Stability and Instability Regions for a Three Term Difference Equation

Petr Tomášek

Abstract The paper discusses stability and instability properties of difference equation $y(n+1) + ay(n-\ell+1) + by(n-\ell) = 0$ with real parameters a, b. Beside known results about its asymptotic stability conditions a deeper analysis of instability properties is introduced. An instability degree of difference equation's solution is introduced in analogy with theory of differential equations. Instability regions of a fixed degree are introduced and described in the paper. It is shown that dislocation of instability regions of various degrees obeys some rules and qualitatively depends on parity of difference equation's order.

Keywords Stability · Instability degree · Linear difference equation

1 Introduction

We are going to consider a three term linear difference equation

$$y(n+1) + ay(n-\ell+1) + by(n-\ell) = 0, \qquad n = 0, 1, 2, \dots, \tag{1}$$

where $a, b \in \mathbb{R}$ and $\ell \in \mathbb{N}$. Our aim is to analyze asymptotic stability and instability conditions for the equation with respect to parameters a, b and ℓ. We recall that a linear difference equation with constant parameters is asymptotically stable if any of its solution tends to zero while n tends to infinity. This property is ensured if and only if all zeros $\lambda_1, \lambda_2, \dots, \lambda_{\ell+1}$ of the equation's characteristic polynomial

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$$P(\lambda) = \lambda^{\ell+1} + a\lambda + b \tag{2}$$

lie inside the unit disk in a complex plane.

While the stability conditions are widely analyzed, the instability counterpart analysis remains submarginal. Since we are going to make a more detailed insight to instability properties of (1), we introduce a degree of instability of difference equation (1) in analogy with theory of delay differential equations (see [9, 11]). The following definition is formulated for (1), but it can be analogously considered also for another difference equations.

Definition 1 A number of polynomial (2) roots λ_k counted with their multiplicities, which satisfy $|\lambda_k| > 1$, is called *instability degree* of Eq. (1).

The degree of instability of (1) splits the parameter's plain (a, b) to disjoint domains. The subject of investigation in this paper is a description and some properties of the instability regions of (1).

2 Stability and Instability Regions

We start with a notion of stability region:

Definition 2 Let $\ell \in \mathbb{N}$. The set S_ℓ of all pairs $(a, b) \in \mathbb{R}^2$ for which (1) is asymptotically stable is called *asymptotic stability region*.

Remark 1 A fundamental necessary restriction for asymptotic stability region location is the following: Let (1) be asymptotically stable. Then $|b| < 1$. The assertion is obvious with respect to the fact that $|b|$ is a modulus of product of all the roots of (2). If (1) is asymptotically stable, then all roots of (2) have modulus lower then 1 and hence for modulus of their product it holds $|b| < 1$. From a graphical point of view it means that S_ℓ must be dislocated within the stripe $-1 < b < 1$ in (a, b) plane.

Remark 2 We can also mention a sufficient condition of asymptotic stability, which constructs the so-called Cohn domain of asymptotic stability, which is in the case of (1) in a form $|a| + |b| < 1$. This condition defines an opened square in (a, b) plane with circumradius one and with vertices situated on axes a and b symmetrically.

In a connection with a description of the region S_ℓ, we can recall necessary and sufficient conditions for (1) to be asymptotically stable. In [5] such conditions had been introduced, but as it was later shown, they were incorrect. It was pointed out in [14] that there was some ambiguity in a proof in [4] for a more general case of trinomial difference equation, but similar one depreciates the result in [5]. Correct conditions were later obtained and can be found in various forms in [2, 3, 12, 13]. The last mentioned recent paper also introduces a generalization of difference equation stability notion called $r-$stability. We present the asymptotic stability conditions for (1) in a form which can be obtained as a conclusion of result introduced in [1].

Lemma 1 *Let $a, b \in \mathbb{R}$ and $\ell \in \mathbb{N}$. If ℓ is odd then (1) is asymptotically stable if and only if $|a| < 1 + b$ and either*

$$b - 1 < |a| \leq 1 - b$$

or

$$|a| > |1 - b|, \quad \ell < \frac{\arccos \frac{-a^2+b^2-1}{2|a|}}{\arccos \frac{-a^2-b^2+1}{2|ab|}}.$$

If ℓ is even then (1) is asymptotically stable if and only if $|b| < 1 + a$ and either

$$a - 1 < |b| \leq 1 - a$$

or

$$|b| > |1 - a|, \quad \ell < \frac{\arccos \frac{-a^2+b^2-1}{2|a|}}{\arccos \frac{-a^2-b^2+1}{2|ab|}}.$$

Definition 3 Let $\ell, k \in \mathbb{N}$. The set $I_{\ell,k}$ of all pairs $(a, b) \in \mathbb{R}^2$ for which (1) has degree of instability k is called *region of the kth degree of instability*.

For a detailed description of S_ℓ and $I_{\ell k}$, $k = 1, 2, \ldots \ell + 1$ we employ the boundary locus technique. We consider $\lambda = e^{\omega i}$, $\omega \in \mathbb{R}$ as a root of polynomial (2), i.e.

$$e^{(\ell+1)\omega i} + ae^{\omega i} + b = 0.$$

Applying the Euler's rule and considering real and imaginary parts separately we get

$$\cos((\ell + 1)\omega) + a\cos(\omega) + b = 0, \tag{3}$$

$$\sin((\ell + 1)\omega) + a\sin(\omega) = 0, \tag{4}$$

respectively. It is enough to consider $\omega \geq 0$ since the left-hand sides of the above equations are even and odd, respectively. In the sequel we determine representation of curves in (a, b) plane of such pairs of parameter (a, b) for which $P(\lambda^*) = 0$. There are only three possible cases to consider with respect to value of ω:

Case 1. For $\omega = 0$ we have a straight line $b = -a - 1$.

Case 2. For $\omega = m\pi, m \in \mathbb{N}$ Eq. (4) is fulfilled trivially and (3) gives straight lines

$$b = a - 1 \quad \text{for} \quad \ell \quad \text{odd},$$
$$b = a + 1 \quad \text{for} \quad \ell \quad \text{even}.$$

Case 3. For $\omega \neq r\pi$, $r \in \mathbb{N}_0$. Equations (4) and (3) give

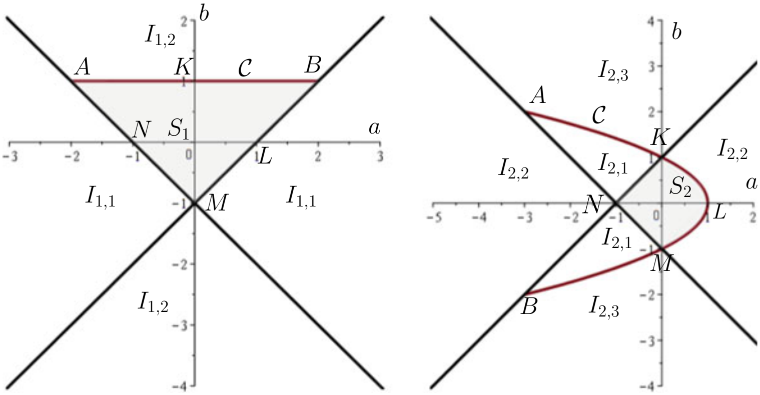

Fig. 1 Stability and instability regions for $\ell = 1$ (left) and $\ell = 2$ (right)

$$a = \frac{-\sin((\ell+1)\omega)}{\sin(\omega)} \tag{5}$$

$$b = \sin((\ell+1)\omega)\cot(\omega) - \cos((\ell+1)\omega) = \frac{\sin(\ell\omega)}{\sin(\omega)}. \tag{6}$$

Considering $\omega \in (0, \pi)$ the above introduced pair of a, b gives parametric expression of a curve $\mathcal{C}$ where some root of $P(\lambda)$ stands on the boundary of the unit circle. The straight lines from cases 1 and 2 together with the curve $\mathcal{C}$ represent the boundary between stability region and regions of various instability degrees.

It is enough to consider $\omega \in (0, \pi)$ to express the boundary curve, since for $\omega \neq r\pi$, $r \in \mathbb{N}_0$ the points of the same curve are obtained. Analyzing the limits of (a, b) for $\omega \to 0^+$ and $\omega \to \pi^-$ we obtain the boundary curve $\mathcal{C}$ endpoints A and B, respectively.

$$A = \lim_{\omega\to 0^+}(a, b) = (-\ell - 1, \ell)$$

and

$$B = \lim_{\omega\to\pi^-}(a, b) = (\ell + 1, \ell) \qquad \text{for } \ell \text{ odd,}$$
$$B = \lim_{\omega\to\pi^-}(a, b) = (-\ell - 1, -\ell) \qquad \text{for } \ell \text{ even.}$$

In the following figures the stability and instability regions are introduced. They are separated by a bold curve $\mathcal{C}$ and bold straight lines corresponding to cases 1 and 2. The asymptotic stability region S_ℓ is highlighted by grey color (Figs. 1 and 2).

To have a better insight into the regions dislocation we introduce some of their properties.

Lemma 2 *The axis b is a part of the asymptotic stability region S_ℓ for $b \in (-1, 1)$ and of instability region of the highest degree $I_{\ell,\ell+1}$ for $b \notin [-1, 1]$.*

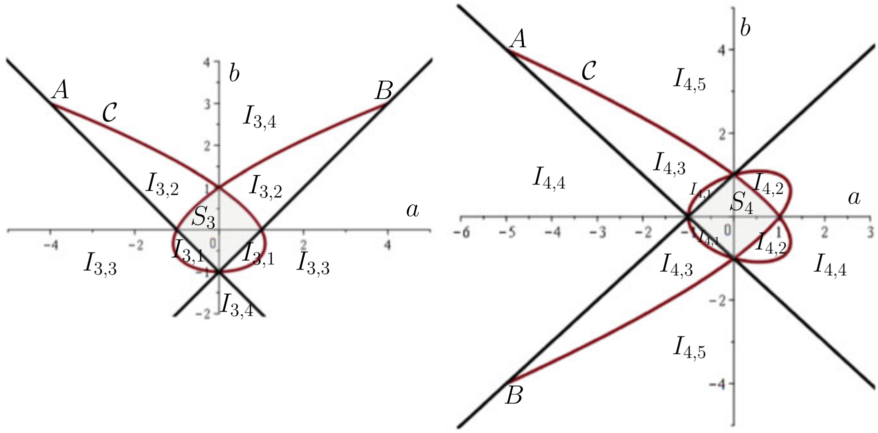

Fig. 2 Stability and instability regions for $\ell = 3$ (left) and $\ell = 4$ (right)

Proof In the case of $a = 0$ the characteristic polynomial $P(\lambda)$ of (1) has the form

$$P(\lambda) = \lambda^{\ell+1} + b$$

which all $\ell + 1$ roots have modulus $|b|^{1/(\ell+1)}$. □

A similar situation occurs for the axis a:

Lemma 3 *The axis a is a part of the asymptotic stability region S_ℓ for $a \in (-1, 1)$ and of instability region of the second highest degree $I_{\ell,\ell}$ for $a \notin [-1, 1]$.*

Proof In the case of $b = 0$ the characteristic polynomial $P(\lambda)$ of (1) has the form

$$P(\lambda) = \lambda^{\ell+1} + a\lambda,$$

which has one root $\lambda = 0$ and all the other ℓ roots have modulus $|a|^{1/\ell}$. □

Now we move our attention to another property, which can be observed from the above figures. The bold straight lines represent the pairs (a, b) for which a real root of modulus one occurs in (2). Thence there is a change just for 1 degree of instability between the neighboring regions on the segments of these lines, where there is no intersection with another curve of boundary. Similarly, on the curve segments of $\mathcal{C}$ free of any intersections points with another part of region boundary a switch for two degrees of instability is expected between the neighboring regions since two complex conjugate roots with unit modulus are present. There is sketched a situation for general odd and even ℓ at Figs. 3 and 4. It can be also observed that curve $\mathcal{C}$ is dislocated within two stripes of width $\sqrt{2}$ which long axes coincide with quadrants symmetry axes. In another words $\mathcal{C} \in T$, where $T = \{(a, b) \in \mathbb{R}^2 : (-a - 1 \le b \le -a + 1) \vee (a - 1 \le b \le a + 1)\}$. This observation can be formulated and proved as follows:

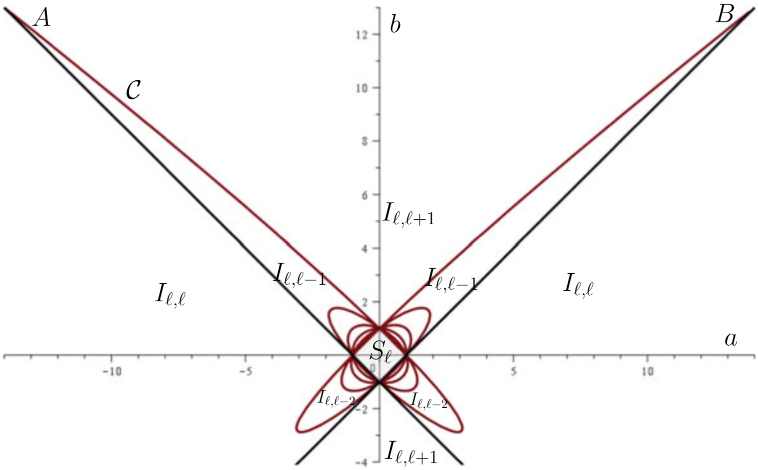

Fig. 3 Stability and instability regions for odd $\ell = 13$

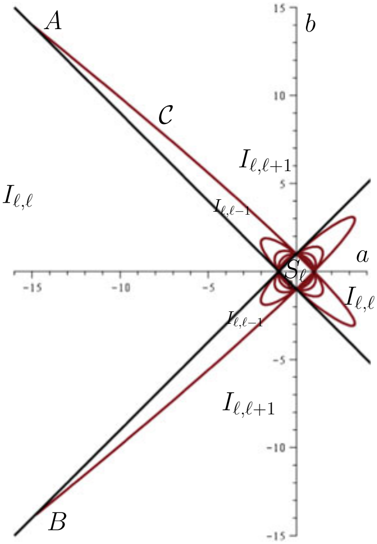

Fig. 4 Stability and instability regions for even $\ell = 14$

Theorem 1 *Let* $\mathcal{C}$ *be curve in plane* (a, b) *defined by (5), (6), where* $\omega \in (0, \pi)$, $\ell \in \mathbb{N}$. *Then* $\mathcal{C} \subset T$, *where* $T = \{(a, b) \in \mathbb{R}^2, |b| \leq |a| + 1, |a| \leq |b| + 1\}$.

Proof From (6) we get $|b| = |a\cos(\omega) + \cos((\ell + 1)\omega)| \leq |a\cos(\omega)| + |\cos((\ell + 1)\omega)|$ which gives $|b| \leq |a| + 1$.

To show $|a| \leq |b| + 1$ we consider (5) which implies $|a| = \left|\frac{\sin((\ell+1)\omega)}{\sin(\omega)}\right| = |b\cos(\omega) + \cos(\ell\omega)| \leq |b\cos(\omega)| + |\cos(\ell\omega)| \leq |b| + 1$. □

Corollary 1 *Let* $U = \{(a, b) \in \mathbb{R}^2, |b| > |a| + 1\}$. *Then* $U \subset I_{\ell,\ell+1}$.

Proof The assertion follows from Lemma 2 and Theorem 1 with respect to dislocation of instability regions boundaries. □

Corollary 2 *Let* $V = \{(a, b) \in \mathbb{R}^2, |a| > |b| + 1\}$. *Then* $V \subset I_{\ell,\ell}$.

Proof The assertion follows from Lemma 3 and Theorem 1 with respect to dislocation of instability regions boundaries. □

From the above assertions it follows that for instability regions up to instability degree $\ell - 1$ it holds $I_{\ell,k} \subset T$, $k = 1, 2, \ldots, \ell - 1$. Naturally, the same holds for the stability region: $S_\ell \subset T$. A general instability degree regions portrait can be assembled in the (a, b) plane with respect to the previous considerations.

Instability regions can be determined by their boundary, which consists of the curve $\mathcal{C}$ segments and eventually segments of straight lines $b = -a - 1$, $b = -a + 1$ and $b = a - 1$. From the parametric Eq. (5), (6) of the curve $\mathcal{C}$ it follows that crossing the curve $\mathcal{C}$ in the point (a, b) in the direction (a, b) (faraway from the origin of the plane) the instability degree rises for two in the points where $\mathcal{C}$ does not intersect itself. In such points the curve $\mathcal{C}$ has just two simple complex conjugate roots. The fact that the conjugate pair of the roots crosses the unit disk boundary outwards follows immediately from the parametric expression of the curve in (a, b). Indeed, consider roots of the characteristic polynomial $P(\lambda)$ on the circle with a general radius ρ, i.e. $\lambda = \rho \exp(i\omega)$. Then using the same steps as in the case of the curve $\mathcal{C}$ description we introduce curve $\mathcal{C}_\subset$, which is connecting pairs (a, b) where roots of $P(\lambda)$ with modulus ρ are presented. Parametric expression of $\mathcal{C}_\subset$ is then

$$a = -\rho^\ell \frac{\sin((\ell + 1)\omega)}{\sin(\omega)}$$
$$b = \rho^{\ell+1} \frac{\sin(\ell\omega)}{\sin(\omega)}.$$

where $\omega \in (0, \pi)$. Considering fixed parameters ω and ℓ a positive perturbation of modulus ρ in any pair $(a, b) \in \mathcal{C}$ moves this point away from the origin in a direction of $(\ell a, (\ell + 1)b)$ vector.

For description of instability regions we split the curve $\mathcal{C}$ to 2ℓ curve segments $\mathcal{C}_i$ given by (5), (6) with $\omega \in J_i = [(i-1)\frac{\pi}{2\ell}, i\frac{\pi}{2\ell}]$, $i = 1, 2, \ldots, 2\ell$. Next we denote $K = (0, 1)$, $L = (1, 0)$, $M = (0, -1)$, $N = (-1, 0)$ points in (a, b) plane (see Fig. 1). These points are boundary points of $\mathcal{C}_i$, $i = 1, 2, \ldots, 2\ell$ curve segments. Notice that they are the only points where $\mathcal{C}$ can intersect itself. The sequence of these points along $\mathcal{C}$ by increasing $\omega \in [0, \pi]$ is $A, K, L, M, [: N, K, L, M :]_{\ell/2-1}, B$ for ℓ even and $A, [: K, L, M, N :]_{(\ell-1)/2}, K, B$ for ℓ odd, where subsequence between symbols $[: \ :]_s$ repeats s times.

Let us consider $\mathcal{C}$ as a function $b = f(a)$ in a suitable neighbourhood of points K, L, M, N. Local analysis of this function enables us to determine which segments $\mathcal{C}_i$ bound the considered instability regions. Particularly the slopes of consequential segments $\mathcal{C}_i, \mathcal{C}_{i+1}$ connected in these points give the sequence of boundaries we cross going along the axis of appropriate quadrant away from the origin. We illustrate the analysis in the point K. In this point there are connected neighbour segment pairs $(\mathcal{C}_{4k+1}, \mathcal{C}_{4k+2}), k = 0, 1, 2, \ldots, k_f$, where $k_f = (\ell - 1)/2$ for ℓ odd and $k_f = \ell/2 - 1$ for ℓ even. For these segment pairs the corresponding points K representations on $\mathcal{C}$ are in $\omega = \omega_k := \frac{(4k+1)\pi}{2\ell}, k = 0, 1, \ldots, k_f$, respectively. From (5), (6) we get

$$f'(a(\omega)) = \frac{\ell \cos(\ell\omega) \sin(\omega) - \sin(\ell\omega) \cos(\omega)}{\sin((\ell + 1)\omega) - (\ell + 1) \cos((\ell + 1)\omega)}. \tag{7}$$

Particularly for $\omega = \omega_k$ we obtain the values representing the slopes of $\mathcal{C}$ in the point K as $f'(a(\omega_k)) = -\cos(\omega_k)/(1 + \ell \sin^2(\omega_k)), \quad k = 0, 1, 2, \ldots, k_f$. Now considering continuous function $\vartheta(u) = -\cos(u)/(1 + \ell \sin^2(u)),\ u \in (0, \pi)$, we have $\vartheta'(u) = (1 + \ell \cos^2(u) + \ell) \sin(u)/(\ell \cos^2(u) - 1 - \ell)^2 > 0$ for $u \in (0, \pi)$. Since $\vartheta(u)$ is increasing in $u \in (0, \pi)$, the studied sequence of the slopes is increasing too. On that account moving along the axis of the first quadrant from the origin we cross the segments $\mathcal{C}_{4k+2},\ k = 0, 1, 2, \ldots, k_f$ in sequence. We recall that each crossing corresponds to the instability degree shift by 2. This gives (in restriction to the first quadrant) that $I_{\ell,2k+2}$ is bounded by $\mathcal{C}_{4k+2}$ and $\mathcal{C}_{4k+6}$, $k = 0, 1, 2, \ldots, k_f - 1$. In the case of odd ℓ the last segment $\mathcal{C}_{4k_f+6}$ connects K with B and therefore the region $I_{\ell,\ell-1}$ is bounded by $\mathcal{C}_{2\ell}, \mathcal{C}_{2\ell-4}$ and straight line segment BL. On the other hand, the increasing slope sequence in K gives in the second quadrant case conclusion that moving along the axis of the quadrant away from the origin we cross the segments $\mathcal{C}_{4k+1},\ k = k_f, k_f - 1, \ldots, 2, 1, 0$ in sequence. Summarizing the previous analysis and considering analogous steps in other points L, M, N enables us to formulate the survey of stability and instability regions given by their boundary.

Theorem 2 *Consider Eq. (1), where $a, b \in \mathbb{R}$ and ℓ is odd integer. Then the stability region S_ℓ boundary and instability regions boundaries are given by the sets of curves*

S_ℓ	$\{\mathcal{C}_2; LM; MN; \mathcal{C}_{2\ell-1}\}$
$I_{\ell,1}$	$\{MN; \mathcal{C}_4\}, \{LM; \mathcal{C}_{2\ell-3}\}$
$I_{\ell,p}$, even $p = 2, 4, \ldots, \ell - 3$	$\{\mathcal{C}_{2p-2}; \mathcal{C}_{2p+2}\}, \{\mathcal{C}_{2\ell-2p+3}; \mathcal{C}_{2\ell-2p-1}\}$
$I_{\ell,m}$, odd $m = 3, 5, \ldots, \ell - 2$	$\{\mathcal{C}_{\ell-2m+6}; \mathcal{C}_{\ell-2m+2}\}, \{\mathcal{C}_{\ell+2m-5}; \mathcal{C}_{\ell+2m-1}\}$
$I_{\ell,\ell-1}$	$\{\mathcal{C}_1; \mathcal{C}_5; AN\}, \{\mathcal{C}_{2\ell}; \mathcal{C}_{2\ell-4}; BL\}$
$I_{\ell,\ell}$	$\{b = -a - 1, a \in (-\infty, -1]; \mathcal{C}_{2\ell-2}; b = a - 1, a \in (-\infty, 0]\}$ $\{b = -a - 1, a \in [0, \infty); \mathcal{C}_3; b = a - 1, a \in [1, \infty)\}$
$I_{\ell,\ell+1}$	$\{ b = -a - 1, a \in (-\infty, -\ell - 1]; \mathcal{C}_1;$ $\mathcal{C}_{2\ell}; b = a - 1, a \in [\ell + 1, \infty)\}$

Notice that the stability and instability regions dislocation is symmetric with respect to the axis b for ℓ odd.

Theorem 3 *Consider Eq. (1), where $a, b \in \mathbb{R}$ and ℓ is even integer. Then the stability region S_ℓ boundary and instability regions boundaries are given by the sets of curves*

S_ℓ	$\{\mathcal{C}_2; \mathcal{C}_{2\ell-1}; MN; NK\}$
$I_{\ell,1}$	$\{MN; \mathcal{C}_4\}$, $\{NK; \mathcal{C}_{2\ell-3}\}$
$I_{\ell,p}$, even $p = 2, 4, \ldots, \ell-2$	$\{\mathcal{C}_{2p-2}; \mathcal{C}_{2p+2}\}$, $\{\mathcal{C}_{2\ell-2p+3}; \mathcal{C}_{2\ell-2p-1}\}$
$I_{\ell,m}$, odd $m = 3, 5, \ldots, \ell-3$	$\{\mathcal{C}_{2m-2}; \mathcal{C}_{2m+2}\}$, $\{\mathcal{C}_{2\ell-2m+3}; \mathcal{C}_{2\ell-2m-1}\}$
$I_{\ell,\ell-1}$	$\{\mathcal{C}_1; \mathcal{C}_5; AN\}$, $\{\mathcal{C}_{2\ell}; \mathcal{C}_{2\ell-4}; NB\}$
$I_{\ell,\ell}$	$\{a = -\lvert b\rvert - 1\}$, $\{b = -a-1, a \in [0,\infty); \mathcal{C}_3; \mathcal{C}_{2\ell-2}; b = a+1, a \in [0,\infty)\}$
$I_{\ell,\ell+1}$	$\{b = -a-1, a \in (-\infty, -\ell-1]; \mathcal{C}_1; b = a+1, a \in [0,\infty)\}$ $\{b = -a-1, a \in [0,\infty); \mathcal{C}_{2\ell}; b = a+1, a \in (-\infty, -\ell-1]\}$

Notice that the stability and instability regions dislocation is symmetric with respect to the axis a for ℓ even.

3 Final Remarks

As it was remarked in the introduction, instability degree regions are not investigated in the literature as wide as the stability regions are, especially in the case of difference equations. As it was shown above, the dislocation of $I_{\ell,k}$ regions obey some rules.

Notice that with respect to the structure of linear difference equations solution there exists a periodic solution of (1) for any point $[a, b]$ from the curve C and from relevant straight line boundaries of instability regions, where a $P(\lambda)$ root with modulus one occurs. Deeper analysis considering this phenomena can be found in [6, 7], where a more general difference system was analyzed from the periodic solution existence point of view.

The introduced considerations have also an impact to the theory of polynomials: a dislocation of pairs (a, b) for which the polynomial (2) has a fixed number of roots inside the unit disk in complex plane, is introduced. There are several kinds of algebraic criteria to determine number of polynomial roots in specified area of complex plane. The description of such criteria including their proofs can be found in [8] or [10]. Most common are questions about location of all characteristic polynomial roots with respect to the left half-plane and unit circle in study of stability of differential equations and difference equations, respectively. But these criteria also enable us to determine a number of roots inside the specified area and outside of it. On that account we can numerically determine the instability degree of studied equation for the given parameters. Then we can develop an instability regions portrait in a computational way. But the above discussion presents another approach: it gives some rules of stability and instability regions dislocation of Eq. (1) in analytical way using boundary locus technique and supplementary considerations. The author believes that research of instability regions dislocation properties can be interesting and fruitful also in another cases of difference equations.

References

1. Čermák, J., Jánský, J., Kundrát, P.: On necessary and sufficient conditions for the asymptotic stability of higher order linear difference equations. J. Differ. Equ. Appl. **18**(11), 1781–1800 (2012)
2. Čermák, J., Jánský, J.: Explicit stability conditions for a linear trinomial delay difference equation. Appl. Math. Lett. **43**, 56–60 (2015)
3. Cheng, S.S., Huang, S.Y.: Alternate derivations of the stability region of a difference equation with two delays. Appl. Math. E-Notes **9**, 225–253 (2009)
4. Dannan, F.: The asymptotic stability of $x(n+k) + ax(n) + bx(n-l) = 0$. J. Differ. Equ. Appl. **10**(6), 589–599 (2004)
5. Dannan, F., Elaydi, S.: Asymptotic stability of linear difference equations of advanced type. J. Comput. Anal. Appl. **6**(2), 173–187 (2004)
6. Györi, I., Horváth, L.: Existence of periodic solutions in al linear higher order system of difference equations. Comput. Math. Appl. **66**, 2239–2250 (2013)
7. Györi, I., Horváth, L.: Utilization of the circulant matrix theory in periodic higher order autonomous difference equations. Int. J. Differ. Equ. **9**, 163–185 (2014)
8. Jury, E.I.: Inners and Stability of Dynamic Systems. Krieger, Malabar (1982)
9. Kolmanovskii, V., Myshkis, A.: Introduction to the Theory and Applications of Functional Differential Equations. Mathematics and Its Applications, vol. 463. Kluwer Academic Publishers, Dordrecht (1999)
10. Marden, M.: Geometry of Polynomials. Mathematical Surveys and Monographs, vol. 3. American Mathematical Society, Providence (1966)
11. Michiels, W., Niculescu, S.I.: Stability and Stabilization of Time-Delay Systems: An Eigenvalue-Based Approach. Advances in Design and Control. SIAM, Philadelphia (2007)
12. Kipnis, M.M., Nigmatulin, R.: Stability of the trinomial linear difference equations with two delays. Autom. Remote Control **65**(11), 1710–1723 (2004)
13. Kipnis, M.M., Nigmatulin, R.: D-decomposition method for stability checking for trinomial linear difference equation with two delays. Int. J. Pure Appl. Math. **111**(3), 479–489 (2016)
14. Ren, H.: Stability analysis of second order delay difference equations. Funkcial. Ekvac. **50**, 405–419 (2007)