Discrete Reaction-Dispersion Equation

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Abstract The paper introduces a discrete analogy of the reaction-diffusion partial differential equation. Both the time and the space are considered to be discrete, the space is represented by a simple graph. The equation is derived from "first principles". Basic qualitative properties, namely, existence and stability of equilibria are discussed. The results are demonstrated on a particular system that can be interpreted as a model of metapopulation on interconnected patches with a deadly boundary. A condition for size of habitat needed for population survival is established.

Keywords Diffusion · Random walk · Graph theory · Stability of equilibria

1 Motivation

An auto-catalytic reaction of a "substance" and its diffusion on one dimensional continuous space is described by the parabolic PDE

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u)\ , \quad t > 0\ ,\ x \in (0, L)\ ; \tag{1}$$

the state variable u represents a substance density, the function f (non-linear, in general) describes the reaction, and D is the diffusivity (the coefficient appearing in the Fickian law). The same equation stands for a model of spatially distributed population—it can be found in any textbook on mathematical ecology or population dynamics, cf. e.g. [4, pp. 265–343]. Let us remind two results concerning the reaction-diffusion equation that are of importance in a context of theoretical ecology.

Let $f : [0, \infty) \to [0, \infty)$ be a continuous function satisfying the conditions $f(0) = 0$, $(u - K)f(u) < 0$ for some positive constant K and $0 < u \neq K$. Then

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the Eq. (1) possesses the spatially homogeneous equilibrium $u \equiv 0$ (the extinction equilibrium). The Eq. (1) with the homogeneous Neumann boundary condition

$$\frac{\partial u(0)}{\partial x} = 0 = \frac{\partial u(L)}{\partial x} \tag{2}$$

has the positive asymptotically stable equilibrium $u \equiv K$ and the extinction equilibrium is unstable. If f is differentiable in a right neighbourhood of 0 with $r = f'(0) > 0$ and $L > \pi\sqrt{D/r}$ then the trivial equilibrium $u \equiv 0$ of the Eq. (1) with the homogeneous Dirichlet boundary condition

$$u(0) = 0 = u(L) \tag{3}$$

is unstable and the boundary value problem (1), (3) possesses an asymptotically stable spatially non-homogeneous equilibrium $u(t, x) = u^*(x)$ such that

$$0 < \max_{0<x<L} u^*(x) = u\left(\tfrac{1}{2}L\right) < K \ .$$

The last case serves as a theoretical explanation of the fact that a spatially distributed population needs some minimal size of its habitat.

The aim of the contribution is to present discrete counterparts of the reminded results. The discrete diffusion (heat) equation is a matter of textbooks, cf. [3, p. 168–172]. Here, the discrete space is a grid and the equation is linear autonomous difference system with Toeplitz matrix. The reaction term can be simply added to such a system. Systems of this type represent accurate discretization of the mentioned PDEs and they were studied not only on grids [2] but, recently, also on general graphs [8]. This approach brings reasonable results. But, by my opinion, this is not the only possibility how to model phenomena consisting of a reaction and dispersal of a "substance" on a discrete space, namely, on networks or graphs.

First, the word "diffusion" takes different meanings in different contexts. Humanities mean by it a very complex and complicated process [7]. In metapopulation ecology, it denotes a form of dispersal, in which the flux of individuals between two patches is proportional to the difference in their densities [1, p. 65]. Such a process coincide with a random walk. A more precise determination distinguishes diffusion from random walk. Diffusion is described by the Laplacian matrix (discrete Laplacian operator), while random walk by a matrix derived from the Markovian one. An important difference between the two processes is that diffusion tends to a state with uniformly distributed values in all nodes, while random walk tends to a state where the values are proportional to node degrees. Hence, the two processes coincide for regular graphs [6, Sects. 6.13 and 6.14]. Because of these terminological reasons, I use the more general word "dispersion" instead of the specific "diffusion".

Second, the process of reaction can be separated (at least mentally) from the one of dispersion. This idea was utilized already in framework of linear models of (meta)population dynamics [5] and it forms a basis for the subsequent considerations.

In the next section, the proposed reaction-dispersion system is derived from mechanistic presuppositions ("first principles"). The dispersion process is represented by the random walk and the model presented is more general than necessary for the rest of the paper. This generality may show that a scope of applicability exceeds the one indicated in the third section where a spatially homogeneous reaction and simple dispersion is considered. Results that can be regarded as analogues to the mentioned results from continuous spatial population dynamics are shown.

The notation used is standard. The vectors are considered to be the column ones, $\mathbf{v} = (v_1, v_2, \ldots, v_k)^{\mathsf{T}}$; we will also denote the entries of the vector $\mathbf{v}$ by $v_i = (\mathbf{v})_i$. The symbol $\operatorname{diag} \mathbf{v}$ denotes the diagonal matrix M with diagonal entries $m_{ii} = v_i$, the symbols $\mathbf{o}$, $\mathbf{1}$, and I denote zero vector, vector with all of entries equal to 1, and identity matrix, respectively.

2 The Model

Of course, a reaction-dispersion equation should describe two processes—a reaction, i.e. constitution and dissociation of a substance or birth and death of individuals forming a population, and dispersion, i.e. random spreading of it to a space. Assume that the space is represented by a simple connected graph $\mathcal{G} = \{N, E\}$, where $N = \{1, 2, \ldots, k\}$ is a set of nodes and $E \subseteq \big\{\{i, j\} : i, j \in N, i \neq j\big\}$ is the set of edges. Let A be the adjacency matrix of the graph $\mathcal{G}$ and denote $\sigma_j = \big|\big\{i : \{i, j\} \in E\big\}\big| = \sum_{i=1}^{k} a_{ij} = \big(\mathbf{1}^{\mathsf{T}}\mathsf{A}\big)_j$ the degree of the node $j \in N$.

Let $x_i(t)$ denote an expected concentration of a substance in the node $i \in N$ at the time t; naturally, we suppose $x_i \geq 0$.

Let us suppose that the reaction take place entirely in the nodes and the dispersion consists in relocation of particles (individuals) along the edges. Let us start with a description of the two processes separately.

2.1 Reaction

Let $\tau_{\mathrm{r}} > 0$ be a "typical reaction time". We assume that the change of concentration in the node $i \in N$ depends on the concentration x_i only, i.e. it is given by the recurrence relation

$$x_i(t + \tau_{\mathrm{r}}) = f_i\big(x_i(t)\big) \,.$$

Here $f_i : [0, \infty) \to [0, \infty)$, $i = 1, 2, \ldots, k$ are functions satisfying

$$f_i(0) = 0 \tag{4}$$

that are non-linear in general. Denoting $\mathbf{f}(\mathbf{x}) = \big(f_1(x_1), f_2(x_2), \ldots, f_k(x_k)\big)^{\mathsf{T}}$, we can write the relation in the simple vector form

$$\mathbf{x}(t + \tau_{\mathrm{r}}) = \mathbf{f}\big(\mathbf{x}(t)\big) \ . \tag{5}$$

2.2 Dispersion

Let us suppose that a particle in a node can either remain in it or it can move to a randomly chosen neighbor node. We can call this process "elementary dispersion event". Let us denote by $\tau_{\mathrm{d}} > 0$ the time necessary for replacement of a particle to a neighbor node and by d, $0 < d \leq 1$, the probability that a particle leaves its node. The adopted assumptions yield the recurrence relations

$$x_i(t + \tau_{\mathrm{d}}) = (1 - d)x_i(t) + \sum_{\{i,j\} \in E} \frac{d}{\sigma_j} x_j(t) \ , \quad i \in N \ ,$$

for the expected concentrations x_i. Let us define the matrix K with the entries $\kappa_{ij} = a_{ij}/\sigma_j$, that is $\mathsf{K} = \mathsf{A}(\operatorname{diag} \mathsf{A}\mathbf{1})^{-1}$. Since

$$\sum_{\{i,j\} \in E} \frac{d}{\sigma_j} x_j(t) = d \sum_{j=1}^{k} \frac{a_{ij}}{\sigma_j} x_j(t) = d\big(\mathsf{K}\mathbf{x}(t)\big)_i \ ,$$

we can rewrite the derived recurrence relation in the vector form

$$\mathbf{x}(t + \tau_{\mathrm{d}}) = \big(\mathsf{I} - d(\mathsf{I} - \mathsf{K})\big)\mathbf{x}(t) \ . \tag{6}$$

The matrix on the right hand side characterizes the described dispersion process. Obviously,

$$\sum_{j=1}^{k} \kappa_{ij} = \sum_{j=1}^{k} \frac{a_{ij}}{\sigma_j} = 1.$$

that is, the matrix K is left (row) stochastic and, subsequently, the dispersion matrix

$$\mathsf{D} = \mathsf{I} - d(\mathsf{I} - \mathsf{K}) \tag{7}$$

is left stochastic as well. The matrix D is used to describe random walk on a graph, cf. comment in the first section and the reference therein.

2.3 Reaction and Dispersion

Let us suppose that the two processes are separated in time. First, the reaction occurs and then it is followed by the dispersion. Such an assumption may be satisfied in population dynamic—e.g., a plant grows, flowers and produces seeds and then the seeds are spread. If this is the case, we can define the time unit (length of projection interval) to be the sum of the "typical reaction and dispersion times", $1 = \tau_\mathrm{r} + \tau_\mathrm{d}$ and, combining (5), (6) and (7), we obtain the non-linear recurrence relation

$$\mathbf{x}(t+1) = \mathbf{x}\big((t+\tau_\mathrm{r})+\tau_\mathrm{d}\big) = \mathsf{D}\mathbf{x}(t+\tau_\mathrm{r}) = \mathsf{D}\mathbf{f}\big(\mathbf{x}(t)\big)\ .$$

Let us note that the above form of the recurrence fixes the "observation times" to moments before "reaction event" and after "dispersion event". This is just an arbitrary choice. The other one leads to the relation $\mathbf{x}(t+1) = \mathbf{f}\big(\mathsf{D}\mathbf{x}(t)\big)$; we will not deal with this option in the present paper.

The situation need not to be simple alternation of reaction and "elementary dispersion event". We can consider the possibility that the reaction is followed by several "elementary dispersions". If the number of them is n then we define the time unit by $1 = \tau_\mathrm{r} + n\tau_\mathrm{d}$ and the process is described by the recurrence relation

$$\mathbf{x}(t+1) = \mathsf{D}^n\mathbf{f}\big(\mathbf{x}(t)\big)\ . \tag{8}$$

2.4 Homogeneous Boundary Conditions

The Neumann boundary condition (2) for the continuous equation (1) states that no amount of the "substance" goes out of the region where it disperses, that is, the process is spatially isolated. The discrete equation (8) is defined on the underlying graph $\mathcal{G}$ only. That is, no "outer space" is considered and hence no additional condition is needed.

The Dirichlet boundary condition (3) determines the concentration on the two boundary points to be zero. This property can be mimic for the discrete equation (8) as well. Let us determine some of the nodes to be the boundary ones and prescribe the zero value of the concentration on them for each time t. This is equivalent to left multiplication of the vector $\mathbf{x}(t)$ by the "boundary" matrix B defined as

$$b_{ij} = \begin{cases} 1, & i = j \notin N_B, \\ 0, & \text{otherwise}, \end{cases}$$

where $N_B \subseteq N$ is the set of boundary nodes. The matrix B is obviously idempotent. Of course, if $N_B = \emptyset$ then the "boundary" matrix $\mathsf{B} = \mathsf{I}$ expresses the "Neumann-type" condition.

Let us suppose first that each "elementary dispersion event" is followed by an immediate "disappearance" of the substance in the boundary nodes. E.g., considering metapopulations we can regard the boundary patches to be be deadly ones. In this case, the matrix describing dispersion equals BD. Due to the condition (4), the identities $\mathbf{x} = \mathsf{B}\mathbf{x}$, $\mathbf{f}(\mathbf{x}) = \mathsf{B}\mathbf{f}(\mathbf{x})$ hold true for any vector $\mathbf{x}$ satisfying the boundary condition $x_i = 0$ for $i \in N_B$. Hence, the discrete reaction-dispersion equation with homogeneous boundary condition can be written in one of the equivalent forms

$$\mathbf{x}(t+1) = (\mathsf{BD})^n \mathbf{f}\big(\mathbf{x}(t)\big) \,, \quad \text{or} \quad \mathbf{x}(t+1) = (\mathsf{BD})^n \mathsf{B}\mathbf{f}\big(\mathbf{x}(t)\big) \,. \tag{9}$$

But the process of "disappearance" of particles/individuals in the boundary nodes may be a matter of the reaction only. If this is the case, the model can be written in one of the equivalent forms

$$\mathbf{x}(t+1) = \mathsf{BD}^n \mathbf{f}\big(\mathbf{x}(t)\big) \,, \quad \text{or} \quad \mathbf{x}(t+1) = \mathsf{BD}^n \mathsf{B}\mathbf{f}\big(\mathbf{x}(t)\big) \,. \tag{10}$$

3 Equilibria and Their Stability

Now, let us consider the system (9) or (10) with $n = 1$, i.e. the system

$$\mathbf{x}(t+1) = \mathsf{BDB}\mathbf{f}\big(\mathbf{x}(t)\big) \,. \tag{11}$$

Let $\mathbf{x}^*$ be the equilibrium of the equation (11), i.e. $\mathsf{BDB}\,\mathbf{f}\,(\mathbf{x}^*) = \mathbf{x}^*$. If the vector function $\mathbf{f}$ is differentiable one can check the stability of the equilibrium using linearization of the system. Since

$$\frac{\partial}{\partial x_j}\big(\mathbf{f}(\mathbf{x}^*)\big)_i = \frac{\partial}{\partial x_j} f_i(x_i^*) = \begin{cases} f_j'(x_j^*), & i = j, \\ 0, & i \neq j, \end{cases}$$

the variation matrix of the system (11) equals $\mathsf{J}(\mathbf{x}^*) = \mathsf{BDB}\mathrm{diag}\,\mathbf{f}'(\mathbf{x}^*)$, where $\mathbf{f}'(\mathbf{x}^*)$ stands for $\big(f_1'(x_1^*), f_2'(x_2^*), \ldots, f_k'(x_k^*)\big)^{\mathsf{T}}$. Consequently, the stability of the equilibrium $\mathbf{x}^*$ is determined by the spectrum of the matrix $\mathsf{BDB}\mathrm{diag}\,\mathbf{f}'(\mathbf{x}^*)$.

Let us pay attention to eigenvalues of the matrices BDB and BKB. Obviously, if $N_B \neq \emptyset$ then the both matrices possesses zero eigenvalues with corresponding eigenvectors $\mathbf{w}$ satisfying $w_i = 0$ for all $i \notin N_B$, i.e. $\mathsf{B}\mathbf{w} = \mathbf{o}$. That is, the geometric multiplicity of zero eigenvalues is at least $|N_B|$. The following statement says something on the others eigenvalues.

Proposition 1 *Let vector* $\mathbf{w}$ *satisfy the boundary condition, i.e.,* $\mathbf{w} = \mathsf{B}\mathbf{w}$. *Then* $\mathbf{w}$ *is the eigenvector of the matrix* BDB *if and only if it is the eigenvector of the matrix* BKB. *The corresponding eigenvalues* $\tilde{\lambda}$ *and* λ *of the matrices* BDB *and* BKB, *respectively, satisfy the equality* $\tilde{\lambda} = 1 - d(1 - \lambda)$.

Proof Let $\mathbf{w} = \mathsf{B}\mathbf{w}$. Then by (7),

$$\mathsf{BDB}\mathbf{w} = \mathsf{B}\big(\mathsf{I} - d(\mathsf{I} - \mathsf{K})\big)\mathsf{B}\mathbf{w} = \mathbf{w} - d\mathbf{w} + d\mathsf{BKB}\mathbf{w} \,.$$

Now, if $\mathsf{BKB}\mathbf{w} = \lambda\mathbf{w}$ then $\mathsf{BDB}\mathbf{w} = \big(1 - d(1-\lambda)\big)\mathbf{w}$, if $\mathsf{BDB}\mathbf{w} = \tilde{\lambda}\mathbf{w}$ then $\big((\tilde{\lambda} - 1 + d)/d\big)\mathbf{w} = \mathsf{BKB}\mathbf{w}$. □

The Proposition 1 allows to prove a statement on stability of particular equilibria of the system (11).

Proposition 2 *Let the reaction function be the same on all non-boundary nodes, i.e.* $\mathbf{f}(\mathbf{x}) = \big(f(x_1), f(x_2), \ldots, f(x_k)\big)^{\mathsf{T}}$, *where* $f : [0,\infty) \to [0,\infty)$ *is differentiable function, and let* $x^*\mathbf{B1}$ *be an equilibrium of the system (11). If*

$$\big|f'(x^*)\big(1 - d(1-\lambda)\big)\big| < 1 \tag{12}$$

for each eigenvalue λ *of the matrix* BKB *such that the corresponding eigenvector* $\mathbf{w}$ *satisfies the boundary condition* $\mathbf{w} = \mathsf{B}\mathbf{w}$, *then the equilibrium is stable.*

If there exists an eigenvalue λ *with corresponding eigenvector* $\mathbf{w}$ *of the matrix* BKB *satisfying the boundary condition and the inequality is reversed, then the equilibrium is unstable.*

Proof We have $\mathsf{B}\mathrm{diag}\,\mathbf{f}'(x^*\mathbf{B1}) = f'(x^*)\mathsf{B}$. Consequently $\mathsf{J}(x^*\mathbf{B1}) = f'(x^*)\mathsf{BDB}$. Hence, if $|f'(x^*)\tilde{\lambda}| < 1$ for all eigenvalues $\tilde{\lambda}$ of the matrix BDB, then the equilibrium is stable. The condition is fulfilled for $\tilde{\lambda} = 0$ such that the corresponding eigenvector $\tilde{\mathbf{w}}$ satisfies $\tilde{\mathbf{w}} \neq \mathsf{B}\tilde{\mathbf{w}} = \mathbf{o}$. Now, the statement follows from the Proposition 1. □

Let us consider the system (11) with symmetric matrix K and no boundary node. Since $\kappa_{ij} = a_{ij}/\sigma_j$, the assumption on symmetry is satisfied if and only if $\sigma_i = \sigma_j$ for all i, j, i.e., for regular graphs only. In this case, a random walk coincides with diffusion. Since the matrix K is left stochastic and symmetric, it is double stochastic and its spectral radius equals 1 and one eigenvalue equals 1. Hence, if $\mathsf{B} = \mathsf{I}$, then the condition (12) holds if and only if it holds for $\lambda = 1$. If x^* is a fixed point of the function f, then

$$\mathsf{D}(x^*\mathbf{1}) = x^*\big(\mathsf{I} - d(\mathsf{I} - \mathsf{K})\big)\mathbf{1} = x^*(\mathbf{1} - d\mathbf{1} + d\mathbf{1}) = x^*\mathbf{1}$$

and $x^*\mathbf{1}$ is an equilibrium of the system (11) with $\mathsf{B} = \mathsf{I}$ and $\mathsf{D} = \mathsf{D}^{\mathsf{T}}$.

Hence, we have obtained an analogy to the statement that, under some condition, the continuous boundary value problem (1), (2) possesses a spatially homogeneous stable equilibrium: Let $K > 0$ and differentiable function $f : [0,\infty) \to [0,\infty)$ satisfy $\big(f(x) - K\big)(x - K) < 0$ for $x \neq K$. Then the system

$$\mathbf{x}(t+1) = \mathsf{D}\mathbf{f}\big(\mathbf{x}(t)\big) \tag{13}$$

with symmetric matrix D has the positive stable equilibrium $K\mathbf{1}$.

The Proposition 2 speaks on stability of spatially homogeneous equilibria of the system (11) with equal "reaction functions" on each node. Due to the property (4), the zero vector $\mathbf{o}$ is a spatially homogeneous equilibrium; if the system is interpreted as a model of metapopulation the vector $\mathbf{o}$ represents the extinction equilibrium. The system (13) without boundary nodes and symmetric matrix D may possess a non-zero spatially homogeneous equilibrium. An example of a system with a non-empty boundary and a non-zero spatially homogeneous equilibrium is presented in the following subsection.

3.1 Particular Case

Let us consider the complete graph with $k \geq 3$ nodes, one of nodes (the last one, say) be the boundary one. Let the reaction be the same on each non-boundary node and let it be of the form

$$f(x) = xg(x) \,,$$

where the function $g : [0, \infty) \to [0, \infty)$ is differentiable with

$$g(0) = r > 1, \quad g'(x) < 0, \quad (x - K)\big(g(x) - 1\big) < 0 \text{ for } x \neq K > 0 \,.$$

Finally, let one reaction be followed by one dispersion event. Now, the system (11) might represent a model of metapopulation on k mutually interconnected patches, one of them is deadly and the others exhibit the same condition for the local populations.

We have $f'(x) = g(x) + xg'(x)$ and

$$\mathsf{B} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \; \mathsf{K} = \begin{pmatrix} 0 & \frac{1}{k-1} & \dots & \frac{1}{k-1} \\ \frac{1}{k-1} & 0 & \dots & \frac{1}{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k-1} & \frac{1}{k-1} & \dots & 0 \end{pmatrix}, \; \mathsf{BKB} = \begin{pmatrix} 0 & \frac{1}{k-1} & \dots & \frac{1}{k-1} & 0 \\ \frac{1}{k-1} & 0 & \dots & \frac{1}{k-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{k-1} & \frac{1}{k-1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

One can easily verify that the matrix BKB has the following eigenvalues with corresponding eigenvectors

$$\lambda_1 = \frac{k-2}{k-1}, \; \mathbf{w_1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = 0 \,, \; \mathbf{w_2} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

$$\lambda_3 = \lambda_4 = \cdots = \lambda_k = \frac{1}{1-k},\ \mathbf{w_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \end{pmatrix},\ \mathbf{w_4} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \end{pmatrix},\ \ldots,\ \mathbf{w_k} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Thus, we have

$$1 - d(1-\lambda_1) = 1 - \frac{d}{k-1}\ , \quad 1 - d(1-\lambda_j) = 1 - \frac{dk}{k-1}\ ,\ j = 3, 4, \ldots, k\ .$$

Obviously, $1 - dk/(k-1) < 1 - d/(k-1)$. The assumptions $d \le 1$ and $k \ge 3$ yield

$$0 \le 1 - \frac{d}{k-1} \quad \text{and} \quad \frac{dk}{k-1} - 1 \le 1 - \frac{d}{k-1}\ ,$$

since $\frac{1}{2}d \le \frac{1}{2} \le (k-1)/(k+1)$. Consequently

$$\left| 1 - \frac{dk}{k-1} \right| \le 1 - \frac{d}{k-1}\ .$$

That is, the criterion of equilibrium stability from the Proposition 2 reads

$$\left| f'(x^*) \right| \left(1 - \frac{d}{k-1} \right) < 1\ .$$

The zero vector $\mathbf{o}$ is one equilibrium (the extinction equilibrium) of the system. The assumption $f'(0) = g(0) = r > 1$ now implies that if

$$r \left(1 - \frac{d}{k-1} \right) < 1\ , \quad \text{i.e.} \quad k < 1 + \frac{dr}{r-1}\ ,$$

then the extinction equilibrium is stable.

Now, let us search conditions for the existence of equilibrium of the form $x^*\mathbf{B1}$ with $x^* > 0$, that is, the equilibrium spatially homogeneous on the non-boundary nodes.

Let us note that under the assumptions listed above, the system (11) becomes

$$x_i(t+1) = (1-d)x_i(t)g\big(x_i(t)\big) + \frac{d}{k-1} \sum_{\substack{\{i,j\} \in E \\ j \ne k}} x_j(t) g\big(x_j(t)\big)\ ,\ i = 1, 2, \ldots, k-1$$

and $x_k(t) \equiv 0$. Consequently

$$g(x^*)\left(1-d+(k-2)\frac{d}{k-1}\right)=1, \quad \text{or,} \quad g(x^*)=\frac{k-1}{k-1-d}\,.$$

Since the function g is strictly decreasing with $g(0)=r$, $g(K)=1$, the last equation possesses unique solution x^* satisfying $0<x^*<K$ if and only if

$$\frac{k-1}{k-1-d}<r\,, \quad \text{i.e.} \quad k>1+\frac{dr}{r-1}\,. \tag{14}$$

If this is the case, we have

$$f'(x^*)=g(x^*)+x^*g'(x^*)=\frac{k-1}{k-1-d}+x^*g'(x^*),$$

$$\left|f'(x^*)\right|\left(1-\frac{d}{k-1}\right)=\left|1+\frac{k-1-d}{k-1}x^*g'(x^*)\right|=\left|1+x^*\frac{g'(x^*)}{g(x^*)}\right|$$

and the criterion of stability becomes

$$-2<x^*\frac{g'(x^*)}{g(x^*)}<0\,.$$

The last inequality is satisfied due the assumption $g'<0$.

Now, we can conclude: If $k<1+dr/(1-r)$ then the extinction equilibrium $\mathbf{o}$ is stable, if $k>1+dr/(1-r)$ then the equilibrium $\mathbf{o}$ is unstable and there exists the interior equilibrium $x^*\mathsf{B1}$ which is stable provided $-2<x^*g'(x^*)/g(x^*)$.

This statement (namely, the second inequality (14)) can be considered to be one of possible analogies to the one that the boundary value problem (1), (3) possesses non-trivial equilibrium just for L "large enough", that is for sufficient "size" k of the space.

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References

1. Caswell, H.: Matrix Population Models. Sinauer Associates, Sunderland (2001)
2. Chow, S.-N., Mallet-Paret, J., Shen, W.: Traveling waves in lattice dynamical systems. J. Differ. Equ. **149**, 248–291 (1998)
3. Elaydi, S.: An Introduction to Difference Equations. Springer, New York (2015)
4. Kot, M.: Elements of Mathematical Ecology. Cambridge University Press, Cambridge (2001)
5. Lebreton, J.-D., Gonzales-Davila, G.: An introduction to models of subdivided populations. J. Biol. Syst. **1**, 389–423 (1993)
6. Newman, M.: Networks. Oxford University Press, Oxford (2018)

7. Rogers, E.M.: Diffusion of Innovations. The Free Press, New York (1983)
8. Stehlík, P.: Exponential number of stationary solutions for Nagumo equations on graphs. J. Math. Anal. Appl. **455**, 1749–1764 (2017)