Strongly Exponentially Separated Linear Difference Equations

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Abstract In the study of linear differential systems, an important concept is that of exponential separation. In a previous paper, we have studied this concept for differential equations. Here we develop the theory for difference equations. Our first aim is to develop a theory which applies to unbounded systems. It turns that in order to have a reasonable theory it is necessary to add the assumption that the angle between the two separated subspaces is bounded below (note this follows automatically for bounded systems). Our second aim is to show that if a bounded linear symplectic system is exponentially separated into two subspaces of the same dimension, then it must have an exponential dichotomy. The theory follows the same lines as the differential equation case with one important difference: for the roughness theorem a different kind of perturbation is needed.

Keywords Linear difference equations · Exponential separation · Exponential dichotomy · Symplectic

1 Introduction

In this paper we study linear difference systems

$$Nowweconsidex(k+1) = A(k)x(k), \quad x \in \mathbb{R}^n \tag{1}$$

where $A(k)$ is an invertible matrix for each k. In the study of such systems, an important concept is that of exponential separation. It is closely related to the concept

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of exponential dichotomy. In this paper we restrict attention to the case where the solution space is split into two exponentially separated subspaces.

Exponential separation for difference equations was first studied in [9, 10]. There it is assumed that the coefficient matrixNow we conside $A(k)$ is bounded in norm together with its inverse. Our first aim here is to develop a theory of exponential separation which applies to unbounded systems. It turns out that in order to have a reasonable theory it is necessary to add the assumption that the angle between the two separated subspaces is bounded below (note this follows automatically for bounded systems). Our second aim is to show that if a bounded linear system (1), where the $A(k)$ is symplectic, is exponentially separated into two subspaces of the same dimension, then it must have an exponential dichotomy. Note that we developed a similar theory for linear differential equations in [2]. Most of the results for invertible difference equations are analogous but there is one important difference which we mention below. Let us remark here that it was already observed in [4] that in the case of exponential separation for differential equations, an additional angle condition is needed for unbounded systems.

Now we summarize the contents of the paper. In Sect. 2, we introduce the basic definitions and examine to what extent the separated subspaces are unique. Then if a system is separated on both half-axes, we describe what must be added to ensure separation on the whole axis. In this section we do not need the additional condition on the angle. In Sect. 3 we introduce the concept of strong exponential separation, which is exponential separation plus the condition that the angle between the separated subspaces be bounded below. Then in Proposition 3, we derive a convenient necessary and sufficient condition for strong exponential separation and use it to show that the condition of strong exponential separation is preserved by the operation of taking the inverse adjoint of a system and that it is implied by exponential dichotomy. In Sect. 4, we mention the result in [9] which shows that when $A(k)$ and its inverse are bounded then exponential separation is equivalent to exponential dichotomy of a shifted equation. In Sect. 5 we show strong exponential separation is robust under small perturbation of the coefficient matrix. In fact, here we see a major difference from the theorem for differential equations and also a difference from the roughness theorem for exponential dichotomy in difference equations. Next in Sect. 6 we study block upper triangular systems. First, we show that if the block upper triangular system is strongly exponentially separated, then the corresponding block diagonal system is strongly exponentially separated. Here no boundedness assumptions are needed. Then, using the perturbation theorem in Sect. 5, we show that if the corresponding block diagonal system is strongly exponentially separated and the off-diagonal blocks are bounded in a certain sense, then the block upper triangular system is strongly exponentially separated. Here again there is a difference from the result for differential equations. Finally in Sect. 7, we show that if a bounded linear symplectic system is exponentially separated into two subspaces of the same dimension, then it must have an exponential dichotomy. An important tool here are the results in Sect. 6 about block upper triangular systems.

2 Exponential Separation in Bounded and Unbounded Systems: General Properties

In this section, we define exponential separation and examine to what extent the separated subspaces are unique. Then if a system is separated on both half-axes (that is, $\mathbb{Z}_+$ or $\mathbb{Z}_-$), we describe what must be added to ensure separation on the whole axis (that is, $\mathbb{Z}$). Note in the case of $\mathbb{Z}_-$, (1) holds for $k \in (-\infty, -1]$ but the solutions are defined for $k \in \mathbb{Z}_-$. However in the sequel we use $\mathbb{Z}_-$ in both cases where, for (1), it is to be understood that $\mathbb{Z}_-$ means $(-\infty, -1]$.

Definition 1 We say system (1) is *exponentially separated* on an infinite interval J of integers if there are nonzero invariant subspaces $V_1(k) \oplus V_2(k) = \mathbb{R}^n$ and positive constants $K \geq 1$ and α such that if $x(k)$ is a nonzero solution in $V_1(k)$ and $y(k)$ a nonzero solution in $V_2(k)$, then

$$\frac{|x(k)|\,|y(m)|}{|x(m)|\,|y(k)|} \leq K e^{-\alpha(k-m)}, \quad k \geq m \text{ in } J.$$

$V_1(k)$ is called the *stable subspace* and $V_2(k)$ the *unstable subspace*. If we denote by $P(k)$ (note that $P(k) \neq 0, \mathbf{I}$) the projection with range $V_1(k)$ and nullspace $V_2(k)$, then this can be written as

$$|\Phi(k,m)\xi|\,|\Phi(m,k)\eta| \leq K e^{-\alpha(k-m)}|\xi|\,|\eta|, \quad k \geq m \text{ in } J \tag{2}$$

for all $\xi \in \mathcal{R}P(m)$ and all $\eta \in \mathcal{N}P(k)$, where $\Phi(k,m)$ is the transition matrix. If $P(k)$ has rank r $(1 \leq r \leq n-1)$, or equivalently $\dim V_1(k) = r$, we say that (1) is exponentially separated with rank r. Note that $P(k)$ has the invariance property $A(k)P(k) = P(k+1)A(k)$ for all k.

Remark 1 It is easy to see that (1) is exponentially separated on $\mathbb{Z}_+$ with subspaces $V_1(k)$, $V_2(k)$ if and only if $x(k+1) = A^{-1}(-k-1)x(k)$ is exponentially separated on $\mathbb{Z}_-$ with subspaces $V_2(-k)$, $V_1(-k)$.

First we make a simple but useful observation.

Proposition 1 *If* (1) *is exponentially separated on* $[T, \infty)$ *for some* $T > 0$*, it is exponentially separated on* $\mathbb{Z}_+$*.*

Proof Let (1) be exponentially separated on $[T, \infty)$ with projection $P(k)$ and constants K, α. Then

$$|\Phi(k,m)\xi|\,|\Phi(m,k)\eta| \leq K e^{-\alpha(k-m)}|\xi|\,|\eta|, \quad k \geq m \geq T \tag{3}$$

for all $\xi \in \mathcal{R}P(m)$ and all $\eta \in \mathcal{N}P(k)$.

There exists M such that $|A(k)|, |A^{-1}(k)| \le M$ for $0 \le k \le T-1$. Then

$$|\Phi(k,m)| \le M^{|k-m|} \quad \text{for} \quad 0 \le k, m \le T$$

so that if $T \ge k \ge m \ge 0$ and $\xi \in \mathcal{R}P(m)$, $\eta \in \mathcal{N}P(k)$,

$$|\Phi(k,m)\xi|\,|\Phi(m,k)\eta| \le M^{2T} e^{\alpha T} e^{-\alpha(k-m)} |\xi|\,|\eta|. \tag{4}$$

Next if $k \ge T \ge m \ge 0$ and $\xi \ne 0$ in $\mathcal{R}P(m)$, $\eta \ne 0$ in $\mathcal{N}P(k)$,

$$\begin{aligned}
&\frac{|\Phi(k,m)\xi|\,|\Phi(m,k)\eta|}{|\xi|\,|\eta|} \\
&= \frac{|\Phi(k,T)\xi_1|\,|\Phi(m,T)\eta_1|}{|\xi|\,|\eta|}, \quad \xi_1 = \Phi(T,m)\xi \in \mathcal{R}P(T),\ \eta_1 = \Phi(T,k)\eta \in \mathcal{N}P(T) \\
&= \frac{|\Phi(k,T)\xi_1|}{|\xi_1|}\frac{|\Phi(T,k)\eta|}{|\eta|} \times \frac{|\Phi(T,m)\xi|}{|\xi|}\frac{|\Phi(m,T)\eta_1|}{|\eta_1|} \\
&\le K e^{-\alpha(k-T)} \times M^{2T} e^{\alpha T} e^{-\alpha(T-m)}, \quad \text{using} \quad (3), (4) \\
&= K M^{2T} e^{\alpha T} e^{-\alpha(k-m)}.
\end{aligned}$$

Together with (3) and (4), this proves the proposition.

Now we examine the extent to which exponentially separated subspaces are unique.

Proposition 2 *For exponentially separated systems on $\mathbb{Z}_+$ the stable subspace is uniquely defined (for a given dimension) and for exponentially separated systems on $\mathbb{Z}_-$ the unstable subspace is uniquely defined. The other subspace can be any complement.*

Proof Consider first $\mathbb{Z}_+$. So we are assuming there is an invariant projection $P(k) \ne 0, \mathbf{I}$ and positive constants K and α such that (2) holds. Let $Q(k)$ be another invariant projection with the same range as $P(k)$.

Suppose $\xi \in \mathcal{R}(Q(m)) = \mathcal{R}(P(m))$ and $\eta \ne 0 \in \mathcal{N}(Q(k))$. Note that $(\mathbf{I} - P(k))\eta \ne 0$, since otherwise $\eta \in \mathcal{R}(P(k)) = \mathcal{R}(Q(k))$ and therefore would be 0 since it is in $\mathcal{N}(Q(k))$ also. Then if $0 \le m \le k$,

$$\begin{aligned}
&|\Phi(k,m)\xi|\,|\Phi(m,k)\eta| \\
&= |\Phi(k,m)\xi|\,|\Phi(m,k)(\mathbf{I} - P(k))\eta| \frac{|\Phi(m,k)(\mathbf{I} - Q(k))\eta|}{|\Phi(m,k)(\mathbf{I} - P(k))\eta|} \\
&\le K e^{-\alpha(k-m)} |\xi|\,|(\mathbf{I} - P(k))\eta| \frac{|\Phi(m,k)(\mathbf{I} - Q(k))\eta|}{|\Phi(m,k)(\mathbf{I} - P(k))\eta|} \\
&= K e^{-\alpha(k-m)} |\xi|\,|\eta| \frac{|\Phi(m,k)(\mathbf{I} - Q(k))\eta|\,|(\mathbf{I} - P(k))\eta|}{|\Phi(m,k)(\mathbf{I} - P(k))\eta|\,|(\mathbf{I} - Q(k))\eta|}.
\end{aligned} \tag{5}$$

Now

$$\begin{aligned}\frac{|\Phi(m,k)(\mathbf{I}-Q(k))\eta|}{|\Phi(m,k)(\mathbf{I}-P(k))\eta|} &= \frac{|\Phi(m,0)(\mathbf{I}-Q(0))\Phi(0,k)\eta|}{|\Phi(m,0)(\mathbf{I}-P(0))\Phi(0,k)\eta|} \\ &\leq 1+\frac{|\Phi(m,0)(P(0)-Q(0))\Phi(0,k)\eta|}{|\Phi(m,0)(\mathbf{I}-P(0))\Phi(0,k)\eta|} \\ &\leq 1+Ke^{-\alpha m}\frac{|(P(0)-Q(0))\Phi(0,k)\eta|}{|(\mathbf{I}-P(0))\Phi(0,k)\eta|} \\ &\leq 1+Ke^{-\alpha m}N,\end{aligned}$$

since $(P(0)-Q(0))\Phi(0,k)\eta \in \mathcal{R}(P(0))$, $(\mathbf{I}-P(0))\Phi(0,k)\eta \in \mathcal{N}(P(0))$ and where

$$N = \sup_{\eta\in\mathcal{N}Q(0),|\eta|=1} \frac{|(P(0)-Q(0))\eta|}{|(\mathbf{I}-P(0))\eta|}.$$

Similarly,

$$\begin{aligned}\frac{|(\mathbf{I}-Q(k))\eta|}{|(\mathbf{I}-P(k))\eta|} &= \frac{|\Phi(k,0)(\mathbf{I}-Q(0))\Phi(0,k)\eta|}{|\Phi(k,0)(\mathbf{I}-P(0))\Phi(0,k)\eta|} \\ &\geq 1-\frac{|\Phi(k,0)(P(0)-Q(0))\Phi(0,k)\eta|}{|\Phi(k,0)(\mathbf{I}-P(0))\Phi(0,k)\eta|} \\ &\geq 1-Ke^{-\alpha k}\frac{|(P(0)-Q(0))\Phi(0,k)\eta|}{|(\mathbf{I}-P(0))\Phi(0,k)\eta|} \\ &\geq 1-Ke^{-\alpha k}N.\end{aligned}$$

So if $k \geq m \geq T = \alpha^{-1}\log(3KN)$,

$$|\Phi(k,m)\xi|\,|\Phi(m,k)\eta| \leq Ke^{-\alpha(k-m)}|\xi|\,|\eta|\frac{1+1/3}{1-1/3} = 2Ke^{-\alpha(k-m)}|\xi|\,|\eta|.$$

Then we get the exponential separation on $\mathbb{Z}_+$ using Proposition 1. Thus $Q(k)$ can also be used as a projection and hence any invariant complement of $\mathcal{R}(P(k))$ can be taken as the unstable subspace.

Now we prove the uniqueness of the stable subspace. Suppose $R(k)$ is an invariant projection with different range from $P(k)$ but with the same rank, with respect to which the system is exponentially separated. Let p be a vector which is in the range of $R(0)$ but not in the range of $P(0)$ and q a vector which is in the range of $P(0)$ but not in the range of $R(0)$. We can take $Q_1(k)$ as an invariant projection with the same range as $P(k)$ with p in the nullspace of $Q_1(0)$ and $Q_2(k)$ as an invariant projection with the same range as $R(k)$ with q in the nullspace of $Q_2(0)$. Now there exist positive constants K_i and α_i such that for $k \geq m \geq 0$,

$$|\Phi(k,m)\xi|\,|\Phi(m,k)\eta| \leq K_1e^{-\alpha_1(k-m)}|\xi|\,|\eta|$$

for all $\xi \neq 0$ in $\mathcal{R}Q_1(m)$ and all $\eta \neq 0$ in $\mathcal{N}Q_1(k)$ and

$$|\Phi(k,m)\xi|\,|\Phi(m,k)\eta| \leq K_2 e^{-\alpha_2(k-m)}|\xi|\,|\eta|$$

for all $\xi \neq 0$ in $\mathcal{R}Q_2(m)$ and all $\eta \neq 0$ in $\mathcal{N}Q_2(k)$. Since q is in the range of $Q_1(0)$ and $\Phi(k,0)p$ is in the nullspace of $Q_1(k)$, it follows that for $k \geq 0$,

$$|\Phi(k,0)q|\,|\Phi(0,k)\Phi(k,0)p| \leq K_1 e^{-\alpha_1 k}|q|\,|\Phi(k,0)p|,$$

and since p is in the range of $Q_2(0)$ and $\Phi(k,0)q$ in the nullspace of $Q_2(k)$, it follows that for $k \geq 0$,

$$|\Phi(k,0)p|\,|\Phi(0,k)\Phi(k,0)q| \leq K_2 e^{-\alpha_2 k}|p|\,|\Phi(k,0)q|.$$

Then, combining these inequalities,

$$|p|\,|\Phi(k,0)q| \leq K_1 e^{-\alpha_1 k} K_2 e^{-\alpha_2 k}|p|\,|\Phi(k,0)q|$$

so that for $k \geq 0$

$$1 \leq K_1 e^{-\alpha_1 k} K_2 e^{-\alpha_2 k},$$

clearly impossible. Thus the stable subspace is unique.

The proof of the Proposition for $\mathbb{Z}_-$ follows using Remark 1.

In the following corollary, we show what additional conditions are needed to ensure that a system which is exponentially separated on both half-axes is also exponentially separated on the whole axis.

Corollary 1 *System (1) is exponentially separated on $\mathbb{Z}$ if and only if it is exponentially separated on $\mathbb{Z}_+$ and $\mathbb{Z}_-$, the respective ranks are the same and the stable subspace on $\mathbb{Z}_+$ and the unstable subspace on $\mathbb{Z}_-$ intersect in $\{0\}$ at $k = 0$.*

Proof Clearly the conditions are necessary.

For the sufficiency, suppose (1) is exponentially separated on $\mathbb{Z}_+$ and $\mathbb{Z}_-$, the respective ranks are the same and the stable subspace on $\mathbb{Z}_+$ and the unstable subspace on $\mathbb{Z}_-$ intersect in $\{0\}$ at $k = 0$. According to Proposition 2, at $k = 0$, we can take the unstable subspace on $\mathbb{Z}_+$ to be the unstable subspace on $\mathbb{Z}_-$ and the stable subspace on $\mathbb{Z}_-$ to be the stable subspace on $\mathbb{Z}_+$. This means we have the same invariant projection $P(k)$ on both $\mathbb{Z}_+$ and $\mathbb{Z}_-$. Then there exist positive constants K and α such that if $\xi \in \mathcal{R}(P(m))$ and $\eta \in \mathcal{N}(P(k))$, we have

$$|\Phi(k,m)\xi|\,|\Phi(m,k)\eta| \leq K e^{-\alpha(k-m)}|\xi|\,|\eta|$$

for $k \geq m \geq 0$ and $0 \geq k \geq m$. Next if $k \geq 0 \geq m$,

$$\begin{aligned}
|\Phi(k,m)\xi|\,|\Phi(m,k)\eta| &= \frac{|\Phi(k,m)\xi|\,|\Phi(0,k)\eta|}{|\Phi(0,m)\xi|}\,\frac{|\Phi(0,m)\xi|\,|\Phi(m,k)\eta|}{|\Phi(0,k)\eta|} \\
&= \frac{|\Phi(k,0)\Phi(0,m)\xi|\,|\Phi(0,k)\eta|}{|\Phi(0,m)\xi|}\,\frac{|\Phi(0,m)\xi|\,|\Phi(m,0)\Phi(0,k)\eta|}{|\Phi(0,k)\eta|} \\
&\leq \frac{Ke^{-\alpha k}|\Phi(0,m)\xi|\,|\eta|}{|\Phi(0,m)\xi|}\,\frac{Ke^{\alpha m}|\xi|\,|\Phi(0,k)\eta|}{|\Phi(0,k)\eta|} \\
&= K^2 e^{-\alpha(k-m)}|\xi|\,|\eta|.
\end{aligned}$$

It follows that (1) is exponentially separated on $\mathbb{Z}$.

3 Strongly Exponentially Separated Systems

In this section we introduce the definition of strong exponential separation. Then we derive a simple necessary and sufficient condition for strong exponential separation and we use it to show that strong exponential separation is preserved by the operation of taking adjoints and also that exponential dichotomy implies strong exponential separation.

First we recall the definitions of kinematic similarity and reducibility.

Definition 2 Systems (1) and $y(k+1) = B(k)y(k)$ are *kinematically similar* if there exists a bounded, invertible matrix function $S(k)$ with bounded inverse such that the transformation $x = S(k)y$ takes (1) into $y(k+1) = B(k)y(k)$, where $B(k) = S^{-1}(k+1)A(k)S(k)$. We refer to the transformation $x = S(k)y$ as a *kinematic similarity*.

Definition 3 System (1) is *reducible* if it is kinematically similar to a block diagonal system

$$y(k+1) = \begin{pmatrix} A_1(k) & 0 \\ 0 & A_2(k) \end{pmatrix} y(k).$$

The following proposition follows from [3] (see also Lemma 1.5.4 in [11]). Note that a projection $P(k)$ is *invariant* for (1) if $A(k)P(k) = P(k+1)A(k)$ for all k.

Proposition 3 *System (1) is reducible if and only if* (1) *has a bounded invariant projection* $P(k) \neq 0, \mathbf{I}$.

Remark 2 Note that the boundedness of $P(k)$ is equivalent to the angle between the range and nullspace of $P(k)$ being bounded below by a positive number.

Remark 3 Here we prove the sufficiency in Proposition 3. If $P(k)$ is invariant and bounded, it follows from [3] that there is a kinematic similarity $S(k)$ such that $P(k) = S(k)PS^{-1}(k)$ with $P = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{pmatrix}$. Then the transformation $x = S(k)y$ takes (1) into a system $y(k+1) = B(k)y(k)$, where

$$\begin{aligned} B(k)P &= B(k)S^{-1}(k)P(k)S(k) \\ &= S^{-1}(k+1)A(k)P(k)S(k) \\ &= S^{-1}(k+1)P(k+1)A(k)S(k) \quad \text{by invariance of } P(k) \\ &= S^{-1}(k+1)P(k+1)S(k+1)B(k) \\ &= PB(k). \end{aligned}$$

So the transformed system has the form

$$y(k+1) = \begin{pmatrix} A_1(k) & 0 \\ 0 & A_2(k) \end{pmatrix} y,$$

and the projection corresponding to $P(k)$ is $S^{-1}(k)P(k)S(k) = P$.

Remark 4 Analogously to Lemma 1 in [7], it can be proved that if $A(k)$ and $A^{-1}(k)$ are bounded and (1) is exponentially separated with corresponding projection $P(k)$, then $P(k)$ is bounded so that (1) is reducible. The example below shows that an unbounded exponentially separated system need not be reducible, in contrast to the case of bounded systems.

Example 1

$$x(k+1) = \begin{pmatrix} e & (e^2-1)e^{k+1} \\ 0 & e^2 \end{pmatrix} x(k). \tag{6}$$

This has the two solutions

$$x(k) = (e^k, 0), \quad y(k) = (e^{3k}, e^{2k}).$$

Using the maximum norm in $\mathbb{R}^2$, we see that $|x(k)| = e^k$ and $|y(k)| = e^{3k}$ for all $k \geq 0$. Then if $k \geq m \geq 0$

$$\frac{|x(k)|\,|y(m)|}{|x(m)|\,|y(k)|} = \frac{e^k\, e^{3m}}{e^m\, e^{3k}} = e^{-2(k-m)}$$

so that the system is exponentially separated on $\mathbb{Z}_+$. Now suppose (6) is reducible. Then it follows that there exists a bounded invariant projection $P(k)$ of rank 1. By direct calculation, it can be shown that there is no such $P(k)$. Indeed if

$$P(0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad P^2(0) = P(0),$$

then with

$$\Phi(k,0) = \begin{pmatrix} e^k & e^{3k}-e^k \\ 0 & e^{2k} \end{pmatrix},$$

we find that the $(2, 1)$ entry in $P(k)$, where by invariance $P(k) = \Phi(k,0)P(0)\Phi(0,k)$, is ce^k. So, if $P(k)$ is bounded, then $c = 0$. Next we find that the $(1, 2)$ entry in $P(k)$ is

$(d-a)e^k + (a+b-d)e^{-k}$. So $a = d$ and it follows that $P(0) = 0$ or the identity and hence cannot have rank 1.

Now we give the definition of strong exponential separation.

Definition 4 System (1) is said to be *strongly exponentially separated* if it is exponentially separated with corresponding subspaces $V_1(k) \oplus V_2(k) = \mathbb{R}^n$ where the angle between $V_1(k)$ and $V_2(k)$ is bounded below by a positive number, or equivalently the projection $P(k)$ on to $V_1(k)$ along $V_2(k)$ is bounded.

Remark 5 Kinematic similarity preserves both exponential separation and strong exponential separation. Indeed, suppose the kinematic similarity $x = S(k)y$ takes (1) into the system $y(k+1) = B(k)y(k)$. Moreover suppose (1) is exponentially separated on an interval J with projection $P(k)$ so that

$$|\Phi(k,m)\xi|\,|\Phi(m,k)\eta| \le Ke^{-\alpha(k-m)}|\xi|\,|\eta|, \quad k \ge m \text{ in } J$$

for all $\xi \in \mathcal{R}P(m)$ and all $\eta \in \mathcal{N}P(k)$. The transition matrix for $y(k+1) = B(k)y(k)$ is $\Psi(k,m) = S^{-1}(k)\Phi(k,m)S(m)$. So if we define the projection $Q(k) = S^{-1}(k)P(k)S(k)$, then $Q(k)$ is invariant with respect to $y(k+1) = B(k)y(k)$ and if $\xi \in \mathcal{R}Q(m)$ and $\eta \in \mathcal{N}Q(k)$, then $S(m)\xi \in \mathcal{R}P(m)$ and $S(k)\eta \in \mathcal{N}P(k)$ so that if $k \ge m$ in J,

$$\begin{aligned}
|\Psi(k,m)\xi|\,|\Psi(m,k)\eta| &= |S^{-1}(k)\Phi(k,m)S(m)\xi|\,|S^{-1}(m)\Phi(m,k)S(k)\eta| \\
&\le N^2|\Phi(k,m)S(m)\xi|\,|\Phi(m,k)S(k)\eta|, \quad \text{with} \quad N = \sup|S^{-1}(k)| \\
&\le N^2Ke^{-\alpha(k-m)}|S(m)\xi|\,|S(k)\eta| \\
&\le M^2N^2Ke^{-\alpha(k-m)}|\xi|\,|\eta|, \quad \text{where} \quad M = \sup|S(k)|.
\end{aligned}$$

Moreover, if $P(k)$ is bounded then $Q(k)$ is bounded also.

Remark 6 In view of the remark before the example, if $A(k)$ and $A^{-1}(k)$ are bounded, exponential separation implies strong exponential separation.

Now we derive a simple criterion for strong exponential separation which is similar to one given in [5] and was proved for the bounded case in [9].

Proposition 4 *A system $x(k+1) = A(k)x(k)$ is strongly exponentially separated on an infinite interval J with projection $P(k)$ if and only if there exist positive constants K and α such that*

$$|\Phi(k,m)P(m)|\,|\Phi(m,k)(\mathbf{I} - P(k))| \le Ke^{-\alpha(k-m)}, \quad m \le k \in J. \tag{7}$$

Proof To prove the sufficiency, note that (7) with $k = m$ implies that

$$|P(k)|\,|\mathbf{I} - P(k))| \le K, \quad k \in J.$$

Since $\mathbf{I} - P(k)$ is a nonzero projection, $|\mathbf{I} - P(k)| \geq 1$ and so

$$|P(k)| \leq K, \quad k \in J.$$

Next observe that if $\xi \in \mathcal{R}P(m)$ and $\eta \in \mathcal{N}P(k)$

$$\begin{aligned} |\Phi(k,m)\xi|\,|\Phi(m,k)\eta| &= |\Phi(k,m)P(m)\xi|\,|\Phi(m,k)(\mathbf{I} - P(k))\eta| \\ &\leq |\Phi(k,m)P(m)|\,|\xi|\,|\Phi(m,k)(\mathbf{I} - P(k))|\,|\eta| \\ &\leq Ke^{-\alpha(k-m)}|\xi|\,|\eta|. \end{aligned}$$

for $k \geq m$. So the sufficiency is proved.

Now we prove the necessity. We are supposing that there are positive constants K and α such that if $\xi \in \mathcal{R}P(m)$ and $\eta \in \mathcal{N}P(k)$, then for $k \geq m$

$$|\Phi(k,m)\xi|\,|\Phi(m,k)\eta| \leq Ke^{-\alpha(k-m)}|\xi|\,|\eta|, \quad |P(k)| \leq K.$$

Then for all ξ and η,

$$\begin{aligned} |\Phi(k,m)P(m)\xi|\,|\Phi(m,k)(\mathbf{I} - P(k))\eta| &\leq Ke^{-\alpha(k-m)}|P(m)\xi|\,|(\mathbf{I} - P(k))\eta| \\ &\leq Ke^{-\alpha(k-m)}|P(m)|\,|\xi|\,|(\mathbf{I} - P(k))|\,|\eta| \\ &\leq K^2(1+K)e^{-\alpha(k-m)}|\xi|\,|\eta|. \end{aligned}$$

Hence

$$|\Phi(k,m)P(m)|\,|\Phi(m,k)(\mathbf{I} - P(k))| \leq K^2(1+K)e^{-\alpha(k-m)}, \quad k \geq m.$$

We use the criterion just derived to show that strong exponential separation is preserved by the operation of taking adjoints.

Corollary 2 *If a system $x(k+1) = A(k)x(k)$ is strongly exponentially separated on an interval J with projection $P(k)$, then so also is its adjoint $x(k+1) = [A^*(k)]^{-1}x(k)$ with projection $\mathbf{I} - P^*(k)$.*

Proof By Proposition 4, there exist an invariant projection $P(k)$ and positive constants K and α such that

$$|\Phi(k,m)P(m)|\,|\Phi(m,k)(\mathbf{I} - P(k))| \leq Ke^{-\alpha(k-m)}, \quad k \geq m,$$

which, using invariance $P(k)\Phi(k,m) = \Phi(k,m)P(m)$, can be rewritten as

$$|P(k)\Phi(k,m)|\,|(\mathbf{I} - P(m))\Phi(m,k)| \leq Ke^{-\alpha(k-m)}, \quad k \geq m.$$

Taking adjoints and using the Euclidean norm, we get

$$|\Phi^*(k,m)P^*(k)|\,|\Phi^*(m,k)(\mathbf{I} - P^*(m))| \leq Ke^{-\alpha(k-m)}, \quad k \geq m.$$

Now the transition matrix for the adjoint system is $\Psi(k,m) = \Phi^*(m,k)$ and so we have

$$|\Psi(k,m)(\mathbf{I} - P^*(m))|\,|\Psi(m,k)P^*(k)| \le Ke^{-\alpha(k-m)}, \quad k \ge m.$$

Thus the adjoint system is strongly exponentially separated with projection $\mathbf{I} - P^*(k)$.

Remark 7 In 2 dimensions this also holds without "strong". For let $x(k) = (x_1(k), x_2(k))^T$ and $y(k) = (y_1(k), y_2(k))^T$ be exponentially separated solutions. Set $X(k) = (x(k), y(k))$. Then $X^*(k)^{-1}$ is a matrix solution of the adjoint system. But

$$X^*(k)^{-1} = (x_1(k)y_2(k) - x_2(k)y_1(k))^{-1}\begin{pmatrix} y_2(k) & -x_2(k) \\ -y_1(k) & x_1(k) \end{pmatrix}.$$

We see that the columns of this are exponentially separated solutions for the adjoint system. However this does not extend to higher dimensions. We give an example of an exponentially separated discrete equation in three dimensions for which the corollary about the adjoint system does not hold.

The system is

$$x(k+1) = \begin{pmatrix} e^3 & 0 & (1-e)e^{k+2} \\ 0 & e^2 & e(1-e) \\ 0 & 0 & e \end{pmatrix} x(k), \quad k \ge 0, \tag{8}$$

for which the transition matrix is

$$\Phi(k,0) = \begin{pmatrix} e^{3k} & 0 & e^{2k}(1-e^k) \\ 0 & e^{2k} & e^k(1-e^k) \\ 0 & 0 & e^k \end{pmatrix}.$$

If P is the projection

$$P := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$\Phi(k,0)P = \begin{pmatrix} 0 & 0 & e^{2k} \\ 0 & e^{2k} & e^k(1-e^k) \\ 0 & 0 & e^k \end{pmatrix}, \quad \Phi(k,0)(\mathbf{I} - P) = \begin{pmatrix} e^{3k} & 0 & -e^{3k} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The space spanned by the columns of $\Phi(k,0)P$ is the

$$\mathrm{span}\{x_1(k), x_2(k)\} \text{ with } x_1(k) = e^{2k}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } x_2(k) = e^k\begin{pmatrix} e^k \\ 1 \\ 1 \end{pmatrix}.$$

With the ℓ^1-norm we have

$$|x_1(k)| = e^{2k} \qquad e^{2k} \le |x_2(k)| \le 3e^{2k}.$$

Next the space spanned by the columns of $\Phi(k,0)(\mathbf{I}-P)$ is

$$\operatorname{span}\{y(k)\} \text{ with } y(k) = e^{3k}\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

Let $x(k) = ax_1(k) + bx_2(k)$ with $a^2+b^2 \ne 0$. Then $(|a|+|b|)e^{2k} \le |x(k)| \le (|a|+3|b|)e^{2k}$. Then for $k \ge 0$, $m \ge 0$,

$$\frac{|x(k)|}{|x(m)|} \le \frac{|a|+3|b|}{|a|+|b|}e^{2(k-m)}.$$

Next

$$\frac{|y(m)|}{|y(k)|} = e^{3(m-k)}.$$

Hence for $k \ge 0$, $m \ge 0$,

$$\frac{|x(k)|}{|x(m)|}\frac{|y(m)|}{|y(k)|} \le \frac{|a|+3|b|}{|a|+|b|}e^{-(k-m)} \le 3e^{-(k-m)}.$$

As a consequence the discrete system (8) is exponentially separated on $\mathbb{Z}_+$ with projection

$$P(k) = \Phi(k,0)P\Phi(0,k) = \begin{pmatrix}0 & 0 & e^k(e^k+1)\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

which is not bounded so that (8) is not strongly exponentially separated.

We prove that the adjoint system is not exponentially separated with projection (at $k=0$)

$$\mathbf{I} - P^*(0) = \mathbf{I} - P^* = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}.$$

If we denote by $\Psi(k,m)$ the transition matrix for the adjoint system, then

$$\Psi(k,0) = \Phi^*(0,k) = \begin{pmatrix}e^{-3k} & 0 & 0\\ 0 & e^{-2k} & 0\\ e^{-k}-e^{-2k} & e^{-k}-e^{-2k} & e^{-k}\end{pmatrix}$$

so that

$$\Psi(k,0)P^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-2k} & 0 \\ e^{-k} & e^{-k} - e^{-2k} & e^{-k} \end{pmatrix}, \quad \Psi(k,0)(\mathbf{I} - P^*) = \begin{pmatrix} e^{-3k} & 0 & 0 \\ 0 & 0 & 0 \\ -e^{-2k} & 0 & 0 \end{pmatrix}.$$

Thus the stable subspace for the adjoint system should be spanned by the solution

$$u(k) = \begin{pmatrix} e^{-3k} \\ 0 \\ -e^{-2k} \end{pmatrix}$$

and we can take the unstable space as the one spanned by the two solutions:

$$v_1(k) = \begin{pmatrix} 0 \\ 0 \\ e^{-k} \end{pmatrix}, \quad v_2(k) = \begin{pmatrix} 0 \\ e^{-2k} \\ -e^{-2k} \end{pmatrix}.$$

Then

$$e^{-2k} \le |u(k)| = e^{-2k}(e^{-k} + 1) \le 2e^{-2k} \quad \text{and} \quad |v_2(k)| = 2e^{-2k}.$$

Now if there was an exponential separation, there would exist positive constants K and α such that for $k \ge m \ge 0$,

$$\frac{|u(k)|}{|u(m)|}\frac{|v_2(m)|}{|v_2(k)|} \le Ke^{-\alpha(k-m)}.$$

However then for $k \ge m \ge 0$,

$$\frac{1}{2} = \frac{e^{-2k}}{2e^{-2m}}\frac{2e^{-2m}}{2e^{-2k}} \le \frac{|u(k)|}{|u(m)|}\frac{|v_2(m)|}{|v_2(k)|} \le Ke^{-\alpha(k-m)}.$$

This is impossible. We conclude that the adjoint system cannot be exponentially separated with projection $\mathbf{I} - P^*$ at $k = 0$.

Next we show that exponential dichotomy implies strong exponential separation. First we give the defintion of exponential dichotomy.

Definition 5 We say system (1) has an *exponential dichotomy* on an infinite interval J of integers if there is an invariant projection $P(k)$ and positive constants $K \ge 1$ and α such that

$$|\Phi(k,m)P(m)| \le Ke^{-\alpha(k-m)}, \quad |\Phi(m,k)(\mathbf{I} - P(k))| \le Ke^{-\alpha(k-m)} \quad k \ge m \text{ in } J,$$

where $\Phi(k,m)$ is the transition matrix. If $P(k)$ has rank r $(0 \le r \le n)$, we say that (1) has an exponential dichotomy with rank r.

Corollary 3 *If a system $x(k+1) = A(k)x(k)$ has an exponential dichotomy on an interval J with projection not equal to* 0 *or* **I***, then it is strongly exponentially separated on J with the same projection.*

Proof The proof follows at once from Proposition 4.

Finally we show that if a system is strongly exponentially separated on $\mathbb{Z}_+$ (resp. $\mathbb{Z}_-$) with a certain invariant projection, then it is strongly exponentially separated with respect to an invariant projection with the same range (resp. nullspace).

Proposition 5 *If system* (1) *is strongly exponentially separated on $\mathbb{Z}_+$ (resp. $\mathbb{Z}_-$) with projection $P(k)$, then it is strongly exponentially separated with respect to an invariant projection $Q(k)$ with the same range (resp. nullspace) as $P(k)$.*

Proof We just prove it for $\mathbb{Z}_+$. By Proposition 2, we know that (1) is exponentially separated with respect to $Q(k)$. All we need to show is that $Q(k)$ is bounded. As in Remark 3, there is a kinematic similarity $x = S(k)y$ taking (1) into a system of the form

$$y(k+1) = \begin{pmatrix} A_1(k) & 0 \\ 0 & A_2(k) \end{pmatrix} y(k),$$

where the projection for the exponential separation is the constant $P = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{pmatrix} = S^{-1}(k)P(k)S(k)$ and so by Proposition 4, there exist positive constants K and α such that

$$|\Phi_1(k,m)|\,|\Phi_2(m,k)| \le Ke^{-\alpha(k-m)}, \quad k \ge m,$$

$\Phi_1(k,m)$ and $\Phi_2(k,m)$ being the respective transition matrices for $y_1(k+1) = A_1(k)y_1(k)$ and $y_2(k+1) = A_2(k)y_2(k)$. If $Q(k)$ is a projection with the same range as $P(k)$, $\tilde{Q}(k) = S^{-1}(k)Q(k)S(k)$ has the same range as $S^{-1}(k)P(k)S(k) = P$ so that

$$\tilde{Q}(k) = \begin{pmatrix} \mathbf{I} & C(k) \\ 0 & 0 \end{pmatrix}$$

for some matrix function $C(k)$, where by the invariance of $\tilde{Q}(k)$,

$$C(k) = \Phi_1(k,0)C(0)\Phi_2(0,k)$$

so that

$$|C(k)| \le |\Phi_1(k,0)|\,|C(0)|\,|\Phi_2(0,k)| \le Ke^{-\alpha k}|C(0)|.$$

So $\tilde{Q}(k)$ is bounded and hence also $Q(k) = S(k)\tilde{Q}(k)S^{-1}(k)$. Note also that we have shown that $|Q(k) - P(k)| \to 0$ as $k \to \infty$ exponentially fast.

4 Exponential Separation and Dichotomy

In Proposition 3 in [9], Papaschinopoulos showed that for a system (1) where $A(k)$ and its inverse are bounded, strong exponential separation is equivalent to the existence of a bounded sequence $p(k) > 0$, where $1/p(k)$ is also bounded, such that the shifted equation

$$x(k+1) = \frac{1}{p(k)} A(k)x(k)$$

has an exponential dichotomy with the same projection. A similar result, in the context of diffeomorphisms on a compact manifold, has been given in [5]. There exponential separation is referred to as "dominated splitting" and exponential dichotomy as "hyperbolicity". It is not known whether or not this result can be extended to the case where $A(k)$ and its inverse are not bounded.

5 Roughness of Strong Exponential Separation

We would like to show, as for differential equations (see [2]), that strong exponential separation is preserved under small perturbations of the coefficient matrix. Now Kalkbrenner [6] has shown that if (1) has an exponential dichotomy and $|B(k)|$ is uniformly small, then $x(k+1) = [A(k) + B(k)]x(k)$ has an exponential dichotomy with projection near that for the unperturbed system. (Such a result was also proved in [8], but under the additional assumptions that $A(k)$ be invertible and $A(k)$ and its inverse be bounded.) However the following example shows that the analogous result is not true, in general, for strong exponential separation.

Example 2 Consider the equation

$$x(k+1) = \begin{pmatrix} e^{-\alpha}a(k) + \delta & 0 \\ 0 & a(k) \end{pmatrix} x(k) \tag{9}$$

with $k \geq 0$, $\delta > 0$ and $a(k) > 0$. Two independent solutions are:

$$u(k) = \left(e^{-\alpha k}\sigma(k) \prod_{j=0}^{k-1} \left(1 + \frac{\delta e^{\alpha}}{a(j)}\right), 0 \right), \quad v(k) = (0, \sigma(k)),$$

where $\sigma(0) = 1$ and

$$\sigma(k) = \prod_{j=0}^{k-1} a(j) \quad \text{for } k \geq 1. \tag{10}$$

Then

$$\frac{|u(k)|}{|u(m)|}\frac{|v(m)|}{|v(k)|} = e^{-\alpha(k-m)}\prod_{j=m}^{k-1}\left(1+\frac{\delta e^{\alpha}}{a(j)}\right).$$

We define $a(k)$ as follows. Let T_i, $i \geq 0$, be a sequence of positive integers such that $T_{i+1} \geq T_i + 2$ (e.g. $T_i = 2i$). Then we define

$$a(k) = \begin{cases} A_i & (i \geq 1,\ T_{2i-1} \leq k < T_{2i}), \\ B_i & (i \geq 0,\ T_{2i} \leq k < T_{2i+1}), \end{cases}$$

where $A_i \to 0$ and $B_i \to \infty$ (e.g. $A_i = 1/i$, $B_i = i$). Then

$$\frac{|u(k)|}{|u(m)|}\frac{|v(m)|}{|v(k)|} = e^{-\alpha(k-m)}\left(1+\frac{\delta e^{\alpha}}{A_i}\right)^{k-m} = e^{[\log(1+\delta e^{\alpha}/A_i)-\alpha](k-m)}. \tag{11}$$

when $T_{2i-1} \leq m \leq k < T_{2i}$. This implies that it is not possible that there exist positive constants K and β such that

$$\frac{|u(k)|}{|u(m)|}\frac{|v(m)|}{|v(k)|} \leq Ke^{-\beta(k-m)}, \quad k \geq m$$

for otherwise, setting $m = T_{2i-1}$ and $k = T_{2i} - 1$ and taking logs,

$$[\log(1+\delta e^{\alpha}/A_i)-\alpha](T_{2i}-1-T_{2i-1}) \leq \log(K) - \beta(T_{2i}-1-T_{2i-1}) \leq \log(K) - \beta,$$

which is impossible since the left side $\to \infty$ as $i \to \infty$. On the other hand, if $T_{2i} \leq m \leq k < T_{2i+1}$ we have

$$\frac{|u(k)|}{|u(m)|}\frac{|v(m)|}{|v(k)|} = e^{-\alpha(k-m)}(1+\delta e^{\alpha}/B_i)^{k-m} = e^{[\log(1+\delta e^{\alpha}/B_i)-\alpha](k-m)}.$$

This implies that it is not possible that there exist positive constants K and β such that

$$\frac{|v(k)|}{|v(m)|}\frac{|u(m)|}{|u(k)|} \leq Ke^{-\beta(k-m)}, \quad k \geq m$$

for otherwise, setting $m = T_{2i}$ and $k = T_{2i+1} - 1$ and taking logs,

$$-(\log(1+\delta e^{\alpha}/B_i)-\alpha)(T_{2i+1}-1-T_{2i}) \leq \log(K) - \beta(T_{2i+1}-1-T_{2i}) \leq \log(K) - \beta,$$

which is impossible since the left side $\to \infty$ as $i \to \infty$.

As a consequence system (9) is not exponentially separated with, at $k = 0$, projection $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ or with projection $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. This also covers all cases where the stable subspace at $k = 0$ is the x-axis, because we may take the unstable subspace at $k = 0$ as the y-axis so that at $k = 0$ the projection is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

This leaves the case that system (9) is exponentially separated with stable subspace at $k = 0$ different from the x-axis. Then we can take the x-axis as the unstable subspace at $k = 0$, that is, we can assume that (9) is exponentially separated with projection $\begin{pmatrix} 0 & \gamma \\ 0 & 1 \end{pmatrix}$ at $k = 0$. Set

$$u(k) = e^{-\alpha k}\sigma(k)\prod_{j=0}^{k-1}\left(1 + \frac{\delta e^{\alpha}}{a(j)}\right), \quad v(k) = \sigma(k),$$

with $\sigma(k)$ as in (10). Then the stable and unstable solutions are, respectively

$$x_s(k) = (\gamma u(k), v(k)), \quad x_u(k) = (u(k), 0)$$

and there should exist positive constants K and β such that

$$\max\left\{|\gamma|, \frac{|v(k)|}{|u(k)|}\right\} = \frac{|x_s(k)|}{|x_u(k)|} \le K e^{-\beta k}, \quad k \ge 0,$$

where we are using the maximum norm in $\mathbb{R}^2$. This implies that $\gamma = 0$. However we have shown the system is not exponentially separated with projection $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ at $k = 0$. This completes the proof that system (9) is not exponentially separated.

In view of this example, we must choose a different kind of perturbation.

Theorem 1 *Suppose (1) is strongly exponentially separated on an interval J with projection $P(k)$. Then if $|B(k)| \le \delta$, where δ is sufficiently small, the perturbed system*

$$x(k+1) = A(k)[\mathbf{I} + B(k)]x(k) \tag{12}$$

is also strongly exponentially separated with projection $Q(k)$ of the same rank. Also there exists a constant N such that

$$|Q(k) - P(k)| \le N\delta.$$

First we prove the perturbation theorem for the special case where the system has been split into its stable and unstable parts and the perturbation is uncoupled. Note that this proof is rather more complicated than the proof of a corresponding lemma for dichotomy would be.

Lemma 1 *Consider the system*

$$x_1(k+1) = A_1(k)[\mathbf{I} + C_1(k)]x_1(k), \quad x_2(k+1) = A_2(k)[\mathbf{I} + C_2(k)]x_2(k).$$

Suppose there exist positive constants K *and* α *such that*

$$|\Phi_1(k,m)|\,|\Phi_2(m,k)| \le Ke^{-\alpha(k-m)}, \quad k \ge m \in J,$$

$\Phi_i(k,m)$ *being the respective transition matrices for* $x_i(k+1) = A_i(k)x_i(k)$. *Then if* $|C_i(k)| \le \delta$, *where* $K\delta < 1$,

$$|\tilde{\Phi}_1(k,m)|\,|\tilde{\Phi}_2(m,k)| \le Ke^{-(\alpha-\log[(1+K\delta)/(1-K\delta)])(k-m)}, \quad k \ge m \in J,$$

where $\tilde{\Phi}_i(k,m)$ *are the respective transition matrices for* $x_i(k+1) = A_i(k)[\mathbf{I} + C_i(k)]x_i(k)$.

Proof Let $x_2(k)$ be a nonzero solution of $x_2(k+1) = A_2(k)x_2(k)$. Then

$$|\Phi_1(k,m)|\,|x_2(m)| = |\Phi_1(k,m)|\,|\Phi_2(m,k)x_2(k)| \le Ke^{-\alpha(k-m)}|x_2(k)|, \quad k \ge m.$$

Next if $x_1(k)$ is a solution of $x_1(k+1) = A_1(k)[\mathbf{I} + C_1(k)]x_1(k)$, then

$$x_1(k) = \Phi_1(k,m)x_1(m) + \sum_{p=m}^{k-1} \Phi_1(k,p)C_1(p)x_1(p)$$

so that

$$|x_1(k)| \le Ke^{-\alpha(k-m)}\frac{|x_2(k)|}{|x_2(m)|}|x_1(m)| + \sum_{p=m}^{k-1} Ke^{-\alpha(k-p)}\frac{|x_2(k)|}{|x_2(p)|}\delta|x_1(p)|.$$

Writing $z(k) = e^{\alpha k}|x_1(k)|/|x_2(k)|$, we get

$$z(k) \le Kz(m) + K\delta\sum_{p=m}^{k-1} z(p)$$

so that for $k \ge m$,

$$z(k) \le K(1+K\delta)^{k-m}z(m).$$

Therefore

$$|x_1(k)| \le Ke^{-(\alpha-\log(1+K\delta))(k-m)}\frac{|x_2(k)|}{|x_2(m)|}|x_1(m)|$$

and hence

$$|\tilde{\Phi}_1(k,m)| \le Ke^{-(\alpha-\log(1+K\delta))(k-m)}\frac{|x_2(k)|}{|x_2(m)|}, \quad k \ge m.$$

That is,

$$|\tilde{\Phi}_1(k,m)|\,|\Phi_2(m,k)x_2(k)| \le Ke^{-(\alpha-\log(1+K\delta))(k-m)}|x_2(k)|, \quad k \ge m.$$

Since $x_2(k)$ is arbitrary, it follows that

$$|\tilde{\Phi}_1(k,m)|\,|\Phi_2(m,k)| \le Ke^{-(\alpha-\log(1+K\delta))(k-m)}, \quad k \ge m.$$

Next let $x_1(k)$ be a solution of $x_1(k+1) = A_1(k)[\mathbf{I} + C_1(k)]x_1(k)$. Then

$$|x_1(k)|\,|\Phi_2(m,k)| = |\tilde{\Phi}_1(k,m)x_1(m)|\,|\Phi_2(m,k)| \le Ke^{-(\alpha-\log(1+K\delta))(k-m)}|x_1(m)|, \quad k \ge m.$$

Now if $x_2(k)$ is a solution of $x_2(k+1) = A_2(k)[\mathbf{I} + C_2(k)]x_2(k)$, then for $k \le m$

$$x_2(k) = \Phi_2(k,m)x_2(m) - \sum_{p=k}^{m-1} \Phi_2(k,p)C_2(p)x_2(p)$$

so that

$$|x_2(k)| \le Ke^{-(\alpha-\log(1+K\delta))(m-k)}\frac{|x_1(k)|}{|x_1(m)|}|x_2(m)| + \sum_{p=k}^{m-1} Ke^{-(\alpha-\log(1+K\delta))(p-k)}\frac{|x_1(k)|}{|x_1(p)|}\delta|x_2(p)|.$$

Writing $z(k) = e^{-(\alpha-\log(1+K\delta))k}|x_2(k)|/|x_1(k)|$, we get

$$z(k) \le Kz(m) + K\delta\sum_{p=k}^{m-1} z(p), \quad k \le m, \quad \text{so that} \quad z(k) \le Kz(m)(1-K\delta)^{k-m}, \quad k \le m$$

and so

$$|x_2(k)| \le Ke^{-(\alpha-\log[(1+K\delta)/(1-K\delta)])(m-k)}\frac{|x_1(k)|}{|x_1(m)|}|x_2(m)|, \quad k \le m.$$

Thus

$$|\tilde{\Phi}_2(k,m)| \le Ke^{-(\alpha-\log[(1+K\delta)/(1-K\delta)])(m-k)}\frac{|x_1(k)|}{|x_1(m)|}, \quad k \le m.$$

That is,

$$|\tilde{\Phi}_1(m,k)x_1(k)|\,|\tilde{\Phi}_2(k,m)| \le Ke^{-(\alpha-\log[(1+K\delta)/(1-K\delta)])(m-k)}|x_1(k)|, \quad k \le m.$$

Since $x_1(k)$ is arbitrary, it follows that

$$|\tilde{\Phi}_1(k,m)|\,|\tilde{\Phi}_2(m,k)| \le Ke^{-(\alpha-\log[(1+K\delta)/(1-K\delta)])(k-m)}, \quad m \le k.$$

Now we prove the general perturbation theorem, using the special case just proved and also Lemma 2. Here we follow the method in the proof on p. 42 of [3]. First we prove a lemma which is essentially well-known (see, for example, Proposition 2.8 in [8]) but we give the proof for the sake of completeness.

Lemma 2 *Let $A(k)$ be an invertible matrix function on $J = \mathbb{Z}$, $\mathbb{Z}_+$ or $\mathbb{Z}_-$. Suppose there exist positive constants K and α such that the transition matrix $\Phi(k, m)$ for (1) satisfies*

$$|\Phi(k, m)| \le Ke^{-\alpha(k-m)}$$

for $k \ge m$ in J. Next let $f(k, x)$ be a function satisfying

$$|f(k, 0)| \le \mu, \quad |f(k, x_1) - f(k, x_2)| \le \theta|x_1 - x_2|$$

for $k \in J$ and $|x_1|$, $|x_2| \le \Delta$. Then if

$$2K\mu \le (1 - e^{-\alpha})\Delta, \quad 2K\theta \le 1 - e^{-\alpha},$$

the equation

$$x(k + 1) = A(k)x(k) + f(k, x(k)), \quad k \in J$$

has a solution $x(k)$ such that $|x(k)| \le 2K\mu/(1 - e^{-\alpha})$.

Proof Let E be the Banach space of bounded sequences $x(k)$ defined for $k \in J$ with the supremum norm $\|\cdot\|_\infty$ and let S be the ball of radius $2K\mu/(1 - e^{-\alpha})$ in E. Then we define a mapping T on S by

$$(Tx)(k) = \sum_{m=b}^{k-1} \Phi(k, m + 1) f(m, x(m)), \quad k \in J,$$

where $b = -\infty$ when $J = \mathbb{Z}$, $\mathbb{Z}_-$ and $b = 0$ when $J = \mathbb{Z}_+$. Note that if $k \in J$,

$$\begin{aligned}|(Tx)(k)| &\le \sum_{m=b}^{k-1} Ke^{-\alpha(k-m-1)}[\mu + \theta\, 2K\mu/(1 - e^{-\alpha})]\\ &\le K(1 - e^{-\alpha})^{-1}[\mu + \theta\, 2K\mu/(1 - e^{-\alpha})]\\ &\le 2K\mu(1 - e^{-\alpha})^{-1}.\end{aligned}$$

Hence T maps S into itself. Next if $x_1(k)$ and $x_2(k)$ are two sequences in S, then for $k \in J_1$,

$$\begin{aligned}|(Tx_1)(k) - (Tx_2)(k)| &\le \sum_{m=b}^{k-1} Ke^{-\alpha(k-m-1)}\theta\|x_1 - x_2\|_\infty\\ &\le K(1 - e^{-\alpha})^{-1}\theta\|x_1 - x_2\|_\infty \le \frac{1}{2}\|x_1 - x_2\|_\infty.\end{aligned}$$

Thus T is a contraction on S and has a unique fixed point $x(k)$ which satisfies $|x(k)| \leq 2K\mu/(1-e^{-\alpha})$ for all $k \in J$ and also

$$x(k) = \sum_{m=b}^{k-1} \Phi(k, m+1) f(m, x(m)), \quad k \in J,$$

from which it follows that for $k \in J$,

$$\begin{aligned} x(k+1) &= \sum_{m=b}^{k} \Phi(k+1, m+1) f(m, x(m)) \\ &= A(k) \sum_{m=b}^{k-1} \Phi(k, m+1) f(m, x(m)) + f(k, x(k)) = A(k)x(k) + f(k, x(k)). \end{aligned}$$

Now we continue with the proof of the theorem.

Proof As in Remark 3, there is a kinematic similarity $x = T(k)y$, $k \in J$, taking (1) into a block diagonal system

$$x(k+1) = \begin{pmatrix} A_1(k) & 0 \\ 0 & A_2(k) \end{pmatrix} x(k).$$

This coefficient matrix commutes with $P = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{pmatrix}$. Also the block diagonal system is strongly exponentially separated with constant projection P so that by Proposition 4 there exist positive constants K and α such that

$$|\Phi_1(k, m)|\,|\Phi_2(m, k)| \leq K e^{-\alpha(k-m)}, \quad m \leq k, \ m, k \in J$$

$\Phi_i(k, m)$ being the respective transition matrices. If we apply the transformation $x = T(k)y$ to the perturbed system (12), we obtain a system with coefficient matrix

$$\begin{aligned} T^{-1}(k+1)A(k)[\mathbf{I} + B(k)]T(k) &= T^{-1}(k+1)A(k)T(k) \\ &\quad + T^{-1}(k+1)A(k)T(k)T^{-1}(k)B(k)T(k) \\ &= \begin{pmatrix} A_1(k) & 0 \\ 0 & A_2(k) \end{pmatrix} + \begin{pmatrix} A_1(k)C_{11}(k) & A_1(k)C_{12}(k) \\ A_2(k)C_{21}(k) & A_2(k)C_{22}(k) \end{pmatrix}, \end{aligned}$$

where

$$T^{-1}(k)B(k)T(k) = \begin{pmatrix} C_{11}(k) & C_{12}(k) \\ C_{21}(k) & C_{22}(k) \end{pmatrix} \quad \text{so that} \quad |C_{ij}(k)| \leq M\delta,$$

M being an upper bound on $|T^{-1}(k)|\,|T(k)|$. We assume

$$2KM\delta < 1.$$

The transformed equation is

$$\begin{aligned} y_1(k+1) &= A_1(k)[y_1(k) + C_{11}(k)y_1(k) + C_{12}(k)y_2(k)], \\ y_2(k+1) &= A_2(k)[y_2(k) + C_{21}(k)y_1(k) + C_{22}(k)y_2(k)]. \end{aligned} \tag{13}$$

Then we use the transformation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = S(k)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & H_{12}(k) \\ H_{21}(k) & \mathbf{I} \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{14}$$

where, provided $H_{12}(k)$, $H_{21}(k)$ are bounded and $|H_{12}(k)H_{21}(k)| \le \gamma < 1$, $S(k)$ is invertible with bounded inverse given by

$$\begin{pmatrix} [\mathbf{I} - H_{12}(k)H_{21}(k)]^{-1} & -[\mathbf{I} - H_{12}(k)H_{21}(k)]^{-1}H_{12}(k) \\ -[\mathbf{I} - H_{21}(k)H_{12}(k)]^{-1}H_{21}(k) & [\mathbf{I} - H_{21}(k)H_{12}(k)]^{-1} \end{pmatrix}.$$

The transformation (14) takes (13) into a block diagonal system

$$\begin{aligned} w_1(k+1) &= A_1(k)[\mathbf{I} + C_{11}(k) + C_{12}(k)H_{21}(k)]w_1(k), \\ w_2(k+1) &= A_2(k)[\mathbf{I} + C_{22}(k) + C_{21}(k)H_{12}(k)]w_2(k) \end{aligned} \tag{15}$$

provided for $k \in J$,

$$H_{12}(k+1)A_2(k)[\mathbf{I} + C_{22}(k) + C_{21}(k)H_{12}(k)] = A_1(k)[(\mathbf{I} + C_{11}(k))H_{12}(k) + C_{12}(k)], \tag{16}$$

$$H_{21}(k+1)A_1(k)[\mathbf{I} + C_{11}(k) + C_{12}(k)H_{21}(k)] = A_2(k)[C_{21}(k) + (\mathbf{I} + C_{22}(k))H_{21}(k)]. \tag{17}$$

First we solve (16). With H_{12} replaced by H, (16) can be rewritten as

$$\begin{aligned} H(k+1) = A_1(k)[\mathbf{I} + C_{11}(k)]H(k)[\mathbf{I} + C_{22}(k) + C_{21}(k)H(k)]^{-1}A_2^{-1}(k) \\ + A_1(k)C_{12}(k)[\mathbf{I} + C_{22}(k) + C_{21}(k)H(k)]^{-1}A_2^{-1}(k), \end{aligned} \tag{18}$$

assuming $\mathbf{I} + C_{22}(k) + C_{21}(k)H(k)$ is invertible, which it will be as long as δ is sufficiently small. Now we know the equation

$$H(k+1) = \mathcal{A}(k)H(k) = A_1(k)H(k)A_2^{-1}(k)$$

is uniformly asymptotically stable because it has transition operator

$$H \to \Psi(k,m)H = \Phi_1(k,m)H\Phi_2(m,k),$$

where

$$|\Psi(k, m)| \leq Ke^{-\alpha(k-m)}, \quad k \geq m.$$

We write (18) as

$$H(k+1) = A_1(k)H(k)A_2^{-1}(k) + f(k, H(k)),$$

where

$$f(k, H) = A_1(k)g(k, H)A_2^{-1}(k) \quad \text{with} \quad g(k, H) = g_1(k, H)g_2^{-1}(k, H),$$

and

$$\begin{aligned} g_1(k, H) &= C_{11}(k)H - HC_{22}(k) - HC_{21}(k)H + C_{12}(k), \\ g_2(k, H) &= \mathbf{I} + C_{22}(k) + C_{21}(k)H. \end{aligned}$$

We see that $|g(k, 0)| \leq (1 - M\delta)^{-1}M\delta$ so that

$$|f(k, 0)| \leq Ke^{-\alpha}(1 - M\delta)^{-1}M\delta$$

in view of the fact that

$$|A_1(k)||A_2^{-1}(k)| = |\Phi_1(k, k-1)||\Phi_2(k-1, k)| \leq Ke^{-\alpha}.$$

Next note that when $|H| \leq 1$,

$$|g_1(k, H)| \leq 4M\delta, \quad |g_2^{-1}(k, H)| \leq (1 - 2M\delta)^{-1}$$

and if $|H_1|, |H_2| \leq 1$,

$$\begin{aligned} |g_1(k, H_1) - g_1(k, H_2)| &\leq 4M\delta|H_1 - H_2|, \\ |g_2^{-1}(k, H_1) - g_2^{-1}(k, H_2)| &= |g_2^{-1}(k, H_1)[g_2(k, H_1) - g_2(k, H_2)]g_2^{-1}(k, H_2)| \\ &\leq (1 - 2M\delta)^{-2}M\delta|H_1 - H_2| \end{aligned}$$

so that

$$\begin{aligned} |g(k, H_1) - g(k, H_2)| &\leq |g_1(k, H_1)|\,|g_2^{-1}(k, H_1) - g_2^{-1}(k, H_2)| \\ &\quad + |g_2^{-1}(k, H_2)|\,|g_1(k, H_1) - g_1(k, H_2)| \\ &\leq [4M\delta(1 - 2M\delta)^{-2}M\delta + (1 - 2M\delta)^{-1}4M\delta]|H_1 - H_2| \\ &= 4M\delta(1 - M\delta)(1 - 2M\delta)^{-2}|H_1 - H_2|. \end{aligned}$$

Then if $|H_1|, |H_2| \leq 1$,

$$|f(k, H_1) - f(k, H_2)| \leq 4MKe^{-\alpha}(1 - M\delta)(1 - 2M\delta)^{-2}\delta|H_1 - H_2|.$$

Now we apply Lemma 2 with $\Delta = 1, \mu = Ke^{-\alpha}(1 - M\delta)^{-1}M\delta, \theta = 4MKe^{-\alpha}(1 - M\delta)(1 - 2M\delta)^{-2}\delta$. Then if

$$2K^2M(1 - M\delta)^{-1}\delta \leq e^{\alpha} - 1, \quad 8K^2M(1 - M\delta)(1 - 2M\delta)^{-2}\delta \leq e^{\alpha} - 1,$$

equation (16) has a solution $H_{12}(k)$ such that $|H_{12}(k)| \leq 2K^2M(e^{\alpha} - 1)^{-1}(1 - M\delta)^{-1}\delta$ for all t.

Now we consider Eq. (17). If we define $\tilde{H}(k) = H_{21}(-k + 1)$, with $k \in 1 - J = \{1 - \ell : \ell \in J\}$, (17) reads:

$$\begin{aligned}
&\tilde{H}(k)A_1(-k)[\mathbf{I} + C_{11}(-k) + C_{12}(-k)\tilde{H}(k+1)] \\
&\quad = A_2(-k)[C_{21}(-k) + (\mathbf{I} + C_{22}(-k))\tilde{H}(k+1)]
\end{aligned}$$

for $k \in 1 - J$, that is,

$$\begin{aligned}
&\tilde{H}(k+1) \\
&= [A_2(-k)(\mathbf{I} + C_{22}(-k)) - \tilde{H}(k)A_1(-k)C_{12}(-k)]^{-1} \\
&\quad \{\tilde{H}(k)A_1(-k)[\mathbf{I} + C_{11}(-k)] - A_2(-k)C_{21}(-k)\} \\
&= A_2(-k)^{-1}\tilde{H}(k)A_1(-k) + f(k, \tilde{H}(k)), \quad k \in 1 - J,
\end{aligned} \tag{19}$$

where

$$f(k, H) = g_2^{-1}(k, H)g_1(k, H),$$

with

$$\begin{aligned}
&g_1(k, H) = p(k, H)C_{11}(-k) - C_{21}(-k) - C_{22}(-k)p(k, H) + p(k, H)C_{12}(k)p(k, H), \\
&g_2(k, H) = \mathbf{I} + C_{22}(-k) - p(k, H)C_{12}(-k), \quad p(k, H) = A_2^{-1}(-k)HA_1(-k).
\end{aligned}$$

Note that if $|H|, |H_1|, |H_2| \leq 1$,

$$\begin{aligned}
&p(k, 0) = 0, \quad |p(k, H_1) - p(k, H_2)| \leq Ke^{-\alpha}|H_1 - H_2| \\
&|g_1(k, H)| \leq M\delta(1 + Ke^{-\alpha})^2, \quad |g_2^{-1}(k, H)| \leq [1 - M\delta(1 + Ke^{-\alpha})]^{-1} \\
&|g_1(k, H_1) - g_1(k, H_2)| \leq 2MKe^{-\alpha}\delta[1 + Ke^{-\alpha}]|H_1 - H_2|, \\
&|g_2^{-1}(k, H_1) - g_2^{-1}(k, H_2)| \leq [1 - M\delta(1 + Ke^{-\alpha})]^{-2}M\delta Ke^{-\alpha}|H_1 - H_2|
\end{aligned}$$

so that if $|H_1|, |H_2| \leq 1$,

$$|f(k, H_1) - f(k, H_2)| \leq \theta|H_1 - H_2|,$$

where

$$\begin{aligned}\theta &= [1 - M\delta(1 + Ke^{-\alpha})]^{-1} 2MKe^{-\alpha}\delta[1 + Ke^{-\alpha}] \\ &\quad + M\delta(1 + Ke^{-\alpha})^2[1 - M\delta(1 + Ke^{-\alpha})]^{-2} M\delta K e^{-\alpha} \\ &\leq (1 - 2KM\delta)^{-2} 4K^2 e^{-\alpha} M\delta.\end{aligned}$$

Also we have

$$|f(k,0)| = |[\mathbf{I} + C_{22}(-k)]^{-1}|\,|C_{21}(-k)| \leq \frac{M\delta}{1 - M\delta}.$$

The transition operator of equation

$$H(k+1) = A_2(-k)^{-1} H(k) A_1(-k), \quad k \in 1 - J,$$

is $\Psi(k,m)H = \Phi_2(1-k, 1-m)H\Phi_1(1-m, 1-k)$ and satisfies

$$|\Psi(k,m)| \leq Ke^{-\alpha(k-m)}$$

for $m \leq k$. Then, applying Lemma 2 with $\mu = \frac{M\delta}{1-M\delta}$, $\theta = \frac{4K^2 e^{-\alpha} M\delta}{(1-2KM\delta)^2}$ and $\Delta = 1$ we see that if

$$M\delta \leq \frac{e^{\alpha} - 1}{2Ke^{\alpha} + e^{\alpha} - 1} \quad \text{and} \quad \frac{8K^3 M\delta}{(1 - 2KM\delta)^2} \leq e^{\alpha} - 1,$$

equation (19) has a solution bounded on $1 - J$ and hence (17) has a solution bounded on J.

In fact, $|H_{21}(k)| \leq 2KM(1 - e^{-\alpha})^{-1}(1 - M\delta)^{-1}\delta$. Then provided

$$|H_{12}(k)H_{21}(k)| \leq 4K^3 e^{-\alpha} M^2 (1 - e^{-\alpha})^{-2}(1 - M\delta)^{-2}\delta^2 < 1,$$

$x = S(k)w$ is a kinematic similarity taking (13) into (15).

Next we show that the transformed Eq. (15) is strongly exponentially separated. Note that

$$|C_{11}(k) + C_{12}(k)H_{21}(k)| \leq \delta_1, \quad |C_{22}(k) + C_{21}(k)H_{12}(k)| \leq \delta_1,$$

where

$$\delta_1 = M\delta(1 + 2KM \max\{1, Ke^{-\alpha}\}(1 - e^{-\alpha})^{-1}(1 - M\delta)^{-1}\delta).$$

Then it follows from Lemma 1 that

$$|\tilde{\Phi}_1(k,m)|\,|\tilde{\Phi}_2(m,k)| \leq Ke^{-(\alpha - \log[(1+K\delta_1)/(1-K\delta_1)])(k-m)}, \quad m \leq k,$$

$\tilde{\Phi}_i(k,m)$ being the respective transition matrices. Provided that

$$\log[(1 + K\delta_1)/(1 - K\delta_1)] < \alpha,$$

it follows that (15) is strongly exponentially separated with constant projection $P = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{pmatrix}$. Hence (12) is strongly exponentially separated with projection $Q(k) = T(k)S(k)PS^{-1}(k)T^{-1}(k)$. Now the projection for the unperturbed system is $P(k) = T(k)PT^{-1}(k)$. It follows that

$$\begin{aligned} |Q(k) - P(k)| &\leq |T(k)|\,|S(k)PS^{-1}(k) - P|\,|T^{-1}(k)| \\ &\leq |T(k)| \left[|S(k) - \mathbf{I}|\,|P|\,|S^{-1}(k)| + |P|\,|S^{-1}(k) - \mathbf{I}|\right] |T^{-1}(k)| \\ &\leq M \left[|S(k) - \mathbf{I}|\,|P|\,|S^{-1}(k)| + |P|\,|S^{-1}(k) - \mathbf{I}|\right] \\ &\leq N\delta \end{aligned}$$

for some constant N, since $|S(k) - \mathbf{I}| = O(\delta)$, $|S^{-1}(k) - \mathbf{I}| = O(\delta)$.

6 Exponential Separation in Block Upper Triangular Systems

In this section we consider a block upper triangular system

$$u(k+1) = A(k)u(k) = \begin{pmatrix} A_{11}(k) & A_{12}(k) & \cdots & A_{1p}(k) \\ 0 & A_{22}(k) & \cdots & A_{2p}(t) \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A_{pp}(k) \end{pmatrix} u(k), \tag{20}$$

where $A_{ii}(t)$ is $n_i \times n_i$, and its associated block diagonal system

$$u(k+1) = \begin{pmatrix} A_{11}(k) & 0 & \cdots & 0 \\ 0 & A_{22}(k) & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A_{pp}(k) \end{pmatrix} u(k). \tag{21}$$

In Proposition 6, we show if (20) is strongly exponentially separated on a half-axis, then (21) is strongly exponentially separated also. Then in Proposition 7, we show the converse of this statement on a half-axis or whole axis but with a boundedness condition. At the end we give some results for diagonal systems and upper triangular systems, where the blocks are scalars.

Proposition 6 *If the block upper triangular system (20) is strongly exponentially separated on a half-axis J, then the projection for the exponential separation of (20) at $k = 0$ can be taken as*

$$P = \begin{pmatrix} P_1 & P_{12} & P_{13} & \cdots & \cdot & P_{1p} \\ 0 & P_2 & P_{23} & \cdots & \cdot & P_{2p} \\ 0 & 0 & P_3 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P_{p-1} & P_{p-1,p} \\ 0 & 0 & 0 & \cdots & \cdot & P_p \end{pmatrix}, \tag{22}$$

where the P_i are projections. Moreover, the block diagonal system (21) is strongly exponentially separated with projection at $k = 0$ given by

$$P = \begin{pmatrix} P_1 & 0 & 0 & \cdots & \cdot & 0 \\ 0 & P_2 & 0 & \cdots & \cdot & 0 \\ 0 & 0 & P_3 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P_{p-1} & 0 \\ 0 & 0 & 0 & \cdots & \cdot & P_p \end{pmatrix}.$$

In the unbounded case, "strongness" is necessary here to get exponential separation. Consider the following example.

Example 3

$$u(k+1) = \begin{pmatrix} 1 & e^k(e-1) \\ 0 & 1 \end{pmatrix} u.$$

This upper triangular system is exponentially separated on $\mathbb{Z}_+$, since we have the two solutions $(1, 0)$ and $(e^k - e, 1)$. However the diagonal system is clearly not exponentially separated. The reason for this is that the exponential separation is not strong.

In order to prove the proposition, we use the following lemma from [2].

Lemma 3 *Consider the Cartesian product of vector spaces*

$$U = U_1 \times U_2 \times \cdots \times U_k,$$

where $k \geq 2$. If V is a subspace of U, there is a projection P on U with range V, which has the form

$$P = \begin{pmatrix} P_1 & P_{12} & P_{13} & \cdots & \cdot & P_{1k} \\ 0 & P_2 & P_{23} & \cdots & \cdot & P_{2k} \\ 0 & 0 & P_3 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P_{k-1} & P_{k-1,k} \\ 0 & 0 & 0 & \cdots & 0 & P_k \end{pmatrix},$$

where P_i is a projection on U_i for $i = 1, \ldots, k$.

Now we prove Proposition 6.

Proof Denote by V the stable (resp. unstable) subspace for (20) at $k = 0$. Then it follows from Lemma 3 that there is a projection P of the form given in the statement of the Proposition which has range V; in the case $\mathbb{Z}_-$, we replace P by $\mathbf{I} - P$ so that the new P has nullspace V. It follows from Proposition 2 that we may take P as the projection for the exponential separation at $k = 0$. The transition matrix for (21) is

$$\tilde{U}(k,m) = \begin{pmatrix} \Phi_1(k,m) & 0 & \cdots & 0 \\ 0 & \Phi_2(k,m) & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \Phi_p(k,m) \end{pmatrix},$$

whereas that for (20) is

$$U(k,m) = \begin{pmatrix} \Phi_1(k,m) & W_{12}(k,m) & \cdots & W_{1p}(k,m) \\ 0 & \Phi_2(k,m) & \cdots & W_{2m}(k,m) \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \Phi_p(k,m) \end{pmatrix},$$

where $\Phi_i(k,m)$ is the transition matrix for $x_i(k+1) = A_{ii}(k)x_i(k)$. We take

$$\tilde{P} = \begin{pmatrix} P_1 & 0 & 0 & \cdots & \cdot & 0 \\ 0 & P_2 & 0 & \cdots & \cdot & 0 \\ 0 & 0 & P_3 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P_{p-1} & 0 \\ 0 & 0 & 0 & \cdots & \cdot & P_p \end{pmatrix}.$$

Then $\tilde{P}(k) = \tilde{U}(k,0)\tilde{P}\tilde{U}(0,k)$ is an invariant projection for (21) and $P(k) = U(k,0)PU(0,k)$ is a projection for the exponential separation of (20). We see that $U(k,0)PU(0,m) = U(k,m)P(m)$ and $\tilde{U}(k)\tilde{P}\tilde{U}^{-1}(m) = \tilde{U}(k,m)\tilde{P}(m)$ only differ in the (i,j)-entries, $i < j$, with those in $\tilde{U}(k,m)\tilde{P}(m)$ being zero. It follows that

$$|\tilde{U}(k,m)\tilde{P}(m)| \le |U(k,m)P(m)|.$$

(Note we use a matrix norm with the property that if $A = [A_{ij}]$ and $B = [B_{ij}]$ are partitioned matrices with $|A_{ij}| \le |B_{ij}|$ for all i, j, then $|A| \le |B|$.) Similarly

$$|\tilde{U}(m,k)(\mathbf{I} - \tilde{P}(k))| \le |U(m,k)(\mathbf{I} - P(k))|.$$

Since by Proposition 4 the strong exponential separation of (20) implies the existence of positive constants K and α such that

$$|U(k,m)P(m)|\,|U(m,k)(\mathbf{I} - P(k))| \le Ke^{-\alpha(k-m)}, \quad m \le k,$$

we conclude that

$$|\tilde{U}(k,m)\tilde{P}(m)|\,|\tilde{U}(m,k)(\mathbf{I}-\tilde{P}(k))| \le Ke^{-\alpha(k-m)}, \quad m \le k.$$

Then the proposition follows from Proposition 4.

Now we prove the converse.

Proposition 7 *Consider system* (20) *on* $J = \mathbb{Z}_+$, $\mathbb{Z}_-$ *or* $\mathbb{Z}$, *where* $A_{ii}^{-1}(k)A_{ij}(k)$ *is bounded for* $i < j$. *Then if the diagonal system* (21) *is strongly exponentially separated, system* (20) *is also strongly exponentially separated.*

Proof We define the matrix

$$S = \begin{pmatrix} \mathbf{I}_{n_1} & 0 & \cdots & 0 \\ 0 & \beta\mathbf{I}_{n_2} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \beta^{p-1}\mathbf{I}_{n_p} \end{pmatrix},$$

where $A_{ii}(k)$ is $n_i \times n_i$ and $\beta > 0$. A simple calculation shows that with $A(k)$ as in (20),

$$S^{-1}A(k)S = A_\beta(k) = \begin{pmatrix} A_{11}(k) & \beta A_{12}(k) & \cdots & \beta^{p-1}A_{1p}(k) \\ 0 & A_{22}(k) & \cdots & \beta^{p-2}A_{2p}(k) \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A_{pp}(k) \end{pmatrix}.$$

This means the constant kinematic similarity $u = Sv$ takes (20) into the system $v(k+1) = A_\beta(k)v(k)$, where

$$A_\beta(k) = \begin{pmatrix} A_{11}(k) & 0 & \cdots & 0 \\ 0 & A_{22}(k) & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & A_{pp}(k) \end{pmatrix} \left[\mathbf{I} + \begin{pmatrix} 0 & \beta A_{11}^{-1}(k)A_{12}(k) & \cdots & \beta^{p-1}A_{11}^{-1}(k)A_{1p}(k) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \beta A_{p-1,p-1}^{-1}(k)A_{p-1,p}(k) \\ 0 & 0 & \cdots & 0 \end{pmatrix} \right]$$

However, when $A_{ii}^{-1}(k)A_{ij}(k)$ is bounded for $i < j$, the last matrix is small when β is small. Then it follows from Theorem 1 that $v(k+1) = A_\beta(k)v(k)$ is strongly exponentially separated if β is sufficiently small. From this it follows that (20) is strongly exponentially separated.

Example 4 These examples show that in Proposition 7 the boundedness condition on $A_{ii}^{-1}(k)A_{ij}(k)$ for $i < j$ is necessary. In both examples the diagonal system is strongly exponentially separated. In the first example the triangular system is not exponentially separated but in the second example it is exponentially separated but not strongly so. Note we use the maximum norm in $\mathbb{R}^2$.

(i) Consider the system

$$u(k+1) = \begin{pmatrix} e & e^k \\ 0 & 1 \end{pmatrix} u(k).$$

Clearly the diagonal system is strongly exponentially separated on $\mathbb{Z}_+$ having the solutions

$$u_s(k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_u(k) = \begin{pmatrix} e^k \\ 0 \end{pmatrix}, \quad k \geq 0.$$

However we will show that the upper triangular system is not exponentially separated. The solution with $u(0) = (a, b)$ is $u(k) = (e^{k-1}(ea + bk), b)$. When $b \neq 0$, we see that $|u_u(k)|/|u(k)| \to 0$ as $t \to \infty$. So if the system is exponentially separated on $\mathbb{Z}_+$, $u_u(k)$ must be the unique stable solution and we may take $w(k) = (ke^{k-1}, 1)$ as the unstable solution. Then there must exist positive constants K and α such that

$$\frac{|u_u(k)|\,|w(m)|}{|u_u(m)|\,|w(k)|} \leq Ke^{-\alpha(k-m)}, \quad k \geq m.$$

That is,

$$\frac{m}{k} \leq Ke^{-\alpha(k-m)}, \quad \text{for any } k \geq m \geq 1.$$

But, with $k = 2m$, $m \geq 1$, this gives

$$\frac{1}{2} \leq Ke^{-\alpha m} \to 0 \quad \text{as } m \to \infty,$$

a contradiction. Hence the system is not exponentially separated.

(ii) Next consider the system:

$$u(k+1) = \begin{pmatrix} \frac{2}{k+1} & \delta \\ 0 & \frac{1}{k+1} \end{pmatrix} u(k).$$

When $\delta = 0$ it is strongly exponentially separated on $\mathbb{Z}_+$ with constant projection $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. The stable solution is $\left(0, \frac{1}{k!}\right)$ and the unstable is $\left(\frac{2^k}{k!}, 0\right)$. When $\delta \neq 0$, the transition matrix is

$$\Phi(k, 0) = \begin{pmatrix} \frac{2^k}{k!} & \frac{2^{k+1}\delta}{k!}\left(1 - \frac{k+2}{2^{k+1}}\right) \\ 0 & \frac{1}{k!} \end{pmatrix}.$$

Let $u(k)$ be the solution $\left(-\delta\frac{k+2}{k!}, \frac{1}{k!}\right)$ and let $v(k)$ be the solution $\left(\frac{2^k}{k!}, 0\right)$. Then, for $k \geq m \geq \delta^{-1} - 2$,

$$\frac{|u(k)|\,|v(m)|}{|u(m)|\,|v(k)|} = \frac{k+2}{m+2} 2^{-(k-m)} \leq e^{(k-m)/(m+2)} 2^{-(k-m)} \leq e^{-(\log 2 - \frac{1}{2})(k-m)}.$$

So this system is exponentially separated. However it is not strongly exponentially separated. To see this, note that taking $u(k) = \left(-\frac{k+2}{k!}, \frac{1}{\delta k!}\right)$ as the stable solution and $v(k) = \left(\frac{2^k}{k!}, 0\right)$ as the unstable corresponds to taking the projection $P(0)$ as

$$P(0) = \begin{pmatrix} 0 & -2\delta \\ 0 & 1 \end{pmatrix}.$$

Then, by invariance, a projection for the exponential separation is

$$P(k) = \Phi(k, 0)P(0)\Phi(0, k) = \begin{pmatrix} 0 & -\delta(k+2) \\ 0 & 1 \end{pmatrix}.$$

Since $|P(k)| \to \infty$ as $k \to \infty$, it follows from Proposition 5 that the system is not strongly exponentially separated.

We finish the section with two propositions about upper triangular systems, that is, block upper triangular systems where the blocks are scalars. We need the following lemma, which gives information about the stable subspace of (21) when it is strongly exponentially separated. Actually we only need it for scalar diagonal systems but prove it for the block diagonal case.

Lemma 4 *Assume (21) is exponentially separated on* $\mathbb{Z}+$ *(resp.* $\mathbb{Z}_-$*). Then the stable (resp. unstable) subspace is a Cartesian product* $V_1 \times V_2 \times \cdots \times V_p$*, where* V_i *is a subspace of* $\mathbb{R}^{n_i}$*. Moreover, if* $1 \le \dim V_i \le n_i - 1$*,* $u_i(k+1) = A_{ii}(k)u_i(k)$ *is exponentially separated with stable (resp. unstable) subspace* V_i *and strongly exponentially separated if (21) is.*

Proof First we prove the case $p = 2$. Define V_1 as the subspace of $\mathbb{R}^{n_1}$ consisting of those ξ for which $(\Phi_1(k, 0)\xi, 0)$ is in the stable (resp. unstable) subspace for (21) and V_2 as the subspace of $\mathbb{R}^{n_2}$ consisting of those η for which $(0, \Phi_2(k, 0)\eta)$ is in the stable (resp. unstable) subspace, where $\Phi_i(k, m)$ is the transition matrix for $u_i(k+1) = A_{ii}(k)u_i(k)$. Then $V_1 \times V_2$ is a subspace of the stable (resp. unstable) subspace for (21) at $k = 0$.

Suppose $z(k) = (u_1(k), u_2(k))$ is a solution of (21) (still with $p = 2$) in the stable (resp. unstable) subspace with $u_1(0) \neq 0, u_2(0) \neq 0$ and suppose there exists α and β with $\alpha\beta \neq 0$ for which $w(k) = (\alpha u_1(k), \beta u_2(k))$ is not in the stable (resp. unstable) subspace. Then we can choose the unstable (resp. stable) space so as to include $w(k)$. Then, in the $\mathbb{Z}_+$ case, there exist positive constants K and γ such that

$$\frac{|z(k)|\,|w(m)|}{|z(m)|\,|w(k)|} \le Ke^{-\gamma(k-m)}, \quad k \ge m.$$

However note that $w(k) = Dz(k)$, where $D = \begin{pmatrix} \alpha\mathbf{I}_{n_1} & 0 \\ 0 & \beta\mathbf{I}_{n_2} \end{pmatrix}$, so that

$$\frac{|w(m)|}{|w(k)|} \ge \frac{|z(m)|}{|z(k)|}\frac{1}{|D|\,|D^{-1}|}.$$

This implies that

$$\frac{1}{|D|\,|D^{-1}|} \le Ke^{-\gamma(k-m)}, \quad k \ge m,$$

which is clearly absurd. In the $\mathbb{Z}_-$ case, there exist positive constants K and γ such that

$$\frac{|w(k)|\,|z(m)|}{|w(m)|\,|z(k)|} \le Ke^{-\gamma(k-m)}, \quad k \ge m.$$

However

$$\frac{|w(k)|}{|w(m)|} \ge \frac{|z(k)|}{|z(m)|}\frac{1}{|D|\,|D^{-1}|}$$

which, as before, leads to an absurdity. So we have proved that if $z(k) = (u_1(k), u_2(k))$ is a solution in the stable (resp. unstable) subspace, then all solutions $(\alpha u_1(k), \beta u_2(k))$ with $\alpha\beta \ne 0$ are in the stable (resp. unstable) subspace. Then we can let $(\alpha, \beta) \to (1, 0)$ and $(0, 1)$ and conclude, by continuity, that $(u_1(k), 0)$ and $(0, u_2(k))$ are in the stable (resp. unstable) subspace. It follows that $z(0) \in V_1 \times V_2$. Thus we have shown that the stable (resp. unstable) subspace at $k = 0$ is $V_1 \times V_2$.

Then we let W_i be subspaces such that $\mathbb{R}^{n_i} = V_i \oplus W_i$. Then by Proposition 2 we may take $W = W_1 \times W_2$ as the unstable (resp. stable) subspace at $k = 0$ for the whole system. Then the conclusion about exponential separation of $u_i(k+1) = A_{ii}(k)u_i(k)$ follows easily. Note also that when (21) is strongly exponentially separated, it follows from Proposition 5 that the angle between $V_1(k) \times V_2(k)$ and $W_1(k) \times W_2(k)$ (here $V_i(k) = \Phi_i(k, 0)V_i$, etc.) is bounded below by a positive number. Hence the angles between the $V_i(k)$ and $W_i(k)$ are bounded below by positive numbers also, and so the systems $u_i(k+1) = A_{ii}(k)u_i(k)$ are strongly exponentially separated also.

The proof for general $p \ge 2$ follows by induction on p using the $p = 2$ case.

Now we give a necessary and sufficient condition that a diagonal system be strongly exponentially separated. As we shall see, strong exponential separation and exponential separation are equivalent for diagonal systems.

Proposition 8 *The diagonal system*

$$u_i(k+1) = a_i(k)u_i(k), \quad i = 1, \ldots, n \tag{23}$$

is exponentially separated on $J = \mathbb{Z}$, $\mathbb{Z}_+$ or $\mathbb{Z}_-$ with rank r if and only if there exists $I \subset \{1, \ldots, n\}$, where $\#I = r$, and there exist constants K and $\alpha > 0$ such that

$$\sum_{p=m}^{k-1}[\log|a_i(p)| - \log|a_j(p)|] \le K - \alpha(k-m) \tag{24}$$

for $i \in I$, $j \notin I$ and $k \ge m$ in J. Morever, if (23) is exponentially separated, it is also strongly exponentially separated.

Proof Suppose (23) is exponentially separated on $\mathbb{Z}_+$ (resp. $\mathbb{Z}_-$). Then by Lemma 4, there exists I such that the stable (resp. unstable) subspace is span$\{e_i : i \in I\}$ (resp. span$\{e_i : i \notin I\}$), where the e_i form the standard basis in $\mathbb{R}^n$. Then we may take the unstable (resp. stable) subspace as span$\{e_i : i \notin I\}$ (resp. span$\{e_i : i \in I\}$). If (23) is exponentially separated on $\mathbb{Z}$, then these must of course be the stable and unstable subspaces. Then if $x_i(k) = \prod_{p=0}^{k-1} a_i(p)e_i$, there exist positive constants K and α such that

$$\frac{|x_i(k)||x_j(m)|}{|x_i(m)||x_j(k)|} \le Ke^{-\alpha(k-m)}, \quad k \ge m \text{ in } J,$$

for $i \in I$, $j \notin I$, from which the inequalities (24) follow with $\log K$ instead of K.

Suppose conversely that inequalities (24) hold. Let P be the projection with range span$\{e_i : i \in I\}$ and nullspace span$\{e_i : i \notin I\}$. Then, for $k \ge m$, if $\Phi(k, m)$ is the transition matrix for (23), $\Phi(k, m)P$ is a diagonal matrix with $\prod_{p=m}^{k-1} a_i(p)$ in the ith position when $i \in I$ and 0 otherwise and $\Phi(m, k)(\mathbf{I} - P)$ is a diagonal matrix with $\prod_{p=m}^{k-1}(1/a_j(p))$ in the jth position when $j \notin I$ and 0 otherwise. If we use the maximum norm for matrices, then for all $k \ge m$ there exist $i \in I$ and $j \notin I$ such that

$$|\Phi(k, m)P|\,|\Phi(m, k)(\mathbf{I} - P)| = \prod_{p=m}^{k-1} |a_i(p)| \prod_{p=m}^{k-1} (1/|a_j(p)|) \le e^K e^{-\alpha(k-m)}.$$

The strong exponential separation follows.

Finally we examine the relation between the strong exponential separation of an upper triangular system and its associated diagonal system.

Proposition 9 *If the upper triangular system*

$$u(k+1) = A(k)u(k) = \begin{pmatrix} a_{11}(k) & a_{12}(k) & \cdots & a_{1n}(k) \\ 0 & a_{22}(k) & \cdots & a_{2n}(k) \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & a_{nn}(k) \end{pmatrix} u$$

is strongly exponentially separated on a half-axis with rank r, then there exists $I \subset \{1, \ldots, n\}$, where $\#I = r$, and there exist constants K and $\alpha > 0$ such that

$$\sum_{p=m}^{k-1} [\log |a_{ii}(p)| - \log |a_{jj}(p)|] \le K - \alpha(k - m)$$

for $i \in I$, $j \notin I$ and $k \ge m$. The converse holds if $\frac{a_{ij}(k)}{a_{ii}(k)}$ is bounded for $i < j$ and on the whole axis also.

Proof Immediate from Propositions 8, 6 and 7.

7 Exponential Separation in Linear Symplectic Systems

In this section we restrict to bounded systems so that exponential separation is equivalent to strong exponential separation. We begin with the observation that, in general, exponential separation does not imply that the *dichotomy spectrum* (or Sacker–Sell spectrum) has at least two components. For example, consider the diagonal system

$$x(k+1) = \begin{pmatrix} a(k) & 0 \\ 0 & e^{\gamma} a(k) \end{pmatrix} x(k)$$

on $\mathbb{Z}_+$, with $0 < \gamma < \beta - \alpha$, where $a(k)$ is a bounded real sequence such that $1/a(k)$ is also bounded satisfying

$$\alpha = \liminf_{k-m\to\infty} \sum_{p=m}^{k-1} \log|a(p)| < \beta = \limsup_{k-m\to\infty} \sum_{p=m}^{k-1} \log|a(p)|.$$

This is clearly exponentially separated. However for any $\lambda > 0$,

$$x(k+1) = \begin{pmatrix} \lambda^{-1} a(k) & 0 \\ 0 & \lambda^{-1} e^{\gamma} a(k) \end{pmatrix} x(k)$$

has an exponential dichotomy if and only if each scalar equation has (see Theorem 3.1 in [12]) and hence if and only if (see Proposition 2.4 in [12])

$$1 \notin [e^{\alpha}/\lambda, e^{\beta}/\lambda], \text{ that is, } \lambda \notin [e^{\alpha}, e^{\beta}]$$

and

$$1 \notin [e^{\alpha+\gamma}/\lambda, e^{\beta+\gamma}/\lambda], \text{ that is, } \lambda \notin [e^{\alpha+\gamma}, e^{\beta+\gamma}].$$

So the dichotomy spectrum of the system is the single interval $[e^{\alpha}, e^{\beta}] \cup [e^{\alpha+\gamma}, e^{\beta+\gamma}] = [e^{\alpha}, e^{\beta+\gamma}]$.

However in this section, we show when the system is bounded linear symplectic and it is exponentially separated with subspaces of the same dimension, the system has an exponential dichotomy with stable and unstable subspaces having the same dimension and hence the dichotomy spectrum has at least two components.

Let

$$x(k+1) = A(k)x(k), \quad x(k) \in \mathbb{R}^{2n} \tag{25}$$

be a linear system, where $A(k)$ is bounded together with its inverse, and is symplectic, that is,

$$A^*(k)\mathcal{J}A(k) = \mathcal{J}, \quad \text{where} \quad \mathcal{J} = \begin{pmatrix} 0 & \mathbf{I}_n \\ -\mathbf{I}_n & 0 \end{pmatrix}.$$

Theorem 2 *Let* (25) *be a linear symplectic system, where* $A(k)$ *is bounded together with its inverse. Suppose* (25) *is exponentially separated on an interval* $J = \mathbb{Z}$, $\mathbb{Z}_+$ *or* $\mathbb{Z}_-$ *such that the stable and unstable subspaces have the same dimension* n. *Then* (25) *has an exponential dichotomy on* J *with stable subspace of dimension* n.

Proof The transition matrix $\Phi(k, m)$ associated with (25) is symplectic. So, as in [2], we have the Iwasawa decomposition

$$\Phi(k, 0) = G(k)R(k),$$

where $G(k)$ is orthogonal and

$$R(k) = \begin{pmatrix} R_{11}(k) & R_{12}(k) \\ 0 & (R_{11}^*(k))^{-1} \end{pmatrix},$$

$R_{11}(k)$ being upper triangular with positive diagonal entries. Note that

$$A(k) = \Phi(k+1, k) = \Phi(k+1, 0)\Phi(0, k) = G(k+1)R(k+1)R^{-1}(k)G^{-1}(k).$$

The transformation $x(k) = G(k)y(k)$ takes system (25) to

$$y(k+1) = B(k)y(k) = G^{-1}(k+1)A(k)G(k)y(k) = R(k+1)R^{-1}(k)y(k). \tag{26}$$

It is clear that $B(k)$ and its inverse are bounded. Of course, it is trivial that $G(k)$ and its inverse are bounded. It follows that (25) and (26) share those dynamical properties preserved by kinematic similarity.

$B(k)$ is a block upper triangular matrix

$$B(k) = R(k+1)R^{-1}(k) = \begin{pmatrix} B_{11}(k) & B_{12}(k) \\ 0 & B_{22}(k) \end{pmatrix},$$

where $B_{11}(k) = R_{11}(k+1)R_{11}^{-1}(k)$ is upper triangular and

$$B_{22}(k) = (R_{11}^*(k+1))^{-1}R_{11}^*(k) = (B_{11}^*)^{-1}(k).$$

Then $B_{22}(k)$ is lower triangular with diagonal entries the reciprocal of those in $B_{11}(k)$.

Since (25), and therefore also (26), is exponentially separated with rank n on an interval $J = \mathbb{Z}$, $\mathbb{Z}_+$ or $\mathbb{Z}_-$, we conclude using Proposition 6 that the block diagonal system

$$y(k+1) = \begin{pmatrix} B_{11}(k) & 0 \\ 0 & B_{22}(k) \end{pmatrix} y(k)$$

is exponentially separated with rank n on J also. We can turn this into an upper triangular system by reversing the order of the last n variables. Then we can conclude, by Proposition 9, that on each half-axis (that is, on J when $J = \mathbb{Z}_+$ or $\mathbb{Z}_-$ and on

$\mathbb{Z}_+$ and $\mathbb{Z}_-$ when $J = \mathbb{Z}$), its diagonal part is also strongly exponentially separated with rank n. Returning to the original variables, this diagonal system has the form

$$y_i(k+1) = a_i(k)y(k), \quad i = 1, \dots, 2n,$$

where $a_{i+n}(k) = 1/a_i(k)$ for $i = 1, \dots, n$. It follows from Proposition 8 that there exist constants K and $\alpha > 0$ such that the set $\{1, 2, \dots, 2n\}$ can be divided into two disjoint subsets I_1 and I_2 of cardinality n with

$$\sum_{p=m}^{k-1} [\log|a_i(p)| - \log|a_j(p)|] \le K - \alpha(k-m), \quad k \ge m$$

when $i \in I_1$ and $j \in I_2$. Suppose we have $i \in I_1$ and $i + n \in I_1$ for some i. We subtract n from those numbers in I_1 or I_2 which exceed n. So there must be a number j, $1 \le j \le n$, which after this subtraction does not appear in I_1. This means $j \in I_2$ and $j + n \in I_2$. Then

$$\sum_{p=m}^{k-1} [\log|a_i(p)| - \log|a_j(p)|] \le K - \alpha(k-m), \quad k \ge m$$

and

$$\sum_{p=m}^{k-1} [\log(|a_{i+n}(p)| - \log|a_{j+n}(p)|] \le K - \alpha(k-m), \quad k \ge m.$$

The latter means that

$$\sum_{p=m}^{k-1} [-\log(|a_i(p)| + \log|a_j(p)|] \le K - \alpha(k-m), \quad k \ge m.$$

and we conclude that

$$-K + \alpha(k-m) \le \sum_{p=m}^{k-1} [\log|a_i(p)| - \log|a_j(p)| \le K - \alpha(k-m)$$

so that

$$\alpha(k-m) \le K, \quad k \ge m,$$

which is clearly impossible. Hence i and $i + n$ cannot both belong to I_1. Similar reasoning shows that they cannot both belong to I_2.

Suppose $i \in I_1$. Then $i + n \in I_2$, where we interpret $i + n$ as $i - n$ when $i > n$. Then

$$\sum_{p=m}^{k-1}[\log|a_i(p)| - (-\log|a_i(p)|] \leq K - \alpha(k-m), \quad k \geq m$$

so that

$$\sum_{p=m}^{k-1} \log|a_i(p)| \leq K/2 - (\alpha/2)(k-m), \quad k \geq m.$$

Next suppose $i \in I_2$. Then $i + n \in I_1$, where we interpret $i + n$ as $i - n$ when $i > n$. Then

$$\sum_{p=m}^{k-1}[-\log|a_i(p)| - \log|a_i(p)|]u \leq K - \alpha(k-m), \quad k \geq m$$

so that

$$\sum_{p=m}^{k-1} \log|a_i(p)| \geq -K/2 + (\alpha/2)(k-m), \quad k \geq m.$$

Hence the diagonal system has an exponential dichotomy on J with projection of rank n when J is a half-axis and on each half-axis when $J = \mathbb{Z}$. Then, by Theorem 4.1 in [12], system (26) and hence (25) has an exponential dichotomy on J with projection of rank n when J is a half-axis and on both half-axes when J is the whole axis. This finishes the proof when $J = \mathbb{Z}_+$ or $\mathbb{Z}_-$.

When $J = \mathbb{Z}$, we know from Corollary 1 that on $\mathbb{Z}_+$ the stable subspace (for the exponential separation) on $\mathbb{Z}_+$ and the unstable subspace on $\mathbb{Z}_-$ intersect in 0 and each has dimension n. However, by Corollary 3, since (25) has an exponential dichotomy with projection not equal to zero or the identity, it is exponentially separated with the same stable and unstable subspaces, the stable (resp. unstable) subspace being unique when $J = \mathbb{Z}_+$ (resp. $\mathbb{Z}_-$). Hence, by the discrete analogue of Proposition 1 in [1], system (25) has an exponential dichotomy on $\mathbb{Z}$ with projection of rank n.

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References

1. Battelli, F., Palmer, K.J.: Criteria for exponential dichotomy for triangular systems. J. Math. Anal. Appl. **428**, 525–543 (2015)
2. Battelli, F., Palmer, K.J.: Strongly exponentially separated linear systems. J. Dyn. Differ. Equ. **31**, 573–600 (2019)
3. Coppel, W.A.: Dichotomies and Stability. Lecture Notes. Springer, Berlin (1978)
4. Cong, N.D., Son, D.T.: An open set of unbounded cocycles with simple Lyapunov spectrum and no exponential separation. Stoch. Dyn. **7**, 335–355 (2007)
5. Fang, C., Gyllenberg, M., Liu, S.: Characterizations of dominated splitting and its relation to hyperbolicity. arXiv: 1209.5595 [math.DS]

6. Kalkbrenner, J.: Exponentielle Dichotomie und chaotische Dynamik nichtinvertierbarer Differenzengleichungen, Dissertation, Universität Augsburg (1994)
7. Palmer, K.J.: Exponential dichotomy, exponential separation and spectral theory for linear systems of ordinary differential equations. J. Differ. Equ. **46**, 324–345 (1982)
8. Palmer, K.J.: Exponential dichotomies, the shadowing lemma and transversal homoclinic points. In: Kirchgraber, U., Walther, H.-O. (eds.) Dynamics Reported, vol. 1, pp. 265–306. B.G. Teubner/Wiley, Stuttgart/Chichester (1988)
9. Papaschinopoulos, G.: Exponential separation, exponential dichotomy and almost periodicity of linear difference equations. J. Math. Anal. Appl. **120**, 276–287 (1986)
10. Papaschinopoulos, G.: A characterization of exponential separation in terms of reducibility for linear difference equations. J. Math. Anal. Appl. **133**, 231–248 (1988)
11. Pötzsche, C.: Nichtautonome Differenzengleichungen mit stationären und invarianten Mannigfaltigkeiten. Diplomarbeit, Universität Augsburg (1998)
12. Pötzsche, C.: Dichotomy spectra of triangular equations. Discret. Contin. Dyn. Syst. Ser. A **36**, 423–450 (2016)