of the sprung mass is simply its gravitational potential energy; its expression is:

$$\mathcal{U}_s = m_s g\left(Z_0 + Z\right) + m_s g\mathbf{e}_3^T\mathbf{R}\mathbf{r}_1\ , \tag{30.20}$$

or, performing the relevant computations

$$\mathcal{U}_s = m_s g\left(Z_0 + Z\right) + m_s g\left[-c\sin\left(\theta_0 + \theta\right) + h\cos\left(\theta_0 + \theta\right)\cos\left(\phi\right)\right]\ . \tag{30.21}$$

Because the model is linearized, the trigonometric functions of small angles may be substituted by their series, truncated after the quadratic term

$$\begin{array}{l}\sin\left(\theta_0 + \theta\right) \approx \theta_0 + \theta\ ,\ \cos\left(\phi\right) \approx 1 - \dfrac{\phi^2}{2}\ , \\ \cos\left(\theta_0 + \theta\right) \approx 1 - \dfrac{\left(\theta_0 + \theta\right)^2}{2} = 1 - \dfrac{\theta_0^2}{2} - \dfrac{\theta^2}{2} - \theta_0\theta\ .\end{array}$$

Neglecting the constant term, it follows that

$$\mathcal{U}_s = m_s g\left[Z - \left(c + h\theta_0\right)\theta - h\frac{\theta^2}{2} - h\frac{\phi^2}{2}\right]\ . \tag{30.22}$$

30.2.3 General Solid Axle Suspension

Geometry of the suspension

A solid axle suspension can be modelled as a secondary rigid body having two degrees of freedom with respect to the main body.

The geometry of the suspension may be simplified by assuming that it is possible to identify a roll center CR in the motion about the reference position. This is a point belonging to the roll axis x and to a plane perpendicular to the ground passing through the centers of the wheels. In the heave motion, point CR belonging to the sprung mass, indicated as CR_s, will not coincide with the corresponding point CR_u belonging to the axle. Assume that the latter moves along a trajectory belonging to the xz plane fixed to the vehicle body. For small displacements of the body, substitute the trajectory with its tangent in CR_s. (Fig. 30.4).

Let the position of CR_s in the frame fixed to the sprung mass be

$$(\overline{CR_s - H}) = \left[0\ 0\ x_u \right]^T . \tag{30.23}$$

If the unit vector tangent to the trajectory of CR_u in CR_s is

$$\mathbf{s} = \left[s_x\ 0\ s_z \right]^T \tag{30.24}$$

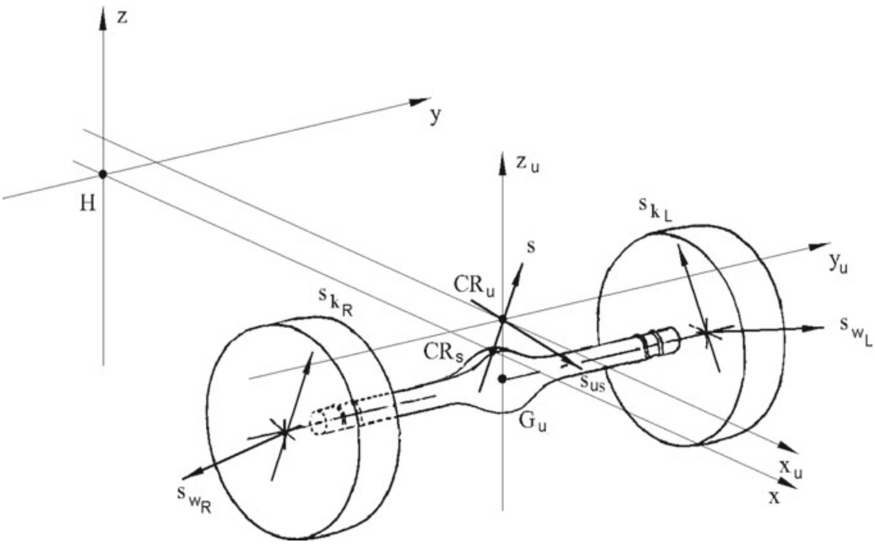

Fig. 30.4 Sketch of an idealized solid axle suspension

and ζ is the distance between these two points, the position of CR_u in the frame fixed to the sprung mass is

$$(\overline{CR_u - H}) = \left[x_u + \zeta s_x \;\; 0 \;\; \zeta s_z \right]^T \approx \left[x_u + \zeta s_x \;\; 0 \;\; \zeta \right]^T . \tag{30.25}$$

The second of the two expressions is justified by the fact that the angle between vector $\mathbf{s}$ and axis z is small.

If the roll rotation of the unsprung mass occurs about an axis parallel to the roll axis x, its rotation matrix is

$$\mathbf{R}_k = \mathbf{R}_1 \mathbf{R}_2 \mathbf{R}_{3k} , \tag{30.26}$$

where the rotation matrices for yaw and pitch rotations are the usual ones, while the roll rotation is

$$\mathbf{R}_{3k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi_k) & -\sin(\phi_k) \\ 0 & \sin(\phi_k) & \cos(\phi_k) \end{bmatrix} .$$

The axis about which the unsprung mass rotates may be different from the roll axis of the vehicle. It is then possible to define a unit vector $\mathbf{s}_k$ (Fig. 30.4) that defines such a rotation axis. If the components of this vector, all functions of ζ, are x_{us}, y_{us} and z_{us}, it is possible to define a matrix $\mathbf{R}_{us}$, that is a function of ζ too, allowing the reference frame of the unsprung mass to be rotated so that its longitudinal axis coincides with the rotation axis of the sprung mass

$$\mathbf{R}_{us}(\zeta) = -\frac{1}{\sqrt{x_{us}^2 + y_{us}^2}} \begin{bmatrix} x_{us}\sqrt{x_{us}^2 + y_{us}^2} & -y_{us} & -z_{us}x_{us} \\ y_{us}\sqrt{x_{us}^2 + y_{us}^2} & x_{us} & -z_{us}y_{us} \\ z_{us}\sqrt{x_{us}^2 + y_{us}^2} & 0 & x_{us}^2 + y_{us}^2 \end{bmatrix} . \tag{30.27}$$

If the deviation of the roll axis of the unsprung mass from the longitudinal direction is small, vector $\mathbf{s}_k$ is contained in the symmetry plane, and the rotation matrix reduces to

$$\mathbf{R}_{us}(\zeta) \approx \begin{bmatrix} 1 & 0 & -z_{us} \\ 0 & 1 & 0 \\ z_{us} & 0 & 1 \end{bmatrix} . \tag{30.28}$$

Translational kinetic energy

The rotation matrix of the unsprung mass is then

$$\mathbf{R}_k = \mathbf{R}_1 \mathbf{R}_2 \mathbf{R}_{us} \mathbf{R}_{3k} . \tag{30.29}$$

Once linearized, the product of matrices $\mathbf{R}_2 \mathbf{R}_{us} \mathbf{R}_{3k}$ is

$$\mathbf{R}_2\mathbf{R}_{us}\mathbf{R}_{3k} \approx \begin{bmatrix} 1 & 0 & \theta+\theta_0-z_{us} \\ 0 & 1 & -\phi_k \\ -\theta-\theta_0+z_{us} & \phi_k & 1 \end{bmatrix} . \qquad (30.30)$$

Let the coordinates of the center of mass G_u of the unsprung mass in the reference position in the reference frame of the vehicle be x_{Gu}, y_{Gu} and z_{Gu} ($y_{Gu} = 0$ for symmetry reasons). Its position in the inertial frame is

$$(\overline{G_u - O'}) = (\overline{H - O'}) + \mathbf{R}_1\mathbf{r} \ , \qquad (30.31)$$

where

$$\mathbf{r} = \mathbf{R}_2 \left[\mathbf{R}_3 \left\{ \begin{array}{c} x_u + \zeta s_x \\ 0 \\ \zeta \end{array} \right\} + \mathbf{R}_{us}\mathbf{R}_{3N} \left\{ \begin{array}{c} x_{Gu} - x_u \\ 0 \\ z_{Gu} \end{array} \right\} \right] , \qquad (30.32)$$

that is, linearizing,

$$\mathbf{r} = \left\{ \begin{array}{c} x_{Gu} - z_{us}z_{Gu} + \theta_0 z_{Gu} + \theta z_{Gu} \\ -\phi_k z_{Gu} \\ \zeta + z_{Gu} + z_{us}\left(x_{Gu} - x_u\right) - x_{Gu}\left(\theta + \theta_0\right) \end{array} \right\} . \qquad (30.33)$$

The height of the center of mass of the kth suspension from the ground may be considered as the sum of a constant value related to the reference position, plus a displacement of the same order of the other small quantities (like Z):

$$Z_{G_u} = Z_{0k} + Z_k = Z_0 + Z + \mathbf{e}_3^T\mathbf{R}_1\mathbf{r} . \qquad (30.34)$$

Performing the relevant computations and linearizing the trigonometric functions of small angles, it follows that

$$Z_{0k} + Z_k = Z_0 + Z - x_{Gu}\left(\theta_0 + \theta\right) + \zeta + z_{Gu} + z_{us}\left(x_{Gu} - x_u\right) . \qquad (30.35)$$

In the reference position, its value is

$$Z_{0k} = Z_0 - x_{Gu}\theta_0 + z_{Gu} + z_{us}\left(x_{Gu} - x_u\right) \qquad (30.36)$$

and then the relationship linking ζ to Z_k is simply

$$\zeta = Z_k - Z + x_{Gu}\theta . \qquad (30.37)$$

The component variable in time Z_k of the Z coordinate of the center of mass of the kth suspension will be assumed to be the generalized coordinate for the vertical displacement of the unsprung mass.

Introducing the linearized value of ζ into the expression for $\mathbf{r}$, it follows that

$$\mathbf{r} = \left\{ \begin{array}{c} x_{Gu} - z_{us} z_{Gu} + \theta_0 z_{Gu} + \theta z_{Gu} \\ -\phi_k z_{Gu} \\ z_{Gu} + z_{us} \left(x_{Gu} - x_u \right) - x_{Gu} \theta_0 + Z_k - Z \end{array} \right\} . \tag{30.38}$$

Its derivative with respect to time, once linearized, is

$$\dot{\mathbf{r}} = \left\{ \begin{array}{c} \dot{\theta} z_{Gu} \\ -\dot{\phi}_k z_{Gu} \\ \dot{Z}_k - \dot{Z} \end{array} \right\} . \tag{30.39}$$

The velocity of the center of mass of the unsprung mass is then

$$\mathbf{V}_{\mathrm{G}_u} = \left[\dot{X} \ \dot{Y} \ \dot{Z} \right]^T + \mathbf{R}_1 \dot{\mathbf{r}} + \dot{\mathbf{R}}_1 \mathbf{r} . \tag{30.40}$$

The velocity $\mathbf{V}_{\mathrm{G}_u}$ in the $x^* y^* z^*$ frame is obtained by premultiplying this expression by $\mathbf{R}_1^T$. Remembering that

$$\mathbf{R}_1^T \mathbf{R}_1 = \mathbf{I} , \mathbf{R}_1^T \dot{\mathbf{R}}_1 = \dot{\psi} \mathbf{S} = \dot{\psi} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{30.41}$$

it follows that

$$\mathbf{V}_{\mathrm{G}_u} = \mathbf{V} + \dot{\mathbf{r}} + \dot{\psi} \mathbf{S} \mathbf{r} . \tag{30.42}$$

The translational kinetic energy of the unsprung mass is then

$$\mathcal{T}_t = \frac{1}{2} m_u \left(\mathbf{V}^T \mathbf{V} + \dot{\mathbf{r}}^T \dot{\mathbf{r}} + \dot{\psi}^2 \mathbf{r}^T \mathbf{S}^T \mathbf{S} \mathbf{r} + 2 \mathbf{V}^T \dot{\mathbf{r}} + 2 \dot{\psi} \mathbf{V}^T \mathbf{S} \mathbf{r} + \mathbf{2} \dot{\psi} \dot{\mathbf{r}}^T \mathbf{S} \mathbf{r} \right) , \tag{30.43}$$

that is, by indicating as r_x, r_y, r_z the components of vector $\mathbf{r}$,

$$\begin{array}{c} \mathcal{T}_t = \frac{1}{2} m_u \left\{ v_x^2 + v_y^2 + v_z^2 + \dot{r}_x^2 + \dot{r}_y^2 + \dot{r}_z^2 + \dot{\psi}^2 \left(r_x^2 + r_y^2 \right) + \right. \\ \left. + 2 \left(v_x \dot{r}_x + v_y \dot{r}_y + v_z \dot{r}_z \right) + 2 \dot{\psi} \left(-v_x r_y + v_y r_x - \dot{r}_x r_y + \dot{r}_y r_x \right) \right\} . \end{array} \tag{30.44}$$

Only the constant and linear terms of $\mathbf{r}$ are present in all terms, except for the term in $v_x \dot{r}_x$. Only the expression of $\dot{r}_x$ containing also the quadratic terms

$$\dot{r}_x = \beta_1 \dot{\theta} + x_{Gu} \dot{\theta} \theta + \beta_3 \left(v_k - v_z \right) + \theta \left(\dot{Z}_k - \dot{Z} \right) + \dot{\theta} \left(Z_k - Z \right) , \tag{30.45}$$

where

$$\beta_1 = z_{Gu} + z_{us} \left(x_{Gu} - x_u \right) + x_u s_x , \beta_3 = \theta_0 + s_x \tag{30.46}$$

and $v_k = \dot{Z}_k$ is the velocity of the kth unsprung mass, needs to be written explicitly.

The following simplified expression is so obtained

$$\begin{aligned}\mathcal{T}_t = \tfrac{1}{2}m_u \{ & v_x^2 + v_y^2 + v_k^2 + \dot\phi_k^2 z_{Gu}^2 + \left(z_{Gu}^2 + x_{Gu}^2\right)\dot\theta^2 + x_{Gu}^2\dot\psi^2 + \\ & -2\dot Z_k \dot\theta x_{Gu} + 2v_x \left[\left(\dot Z_k - \dot Z\right)\beta_3 + \dot\theta\beta_1 + \dot\theta\left(Z_u - Z\right)\right. \\ & \left. +\theta\left(\dot Z_k - \dot Z\right) + x_{Gu}\theta\dot\theta + \dot\psi\phi_k z_{Gu}\right] + \\ & +2v_y\left[-\dot\phi_k z_{Gu} + \dot\psi x_{Gu}\right] + 2\dot\psi\dot\phi_k z_{Gu} x_{Gu}\} .\end{aligned} \tag{30.47}$$

Angular velocity of the wheels

If the axle did not rotate with respect to the body about its longitudinal axis, its absolute angular velocity about its longitudinal axis, as expressed in its own reference frame, is

$$\Omega_k = \mathbf{R}_{us}^T \Omega = \left\{ \begin{array}{c} \Omega_x + z_{us}\Omega_z \\ \Omega_y \\ \Omega_z - z_{us}\Omega_x \end{array} \right\} . \tag{30.48}$$

In reality, the unsprung mass is free to rotate about that axis and its angular velocity is

$$\Omega_k = \left\{ \begin{array}{c} \dot\phi_k \\ \Omega_y \\ \Omega_z - z_{us}\Omega_x \end{array} \right\} . \tag{30.49}$$

The rotation and steering motion of each wheel are taken into account independently. Let the rotation angles of the wheels be χ_R and χ_L and the steering angles be δ_R and δ_L (R and L designate the right and left wheel of the axle).

If the wheel's rotation axis coincided with axis y_u of the unsprung mass, and both were parallel to the y axis of the axle, the angular velocity of the ith wheel in the reference frame of the kth unsprung mass would be

$$\Omega_{wi} = \Omega_k + \dot\chi_i \mathbf{e}_2 \quad (i = L, R). \tag{30.50}$$

Generally speaking, the direction of the rotation axis of the wheel may be different (although usually not by much except for steering) from that of the y axis, making it possible to define the unit vector of the rotation axis $\mathbf{e}_{wi}$ in the reference frame of the unsprung mass. Such a unit vector does not depend upon the position of the suspension and thus is a function neither of ζ nor of ϕ_k. It follows that

$$\Omega_{wi} = \Omega_k + \dot\chi_i \mathbf{e}_{wi} \quad (i = L, R). \tag{30.51}$$

The position of the rotation axis may be defined by introducing a rotation matrix $\mathbf{R}_{wi}$, allowing us to pass from the reference frame of the unsprung mass to a frame whose y axis coincides with the rotation axis of the wheel

$$\mathbf{e}_{wi} = \mathbf{R}_{wi}\mathbf{e}_2 . \tag{30.52}$$

If x_w, y_w are z_w the components of unit vector $\mathbf{e}_{wi}$,[3] the value of the rotation matrix $\mathbf{R}_w$ is

$$\mathbf{R}_{wi} = \frac{1}{\sqrt{x_w^2 + y_w^2}} \begin{bmatrix} y_w & x_w\sqrt{x_w^2 + y_w^2} & -x_w z_w \\ -x_w & y_w\sqrt{x_w^2 + y_w^2} & -y_w z_w \\ 0 & z_w\sqrt{x_w^2 + y_w^2} & x_w^2 + y_w^2 \end{bmatrix} . \tag{30.53}$$

The rotation axis of the wheel is usually little inclined with respect to the horizontal direction. The trigonometric functions of the rotation axis included in matrix $\mathbf{R}_{wi}$ may be linearized. It follows thus

$$\mathbf{R}_{wi} = \begin{bmatrix} 1 & x_w & 0 \\ -x_w & 1 & -z_w \\ 0 & z_w & 1 \end{bmatrix} , \tag{30.54}$$

where x_w coincides with the steering angle of the wheel (when the axle does not steer) with its sign changed (this angle is usually due to toe in and is very small), while z_w coincides with the camber angle of the wheel and is also small. For symmetry reasons, it follows that

$$x_{w_R} = -x_{w_L} \; , \; z_{w_R} = -z_{w_L} \; . \tag{30.55}$$

The angular velocity of the wheel in its reference frame, instead of the frame of the unsprung mass, is

$$\mathbf{\Omega}_{wi} = \mathbf{R}_{wi}{}^T \mathbf{\Omega}_k + \dot{\chi}\mathbf{e}_2 \; . \tag{30.56}$$

If the wheel steers, the reference frame of the ith wheel will no longer be parallel to the frame $x_u y_u z_u$ of the unsprung mass, but will be rotated by a steering angle δ_i. Assume that the kingpin axis of the wheel is parallel to axis z_u and define a further rotation matrix

$$\mathbf{R}_{4i} = \begin{bmatrix} \cos(\delta_i) & -\sin(\delta_i) & 0 \\ \sin(\delta_i) & \cos(\delta_i) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{30.57}$$

In this case, the kingpin axis is generally not parallel to the z axis of the axle. If $\mathbf{e}_k$ is the unit vector of the kingpin axis (its components will be indicated as x_k, y_k and z_k[4]), which in solid axle suspensions may be considered as fixed, the rotation matrix $\mathbf{R}_{ki}$ allowing the reference frame of the unsprung mass to be rotated so that its z_u axis coincides with the kingpin axis of the ith wheel is

$$\mathbf{R}_{ki} = \frac{1}{\sqrt{x_w^2 + z_w^2}} \begin{bmatrix} z_w & -x_w y_w & x_w\sqrt{x_w^2 + z_w^2} \\ 0 & \left(x_w^2 + z_w^2\right) & y_w\sqrt{x_w^2 + z_w^2} \\ -x_w & -z_w y_w & z_w\sqrt{x_w^2 + z_w^2} \end{bmatrix} . \tag{30.58}$$

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[3] Obviously $\sqrt{x_w^2 + y_w^2 + z_w^2} = 1$.

[4] Obviously $\sqrt{x_k^2 + y_k^2 + z_k^2} = 1$.

Usually the longitudinal inclination angle (the pitch angle of the kingpin axis) and the transversal inclination angle (the roll angle of the kingpin axis), are quite small, and the rotation matrix $\mathbf{R}_k$ reduces to

$$\mathbf{R}_{ki} \approx \begin{bmatrix} 1 & 0 & x_k \\ 0 & 1 & y_k \\ -x_k & -y_k & 1 \end{bmatrix} , \tag{30.59}$$

where x_k and y_k coincide respectively with the longitudinal inclination angle (not larger than about 1°) and the transversal inclination angle changed in sign (usually not larger than about 10°). For symmetry reasons, it follows that

$$x_{k_R} = x_{k_L} , \; y_{k_R} = -y_{k_L} . \tag{30.60}$$

The angular velocity of the wheel in the reference frame of the sprung mass is then

$$\Omega_{wi} = \boldsymbol{\Omega}_k + \dot{\delta}_i \mathbf{R}_{ki} \mathbf{e}_3 + \dot{\chi}_i \mathbf{R}_{ki} \mathbf{R}_{4i} \mathbf{R}_{ki}^T \mathbf{R}_{wi} \mathbf{e}_2 . \tag{30.61}$$

To obtain the angular velocity of the wheel in its own reference frame, Eq. (30.61) must be premultiplied by $(\mathbf{R}_{ki}\mathbf{R}_{4i}\mathbf{R}_{ki}^T\mathbf{R}_{wi})^T$. Remembering that

$$\mathbf{R}_{4i}\mathbf{e}_3 = \mathbf{e}_3 ,$$

it follows that

$$\Omega_{wi} = \dot{\chi}\mathbf{e}_2 + \dot{\delta}\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2\boldsymbol{\Omega}_k. \tag{30.62}$$

where

$$\boldsymbol{\alpha}_1 = \mathbf{R}_{wi}^T \mathbf{R}_{ki} \mathbf{e}_3 , \; \boldsymbol{\alpha}_2 = \mathbf{R}_{wi}^T \mathbf{R}_{ki} \mathbf{R}_{4i}^T \mathbf{R}_{ki}^T . \tag{30.63}$$

It must be remembered that in a suspension there are two matrices $\mathbf{R}_{wi}$ and $\mathbf{R}_{ki}$ ($i = L, R$), one for each wheel.

Rotational kinetic energy

Because the wheel is a gyroscopic body (two of the principal moments of inertia are equal to each other), with a principal axis of inertia coinciding with the rotation axis, its inertia tensor has a peculiar form

$$\mathbf{J}_w = \text{diag}\left(\left[\, J_{tw} \;\; J_{pw} \;\; J_{tw} \,\right]\right) , \tag{30.64}$$

where J_{pw} and J_{tw} are respectively the polar and transversal moment of inertia of the wheel.

The rotational kinetic energy of the ith wheel is

$$\begin{aligned}\mathcal{T}_{wri} = \tfrac{1}{2}\boldsymbol{\Omega}_k^T\boldsymbol{\alpha}_2^T\mathbf{J}_w\boldsymbol{\alpha}_2\boldsymbol{\Omega}_k + \tfrac{1}{2}\dot{\chi}^2\mathbf{e}_2^T\mathbf{J}_w\mathbf{e}_2 + \tfrac{1}{2}\dot{\delta}^2{}_1^T\mathbf{J}_w\boldsymbol{\alpha}_1 + \\ +\dot{\chi}\dot{\delta}\mathbf{e}_2^T\mathbf{J}_w\boldsymbol{\alpha}_1 + \dot{\chi}\mathbf{e}_2^T\mathbf{J}_w\boldsymbol{\alpha}_2\boldsymbol{\Omega}_k + \dot{\delta}\boldsymbol{\alpha}_1^T\mathbf{J}_w\boldsymbol{\alpha}_2\boldsymbol{\Omega}_k \ .\end{aligned} \tag{30.65}$$

The first term is the rotational kinetic energy due to the angular velocity of the unsprung mass.

By neglecting the first term, which will later be included in the kinetic energy of the axle, and by linearizing and introducing the linearized expressions of the kinematic equations, the rotational kinetic energy of the ith wheel reduces to

$$\mathcal{T}_{wri} = \frac{1}{2}\dot{\chi}_i^2 J_{pw} + \frac{1}{2}\dot{\delta}_i^2 J_{tpw} + \dot{\chi}_i\dot{\delta}_i J_{pw}\left(y_k + z_w\right) + \tag{30.66}$$

$$+\dot{\chi}_i J_{pw}\left[\dot{\theta} + \phi\dot{\psi} + \left(x_w - \delta_i\right)\dot{\phi}_k + z_w\dot{\psi}\right] + \dot{\delta} J_{tw}\dot{\psi} \ . \tag{30.67}$$

The first term that was neglected above can be inserted into the rotational kinetic energy of the unsprung mass $\mathcal{T}_{ur}$:

$$\mathcal{T}_{ur} = \tfrac{1}{2}\boldsymbol{\Omega}_k^T\mathbf{J}_u\boldsymbol{\Omega}_k, \tag{30.68}$$

if the inertia tensor $\mathbf{J}_u$ also includes the inertia of the wheels, assumed to be non-rotating and non-steering.

Operating in this way, the variation of the inertia of the unsprung mass at the changing steering angle is neglected, but this approximation is acceptable. The inertia tensor of the unsprung mass has a structure similar to that of the sprung mass, because the suspension also has a symmetry plane coinciding with the $x_u z_u$ plane.

Performing the relevant computations, it follows that

$$\mathcal{T}_{ur} = \tfrac{1}{2}J_{x_u}\Omega_{xk}^2 + \tfrac{1}{2}J_{y_u}\Omega_{yk}^2 + \tfrac{1}{2}J_{z_u}\Omega_{zk}^2 - J_{xz_u}\Omega_{xk}\Omega_{zk} \tag{30.69}$$

and, by linearizing and including the linearized kinematic equations, the simple expression is obtained

$$\mathcal{T}_{ur} = \tfrac{1}{2}J_{x_u}^2\dot{\phi}_k + \tfrac{1}{2}J_{y_u}\dot{\theta}^2 + \tfrac{1}{2}J_{z_u}\dot{\psi}^2 + J_{xz_u}\dot{\phi}\dot{\psi} \ . \tag{30.70}$$

The total rotation kinetic energy of the axle is then

$$\mathcal{T}_{urt} = \mathcal{T}_{ur} + \mathcal{T}_{wrR} + \mathcal{T}_{wrL} \ . \tag{30.71}$$

Total kinetic energy

The kinetic energy of the axle is then

$$\mathcal{T}_{wri} = \dot{\chi}_i J_{pw}\left[\dot{\theta} + \phi\dot{\psi} + z_w\dot{\psi}\right] \ , \tag{30.72}$$

$$\begin{array}{c}\mathcal{T}_u = \frac{1}{2}m_u\left(v_x^2+v_y^2+v_k^2\right)+\frac{1}{2}\beta_{11}\dot{\theta}^2+\frac{1}{2}\beta_{12}\dot{\psi}^2+\frac{1}{2}\beta_{13}\dot{\phi}_k+\\ +\beta_{14}\dot{\psi}\dot{\phi}_k+\frac{1}{2}\dot{\chi}_s^2 J_{pw}+\frac{1}{2}\dot{\delta}_s^2 J_{tw}+\frac{1}{2}\dot{\chi}_R^2 J_{pw}+\frac{1}{2}\dot{\delta}_R^2 J_{tw}+\\ +\dot{\chi}_s\dot{\delta}_s J_{pw}\left(y_k+z_w\right)-\dot{\chi}_R\dot{\delta}_R J_{pw}\left(y_k+z_w\right)-\beta_{16}v_k\dot{\theta}\\ +m_u v_x\left(\dot{\theta}\beta_1+\dot{\theta}Z_k-\dot{\theta}Z+\theta v_k-\theta v_z+\beta_3 v_k-\beta_3 v_z+\right.\\ \left.+x_{Gu}\theta\dot{\theta}+\beta_5\dot{\psi}\phi_k\right)+m_u v_y\left(-\beta_5\dot{\phi}_k+x_{Gu}\dot{\psi}\right)+\\ +\beta_{19}\dot{\delta}_L\dot{\psi}+\beta_{19}\dot{\delta}_R\dot{\psi}+\dot{\chi}_L\beta_{20}\left(x_w-\delta_s\right)\dot{\phi}_k-\dot{\chi}_R\beta_{20}\left(x_w+\delta_R\right)\dot{\phi}_k+\\ +\dot{\chi}_L J_{pr}\left(\dot{\theta}+\phi\dot{\psi}+z_w\dot{\psi}\right)+\dot{\chi}_R J_{pr}\left(\dot{\theta}+\phi\dot{\psi}-z_w\dot{\psi}\right)\ ,\end{array} \tag{30.73}$$

where

$$\begin{array}{c}\beta_5 = z_{Gu}\ ,\ \beta_{11} = m_u\left(z_{Gu}^2+x_{Gu}^2\right)+J_{y_u}\ ,\ \beta_{12} = m_u x_{Gu}^2+J_{z_u},\\ \beta_{13} = m_u z_{Gu}^2+J_{x_u}\ ,\ \beta_{14} = m_u z_{Gu}x_{Gu}-J_{xz_u}\ ,\\ \beta_{16} = m_u x_{Gu}\ ,\ \beta_{19} = J_{tw}\ ,\ \beta_{20} = J_{pw}.\end{array}$$

Potential energy

The height of the center of mass of the axle on the ground is

$$h_u = \mathbf{e}_3^T\left(\overline{\mathrm{G_N - O'}}\right) = \mathbf{e}_3^T\left(\overline{\mathrm{H - O'}}\right)+\mathbf{e}_3^T\mathbf{R}_1\mathbf{r}\ , \tag{30.74}$$

i.e.,

$$h_u = Z_0 + Z + r_z\ , \tag{30.75}$$

The expression of r_z, obtained by approximating vector $\mathbf{r}$ with its Taylor series truncated after the quadratic term in the small quantities and cancelling the constant terms that do not influence the equations of motion, is

$$r_z = Z_k - Z - \theta\beta_{22} - \frac{1}{2}\theta^2 z_{Gu} - \frac{1}{2}\phi_k^2 z_{Gu}, \tag{30.76}$$

where

$$\beta_{22} = z_{Gu}\left(\theta_0 - z_{us}\right)\ .$$

The gravitational potential energy

$$\mathcal{U}_g = m_u g h_u \tag{30.77}$$

is then

$$\mathcal{U}_g = m_u g\left(Z_k - \theta\beta_{22} - \frac{1}{2}\theta^2 z_{Gu} - \frac{1}{2}\phi_k^2 z_{Gu}\right)\ . \tag{30.78}$$

On each of the springs of the suspension it is possible to identify two points: one of these (A) is fixed to the body, while the other (B) is fixed to the axle. Considering $\mathbf{r}_A$ and $\mathbf{r}_B$ as the vectors defining their positions in the frame of the sprung mass ($\mathbf{r}_A$ is constant, while $\mathbf{r}_B$ depends on ζ and ϕ_k), it is possible to compute the shortening of the spring and then its elastic potential energy. In a similar way, it is possible to compute the potential energy of possible anti-roll bars applied to the axle.

In a linearized model of the suspension, the spring system linking the two rigid bodies can be reduced to a spring with stiffness K_ζ reacting to linear displacements, and a torsional spring with stiffness K_ϕ reacting to the relative rotation between sprung and unsprung masses $\phi - \phi_k$.

The general expression of the elastic potential energy of the whole suspension is

$$\mathcal{U}_m = \frac{1}{2}K_\zeta\left(\zeta + \zeta_0\right)^2 + \frac{1}{2}K_\phi\left(\phi - \phi_k\right)^2 . \tag{30.79}$$

Note that for symmetry reasons the stiffness for heave and roll motions are fully independent from each other.

By introducing the expression for ζ as a function of the generalized coordinates, it follows that

$$\begin{aligned} \mathcal{U}_m = {} & \tfrac{1}{2}K_{11}\left(Z + Z_0\right)^2 + \tfrac{1}{2}K_{22}\left(Z_k + Z_{k0}\right)^2 + \tfrac{1}{2}K_{33}\left(\theta + \theta_0\right)^2 + \\ & -K_{12}\left(Z + Z_0\right)\left(Z_k + Z_{k0}\right) - K_{13}\left(Z + Z_0\right)\left(\theta + \theta_0\right) + \\ & +K_{23}\left(Z_k + Z_{k0}\right)\left(\theta + \theta_0\right) + \tfrac{1}{2}K_\phi\phi^2 + \tfrac{1}{2}K_\phi\phi_k^2 - K_\phi\phi\phi_k , \end{aligned} \tag{30.80}$$

where constants K_{ij} depend on both the elastic and geometric characteristics of the suspension.

This expression of the potential energy, along with the expression of the kinetic energy seen above, takes implicitly into account the actual trajectory of the suspension, or better, because the model is linearized, the tangent to the trajectory in the reference position. On the other hand, using a model of this kind makes it impossible to account for the deformation in longitudinal and lateral directions, and for the yaw and pitch compliance of the suspension.

In a similar way, the general expression of the elastic potential energy due to the deformation of the tires of the suspension is

$$\mathcal{U}_p = \frac{1}{2}K_{pz}\left(Z_k + Z_{k0}\right)^2 + \frac{1}{2}K_{p\phi}\phi_k^2 . \tag{30.81}$$

If K_p is the stiffness of a single tire and t is the track of the axle, for an axle with two wheels it follows that

$$K_{pz} = 2K_p \, , \; K_{p\phi} = \frac{1}{2}t^2 K_p \, . \tag{30.82}$$

Dissipation function

Operating with the same method used for the elastic potential energy of the suspension, the dissipation function due to the shock absorbers is

$$\mathcal{F}_a = \frac{1}{2}c_{11}v_z^2 + \frac{1}{2}c_{22}v_k^2 + \frac{1}{2}c_{33}\dot{\theta}^2 + \frac{1}{2}c_\phi\dot{\phi}^2 + \frac{1}{2}c_\phi\dot{\phi}_k^2 + \\ -c_{12}v_zv_k - c_{13}v_z\dot{\theta} + c_{23}v_k\dot{\theta} - c_\phi\dot{\phi}\dot{\phi}_k, \tag{30.83}$$

where constants c_{ij} depend both on the characteristics of the dampers and on the geometry of the suspension.

The dissipation function due to the damping of tires can be computed in the same way:

$$\mathcal{F}_p = \frac{1}{2}c_{pz}v_k^2 + \frac{1}{2}c_{p\phi}\Omega_k^2 \,. \tag{30.84}$$

30.2.4 General Independent Suspension

Geometry of the suspension

An axle with independent suspensions will be assumed to be made with two suspensionsthat are the mirror image of each other. Some parameters are identical for the two suspensions (for instance the mass m_i, some geometrical characteristics, some angles, etc.); others will be identical in modulus but with opposite sign (for instance, the product of inertia J_{xy}, some angles, etc.). In the latter case, reference will be made to the left suspension (subscript $i = L$), while the characteristics of the right suspension (subscript $i = R$) will have opposite sign.

The simplest case, although only an ideal one, is a suspension in which the unsprung masses can only move along a straight line in the direction of the z axis of the sprung mass. The sprung mass is the main body, while the single suspension (wheel, hub and all parts attached to it) is the secondary body (Fig. 30.5). The suspension is constrained to the main body by a prismatic guide whose axis is parallel to the z axis. The reference point C is on the axis of the guide and the distance ζ

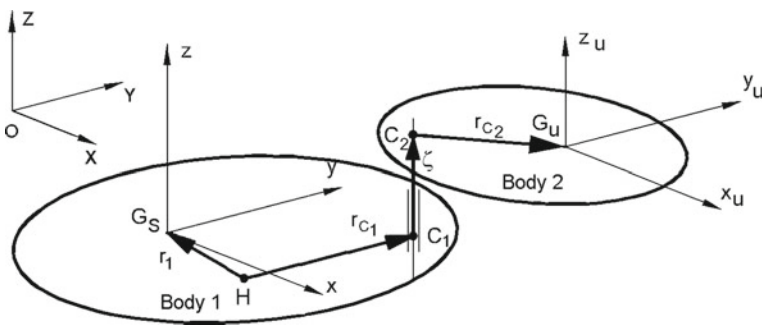

Fig. 30.5 Sketch of an idealized independent suspension

between the position C_1 of C belonging to the sprung mass and C_2 belonging to the unsprung mass is taken as a generalized coordinate. Obviously in the reference position with $\zeta = 0$, C_1 coincides with C_2. Moreover, assume that the directions of the axes of frame $x_u y_u z_u$ coincide with those of the axes of frame xyz.

Consider as reference point for translational coordinates the same point H already used to compute the kinetic energy of the sprung mass. The position of the center of mass of the suspension G_u in the inertial frame is

$$\overline{(G_u - O')} = \overline{(H - O')} + \mathbf{R}\,(\mathbf{r}_{C1} + \mathbf{r}_{Gu} + \zeta\mathbf{e}_3) = \overline{(H - O')} + \mathbf{R}\mathbf{r}_2\,, \tag{30.85}$$

where $\mathbf{R}$ is the rotation matrix defining the position of the xyz frame with respect to the inertial frame $X_i Y_i Z_i$, $\mathbf{e}_3$ is the unit vector of the z axis and

$$\mathbf{r}_2 = \mathbf{r}_{C1} + \mathbf{r}_{C2} + \zeta\mathbf{e}_3\,. \tag{30.86}$$

If the steering of the wheel is accounted for, a part of the unsprung mass may rotate about the kingpin axis, which in the simplified model may be assumed to be parallel to z_u, and thus to the z, axis. The wheel also rotates about its own axis, which may be assumed to be parallel to the y axis when the steering angle is zero.

However, this model of independent suspension is too simple. No modern car has suspensions made by prismatic guides parallel to the z axis of the unsprung mass, nor is the kingpin axis parallel to the same axis, while the rotation of the wheels does not occur about an axis parallel to the y axis.

Each suspension has its own specific kinematics (as an example, an SLA suspension is shown in Fig. 30.6), or better, its own elasto-kinematics, because the various elements of the suspensions are rigid bodies only as a first approximation. However, while the exact elasto-kinematics is important in assessing the position of the wheel with respect to the ground and thus the forces they exchange, its effects on the inertia reactions of the various components of the suspension are usually very limited. It is then possible to neglect the deformation of the various links in the computation of the inertial part of the equations of motion and to introduce them later in the computation of the forces due to the tires.

If the deformation of the linkages is neglected, it is possible to define the trajectory of all points of the suspension in a reference frame fixed to the sprung mass. The trajectory of the center of mass, for instance, may be expressed by a function

$$\mathbf{r}_2 = \mathbf{r}_2(\zeta)\,, \tag{30.87}$$

where ζ is a generalized coordinate that defines the position of the unsprung mass, with reference to a given position. In the extremely simplified case seen above, coordinate ζ is nothing other than the displacement of the unsprung mass in the z direction, and function $\mathbf{r}_2(\zeta)$ is the linear function expressed by Eq. (30.86).

The function expressed by Eq.(30.87) can be developed in McLaurin series about the reference position and the terms of an order higher than the first may be neglected. The position of the center of mass G_u of the suspension with respect to point H is

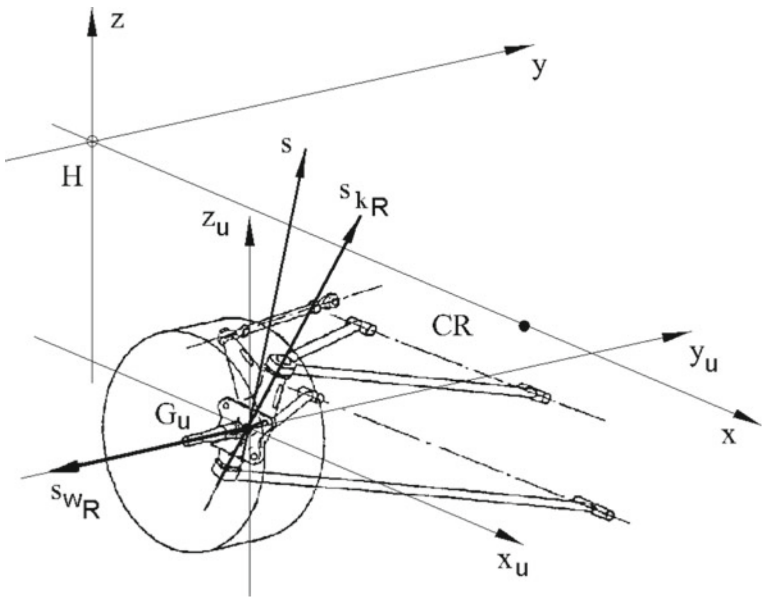

Fig. 30.6 Sketch of an SLA suspension

$$\overline{(G_u\!-\!H)} = \mathbf{r}_2 = \mathbf{r}_{2_0} + \left(\frac{d\mathbf{r}_2}{d\zeta}\right)_{\zeta=\zeta_0} \zeta \,. \tag{30.88}$$

Equation (30.88) coincides with Eq. (30.86) if vector

$$\mathbf{s}_0 = \left(\frac{d\mathbf{r}_2}{d\zeta}\right)_{\zeta=\zeta_0}$$

is substituted for $\mathbf{e}_3$, the unit vector of the z axis. The components of vector $\mathbf{r}_{2_0}$ will be indicated as x_{Gu}, y_{Gu} and z_{Gu}.

As an example, in the case of a trailing arms suspension hinged about an axis parallel to the y axis (Fig. 30.7) and with point C on the hinge axis in the oscillation plane of the center of mass, ζ_0 is the angle line CG_u makes with the x axis in the reference position, and coordinate ζ' is the angle the suspension rotates with respect to that position. The position of the center of mass may be thus defined by the function

$$\mathbf{r}_2 = \begin{Bmatrix} x_0 + d\cos\left(\zeta' + \zeta_0\right) \\ y_0 \\ z_0 - d\sin\left(\zeta' + \zeta_0\right) \end{Bmatrix} . \tag{30.89}$$

If ζ' is small, the truncated series yielding the position of G_u is

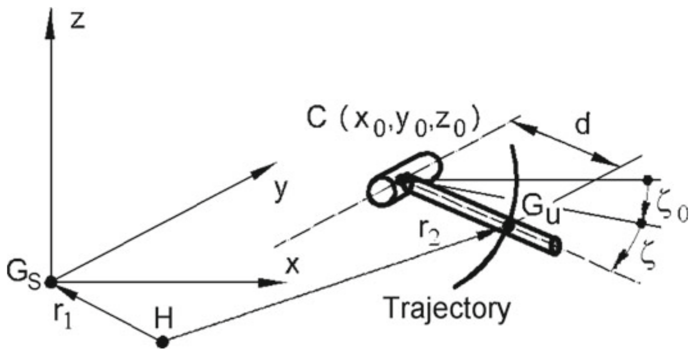

Fig. 30.7 Sketch of a trailing arm suspension

$$\mathbf{r}_2 = \begin{Bmatrix} x_0 + d\cos(\zeta_0) \\ y_0 \\ z_0 - d\sin(\zeta_0) \end{Bmatrix} + \begin{Bmatrix} -\sin(\zeta_0) \\ 0 \\ -\cos(\zeta_0) \end{Bmatrix} d\zeta' \, . \tag{30.90}$$

Or, to give coordinate ζ the meaning of a displacement in the z direction of the unsprung mass, it is possible to state

$$\zeta = -\cos(\zeta_0)\, d\zeta' \, , \tag{30.91}$$

and then

$$\mathbf{r}_2 = \begin{Bmatrix} x_0 + d\cos(\zeta_0) \\ y_0 \\ z_0 - d\sin(\zeta_0) \end{Bmatrix} + \mathbf{s}_0\zeta \, , \tag{30.92}$$

where

$$\mathbf{s}_0 = \left[\tan(\zeta_0)\ 0\ 1\right]^T \, . \tag{30.93}$$

The generalized coordinate for the ith unsprung mass (right or left) may be the height on the ground of its center of mass instead of ζ. Such height is simply

$$c = \mathbf{e}_3^T \left[(\overline{\mathrm{H} - \mathrm{O'}}) + \mathbf{R}\mathbf{r}_2\right] \ (i = L, R) \, . \tag{30.94}$$

Also, $Z_{\mathrm{G}u_i}$ may be considered as the sum of a value taken at the reference position plus a displacement, one that is of the same order as the other small quantities (like Z):

$$Z_{\mathrm{G}u_i} = Z_{\mathrm{G}u_0} + Z_{u_i} = Z_0 + Z + \mathbf{e}_3^T\mathbf{R} \begin{Bmatrix} x_{Gu} + \zeta s_x \\ y_{Gu} + \zeta s_y \\ z_{Gu} + \zeta s_z \end{Bmatrix} \, . \tag{30.95}$$

Note that, owing to symmetry, $Z_{\mathrm{G}u_0}$, x_{Gu}, z_{Gu}, s_x, and s_z are equal for the two suspensions of the same axle, while y_{Gu} and s_y have opposite signs. In the following, as already stated, the signs of the left suspension (that with positive y_{Gu}) will be taken as a reference.

In the reference position it follows that

$$Z_{0_u} = Z_0 - x_{Gu}\theta_0 + z_{Gu} \ . \tag{30.96}$$

By performing the computations, linearizing the trigonometric functions of the small angles and assuming that $s_z = 1$ and that s_x and s_y are small, it follows that

$$Z_{u_i} = Z - x_{Gu}\theta + y_{Gu}\phi + \zeta \tag{30.97}$$

and thus the relationship linking ζ to Z_u is simply

$$\zeta = Z_{u_i} - Z + x_{Gu}\theta - y_{Gu}\phi \ . \tag{30.98}$$

Rotation of the wheels

Assume that the rotation axis of the wheels is fixed to the unsprung mass. Let χ be the angle of rotation of the wheel and $\dot{\chi}$ its angular velocity in a frame fixed to the suspension. The signs of angular velocities have been defined in such a way that, when $\dot{\chi}$ is positive, the wheel rotates in a direction that is consistent with a positive velocity v_x of the vehicle.

If the direction of the rotation axis of the wheel coincides with that of axis y_u of the unsprung mass, and then is parallel to the y axis of the vehicle (whose unit vector is $\mathbf{e}_2$), the absolute angular velocity of the wheel is, in the reference frame of the sprung mass,

$$\Omega_w = \Omega + \dot{\chi}\mathbf{e}_2 = \begin{Bmatrix} \Omega_x \\ \Omega_y + \dot{\chi} \\ \Omega_z \end{Bmatrix} \ . \tag{30.99}$$

However, the rotation axis of the wheel usually has a direction that may be different (only slightly, when the wheel is not steered) from the y axis, so that it is possible to define the unit vector of the rotation axis $\mathbf{e}_w$ in the reference frame of the sprung mass. Obviously such a vector depends on the position of the suspension and is then a function of ζ:

$$\mathbf{e}_w = \mathbf{e}_w(\zeta) \ . \tag{30.100}$$

As an alternative, the position of the rotation axis can be defined by stating a yaw angle ψ_w (steering due to the motion of the suspension) and a roll angle ϕ_w of the rotation axis of the wheel. A rotation matrix $\mathbf{R}_w$ is then written to pass from the reference frame of the sprung mass to a frame whose y axis coincides with the rotation axis of the wheel. Since the two ways of describing the position of the rotation axis must yield the same results, it follows that

$$\mathbf{e}_w = \mathbf{R}_w \mathbf{e}_2 \ . \tag{30.101}$$

With x_w, y_w and z_w being the components[5] of unit vector $\mathbf{e}_w$, the rotation matrix $\mathbf{R}_w$ is still expressed by Eq. (30.53). Because the rotation axis of the wheel is usually not far from horizontal, the trigonometric functions of ψ_w and ϕ_w can be linearized. Eq. (30.54), repeated here, still holds

$$\mathbf{R}_w = \begin{bmatrix} 1 & x_w & 0 \\ -x_w & 1 & -z_w \\ 0 & z_w & 1 \end{bmatrix} .$$

The angular velocity of the wheel in the reference frame of the sprung mass is

$$\boldsymbol{\Omega}_w = \boldsymbol{\Omega} + \dot{\chi} \mathbf{e}_w (\zeta) \ . \tag{30.102}$$

Actually, it is expedient to write the components of the angular velocity of the wheel in the frame fixed to the wheel instead of that fixed to the sprung mass.

The angular velocity of the wheel in its own reference frame is

$$\boldsymbol{\Omega}_w = \mathbf{R}_w{}^T \boldsymbol{\Omega} + \dot{\chi} \mathbf{e}_2 \ . \tag{30.103}$$

Steering

If the wheel steers, the reference frame of the wheel will no longer be parallel to frame $x_u y_u z_u$ of the unsprung mass, but will be rotated by a steering angle δ. Here two different approaches are possible: δ may be one of the variables of motion (free controls approach), or a constant or a known variable (locked controls approach).

Assume that the kingpin axis of the wheel is parallel to the z axis of the unsprung mass and define a further rotation matrix

$$\mathbf{R}_4 = \begin{bmatrix} \cos(\delta) & -\sin(\delta) & 0 \\ \sin(\delta) & \cos(\delta) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{30.104}$$

Assuming that the direction of the rotation axis does not change with the heave motion and is parallel to the z axis, the rotation velocity of the wheel, referred to its own reference frame, is

$$\Omega_w = \dot{\chi} \mathbf{e}_2 + \dot{\delta} \mathbf{e}_3 + \mathbf{R}_4^T \boldsymbol{\Omega} \ . \tag{30.105}$$

[5] Obviously, they are functions of ζ and $\sqrt{x_w^2 + y_w^2 + z_w^2} = 1$.

Actually, the kingpin axis moves with changing ζ and in general is not parallel to the z axis of the unsprung mass, but its direction is defined by the unit vector $\mathbf{e}_k$, which is a function of ζ:

$$\mathbf{e}_k = \mathbf{e}_k(\zeta) \ . \tag{30.106}$$

The rotation matrix $\mathbf{R}_k$ to rotate the reference frame of the unsprung mass so that its z axis coincides with the kingpin axis

$$\mathbf{e}_k = \mathbf{R}_k \mathbf{e}_3 \ , \tag{30.107}$$

can also be written.

By defining a pitch angle θ_k of the kingpin axis (coinciding with the longitudinal inclination angle) and a roll angle ϕ_k (coinciding with the transversal inclination angle), and by indicating with x_k, y_k and z_k the components[6] of the unit vector $\mathbf{e}_k$, the rotation matrix $\mathbf{R}_k$ is still expressed by Eq. (30.58). If θ_k and ϕ_k are small angles, matrix $\mathbf{R}_k$ reduces to:

$$\mathbf{R}_k \approx \begin{bmatrix} 1 & 0 & x_k \\ 0 & 1 & y_k \\ -x_k & -y_k & 1 \end{bmatrix} . \tag{30.108}$$

The velocity of the wheel in the reference frame of the sprung mass is then

$$\Omega_w = \mathbf{\Omega} + \dot{\delta}\mathbf{e}_k + \dot{\chi}\mathbf{R}_k\mathbf{R}_4\mathbf{R}_k^T\mathbf{e}_w \ . \tag{30.109}$$

To write it in the principal reference frame of the wheel, the expression of the angular velocity must be multiplied by $\left(\mathbf{R}_k\mathbf{R}_4\mathbf{R}_k^T\mathbf{R}_w\right)^T$:

$$\Omega_{wi} = \dot{\chi}\mathbf{e}_2 + \dot{\delta}\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2\mathbf{\Omega}, \tag{30.110}$$

where

$$\boldsymbol{\alpha}_1 = \mathbf{R}_{wi}^T\mathbf{R}_{ki}\mathbf{R}_4^T\mathbf{e}_3 \ , \ \boldsymbol{\alpha}_2 = \mathbf{R}_{wi}^T\mathbf{R}_{ki}\mathbf{R}_{4i}^T\mathbf{R}_{ki}^T \ . \tag{30.111}$$

The steering angle so defined does not coincide exactly with the steering angle defined in the preceding chapters, because it also has components along axes x and y.

Translational kinetic energy

The position of the center of mass of the unsprung mass in the inertial reference frame is

$$(\overline{\mathrm{G}_u - \mathrm{O'}}) = (\overline{\mathrm{H} - \mathrm{O'}}) + \mathbf{R}_1\mathbf{r} \ , \tag{30.112}$$

where

[6] Obviously, they are functions of ζ and $\sqrt{x_k^2 + y_k^2 + z_k^2} = 1$.

$$\mathbf{r} = \mathbf{R}_2\mathbf{R}_3 \left\{ \begin{array}{c} x_{Gu} + \zeta s_x \\ y_{Gu} + \zeta s_y \\ z_{Gu} + \zeta \end{array} \right\} . \tag{30.113}$$

By linearizing the expression for $\mathbf{r}$ and substituting its value for ζ, it follows that

$$\mathbf{r} = \left\{ \begin{array}{c} x_{Gu} + \theta_0 z_{Gu} + \theta z_{Gu} \\ y_{Gu} - \phi z_{Gu} \\ z_{Gu} - \theta_0 x_{Gu} + Z_u - Z \end{array} \right\} . \tag{30.114}$$

Its derivative, again approximated to the first-order term in the small quantities, is

$$\dot{\mathbf{r}} = \left\{ \begin{array}{c} \dot{\theta} z_{Gu} \\ -\dot{\phi} z_{Gu} \\ \dot{Z}_u - \dot{Z} \end{array} \right\} . \tag{30.115}$$

The speed of the center of mass of the unsprung mass, written in the inertial frame, is still expressed by Eq. (30.40) while the expression of the translational kinetic energy of the unsprung mass is identical to that seen for the rigid axle suspension (Eq. (30.44), where r_x, r_y, r_z are the components of vector $\mathbf{r}$). The equation is repeated here:

$$\begin{array}{c} \mathcal{T}_t = \frac{1}{2} m_{ui} \left\{ v_x^2 + v_y^2 + v_z^2 + \dot{r}_x^2 + \dot{r}_y^2 + \dot{r}_z^2 + \dot{\psi}^2 \left(r_x^2 + r_y^2 \right) + \right. \\ \left. +2 \left(v_x \dot{r}_x + v_y \dot{r}_y + v_z \dot{r}_z \right) + 2\dot{\psi} \left(-v_x r_y + v_y r_x - \dot{r}_x r_y + \dot{r}_y r_x \right) \right\} . \end{array}$$

To obtain an expression containing all terms up to the quadratic in small quantities, the linearized expression of the components of $\mathbf{r}$ and their derivatives may be used, except for the term in $v_x \dot{r}_x$, where the quadratic terms must also be used

$$\dot{r}_x = \beta_1 \dot{\theta} + x_{Gu} \dot{\theta} \theta + \beta_3 (v_u - v_z) + \theta (v_u - v_z) + \dot{\theta} (Z_u - Z) - y_{Gu} s_x \dot{\phi} \tag{30.116}$$

where

$$\beta_1 = x_{Gu} s_x + z_{Gu} \, , \; \beta_3 = \theta_0 + s_x \, . \tag{30.117}$$

By linearizing the kinematic equations (30.5) and indicating with v_{ui} the derivative of Z_{ui}, the following expression of the translational kinetic energy of a single independent suspension is obtained:

$$\begin{array}{c} \mathcal{T}_{ti} = \frac{1}{2} m_{ui} \left(v_x^2 + v_y^2 + v_{ui}^2 + \dot{\phi}^2 z_{Gu}^2 + \dot{\theta}^2 z_{Gu}^2 \right) + \\ + m_{ui} v_x \left[\beta_1 \dot{\theta} + x_{Gu} \dot{\theta} \theta + \beta_3 (v_{ui} - v_z) + \right. \\ \left. + \theta (v_{ui} - v_z) + \dot{\theta} (Z_{ui} - Z) - y_{Gu} s_x \dot{\phi} - \dot{\psi} y_{Gu} + z_{Gu} \phi \dot{\psi} \right] + \\ + m_{ui} v_y \left(\dot{\psi} x_{Gu} - \dot{\phi} z_{Gu} \right) - m_{ui} \dot{\psi} \dot{\theta} z_{Gu} y_{Gu} + m_{ui} \dot{\psi} \dot{\phi} z_{Gu} x_{Gu} \, . \end{array} \tag{30.118}$$

Note that operating in this way the translational kinetic energy linked with steering has been neglected. This would be correct if the center of mass of the steering part of the suspension lies on the kingpin axis; however, the small error so introduced may be at least partially compensated for by introducing the moment of inertia of the steering parts about the kingpin axis instead of about a baricentric axis.

Rotational kinetic energy

The rotational kinetic energy of the wheel is

$$\begin{aligned}\mathcal{T}_{wr} = &\tfrac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{\alpha}_2^T\mathbf{J}_w\boldsymbol{\alpha}_2\boldsymbol{\Omega}+\tfrac{1}{2}\dot{\delta}^2\boldsymbol{\alpha}_1^T\mathbf{J}_w\boldsymbol{\alpha}_1+\dot{\delta}\boldsymbol{\alpha}_1^T\mathbf{J}_w\boldsymbol{\alpha}_2\boldsymbol{\Omega}+\\ &+\dot{\chi}\dot{\delta}\mathbf{e}_2^T\mathbf{J}_w\boldsymbol{\alpha}_1+\tfrac{1}{2}\dot{\chi}^2\mathbf{e}_2^T\mathbf{J}_w\mathbf{e}_2+\dot{\chi}\mathbf{e}_2^T\mathbf{J}_w\boldsymbol{\alpha}_2\boldsymbol{\Omega}\,.\end{aligned} \tag{30.119}$$

Because the wheel is a gyroscopic body (two of its principal moments of inertia are equal to each other), with one of its principal axes of inertia coinciding with the rotation axis, its inertia matrix is diagonal and has a particular form

$$\mathbf{J}_w = \text{diag}\left(\left[\begin{array}{ccc} J_{tw} & J_{pw} & J_{tw}\end{array}\right]\right)\,, \tag{30.120}$$

where J_{pw} and J_{tw} are the polar and transversal moments of inertia, respectively.

The rotational kinetic energy of the non-rotating parts of the suspensions is

$$\mathcal{T}_{nr} = \frac{1}{2}\dot{\delta}^2\boldsymbol{\alpha}_1^T\mathbf{J}_m\boldsymbol{\alpha}_1+\frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{\alpha}_2^T\mathbf{J}_m\boldsymbol{\alpha}_2\boldsymbol{\Omega}+\dot{\delta}\boldsymbol{\alpha}_1^T\mathbf{J}_m\boldsymbol{\alpha}_2\boldsymbol{\Omega}\,.$$

where $\mathbf{J}_m$ is the inertia tensor of the non-rotating parts of the unsprung mass.

The first three terms of Eq. (30.119) may be directly included in the expression of $\mathcal{T}_{nr}$ if the inertia of the wheels is included in tensor $\mathbf{J}_m$.

Remembering that all angular velocities except for $\dot{\chi}$ are small quantities, the expression of the kinetic energy truncated to the second-order terms is fairly simplified.

Stating that $\mathbf{J}_m$ is the inertia tensor of the unsprung mass, which in general has no symmetry property, and remembering the peculiar structure of the inertia tensor of the wheel, the rotational kinetic energy of the unsprung mass (i.e., of one of the two unsprung masses of the axle) is

$$\begin{aligned}\mathcal{T}_{ur} = &\tfrac{1}{2}J_{pw}\dot{\chi}^2+\tfrac{1}{2}\dot{\delta}^2 J_{mz}+\tfrac{1}{2}\dot{\phi}^2 J_{mx}+\tfrac{1}{2}\dot{\theta}^2 J_{my}+\\ &+\tfrac{1}{2}\dot{\psi}^2 J_{mz}-\dot{\phi}\dot{\theta}J_{mxy}-\dot{\phi}\dot{\psi}J_{mxz}-\dot{\theta}\dot{\psi}J_{myz}+\\ &-\dot{\delta}\dot{\phi}J_{mxz}-\dot{\delta}\dot{\theta}J_{myz}+\dot{\delta}\dot{\psi}J_{mz}+\dot{\chi}\dot{\delta}J_{pw}\left(y_k+z_w\right)+\\ &+\dot{\chi}\dot{\phi}J_{pw}\left(x_w-\delta\right)+\dot{\chi}J_{pw}\left[\dot{\theta}+\dot{\psi}\left(z_w+\phi\right)\right].\end{aligned} \tag{30.121}$$

As already stated, this expression is approximated for various reasons, and also because the parts of the suspension that do not steer have been neglected.

Total kinetic energy of the axle

Because different types of suspensions may be used on the same vehicle for front and rear axles, the equations of motion are best written with reference to coordinates that may be used for both rigid axle and independent suspensions. Consider the general kth axle made of two independent suspensions and assume

$$\begin{cases} Z_k = \dfrac{Z_{uL} + Z_{uR}}{2} , \\ \phi_k = \dfrac{Z_{uL} - Z_{uR}}{d_0} , \end{cases} \tag{30.122}$$

where as usual subscripts L and R designate the left and right suspension and d_0 is an arbitrary length, for instance the distance between the centers of mass of the two suspensions. Coordinate Z_k coincides with the vertical displacement of the center of mass of the system made by the two suspensions of the axle, and ϕ_k is the roll rotation of a line passing through the two centers of mass (if d_0 is their distance). The coordinates are then the same used for rigid axles.

Taking into account the symmetry of the two suspensions, some terms in the kinetic energy are equal in modulus but have opposite signs (for instance $\beta_2 y_{Gu}\dot{\phi}$) and then cancel each other. Remembering that $m_u = 2m_{ui}$, substituting the coordinates Z_k and ϕ_k to Z_{ui}, the total kinetic energy of the system made by the two suspensions is

$$\begin{gathered} \mathcal{T}_u = \tfrac{1}{2}m_u\left(v_x^2 + v_y^2 + v_k^2\right) + \tfrac{1}{2}\beta_{10}\dot{\phi}^2 + \tfrac{1}{2}\beta_{11}\dot{\theta}^2 + \tfrac{1}{2}\beta_{12}\dot{\psi}^2 + \\ \tfrac{1}{2}\beta_{13}\dot{\phi}_k^2 + \beta_{15}\dot{\phi}\dot{\psi} + \tfrac{1}{2}J_{pw}\dot{\chi}_L^2 + \tfrac{1}{2}J_{pw}\dot{\chi}_R^2 + \tfrac{1}{2}\dot{\delta}_L^2 J_{mz} + \tfrac{1}{2}\dot{\delta}_R^2 J_{mz} + \\ +\dot{\chi}_L\dot{\delta}_L J_{pw}\left(y_k + z_w\right) - \dot{\chi}_R\dot{\delta}_R J_{pw}\left(y_k + z_w\right) + \\ +m_u v_x\left(\beta_1\dot{\theta} + \dot{\theta}Z_k - \dot{\theta}Z + \theta v_k - \theta v_z + \beta_3 v_k - \beta_3 v_z + \right. \\ \left. +\beta_4\dot{\psi}\phi + x_{Gu}\dot{\theta}\theta\right) + m_u v_y\left(x_{Gu}\dot{\psi} - \beta_4\dot{\phi}\right) - \beta_{17}\dot{\delta}_L\dot{\phi} - \beta_{17}\dot{\delta}_R\dot{\phi} + \\ -\beta_{18}\dot{\delta}_L\dot{\theta} + \beta_{18}\dot{\delta}_R\dot{\theta} + \beta_{19}\dot{\delta}_L\dot{\psi} + \beta_{19}\dot{\delta}_R\dot{\psi} + \dot{\chi}_L\dot{\phi}\beta_{21}\left(x_w - \delta_L\right) + \\ -\dot{\chi}_R\dot{\phi}\beta_{21}\left(x_w + \delta_R\right) + \dot{\chi}_L J_{pw}\left[\dot{\theta} + \dot{\psi}\left(z_w + \phi\right)\right] + \dot{\chi}_R J_{pw}\left[\dot{\theta} + \dot{\psi}\left(-z_w + \phi\right)\right] , \end{gathered} \tag{30.123}$$

where

$$\begin{gathered} \beta_4 = z_{Gu} ,\ \beta_{10} = m_u z_{Gu}^2 + 2J_{mx} ,\ \beta_{11} = m_u z_{Gu}^2 + 2J_{my} , \\ \beta_{12} = 2J_{mz} ,\ \beta_{13} = \tfrac{1}{2}m_u d_0^2 ,\ \beta_{15} = m_u z_{uG} x_{Gu} - 2J_{mxz} , \\ \beta_{17} = J_{mxz} ,\ \beta_{18} = J_{myz} ,\ \beta_{19} = J_{mz} ,\ \beta_{21} = J_{pw} . \end{gathered}$$

Potential energy

The height on the ground of the center of mass of one of the two suspensions is still expressed by Eq. (30.75):

$$h_u = Z_0 + Z + r_z .$$

The expression of r_z, obtained by approximating vector $\mathbf{r}$ with its series truncated after quadratic terms in the small quantities and eliminating the constant terms that do not affect the equations of motion, is

$$r_z = z_{Gu} - \frac{1}{2}\theta_0 z_{Gu} + Z_{ui} - Z - \theta\theta_0 z_{Gu} - \frac{1}{2}\theta^2 z_{Gu} - \frac{1}{2}\phi^2 z_{Gu}. \tag{30.124}$$

Neglecting constant terms, the gravitational potential energy of one of the two independent suspensions is

$$\mathcal{U}_{gi} = m_{ui} g \left(-\theta\theta_0 z_{Gu} + Z_{ui} - \frac{1}{2}\theta^2 z_{Gu} - \frac{1}{2}\phi^2 z_{Gu} \right) . \tag{30.125}$$

The potential energy of the system made by the two suspensions is then

$$\mathcal{U}_g = m_u g \left(Z_k - \theta\beta_{22} - \frac{1}{2}\theta^2 z_{Gu} - \frac{1}{2}\phi^2 z_{Gu} \right) \tag{30.126}$$

where

$$\beta_{22} = \theta_0 z_{Gu} .$$

The expressions for the linearized elastic potential energy of the axle and the tires, as well as those of the dissipation functions, are those already seen for rigid axles. Obviously, the expressions of the coefficients describing the various stiffnesses and damping coefficients are different and must be computed in each case from the mechanical and geometrical characteristics of the suspensions, but in the linearized approach they are at any rate constant.

30.2.5 Comparison Between Independent and Rigid Axle Suspensions

As already stated, the generalized coordinates here chosen may be used for both types of suspensions. The general expression of the kinetic energy of the axle is

$$\begin{aligned}
\mathcal{T}_u = {} & \tfrac{1}{2} m_u \left(v_x^2 + v_y^2 + v_k^2 \right) + \tfrac{1}{2}\beta_{10}\dot{\phi}^2 + \tfrac{1}{2}\beta_{11}\dot{\theta}^2 + \tfrac{1}{2}\beta_{12}\dot{\psi}^2 + \tfrac{1}{2}\beta_{13}\dot{\phi}_k^2 + \\
& + \beta_{14}\dot{\psi}\dot{\phi}_k + \beta_{15}\dot{\psi}\dot{\phi} + \tfrac{1}{2}\dot{\chi}_L^2 J_{pw} + \tfrac{1}{2}\dot{\delta}_L^2 J_{tw} + \tfrac{1}{2}\dot{\chi}_R^2 J_{pw} + \tfrac{1}{2}\dot{\delta}_R^2 J_{tw} + \\
& + \dot{\chi}_L \dot{\delta}_L J_{pw} \left(y_k + z_w \right) - \dot{\chi}_R \dot{\delta}_R J_{pw} \left(y_k + z_w \right) - \beta_{16} \dot{Z}_k \dot{\theta} \\
& + m_u v_x \left(\dot{\theta}\beta_1 + \dot{\theta} Z_k - \dot{\theta} Z + \theta v_k - \theta v_z + \beta_3 v_k - \beta_3 v_z + \right. \\
& \left. + \beta_4 \phi\dot{\psi} + x_{Gu}\dot{\theta}\theta + \beta_5 \phi_k \dot{\psi} \right) + m_u v_y \left(-\beta_5 \dot{\phi}_k + x_{Gu}\dot{\psi} - \beta_4 \dot{\phi} \right) + \\
& - \beta_{17}\dot{\delta}_L\dot{\phi} - \beta_{17}\dot{\delta}_R\dot{\phi} - \beta_{18}\dot{\delta}_L\dot{\theta} + \beta_{18}\dot{\delta}_R\dot{\theta} + \beta_{19}\dot{\delta}_L\dot{\psi} + \beta_{19}\dot{\delta}_R\dot{\psi} + \\
& + \dot{\chi}_L \beta_{20} \left(x_w - \delta_L \right) \dot{\phi}_k - \dot{\chi}_R \beta_{20} \left(x_w + \delta_R \right) \dot{\phi}_k + \dot{\chi}_L \dot{\phi} \beta_{21} \left(x_w - \delta_L \right) + \\
& - \dot{\chi}_R \dot{\phi} \beta_{21} \left(x_w + \delta_R \right) + \dot{\chi}_L J_{pw} \left(\dot{\theta} + \phi\dot{\psi} + z_w \dot{\psi} \right) + \dot{\chi}_R J_{pw} \left(\dot{\theta} + \phi\dot{\psi} - z_w \dot{\psi} \right) .
\end{aligned} \tag{30.127}$$

Some coefficients β_i vanish in the case of rigid axles (for instance β_{10} or β_{15}), while others vanish in independent suspensions (for instance β_{14} or β_{16}).

In a similar way, the gravitational potential energy can be written in the form

$$\mathcal{U}_g = m_u g \left(Z_u - \theta\beta_{22} - \frac{1}{2}\theta^2 z_{Gu} - \frac{1}{2}\phi^2\beta_{24} - \frac{1}{2}\phi_u^2\beta_{23} \right) \tag{30.128}$$

where

$$\beta_{23} = z_{Gu} \ , \ \beta_{24} = 0 \tag{30.129}$$

in the case of rigid axles, and

$$\beta_{23} = 0 \ , \ \beta_{24} = z_{Gu} \tag{30.130}$$

for independent suspensions.

30.2.6 Lagrangian Function of the Whole Vehicle

The Lagrangian function $\mathcal{L} = \mathcal{T} - \mathcal{U}$ of the whole vehicle can thus be computed without any difficulty:

$$\begin{array}{c}
\mathcal{L} = \frac{1}{2}m\left(v_x^2 + v_y^2\right) + \frac{1}{2}m_s v_z^2 + \frac{1}{2}J_x\dot{\phi}^2 + \frac{1}{2}J_y\dot{\theta}^2 + \frac{1}{2}J_z\dot{\psi}^2 + \\
-J_{xz}\dot{\psi}\dot{\phi} - m_s c v_z\dot{\theta} + v_x\dot{\theta}J_{s1} - J_{s3}v_y\dot{\phi} + J_{s3}v_x\phi\dot{\psi} + \\
+\sum_{\forall k}\left[\frac{1}{2}m_u v_k^2 + \frac{1}{2}\beta_{13}\dot{\phi}_k^2 + \beta_{14}\dot{\psi}\dot{\phi}_k + \frac{1}{2}\dot{\chi}_L^2 J_{pw} + \frac{1}{2}\dot{\delta}_L^2 J_{tw} + \frac{1}{2}\dot{\chi}_R^2 J_{pw} + \right.\\
+\frac{1}{2}\dot{\delta}_R^2 J_{tw} + \dot{\chi}_L\dot{\delta}_L J_{pw}\left(y_k + z_w\right) - \dot{\chi}_R\dot{\delta}_R J_{pw}\left(y_k + z_w\right) - \beta_{16}v_k\dot{\theta} \\
+m_u v_x\left(\dot{\theta}Z_k - \dot{\theta}Z + \theta v_k - \theta v_z + \beta_3 v_k - \beta_3 v_z + \beta_5\phi_k\dot{\psi}\right) + \\
-m_u v_y\beta_5\dot{\phi}_k - \beta_{17}\dot{\delta}_L\dot{\phi} - \beta_{17}\dot{\delta}_R\dot{\phi} - \beta_{18}\dot{\delta}_L\dot{\theta} + \beta_{18}\dot{\delta}_R\dot{\theta} + \\
+\beta_{19}\dot{\delta}_L\dot{\psi} + \beta_{19}\dot{\delta}_R\dot{\psi} + \dot{\chi}_L\beta_{20}\left(x_w - \delta_L\right)\dot{\phi}_k - \dot{\chi}_R\beta_{20}\left(x_w + \delta_R\right)\dot{\phi}_k + \\
+\dot{\chi}_L\dot{\phi}\beta_{21}\left(x_w - \delta_L\right) - \dot{\chi}_R\dot{\phi}\beta_{21}\left(x_w + \delta_R\right) + \dot{\chi}_L J_{pw}\left(\dot{\theta} + \phi\dot{\psi} + z_w\dot{\psi}\right) + \\
\left.+\dot{\chi}_R J_{pw}\left(\dot{\theta} + \phi\dot{\psi} - z_w\dot{\psi}\right)\right] - m_s g Z + M_{g1}\theta + \frac{1}{2}M_{g2}\theta^2 + \frac{1}{2}M_{g3}\phi^2 + \\
-g\sum_{\forall k} m_u\left(Z_u - \frac{1}{2}\beta_{23}\phi_k^2\right) - \sum_{\forall k}\left(\frac{1}{2}K_{11}\left(Z + Z_0\right)^2 + \right.\\
+\frac{1}{2}\left(K_{22} + K_{pz}\right)\left(Z_k + Z_{k0}\right)^2 + \frac{1}{2}K_{33}\left(\theta + \theta_0\right)^2 + \\
-K_{12}\left(Z + Z_0\right)\left(Z_k + Z_{k0}\right) - K_{13}\left(Z + Z_0\right)\left(\theta + \theta_0\right) + \\
\left.+K_{23}\left(Z_k + Z_{k0}\right)\left(\theta + \theta_0\right) + \frac{1}{2}K_\phi\phi^2 + \frac{1}{2}\left(K_\phi + K_{p\phi}\right)\phi_k^2 - K_\phi\phi\phi_k\right) \ ,
\end{array} \tag{30.131}$$

where

$$\begin{array}{c}
m = m_s + \sum_{\forall k} m_k \ , \ J_x = J_{x_L} + m_s h^2 + \sum_{\forall k}\beta_{10} \\
J_y = J_{y_L} + m_s\left(h^2 + c^2\right) + \sum_{\forall k}\beta_{11} \ , \ J_z = J_{z_L} + m_s c^2 + \sum_{\forall k}\beta_{12} \\
J_{xz} = J_{xz_L} - m_s ch - \sum_{\forall k}\beta_{15} \ , \ J_{s1} = -m_s\left(c\theta_0 - h\right) + \sum_{\forall k} m_k\beta_1 \\
J_{s3} = m_s h + \sum_{\forall k} m_k\beta_4 \ , \ M_{g1} = m_s g\left(c + h\theta_0\right) + g\sum_{\forall k} m_k\beta_{22} \ , \\
M_{g2} = m_s gh + g\sum_{\forall k} m_k z_{Gu} \ , \ M_{g3} = m_s gh + g\sum_{\forall k} m_k\beta_{24} \ .
\end{array} \tag{30.132}$$

30.3 Model with 10 Degrees of Freedom with Locked Controls

Consider a vehicle moving at a stated speed with a stated steering angle and neglect the longitudinal slip of the wheels. The forward speed V, which in the linearized approach (small value of the sideslip angle β) coincides with v_x, and its derivative $\dot{V}$ are imposed, and so are the steering angles of the wheelsand their derivatives. The angular velocity of the wheels is simply

$$\dot{\chi}_i = \frac{V}{R_{e_i}} \tag{30.133}$$

where R_{e_i} is the effective rolling radius. This expression is approximated even if the rolling radius corresponding with the actual longitudinal slip was used, because the speed of the centers of the wheels does not coincide with the velocity V of the center of mass of the vehicle. Nonetheless, if motion takes place in conditions allowing the equations to be linearized, such assumptions can be accepted.

Moreover, assume that the derivatives $\dot{\delta}_i$ of the steering angles are vanishingly small, either because the steering angles are actually locked at a constant value or because the dynamic effects of their variation are negligible

30.3.1 Expression of the Lagrangian Function and its Derivatives

The expression of the Lagrangian function is much simplified and may be written as

$$\begin{gathered}
\mathcal{L} = \tfrac{1}{2}m_e v_x^2 + \tfrac{1}{2}m v_y^2 + \tfrac{1}{2}m_s v_z^2 + \tfrac{1}{2}J_x\dot{\phi}^2 + \tfrac{1}{2}J_y\dot{\theta}^2 + \tfrac{1}{2}J_z\dot{\psi}^2 + \\
-J_{xz}\dot{\psi}\dot{\phi} - m_s c v_z\dot{\theta} + v_x\dot{\theta}J_{s2} - J_{s3}v_y\dot{\phi} + J_{s3}v_x\phi\dot{\psi} + \\
+\textstyle\sum_{\forall k}\left\{\tfrac{1}{2}m_k v_k^2 + \tfrac{1}{2}\beta_{13}\dot{\phi}_k^2 + \beta_{14}\dot{\psi}\dot{\phi}_k - \beta_{16}v_k\dot{\theta} - m_k v_y\beta_5\dot{\phi}_k + \right. \\
+m_k v_x\left(\dot{\theta}Z_k - \dot{\theta}Z + \theta v_k - \theta v_z + \beta_3 v_k - \beta_3 v_z + \beta_5\phi_k\dot{\psi}\right) + \\
\left. +2\tfrac{v_x}{R_e}\left[-\beta_{20}\delta\dot{\phi}_k - \beta_{21}\delta\dot{\phi} + J_{pr}\phi\dot{\psi}\right]\right\} - m_s g Z + M_{g1}\theta + \tfrac{1}{2}M_{g2}\theta^2 + \\
+\tfrac{1}{2}M_{g3}\phi^2 - g\textstyle\sum_{\forall k} m_k\left(Z_k - \tfrac{1}{2}\beta_{23}\phi_k^2\right) - \sum_{\forall k}\left(\tfrac{1}{2}K_{11}(Z+Z_0)^2 + \right. \\
+\tfrac{1}{2}\left(K_{22}+K_{pz}\right)(Z_k+Z_{k0})^2 + \tfrac{1}{2}K_{33}(\theta+\theta_0)^2 + \\
-K_{12}(Z+Z_0)(Z_k+Z_{k0}) - K_{13}(Z+Z_0)(\theta+\theta_0) + \\
\left. +K_{23}(Z_k+Z_{k0})(\theta+\theta_0) + \tfrac{1}{2}K_\phi\phi^2 + \tfrac{1}{2}\left(K_\phi + K_{p\phi}\right)\phi_k^2 - K_\phi\phi\phi_k\right) ,
\end{gathered} \tag{30.134}$$

where the equivalent mass and J_{s2} are

$$m_e = m + 2\sum_{\forall k} J_{pw}\frac{1}{R_e^2} \ , \ J_{s2} = J_{s1} + 2\sum_{\forall k}\frac{J_{pw}}{R_e} \tag{30.135}$$

(coefficients 2 come from the assumption that each axle has two wheels) and δ is the average steering angle of the axle

$$\delta = \frac{\delta_L + \delta_R}{2} \ . \tag{30.136}$$

The derivatives of the Lagrangian function with respect to the generalized velocities and coordinates are

$$\frac{\partial \mathcal{L}}{\partial v_x} = m_e v_x + \dot{\theta} J_{s2} \ . \tag{30.137}$$

Note that the expression of this derivative has been further linearized by cancelling the terms of the same order of the squares of small quantities. For instance, the term in $\beta_3 v_k$ was cancelled because both β_3 and v_k are small quantities. Moreover, the use of the equivalent mass, which includes only the contribution to inertia due to the wheels, may be criticized because the transmission has not been modelled. Physically, this corresponds to considering the vehicle as pushed forward by an external force in the x direction, as in jet propelled record vehicles, instead of propelled by the driving torque applied to the wheels (or slowed by the braking torque).

$$\frac{\partial \mathcal{L}}{\partial v_y} = m v_y - J_{s3}\dot{\phi} - \sum_{\forall k} m_k \beta_5 \dot{\phi}_k, \tag{30.138}$$

$$\frac{\partial \mathcal{L}}{\partial v_z} = m_s \dot{Z} - m_s c \dot{\theta} - v_x \sum_{\forall k} m_k \left(\theta + \beta_3\right), \tag{30.139}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = J_x \dot{\phi} - J_{xz} \dot{\psi} - J_{s3} v_y - 2 \sum_{\forall k} \frac{v_x}{R_e} \beta_{21} \delta, \tag{30.140}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = J_y \dot{\theta} - m_s c \dot{Z} + v_x J_{s2} + \sum_{\forall k} \left[-\beta_{16} \dot{Z}_k + m_k v_x \left(Z_k - Z\right)\right], \tag{30.141}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = J_z \dot{\psi} - J_{xz} \dot{\phi} + J_{s3} v_x \phi + \sum_{\forall k} \left(\beta_{14} \dot{\phi}_k + m_k v_x \beta_5 \phi_k + 2 \frac{v_x}{R_e} J_{pw} \phi\right), \tag{30.142}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{Z}_k} = m_k \dot{Z}_k - \beta_{16k} \dot{\theta} + m_k v_x \left(\theta + \beta_{3k}\right) \text{ for } k = 1, 2, \tag{30.143}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}_k} = \beta_{13k} \dot{\phi}_k + \beta_{14k} \dot{\psi} - m_k v_y \beta_{5k} - 2 \frac{v_x}{R_{ek}} \beta_{20k} \delta_k \text{ for } k = 1, 2, \tag{30.144}$$

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{\partial \mathcal{L}}{\partial Y} = \frac{\partial \mathcal{L}}{\partial \psi} = 0, \tag{30.145}$$

$$\frac{\partial \mathcal{L}}{\partial Z} = -m_s g - \sum_{\forall k} \left[m_k v_x \dot{\theta} + K_{11} (Z + Z_0) + \right.$$
$$\left. -K_{12} (Z_k + Z_{0k}) - K_{13} (\theta + \theta_0) \right], \tag{30.146}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = M_{g1} + M_{g2}\theta + \sum_{\forall k} \left[m_u v_x \left(\dot{Z}_k - \dot{Z} \right) + \right.$$
$$\left. -K_{33} (\theta + \theta_0) + K_{13} (Z + Z_0) - K_{23} (Z_k + Z_{k0}) \right], \tag{30.147}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = J_{s3} v_x \dot{\psi} + M_{g3}\phi + \sum_{\forall k} \left(2 \frac{v_x}{R_e} J_{pw} \dot{\psi} - K_{\phi}\phi + K_{\phi}\phi_k \right), \tag{30.148}$$

$$\frac{\partial \mathcal{L}}{\partial Z_k} = -g m_k + m_k v_x \dot{\theta} - \left(K_{22k} + K_{pzk} \right) (Z_k + Z_{k0})$$
$$+ K_{12k} (Z + Z_0) - K_{23k} (\theta + \theta_0), \tag{30.149}$$

$$\frac{\partial \mathcal{L}}{\partial \phi_k} = + m_k v_x \beta_{5k} \dot{\psi} + g m_k \beta_{23k} \phi_k - \left(K_{\phi k} + K_{p\phi k} \right) \phi_k + K_{\phi k} \phi. \tag{30.150}$$

The last two derivatives must be computed for the various axles ($k = 1, 2$ for a two-axles vehicle).

30.3.2 Kinematic Equations

Even if velocity V is stated, all 10 equations of motion must be written, because the generalized coordinates are 10. When all equations of motion have been obtained, it will be possible to state that the forward velocity is known and one of the equations can be eliminated.

The generalized coordinates for a two-axles vehicle are then

$$\mathbf{q} = \left[X \ Y \ Z \ \phi \ \theta \ \psi \ z_1 \ \phi_1 \ z_2 \ \phi_2 \right]^T. \tag{30.151}$$

The vector containing the generalized velocities $\mathbf{w}$ is

$$\mathbf{w} = \left[v_x \ v_y \ v_z \ v_\phi \ v_\theta \ v_\psi \ v_1 \ v_{\phi 1} \ v_2 \ v_{\phi 2} \right]^T. \tag{30.152}$$

Remark 30.1 Velocities $\mathbf{w}$ are referred neither to a body-fixed frame nor to an inertial frame. Linear velocities are referred to the intermediate frame $x^* y^* z$, while the generalized velocities related to angular coordinates are the derivatives of Tait-Bryan angles. This may make the analysis more complicated, but only to a point.

The relationship linking the velocities to the derivatives of the coordinates is, as usual

$$\mathbf{w} = \mathbf{A}^T \dot{\mathbf{q}} \,,$$

where

$$\mathbf{A} = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & \mathbf{0} \\ \sin(\psi) & \cos(\psi) & \\ \mathbf{0} & & \mathbf{I} \end{bmatrix} \tag{30.153}$$

and $\mathbf{I}$ is an identity matrix of size 8×8.

The kinematic equations are the inverse transformation

$$\dot{\mathbf{q}} = \mathbf{A}^{-\mathbf{T}}\mathbf{w} = \mathbf{B}\mathbf{w} \,. \tag{30.154}$$

$\mathbf{A}$ is in this case a rotation matrix, so that

$$\mathbf{A}^{-1} = \mathbf{A}^T \,, \mathbf{B} = \mathbf{A} \,. \tag{30.155}$$

The equation of motion in the state space is made by the 10 equations of motion, plus the 10 kinematic equations and is then Eq. (A.101):

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\left\{\dfrac{\partial \mathcal{L}}{\partial w}\right\}\right) + \mathbf{B}^{\mathbf{T}}\Gamma\left\{\dfrac{\partial \mathcal{L}}{\partial w}\right\} - \mathbf{B}^T\left\{\dfrac{\partial \mathcal{L}}{\partial q}\right\} + \left\{\dfrac{\partial \mathcal{F}}{\partial w}\right\} = \mathbf{B}^T\mathbf{Q} \,, \\ \{\dot{q}_i\} = \mathbf{B}\{w_i\} \,. \end{cases} \tag{30.156}$$

The column matrix $\mathbf{B}^T\mathbf{Q}$ containing the 10 components of the vector of the generalized forces will be computed later by writing the virtual work of the forces acting on the system. In the following equations its elements will be indicated with Q_x, Q_y, Q_z, Q_ϕ, Q_θ, Q_ψ, Q_{zk}, $Q_{\phi k}$.

The most complicated part of the computation is writing matrix $\mathbf{B}^{\mathbf{T}}\Gamma$. By performing fairly intricate computations, following the procedure described in Appendix A, it follows that

$$\mathbf{B}^{\mathbf{T}}\boldsymbol{\Gamma} = \begin{bmatrix} \begin{bmatrix} 0 & -\dot{\psi} \\ \dot{\psi} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -v_y & v_x \end{bmatrix} & \mathbf{0}_{6\times 8} \\ \mathbf{0}_{4\times 2} & \mathbf{0}_{4\times 8} \end{bmatrix} .$$

By using the expressions of the derivatives with respect to the generalized velocities seen above, and differentiating again with respect to time, it follows that

$$\frac{\partial}{\partial t}\left(\left\{\frac{\partial \mathcal{L}}{\partial w}\right\}\right) =$$
$$= \left\{\begin{array}{l}
m_e \dot{v}_x + \ddot{\theta} J_{s2} \\
m\dot{v}_y - J_{s3}\ddot{\phi} - \sum_{\forall k} m\beta_{5k}\ddot{\phi}_k \\
m_s\ddot{Z} - m_s c\ddot{\theta} - \dot{v}_x \sum_{\forall k} m_k (\theta + \beta_{3k}) - v_x\dot{\theta}\sum_{\forall k} m_k \\
J_x\ddot{\phi} - J_{xz}\ddot{\psi} - J_{s3}\dot{v}_y - 2\dot{v}_x \sum_{\forall k} \frac{1}{R_{ek}}\beta_{21k}\delta_k \\
J_y\ddot{\theta} - m_s c\ddot{Z} + \dot{v}_x J_{s2} + \sum_{\forall k}\left[-\beta_{16k}\ddot{Z}_k + m_k\dot{v}_x (Z_k - Z) + \right.\\
\quad \left. + m_k v_x \left(\dot{Z}_k - \dot{Z}\right)\right] \\
J_z\ddot{\psi} - J_{xz}\ddot{\phi} + J_{s3}\dot{v}_x\phi + J_{s3}v_x\dot{\phi} + \sum_{\forall k}\left[\beta_{14k}\ddot{\phi}_k + m_k\dot{v}_x\beta_{5k}\phi_k + \right.\\
\quad \left. + m_k v_x\beta_{5k}\dot{\phi}_k + 2\dot{v}_x\frac{1}{R_{ek}}J_{pwk}\phi + 2v_x\frac{1}{R_{ek}}J_{pwk}\dot{\phi}\right] \\
m_k\ddot{Z}_k - \beta_{16k}\ddot{\theta} + m_k\dot{v}_x(\theta + \beta_{3k}) + m_k v_x\dot{\theta} \\
\beta_{13k}\ddot{\phi}_k + \beta_{14k}\ddot{\psi} - m_k\dot{v}_y\beta_{5k} - 2\dot{v}_x\frac{1}{R_{ek}}\beta_{20k}\delta_k
\end{array}\right\} . \tag{30.157}$$

The last two equations refer to the coordinates of the axles, and then must be repeated for $k = 1,\ 2$.

$$\mathbf{B}^T\mathbf{\Gamma}\left\{\frac{\partial \mathcal{L}}{\partial w}\right\} = \left[-\dot{\psi}\frac{\partial \mathcal{L}}{\partial v_y}\ \dot{\psi}\frac{\partial \mathcal{L}}{\partial v_x}\ 0\ 0\ 0\ -v_y\frac{\partial \mathcal{L}}{\partial v_x} + v_x\frac{\partial \mathcal{L}}{\partial v_y}\ 0\ 0\ 0\ 0\right]^T . \tag{30.158}$$

By introducing the values of the derivatives and linearizing, it follows that

$$\mathbf{B}^T\mathbf{\Gamma}\left\{\frac{\partial \mathcal{L}}{\partial w}\right\} = \left\{\begin{array}{c}
\left\{\begin{array}{l} 0 \\ m_e v_x\dot{\psi}\end{array}\right\} \\
\mathbf{0}_{3\times 1} \\
v_x\left[-J_{s3}\dot{\phi} - \sum_{\forall k}\left(2v_y J_{pwk}\frac{1}{R_e^2} + m_k\beta_5\dot{\phi}_k\right)\right] \\
\mathbf{0}_{4\times 1}
\end{array}\right\} . \tag{30.159}$$

Finally:

$$\mathbf{B}^T\left\{\frac{\partial \mathcal{L}}{\partial q}\right\} = \left\{\begin{array}{l}
0 \\
0 \\
-m_s g - \sum_{\forall k}\left[m_k v_x\dot{\theta} + K_{11}(Z + Z_0) + \right.\\
\quad \left. - K_{12}(Z_k + Z_{k0}) - K_{13}(\theta + \theta_0)\right] \\
J_{s3}v_x\dot{\psi} + M_{g3}\phi + \sum_{\forall k}\left(2v_x\frac{1}{R_e}J_{pwk}\dot{\psi} - K_{\phi}\phi + K_{\phi}\phi_k\right) \\
M_{g1} + M_{g2}\theta + \sum_{\forall k}\left[m_u v_x\left(\dot{Z}_k - \dot{Z}\right) - K_{33}(\theta + \theta_0)\right.\\
\quad \left. + K_{13}(Z + Z_0) - K_{23}(Z_k + Z_{k0})\right] \\
0 \\
-gm_k + m_k v_x\dot{\theta} - \left(K_{22k} + K_{pzk}\right)(Z_k + Z_{k0}) + \\
\quad + K_{12k}(Z + Z_0) - K_{23k}(Z + Z_0) \\
-m_k v_x\beta_{5k}\dot{\psi} - gm_{uk}\beta_{23k}\phi_k - K_{\phi k}\phi_k + K_{\phi k}\phi - K_{p\phi k}\phi_k
\end{array}\right\} . \tag{30.160}$$

The last two equations refer to the axles and must be repeated for $k = 1,\ 2$.
The derivatives of the dissipation function are

$$\left\{\frac{\partial \mathcal{F}}{\partial w}\right\} = \left\{\begin{array}{l} 0 \\ 0 \\ \sum_{\forall k}\left(c_{11k}\dot{Z} - c_{12k}\dot{Z}_k - c_{13k}\dot{\theta}\right) \\ \sum_{\forall k}\left(c_{\phi k}\dot{\phi} - c_{\phi k}\dot{\phi}_k\right) \\ \sum_{\forall k}\left(c_{33k}\dot{\theta} - c_{13k}\dot{Z} + c_{23k}\dot{Z}_k\right) \\ 0 \\ \left(c_{22k} + c_{pzk}\right)\dot{Z}_k - c_{12k}\dot{Z} + c_{23k}\dot{\theta} \\ \left(c_{\phi k} + c_{p\phi k}\right)\dot{\phi}_k - c_{\phi k}\dot{\phi} \end{array}\right\} . \tag{30.161}$$

30.3.3 Equations of Motion

First equation: longitudinal translation

By introducing the forward velocity of the vehicle V instead of v_x, the first equation becomes

$$m_e\dot{V} + \ddot{\theta}J_{s2} = Q_x . \tag{30.162}$$

Second equation: lateral translation

$$m\dot{v}_y + m_eV\dot{\psi} - J_{s3}\ddot{\phi} - \sum_{\forall k} m_u\beta_5\ddot{\phi}_k = Q_y . \tag{30.163}$$

Third equation: vertical translation

$$\begin{array}{c} m_s\ddot{Z} - m_sc\ddot{\theta} + \sum_{\forall k}\left[K_{11}\left(Z + Z_0\right) - K_{12}\left(Z_k + Z_{k0}\right) + \right. \\ -K_{13}\left(\theta + \theta_0\right) + c_{11k}\dot{Z} - c_{12k}\dot{Z}_k + \\ \left. -c_{13k}\dot{\theta} - \dot{V}m_k\left(\theta + \beta_{3k}\right)\right] = -m_sg + Q_z . \end{array} \tag{30.164}$$

Fourth equation: roll rotation

$$\begin{array}{c} J_x\ddot{\phi} - J_{xz}\ddot{\psi} - J_{s3}\dot{v}_y - J_{s3}V\dot{\psi} - M_{g3}\phi + \sum_{\forall k}\left(-2\dot{V}\frac{1}{R_{ek}}\beta_{21k}\delta_k + \right. \\ \left. -2v_x\frac{1}{R_e}J_{pwk}\dot{\psi} + K_\phi\phi - K_\phi\phi_k + c_{\phi k}\dot{\phi} - c_{\phi k}\dot{\phi}_k\right) = Q_\phi . \end{array} \tag{30.165}$$

Fifth equation: pitch rotation

$$\begin{array}{c} J_y\ddot{\theta} - m_sc\ddot{Z} + \dot{V}J_{s2} - M_{g2}\theta + \sum_{\forall k}\left[-\beta_{16}\ddot{Z}_k + \right. \\ +m_k\dot{V}\left(Z_k - Z\right) + K_{33}\left(\theta + \theta_0\right) - K_{13}\left(Z + Z_0\right) + \\ \left. +K_{23}\left(Z_k + Z_{k0}\right) + c_{33k}\dot{\theta} - c_{13k}\dot{Z} + c_{23k}\dot{Z}_k\right] = M_{g1} + Q_\theta . \end{array} \tag{30.166}$$

Sixth equation: yaw rotation

$$\begin{array}{c}J_z\ddot{\psi}-J_{xz}\ddot{\phi}+J_{s3}\dot{V}\phi+\sum_{\forall k}\left[\beta_{14k}\ddot{\phi}_k+m_k\dot{V}\beta_{5k}\phi_k+\right.\\ \left.+2\dot{V}\frac{1}{R_e}J_{pwk}\phi+2V\frac{1}{R_{ek}}J_{pwk}\dot{\phi}-2Vv_yJ_{pwk}\frac{1}{R_{ek}^2}\right]=Q_\psi\ .\end{array} \tag{30.167}$$

Seventh and ninth equations: translation of axles

$$\begin{array}{c}m_k\ddot{Z}_k-\beta_{16k}\ddot{\theta}+m_k\dot{V}\left(\theta+\beta_{3k}\right)+\\ +\left(K_{22k}+K_{pzk}\right)\left(Z_k+Z_{k0}\right)-K_{12k}\left(Z+Z_0\right)+\\ +K_{23k}\left(\theta+\theta_0\right)+\left(c_{22k}+c_{pzk}\right)\dot{Z}_k-c_{12k}\dot{Z}+c_{23k}\dot{\theta}=-gm_k+Q_{zk}\ .\end{array} \tag{30.168}$$

Eighth and tenth equations: rotation of axles

$$\begin{array}{c}\beta_{13k}\ddot{\phi}_k+\beta_{14k}\ddot{\psi}-m_k\dot{v}_y\beta_{5k}-2\dot{V}\frac{1}{R_{ek}}\beta_{20k}\delta_k-m_kv_x\beta_{5k}\dot{\psi}-gm_k\beta_{23k}\phi_k+\\ +\left(K_{\phi k}+K_{p\phi k}\right)\phi_k-K_{\phi k}\phi+\left(c_{\phi k}+c_{p\phi k}\right)\dot{\phi}_k-c_{\phi k}\dot{\phi}=Q_{\phi k}\ .\end{array} \tag{30.169}$$

30.3.4 Sideslip Angles of the Wheels

The sideslip angles of the wheelscan be computed directly from the components of the speed of the centers of the wheel-ground contact zone in the x^*y^*z frame. In the case of a solid axle suspension, the position of the center of the contact zone may be computed with the methods used for the center of mass of the axle, by substituting the coordinates of the center of the contact area x_{Cn}, y_{Cn}, z_{Cn} for those of the center of mass x_{Gu}, 0, z_{Gu} in Eq. (30.32):

$$\overline{(\mathrm{C}_u - \mathrm{O'})} = \overline{(\mathrm{H} - \mathrm{O'})} + \mathbf{R}_1\mathbf{r}\ , \tag{30.170}$$

where

$$\mathbf{r}=\mathbf{R}_2\left[\mathbf{R}_3\left\{\begin{array}{c}x_u+\zeta s_x\\ 0\\ \zeta\end{array}\right\}+\mathbf{R}_{us}\mathbf{R}_{3k}\left\{\begin{array}{c}x_{Cu}\\ y_{Cu}\\ z_{Cu}\end{array}\right\}\right]\ . \tag{30.171}$$

Obviously, vector x_{Cu}, y_{Cu}, z_{Cu} must be expressed in the same reference frame in which the coordinates of the center of mass x_{Gu}, 0, z_{Gu} were expressed. This procedure is approximate, because the deformations of the tire are neglected, but the approximation is not greater than those already introduced in the linearized model.

By introducing the linearized expression for ζ, it follows that

$$\mathbf{r}=\left\{\begin{array}{c}x_{Ct}+\theta z_{Cu}\\ y_{Cu}-\phi_k z_{Cu}\\ z_{Ct}+Z_k-Z-x_{Cu}\theta\end{array}\right\}\ , \tag{30.172}$$

where

$$x_{Ct} = x_u + x_{Cu} - z_{us}x_{Cu} + \theta_0 z_{Cu} \ ,$$
$$z_{Ct} = z_{Cu} + z_{us}x_{Cu} - \theta_0 x_{Cu}. \tag{30.173}$$

In the case of independent suspensions, it follows (Eq. 30.113) that:

$$\mathbf{r} = \mathbf{R}_2\mathbf{R}_3 \left\{ \begin{array}{c} x_{Cu} + \zeta s_x \\ y_{Cu} + \zeta s_y \\ z_{Cu} + \zeta \end{array} \right\} \ . \tag{30.174}$$

By linearizing the expression for $\mathbf{r}$ and introducing the value of ζ, it follows that

$$\mathbf{r} = \left\{ \begin{array}{c} x_{Ct} + \theta z_{Cu} \\ y_{Cu} - \phi z_{Cu} \\ z_{Ct} + Z_k - Z \end{array} \right\} \ , \tag{30.175}$$

where

$$x_{Ct} = x_{Cu} + \theta_0 z_{Cu} \ , z_{Ct} = z_{Cu} - \theta_0 x_{Cu}. \tag{30.176}$$

Note that the meaning of symbols x_{Cu}, y_{Cu}, z_{Cu} is different for the two suspension types.

To express $\mathbf{r}$ with a single equation, that holds in all cases, it is possible to write

$$\mathbf{r} = \left\{ \begin{array}{c} x_{Ct} + \theta z_{Cu} \\ y_{Cu} - \phi z_1 - \phi_k z_2 \\ z_{Ct} + Z_k - Z \end{array} \right\} \ , \tag{30.177}$$

where

$$z_1 = 0 \ , z_2 = z_{Cu}, \tag{30.178}$$

in the case of solid axles, and

$$z_1 = z_{Cn} \ , z_2 = 0, \tag{30.179}$$

in the case of independent suspensions

The velocity of the center of the contact area, expressed in the inertial frame, is

$$V_{C_N} = \left[\ \dot{X} \ \dot{Y} \ \dot{Z} \ \right]^T + \mathbf{R}_1\dot{\mathbf{r}} + \dot{\mathbf{R}}_1\mathbf{r} \ . \tag{30.180}$$

By premultiplying the velocity by $\mathbf{R}_1^T$ it is possible to obtain its value in the $x^*y^*z^*$ frame. Remembering Eq. (30.41), it follows that

$$V_{Cu} = \mathbf{V} + \dot{\mathbf{r}} + \dot{\psi}\mathbf{S}\mathbf{r} \ , \tag{30.181}$$

where the linearized expression for the derivative of **r** with respect to time is

$$\dot{\mathbf{r}} = \begin{Bmatrix} \dot{\theta} z_{Cu} \\ -\dot{\phi} z_1 - \dot{\phi}_k z_2 \\ \dot{Z}_k - \dot{Z} - x_{Cu}\dot{\theta} \end{Bmatrix} . \tag{30.182}$$

As a first approximation, it is possible to assume that the position of the center of the contact area of the ith tire P_i coincides with the projection on the ground of the center of the wheel. The velocity of the center of the contact area is then

$$V_{Pi} = V_{Cu} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{Bmatrix} v_x + \dot{r}_x - \dot{\psi} r_y \\ v_y + \dot{r}_y + \dot{\psi} r_x \\ 0 \end{Bmatrix} . \tag{30.183}$$

By performing the relevant computations and linearizing, it follows that

$$V_{Pi} = \begin{Bmatrix} v_x + \dot{\theta} z_{Cu} - \dot{\psi} y_{Cu} \\ v_y - \dot{\phi} z_1 - \dot{\phi}_k z_2 + \dot{\psi} x_t \\ 0 \end{Bmatrix} . \tag{30.184}$$

Because the mid-plane of the wheel is rotated by the steering angle δ_k (possibly increased by $(\delta_k)_{,\phi} (\phi - \phi_k)$ to account for roll steer) with respect to the x^*z plane, the usual linearizations allow writing

$$\alpha_k = \frac{v_y}{V} + \dot{\psi}\frac{x_{tk}}{V} - \dot{\phi}\frac{z_{1k}}{V} - \dot{\phi}_k\frac{z_{2k}}{V} - \delta_k - (\delta_k)_{,\phi} (\phi - \phi_k) , \tag{30.185}$$

where subscript k refers to the axle.

The two wheels of the same axle are then at the same sideslip angle, as was the case for the rigid vehicle. Linearization again allows us to work in terms of axles instead of single wheels. The terms in $\dot{\phi}$ and $\dot{\phi}_k$ are usually small and will be neglected in the following equations.

The sideslip angles are then

$$\alpha_k = \frac{v_y}{V} + \dot{\psi}\frac{x_{tk}}{V} - \delta_k - (\delta_k)_{,\phi} (\phi - \phi_k) . \tag{30.186}$$

This expression coincides with that obtained for the rigid vehicle, to which roll steer has been added.

30.3.5 Generalized Forces

The generalized forces Q_k to be introduced into the equations of motion include only the forces due to tires, aerodynamic forces and possible forces that may be applied to the vehicle.

The virtual displacement of the left (right) wheel of the kth axle has an expression similar to Eq. (30.184):

$$\{\delta s_{Pk_{L(R)}}\}_{x^*y^*z} = \left\{ \begin{array}{c} \delta x^* + \delta\theta z_{Cu} - \delta\psi y_{Cu} \\ \delta y^* - \delta\phi z_1 - \delta\phi_k z_2 + \delta\psi x_t \\ 0 \end{array} \right\} . \tag{30.187}$$

If coupling between vertical and horizontal displacements of the suspension must be accounted for, a term

$$\left(\frac{\partial x}{\partial z}\right)_k (\delta Z - x_t \delta\theta) = (x_k)_{,z}(\delta Z - x_t \delta\theta)$$

must be added to the x^* component of the virtual displacement.

If the forces exerted by the tire in the direction of the x^* and y^* axes are

$$F_x{}^* = F_{x\,p} \cos\left[\delta_i - (\delta_i)_{,\phi}\phi\right] - F_{y_p} \sin\left[\delta_i - (\delta_i)_{,\phi}\phi\right] ,$$

$$F_y{}^* = F_{x\,p} \sin\left[\delta_i - (\delta_i)_{,\phi}\phi\right] + F_{y_p} \cos\left[\delta_i - (\delta_i)_{,\phi}\phi\right] ,$$

and assuming that the longitudinal forces acting on the wheels of any axle are equal (if they are not, it is not difficult to add a yawing torque about the z axis), the expression of the virtual work is

$$\begin{aligned} \delta\mathcal{L}_k = {} & \delta x^* F_x{}^* + \delta Z (x_k)_{,z} F_x{}^* + \delta\theta F_x{}^* \left(z_{Cu} - (x_k)_{,z} x_t\right) + \\ & + \delta y^* F_y{}^* - \delta\phi F_y{}^* z_{1k} - \delta\phi_k F_y{}^* z_{2k} + \delta\psi \left\{F_y{}^* x_{tk} + M_z\right\} . \end{aligned} \tag{30.188}$$

The generalized forces may be obtained by differentiating the virtual work with respect to the virtual displacements δx^*, δy^*, $\delta\theta$, etc. The first two generalized forces are true forces directed along axes x^* and y^* and a suitable rotation matrix can be used to obtain the forces along the axes of the inertial frame.

Force $F_{y_{p_i}}$ on the ith tire may be expressed as a linear function of the sideslip and camber angles α_k and

$$\gamma_{0k} + (\gamma_k)_{,\phi}\phi + (\gamma_k)_{,\phi_k}\phi_k + (\gamma_k)_{,z}(Z - Z_k) .$$

Terms γ_{0k} and $(\gamma_k)_{,z}(Z - Z_k)$ for the two wheels of any axle cancel each other in the linearized model, because they produce equal and opposite forces. The side forces applied on each axle are then

$$F_{y\,pk} = -C_k\alpha_k + C_{\gamma k}\left[(\gamma_k)_{,\phi}\phi + (\gamma_k)_{,\phi k}\phi_k\right] \ , \tag{30.189}$$

where both C_k and $C_{\gamma k}$ are referred to the whole axle.

Assume that aerodynamic forces are applied to the center of mass of the sprung mass. The virtual displacement of such a point in the x^*y^*z frame is

$$\{\delta s_{G_s}\}_{x^*y^*z} = \left\{\begin{array}{c} \delta x^* + h\delta\theta + h\phi\delta\psi \\ \delta y^* - h\delta\phi + (c + h\theta_0 + h\theta)\,\delta\psi \\ \delta Z - c\delta\theta \end{array}\right\} . \tag{30.190}$$

The virtual work of aerodynamic forces and moments is

$$\begin{array}{c} \delta\mathcal{L}_a = F_{xa}\delta x^* + F_{y_a}\delta y^* + F_{z_a}\delta Z + \\ +\left(M'_{xa} - F_{y_a}h\right)\delta\phi + \left(F_{x_a}h - F_{z_a}c + M'_{y_a} + M'_{za}\phi\right)\delta\theta + \\ +\left[F_{x_a}h\phi + F_{y_a}(c + h\theta_0 + h\theta) - M'_{y_a}(\theta_0 + \theta - \phi) + M'_{za}\right]\delta\psi \ . \end{array} \tag{30.191}$$

It is also possible to directly obtain the generalized forces by differentiating the virtual work with respect to the virtual displacements.

Because the aerodynamic forces are applied in the center of mass of the sprung mass instead of the center of mass of the vehicle, the aerodynamic moments referred to the former must be substituted for those defined in the usual way, that is, with reference to the center of mass of the vehicle:

$$\left\{\begin{array}{l} M_{xa} = M'_{xa} - F_{y_a}h \ , \\ M_{y_a} = M'_{y_a} + F_{xa}h \ , \\ M_{za} = M'_{za} - F_{y_a}(c + h\theta_0) \ . \end{array}\right. \tag{30.192}$$

Due to the linearization of the vehicle, force F_{xa} may be considered as a constant, while F_{y_a}, M_{xa} and M_{za} may be considered as linear with angle β_a (if there is no side wind, with angle β), while F_{za} and M_{y_a} may be considered as linear with angle θ. Neglecting small terms, it follows that

$$\delta\mathcal{L}_{aer} = F_{xa}\delta x^* + \tfrac{\partial F_{y_a}}{\partial\beta}\beta\delta y^* + \tfrac{\partial F_{za}}{\partial\theta}\theta\delta Z + \tag{30.193}$$

$$+\left(\tfrac{\partial M_{xa}}{\partial\beta} - \tfrac{\partial F_{y_a}}{\partial\beta}\beta h\right)\delta\phi + \tfrac{\partial M_{y_a}}{\partial\theta}\delta\theta + \left(F_{xa}h\phi + \tfrac{\partial M_{za}}{\partial\beta}\beta\right)\delta\psi \ .$$

The vector of the generalized forces may be obtained by differentiating the virtual work with respect to the virtual displacements, and eliminating the terms containing generalized forces multiplied by variables of motion, which would lead to nonlinear terms once the generalized forces are expressed as functions of the same variables

$$\mathbf{Q} = \left\{ \begin{array}{c} F_{x1} + F_{x2} + F_{xa} \\ \sum_{\forall k} \left\{ -C_k \alpha_k + C_{\gamma k} \left[(\gamma_k)_{,\phi} \phi + (\gamma_k)_{,\phi k} \phi_k \right] \right\} + \frac{\partial F_{y_a}}{\partial \beta} \beta \\ -(x_{i1})_{,z} F_{x1} - (x_{i2})_{,z} F_{x2} + \frac{\partial F_{z_a}}{\partial \theta} \theta + \left(F_{za} \right)_{\theta=0} \\ \sum_{\forall k} z_{1,k} \left\{ -C_k \alpha_k + C_{\gamma k} \left[(\gamma_k)_{,\phi} \phi + (\gamma_k)_{,\phi k} \phi_k \right] \right\} - \frac{\partial M_{xa}}{\partial \beta} - \frac{\partial F_{y_a}}{\partial \beta} \beta h \\ F_{x1} \left(z_{Cn1} - (x_{i1})_{,z} x_{t1} \right) + F_{x2} \left(z_{Cn2} - (x_{i2})_{,z} x_{t2} \right) + \frac{\partial M_{y_a}}{\partial \theta} \theta + \\ + \left(M_{y_a} \right)_{\theta=0} \\ \sum_{\forall k} \left(\frac{\partial M_{z1}}{\partial \alpha} \alpha_1 + x_{w_{iA}} \left\{ -C_k \alpha_k + C_{\gamma k} \left[(\gamma_k)_{,\phi} \phi + (\gamma_k)_{,\phi k} \phi_k \right] \right\} \right) + \\ + F_{x_a} h \phi + \frac{\partial M_{z_a}}{\partial \beta} \beta \\ 0 \\ z_{2,1} \left\{ -C_1 \alpha_1 + C_{\gamma 1} \left[(\gamma_1)_{,\phi} \phi + (\gamma_1)_{,\phi 1} \phi_1 \right] \right\} \\ 0 \\ z_{2,2} \left\{ -C_2 \alpha_2 + C_{\gamma 2} \left[(\gamma_2)_{,\phi} \phi + (\gamma_k)_{,\phi 2} \phi_2 \right] \right\} \end{array} \right\} . \tag{30.194}$$

The term

$$F_{x1} \left[\delta_1 - (\delta_1)_{,\phi} (\phi - \phi_1) \right] + F_{x2} \left[\delta_2 - (\delta_2)_{,\phi} \phi (\phi - \phi_2) \right]$$

should be included in the generalized force Q_y. It results from the component of the longitudinal force of the tire in the direction of the y axis of the vehicle due to the steering angle. It is a small term, owing to the small size of the longitudinal force F_x when compared to the cornering stiffness, and is usually neglected. A similar term should also be included in Q_ψ, but is usually neglected as well.

The vector of the generalized forces so obtained may be used directly in the equation of motion, because it is referred to the pseudo-coordinates x^*, y^* and to coordinates Z, ϕ, θ, ψ, etc.

30.3.6 Final Form of the Equations of Motion

Remembering that the steering angles are small, it is easy to pass from the forces expressed in a frame fixed to the vehicle to one fixed to the tires. The equations of motion then take their form.

First equation: longitudinal translation

$$m_e \dot{V} + \ddot{\theta} J_{s2} = F_{x1} + F_{x2} - \frac{1}{2}\rho V^2 S C_x \ . \tag{30.195}$$

Second equation: lateral translation

The sideslip angles, and then the cornering forces, may be easily expressed as functions of the variables of motion. Assuming that the steering angles of the axles are proportional to a reference value δ through constants K'_k:

$$\delta_k = K'_k \delta \tag{30.196}$$

and adding a side force F_{ye} applied to the vehicle, it follows that

$$Q_y = Y_v v_y + Y_w \dot{\psi} + Y_\phi \phi + Y_{\phi 1} \phi_1 + Y_{\phi 2} \phi_2 + Y_\delta \delta + F_{y_e} \ , \tag{30.197}$$

where

$$\begin{cases} Y_v = -\frac{1}{V} \sum_{\forall k} C_k + \frac{1}{2}\rho V_a S (C_y)_{,\beta} \ , \\ Y_w = -\frac{1}{V} \sum_{\forall k} x_{wk} C_k \ , \\ Y_\phi = \sum_{\forall k} C_i (\delta_k)_{,\phi} + \sum_{\forall k} C_{\gamma k} (\gamma_k)_{,\phi} \ , \\ Y_{\phi k} = C_{\gamma k} (\gamma_k)_{,\phi} \ , \\ Y_\delta = \sum_{\forall k} K'_k C_k \ . \end{cases} \tag{30.198}$$

The second equation then becomes

$$\begin{gathered} m \dot{v}_y - J_{s3} \ddot{\phi} - \sum_{\forall k} m_k \beta_{5k} \ddot{\phi}_k = Y_v v_y + Y_{\dot{\psi}} \dot{\psi} + \\ + Y_\phi \phi + \sum_{\forall k} Y_{\phi_k} \phi_k + Y_\delta \delta + F_{y_e} \ , \end{gathered} \tag{30.199}$$

where

$$Y_{\dot{\psi}} = Y_w - m_e V \ . \tag{30.200}$$

Third equation: vertical translation

By introducing the generalized forces into the third equation, it follows that

$$\begin{gathered} m_s \ddot{Z} - m_s c \ddot{\theta} + Z_{\dot{z}} \dot{Z} + Z_{\dot{\theta}} \dot{\theta} + Z_{\dot{z}1} \dot{Z}_1 + Z_{\dot{z}2} \dot{Z}_2 + Z_z Z + Z_\theta \theta + \\ + Z_{z1} Z_1 + Z_{z2} Z_2 + \sum_{\forall k} (K_{11k} Z_0 - K_{12k} Z_{0k} - K_{13} \theta_0) = -m_s g + \\ + \dot{V} \sum_{\forall k} m_k \beta_{3k} + \frac{1}{2}\rho V^2 S (C_Z)_{\theta=0} - (x_{i1})_{,z} F_{x1} - (x_{i2})_{,z} F_{x2} \ , \end{gathered} \tag{30.201}$$

where

$$\begin{cases} Z_{\dot{z}} = \sum_{\forall k} c_{11k} \; , \; Z_{\dot{\theta}} = -\sum_{\forall k} c_{13k} \; , \\ Z_{\dot{z}k} = -c_{12k} \; , \; Z_z = \sum_{\forall k} K_{11k} \; , \\ Z_\theta = -\sum_{\forall k} K_{13k} - \frac{1}{2}\rho V^2 S (C_z)_{,\theta} - \dot{V} \sum_{\forall k} m_k \; , \\ Z_{zk} = -K_{12k} \; . \end{cases} \tag{30.202}$$

Fourth equation: roll rotation

Operating with the same methods used for the second equation, and linearizing the kinematic equations, the generalized forces may be written as functions of the variables of motion. The final form of the fourth equation can thus be obtained,

$$\begin{aligned} J_x \ddot{\phi} - J_{xz} \ddot{\psi} - J_{s3} \dot{v}_y = L_v v_y + L_{\dot{\psi}} \dot{\psi} + L_{\dot{\phi}} \dot{\phi} + L_\phi \phi + \\ + \sum_{\forall k} L_{\dot{\phi} k} \dot{\phi}_k + \sum_{\forall k} L_{\phi k} \phi_k + L_\delta \delta \; , \end{aligned} \tag{30.203}$$

where

$$\begin{cases} L_v = -\frac{1}{2}\rho V_a S \left[h (C_y)_{,\beta} + t (C_{M_x})_{,\beta} \right] - \frac{1}{V} \sum_{\forall k} C_k z_{1k} \; , \\ L_{\dot{\psi}} = -\frac{1}{V} \sum_{\forall k} x_{wk} z_{1k} C_k + \sum_{\forall k} 2V \frac{1}{R_{ek}} J_{pwk} + J_{s3} V \; , \\ L_{\dot{\phi}} = -\sum_{\forall k} c_{\phi k} \; , \\ L_\phi = M_{g3} - \sum_{\forall k} \left[K_{\phi k} + C_k (\delta_k)_{,\phi} z_{1k} + C_{\gamma k} (\gamma_k)_{,\phi} z_{1k} \right] \; , \\ L_{\dot{\phi} k} = c_{\phi k} \; , \\ L_{\phi k} = K_{\phi k} - C_{\gamma k} (\gamma_k)_{,\phi k} z_{1k} + C_k (\delta_k)_{,\phi} z_{1k} \; , \\ L_\delta = \sum_{\forall k} \left(K'_k C_k z_{1k} + 2 \frac{\dot{V}}{R_e} \beta_{21} \right) \; . \end{cases} \tag{30.204}$$

Fifth equation: pitch rotation

$$\begin{aligned} J_y \ddot{\theta} - m_s c \ddot{Z} + \dot{V} J_{s2} - \sum_{\forall k} \beta_{16k} \ddot{Z}_k + M_{\dot{z}} \dot{Z} + M_{\dot{\theta}} \dot{\theta} + \sum_{\forall k} M_{\dot{z}k} \dot{Z}_k + \\ + M_z Z + M_\theta \theta + \sum_{\forall k} M_{zk} Z_u + K_{33k} \theta_0 - K_{13k} Z_0 + K_{23k} Z_{k0} = M_{g1} + \\ + \left(M_{y_{aer}} \right)_{\theta=0} - \sum_{\forall k} F_{x1} \left(z_{Cu1} - (x_{i1})_{,z} x_{t1} \right) + F_{x2} \left(z_{Cu2} - (x_{i2})_{,z} x_{t2} \right) \; , \end{aligned} \tag{30.205}$$

where

$$\begin{cases} M_{\dot{z}} = Z_{\dot{\theta}} = -\sum_{\forall k} c_{13k} \; , \; M_{\dot{\theta}} = \sum_{\forall k} c_{33k} \; , \\ M_{\dot{z}k} = c_{23k} \; , \; M_z = -\sum_{\forall k} K_{13k} - m_k \dot{V} \; , \\ M_\theta = -\frac{1}{2}\rho V_a^2 S (C'_{M_y})_{,\theta} - M_{g2} + \sum_{\forall k} K_{33k} \; , \\ M_{zk} = K_{23k} + m_k \dot{V} \; . \end{cases} \tag{30.206}$$

Sixth equation: yaw rotation

$$J_z\ddot{\psi} - J_{xz}\ddot{\phi} + \sum_{\forall k}\beta_{14k}\ddot{\phi}_k = N_v v_y + N_{\dot{\psi}}\dot{\psi} + N_{\dot{\phi}}\dot{\phi} + \\ + N_\phi \phi + \sum_{\forall k} N_{\phi k}\phi_k + N_\delta \delta + M_{ze} \; , \tag{30.207}$$

where

$$\begin{cases} N_v = +\frac{1}{2}\rho V_a S l (C'_{M_z})_{,\beta} + \frac{1}{V}\sum_{\forall k}\left[-x_{wk}C_k + (M_{zk})_{,\alpha} + 2J_{pwk}\left(\frac{V}{R_e}\right)^2\right] \; , \\ N_{\dot{\psi}} = \frac{1}{V}\sum_{\forall k}\left[x_{wk}^2 C_k + x_{wk}(M_{z_k})_{,\alpha}\right] \; , \\ N_{\dot{\phi}} = -2V\sum_{\forall k}\frac{1}{R_e}J_{pwk} \; , \\ N_\phi = -J_{s3}\dot{V} - \frac{1}{2}\rho V^2 S h C_x + \sum_{\forall k}\left[x_{wk}C_k(\delta_k)_{,\phi} + x_{wk}C_{\gamma k}(\gamma_k)_{,\phi}\right. \\ \qquad\qquad \left. -(M_{zk})_{,\alpha}(\delta_k)_{,\phi} - 2\dot{V}\frac{1}{R_e}J_{pwk}\right] \; , \\ N_{\phi k} = x_{wk}C_{\gamma k}(\gamma_k)_{,\phi} - m_k\dot{V}\beta_{5k} - x_{wk}C_k(\delta_k)_{,\phi} + (M_{z_k})_{,\alpha}(\delta_k)_{,\phi} \; , \\ N_\delta = \sum_{\forall k}\left[x_{wk}K'_k C_k - (M_{zk})_{,\alpha}\right] \; . \end{cases} \tag{30.208}$$

Seventh and ninth equations: translations of axles

$$m_k\ddot{Z}_k - \beta_{16k}\ddot{\theta} + Z_{kzk}Z_k + Z_{zk}Z + M_{zk}\theta + \\ + Z_{k\dot{z}k}\dot{Z}_k + Z_{\dot{z}k}\dot{Z} + M_{\dot{z}k}\dot{\theta} + \left(K_{22k} + K_{pzk}\right)Z_{k0} + \\ - K_{12k}Z_0 + K_{23k}\theta_0 = -m_k\dot{V}\beta_{3k} - g m_k \; , \tag{30.209}$$

where

$$Z_{k\dot{z}k} = c_{22k} + c_{pzk} \; , \; Z_{kzk} = K_{22k} + K_{pzk} \tag{30.210}$$

and the other coefficients have already been defined.

Eighth and tenth equation: rotation of axles

$$\beta_{13k}\ddot{\phi}_k + \beta_{14k}\ddot{\psi} - m_{uk}\dot{v}_y\beta_{5k} = \\ = L_{k\beta}\beta + L_{k\dot{\psi}}\dot{\psi} + L_{k\phi}\phi + L_{k\dot{\phi}}\dot{\phi} + L_{k\phi k}\phi_i + L_{k\dot{\phi}k}\dot{\phi}_i + L_{k\delta}\delta \; ,$$

where

$$\begin{cases} L_{kv} = -\frac{1}{V} z_{2k} C_k \ , \\ L_{k\dot{\psi}} = +m_k V \beta_{5k} - \frac{1}{V} x_{wk} z_{2k} C_k \ , \\ L_{k\dot{\phi}} = L_{\dot{\phi}k} \ , \\ L_{k\dot{\phi}k} = -\left(c_{\phi k} + c_{p\phi k}\right) \ , \\ L_{k\phi} = +K_{\phi k} - z_{2k}\left[C_k(\delta_k)_{,\phi} + C_{\gamma k}(\gamma_k)_{,\phi}\right] \ , \\ L_{k\phi k} = +g m_k \beta_{23k} - \left(K_{\phi k} + K_{p\phi k}\right) - z_{2k}\left[C_{\gamma k}(\gamma_k)_{,\phi} - C_k(\delta_k)_{,\phi}\right] \ , \\ L_{k\delta} = 2K'_k \dot{V} \frac{1}{R_{ek}} \beta_{20k} \ . \end{cases} \tag{30.211}$$

30.3.7 Handling-Comfort Uncoupling

The 10 equations of motion ($6 + 2n$ equations in the generic case of a vehicle with n axles) obtained in the previous section constitute a set of linear second-order differential equations, even if the order of such a set is only 17 ($9 + 4n$) because three of the unknowns, namely x^*, y^* and ψ, are present only with their derivatives V, v_y and $\dot{\psi}$.

However, a detailed examination of such equations shows clearly that, if the speed V of the vehicle (which in the linearized model may be confused with its component v_x along the x^* axis) is a known function of time, the equations form two completely uncoupled sets of 5 ($3 + n$) equations each.

The first set contains only the generalized coordinates y^*, ψ, ϕ and ϕ_k (y^* is not a true but a pseudo-coordinate): as a consequence, it deals with the lateral behavior of the vehicle, or, as is usually said, its handling.

The second set contains the generalized coordinates x^*, Z, θ and Z_k, dealing with the "suspension motion" of the vehicle—its ride behavior. This set can be further uncoupled by separating the first equation, that regarding x^* coordinate (i.e. dealing with the longitudinal dynamics of the vehicle), and the following ($2 + n$) equations containing coordinates Z, θ and Z_k which allow ride comfort in a proper sense to be studied.

This uncoupling is an interesting result, even if it is strictly linked with a number of assumptions and, as a consequence, becomes inapplicable if one of them is dropped. The first assumption is the existence of a plane of symmetry, the xz plane. Usually the lack of inertial symmetry of the structure and the differences between the characteristics of the individual springs and shock absorbers located at opposite sides of the vehicle are small enough to be neglected. However, it can happen that the payload of the vehicle is placed asymmetrically, leading to a position of the centre of mass outside the symmetry plane and to non-vanishing moments of inertia J_{xy} and J_{yz}.

A second assumption is that of a perfect linearity of the behavior of the springs and shock absorbers. The linearity of the elastic behavior of springs and tires is an acceptable assumption in the motion about any equilibrium position, provided that its amplitude is small. The nonlinearity of the shock absorbers, on the other hand, cannot in principle be neglected even in the motion "in the small" if their force-velocity characteristic is unsymmetrical, because in the jounce and rebound movements they act with different damping coefficients even if the amplitude of the motion tends to zero. This issue has already been dealt with in detail in Part IV.

A third assumption regards angles β, α_k, θ_0, θ, ϕ and ϕ_k, which must be small enough to allow the linearization of their trigonometric functions. This assumption holds only for small displacements from the equilibrium position and also depends on the characteristics of the vehicle: The harder the suspensions, the more extended the range in which this assumption holds. In general, the mentioned angles are small enough in all normal driving conditions, except for vehicles with two wheels that may operate with large roll angles.

The linearization of the tire behavior in terms of the generation of longitudinal and cornering forces and aligning torques is not strictly required for uncoupling: Even if nonlinear laws $F_y(\alpha)$, $F_y(\gamma)$, $M_z(\alpha)$, etc. are introduced into the equations of motion, the two sets of equations for handling and ride would remain uncoupled, although nonlinear. This last statement is important, because the linear model for the behavior of the tires holds only for values of angles α and γ far smaller than those allowing the trigonometric functions to be linearized.

The kinetic energy linked with wheel rotation was taken into account in the model, with gyroscopic torques due to the wheels included in the equations. If their plane of rotation is close to the xz plane, this effect does not prevent uncoupling.

Some assumptions have been made on the modelling of the suspensions that are better suited for solid axles than for independent suspensions. While unavoidable kinematic errors cannot be accounted for in this way, it will be shown that this does not affect uncoupling.

The interaction between cornering forces and loads in the x and z direction on the tires should actually couple all equations. If the same approximated approach used for rigid vehicles is also adopted in the present case, however, it is possible to resort to uncoupled equations.

The uncoupled model, even if it represents only a first approximation, is important for two reasons. First, it sheds light on the actual behavior of road vehicles and gives a theoretical foundation to the practice of using separate approximate models for the study of handling and ride characteristics. Second, simple linearized models, allowing closed form solutions to be obtained, are well suited for optimization and parametric studies.

Clearly, there is no need to uncouple the equations. Comprehensive, detailed nonlinear models can be used if numerical simulations are performed. The limit in this case may well be the unavailability of good estimates of the numerical values of many parameters that must be entered into the equations.

30.3.8 Handling of a Vehicle on Elastic Suspensions

The explicit formulation of the mathematical model for the handling of a vehicle with two axles is then

$$\mathbf{M}_1\ddot{\mathbf{q}}_1 + \mathbf{C}_1\dot{\mathbf{q}}_1 + \mathbf{K}_1\mathbf{q}_1 = \mathbf{F}_1 \ , \tag{30.212}$$

where

$$\mathbf{q}_1 = \left[\, y^* \ \psi \ \phi \ \phi_1 \ \phi_2 \,\right]^T ,$$

$$\mathbf{M}_1 = \begin{bmatrix} m & 0 & -J_{s3} & -m_1\beta_{5,1} & -m_2\beta_{5,2} \\ & J_z & -J_{xz} & \beta_{14,1} & \beta_{14,2} \\ & & J_x & 0 & 0 \\ & & & \beta_{13,1} & 0 \\ & \text{symm.} & & & \beta_{13,2} \end{bmatrix} ,$$

$$\mathbf{C}_1 = \begin{bmatrix} -Y_v & -Y_{\dot{\psi}} & 0 & 0 & 0 \\ -N_v & -N_{\dot{\psi}} & -N_{\dot{\phi}} & 0 & 0 \\ -L_v & -L_{\dot{\psi}} & -L_{\dot{\phi}} & -L_{\dot{\phi}_1} & -L_{\dot{\phi}_2} \\ -L_{1v} & -L_{1\dot{\psi}} & -L_{\dot{\phi}_1} & -L_{1\dot{\phi}1} & 0 \\ -L_{2v} & -L_{2\dot{\psi}} & -L_{\dot{\phi}_2} & 0 & -L_{2\dot{\phi}2} \end{bmatrix} ,$$

$$\mathbf{K}_1 = \begin{bmatrix} 0 & 0 & -Y_{\phi} & -Y_{\phi_1} & -Y_{\phi_2} \\ 0 & 0 & -N_{\phi} & -N_{\phi_1} & -N_{\phi_2} \\ 0 & 0 & -L_{\phi} & -L_{\phi_1} & -L_{\phi_2} \\ 0 & 0 & -L_{1\phi} & -L_{1\phi1} & 0 \\ 0 & 0 & -L_{2\phi} & 0 & -L_{1\phi2} \end{bmatrix} ,$$

$$\mathbf{F}_1 = \delta \left[\, Y_\delta \ N_\delta \ L_\delta \ L_{1\delta} \ L_{2\delta} \,\right]^T + \left[\, F_{y_e} \ M_{z_e} \ 0 \ 0 \ 0 \,\right]^T \ .$$

As already stated, coordinates y^* and ψ are present only with their derivatives[7]: The order of the set of differential equations is then 8 instead of 10.

The mass matrix is symmetrical, as can be readily predicted. The other two matrices are not symmetrical; for instance, the damping matrix $\mathbf{C}_1$ contains symmetrical terms, like $L_{\dot{\phi}_1}$ and $L_{\dot{\phi}_2}$ that are linked with the roll damping of the axles, and skew symmetric terms, like

$$2V \frac{1}{R_{ek}} J_{pwk} \ ,$$

due to the gyroscopic moment of the wheels of each axle, contained in $L_{\dot{\psi}}$ and, with opposite sign, in $N_{\dot{\phi}}$. The other terms in J_{pwk} are not due to gyroscopic effects but wheel acceleration (terms in L_δ and $L_{k\delta}$) or the equivalent mass (term in N_v), and thus have no particular symmetry properties. Other terms due to generalized forces,

[7] They have no physical meaning.

such as the terms present in the stiffness matrix, are neither symmetrical nor skew symmetrical.

Even if it is possible to separate the symmetrical and the skew-symmetrical parts of the various matrices (so defining a gyroscopic and a circulatory matrix), the advantages so obtained do not justify the work.

If a state-space approach is used (Eq. (A.5)), by introducing the state variables $p = \dot{\phi}$, $p_1 = \dot{\phi}_1$, $p_2 = \dot{\phi}_2$ and $r = \dot{\psi}$, the relevant vectors and matrices are:

– State vector

$$\mathbf{z} = \left[v \ r \ p \ p_1 \ p_2 \ \phi \ \phi_1 \ \phi_2 \right]^T , \tag{30.213}$$

– Dynamic matrix

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}_1^{-1}\mathbf{C}_1 & -\mathbf{M}_1^{-1}\mathbf{K}_1^* \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} , \tag{30.214}$$

where $\mathbf{K}_1^*$ is matrix $\mathbf{K}_1$ with the first two columns cancelled.

– input gain matrix

$$\mathbf{B} = \begin{bmatrix} -\mathbf{M}_1^{-1} \begin{bmatrix} Y_\delta & 1 & 0 \\ N_\delta & 0 & 1 \\ L_\delta & 0 & 0 \\ L_{1_\delta} & 0 & 0 \\ L_{2_\delta} & 0 & 0 \end{bmatrix} \\ [0]_{3\times 3} \end{bmatrix} , \tag{30.215}$$

– input vector

$$\mathbf{u} = \left[\delta \ F_{y_e} \ M_{z_e} \right]^T . \tag{30.216}$$

This approach may be used for the study of the stability of the vehicle or for computing its response to the various inputs, as previously seen for rigid vehicle models. This model is only marginally more complex if numerical solutions are searched.

Even if the complexity of the model is not a factor, it is interesting to perform a simplification allowing one to reduce its size without sacrificing its applicability to actual problems. Because the stiffness of the tires in the z direction is much higher than that of the suspensions, their compliance becomes important only in high frequency motions, much higher than the frequencies involved in the handling of the vehicle. As a consequence, if the compliance of the tires is neglected, which amounts to stating that ϕ_1 and ϕ_2 and their derivatives are vanishingly small, the model reduces to a set of three equations (four first-order equations in the state-space approach) that retains most of the features of the complete, five equation set.

Because yawing moments due to load shift were not included in the present model, the three equations of motion may be obtained directly from those of the previous model by stating

$$\phi_1 = \phi_2 = 0 .$$

The sideslip angle β is often used in handling models instead of the lateral velocity v_y as a variable of motion, and the yaw velocity is indicated as r. Remembering that

$$v_y = V\beta , \dot{\psi} = r ,$$

it follows that

$$\begin{cases} mV\dot{\beta} - J_{s3}\ddot{\phi} = \left(VY_v - m\dot{V}\right)\beta + Y_{\dot{\psi}}r + Y_\phi\phi + Y_\delta\delta + F_{y_e} , \\ J_z\dot{r} - J_{xz}\ddot{\phi} = N_vV\beta + N_{\dot{\psi}}r + N_{\dot{\phi}}\dot{\phi} + N_\phi\phi + N_\delta\delta + M_{ze} , \\ J_x\ddot{\phi} - J_{xz}\dot{r} - J_{s3}V\dot{\beta} = \left(L_vV + J_{s3}\dot{V}\right)\beta + L_{\dot{\psi}}r + L_{\dot{\phi}}\dot{\phi} + L_\phi\phi + L_\delta\delta , \end{cases} \tag{30.217}$$

where terms Y_vV, N_vV and L_vV are often written as Y_β, N_β and L_β.

If the terms in $\dot{V}$ are dropped, the same set of equations frequently described in the literature[8] is obtained. There is, however, a difference: The model described here is obtained from the complete model of the vehicle with elastic suspensions through uncoupling and controlled simplifications, while that model is obtained through a number of more or less arbitrary assumptions. Moreover, this model accounts for the rotation of the wheels.

The study of either the stability or the response to a steering input or external force or moment is straightforward and follows the same lines seen for the rigid vehicle. Here the presence of an equation containing the first and second derivative of a generalized coordinate ϕ together with the coordinate itself may induce an oscillatory behavior. If roll oscillations are strongly coupled with those of the other variables of the motion (namely β and r), as may be caused by roll steer, the overall behavior may become strongly oscillatory and dynamic stability may be decreased.

The steady-state response of the vehicle is easily obtained from the following set of algebraic equations:

$$\begin{cases} m\dfrac{V^2}{R} = Y_\beta\beta + Y_{\dot{\psi}}\dfrac{V}{R} + Y_\phi\phi + Y_\delta\delta + F_{y_e} , \\ 0 = N_\beta\beta + N_{\dot{\psi}}\frac{V}{R} + N_\phi\phi + N_\delta\delta + M_{z_e} , \\ -J_{s3}\dfrac{V^2}{R} = L_\beta\beta + L_{\dot{\psi}}\dfrac{V}{R} + L_\phi\phi + L_\delta\delta , \end{cases} \tag{30.218}$$

where the steady-state curvature of the trajectory

$$\frac{1}{R} = \frac{r}{V}$$

has been explicitly introduced.

[8] See for example W. Steeds, *Mechanics of Road Vehicles*, ILIFFE & Sons, London, 1960.

By solving equations (30.218) in $1/R$ and neglecting external forces, the path curvature gain $1/R\delta$ is readily obtained,

$$\frac{1}{R\delta} = \frac{DC - AE}{V(BC - AF)} , \tag{30.219}$$

where

$$\begin{array}{ll} A = N_\beta L_\phi - N_\phi L_\beta, & B = J_{s3} V N_\phi - N_{\dot{\psi}} L_\phi + N_\phi L_{\dot{\psi}}, \\ C = Y_\beta N_\phi - Y_\phi N_\beta, & D = N_\delta L_\phi - N_\phi L_\delta, \\ E = Y_\delta N_\phi - Y_\phi N_\delta, & F = mVN_\phi - Y_{\dot{\psi}} N_\phi + Y_\phi N_{\dot{\psi}} . \end{array}$$

30.3.9 Ride Comfort

The explicit formulation of the mathematical model for ride comfort of a vehicle with two axles is

$$\mathbf{M}_2 \ddot{\mathbf{q}}_2 + \mathbf{C}_2 \dot{\mathbf{q}}_2 + \mathbf{K}_2 \mathbf{q}_2 + \mathbf{K}_{2st} \mathbf{q}_{2st} = \mathbf{F}_2 + \mathbf{F}_{2st} , \tag{30.220}$$

where

$$\mathbf{q}_2 = \left[x^* \; Z \; \theta \; Z_1 \; Z_2 \right]^T ,$$

$$\mathbf{q}_{2st} = \left[0 \; Z_0 \; \theta_0 \; Z_{10} \; Z_{20} \right]^T ,$$

$$\mathbf{M}_2 = \begin{bmatrix} m_{at} & 0 & J_{s2} & 0 & 0 \\ & m_s & -m_s c & 0 & 0 \\ & & J_y & -\beta_{16,1} & -\beta_{16,2} \\ & & & m_1 & 0 \\ & \text{symm.} & & & m_2 \end{bmatrix} ,$$

$$\mathbf{C}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & Z_{\dot{z}} & Z_{\dot{\theta}} & Z_{\dot{z}1} & Z_{\dot{z}2} \\ 0 & & M_{\dot{\theta}} & M_{\dot{z}1} & M_{\dot{z}2} \\ 0 & & & Z_{1\dot{z}1} & 0 \\ 0 & & \text{symm.} & & Z_{2\dot{z}2} \end{bmatrix} ,$$

$$\mathbf{K}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & Z_z & Z_\theta & Z_{z1} & Z_{z2} \\ 0 & M_z & M_\theta & M_{z1} & M_{z2} \\ 0 & Z_{z1} & M_{z1} & Z_{1z1} & 0 \\ 0 & Z_{z2} & M_{z3} & 0 & Z_{2z2} \end{bmatrix}$$

$$\mathbf{K}_{st} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & K_{11} & K_{13} & -K_{12,1} & -K_{12,2} \\ 0 & & K_{33} & K_{23,1} & K_{23,2} \\ 0 & & & K_{22,1} & 0 \\ 0 & & \text{symm.} & & K_{22,2} \end{bmatrix},$$

$$\mathbf{F}_2 = \left\{ \begin{array}{c} F_{x1} + F_{x2} - \frac{1}{2}\rho V^2 S C_x \\ \frac{1}{2}\rho V^2 S (C_Z)_{\theta=0} + \dot{V} \sum_{\forall k} \left[m_k \beta_{3k} - (x_{ik})_{,z} F_{xk} \right] \\ \left(M_{y_{aer}}\right)_{\theta=0} - \sum_{\forall k} F_{xk} \left(z_{Cuk} - (x_{ik})_{,z} x_{tk} \right) \\ -m_1 \dot{V} \beta_{3,1} \\ -m_2 \dot{V} \beta_{3,2} \end{array} \right\},$$

$$\mathbf{F}_{2st} = \left[0 \ -m_s g \ M_{g1} \ -g m_1 \ -g m_2 \right]^T .$$

In the reference condition, all variables of motion included in vector $\mathbf{q}_2$ vanish. It then follows that

$$\mathbf{K}_{2st} \mathbf{q}_{2st} = \mathbf{F}_{2st} . \tag{30.221}$$

Equation (30.221) allows the values of Z_0, θ_0, etc.,—the static equilibrium condition—to be computed. Although it is arbitrary to use the linearized equation for computing the static equilibrium condition, because Z_0 and Z_{k0} are not, generally, small quantities, this approximation influences the reference condition so obtained but is immaterial for the study of the small oscillations about that condition and thus does not detract from the dynamic study in the small here shown.

By introducing Eq. (30.221) into Eq. (30.220), it follows that

$$\mathbf{M}_2 \ddot{\mathbf{q}}_2 + \mathbf{C}_2 \dot{\mathbf{q}}_2 + \mathbf{K}_2 \mathbf{q}_2 = \mathbf{F}_2 . \tag{30.222}$$

The mass and damping matrices are symmetrical. The stiffness matrix is symmetrical except for the terms in position 23 and 32: in Z_θ a term of aerodynamic origin is present, due to changes to aerodynamic lift caused by the pitch angle that is absent from M_z. A similar term in M_z would denote a change in the pitching moment due to vertical displacements that does not exist.

The first equation of the second set of five differential equations, that related to the longitudinal dynamics, is weakly coupled with the others and may be written in the form

$$\dot{V} = \frac{J_s \ddot{\theta} + \sum_{\forall i} F_{xi} + \frac{1}{2}\rho V_a^2 S C_x}{m} . \tag{30.223}$$

By introducing Eq. (30.223) into the other equations, the following set of four equations describing the suspension motions of a vehicle with two axles is obtained,

$$\mathbf{M}_3 \ddot{\mathbf{q}}_3 + \mathbf{C}_3 \dot{\mathbf{q}}_3 + \mathbf{K}_3 \mathbf{q}_3 = \mathbf{F}_3 , \tag{30.224}$$

where

$$\mathbf{q}_3 = \left[Z \ \theta \ Z_1 \ Z_2 \right]^T ,$$

$$\mathbf{M}_3 = \begin{bmatrix} m_s & -m_s c & 0 & 0 \\ & J_y - \dfrac{J_{s2}^2}{m_{at}} & 0 & 0 \\ & & m_1 & 0 \\ & \text{symm.} & & m_2 \end{bmatrix} ,$$

matrices $\mathbf{C}_3$ and $\mathbf{K}_3$ coincide with matrices $\mathbf{C}_2$ and $\mathbf{K}_2$ without the first row and column, and

$$\mathbf{F}_3 = \begin{Bmatrix} \frac{1}{2}\rho V^2 S(C_Z)_{\theta=0} + \dot{V} \sum_{\forall k} \left[m_k \beta_{3k} - (x_{ik})_{,z} F_{xk} \right] \\ \frac{1}{2}\rho V_a^2 S \left[-\frac{J_{s2}}{m_{at}} C_x + l(C_{M_y})_{\theta_0} \right] - \sum_{\forall k} F_{xk} \left(z_{Cuk} - (x_{ik})_{,z} x_{tk} + \frac{J_{s2}}{m_{at}} \right) \\ -m_1 \dot{V} \beta_{3,1} \\ -m_2 \dot{V} \beta_{3,2} \end{Bmatrix} .$$

The expression for the generalized forces $\mathbf{F}_3$ was obtained assuming that the reference configuration corresponds to the static equilibrium position with the vehicle at standstill and with no force F_{x_i}. In such a condition, all generalized coordinates are equal to zero, because they were defined as displacements from the same condition.

The equations were written with reference to coordinate Z, i.e., to the changes of the vertical displacement of point H in Fig. 30.3, which results in an inertial coupling. To study ride comfort it is better to refer to the vertical displacements of point H′, so that the equations of motion have no inertial coupling, i.e. the mass matrix is diagonal. By introducing the coordinate

$$z_s = Z - c(\theta + \theta_0)$$

of point H′, the mass matrix becomes

$$\mathbf{M}_3 = \begin{bmatrix} m_s & 0 & 0 & 0 \\ & J_y^* & 0 & 0 \\ & & m_1 & 0 \\ & symm. & & m_2 \end{bmatrix} , \qquad (30.225)$$

where

$$J_y^* = J_y - c^2 m_s - \frac{J_{s2}^2}{m_e} .$$

The damping and stiffness matrices are unchanged, provided that the distances x_i of wheels, springs and dampers are substituted by $x_i - c$. All matrices, except the stiffness matrix, remain symmetrical.

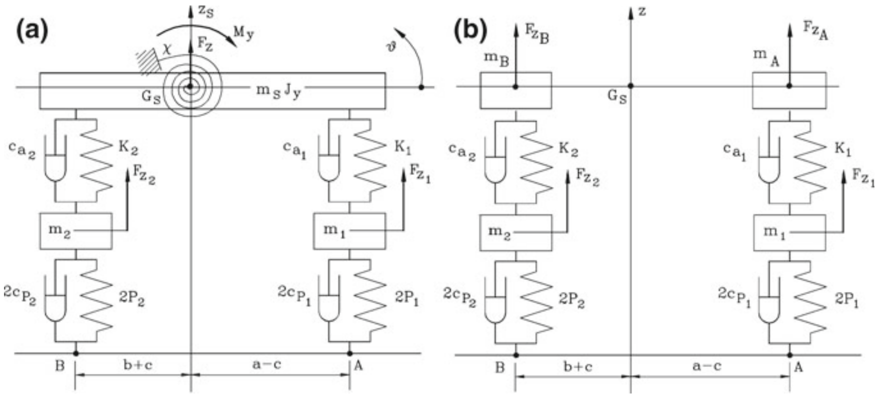

Fig. 30.8 **a** Model with four degrees of freedom for the study of ride comfort; **b** model in which the sprung mass is simulated by two separate masses. Lengths a and b are the same as for the rigid vehicle, $a = x_1$ and $b = -x_2$. Note that the longitudinal positions of the springs and shock absorbers are assumed to be coincident ($x_i = x_{m_i} = x_{a_i}$)

If the aerodynamic term causing the lack of symmetry of the stiffness matrix is neglected, which introduces only a small error because of the small size of the term, the system may be sketched as in Fig. 30.8a. The vehicle is modelled as a beam with elastic and damped supports, connected to the ground through the unsprung masses.

The quasi-static equilibrium attitude of the vehicle, which is different from the reference position because it takes into account both longitudinal forces on the tires and aerodynamic forces, can be immediately obtained from the steady-state solution of Eq. (30.224). Even if the acceleration of the vehicle does not appear explicitly in the equations, it is accounted for through forces F_{xi}.

The dynamic response of the vehicle to motion on uneven road is easily computed by assuming that points A and B move in a vertical direction with laws

$$h_A(t) = h(Vt) \text{ and } h_B(t) = h(Vt + l) ,$$

where $h(x)$ is a function expressing the road profile. This amounts to exciting the two masses m_1 and m_2 with two forces equal to

$$K_{pz1} h_A(t) + c_{pz_1} \dot{h}_A(t) \text{ and } K_{pz2} h_B(t + \tau) + c_{pz2} \dot{h}_B(t + \tau)$$

respectively, where $\tau = V/l$ is the delay due to the wheelbase.

30.3.10 Conclusions

The linearized model with 10 degrees of freedom for a vehicle with two axles, or more generally the model with $6 + 2n$ degrees of freedom for a vehicle with n axles, splits into three separate models, namely

- *Model for longitudinal behavior*, or performance model. The model includes a single degree of freedom, coordinate x^* (or better the forward speed V, because on a curved trajectory x^* is not a true coordinate), and allows the relationship between the longitudinal forces at the wheel-road contact and the vehicle speed to be computed, along with acceleration and braking performance. A detailed model of the tires may be introduced if their longitudinal slip is accounted for, as well as a model of the transmission and possibly of the engine. A model of the driver, intended as controller of the longitudinal motion through the accelerator and brake pedals, may also be introduced. Note that, owing to linearization, the longitudinal behavior on a curved trajectory coincides with that on straight road and thus with that studied in Chap. 23.
- *Model for lateral behavior*, or handling model. This model includes the degrees of freedom of lateral displacement (or better, lateral velocity, for the reasons cited above) and the yaw angle, which are the same degrees of freedom seen for the study of the handling of a rigid vehicle, plus the degrees of freedom related to the rolling of the vehicle body and of the axles. In such a model the input is the steering angle, but this can be easily modified to study the motion with free controls, possibly introducing a driver model as a steering controller as well. It has been assumed that the variations of the steering angle are slow enough to neglect its derivative $\dot{\delta}$. The presence of gyroscopic torques has no effect on the uncoupling between handling and comfort models, because a mass rotating about the y axis couples yaw and roll motions, both belonging to this model.
- *Model for suspension motions*, or ride comfort model. This model includes the degrees of freedom for vertical motion of the body (heave motion) and the axles, plus the pitch angle. Uncoupling between the longitudinal and comfort model is not complete, as shown by the pitching motions due to braking (dive) or driving (lift or squat). In the present chapter, the changes of longitudinal acceleration have been assumed to occur slowly. The changes of pitch angle may therefore be considered as a quasi static phenomenon, introduced into the equations by the longitudinal tire-road contact forces. In cases where longitudinal forces change quickly, ride comfort and longitudinal behavior (as well as transmission behavior) must be studied jointly.

30.4 Models of Deformable Vehicles

The assumption that the vehicle body can be considered as a rigid body is clearly an approximation that may be, in some cases, quite rough. This is particularly true of industrial vehicles and some passenger vehicles, such as open cars, whose stiffness is lower than usual.

If the body of the vehicle is not stiff, the position of any point P in the same inertial reference frame $OX_iY_iZ_i$ shown in Fig. 30.3 and already used to study the model based on rigid bodies, may be written in the form

$$\overline{(\text{P}-\text{O'})} = \overline{(\text{P}_\text{u}-\text{O'})} + \mathbf{s}_\text{P} \,, \tag{30.226}$$

where $\overline{(\text{P}_\text{u}-\text{O'})}$ is the position of point P obtained by neglecting the deformation of the body and

$$\mathbf{s}_\text{P} = [u_x, u_y, u_z]_\text{P}^T \,, \tag{30.227}$$

is the displacement function of time, due to compliance.

The position of P in undeflected conditions may be expressed by an equation similar to Eq. (30.11), and then

$$\overline{(\text{P}-\text{O'})} = \overline{(\text{H}-\text{O'})} + \mathbf{R}\left(\mathbf{r}_\text{P} + \mathbf{s}_\text{P}\right) \,, \tag{30.228}$$

where vector

$$\mathbf{r}_\text{P} = [x,\ y,\ z]_\text{P}^T \,, \tag{30.229}$$

is the vector, independent from time, leading from point H to point P, expressed in the reference frame of the sprung mass.

The velocity of point P is then

$$\mathbf{V}_\text{P} = \left[\,\dot{X}\ \dot{Y}\ \dot{Z}\,\right]^T + \dot{\mathbf{R}}\left(\mathbf{r}_\text{P} + \mathbf{s}_\text{P}\right) + \mathbf{R}\dot{s}_\text{P} \tag{30.230}$$

or, by introducing the velocity $\mathbf{V}$ in the reference frame x^*y^*z:

$$\mathbf{V}_\text{P} = \mathbf{R}_1\mathbf{V} + \dot{\mathbf{R}}\left(\mathbf{r}_\text{P} + \mathbf{s}_\text{P}\right) + \mathbf{R}\dot{s}_\text{P} \,. \tag{30.231}$$

By remembering that pitch and roll angles are small, and introducing matrix

$$\mathbf{R}_{23} = \mathbf{R}_2\mathbf{R}_3 \approx \begin{bmatrix} 1 & 0 & \theta_0 + \theta \\ 0 & 1 & -\phi \\ -\theta_0 - \theta & \phi & 1 \end{bmatrix} ,$$

it is possible to write

$$\mathbf{V}_\text{P} = \mathbf{R}_1\left[\mathbf{V} + \left(\dot{\mathbf{R}}_{23} + \dot{\psi}\mathbf{S}\mathbf{R}_{23}\right)\left(\mathbf{r}_\text{P} + \mathbf{s}_\text{P}\right) + \mathbf{R}_{23}\dot{\mathbf{s}}_\text{P}\right] \,, \tag{30.232}$$

where

$$\mathbf{S} \approx \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

The kinetic energy of the infinitesimal element of massat coordinates x, y, z is

$$\begin{gathered} d\mathcal{T} = \frac{1}{2}dm \Big[\mathbf{V}^T\mathbf{V} + \mathbf{r}_\mathrm{P}^T \left(\dot{\mathbf{R}}_{23} + \dot{\psi}\mathbf{S}\mathbf{R}_{23}\right)^T \left(\dot{\mathbf{R}}_{23} + \dot{\psi}\mathbf{S}\mathbf{R}_{23}\right)\mathbf{r}_\mathrm{P} + \\ +2\mathbf{V}^T \left(\dot{\mathbf{R}}_{23} + \dot{\psi}\mathbf{S}\mathbf{R}_{23}\right)\mathbf{r}_\mathrm{P} + \mathbf{s}_\mathrm{P}^T \left(\dot{\mathbf{R}}_{23} + \dot{\psi}\mathbf{S}\mathbf{R}_{23}\right)^T \left(\dot{\mathbf{R}}_{23} + \dot{\psi}\mathbf{S}\mathbf{R}_{23}\right)\mathbf{s}_\mathrm{P} + \\ +2\mathbf{V}^T \left(\dot{\mathbf{R}}_{23} + \dot{\psi}\mathbf{S}\mathbf{R}_{23}\right)\mathbf{s}_\mathrm{P} + \dot{\mathbf{s}}_\mathrm{P}^T\mathbf{R}_{23}^T\mathbf{R}_{23}\dot{\mathbf{s}}_\mathrm{P} + 2\mathbf{V}^T\mathbf{R}_{23}\dot{\mathbf{s}}_\mathrm{P} + \\ +2\dot{\mathbf{s}}_\mathrm{P}^T\mathbf{R}_{23}^T \left(\dot{\mathbf{R}}_{23} + \dot{\psi}\mathbf{S}\mathbf{R}_{23}\right)(\mathbf{r}_\mathrm{P} + \mathbf{s}_\mathrm{P})\Big] . \end{gathered} \tag{30.233}$$

The sum of the first three terms (those not containing the deformation $\mathbf{s}_\mathrm{P}$ or its derivatives) is the kinetic energy $d\mathcal{T}_R$ of the same mass element in a rigid motion. The other terms may be greatly simplified if the products of more than two small quantities are neglected. Because the deformations $\mathbf{s}_\mathrm{P}$ and their derivatives are small quantities, it follows that

$$\begin{gathered} d\mathcal{T} = d\mathcal{T}_R + \frac{1}{2}dm \left(\dot{u}_x^2 + \dot{u}_y^2 + \dot{u}_z^2\right) + dm\, v_x \left[\dot{\theta}u_z - \dot{\psi}u_y + \right. \\ \left. +\dot{u}_x + (\theta + \theta_o)\,\dot{u}_z\right] + dm\,\dot{u}_x\left(\dot{\theta}z - \dot{\psi}y\right) + \\ +dm\,\dot{u}_y\left(v_y - \dot{\phi}z + \dot{\psi}x\right) + dm\,\dot{u}_z\left(v_z - \dot{\theta}x + \dot{\phi}y\right) . \end{gathered} \tag{30.234}$$

The deformation of the sprung mass can be expressed in terms of its modal coordinates, i.e., as a linear combination of the eigenfunctions of the undamped system. Note that this remains true even if the sprung mass is damped and even for nonlinear systems. Because the sprung mass has a plane of symmetry (in the present case the xz plane), its modes may be subdivided into symmetrical and skew-symmetrical modes, designated by subscripts s and a in the following equations.

The displacements $\mathbf{s}_\mathrm{P} = [u_x, u_y, u_z]^T$ of point P(x, y, z) can thus be expressed as

$$\left\{\begin{array}{c} u_x \\ u_y \\ u_z \end{array}\right\} = \mathbf{Q}_s(x, y, z)\boldsymbol{\eta}_s(t) + \mathbf{Q}_a(x, y, z)\boldsymbol{\eta}_a(t) , \tag{30.235}$$

where $\mathbf{Q}_i(x, y, z)$ are matrices containing the eigenfunctions while $\boldsymbol{\eta}_i(t)$ are vectors containing the modal coordinates. Equation (30.235) is exact only if an infinity of eigenfunctions and modal coordinates are considered; however, a very good approximation is usually obtained by taking into account a small number of modes, particularly if the system is linear and lightly damped. The modes considered in the equation are those of the free structure; the rigid-body modes need not be considered because they have already been included in the rigid-body analysis already performed.

Instead of using the eigenfunctions, a set of arbitrary functions of the space coordinates may be used, as is common in the assumed modes methods for structural analysis; in this case, however, the number of coordinates needed to obtain a good approximation is higher and depends on the choice of the arbitrary functions: Moreover, the mass and stiffness matrices are not diagonal, as they are when using the eigenfunctions.

Let A and B be two points located in symmetrical positions with respect to the xz plane. It follows that

$$u_{x_A} = u_{x_B} \ , \ u_{y_A} = -u_{y_B} \ , u_{z_A} = u_{z_B}$$

for symmetrical modes and

$$u_{x_A} = -u_{x_B} \ , \ u_{y_A} = u_{y_B} \ , u_{z_A} = -u_{z_B}$$

for skew-symmetrical ones. As a consequence of symmetry, some relevant integrals extended to the whole unsprung mass may be written in a much simplified form

$$\begin{gathered}
\int_m \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} dm = \begin{Bmatrix} \mathcal{A} \\ \mathbf{0} \\ \mathcal{C} \end{Bmatrix} \boldsymbol{\eta}_s + \begin{Bmatrix} \mathbf{0} \\ \mathcal{B} \\ \mathbf{0} \end{Bmatrix} \boldsymbol{\eta}_a \ , \\
\int_m x \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} dm = \begin{Bmatrix} \mathcal{N} \\ \mathbf{0} \\ \mathcal{F} \end{Bmatrix} \boldsymbol{\eta}_s + \begin{Bmatrix} \mathbf{0} \\ \mathcal{D} \\ \mathbf{0} \end{Bmatrix} \boldsymbol{\eta}_a \ , \\
\int_m y \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} dm = \begin{Bmatrix} \mathbf{0} \\ \mathcal{H} \\ \mathbf{0} \end{Bmatrix} \boldsymbol{\eta}_s + \begin{Bmatrix} \mathcal{E} \\ \mathbf{0} \\ \mathcal{G} \end{Bmatrix} \boldsymbol{\eta}_a \ , \\
\int_m z \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} dm = \begin{Bmatrix} \mathcal{I} \\ \mathbf{0} \\ \mathcal{L} \end{Bmatrix} \boldsymbol{\eta}_s + \begin{Bmatrix} \mathbf{0} \\ \mathcal{M} \\ \mathbf{0} \end{Bmatrix} \boldsymbol{\eta}_a \ , \\
\int_m \left(u_x^2 + u_y^2 + u_z^2\right) dm = \boldsymbol{\eta}_s^T \overline{\mathbf{M}}_s \boldsymbol{\eta}_s + {}_a^T \overline{\mathbf{M}}_a \boldsymbol{\eta}_a \ ,
\end{gathered} \tag{30.236}$$

where the diagonal matrices $\overline{\mathbf{M}}_s$ and $\overline{\mathbf{M}}_a$ are the modal mass matrices for symmetrical and skew-symmetrical modes, and where matrices from $\mathcal{A}$ to $\mathcal{N}$ are row matrices whose size is $1 \times n$, where n is the number of modal coordinates (either symmetrical or skew-symmetrical) that are considered.

By integrating Eq. (30.234), the kinetic energy of the sprung mass reduces to

$$\begin{gathered}
\mathcal{T} = \mathcal{T}_R + \tfrac{1}{2}\dot{\boldsymbol{\eta}}_s^T \overline{\mathbf{M}}_s \dot{\boldsymbol{\eta}}_s + \tfrac{1}{2}{}_a^{\cdot T} \overline{\mathbf{M}}_a \dot{\boldsymbol{\eta}}_a + \dot{\psi}\left(\mathcal{D} - \mathcal{E}\right)\dot{\boldsymbol{\eta}}_a + \dot{\theta}\left(\mathcal{I} - \mathcal{F}\right)\dot{\boldsymbol{\eta}}_s + \\
+ \dot{\phi}\left(\mathcal{G} - \mathcal{M}\right)\dot{\boldsymbol{\eta}}_a + V\left[\mathcal{A}\dot{\boldsymbol{\eta}}_s + \dot{\theta}\mathcal{C}\boldsymbol{\eta}_s + \left(\theta + \theta_0\right)\mathcal{C}\dot{\boldsymbol{\eta}}_s - \dot{\psi}\mathcal{B}\boldsymbol{\eta}_{\mathbf{a}}\right] + v_y \mathcal{B}\mathbf{P_a} + v_z \mathcal{C}\dot{\boldsymbol{\eta}}_s \ .
\end{gathered} \tag{30.237}$$

The gravitational potential energy of the sprung mass may be expressed in the form

$$\mathcal{U}_{gS} = g \int_m Z_p dm = g \int_m \mathbf{e}_3^T \left[(\overline{\text{H}-\text{O'}}) + \mathbf{R} \left(\mathbf{r}_\text{P} + \mathbf{s}_\text{P} \right) \right] dm \; . \tag{30.238}$$

It then follows that

$$\mathcal{U}_{gS} = g \left(Z + Z_0 \right) m_s + g \int_m \mathbf{e}_3^T \mathbf{R}_{23} \mathbf{r}_\text{P} dm + g \int_m \mathbf{e}_3^T \mathbf{R}_{23} s_\text{P} dm \; . \tag{30.239}$$

The first two terms are the potential energy $\mathcal{U}_{gR}$ of the rigid body computed above. By introducing the linearized expression of matrix $\mathbf{R}_{23}$ it follows that

$$\mathcal{U}_{gS} = \mathcal{U}_{gR} + g \int_m \left[- \left(\theta + \theta_0 \right) u_x + \phi u_y + u_z \right] dm \; , \tag{30.240}$$

or, by introducing the modal coordinates to express the deformation of the vehicle body,

$$\mathcal{U}_{gS} = \mathcal{U}_{gR} + g \left[- \left(\theta + \theta_0 \right) \mathcal{A}_s + \phi \mathcal{B}_a + \mathcal{C}_s \right] .$$

The deformation potential energy of the springs of the kth suspension is still expressed by Eq. (30.79), to which the terms linked with the deformation modes are added. Assuming that the points of attachment of the springs of the left (right) kth suspension are x_i, $\pm y_i$ and z_i, it follows that

$$\mathcal{U}_{mk} = \frac{1}{2} K_\zeta \left(\zeta + \zeta_0 - \mathbf{Q}_{zsk} \boldsymbol{\eta}_s \right)^2 + \frac{1}{2} K_\phi \left(\phi - \phi_k - \mathbf{Q}_{zak} \boldsymbol{\eta}_a \right)^2 \; , \tag{30.241}$$

where $\mathbf{Q}_{zsk}$ and $\mathbf{Q}_{zak}$ are the parts of the matrices of the eigenfunctions for symmetrical and skew-symmetrical modes linked to u_z computed at the point of coordinates x_i, y_i, z_i.

The potential energy of the kth suspension is then

$$\mathcal{U}_{mk} = \mathcal{U}_{mR} + \tfrac{1}{2} \boldsymbol{\eta}_s^T \mathbf{K}_{44} \boldsymbol{\eta}_s + \mathbf{K}_{14} \left(Z + Z_0 \right) \boldsymbol{\eta}_s + \mathbf{K}_{24} \left(Z_k + Z_{k0} \right) \boldsymbol{\eta}_s +$$

$$+ \mathbf{K}_{34} \left(\theta + \theta_0 \right) \boldsymbol{\eta}_s + \tfrac{1}{2} K_\phi \boldsymbol{\eta}_a^T \mathbf{Q}_{zak}^T \mathbf{Q}_{zak} \boldsymbol{\eta}_a - K_\phi \phi \mathbf{Q}_{zak} \boldsymbol{\eta}_a + K_\phi \phi_k \mathbf{Q}_{zak} \boldsymbol{\eta}_a \; , \tag{30.242}$$

where $\mathbf{K}_{44}$ is a square matrix of size $n_s \times n_s$ (where n_s is the number of symmetrical modes considered), while $\mathbf{K}_{14}$, $\mathbf{K}_{24}$ and $\mathbf{K}_{34}$ are row matrices of size $1 \times n_s$.

The deformation potential energy of the tires of the kth suspension is expressed by Eq. (30.81) without any change. The potential energy due to the deformation of the sprung mass is obviously

$$\mathcal{U}_R = \frac{1}{2} \boldsymbol{\eta}_s^T \overline{\mathbf{K}}_s \boldsymbol{\eta}_s + \frac{1}{2} \boldsymbol{\eta}_a^T \overline{\mathbf{K}}_a \boldsymbol{\eta}_a \; , \tag{30.243}$$

where $\overline{\mathbf{K}}_s$ and $\overline{\mathbf{K}}_a$ are the modal stiffness matrices.

In a similar way, the Raleigh dissipation functionof the shock absorbers of the kth suspension is

$$\begin{aligned}\mathcal{F}_{ak} = \mathcal{F}_{akR} + \tfrac{1}{2}\dot{\boldsymbol{\eta}}_s^T \mathbf{c}_{44}\dot{\boldsymbol{\eta}}_s + \mathbf{c}_{14}\dot{Z}\dot{\boldsymbol{\eta}}_s + \mathbf{c}_{24}\dot{Z}_k\dot{\boldsymbol{\eta}}_s + \mathbf{c}_{34}\dot{\theta}\dot{\boldsymbol{\eta}}_s +\\ +\tfrac{1}{2}c_\phi\dot{\boldsymbol{\eta}}_a^T \mathbf{Q}_{zaka}^T \mathbf{Q}_{zak}\dot{\boldsymbol{\eta}}_a - c_\phi\dot{\phi}\mathbf{Q}_{zak}\dot{\mathbf{P}}_a + c_\phi\dot{\phi}_k\mathbf{Q}_{zaka}\dot{\boldsymbol{\eta}}_a\ ,\end{aligned} \tag{30.244}$$

where $\mathcal{F}_{akR}$ is the function seen before for the rigid vehicle, matrices $\mathbf{c}_{ij}$ are similar to matrices $\mathbf{K}_{ij}$ seen above and $\mathbf{Q}_{zaka}$ is similar to $\mathbf{Q}_{zak}$, but referred to the points where the shock absorbers are attached.

The Raleigh dissipation function for the tires (Eq. 30.84) is unchanged, while that linked with deformation modes of the sprung mass is

$$\mathcal{F}_R = \frac{1}{2}\dot{\boldsymbol{\eta}}_s^T\overline{\mathbf{C}}_s\dot{\boldsymbol{\eta}}_s + \frac{1}{2}\dot{\boldsymbol{\eta}}_a^T\overline{\mathbf{C}}_a\dot{\boldsymbol{\eta}}_a\ , \tag{30.245}$$

where $\overline{\mathbf{C}}_s$ and $\overline{\mathbf{C}}_a$ are the modal damping matrices. Note that the last expression is just an approximation, because modal damping matrices are not diagonal and may couple the various modes. Other approximations are linked to the way the presence of suspensions has been accounted for, but such approximations are similar to those already seen for the other linearized models.

If the virtual work of the external forces is computed neglecting displacements due to deformation modes, the expressions of the generalized forces are the same as those used for models based on rigid bodies. Such an assumption is well suited to the present linearized model, where the exact kinematics of suspensions has not been taken into account.

A detailed inspection of the expressions of the kinetic and potential energies and of the dissipation function shows that, if the forward velocity V is assumed to be a known function, the equations of motion divide into two separate sets, exactly as they do when the compliance of the sprung mass is neglected.

30.4.1 Handling Model

A first set of equations contains generalized coordinates y^*, ψ, ϕ, ϕ_i and $\boldsymbol{\eta}_a$. If the vehicle has n axles and n_a skew-symmetrical modes are considered, their number is $3 + n + n_a$. The differential equations modelling the lateral behavior are

$$\begin{aligned}\begin{bmatrix}\mathbf{M}_1 & \mathbf{M}_{a1}^T\\ \mathbf{M}_{a1} & \overline{\mathbf{M}}_a\end{bmatrix}\begin{Bmatrix}\ddot{\mathbf{q}}_1\\ \ddot{\eta}_a\end{Bmatrix} + \begin{bmatrix}\mathbf{C}_1 & \mathbf{C}_{a1}^T\\ \mathbf{C}_{a1} & \mathbf{C}_{aa}\end{bmatrix}\begin{Bmatrix}\dot{\mathbf{q}}_1\\ \dot{\eta}_a\end{Bmatrix} +\\ +\begin{bmatrix}\mathbf{K}_1 & \mathbf{K}_{1a}\\ \mathbf{K}_{a1} & \mathbf{K}_{aa}\end{bmatrix}\begin{Bmatrix}\mathbf{q}_1\\ \eta_a\end{Bmatrix} = \begin{Bmatrix}\mathbf{F}_1\\ \mathbf{0}\end{Bmatrix}\ ,\end{aligned} \tag{30.246}$$

where $\mathbf{q}_1$, $\mathbf{M}_1$, $\mathbf{C}_1$, $\mathbf{K}_1$ and $\mathbf{F}_1$ are the same vectors and matrices seen in Eq. (30.212), and

$$\begin{aligned}
\mathbf{M}_{a1} &= \left[\mathcal{B}^T\ \mathcal{D}^T - \mathcal{E}^T\ \mathcal{G}^T - \mathcal{M}^T\ \mathbf{0}\ \mathbf{0}\right] ,\\
\mathbf{C}_{a1} &= \left[\mathbf{0}\ -V\mathcal{B}^T - \textstyle\sum_{\forall k} c_{\phi k}\mathbf{Q}_{zaka}^T\ c_{\phi 1}\mathbf{Q}_{za1a}^T\ c_{\phi 2}\mathbf{Q}_{za2a}^T\right] ,\\
\mathbf{C}_{aa} &= \overline{\mathbf{C}}_a + \sum_{\forall k} c_k \mathbf{Q}_{zaka}^T \mathbf{Q}_{zaka} ,\\
\mathbf{K}_{1a} &= \left[\mathbf{0}\ -\dot{V}\mathcal{B}\ g\mathcal{B} - \textstyle\sum_{\forall k} k_{\phi k}\mathbf{Q}_{zak}\ k_{\phi 1}\mathbf{Q}_{za1}\ k_{\phi 2}\mathbf{Q}_{za2}\right]^T ,\\
\mathbf{K}_{a1} &= \left[\mathbf{0}\ \mathbf{0}\ g\mathcal{B}^T - \textstyle\sum_{\forall k} k_{\phi k}\mathbf{Q}_{zak}^T\ k_{\phi 1}\mathbf{Q}_{za1}^T\ k_{\phi 2}\mathbf{Q}_{za2}^T\right] ,\\
\mathbf{K}_{aa} &= \overline{\mathbf{K}}_a + \sum_{\forall k} k_{\phi k} \mathbf{Q}_{zak}^T \mathbf{Q}_{zak} .
\end{aligned}$$

The expressions of the various matrices refer to a two-axle vehicle, but they may be easily generalized.

As in the previous models, coordinates y^* and ψ are present only with their derivatives: the order of the differential set of equations is then $4 + 2n + 2n_a$.

30.4.2 Ride Comfort Model

The second set of equations contains generalized coordinates x^*, Z, θ, Z_i and $\boldsymbol{\eta}_s$. If n_s symmetrical modes are included in the model, they are $3 + n + n_s$.

In this case the first equation, that describing longitudinal dynamics, is weakly coupled with the others, and may be studied separately. Its expression is still Eq. (30.223), with the term $\mathcal{A}\ddot{\boldsymbol{\eta}}_s$ added to the left side.

Neglecting the deformation corresponding to the static equilibrium condition (which may be computed using an equation of the type of Eq. (30.221), in which the modal coordinates corresponding to the static deformation and terms like $g\mathcal{C}$ are included), the set of $2 + n + n_s$ equations, remaining after separating the first equation, describes the suspension motions of the vehicle:

$$\begin{bmatrix} \mathbf{M}_3 & \mathbf{M}_{s3}^T \\ \mathbf{M}_{s3} & \overline{\mathbf{M}}_s \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_3 \\ \ddot{\boldsymbol{\eta}}_s \end{Bmatrix} + \begin{bmatrix} \mathbf{C}_3 & \mathbf{C}_{s3}^T \\ \mathbf{C}_{s3} & \mathbf{C}_{ss} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_3 \\ \dot{\boldsymbol{\eta}}_s \end{Bmatrix} + \\ + \begin{bmatrix} \mathbf{K}_3 & \mathbf{K}_{s3}^T \\ \mathbf{K}_{s3} & \mathbf{K}_{ss} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_3 \\ \boldsymbol{\eta}_s \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_3 \\ \mathbf{F}_s \end{Bmatrix} , \tag{30.247}$$

where $\mathbf{q}_3$, $\mathbf{M}_3$, $\mathbf{C}_3$, $\mathbf{K}_3$ and $\mathbf{F}_3$ are the same matrices and vectors seen in Eq. (30.224), and

$$\mathbf{M}_{s3} = \left[\mathcal{C}^T \; \mathcal{I}^T - \mathcal{F}^T \; \mathbf{0} \; \mathbf{0} \right] ,$$
$$\mathbf{C}_{s3} = \left[\sum_{\forall k} \mathbf{c}_{14k}^T \; \sum_{\forall k} \mathbf{c}_{34k}^T \; \mathbf{c}_{24,1}^T \; \mathbf{c}_{24,2}^T \right]^T ,$$
$$\mathbf{C}_{ss} = \overline{\mathbf{C}}_s + \sum_{\forall i} \mathbf{c}_{44k} ,$$
$$\mathbf{K}_{s3} = \left[\sum_{\forall k} \mathbf{K}_{14k}^T \; \dot{V}\mathcal{C}^T - g\mathcal{A}^T + \sum_{\forall i} \mathbf{K}_{34k}^T \; \mathbf{K}_{24,1}^T \; \mathbf{K}_{24,2}^T \right]^T ,$$
$$\mathbf{K}_{ss} = \overline{\mathbf{K}}_s + \sum_{\forall i} \mathbf{K}_{44k} ,$$
$$\mathbf{F}_s = -\dot{V}\mathcal{C}\theta_0 .$$

All matrices are symmetrical, except for the stiffness matrix resulting from the usual aerodynamic term included in $\mathbf{K}_3$. All aerodynamic forces have been assumed to be independent from the deformation modes: if this assumption were abandoned, the equations would change, but uncoupling would still hold.

30.4.3 Uncoupling of the Equations of Motion

Symmetrical and skew-symmetrical deformation modes thus play a very different role in the dynamic behavior of the vehicle. The first, like bending modes in the xz plane, affect riding comfort but have no importance in the study of handling. The most important skew-symmetrical modes are those related to torsional deformations. The significance of their influence on handling, particularly in sport cars and above all in Formula 1 racers, is well known. Transversal bending can have a similar effect.

Modal matrices $\overline{\mathbf{M}}_s$, $\overline{\mathbf{M}}_a$, $\overline{\mathbf{K}}_s$ and $\overline{\mathbf{K}}_a$ are diagonal, and describe the dynamic behavior of the vehicle body as a free compliant body. Usually the damping linked to the deformation modes of the body is not large, and it may be considered as an undamped, or at most a lightly damped, system. Neglecting damping, the natural frequencies of the symmetrical and skew-symmetrical modes are

$$\Omega_{sj} = \sqrt{\frac{\overline{\overline{\mathbf{K}}}_{sj}}{\overline{\overline{\mathbf{M}}}_{sj}}} \, , \Omega_{aj} = \sqrt{\frac{\overline{\overline{\mathbf{K}}}_{aj}}{\overline{\overline{\mathbf{M}}}_{aj}}} \, . \tag{30.248}$$

If the coupling matrices $\mathbf{M}_{s3}$, $\mathbf{M}_{a1}$, $\mathbf{C}_{s3}$, $\mathbf{C}_{a1}$, etc. were negligible, the dynamic behavior of the vehicle could be studied by separating the dynamic behavior of the rigid vehicle on elastic suspensions (studied in the proceeding sections) from the dynamic behavior of the vehicle body, considered as a compliant body free in space.

In the case of passenger cars, the natural frequencies of deformation modes are much higher than those typical of a vehicle on elastic suspensions (as already stated, the typical frequencies of the sprung mass are slightly above 1 Hz while those linked to the unsprung masses are at most 8–10 Hz). Coupling between the dynamic behavior of the vehicle as made of rigid bodies and of the compliant vehicle body is weak, and the vibration of the latter influences acoustic comfort more than handling or ride comfort.

Open cars and vans are often an exception: the stiffness of their bodies is lower, particularly in torsion and bending in the symmetry plane, and the related natural frequencies are little different than those linked with the behavior of the vehicle as a whole. The torsional deformation of the chassis and the body may have a strong effect on handling, usually making it worse. Bending of the body in its plane may have some effect on comfort, even if it is impossible to determine in general whether it improves or worsens it.

In the case of industrial vehicles, particularly trucks, some natural frequencies of deformation modes are usually low and the related modes may strongly interfere with handling modes (if skew-symmetrical) or with comfort modes (if symmetrical).

30.5 Articulated Vehicles

Consider an articulated vehicle, such as a tractor and a trailer or a semi-trailer. As already stated, it is possible to study its behavior using a multibody model, if its parts may be considered rigid. The number of degrees of freedom is six for each part constituting the body of the vehicle, plus a further degree of freedom for each independent suspension and two degrees of freedom for each solid axle suspension, minus the number of degrees of freedom constrained by the links connecting the various parts constituting the body.

As an example, the articulated truck in Fig. 30.1d has 21 degrees of freedom if the hitch connecting tractor and trailer may be modelled as a spherical hinge (24 degrees of freedom for the two rigid bodies and the 6 solid axles, minus 3 degrees of freedom constrained by the hinge). If the hitch were modelled as a cylindrical hinge with its axis in the vertical direction, the number of degrees of freedom would reduce to 19 (the hinge would constrain 5 of them instead of 3), but to constrain pitch and roll rotations of the trailer with respect to the tractor, the moments about the x and y axes the hinge would experience (and with negligible deformations, otherwise some deformation degrees of freedom would be needed) would be extremely high, creating a non-viable solution.

In normal operation the pitch, roll and yaw angles of the trailer with respect to the tractor are small and, as in the case of articulated vehicles it is possible to use linearized models. An articulated vehicle made by two rigid bodies plus compliant suspensions may be thought of as a single compliant system, whose deformation consists in the relative motion of the two bodies about the hinge. The rigid body modes of this compliant body may be considered similar to the deformation modes seen above, the only difference being that the relevant natural frequencies are zero, because the modal stiffness vanishes. This is obvious because there are no elastic systems applying restoring moments at the hitch.

The rigid body modes may also be subdivided into symmetrical and skew-symmetrical modes. In the case of an articulated truck the yaw rotation of the trailer (like angle θ in the model seen in Sect. 25.15 and the rolling motion of the trailer are skew-symmetric modes and thus couple with handling, while pitching rotations of the trailer couple with ride comfort. In the case of the truck and trailer system in Fig. 25.41a, it is possible to assume that the connection of the draw bar to the trailer and that between the dolly and the trailer are spherical hinges, even if the roll rotation between the trailer and the dolly may in some cases be considered locked. The pitch rotation of the dolly and the trailer thus enter the comfort model, while all yaw and roll rotations enter the handling model.

30.6 Gyroscopic Moments and Other Second Order Effects

Gyroscopic moments due to wheel rotation were examined in the 10 degrees of freedom model for isolated vehicles with two axles. Within the frame of a linearized model, they have no effect on handling-comfort uncoupling and they enter only into the handling model. Gyroscopic moments are automatically present when the equations of motion are obtained through Lagrange equations, provided that the angular velocity of all rotating parts of the model is considered.

To evaluate the impact of gyroscopic moments caused by the wheels on handling, it is possible to assume, at least in the case of solid axle suspensions, that the angular velocity of the ith wheel $\dot{\chi}_i$ lies on axis y_k of the kth unsprung mass. Any angular velocity of the vehicle about the x_k and z_k axes will produce a gyroscopic moment due to the ith wheel that may be expressed in the x_k, y_k, z_k frame as

$$M_g = \dot{\chi}_i J_{pi} \left\{ \begin{array}{c} \dot{\psi} \\ 0 \\ -\dot{\phi}_k \end{array} \right\} = J_{pi} \frac{V}{R_{ei}} \left\{ \begin{array}{c} \dot{\psi} \\ 0 \\ -\dot{\phi}_i \end{array} \right\} , \tag{30.249}$$

where the angular velocity of the wheel has been assumed to be linked with the forward velocity by the usual relationship

$$\dot{\chi}_i = \frac{V}{R_{ei}} . \tag{30.250}$$

As previously stated, the terms due to the gyroscopic moment are present in $L_{\dot{\psi}}$ and, with opposite sign, in $L_{\dot{\phi}k}$. In steady state motion, gyroscopic moments are usually quite small. Neglecting the camber angle, $\dot{\psi} = V/R$ (where R is the radius of the trajectory) and $\dot{\phi}_i = 0$. The component M_{g_z} of the gyroscopic moment vanishes, while

$$M_{g_x} = \frac{V^2}{R} \sum_{\forall i} \frac{J_{pi}}{R_{ei}} , \tag{30.251}$$

where the sum extends to all wheels. Gyroscopic moment is thus proportional to the centrifugal acceleration V^2/R, which is limited. If the sum of the polar moments of inertia of the wheels of an axle and R_e are equal, for instance, to 6 kg m^2 and 0.5 m respectively, values related to an industrial vehicle, and the centrifugal acceleration is 5 m/s^2, a gyroscopic moment of 60 Nm is obtained. If the track of the axle is 1.4 m, the load transfer due to the gyroscopic moment of the two wheels is roughly 43 N.

Gyroscopic wheel moments may, however, be more important in non-stationary conditions and, above all, can affect to a large extent the dynamics of the steering system: Their effect on free-control dynamics may thus be important. In solid axle suspensions of steering axles, strong reactions on the steering wheel due to gyroscopic wheel moments may be caused by travelling on uneven road. They can cause severe discomfort and make driving difficult.

Gyroscopic moments due to the engine or other rotating elements of the vehicle are usually less important, except in particular cases (usually not related with road vehicles), such as that of electric railway engines, in which they can cause an increase of wear of the wheel rims. As a last consideration, a mass rotating about an axis parallel to the z axis couples roll and pitch motions, making uncoupling between handling and comfort impossible even if the assumptions of small displacements, linearity and symmetry hold. This effect may even be exploited, as in the case of flywheel stabilizers in ships, where coupling allows the larger pitch moment of inertia to be used to limit roll oscillations. Also, a mass rotating about the x-axis has a coupling effect between pitch and yaw, while a mass rotating about the y-axis couples roll and yaw, already coupled in the handling behavior.

Other second-order dynamic effects may be of some importance. In non-stationary motion, for example, the angular acceleration of rotating masses may produce inertia torques of non-negligible size. Some of these effects have been included in the model seen above and are included in the terms containing $\dot{V}$ (because of the relationship assumed between the forward velocity V and the wheel velocity $\dot{\chi}$, the acceleration $\ddot{\chi}$ is proportional to $\dot{V}$).

In the previous models some second order effects have been neglected. For instance, the transmission of the driving torque to the wheels may cause a reaction torque that, being exerted between the parts constituting the vehicle, has no effect on its global dynamics, at least as a first approximation. This torque may, however, modify the configuration of the vehicle and affect the forces the vehicle exchanges with the ground or, although to a much lesser extent, with the air. In a vehicle with longitudinal engine and rear wheel drive with the differential on a solid axle, the driving torque causes a small roll angular displacement between the vehicle body and the solid axle. The small roll angle may induce roll steer that may affect handling. These effects are usually neglected, because they are small, but there is no difficulty in introducing them into the model.

A larger effect may be caused by the reaction torque exerted on suspensions when not directly transferred to the body by the suspension linkages. Instead it loads the suspension springs, causing lifting or sinking of the attachment points, as seen in *antidive*, *antilift* or *antisquat* configurations. For this effect to be present, the torque must be applied to the unsprung mass. For driving torques this occurs only in the case of live axles, while in braking torques the brakes are almost always located on the unsprung masses, the only exceptions being the little used layout in which they are placed close to the differential gear in De Dion axles, or when driving wheel suspension is independent. In such cases the suspension layout must allow a vertical movement when a torque is applied to it, i.e. the derivative $\partial z/\partial M_y$ must be other than zero, an example being that of trailing arm suspensions.

Similarly it is possible to take into account the deformation of unsprung masses without changing the general conclusion: This is important because configurations based on the compliance of the unsprung masses are increasingly common. With solid axles it is easy to evaluate which deformation modes of the unsprung masses are symmetrical, and thus are to be included in comfort dynamics, and which are skew-symmetrical and affect handling. In a suspension in which leaf springs are used as guiding elements, for instance, the lateral compliance of the springs gives way to skew symmetrical modes and influences handling, while their S deformation about the y axis is a symmetrical deformation and thus couples with comfort dynamics, or better longitudinal dynamics. The longitudinal compliance of the suspension may strongly affect comfort.

In the case of independent suspensions, the suspension of the whole axle, with its two rigid-body degrees of freedom, must be considered. The whole axle must be studied as well for deformation modes, as was done in Eq. (30.122). In this way it is again possible to distinguish between symmetrical and skew-symmetrical modes.

But uncoupling is a more general feature still. The above considerations may be applied to vehicles with two wheels, the only exceptions being that the roll angle can easily take values beyond the range in which linearization of trigonometric functions applies, while the lateral movements of the driver, aimed at displacing the centre of mass and producing unsymmetrical aerodynamic forces, can destroy the symmetry on which uncoupling is based.

No particular assumption about the nature of the forces supporting the vehicle has been made. The same uncoupling also holds for vehicles supported by hydrostatic, aerostatic or aerodynamic forces. In the first case, the assumption of the existence of a roll axis of the suspension is replaced by the assumption of a roll axis fixed to the hull in its undeflected configuration. The small roll oscillations are thus demonstrated to be uncoupled with pitch and bounce motions, which are coupled with each other. In the case of aircraft, roll and yaw oscillations are known to be coupled (dutch roll) while bounce and pitch oscillations are also coupled with each other. Even the presence of aerodynamic forces due to the deformations of the structure does not change the overall picture, provided that they can be assumed to depend linearly on the modal coordinates $\boldsymbol{\eta}_s$ and $\boldsymbol{\eta}_a$.