A Note on Transformations of Independent Variable in Second Order Dynamic Equations

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Abstract The main purpose of this paper is to show how a transformation of independent variable in dynamic equations combined with suitable statements on a general time scale can yield new results or new proofs to known results. It seems that this approach has not been extensively used in the literature devoted to dynamic equations. We present, in particular, two types of applications. In the first one, an original dynamic equation is transformed into a simpler equation. In the second one, a dynamic equation in a somehow critical setting is transformed into a noncritical case. These ideas will be demonstrated on problems from oscillation theory and asymptotic theory of second order linear and nonlinear dynamic equations.

Keywords Transformation · Chain rule · Dynamic equation · Time scale · Oscillation · Asymptotic formulae

1 Introduction

It is well known that the chain rule in the "pure" form $(f \circ g)^{\Delta}(t) = f'(g(t))g^{\Delta}(t)$ does not hold on a general time scale $\mathbb{T}$, even if the derivative f' and the delta derivative g^{Δ} exist, see, e.g., [3]. This is the reason why the transformation of independent variable, a useful tool in the theory of differential equations, is not fully at our disposal for dynamic equations. Naturally, the problems can occur also with using the substitution method in delta integrals. There exist variants of chain rule on time scales, such as $(f \circ g)^{\Delta}(t) = \left[\int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))\,\mathrm{d}h\right] g^{\Delta}(t)$ [5, Theorem 1.90] involving the classical, say Riemann, integral, or $(f \circ g)^{\Delta}(t) = f'(g(\xi))g^{\Delta}(t)$ [5, Theorem 1.87] with an unspecified value ξ coming from the Lagrange mean value theorem; they are however unsuitable for the use in many situations. Another version of the chain rule, $(f \circ g)^{\Delta} = (f^{\tilde{\Delta}} \circ g)g^{\Delta}$ [5, Theorem 1.93], involves two

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generally different time scales $\mathbb{T}$, $\widetilde{\mathbb{T}}$, which are related through the inner function g by $\widetilde{\mathbb{T}} = g(\mathbb{T})$. The problem with applications of the last variant, when being interested in the transformation of independent variable, is that a dynamic equation on a time scale is transformed into a dynamic equation on a different time scale. On the other hand, if we take into account that nowadays a lot of results for dynamic equations hold on a general time scale, we can successfully use this approach, for instance, in the below described way.

As it is well known, transformations of independent and dependent variables in differential equations are useful, among others, when it is possible to transform a given differential equation into a differential equation which is in some way simpler. In this paper, we deal with dynamic equations and use some existing results that are valid on a general time scale to obtain new results or new proofs of known statements. We consider, in particular, two types of applications. In one we transform certain second order dynamic equations with a general coefficient at the leading term into dynamic equations of a similar type, on a different time scale, but with the leading coefficient equaling to one. In the other type of application we transform a dynamic equation under somehow critical setting (this will be specified later) into a dynamic equation on a different time scale under certain non-critical setting which can be handled by existing results.

Some analysis of basic aspects related to transformations of difference and dynamic equations that are close to our topic has already occurred in the literature. Transformations in linear Hamiltonian systems on time scales are treated in [3], Sturm–Liouville expressions on Sturmian time scales (the time scales that contain only isolated or l-d/r-d points) are, from this point of view, studied in [4], and the transformations for even order difference operators are considered in [25]. On the other hand, it seems that the ideas from our paper, although being practically known in the differential equations case, have not been extensively applied in dynamic equations in the literature, in spite of availability of various results on a general time scale.

For an outline of the first mentioned type of applications, let us give one problem from oscillation theory. Let us say we have oscillation criteria (for definiteness, the so-called Hille–Nehari criteria, see Sect. 2) for the difference equation $\Delta^2 y_k + p_k y_{k+1} = 0$ at disposal, and we are interested in criteria for the more general equation $\Delta(r_k \Delta y_k) + p_k y_{k+1} = 0$. It is problematic, in contrast to the corresponding differential equations case, to transform the latter form of the equation into the former one. Thus one would say that we have to analyse the difference equation directly in the more general form. However, there is also other possibility which is characteristic for what we do in this paper. If we know the criteria for the dynamic equation of the form $y^{\Delta\Delta} + p(t)y^\sigma = 0$ on a general time scale, then a suitable transformation of independent variable (which preserves oscillation properties) in the equation $\Delta(r_k \Delta y_k) + p_k y_{k+1} = 0$ can transform it into the former dynamic equation, which is then examined by existing results. Note that historically, the Hille–Nehari type criteria for the former, and simpler, difference equation were obtained as first, see, e.g., [7, 10]. It is worthy of mention also the following interesting fact related to this discrete oscillation problem. The Hille–Nehari type criteria for the simpler equation

$\Delta^2 y_k + p_k y_{k+1} = 0$ involves the well known critical constant $1/4$. But passing to the more general case, to the criteria for the equation $\Delta(r_k \Delta y_k) + p_k y_{k+1} = 0$, we find out that the critical constant can have a value different from $1/4$, in contrast to the differential equations case, and depends on the coefficient r. In fact, as we will see later, for dynamic equations we reveal also its dependence on the graininess of time scale.

To outline the second type of applications, we consider the difference equation $\Delta(r_k \Delta y_k) = p_k y_{k+1}$, where $r, p > 0$. It is known [18] that under the condition (which in fact can be weakened) $\lim_{k\to\infty} k\Delta p_k/p_k = \delta$ we are able to establish quite precise asymptotic formulae via the discrete theory of regular variation provided $\delta \neq -1$. The critical case $\delta = -1$ leads to a somehow delicate setting; it turns out that a suitable transformation of independent variable can help here, and brings us to dynamic equations which are in a non-critical setting in the above sense.

The paper is organized as follows. In the next section we deal with the so-called Hille–Nehari criteria for half-linear dynamic equations; we give a new proof to existing results. New results are obtained in Sect. 3 where we derive oscillation criteria for nonlinear dynamic equations. In Sect. 4 we study two variants of Euler type equations for which we establish the values of their oscillation constant. In addition to transformation of independent variable, a transformation of dependent variable plays a role, too. In the last section, we present also a new result. We do an asymptotic analysis of linear dynamic equations and establish asymptotic formulae for solutions in a critical case which is missing in the existing literature.

Let, as usually, $\mathbb{T}$ denote a time scale, which is assumed to be unbounded from above in our paper. We use the standard time scale notation, see [5, 6]. In particular, the symbols σ, μ, f^σ, f^Δ, $\int_a^b f(s)\,\Delta s$, $[a,b]_\mathbb{T}$, and $C_{\rm rd}$ stand for forward jump operator, graininess, $f \circ \sigma$, delta derivative, delta integral, a time scale interval, and the class of rd-continuous functions, respectively. The symbols $\widetilde{\sigma}$, $\widetilde{\mu}$, $f^{\widetilde{\Delta}}$, $\int_a^b f(s)\,\widetilde{\Delta} s$ have an analogous meaning, with being associated to a time scale $\widetilde{\mathbb{T}}$. For functions defined on $\mathbb{T}$ we denote: $f(t) \sim g(t)$ as $t \to \infty$ if $\lim_{t\to\infty} f(t)/g(t) = 1$; $f(t) = o(g(t))$ as $t \to \infty$ if $\lim_{t\to\infty} f(t)/g(t) = 0$; $f(t) = O(g(t))$ as $t \to \infty$ if $\exists c \in (0,\infty)$ such that $|f(t)| \le c|g(t)|$ for large $t \in \mathbb{T}$.

2 Oscillation of Half-Linear Dynamic Equations

We start with a simple observation concerning Hille–Nehari type oscillation criteria for half-linear dynamic equations. For information about these and other criteria for linear and half-linear differential and difference equations see [1, 2, 8, 24]. The result presented in the next theorem is actually known, see [16]. However, as already indicated, our aim is to demonstrate how the ideas based on a transformation can serve to establish a new proof where the result for the original equation will be obtained from the result for a simpler equation. A similar approach can be used whenever criteria for half-linear dynamic equations (including functional ones) or some other types of second order equations with the leading coefficient 1 are at disposal.

Consider the half-linear dynamic equation

$$(r(t)\Phi(y^{\Delta}))^{\Delta} + p(t)\Phi(y^{\sigma}) = 0, \tag{1}$$

where $\Phi(u) = |u|^{\alpha-1}\text{sgn}u$ with $\alpha > 1$, and assume $p \in C_{\text{rd}}([a,\infty)_{\mathbb{T}}, \mathbb{R})$, $1/r \in C_{\text{rd}}([a,\infty)_{\mathbb{T}}, (0,\infty))$, $\int_a^\infty r^{1-\beta}(s)\,\Delta s = \infty$, $1/\alpha + 1/\beta = 1$, and $\int_t^\infty p(s)\,\Delta s$ exists, is nonnegative and eventually nontrivial for large t. Thanks to the Sturm type separation result, one (nontrivial) solution of (1) is oscillatory (i.e., it is neither eventually positive nor eventually negative) if and only if all solutions are oscillatory. Hence, we can classify Eq. (1) as oscillatory (all its solutions are oscillatory) or nonoscillatory (all its solutions are nonoscillatory).

Note that the constants which appear on the right-hand sides of the criteria in Theorem 1 depend on the graininess of time scale and the coefficient r. Denote

$$R_\alpha(t) = \int_a^t r^{1-\beta}(s)\,\Delta s, \quad M_* = \liminf_{t\to\infty} \frac{\mu(t)r^{1-\beta}(t)}{R_\alpha(t)}, \quad M^* = \limsup_{t\to\infty} \frac{\mu(t)r^{1-\beta}(t)}{R_\alpha(t)},$$

and

$$\gamma_\alpha(x) = \lim_{t\to x}\left(\frac{(t+1)^{\frac{\alpha-1}{\alpha}} - 1}{t}\right)^{\alpha-1}\left(1 - \frac{1-(t+1)^{-\frac{(\alpha-1)^2}{\alpha}}}{1-(t+1)^{1-\alpha}}\right).$$

Examples of particular settings are presented in Remark 1 and Sect. 4. The form of the constant $\gamma_\alpha(x)$ is related to the roots of a certain algebraic equation which is associated to a generalized Riccati type dynamic equation [16]; here the generalized Riccati dynamic equation is the first order nonlinear equation arising from (1) through the substitution $w = r\Phi(y^\Delta/y)$ [1, 13, 16].

Theorem 1 *If*

$$\liminf_{t\to\infty} R_\alpha^{\alpha-1}(t)\int_t^\infty p(s)\,\Delta s > \gamma_\alpha(M_*), \tag{2}$$

then (1) *is oscillatory. If*

$$\limsup_{t\to\infty} R_\alpha^{\alpha-1}(t)\int_t^\infty p(s)\,\Delta s < \gamma_\alpha(M^*), \tag{3}$$

then (1) *is nonoscillatory.*

Proof Let y be a solution of (1). Set $u(s) = y(t)$, $s = \tau(t)$, where $\tau : \mathbb{T} \to \mathbb{R}$, being more precisely defined later, is strictly increasing. Denote $\widetilde{\mathbb{T}} = \{\tau(t) : t \in \mathbb{T}\}$. Here we assume that the delta derivatives which are involved in our computations exist; later we will see that it indeed holds under our particular setting. Moreover, our τ will always be at least in C^1_{rd} (so, in particular, will be continuous) and our $\widetilde{\mathbb{T}}$ will be (an unbounded) time scale. In view of the chain rule [5, Theorem 1.93], we have $y^\Delta = (u^{\widetilde{\Delta}} \circ \tau)\tau^\Delta$. Using the chain rule again,

$$\begin{aligned}(r\Phi(y^\Delta))^\Delta &= \left(r\Phi(\tau^\Delta)\Phi(u^{\widetilde{\Delta}} \circ \tau)\right)^\Delta = \left[[(r\Phi(\tau^\Delta)) \circ \tau^{-1} \circ \tau]\,\Phi(u^{\widetilde{\Delta}} \circ \tau)\right]^\Delta \\ &= \left[[(r\Phi(\tau^\Delta)) \circ \tau^{-1}]\Phi(u^{\widetilde{\Delta}})\right]^{\widetilde{\Delta}} \circ \tau\, \tau^\Delta.\end{aligned} \tag{4}$$

Thanks to the properties of τ, we have $\tau \circ \sigma = \widetilde{\sigma} \circ \tau$, and so $(u \circ \tau)^\sigma = u^{\widetilde{\sigma}} \circ \tau$. Therefore, in view of (4), u satisfies the equation

$$\left(\widetilde{r}(s)\Phi(u^{\widetilde{\Delta}})\right)^{\widetilde{\Delta}} + \widetilde{p}(s)\Phi(u^{\widetilde{\sigma}}) = 0 \tag{5}$$

on $\widetilde{\mathbb{T}}$, where

$$\widetilde{r} = (r\Phi(\tau^\Delta)) \circ \tau^{-1} \quad \text{and} \quad \widetilde{p} = \frac{p}{\tau^\Delta} \circ \tau^{-1}.$$

Now we set $\tau = R_\alpha$. Then $\widetilde{\mathbb{T}} = \tau(\mathbb{T})$ is an unbounded time scale and, in particular, the interval $[a, \infty)_\mathbb{T}$ is transformed into the interval of the form $[\widetilde{a}, \infty)_{\widetilde{\mathbb{T}}}$. Further, $\tau^\Delta = r^{1-\beta}$, thus $\widetilde{r} = (rr^{(\alpha-1)(1-\beta)}) \circ \tau^{-1} = (r/r) \circ \tau^{-1} = 1$. From [16] we know that (5) is oscillatory provided $\liminf_{s\to\infty} s^{\alpha-1} \int_s^\infty \widetilde{p}(\eta)\, \widetilde{\Delta}\eta > \gamma_\alpha(\widetilde{M}_*)$ and nonoscillatory provided $\limsup_{s\to\infty} s^{\alpha-1} \int_s^\infty \widetilde{p}(\eta)\, \widetilde{\Delta}\eta < \gamma_\alpha(\widetilde{M}^*)$, where $\widetilde{M}_* = \liminf_{s\to\infty} \widetilde{\mu}(s)/s$, $\widetilde{M}^* = \limsup_{s\to\infty} \widetilde{\mu}(s)/s$, and the integral $\int_s^\infty \widetilde{p}(\eta)\, \widetilde{\Delta}\eta$ is nonnegative and eventually nontrivial for large s. We have

$$\begin{aligned}\frac{\widetilde{\mu}(s)}{s} &= \frac{\widetilde{\sigma}(s) - s}{s} = \frac{(\widetilde{\sigma} \circ R_\alpha)(t) - R_\alpha(t)}{R_\alpha(t)} \\ &= \frac{R_\alpha^\sigma(t) - R_\alpha(t)}{R_\alpha(t)} = \frac{\mu(t)R_\alpha^\Delta(t)}{R_\alpha(t)} = \frac{\mu(t)r^{1-\beta}(t)}{R_\alpha(t)},\end{aligned}$$

and so $\widetilde{M}_* = M_*$ and $\widetilde{M}^* = M^*$. Further, applying the substitution method in delta integrals [5, Theorem 1.98], see also [6, Theorem 5.40], we obtain

$$\begin{aligned}s^{\alpha-1}\int_s^S \widetilde{p}(\eta)\, \widetilde{\Delta}\eta &= R_\alpha^{\alpha-1}(t)\int_{R_\alpha(t)}^{R_\alpha(T)} \left(\frac{p}{R^\Delta} \circ R^{-1}\right)(\eta)\, \widetilde{\Delta}\eta \\ &= R_\alpha^{\alpha-1}(t)\int_t^T \frac{p(\xi)}{R^\Delta(\xi)} R^\Delta(\xi)\, \Delta\xi = R_\alpha^{\alpha-1}(t)\int_t^T p(\xi)\, \Delta\xi,\end{aligned}$$

where $S = R_\alpha(T)$, $T \in [t, \infty)_\mathbb{T}$. Letting T to ∞, we get

$$s^{\alpha-1}\int_s^\infty \widetilde{p}(\eta)\, \widetilde{\Delta}\eta = R_\alpha^{\alpha-1}(t)\int_t^\infty p(\xi)\, \Delta\xi.$$

Since our transformation preserves (non)oscillation of the equation, the statement is now clear. □

Remark 1 (i) Let $M := M_* = M^*$. Then the constants on the right-hand sides of (2) and (3) are the same and we have

$$\gamma_\alpha(M) = \begin{cases} \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} & \text{if } M = 0, \\ \left(\frac{(M+1)^{\frac{\alpha-1}{\alpha}}-1}{M}\right)^{\alpha-1}\left(1-\frac{1-(M+1)^{-\frac{(\alpha-1)^2}{\alpha}}}{1-(M+1)^{1-\alpha}}\right) & \text{if } 0 < M < \infty, \\ 0 & \text{if } M = \infty. \end{cases}$$

For example, if $\mu(t) = 0$ or $r(t) = 1$ with $\mu(t) = o(t)$ as $t \to \infty$, then $M_* = M^* = 0$. If $\mu(t) = (q-1)t$, $q > 1$, (as in q-calculus), then $M_* = M^* = q - 1 > 0$. In the case corresponding to linear equations we have

$$\gamma_2(M) = \begin{cases} \frac{1}{4} & \text{if } M = 0, \\ \frac{1}{(\sqrt{M+1}+1)^2} & \text{if } 0 < M < \infty, \\ 0 & \text{if } M = \infty. \end{cases}$$

In particular, $\gamma_2(0) = 1/4$, which is the well known constant from oscillation theory of linear DEs, ee e.g. [24]. See also [7, 10] for the linear discrete case where $r(t) = 1$. We again emphasize that if $r(t) \not\equiv 1$, then even in the difference equation case, the value $1/4$ does not need to be maintained; taking, e.g., $r(t) = 2^{-t}$, we get $M = 2$, and so $\gamma_2(M) = (\sqrt{2}+1)^{-2}$. Further, the constant $\gamma_2(M)$ can differ from $1/4$ also when $r(t) = 1$. For instance, if $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$, $q > 1$, then $M = q - 1$, and so $\gamma_2(M) = (\sqrt{q}+1)^{-2}$.

(ii) In view of the previous remark, as very special cases of Theorem 1, we get the following criteria. Let $\int_t^\infty p(s)\,\Delta s \geq 0$ for large t. Assuming $\mathbb{T} = \mathbb{R}$, $\alpha = 2$, $r(t) = 1$, if $\liminf_{t\to\infty} t\int_t^\infty p(s)\,\mathrm{d}s > 1/4$ ($\limsup_{t\to\infty} t\int_t^\infty p(s)\,\mathrm{d}s < 1/4$), then $y'' + p(t)y = 0$ is oscillatory (nonoscillatory), cf. [24]. Assuming $\mathbb{T} = \mathbb{Z}$, $\alpha = 2$, $r(t) = 1$, if $\liminf_{t\to\infty} t\sum_{j=t}^\infty p(j) > 1/4$ ($\limsup_{t\to\infty} t\sum_{j=t}^\infty p(j) < 1/4$), then $\Delta^2 y(t) + p(t)y(t+1) = 0$ is oscillatory (nonoscillatory), cf. [7, 10].

(iii) In the previous proof we used the criteria from [16]. The proof of those results is based on the function sequence technique combined with Riccati type transformation. Note that another possibility how to prove those criteria could be, for example, to combine the information about the oscillation constant of a certain Euler type half-linear dynamic equation (provided we have it at disposal) with integral comparison theorem, see [14, Theorem 11] and [16, Sect. 7].

(iv) Looking at the conditions posed on the coefficients of (1) in Theorem 1, a natural problem arises out, namely to obtain analogous criteria when the integral $\int_a^\infty r^{1-\beta}(s)\,\Delta s$ converges. The trouble in this case is that the same transformation as in the previous proof transforms the range of definition into a bounded set. We do not aim to treat this problem in our paper. Note only that in the linear case, we can use the transformation of dependent variable $y = hu$, $h(t) = \int_t^\infty 1/r(s)\,\Delta s$

(similarly as below in Sect. 4), where the original equation is transformed into an equation of the same type, but with the divergent integral of the reciprocal of its leading term. This approach is not at our disposal in the half-linear case, since it requires the linearity of solution space. However, we can think—and we believe it could work—about replacement for half-linear equations in the sense that the corresponding transformation is made in terms of the associated generalized Riccati equations; the nonlinear term in the Riccati equation associated to the original half-linear equation is somehow quadrated in asymptotic sense. For some applications of this idea in the differential equations case see e.g. [9, 19]. Other possibility is to utilize the so-called reciprocity principle (see e.g. [1]); the original equation is transformed via the relation $u = r\Phi(y^{\Delta})$ into an equation of the same type, where the new equation satisfies the assumption of the divergence of the integral containing the leading term. This approach however requires to overcome some technical problems since the delta derivative and the jump operator do not commute on a general time scale; a possibility is to consider the transformed equation in an integral form or to work with systems of two first order equations. Finally note that Hille–Nehari type criteria under the condition $\int_a^\infty r^{1-\beta}(s)\,\Delta s < \infty$ are directly in this setting proved in [17] via the function sequence technique involving a weighted Riccati transformation. For further information related to oscillation and other qualitative properties of half-linear equations see [8] (differential equations case), [2, 12] (difference equations case), and [1, 13] (dynamic equations case).

3 Oscillation of Nonlinear Dynamic Equations

Here we prove sharp criteria, which generalize existing ones (see Remark 2), for nonlinear dynamic equations of the form

$$(r(t)y^{\Delta})^{\Delta} + p(t)f(y) = 0, \tag{6}$$

where $p \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}}, \mathbb{R})$ $1/r \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}}, (0,\infty))$, $\int_a^\infty 1/r(s)\,\Delta s = \infty$, and f is a continuous function on $\mathbb{R}$ satisfying $xf(x) > 0$ for $x \neq 0$. Denote $R(t) = \int_a^t 1/r(s)\,\Delta s$.

Theorem 2 *(a) If there exists $\lambda \in \mathbb{R}$ with $\lambda > 1/4$ such that*

$$R(t)R^{\sigma}(t)r(t)p(t)\frac{f(x)}{x} \geq \lambda \tag{7}$$

for $t \in [a,\infty)_{\mathbb{T}}$ large and $|x|$ large, then all nontrivial solutions of (6) are oscillatory.

(b) If

$$R(t)R^{\sigma}(t)r(t)p(t)\frac{f(x)}{x} \leq \frac{1}{4} \tag{8}$$

for $t \in [a, \infty)_{\mathbb{T}}$ *large and* $x > 0$ *or* $x < 0$ *with* $|x|$ *large, then* (6) *has a nonoscillatory solution.*

Proof (a) Let y be a solution of (6). Set $u(s) = y(t)$, $s = R(t)$. Then, using the arguments similar to those in the proof of Theorem 1, we get that u satisfies the equation

$$u^{\widetilde{\Delta}\widetilde{\Delta}} + \widetilde{p}(s) f(u) = 0 \tag{9}$$

on the time scale $\widetilde{\mathbb{T}} = \{R(t) : t \in \mathbb{T}\}$, where $\widetilde{p} = (p/R^{\Delta}) \circ R^{-1} = (pr) \circ R^{-1}$. The coefficient in the leading term of (9) is equal to 1 since $rR^{\Delta} = r/r = 1$. The result now follows from the transformation relations and [22, Theorem 5.1] applied to Eq. (9); that theorem says that all nontrivial solutions of (9) are oscillatory provided there is $\lambda > 1/4$ such that $s\widetilde{\sigma}(s)\widetilde{p}(s) f(x)/x \geq \lambda$ for large $s \in \widetilde{\mathbb{T}}$ and $|x|$.

(b) The proof is similar to that of part (a); here we apply [22, Theorem 5.2] to transformed equation (9). □

Remark 2 (i) If $r(t) = 1$ and $p(t) = 1/(r(t)R(t)R^{\sigma}(t))$ with $a = 0$, i.e., $p(t) = 1/(t\sigma(t))$, then Theorem 2-(a) reduces to [22, Theorem 1.1] and Theorem 2-(b) reduces to [22, Theorem 1.2]. If $p = 1/(rRR^{\sigma})$, then the left-hand sides of (7) and (8) read as $f(x)/x$ and depend only on f. If $p = 1/(rR^2)$, then (6) can be seen as a "more natural time scale discretization" (when compared with the setting $p = 1/(rRR^{\sigma})$) of the differential equation $(r(t)y')' + f(y)/(r(t)R^2(t)) = 0$ on $\mathbb{T} = \mathbb{R}$. Conditions (7) and (8) read as $f(x)/x \geq \lambda R(t)/R^{\sigma}(t)$ and $f(x)/x \leq R(t)/(4R^{\sigma}(t))$, respectively, and we see how a larger graininess is "more favorable" to oscillation in this case.

(ii) For more information on the criteria of the type presented in Theorem 2 see [23] (in differential equations case) and [26] (in difference equations case). See also the last paragraph of the next section.

4 Oscillation Constants for Euler Type Linear Dynamic Equations and Their Perturbations

In this section we establish the so-called oscillation constant for two variants of Euler type dynamic equation. By oscillation constant of the equation $y^{\Delta\Delta} + p(t;\lambda)y^{\sigma} = 0$ we mean the number λ_0 such that the equation is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. For other equations we define this concept similarly. As indicated in several points in this paper, Euler type equations (or their perturbations) are important for comparison purposes, see also e.g. [2, 8, 16, 22–24, 26]. By (non)oscillation of the equation we mean (non)oscillation of all its nontrivial solutions.

It is worthy of note that while in Theorem 3, the oscillation constant has the fixed value 1/4, the oscillation constant for the equation considered in Theorem 4 depends on the coefficient r and the graininess of a time scale.

Consider first the equation

$$(r(t)y^{\Delta})^{\Delta} + \lambda p(t)y = 0, \tag{10}$$

where $1/r \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}},(0,\infty))$ and $p(t)$ will be specified in Theorem 3. An interesting fact is that for both equations considered in Theorem 3 we can write their general solutions, see Remark 3. The Sturmian theory (in particular, the separation result) does not hold in general for equations of the form (10), in contrast to equations of the form (11). This means that we have not guaranteed the implication: one solution is (non)oscillatory implies all solutions are (non)oscillatory. In spite of this fact, under our special setting, we state the "true" oscillation constant which is defined as above, i.e., via (non)oscillation of equation.

The next result is new, it is an improvement of [22, Proposition 2.3]. Actually, Theorem 3-(a) can be obtained as an immediate consequence of Theorem 2. However, we offer an alternative way of the proof—it is based on a suitable transformation.

Theorem 3 *(a) Let* $\int_a^\infty 1/r(s)\,\Delta s=\infty$. *Then Eq.* (10) *with*

$$p(t)=\frac{1}{r(t)R(t)R^\sigma(t)},\quad R(t)=\int_a^t\frac{1}{r(s)}\,\Delta s,$$

has the oscillation constant $\lambda=1/4$.

(b) Let $\int_a^\infty 1/r(s)\,\Delta s<\infty$. *Then Eq.* (10) *with*

$$\frac{1}{p(t)}=r(t)R_c(t)R_c^\sigma(t)\left(1-\frac{R_c(t)}{R_c(a)}\right)\left(1-\frac{R_c^\sigma(t)}{R_c(a)}\right),\quad R_c(t)=\int_t^\infty\frac{1}{r(s)}\,\Delta s,$$

has the oscillation constant $\lambda=1/4$.

Proof (a) Let y be a solution of (10). Set $u(s)=y(t)$ and $s=R(t)$. Then, similarly as in the proof of Theorem 1, we get that u satisfies the equation $(\widetilde{r}(s)u^{\widetilde{\Delta}})^{\widetilde{\Delta}}+\lambda\widetilde{p}(s)u=0$ on the time scale $\widetilde{\mathbb{T}}=\{R(t):t\in\mathbb{T}\}$, where $\widetilde{r}(s)=(rR^\Delta)\circ R^{-1}(s)=1$ and $\widetilde{p}(s)=(p/R^\Delta)\circ R^{-1}(s)=(1/(RR^\sigma))\circ R^{-1}(s)=1/(s\widetilde{\sigma}(s))$, i.e., the equation $u^{\widetilde{\Delta}\widetilde{\Delta}}+(\lambda/(s\widetilde{\sigma}(s)))u=0$. Applying [22, Proposition 2.3] and the ideas of its proof, we obtain that the oscillation constant of the latter equation is $\lambda=1/4$. From the transformation relations it is clear that $\lambda=1/4$ is the oscillation constant also for original Eq. (10).

(b) First we introduce the dynamic operators $\mathcal{L}[y]=(ry^\Delta)^\Delta+py$ (note that here r and p can be general) and $\widehat{\mathcal{L}}[y]=(\widehat{r}y^\Delta)^\Delta+\widehat{p}y$, where $\widehat{r}=rhh^\sigma$ and $\widehat{p}=phh^\sigma$. It can be shown that if $h=R_c$, but also if $h=R$, then $h^\sigma\mathcal{L}[hz]=\widehat{\mathcal{L}}[z]$ for a sufficiently smooth z. Consequently, if y is a solution of (10) and we set $y=hz$, where $h=R_c$, then z satisfies the equation $(\widehat{r}(t)z^\Delta)^\Delta+\widehat{p}(t)z=0$, where $\widehat{r}=rR_cR_c^\sigma$ and

$$\widehat{p}(t)=\frac{\lambda}{r(t)(1-R_c(t)/R_c(a))(1-R_c^\sigma(t)/R_c(a))}.$$

We have

$$\int_a^t\frac{1}{\widehat{r}(s)}\,\Delta s=\int_a^t\left(\frac{1}{R_c(s)}\right)^\Delta\Delta s=\frac{1}{R_c(t)}-\frac{1}{R_c(a)}\to\infty$$

as $t \to \infty$. Denote $\widehat{R}(t) = \int_a^t 1/\widehat{r}(s)\,\Delta s$. Then

$$\frac{\lambda}{\widehat{r}(t)\widehat{R}(t)\widehat{R}^\sigma(t)} = \frac{\lambda}{r(t)R_c(t)R_c^\sigma(t)(1/R_c(t) - 1/R_c(a))(1/R_c^\sigma(t) - 1/R_c(a))}$$
$$= \frac{\lambda}{r(t)(1 - R_c(t)/R_c(a))(1 - R_c^\sigma(t)/R_c(a))} = \widehat{p}(t).$$

Now we can apply part (a) to the equation $(\widehat{r}(t)z^\Delta)^\Delta + \widehat{p}(t)z = 0$ to obtain that its oscillation constant is $\lambda = 1/4$ and, in view of the transformation relations, it is also the oscillation constant of (10). □

Remark 3 (i) Note that for the coefficient $p(t)$ in part (b) of the previous theorem we have $p(t) \sim 1/(r(t)R_c(t)R_c^\sigma(t))$ as $t \to \infty$. This means that for typical (namely asymptotic) comparison purposes involving (10) the coefficient $p(t)$ in the setting of (b) can be replaced by $1/(r(t)R_c(t)R_c^\sigma(t))$.

(ii) It is worthy of note that for the equations considered in Theorem 3 (in contrast for those in Theorem 4), we can establish the exact form of a general solution. We omit details. Let us note just that the arguments for such a statement are based on the transformation of independent variable in case (a) and the transformation of independent and dependent variable in case (b) and the knowledge (see [11]) of the general solution for the equation $u^{\widetilde{\Delta}\widetilde{\Delta}} + (\lambda/(s\widetilde{\sigma}(s)))u = 0$.

In the next theorem we consider a different variant of Euler type equation, namely

$$(r(t)y^\Delta)^\Delta + \lambda p(t)y^\sigma = 0, \tag{11}$$

where $1/r \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}}, (0,\infty))$ and $p(t)$ will be specified in Theorem 4. As we will see, the difference between (10) and (11) is only seemingly slight—notice how the form of the oscillation constant is affected. The next result is known, but here we offer an alternative approach to its proof.

Theorem 4 *(a) Let $\int_a^\infty 1/r(s)\,\Delta s = \infty$ and the limit*

$$\lim_{t\to\infty} \frac{\mu(t)}{r(t)\int_a^t 1/r(s)\,\Delta s} =: M \in [0,\infty) \cup \{\infty\}$$

exist. Then equation (11) with

$$p(t) = \frac{1}{r(t)R(t)R^\sigma(t)}, \quad R(t) = \int_a^t \frac{1}{r(s)}\,\Delta s,$$

has the oscillation constant $\lambda = (\sqrt{M+1}+1)^{-2}$ provided $M < \infty$. If $M = \infty$, then this equation is oscillatory for all $\lambda > 0$ (thus it is strongly oscillatory) and nonoscillatory otherwise.

(b) Let $\int_a^\infty 1/r(s)\,\Delta s < \infty$ *and the limit*

$$\lim_{t\to\infty} \frac{\mu(t)}{r(t)\int_{\sigma(t)}^\infty 1/r(s)\,\Delta s} =: N \in [0,\infty)\cup\{\infty\}$$

exist. Then Eq. (11) *with*

$$p(t) = \frac{1}{r(t)(R_c^\sigma(t))^2}, \quad R_c(t) = \int_t^\infty \frac{1}{r(s)}\,\Delta s,$$

has the oscillation constant $\lambda = (\sqrt{N+1}+1)^{-2}$ *provided* $N < \infty$. *If* $N = \infty$, *then this equation is oscillatory for all* $\lambda > 0$ *(thus it is strongly oscillatory) and nonoscillatory otherwise.*

Proof (a) Let y be a solution of (11). Set $u(s) = y(t)$ and $s = R(t)$. Then by the arguments similar to those in the proof of Theorem 1, u can be shown to satisfy the equation $u^{\widetilde{\Delta}\widetilde{\Delta}} + (\lambda/(s\widetilde{\sigma}(s)))u^{\widetilde{\sigma}} = 0$ on the time scale $\widetilde{\mathbb{T}} = \{R(t) : t \in \mathbb{T}\}$. From [15] we know that the oscillation constant for this equation is $\lambda = (\sqrt{M_0+1}+1)^{-2}$, where $M_0 := \lim_{s\to\infty} \widetilde{\mu}(s)/s \in [0,\infty)$. Since (assuming here that $M < \infty$)

$$\frac{\widetilde{\mu}(s)}{s} = \frac{\widetilde{\sigma}(s)-s}{s} = \frac{R^\sigma(t)-R(t)}{R(t)} = \frac{\mu(t)R^\Delta(t)}{R(t)} = \frac{\mu(t)}{r(t)R(t)}.$$

the (finite) limit M_0 indeed exists and we have

$$M_0 = \lim_{s\to\infty} \frac{\widetilde{\mu}(s)}{s} = \lim_{t\to\infty} \frac{\mu(t)}{r(t)R(t)} = M.$$

Thus the oscillation constant for the original Eq. (11) is $\lambda = (\sqrt{M+1}+1)^{-2}$. Oscillation of (11) for all $\lambda > 0$ when $M = \infty$ follows from strong oscillation (see [15]) of the equation $u^{\widetilde{\Delta}\widetilde{\Delta}} + (\lambda/(s\widetilde{\sigma}(s)))u^{\widetilde{\sigma}} = 0$ when $M_0 = \infty$. Alternatively, the result can be proved by a direct application of Theorem 1.

(b) The statement can be proved via transforming the equation under consideration into an equation satisfying the setting of (a) in the following sense. If we denote $\mathcal{L}_s[y] = (ry^\Delta)^\Delta + py^\sigma$ and $\widehat{\mathcal{L}}_s[y] = (\widehat{r}y^\Delta)^\Delta + \widehat{p}y^\sigma$, where $\widehat{r} = rhh^\sigma$ and $\widehat{p} = h^\sigma\mathcal{L}_s[h]$, then $h^\sigma\mathcal{L}_s[hz] = \widehat{\mathcal{L}}_s[z]$. Therefore, being y a solution of $\mathcal{L}_s[y] = 0$ and setting $y = hz, h \neq 0$, we get that z is a solution of $\widehat{\mathcal{L}}_s[z] = 0$. Since $\int_a^\infty 1/\widehat{r}(s)\,\Delta s = \infty$ when $h = R_c$, we find ourselves in the setting of (a). The details are left to the reader. □

Remark 4 We emphasize that the fact whether we consider or not the jump operators in the coefficient $\lambda/(t\sigma(t))$ of the equations which appear in the proofs (or in their generalized versions presented in the theorems) plays an important role. Let us demonstrate it on the time scale $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$, $q > 1$, for the equation $y^{\Delta\Delta} + \frac{\lambda}{t\sigma(t)}y^\sigma = 0$ and its variants. Since $M = \lim_{t\to\infty} \mu(t)/t = q-1$, the oscillation constant for this equation is $\lambda = \frac{1}{(\sqrt{q}+1)^2}$ and it (linearly) decreases to zero as $q \to \infty$. The oscillation constant for the q-difference equation $y^{\Delta\Delta} + \frac{\lambda}{(\sigma(t))^2} y^\sigma = 0$ is $\lambda = \frac{q}{(\sqrt{q}+1)^2}$ and it increases to 1 as $q \to \infty$. Finally, the oscillation constant for the q-difference equation $y^{\Delta\Delta} + \frac{\lambda}{t^2} y^\sigma = 0$ is $\lambda = \frac{1}{q(\sqrt{q}+1)^2}$ and it (quadratically) decreases to 0 as $q \to \infty$. In all these cases the oscillation constant tends to $\frac{1}{4}$ as $q \to 1+$.

Perturbed Euler type equations In view of the previous considerations a natural problem arises out: to consider an Euler type equation where the parameter in the coefficient reaches its critical value and to study how perturbations of this term affect oscillatory properties of the equation; note that there can be revealed the relation of this critical setting with the double root case of the associated algebraic equation in some instances. Let us recall that a suitable combination of transformations of dependent and independent variable, precisely, $s = \ln t$ and $u(s) = t^{-1/2} y(t)$, can transform the perturbed Euler differential equation (the so-called Riemann–Weber equation)

$$y'' + \frac{1}{t^2}\left(\frac{1}{4} + \frac{\lambda}{\ln^2 t}\right) y = 0$$

into the equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}s^2} + \frac{\lambda}{s^2} u = 0.$$

This trick can be applied repeatedly, thus equations such as

$$y'' + \frac{1}{t^2}\left(\frac{1}{4} + \frac{1}{4}\sum_{k=1}^{n-1} \frac{1}{\mathrm{Ln}_k^2 t} + \frac{\lambda}{\mathrm{Ln}_n^2 t}\right) y = 0,$$

where $\mathrm{Ln}_k t = \prod_{j=1}^{k} \ln_j t$ and $\ln_{j+1} t = \ln(\ln_j t)$, can be treated. We believe that utilizing suitable transformations of dependent and independent variable in combination with existing theory on time scales, will enable us to examine, for example, the equation

$$y^{\Delta\Delta} + \frac{1}{t\sigma(t)}\left(\omega_0 + \frac{\lambda}{\ell(t)\ell^\sigma(t)}\right) y^\sigma = 0 \tag{12}$$

or equations of similar forms, where ω_0 is the oscillation constant of the equation $y^{\Delta\Delta} + (\omega/t\sigma(t))y^\sigma = 0$ and ℓ is a function which is somehow related to a logarithmic function. For instance, we conjecture that under the assumption $\mu(t) = o(t)$, the substitutions $y(t) = (u(t)/2)\int_a^t s^{-1/2}\,\Delta s$, $z(s) = u(t)$, $s = \ell(t) = \int_a^t (1/s)\,\Delta s$ transforms (12) with $\omega_0 = 1/4$ into the equation

$$\left((1 + o(1)) z^{\tilde{\Delta}}\right)^{\tilde{\Delta}} + (1 + o(1)) \frac{\lambda}{s\tilde{\sigma}(s)} z^{\tilde{\sigma}} = 0$$

on the time scale $\widetilde{\mathbb{T}} = \ell(\mathbb{T})$, and this equation can be further transformed introducing new independent variable η (similarly as in the proof of Theorem 1) into the equation

$$w^{\overline{\Delta}\,\overline{\Delta}} + (1 + o(1))\frac{\lambda}{\eta\overline{\sigma}(\eta)} w^{\overline{\sigma}} = 0$$

on a certain (unbounded) time scale $\overline{\mathbb{T}}$. Having at disposal information about perturbed Euler type dynamic equations, we can apply them, in combination with some comparison principle, to obtain (non)oscillation criteria for related linear or nonlinear dynamic equations, see, e.g. [1, 2, 14, 16, 22]. Note that there are also other equations, which can be understood as a perturbation of the equation under a certain critical setting, for example,

$$(ty^{\Delta})^{\Delta} + \frac{L(t)}{t} y^{\sigma} = 0, \tag{13}$$

where L varies slowly in some way (see Remark 5); likewise this setting corresponds somehow with the above mentioned critical double root case. The applications are expected not only in oscillation theory, but also in asymptotic theory. Indeed, the setting which is considered in below given Theorem 5 (see also Remark 5-(ii), (vi)), where we deal with asymptotic formulae, includes Eq. (13) as a special case.

5 Asymptotic Formulae

Consider the equation

$$(r(t)y^{\Delta})^{\Delta} = p(t)y^{\sigma}, \tag{14}$$

where $p, 1/r \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}}, (0,\infty))$. This equation is nonoscillatory (see e.g. [1]) and any its nontrivial solution is eventually monotone (see [21]). Thus the set of eventually positive solutions of (14) consists of eventually positive decreasing solutions and eventually positive increasing solutions. Both these classes are nonempty [21]. Next we prove a new result which can cover the missing case in [21, Theorem 4], see also Remark 5-(ii) below.

Theorem 5 *Let $\tau(t) = \int_a^t 1/s\,\Delta s$ and $\Psi(t) = t\tau(t)p(t)/r(t)$. Assume that $\mu(t) = o(t\tau(t))$ as $t \to \infty$,*

$$\exists q \in C^1_{\mathrm{rd}} \text{ such that } q(t) \sim tp(t),\ \frac{q^{\Delta}(t)}{q(t)} t\tau(t) \to \gamma \text{ as } t \to \infty, \tag{15}$$

and

$$\lim_{t\to\infty} (t\tau(t))^2 \frac{p(t)}{r(t)} = 0. \tag{16}$$

(a) Let $\int_a^\infty 1/r(s)\,\Delta s = \infty$ *and* $\gamma < -1$. *Then any eventually positive decreasing solution* y *of* (14) *satisfies* $t\tau(t)y^\Delta(t)/y(t) \to 0$, $r(t)y^\Delta(t) \to 0$ *as* $t \to \infty$, *and one has:*

(a1) If $\int_a^\infty \Psi(s)\,\Delta s = \infty$, *then*

$$y(t) = \exp\left\{\int_a^t (1 + o(1))\frac{\Psi(s)}{\gamma + 1}\,\Delta s\right\} \tag{17}$$

and $y(t) \to 0$ *as* $t \to \infty$.

(a2) If $\int_a^\infty \Psi(s)\,\Delta s < \infty$, *then*

$$y(t) = \ell_y \exp\left\{-\int_t^\infty (1 + o(1))\frac{\Psi(s)}{\gamma + 1}\,\Delta s\right\} \tag{18}$$

and $y(t) \to \ell_y \in (0, \infty)$ *as* $t \to \infty$.

(b) Let $\int_a^\infty 1/r(s)\,\Delta s < \infty$ *and* $\gamma > -1$. *Then any eventually positive increasing solution* y *of* (14) *satisfies* $t\tau(t)y^\Delta(t)/y(t) \to 0$, $r(t)y^\Delta(t) \to \infty$ *as* $t \to \infty$, *and one has:*

(b1) If $\int_a^\infty \Psi(s)\,\Delta s = \infty$, *then* (17) *holds and* $y(t) \to \infty$ *as* $t \to \infty$.
(b2) If $\int_a^\infty \Psi(s)\,\Delta s = \infty$, *then* (18) *holds and* $y(t) \to \ell_y \in (0, \infty)$ *as* $t \to \infty$.

Proof Let y be a solution of (14). Set $u(s) = y(t)$, $s = \tau(t)$, where τ can be, at this moment, a general strictly increasing function in $C^1_{\mathrm{rd}}(\mathbb{T})$ with $\widetilde{\mathbb{T}} = \tau(\mathbb{T})$ unbounded from above. Then, similarly as in the proof of Theorem 1, u satisfies the equation

$$(\widetilde{r}(s)u^{\widetilde{\Delta}})^{\widetilde{\Delta}} = \widetilde{p}(s)u^{\widetilde{\sigma}}, \tag{19}$$

where $\widetilde{r} = (r\tau^\Delta) \circ \tau^{-1}$ and $\widetilde{p} = (p/\tau^\Delta) \circ \tau^{-1}$. Applying the substitution method in delta integrals [5, Theorem 1.98], we have, with $S = \tau(T)$ and $\widetilde{a} = \tau(a)$,

$$\int_{\widetilde{a}}^S \frac{1}{\widetilde{r}(s)}\,\widetilde{\Delta} s = \int_{\tau(a)}^{\tau(T)} \left(\frac{1}{r\tau^\Delta} \circ \tau^{-1}\right)(s)\,\widetilde{\Delta} s = \int_a^T \frac{\tau^\Delta(t)}{r(t)\tau^\Delta(t)}\,\Delta t = \int_a^T \frac{1}{r(t)}\,\Delta t.$$

Hence, in particular, $\int_{\widetilde{a}}^\infty 1/\widetilde{r}(s)\,\widetilde{\Delta} s$ converges if and only if $\int_a^\infty 1/r(s)\,\Delta s$ converges. Further, with $s = \tau(t)$, using the chain rule [5, Theorem 1.93] and the formula for delta derivative of the inverse [5, Theorem 1.97], we obtain

$$\begin{aligned} \frac{s(q \circ \tau^{-1})^{\widetilde{\Delta}}(s)}{(q \circ \tau^{-1})(s)} &= \frac{s(q^\Delta \circ \tau^{-1})(s)(\tau^{-1})^{\widetilde{\Delta}}(s)}{(q \circ \tau^{-1})(s)} \\ &= \frac{s(q^\Delta \circ \tau^{-1})(s)}{(q \circ \tau^{-1})(s)(\tau^\Delta \circ \tau^{-1})(s)} = \frac{\tau(t)q^\Delta(t)}{\tau^\Delta(t)q(t)}. \end{aligned} \tag{20}$$

Set $\tau(t) = \int_a^t 1/s\,\Delta s$. Then $\lim_{t\to\infty} \tau(t) = \infty$ by [22, Lemma 2.1] and $\tau^\Delta(t) = 1/t$. Hence, in view of (15) and (20),

$$\frac{s(q\circ\tau^{-1})^{\widetilde{\Delta}}(s)}{(q\circ\tau^{-1})(s)} = t\tau(t)\frac{q^\Delta(t)}{q(t)} \to \gamma \tag{21}$$

as $t\to\infty$ (i.e., as $s\to\infty$). By (15), we also have

$$\widetilde{p}(s) = \left(\frac{p}{\tau^\Delta}\circ\tau^{-1}\right)(s) = \tau^{-1}(s)p(\tau^{-1}(s)) \sim (q\circ\tau^{-1})(s) \tag{22}$$

as $s\to\infty$. Relations (21) and (22) mean that

$$\widetilde{p}\in\mathcal{RV}_{\widetilde{\mathbb{T}}}(\gamma), \tag{23}$$

where $\mathcal{RV}_{\widetilde{\mathbb{T}}}(\gamma)$ denotes the class of regularly varying functions of index γ on the time scale $\widetilde{\mathbb{T}}$, see, e.g., [21] and also Remark 5-(i). Further,

$$\frac{s^2\widetilde{p}(s)}{\widetilde{r}(s)} = s^2\left(\frac{p}{\tau^\Delta r\tau^\Delta}\right)\circ\tau^{-1}(s) = \left(\frac{\tau(t)}{\tau^\Delta(t)}\right)^2\frac{p(t)}{r(t)} = (t\tau(t))^2\frac{p(t)}{r(t)}.$$

Hence, in view of (16),

$$\lim_{s\to\infty}\frac{s^2\widetilde{p}(s)}{\widetilde{r}(s)} = 0. \tag{24}$$

Finally,

$$\frac{\widetilde{\mu}(s)}{s} = \frac{\widetilde{\sigma}(s)-s}{s} = \frac{\widetilde{\sigma}(\tau(t))-\tau(t)}{\tau(t)} = \frac{\tau^\sigma(t)-\tau(t)}{\tau(t)} = \frac{\mu(t)\tau^\Delta(t)}{\tau(t)} = \frac{\mu(t)}{t\tau(t)} \to 0 \tag{25}$$

as $t\to\infty$ (i.e., as $s\to\infty$). Conditions (23)–(25) guarantee that the assumptions of [21, Theorem 4] are fulfilled for Eq. (19), and thus we can apply that result in the next steps.

Consider now case (a), thus we assume that $\gamma < -1$ and $\int_a^\infty 1/r(s)\,\Delta s = \infty$. Take an eventually positive decreasing solution y of (14). Since $u\circ\tau = y$ and $y^\Delta = (u^{\widetilde{\Delta}}\circ\tau)\tau^\Delta$, u is eventually positive decreasing solution of (19). By [21, Theorem 4-(i)], $u\in\mathcal{NSV}_{\widetilde{\mathbb{T}}}$, i.e., u is normalized slowly varying on $\widetilde{\mathbb{T}}$, i.e., $su^{\widetilde{\Delta}}(s)/u(s)\to 0$ as $s\to\infty$. Consequently, $y\circ\tau^{-1}\in\mathcal{NSV}_{\widetilde{\mathbb{T}}}$. Therefore, with using the ideas similar to those in (20),

$$t\tau(t)\frac{y^\Delta(t)}{y(t)} = \frac{s(y\circ\tau^{-1})^{\widetilde{\Delta}}(s)}{(y\circ\tau^{-1})(s)} \to 0$$

as $s\to\infty$. Further, with $S=\tau(T)$ and $\widetilde{a}=\tau(a)$, applying the substitution method in delta integrals, we obtain

$$\int_{\widetilde{a}}^{S} \frac{s\widetilde{p}(s)}{\widetilde{r}(s)}\,\widetilde{\Delta}s = \int_{\tau(a)}^{\tau(T)} s\left(\frac{p}{\tau^{\Delta} r \tau^{\Delta}}\right)\circ\tau^{-1}(s)\,\widetilde{\Delta}s$$
$$= \int_{a}^{T} \frac{\tau(t)p(t)}{(\tau^{\Delta}(t))^2 r(t)}\tau^{\Delta}(t)\,\Delta t = \int_{a}^{T} \frac{\tau(t)p(t)}{\tau^{\Delta}(t)r(t)}\,\Delta t = \int_{a}^{T} \Psi(t)\,\Delta t. \tag{26}$$

In particular, $\int_{\widetilde{a}}^{\infty} s\widetilde{p}(s)/\widetilde{r}(s)\,\widetilde{\Delta}s$ converges if and only if $\int_{a}^{\infty} \Psi(t)\,\Delta t$ converges. Assume that $\int_{a}^{\infty} \Psi(t)\,\Delta t = \infty$. Then $\int_{\widetilde{a}}^{\infty} s\widetilde{p}(s)/\widetilde{r}(s)\,\widetilde{\Delta}s = \infty$, and, by [21, Theorem 4-(i)],

$$u(s) = \exp\left\{\int_{\widetilde{a}}^{s} (1+o(1))\frac{\eta\widetilde{p}(\eta)}{(\gamma+1)\widetilde{r}(\eta)}\,\widetilde{\Delta}\eta\right\} \tag{27}$$

and $u(s) \to 0$ as $s \to \infty$. From $y(t) = u(s)$, (26), and (27), we get (17). If $\int_{a}^{\infty} \Psi(t)\,\Delta t < \infty$, then $\int_{\widetilde{a}}^{\infty} s\widetilde{p}(s)/\widetilde{r}(s)\,\widetilde{\Delta}s < \infty$, and, by [21, Theorem 4-(i)],

$$u(s) = \ell_u \exp\left\{-\int_{s}^{\infty} (1+o(1))\frac{\eta\widetilde{p}(\eta)}{(\gamma+1)\widetilde{r}(\eta)}\,\widetilde{\Delta}\eta\right\} \tag{28}$$

as $s \to \infty$ with $\ell_u := \lim_{s\to\infty} u(s) \in (0,\infty)$. Similarly as before, using now (26) and (28), we get (18). Moreover, $\ell_y = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} u(\tau(t)) = \lim_{s\to\infty} u(s) = \ell_u$.

(b) This part can be proved similarly as part (a); we apply [21, Theorem 4-(ii)] to transformed equation (19). □

Remark 5 (i) A closer examination of the proof shows that condition (15) can (equivalently) be replaced by $\tau^{-1}\cdot(p\circ\tau^{-1}) \in \mathcal{RV}_{\widetilde{\mathbb{T}}}(\gamma)$. Here, $\mathcal{RV}_{\mathbb{T}}(\vartheta)$ denotes the class of regularly varying functions of index ϑ on time scale $\mathbb{T}$. An rd-continuous positive function f belongs to $\mathcal{RV}_{\mathbb{T}}(\vartheta)$ if and only if there is $g \in (C^1_{\mathrm{rd}}(\mathbb{T}),(0,\infty))$ such that $f(t) \sim g(t)$ and $tg^{\Delta}(t)/g(t) \to \vartheta$ as $t \to \infty$, see [21].

(ii) Let us consider Eq. (14) and assume that $p \in \mathcal{RV}_{\mathbb{T}}(\delta)$. Notice that [21, Theorem 4] (which was applied in the previous proof to the transformed equation) requires $\delta \neq -1$. A natural problem is therefore to consider the critical case $\delta = -1$. This setting is somehow delicate and the method of the proof of [21, Theorem 4] does not work. However, the previous theorem enables us to treat this case, as the following example shows. Let $\mu(t) = o(t)$ as $t \to \infty$, $p(t) = L(t)/t$, where $L(t) = (\ln t)^{-2}(\ln(\ln t))^{-\omega}$ with $\omega > 0$, and $r(t) = t$. If $\tau(t)$ is as in the theorem, then $\tau(t) \sim \ln t$ as $t \to \infty$, see the next item (iii). We have $p \in \mathcal{RV}_{\mathbb{T}}(-1)$, $\int_{a}^{\infty} 1/r(s)\,\Delta s = \infty$, and

$$(t\tau(t))^2\frac{p(t)}{r(t)} \sim (t\ln t)^2\frac{p(t)}{r(t)} = \frac{1}{(\ln(\ln t))^{\omega}} \to 0$$

as $t \to \infty$. Further, as $s \to \infty$,

$$\tau^{-1}(s)p(\tau^{-1}(s)) = \frac{1}{(\ln\tau^{-1}(s))^2(\ln(\ln\tau^{-1}(s)))^{\omega}} \sim \frac{1}{s^2\ln^{\omega} s} \in \mathcal{RV}_{\widetilde{\mathbb{T}}}(-2).$$

In view of the previous item (i), condition (15) is fulfilled with $\gamma = -2 < -1$. Thus Theorem 5-(a) can be applied and we get that any eventually positive decreasing solution y of (14) satisfies $t \ln t\, y^{\Delta}(t)/y(t) \to 0$, $r(t)y^{\Delta}(t) \to 0$ as $t \to \infty$, and obeys one of the formulae (17) or (18), according to whether $\int_a^{\infty} \Psi(s)\,\Delta s = \infty$ or $\int_a^{\infty} \Psi(s)\,\Delta s < \infty$, respectively, where, because of $\tau(t) \sim \ln t$ as $t \to \infty$, the function Ψ can be taken as

$$\Psi(t) = \frac{1}{t \ln t\, (\ln \ln t)^{\omega}}.$$

(iii) The condition $\mu(t) = o(t\tau(t))$ as $t \to \infty$ in Theorem 5, where $\tau(t) = \int_a^t 1/s\,\Delta s$, allows us to cover any time scale with a bounded graininess, for example, $\mathbb{R}$, $\mathbb{Z}$, $h\mathbb{Z}$, or the harmonic numbers $\mathbb{H} := \{\sum_{j=1}^{k-1} 1/j : k \in \mathbb{N}\}$, but also some time scales with an unbounded graininess such as $\mathbb{N}^{\alpha} := \{n^{\alpha} : n \in \mathbb{N}\}$ with $\alpha > 1$ (here we have $\mu(t) = \alpha t^{(\alpha-1)/\alpha} + O(t^{-1/\alpha})$) or the quantum calculus case $q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$ with $q > 1$ (here we have $\mu(t) = (q-1)t$). If we strengthen the condition $\mu(t) = o(t\tau(t))$ to the condition $\mu(t) = o(t)$ as $t \to \infty$ (which is satisfied, e.g., for $\mathbb{R}$, $\mathbb{Z}$, $h\mathbb{Z}$, $\mathbb{H}$, $\mathbb{N}^{\alpha}$), then $\tau(t) \sim \ln t$ as $t \to \infty$. Indeed, by the time scale L'Hospital rule and the Lagrange mean value theorem, we have

$$\lim_{t\to\infty} \frac{\ln t}{\tau(t)} = \lim_{t\to\infty} \frac{(\ln t)^{\Delta}}{1/t} = \lim_{t\to\infty} \frac{t}{\xi(t)},$$

where $t \le \xi(t) \le \sigma(t)$. Further,

$$1 = \frac{t}{t} \le \frac{\xi(t)}{t} \le \frac{\sigma(t)}{t} = \frac{t + \mu(t)}{t} = 1 + \frac{\mu(t)}{t}.$$

Since $\mu(t) = o(t)$, we get $\lim_{t\to\infty} t/\xi(t) = 1$ and so $\lim_{t\to\infty} \ln t/\tau(t) = 1$ Consequently, the formulae in Theorem 5 can be rewritten as follows: Instead of $\tau(t)$ in $\Psi(t)$, (15), and (16), we can write directly $\ln t$. Finally note that if $\mathbb{T} = q^{\mathbb{N}_0}$, then $\lim_{t\to\infty} \ln t/\tau(t) = \lim_{t\to\infty}[(\ln q)/t]/[(q-1)/t] = \ln q/(q-1)$, and so $\tau(t) \sim (q-1)\ln t/\ln q$ as $t \to \infty$.

(iv) Consider the transformation involving $\tau(t) = \int_a^t 1/s\,\Delta s$, as in the proof of Theorem 5. Then q-difference equations (that is, equations defined on $\mathbb{T} = q^{\mathbb{N}_0}$) are actually transformed into difference equations since $\tau(t) = \int_1^t 1/s\,\Delta s = \sum_{s\in[1,t)_{\mathbb{T}}} (1/s)(q-1)s = \sum_{s\in[1,t)_{\mathbb{T}}}(q-1) = \sum_{j=0}^{k-1}(q-1) = (q-1)k$, where $t = q^k$. Further, difference equations are by the same form of the substitution transformed into dynamic equations on the harmonic numbers. Relations between difference and q-difference equations are utilized in [20, Theorems 7.1–7.4], where asymptotic properties of solutions to q-difference equations under the critical setting are studied.

(v) In the proof of Theorem 5 we have used the transformation involving mapping τ in the special form $\tau(t) = \int_a^t 1/s\,\Delta s$. This suggests an idea to consider a transformation in a general form which could lead to a generalization of the existing results and a refinement of the concept of regular variation on time scales.

(vi) We believe that a combination of suitable transformations (see also the last paragraph in Sect. 4) along with existing results will enable us to examine asymptotics of other equations in critical cases which are close, for instance, to Euler type or Riemann–Weber type equations (see (12)), such as the equation $y^{\Delta\Delta} + p(t)y^{\sigma} = 0$, with $\mu(t) = o(t)$, where $t^2 p(t) \to 1/4$ and $(1/4 - t^2 p(t)) \ln^2 t \to 0$ as $t \to \infty$, and $|1/4 - t^2 p(t)|$ belongs to a suitable subclass of slowly varying functions on $\mathbb{T}$.

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