Assignability of Lyapunov Spectrum for Discrete Linear Time-Varying Systems

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Abstract We discuss relations between the four formulations of the problem of assignability of the Lyapunov spectrum for discrete linear time-varying systems by a time-varying feedback. For two of them: global assignability and proportional local assignability, we have already [2–4] obtained sufficient conditions in terms of uniform complete controllability and certain asymptotic properties of the free system. In the present paper we discuss the assumptions of our papers and demonstrate the use of the obtained conditions by numerical examples. We also compare our results with the classical pole placement problem. Finally, we formulate a couple of directions for further research in this area.

Keywords Discrete linear time-varying system · Lyapunov spectrum · Pole placement problem · Controllability

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1 Introduction

One of the main methods of designing the control strategy for linear systems with time-invariant coefficients is pole placement method [17]. This method is based on the selection of feedback in such a way that the poles of the closed-loop system are in advance given points on the complex plane. The theoretical basis of this method is the following classical theorem [8, p. 458] (see also [7]): the pair $(A, B) \in \mathbb{R}^{s\times s} \times \mathbb{R}^{s\times t}$ is completely controllable if and only if for any set $\Lambda = \{\mu_1, \mu_2, \ldots, \mu_s\}$ of arbitrary s complex numbers such that $\overline{\Lambda} = \{\overline{\mu}_1, \overline{\mu}_2, \ldots, \overline{\mu}_s\} = \Lambda$, there exists a constant matrix $U \in \mathbb{R}^{t\times s}$ such that the eigenvalues of $A + BU$ form the set Λ. At the same time, for such systems, relationships between the location of the poles and dynamic properties such as stability, stability and oscillation degrees, and the size of overshoot are well known and described in the literature [1].

This problem for time-varying systems is much more complex and less studied. In the literature most of the results are for continuous-time systems and they are summarized in [14]. For time-varying systems, there are various concepts of stability (asymptotic, uniform, exponential, etc. [13]). Similarly, the concept of controllability of such systems can be understood in several different ways (uniform, complete, output, etc. [12]). Moreover, for time-varying systems, we do not have an obvious notion which fully matches the concept of the poles of time-invariant systems. The Lyapunov exponents play, to a certain extent, the same role as the logarithms of absolute values of poles of discrete-time systems and real parts of poles of continuous-time systems.

The problems of assignability of the Lyapunov spectrum for discrete-time systems were considered by us in [2–4]. In this article, we summarize these results and compare time-invariant versions of them with pole placement theorem, give some examples and formulate directions of further research.

The paper is organized as follows. In Sect. 2, we introduce the basic notation and definitions. We present four different formulations of the assignability of the Lyapunov spectrum: global, proportional global, local and proportional local assignability. In Sect. 3 we recall our previous results from [2–4], discuss the assumptions and relations between them and present two examples to depict the relation between global and proportional local assignability, and uniform complete controllability. The work is ended with conclusions and formulations of some open questions.

2 Problem Statement

Let $\mathbb{R}^s$ be the s-dimensional Euclidean space with a fixed orthonormal basis and the Euclidean norm $\|\cdot\|$. By $\mathbb{R}^{s\times t}$ we will denote the space of all real matrices of the size $s \times t$ with the spectral norm, i.e. with the operator norm generated in $\mathbb{R}^{s\times t}$ by Euclidean norms in $\mathbb{R}^s$ and $\mathbb{R}^t$, respectively; $I \in \mathbb{R}^{s\times s}$ is the identity matrix. For any sequence $F = \big(F(n)\big)_{n\in\mathbb{N}} \subset \mathbb{R}^{s\times t}$ we define

$$\|F\|_\infty = \sup_{n\in\mathbb{N}} \|F(n)\|.$$

A bounded sequence $\big(L(n)\big)_{n\in\mathbb{N}} \subset \mathbb{R}^{s\times s}$ of invertible matrices such that $\big(L^{-1}(n)\big)_{n\in\mathbb{N}}$ is bounded, will be called the Lyapunov sequence. By $\mathbb{R}^s_\leq$ we denote the set of all nondecreasing sequences of s real numbers. For a fixed sequence $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{R}^s_\leq$ and any $\delta > 0$ let us denote by $O_\delta(\mu)$ the set of all sequences $\nu = (\nu_1, \ldots, \nu_s) \in \mathbb{R}^s_\leq$ such that $\max_{j=1,\ldots,s} |\nu_j - \mu_j| < \delta$.

We consider a discrete linear time-varying system

$$x(n+1) = A(n)x(n) + B(n)u(n), \quad n \in \mathbb{N}, \tag{1}$$

where $A = (A(n))_{n\in\mathbb{N}} \subset \mathbb{R}^{s\times s}$ is a Lyapunov sequence, $B = (B(n))_{n\in\mathbb{N}} \subset \mathbb{R}^{s\times t}$ is a bounded sequence, and $u = (u(n))_{n\in\mathbb{N}} \subset \mathbb{R}^t$ is a control sequence.

We denote the transition matrix of the free system

$$x(n+1) = A(n)x(n) \tag{2}$$

by $\Phi_A(n,m)$, $n, m \in \mathbb{N}$, and by $(x(n, x_0))_{n\in\mathbb{N}}$ its solution with the initial condition $x(1, x_0) = x_0$.

For $x_0 \in \mathbb{R}^s$, $x_0 \neq 0$, the Lyapunov exponent $\lambda(x_0)$ of the solution $x = (x(n, x_0))_{n\in\mathbb{N}}$ is defined as

$$\lambda(x_0) = \lambda[x] \doteq \limsup_{n\to\infty} \frac{1}{n} \ln \|x(n, x_0)\|$$

and $\lambda(0) \doteq -\infty$. It is well known (see [6], [9, pp. 51–52]) that if $A = (A\,(n))_{n\in\mathbb{N}}$ is a Lyapunov sequence, then the set of the Lyapunov exponents of all nontrivial solutions of system (2) contains at most s elements, say

$$-\infty < \Lambda_1(A) < \Lambda_2(A) < \cdots < \Lambda_q(A) < \infty,$$

where $q \leq s$. For each $i \in \{1, \ldots, q\}$, we consider the linear subspace

$$E_i = \left\{x_0 \in \mathbb{R}^s : \lambda(x_0) \leq \Lambda_i(A)\right\} \subset \mathbb{R}^s.$$

We also set $E_0 = \{0\}$. The multiplicity s_i of the Lyapunov exponent $\Lambda_i(A)$ is defined as

$$\dim E_i - \dim E_{i-1}, \quad i = 1, \ldots, q.$$

Note that $s_1 + \cdots + s_q = s$. The sequence of s numbers

$$\big(\Lambda_1(A), \ldots, \Lambda_1(A), \ldots, \Lambda_q(A), \ldots, \Lambda_q(A)\big),$$

where each Lyapunov exponent $\Lambda_i(A)$ appears s_i times, is called the Lyapunov spectrum of (2) (see [6], [9, p. 57]) and is denoted by

$$\lambda(A) = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_s(A)) .$$

We assume that the Lyapunov spectrum is numbered in nondecreasing order, i.e. $\lambda(A) \in \mathbb{R}^s_{\leq}$.

For any bounded sequence $U = (U(n))_{n\in\mathbb{N}} \subset \mathbb{R}^{t\times s}$, we consider a linear feedback control

$$u(n) = U(n)x(n), \quad n \in \mathbb{N}$$

for system (1). We identify this control u with the sequence U and call this sequence U a feedback control for system (1).

Definition 1 A bounded sequence

$$U = (U\ (n))_{n\in\mathbb{N}} \subset \mathbb{R}^{t\times s}$$

is said to be an admissible feedback control for system (1) if the sequence

$$(A\ (n) + B\ (n)\ U\ (n))_{n\in\mathbb{N}}$$

is a Lyapunov sequence.

Let $U = (U(n))_{n\in\mathbb{N}}$ be any admissible feedback control for system (1). Then, for the closed-loop system

$$x(n+1) = (A(n) + B(n)U(n))\, x(n) \tag{3}$$

we can define the Lyapunov spectrum

$$\lambda(A + BU) = \big(\lambda_1(A + BU), \ldots, \lambda_s(A + BU)\big) \in \mathbb{R}^s_{\leq}.$$

The next definition expresses one of the possible way of formulation of the Lyapunov spectrum assignability problem.

Definition 2 The Lyapunov spectrum of system (3) is called globally assignable if for each $\mu \in \mathbb{R}^s_{\leq}$ there exists an admissible feedback control U such that

$$\lambda(A + BU) = \mu. \tag{4}$$

In this definition there is in general no bound on the norm of the feedback control. In some practical applications it is desirable to have a bound on the control which tends to zero in case the placed Lyapunov spectrum tends to the Lyapunov spectrum of the free system. This requirement is the base for the following definition.

Definition 3 The Lyapunov spectrum of system (3) is called proportionally globally assignable if for all $\Delta > 0$ there exists $\ell = \ell(\Delta) > 0$ such that for any sequence $\mu = \big(\mu_1, \ldots, \mu_s\big) \in O_\Delta\big(\lambda(A)\big)$ there exists an admissible feedback control U, satisfying the estimate

$$\|U\|_\infty \leq \ell \max_{j=1,\ldots,s} |\lambda_j(A) - \mu_j| \tag{5}$$

and such that equality (4) is satisfied.

One may also consider the local version of assignability of the Lyapunov spectrum.

Definition 4 The Lyapunov spectrum of system (3) is called locally assignable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mu \in O_\delta\big(\lambda(A)\big)$ there exists an admissible feedback control U such that

$$\lambda(A + BU) = \mu \ \text{ and } \ \|U\|_\infty < \varepsilon.$$

Definition 5 The Lyapunov spectrum of system (3) is called proportionally locally assignable if there exist $\ell > 0$ and $\delta > 0$ such that for all $\mu \in O_\delta\big(\lambda(A)\big)$ there exists an admissible feedback control U, such that estimate (5) and equality (4) are satisfied.

All the proposed definitions of the assignability problem were formulated for continous-time systems in [14] and our Definitions 2, 4 and 5 are direct translations of their continuous counterparts. However, the direct translation of definition of proportional global assignability is as follows: the Lyapunov spectrum of system (3) is called proportionally globally assignable if there exists $\ell > 0$ such that for all $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{R}^s_\leq$ there exists a feedback control U, satisfying (4) and (5).

The next example justifies our modification.

Example 6 Let us consider a linear discrete-time control system

$$x(n+1) = x(n) + u(n). \tag{6}$$

Here the matrices $A(n)$, $B(n)$ have the sizes 1×1 and $A(n) = B(n) = 1$ for all n. Therefore, for the transition matrix of the free system

$$x(n+1) = x(n)$$

we have $\Phi_A(n, m) = 1$ for all n, m. Thus, system (6) is uniformly completely controllable with $K = 1$ (see Definition 7 below). Since every solution $x(n, x_0)$ of the free system is constant, it follows that the Lyapunov spectrum coincides with 0. Let us close system (6) by a feedback $u(n) = U(n)x(n)$. Then we get a system

$$x(n+1) = (1 + U(n))\, x(n). \tag{7}$$

By the Theorem 4.7 from [2] the Lyapunov spectrum of system (7) is globally assignable, so for every $\alpha \in \mathbb{R}$ we can construct a control U, such that the Lyapunov spectrum of system (7) coincides with the number α. Let us find out whether it is possible to find a number $\ell > 0$, such that for all $\alpha > 0$ there exists a control U for which we have

$$\lambda(A + BU) = \alpha, \quad \|U\|_\infty \le \ell\alpha. \tag{8}$$

Here we restrict ourselves to the consideration of positive numbers α, since below we will prove that even for this case it is impossible. Suppose that for each $\alpha > 0$ there exists a control U for which both conditions (8) are satisfied. Then for an arbitrary nontrivial solution $x_U(n, x_0)$ of system (7) we have estimates

$$\alpha = \lambda(A + BU) = \limsup_{n\to\infty} \frac{1}{n} \ln |x_U(n, x_0)|$$
$$= \limsup_{n\to\infty} \frac{1}{n} \ln\Big|\prod_{j=1}^{n-1}\big(1 + U(j)\big)x_0\Big| \le \limsup_{n\to\infty} \frac{1}{n} \ln \prod_{j=1}^{n-1}\big(1 + |U(j)|\big)|x_0|$$
$$\le \limsup_{n\to\infty} \frac{1}{n} \ln \prod_{j=1}^{n-1}\big(1 + \ell\alpha\big)|x_0| = \limsup_{n\to\infty} \frac{1}{n} \ln\big(1 + \ell\alpha\big)^{n-1}|x_0| = \ln(1 + \ell\alpha).$$

Thus, there exists $\ell > 0$ such that for each $\alpha > 0$ the inequality $\alpha \le \ln(1 + \ell\alpha)$ holds, that is, $e^\alpha \le 1 + \ell\alpha$. But this is impossible, since the exponential function grows faster than any linear function. But if we choose an arbitrary $\Delta > 0$, then there exists an $\ell = \ell(\Delta) > 0$ such that for each $\alpha \in \mathbb{R}$, $|\alpha| < \Delta$ there exists a control U for which the conditions (8) are satisfied. Here we can take $U(n) = e^\alpha - 1$. Then

$$\|U\|_\infty = |e^\alpha - 1| \le e^{|\alpha|} - 1 \le \ell|\alpha|,$$

where $\ell = \frac{e^\Delta - 1}{\Delta}$.

In our further consideration we will present some conditions for solvability of assignability problems of the Lyapunov spectrum for discrete-time systems. Uniform complete controllability is the first of these conditions.

Definition 7 ([10]) System (1) is called uniformly completely controllable if there exist $K \in \mathbb{N}$ and $\gamma > 0$ such that

$$W(k_0, k_0 + K) \ge \gamma I,$$

for all $k_0 \in \mathbb{N}$, where

$$W(k, n) \doteq \sum_{j=k}^{n-1} \Phi_A(k, j+1)B(j)B^T(j)\Phi_A^T(k, j+1)$$

is the Kalman controllability matrix.

3 Comparisons and Discussions of Assignability Problems

The next theorem presents a sufficient condition for global assignability of the Lyapunov spectrum.

Theorem 8 ([2]) *If system (1) is uniformly completely controllable, then the Lyapunov spectrum of system (3) is globally assignable.*

The next example taken from [3] shows that the global assignability of the Lyapunov spectrum does not imply in general the uniform complete controllability. Therefore, uniform complete controllability is only a sufficient, but not a necessary, condition for the global assignability.

Example 9 ([3]) Let us define a sequence $(n_k)_{k\in\mathbb{N}}$ by the recurrent formulae

$$n_1 = 1, \quad n_{2m} = m n_{2m-1}, \quad n_{2m+1} = m + n_{2m}$$

for all $m \in \mathbb{N}$, define

$$b(n) = \begin{cases} 1 \text{ for } & n = 1, \\ 1 \text{ for } n \in [n_{2m-1}, n_{2m} - 1], \\ 0 \text{ for } n \in [n_{2m}, n_{2m+1} - 1], \end{cases}$$

for $m = 2, 3, \ldots$, and consider the scalar linear control equation

$$x(n+1) = x(n) + b(n)u(n). \tag{9}$$

Equation (9) is not uniformly completely controllable. Indeed, for each $K \in \mathbb{N}$, there exists a number $m \doteq K$ such that the Kalman controllability matrix of Eq. (9) is equal to zero on the interval $[n_{2m}, n_{2m} + K]$:

$$W(n_{2m}, n_{2m} + K) = W(n_{2m}, n_{2m} + m) = W(n_{2m}, n_{2m+1}) = \sum_{j=n_{2m}}^{n_{2m+1}-1} b^2(j) = 0.$$

The closed-loop equation corresponding to Eq. (9) has the form

$$x(n+1) = \big(1 + b(n)U(n)\big)x(n), x \in \mathbb{R},\ U \in \mathbb{R},\ n \in \mathbb{N}. \tag{10}$$

Now, let us show that the above equation has the global assignability property of the Lyapunov spectrum. Fix any $\alpha \in \mathbb{R}$, denote $\beta = e^\alpha - 1$ and define $U(n) \equiv \beta$, $n \in \mathbb{N}$. The Lyapunov exponent of each nontrivial solution of Eq. (10) with the defined admissible U coincides with the upper mean value of the function $1 + \beta b(\cdot)$, that is, with the value

$$\mu \doteq \limsup_{n\to\infty} n^{-1} \sum_{j=1}^{n-1} \ln\big(1 + \beta b(j)\big).$$

Our aim is to prove that $\mu = \alpha$.

Put $\varphi(1) = 0$ and

$$\varphi(n) = \frac{1}{n} \sum_{j=1}^{n-1} \ln\bigl(1 + \beta b(j)\bigr)$$

for natural number $n > 1$.

It is clear that

$$\ln\bigl(1 + \beta b(n)\bigr) = \begin{cases} \alpha \text{ for } & n = 1, \\ \alpha \text{ for } n \in [n_{2m-1}, n_{2m} - 1], \\ 0 \text{ for } n \in [n_{2m}, n_{2m+1} - 1], \end{cases}$$

for $m = 2, 3, \ldots$.

Let $\alpha \geq 0$. Then,

$$0 \leq \ln(1 + \beta b(n)) \leq \alpha$$

and therefore, $\varphi(n) \leq \alpha$ for all $n \in \mathbb{N}$. Hence, $\mu \leq \alpha$. By the definition of the sequence $(n_k)_{k\in\mathbb{N}}$ we know that the sequence $(n_k)_{k\in\mathbb{N}}$ is strictly increasing for $k \geq 2$, tends to $+\infty$ and satisfies the relations

$$\lim_{m\to\infty} \frac{n_{2m-1}}{n_{2m}} = \lim_{m\to\infty} \frac{1}{m} = 0,$$

$$\lim_{m\to\infty} \frac{m}{n_{2m}} = \lim_{m\to\infty} \frac{1}{n_{2m-1}} = 0$$

and

$$\lim_{m\to\infty} \frac{n_{2m}}{n_{2m+1}} = \lim_{m\to\infty} \frac{1}{1 + m/n_{2m}} = 1.$$

Therefore,

$$\mu \geq \limsup_{m\to\infty} \varphi(n_{2m}) = \limsup_{m\to\infty} \frac{1}{n_{2m}} \sum_{j=1}^{n_{2m}-1} \ln\bigl(1 + \beta b(j)\bigr)$$

$$\geq \limsup_{m\to\infty} \frac{1}{n_{2m}} \sum_{j=n_{2m-1}}^{n_{2m}-1} \alpha = \alpha \lim_{m\to\infty} \frac{n_{2m} - n_{2m-1}}{n_{2m}} = \alpha.$$

Thus, $\mu = \alpha$.

Now let $\alpha \leq 0$. Then,

$$0 \geq \ln(1 + \beta b(n)) \geq \alpha$$

and therefore $0 \geq \varphi(n) \geq \alpha$ for all $n \in \mathbb{N}$. Hence, $\mu \geq \alpha$.

On the other hand, for each $k \in [n_{2m-1}, n_{2m}]$ with any natural number $m > 1$, we have

$$\begin{aligned}\varphi(k) &= \frac{1}{k}\Big(\sum_{j=1}^{n_{2m-1}-1} \ln\big(1+\beta b(j)\big) + (k-n_{2m-1})\alpha\Big) \\ &= k^{-1}\big(\varphi(n_{2m-1})n_{2m-1} + (k-n_{2m-1})\alpha\big) \\ &= \frac{n_{2m-1}}{k}\varphi(n_{2m-1}) + \alpha\frac{k-n_{2m-1}}{k} \le \varphi(n_{2m-1}).\end{aligned}$$

In addition, for $k = n_{2m}$, we obtain

$$\varphi(n_{2m}) \le \alpha\frac{n_{2m}-n_{2m-1}}{n_{2m}} = \alpha\left(1-\frac{1}{m}\right). \tag{11}$$

For each $k \in [n_{2m}, n_{2m+1}]$ with any $m \in \mathbb{N}$, we also have

$$\varphi(k) = k^{-1}\varphi(n_{2m})n_{2m} \le n_{2m+1}^{-1}\varphi(n_{2m})n_{2m} = \varphi(n_{2m+1}). \tag{12}$$

Thus, $\varphi(k) \le \varphi(n_{2m+1})$ for all $k \in [n_{2m}, n_{2m+2}]$. Moreover, from (11) and (12), we get

$$\varphi(n_{2m+1}) = \frac{n_{2m}}{n_{2m+1}}\varphi(n_{2m}) \le \alpha\frac{n_{2m}}{n_{2m+1}}\left(1-\frac{1}{m}\right),$$

so

$$\varphi(k) \le \alpha\frac{n_{2m}}{n_{2m+1}}\left(1-\frac{1}{m}\right)$$

for all $k \in [n_{2m}, n_{2m+2}]$. Note that

$$\lim_{m\to\infty} \alpha\frac{n_{2m}}{n_{2m+1}}\left(1-\frac{1}{m}\right) = \alpha.$$

Put

$$r(k) = \begin{cases} 0, & k = 1, \\ \dfrac{\alpha n_{2m}}{n_{2m+1}}\left(1-\dfrac{1}{m}\right), & k \in [n_{2m}, n_{2m+2}-1]. \end{cases}$$

It is clear that $r(k) \to \alpha$ as $k \to \infty$. Since $\varphi(k) \le r(k)$ for all $k \in \mathbb{N}$, we have

$$\mu = \limsup_{k\to\infty} \varphi(k) \le \limsup_{k\to\infty} r(k) = \alpha.$$

Therefore, $\mu = \alpha$.

Thus, Eq. (10) with the defined control U has the Lyapunov spectrum consisting of α, and the Lyapunov spectrum of the Eq. (10) is globally assignable.

To present a deeper relation between global assignability and uniform complete controllability let us introduce the concept of Bebutov hull of a sequence. For any bounded sequence $F_0 = (F_0(n))_{n\in\mathbb{N}} \subset \mathbb{R}^{q\times r}$ and any $m \in \mathbb{N}$, let us consider a sequence $F_m = (F_m(n))_{n\in\mathbb{N}}$, where $F_m(n) = F_0(n+m)$ is a shift of $F_0(n)$ by m. Let us denote by $\mathfrak{R}(F_0)$ the closure in the topology of pointwise convergence on $\mathbb{N}$ of the set $\{F_m(\cdot) : m \in \mathbb{N}\}$. It is well known that $\mathfrak{R}(F_0)$ is metrizable by means of the metric

$$\varrho(F, \widehat{F}) = \sup_{n\in\mathbb{N}} \min\{\|F(n) - \widehat{F}(n)\|, n^{-1}\}.$$

The space $(\mathfrak{R}(F_0), \varrho)$ is compact [15, p. 34] and it is called the Bebutov hull of the sequence F_0 (see [11, p. 32], [16]).

Let us identify system (1) with the sequence $(A, B) = \big(A(n), B(n)\big)_{n\in\mathbb{N}} \subset \mathbb{R}^{s\times(s+t)}$. The space $\mathfrak{R}(A, B)$ will be called the Bebutov hull of system (1).

Theorem 10 ([3]) *System (1) is uniformly completely controllable if and only if for each system from $\mathfrak{R}(A, B)$ the corresponding closed-loop system has globally assignable Lyapunov spectrum.*

For a given system (1), which is not uniformly completely controllable, the problem of finding a system from $\mathfrak{R}(A, B)$ such that corresponding closed-loop system does not have assignable Lyapunov spectrum is in general a difficult task. The proof of Theorem 10 does not give a recipe to find a "bad" system from the hull, but only establishes the fact of its existence. The example below presents this "bad" system explicitly.

Example 11 ([3]) Let us consider a linear control system

$$x(n+1) = A_0(n)x(n) + B_0(n)u(n), \;\; x \in \mathbb{R}^2, \;\; u \in \mathbb{R}^2, \;\; n \in \mathbb{N}, \tag{13}$$

where

$$A_0(n) = I \in \mathbb{R}^{2\times 2}, \quad B_0(n) = \begin{pmatrix} 1 & 0 \\ 0 & b(n) \end{pmatrix},$$

and the sequence $b(n)$ is defined in Example 9. Since the Kalman controllability matrix of system (13) has the form

$$W_0(k, n) = \sum_{j=k}^{n-1} B_0(j)B_0^T(j) = \begin{pmatrix} 1 & 0 \\ 0 & \sum_{j=k}^{n-1} b^2(j) \end{pmatrix},$$

it follows that for each $K \in \mathbb{N}$ there exists a number $m \doteq K$ such that

$$W_0(n_{2m}, n_{2m} + K) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It means that system (13) is not uniformly completely controllable.

We will show that the hull of this system contains the system

$$x(n+1) = A_0(n)x(n) + B(n)u(n), \tag{14}$$

where

$$B(n) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In fact, let us consider the sequence $\big(n_{2m}\big)_{m\in\mathbb{N}}$ and any $n \in \mathbb{N}$. Then we have

$$\|B_{n_{2m}}(n) - B(n)\| = |b(n_{2m} + n)|.$$

For any $m > n$ the following inequalities

$$n_{2m} < n_{2m} + n < n_{2m} + m = n_{2m+1}$$

hold, therefore $b(n_{2m} + n) = 0$. It means, that

$$\lim_{m\to\infty} \|B_{n_{2m}}(n) - B(n)\| = 0$$

for any $n \in \mathbb{N}$, what implies that $\big(A_0, B\big) \in \mathfrak{R}\big(A_0, B_0\big)$.

Now, we will show that the Lyapunov spectrum of the closed-loop system

$$x(n+1) = \big(A_0 + BU(n)\big)x(n) \tag{15}$$

is not assignable. In fact, for any feedback control $U(n) = \{u_{ij}(n)\}_{i,j=1,2}$ the coefficient matrix of the closed-loop system (15) has the following form

$$A_0 + BU(n) = \begin{pmatrix} 1 + u_{11}(n) & u_{12}(n) \\ 0 & 1 \end{pmatrix}.$$

For the second coordinate $x_2(n)$ of any solution $x(n)$ of system (15) we have the equality

$$x_2(n+1) = x_2(n), \quad n \in \mathbb{N},$$

which means that the second coordinate is constant. It is clear, that every fundamental system of solutions of system (15) contains a solution with the nonzero second coordinate and for this solution we have $\lambda[x] \geq \lambda[x_2] = 0$. It means that for any admissible feedback control U the Lyapunov spectrum of system (15) contains a nonnegative number and therefore the Lyapunov spectrum of system (15) is not assignable. Moreover, the stationarity of the second coordinate of any solution of this system, when choosing the arbitrary matrix control U, implies that system (15) is not stabilizable. Thus, system (14) is the "bad" system from the hull of system (13).

To conclude the example, we show that the Lyapunov spectrum of the original system

$$x(n+1) = \big(A_0(n) + B_0(n)U(n)\big)x(n) \tag{16}$$

is assignable.

The Lyapunov spectrum of the free system

$$x(n+1) = A_0(n)x(n)$$

coincides with the sequence $(0, 0)$. Let us fix any numbers $\alpha_1 \leq \alpha_2$, denote $\beta_i = e^{\alpha_i} - 1$, $i = 1, 2$, and apply to system (16) the feedback control

$$U(n) = \operatorname{diag}(\beta_1, \beta_2).$$

Then the closed-loop system (16) has the diagonal form

$$x(n+1) = \operatorname{diag}\big(1 + \beta_1, 1 + b(n)\beta_2\big)x(n), \tag{17}$$

and therefore its Lyapunov spectrum consists of upper mean values of the diagonal elements [9, p. 55], i.e. of the numbers

$$\mu_1 = \limsup_{n\to\infty} n^{-1} \sum_{j=1}^{n-1} \ln(1 + \beta_1) = \alpha_1,$$

$$\mu_2 = \limsup_{n\to\infty} n^{-1} \sum_{j=1}^{n-1} \ln\big(1 + \beta_2 b(j)\big) = \alpha_2.$$

Here the second equality follows from Example 9. It means that the spectrum of system (16) is globally assignable.

Now we will present a result about local proportional assignability of the spectrum of system (3). It will be expressed in terms of certain concepts from the asymptotic theory of linear systems, which are defined below.

Definition 12 ([9, p. 63]) System (2) is called regular (in the Lyapunov sense) if the following equality

$$\sum_{i=1}^{s} \lambda_i(A) = \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} \ln|\det A(j)|$$

holds.

Definition 13 (see [9, p. 100], [10, p. 15]) Let $\big(L(n)\big)_{n\in\mathbb{N}} \subset \mathbb{R}^{s\times s}$ be a Lyapunov sequence. A linear transformation

$$y = L(n)x, \quad n \in \mathbb{N}, \tag{18}$$

of the space $\mathbb{R}^s$ is called a Lyapunov transformation.

Definition 14 (*see* [10, p. 15]) We say that system (2) is dynamically equivalent to the system

$$y(n+1) = C(n)y(n), \quad n \in \mathbb{N}, \; y \in \mathbb{R}^s, \tag{19}$$

if there exists a Lyapunov transformation (18) which connects these systems, i.e. for every solution $x(n)$ of system (2) the function $y(n) = L(n)x(n)$ is a solution of system (19) and for every solution $y(n)$ of system (19) the function $x(n) = L^{-1}(n)y(n)$ is a solution of system (2).

Definition 15 System (2) is called diagonalizable if it is dynamically equivalent to a system (19) with a diagonal matrix $C = \big(C(n)\big)_{n\in\mathbb{N}}$.

Definition 16 ([5]) The Lyapunov spectrum of system (2) is called stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any Lyapunov sequence $R = (R(n))_{n\in\mathbb{N}} \subset \mathbb{R}^{s\times s}$ the inequality $\|R - I\|_\infty < \delta$ implies that $\lambda(AR) \in O_\varepsilon\big(\lambda(A)\big)$, where $\lambda(AR)$ is the Lyapunov spectrum of the multiplicatively perturbed system

$$z(n+1) = A(n)R(n)z(n), \quad n \in \mathbb{N}, \; z \in \mathbb{R}^s.$$

Theorem 17 ([4]) *Let system (1) be uniformly completely controllable and assume that at least one of the following conditions holds:*

1. *system (2) is regular;*
2. *system (2) is diagonalizable;*
3. *the Lyapunov spectrum of system (2) is stable.*

Then the Lyapunov spectrum of system (3) is proportionally locally assignable.

Let us compare the form of Theorems 8, 10 and 17 for time-invariant systems to the classical pole placement theorem cited in the introduction section.

The main differences are as follows:

(i) the problems are posed and solved for systems, not for a pair of matrices;
(ii) in our problem the Lyapunov spectrum is assigned, which coincides with the logarithms of the absolute values of eigenvalues of the coefficient matrix of the system, and not with the usual spectrum of this matrix as in the classical statement of the problem;
(iii) in case of Theorem 17 the assigned values of the spectrum lie in some neighborhood of the spectrum of the original system and not in the whole set of possible values of the spectrum;

(iv) in case of Theorem 17 there is a Lipschitz-type estimate of the norm of the feedback control needed to shift the spectrum by a given value from the original one;
(v) the feedback U constructed by us to assign the spectrum may depend on time;
(vi) in contrast to the classical theorem, our result provides only sufficient conditions for the assignability of the spectrum of system (3).

4 Conclusions and Open Problems

In this paper we formulated four statements of the problem of assignability of the Lyapunov spectrum for discrete linear time-varying systems: global, proportional global, local and proportional local assignability. We showed in [2] and [3] that uniform complete controllability is a sufficient but not a necessary condition for global assignability but it is a necessary and sufficient condition for global assignability of the spectrum of any system from the Bebutov hull of the original system. In [4] we also showed that diagonalizability, as well as Lyapunov regularity or stability of Lyapunov spectrum ensures the solvability of the problem of proportional local assignability for any uniformly completely controllable system. The proof of this result does not give reasons to suppose that the property of diagonalizability, regularity or stability of the Lyapunov exponents are necessary for the proportional local assignability or even close to those. The instability of the Lyapunov exponents of the original system means that the Lyapunov spectrum, considered as a function defined on the space of systems with the topology of the uniform convergence on $\mathbb{N}$, has a discontinuity at the point corresponding to the system under consideration, i.e. for arbitrarily small perturbations some of exponents may vary considerably having the so-called jumps. In this case, if the free system is neither diagonalizable nor regular, some of the corresponding control systems may not have the property of local proportional assignability of the exponents. However, even the construction of examples of such systems, not mentioning the study of the assignability of their exponents, is a difficult task that must be further investigated. The problem of the necessity of the condition of uniform complete controllability for local proportional assignability of the Lyapunov spectrum is also unsolved in general case. The problems of finding conditions for global proportional and local assignability remain open.

It is clear from the definitions that:

(1) global proportional assignability implies global assignability;
(2) local proportional assignability implies local assignability;
(3) global proportional assignability implies local proportional assignability.

The other relations between the proposed definitions of assignability are unknown.

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