Linearized Oscillation Theory for a Nonlinear Nonautonomous Difference Equation

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Abstract We review some theorems and mistakes in linearized oscillation results for difference equations with variable coefficients and constant delays, as well as develop linearized oscillation theory when delays are also variable. Main statements are applied to discrete models of population dynamics. In particular, oscillation of generalized Pielou, Ricker and Lasota–Wazewska equations is considered.

Keywords Difference equations · Variable delays · Linearized oscillation · Pielou equation · Ricker model · Lasota–Wazewska equation

1 Introduction

Linearized theory usually relates properties of nonlinear equations to their linearized versions. Results connecting oscillation properties of a nonlinear delay difference equation

$$x(n+1) - x(n) + \sum_{k=1}^{m} r_k(n) f_k[x(n-\tau_k)] = 0 \quad \text{for } n \geq n_0, \tag{1}$$

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where $\{r_k(n)\} \subset \mathbb{R}_0^+ := [0, \infty)$, $\tau_k \in \mathbb{N}_0 := \{0, 1, \ldots\}$, $k = 1, 2, \ldots, m$, with the linear equation

$$x(n+1) - x(n) + \sum_{k=1}^{m} r_k(n)x(n-\tau_k) = 0 \quad \text{for } n \geq n_0 \tag{2}$$

go back to 1990ies [14]. Here in (1), $uf_k(u) > 0$ for $u \in \mathbb{R}\backslash\{0\}$ and either $\liminf\limits_{u\to 0} \dfrac{f_k(u)}{u} = 1$ or $|f_k(u)| \geq |u|$ in some neighborhood of the origin for any $k = 1, 2, \ldots, m$.

In [14, Theorem 6], under natural assumptions, Yan and Qian stated that oscillation of (2) implies oscillation of (1). However, in [10, p. 478], Tang and Yu disproved the statement by Yan and Qian by constructing a counterexample in the case of a single constant delay to show that (1) may have a nonoscillatory solution while every solution of (2) is oscillatory. This happens in the critical case when the variable coefficient is close to the boundary of the oscillation domain, see [11] for more details.

It is mentioned in [9] that to explore linearized oscillation of the autonomous equation

$$x(n+1) - x(n) + \sum_{k=1}^{m} p_k f_k[x(n-\tau_k)] = 0 \quad \text{for } n \geq n_0, \tag{3}$$

it is sufficient to imply limitations on nonlinear functions in a small neighbourhood of zero.

Proposition 1 ([9, p. 570]) *Assume that $p_k \in \mathbb{R}^+ := (0, \infty)$, $f_k \in C(\mathbb{R}, \mathbb{R})$ satisfies $uf_k(u) > 0$ for all $u \in (-\delta, \delta)\backslash\{0\}$, where $\delta \in \mathbb{R}^+$, and $\lim\limits_{u\to\infty} \dfrac{f_k(u)}{u} = 1$ for $k = 1, 2, \ldots, m$. Every solution of (3) is oscillatory if and only if every solution of the autonomous equation*

$$x(n+1) - x(n) + \sum_{k=1}^{m} p_k x(n-\tau_k) = 0 \quad \textit{for } n \geq n_0 \tag{4}$$

is oscillatory.

However, a slight modification of [9, p. 574] disproves this statement.

Example 1 Let $\tau \in \mathbb{N}$ and $p \in [1, \infty)$. Consider the nonlinear equation

$$x(n+1) - x(n) + p\sin\big(x(n-\tau)\big) = 0 \quad \text{for } n \geq 0. \tag{5}$$

As any eventually positive solution of the linearized equation

$$x(n+1) - x(n) + px(n-\tau) = 0 \quad \text{for } n \geq 0$$

should be monotone decreasing for $n \geq n_0$, we get $x(n_0+1) < 0$. Similarly, there are no eventually negative solutions. However, (5) has an infinite number of constant nonoscillatory solutions $\{x(n)\} = \{j\pi\}$, $j = \pm 1, \pm 2, \ldots$, both positive and negative.

Reducing conditions to a small neighbourhood of zero as in Proposition 1 is allowed only when all nonoscillatory solutions $\{x(n)\}$ tend to zero as $n \to \infty$.

The purpose of the present paper is to establish connections between oscillation properties of the nonlinear equation with variable coefficients and delays

$$x(n+1) - x(n) + \sum_{k=1}^{m} r_k(n) f_k\big[x\big(h_k(n)\big)\big] = 0 \quad \text{for } n \geq n_0 \tag{6}$$

and the linear equation

$$x(n+1) - x(n) + \sum_{k=1}^{m} r_k(n) x\big(h_k(n)\big) = 0 \quad \text{for } n \geq n_0, \tag{7}$$

as well as to apply the obtained results to some models of population dynamics models.

We consider (6) under some the following assumptions:

(A1) For $k = 1, 2, \ldots, m$, $\{r_k(n)\} \subset \mathbb{R}_0^+$.
(A2) For $k = 1, 2, \ldots, m$, $\{h_k(n)\} \subset \mathbb{Z}$, $h_k(n) \leq n$, $n \geq n_0$ and $\lim\limits_{n\to\infty} h_k(n) = \infty$.
(A3) For $k = 1, 2, \ldots, m$, $f_k \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ satisfies $u f_k(u) > 0$ for all $u \in \mathbb{R}\backslash\{0\}$.
(A4) $\lim\limits_{u\to 0} \dfrac{f_k(u)}{u} = 1$ for $k = 1, 2, \ldots, m$.
(A5) There exists $\delta \in \mathbb{R}^+$ such that either

$$0 \leq f_k(u) \leq u \quad \text{for all } u \in [0, \delta] \text{ and } k = 1, 2, \ldots, m, \tag{8}$$

or

$$0 \geq f_k(u) \geq u \quad \text{for all } u \in [-\delta, 0] \text{ and } k = 1, 2, \ldots, m. \tag{9}$$

For all results concerning (6), we assume that (A1) and (A2) hold. Define

$$n_{-1} := \min_k \min\{h_k(n) : \ n \geq n_0\},$$

which exists and is finite by (A2). By a solution of (6), we mean a sequence $\{x(n)\}_{n=n_{-1}}^{\infty}$ for which

$$x(n+1) = x(n) - \sum_{k=1}^{m} r_k(n) f_k\big[x\big(h_k(n)\big)\big] \quad \text{for } n = n_0, n_0+1, \ldots \ .$$

It is well known that (6) has a unique solution satisfying the initial condition

$$x(n) = \varphi_{n-n_{-1}} \quad \text{for } n = n_{-1}, n_{-1}+1, \dots, n_0,$$

where $\varphi_0, \varphi_1, \dots, \varphi_{n_0-n_{-1}}$ are prescribed real numbers.

A solution $\{x(n)\}$ of (6) is said to be *oscillatory* if $x(n)$ are neither eventually positive nor eventually negative. Equation (6) is *oscillatory* if all its solutions are oscillatory. Otherwise, (6) is called *nonoscillatory*.

After linearization, we have to apply oscillation results for linear equation (7), so we present below some of them.

Proposition 2 ([7, Corollary 7.1.1]) *Assume that* $p_k \in \mathbb{R}$ *and* $\tau_k \in \mathbb{Z}$ *for* $k = 1, 2, \dots, m$. *Linear autonomous equation* (4) *is oscillatory if and only if the characteristic equation*

$$\lambda - 1 + \sum_{k=0}^{m} p_k \lambda^{-\tau_k} = 0 \tag{10}$$

has no positive roots.

Proposition 3 ([12, Theorem 1]) *Let (A1) and (A2) hold and*

$$\liminf_{n\to\infty} \inf_{\lambda\in(0,1)} \left\{ \sum_{k=1}^{m} \frac{r_k(n)}{\lambda(1-\lambda)^{n-h_k(n)}} \right\} > 1.$$

Then, (7) *is oscillatory.*

Proposition 4 ([15, Corollary 3]) *Assume that (A1) and (A2) are satisfied and there exist* $\lambda_0 \in (0, 1)$ *and* $n_1 \geq n_0$ *such that*

$$\sum_{k=1}^{m} \frac{r_k(n)}{\lambda_0(1-\lambda_0)^{n-h_k(n)}} \leq 1 \quad \textit{for} \ \ n \geq n_1.$$

Then, (7) *is nonoscillatory.*

Denoting

$$\{\mu(n)\} = \left\{ \frac{u(n)}{1-u(n)} \right\}$$

in [3, Theorem 2.1], we get the following.

Proposition 5 ([15, Corollary 3]) *If (A1) and (A2) hold, the following statement are equivalent.*

(i) Equation (7) *is nonoscillatory.*
(ii) There exists $n_0 \in \mathbb{Z}$ *such that the inequality*

$$x(n+1) - x(n) + \sum_{k=1}^{m} r_k(n) x\big(h_k(n)\big) \leq 0$$

has an eventually positive solution for $n \geq n_0$ *and/or*

$$x(n+1) - x(n) + \sum_{k=1}^{m} r_k(n) x\big(h_k(n)\big) \geq 0$$

has an eventually negative solution for $n \geq n_0$.

(iii) *There exist* $n_0 \in \mathbb{Z}$ *and a sequence* $\{\mu(n)\} \subset \mathbb{R}_0^+$ *such that*

$$\mu(n) \geq \sum_{k=1}^{m} r_k(n) \prod_{j=h_k(n)}^{n} [1 + \mu(j)] \quad \text{for } n \geq n_1, \tag{11}$$

where $n_1 \geq n_0$ *satisfies* $h_k(n) \geq n_0$ *for all* $n \geq n_1$ *and* $k = 1, 2, \ldots, m$.

Remark 1 Proposition 4 follows from Proposition 5 with $\{\mu(n)\} \equiv \left\{\dfrac{\lambda_0}{1-\lambda_0}\right\}$.

Next, we quote an oscillation result for nonlinear equation (6).

Proposition 6 ([7, Theorem 7.4.1]) *Assume that (A4) and (A5) hold,* $p_k \in \mathbb{R}^+$ *and* $\tau_k \in \mathbb{N}_0$ *for* $k = 1, 2, \ldots, m$. *The autonomous nonlinear equation*

$$x(n+1) - x(n) + \sum_{k=1}^{m} p_k f_k[x(n-\tau_k)] = 0 \quad \text{for } n \geq n_0 \tag{12}$$

is oscillatory if and only if the autonomous linear equation

$$x(n+1) - x(n) + \sum_{k=1}^{m} p_k x(n-\tau_k) = 0 \quad \text{for } n \geq n_0$$

is oscillatory.

Proposition 7 *Assume (A4) and (A5). The autonomous nonlinear equation* (12) *is oscillatory if and only if the characteristic equation* (10) *has no positive roots.*

By analyzing the characteristic equation

$$\lambda - (1-p) + q\lambda^{-\tau} = 0,$$

we get the following result.

Proposition 8 *Assume* $f, g \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ *satisfy (A4) and (A5),* $p \in [0, 1)$, $q \in \mathbb{R}_0^+$ *and* $\tau \in \mathbb{N}_0$. *The autonomous nonlinear equation*

$$x(n+1) - x(n) + pf[x(n)] + qg[x(n-\tau)] = 0 \quad \text{for } n \geq n_0$$

is oscillatory if and only if

$$q(\tau+1)^{\tau+1} > (1-p)^{\tau+1}\tau^{\tau}.$$

Further, Sect. 2 contains known and some new auxiliary statements that will be required to prove main results in Sect. 3. In Sect. 4, we apply linearization theorems to equations of mathematical biology generalizing Pielou's equation, Ricker's model and Lasota–Wazewska equation. Some final comments are presented in Sect. 5.

2 Auxiliary Results

We will assume without further mentioning that (A1)–(A3) hold. Let us start with the statement that under (A1)–(A3), unlike Example 1, there is a nonoscillatory solution which does not tend to zero if and only if the series of the sum of coefficients converges.

Theorem 1 *The following statements are equivalent.*

(i) $\displaystyle\sum_{j}^{\infty}\sum_{k=1}^{m} r_k(j) < \infty.$

(ii) Equation (6) *has a nonoscillatory solution* $\{x(n)\}$ *such that* $\lim_{n\to\infty} x(n) \neq 0.$

Proof (i) $\Longrightarrow$ (ii): Pick $L > 0$ and denote $M := \max_k \max_{L\le x\le 2L}\{f_k(x)\}$. Since the series $\sum_{j}^{\infty}\sum_{k=1}^{m} r_k(j)$ converges, we can find $n_1 \ge n_0$ such that

$$\sum_{j=n}^{\infty}\sum_{k=1}^{m} r_k(j) \le \frac{L}{M} \quad \text{for } n \ge n_1.$$

By (A2), there is $n_2 \ge n_1$ such that $h_k(n) \ge n_1$ for $n \ge n_2$. Define $x_0(n) :\equiv 1$ for $n \ge n_1$ and $\{x_\ell(n)\}$ for $\ell \in \mathbb{N}$ as

$$x_\ell(n) := \begin{cases} L + \displaystyle\sum_{j=n}^{\infty}\sum_{k=1}^{m} r_k(j) f_k\big[x_{\ell-1}\big(h_k(j)\big)\big], & n \ge n_2, \\ 2L, & n_1 \le n \le n_2. \end{cases}$$

Inductive arguments yield that $2L \ge x_\ell(n) \ge x_{\ell+1}(n) \ge L$ for $n \ge n_1$ and $\ell \in \mathbb{N}$. Define $\{x(n)\}$ by $x(n) := \lim_{\ell\to\infty} x_\ell(n)$ for $n \ge n_1$.

Then, we see that $\{x(n)\}$ is a positive solution of (6) satisfying $\lim_{n\to\infty} x(n) = L > 0$.

(ii) $\Longrightarrow$ (i): We may suppose without loss of generality that $\{x(n)\}$ is an eventually positive solution of (6) such that $\lim_{n\to\infty} x(n) \neq 0$. By (A1)—(A3) and (6), $\{x(n)\}$ is eventually nonincreasing. Then, $\lim_{n\to\infty} x(n) =: L > 0$. We can find $n_1 \geq n_0$ such that $x(n) > 0$ and $\frac{L}{2} \leq x(h_k(n)) \leq \frac{3L}{2}$ for all $n \geq n_1$. Set $m := \min_k \min_{|L-x|\leq\frac{L}{2}} \{f_k(x)\}$, then $m > 0$ by (A3). Summing (6) from n_1 to $(n-1)$, we get

$$0 = x(n) - x(n_1) + \sum_{j=n_1}^{n-1} \sum_{k=1}^{m} r_k(j) f_k\big[x\big(h_k(n)\big)\big] \quad \text{for all } n \geq n_1,$$

which yields

$$\sum_{j=n_1}^{n-1} \sum_{k=1}^{m} r_k(j) \leq \frac{x(n_1)}{m} \quad \text{for all } n \geq n_1.$$

This proves (i) provided that $\{x(n)\}$ is eventually positive. The case of $\{x(n)\}$ being eventually negative is similar and thus is omitted.

Let us illustrate that in Theorem 1 the limit assumption on variable delays in (A2) is necessary, as well as continuity and the sign condition on f in (A3), with two examples.

Example 2 The equation

$$x(n+1) - x(n) + x(-1) = 0 \quad \text{for } n \geq 0$$

with the initial conditions $x(-1) = -1$ and $x(0) = 0$ has an eventually positive solution $\{x(n)\} = \{n\}$, which does not tend to zero, because the delay obviously does not satisfy $\lim_{n\to\infty} h(n) = \infty$ of (A2).

Example 3 For the equation

$$x(n+1) - x(n) + 2f[x(n-1)] = 0 \quad \text{for } n \geq 0, \tag{13}$$

where f is either

$$f_1(u) := \begin{cases} u, & u \leq 1 \\ \frac{1}{8}(u-1), & u > 1 \end{cases} \quad \text{or} \quad f_2(u) := \begin{cases} \frac{1}{2} - \left|\frac{1}{2} - u\right|, & u \leq 1 \\ \frac{1}{8}(u-1), & u \geq 1, \end{cases}$$

all the conditions but one (continuity for f_1 or sign condition for f_2) in (A3) are satisfied. However, all solutions of its linearized counterpart are obviously oscillatory, while $\{x(n)\} = \{1 + 2^{-n}\}$ is a positive solution of (13) with the initial conditions $x(-1) = 3$ and $x(0) = 2$ since

$$x(n+1) = 1 + 2^{-(n+1)} = 1 + 2^{-n} - \frac{1}{2}2^{-n} = x(n) - 2f_1[x(n-1)] \quad \text{for } n = 0, 1, \ldots .$$

With $f = f_2$, it is easy to verify that $\{x(n)\} = \{1\}$ is also a solution of (13) with the initial conditions $x(-1) = 1$ and $x(0) = 1$.

Now, consider the condition

$$\sum_{j}^{\infty} \sum_{k=1}^{m} r_k(j) = \infty. \tag{14}$$

Corollary 1 *Every nonoscillatory solution* $\{x(n)\}$ *of* (6) *tends to zero as* $n \to \infty$ *if and only if* (14) *holds.*

Remark 2 If (6) is oscillatory, then (14) holds.

Lemma 1 *Assume that every nonoscillatory solution* $\{x(n)\}$ *of* (7) *satisfies* $\lim_{n\to\infty} x(n) = 0$. *Then, there exists a solution* $\{\mu(n)\}$ *of inequality* (11) *satisfying*

$$\lim_{n\to\infty} \left(\prod_{j}^{n-1} [1 + \mu(j)] \right)^{-1} = 0.$$

Proof Without loss of generality, let $\{x(n)\}$ be an eventually positive solution of (7) such that $x(n) \to 0$ as $n \to \infty$. Then, we can find $n_1 \geq n_0$ such that $x(n) > 0$ and $x(h_k(n)) > 0$ for all $n \geq n_1$ and $k = 1, 2, \ldots, m$. From (7), $x(n+1) \leq x(n)$ for all $n \geq n_1$. Now, define $\mu(n) := \dfrac{x(n)}{x(n+1)} - 1 \geq 0$ for $n \geq n_1$, then

$$x(n) = x(n_1) \left(\prod_{j=n_1}^{n-1} [1 + \mu(j)] \right)^{-1} \quad \text{for } n \geq n_1. \tag{15}$$

Substituting (15) into (7), we get

$$\mu(n) = \sum_{k=1}^{m} r_k(n) \prod_{j=h_k(n)}^{n} [1 + \mu(j)] \quad \text{for } n \geq n_2,$$

where $n_2 \geq n_1$ is such that $h_k(n) \geq n_1$ for all $n \geq n_1$ and $k = 1, 2, \ldots, m$, i.e., $\{\mu(n)\}$ satisfies (11) with equality. Further, we have

$$\lim_{n\to\infty} \left(\prod_{j=n_1}^{n-1} [1 + \mu(j)] \right)^{-1} = \lim_{n\to\infty} \frac{x(n)}{x(n_1)} = 0.$$

This completes the proof.

3 Main Results

Our first main result states that oscillation of the perturbed linear equation implies oscillation of the nonlinear equation.

Theorem 2 *Assume that (A4) holds, and there exists* $\theta \in (0, 1)$ *such that the linear equation*

$$x(n+1) - x(n) + \theta \sum_{k=1}^{m} r_k(n) x\big(h_k(n)\big) = 0 \quad \text{for } n \geq n_0 \tag{16}$$

is oscillatory. Then, (6) *is oscillatory.*

Proof Assume the contrary that $\{x(n)\}$ is a nonoscillatory solution of (6). First, suppose that $\{x(n)\}$ is eventually positive. Then, we can find $n_1 \geq n_0$ such that $x(n) > 0$ and $x(h_k(n)) > 0$ for all $n \geq n_1$ and $k = 1, 2, \ldots, m$. From (6), $x(n+1) \leq x(n)$ for all $n \geq n_1$. By Corollary 1, we see that $\lim_{n\to\infty} x(n) = 0$. Thus, we can find $n_2 \geq n_1$ such that

$$f_k\big[x\big(h_k(n)\big)\big] \geq \theta x\big(h_k(n)\big) \quad \text{for all } n \geq n_2 \text{ and } k = 1, 2, \ldots, m.$$

From (6), we obtain the inequality

$$x(n+1) - x(n) + \theta \sum_{k=1}^{m} r_k(n) x\big(h_k(n)\big) \leq 0 \quad \text{for all } n \geq n_2.$$

By Proposition 5, Eq. (16) also has a nonoscillatory solution. The case where $\{x(n)\}$ is eventually negative is similar, which concludes the proof.

Next, we show that oscillation of a nonlinear equation implies oscillation of its linearized counterpart.

Theorem 3 *Assume that (A5) holds and* (6) *is oscillatory. Then,* (7) *is also oscillatory.*

Proof Assume the contrary, let $\{x(n)\}$ be a nonoscillatory solution of (7). By Corollary 1 and Remark 2, we have $\lim_{n\to\infty} x(n) = 0$. By Proposition 5, there exists a positive sequence $\{\mu_0(n)\}$ such that

$$\mu_0(n) \geq \sum_{k=1}^{m} r_k(n) \prod_{j=h_k(n)}^{n} [1 + \mu_0(j)] \quad \text{for all } n \geq n_1,$$

where $n_1 \geq n_0$. Thus, by Lemma 1, we have

$$\lim_{n\to\infty} \left(\prod_{j=n_1}^{n-1} [1 + \mu(j)] \right)^{-1} = 0.$$

First, suppose that there exists $\delta \in \mathbb{R}^+$ satisfying (A5) with (8). We can find $n_2 \geq n_1$ such that $\left(\prod_{j=n_1}^{n-1}[1+\mu(j)]\right)^{-1} \leq \delta$ for all $n \geq n_2$. By (A2), we can find $n_3 \geq n_2$ such that $h_k(n) \geq n_2$ for all $n \geq n_3$ and $k = 1, 2, \ldots, m$. Note that $\left(\prod_{j=n_1}^{h_k(n)-1}[1+\mu(j)]\right)^{-1}$ $\leq \delta$ for all $n \geq n_3$ and $k = 1, 2, \ldots, m$. Define $\{\mu_\ell(n)\}$ by

$$\mu_\ell(n) = \sum_{k=1}^{m} r_k(n) f_k \left[\left(\prod_{j=n_1}^{h_k(n)-1}[1+\mu_{\ell-1}(j)]\right)^{-1}\right] \prod_{j=n_1}^{n}[1+\mu_{\ell-1}(j)]$$

for any $n \geq n_3$ and $\ell \in \mathbb{N}$. Clearly, $\mu_0(n) \geq \mu_1(n) \geq \cdots \geq \mu_\ell(n) \geq \mu_{\ell+1}(n) > 0$ for $n \geq n_3$ and $\ell \in \mathbb{N}$. Let $\mu(n) := \lim_{\ell\to\infty} \mu_\ell(n)$ for $n \geq n_3$, and define

$$y(n) := \left(\prod_{j=n_3}^{n-1}[1+\mu(j)]\right)^{-1} \quad \text{for } n \geq n_3.$$

Then $\{y(n)\}$ is an eventually positive solution of (6). If (A5) holds with (9), we can proceed similarly and show that (6) has an eventually negative solution, which completes the proof.

4 Applications

Discrete population models are usually constructed assuming that per capita production rate g is density-dependent $x(n+1) - x(n) = x(n)g[x(n)]$. However, this rate may depend on population size at one of the previous stages $x(n+1) - x(n) = x(n)g[x(h(n))]$. To account for reference population sizes at different moments in the past, either additive

$$x(n+1) - x(n) = x(n)\sum_{k=1}^{m} r_k(n) g_k\big[x\big(h_k(n)\big)\big] \quad \text{for } n \geq n_0$$

or multiplicative

$$x(n+1) - x(n) = x(n)\prod_{k=1}^{m} r_k(n) g_k\big[x\big(h_k(n)\big)\big] \quad \text{for } n \geq n_0$$

extensions can be considered.

4.1 Pielou's Equation with Several Arguments

First, consider the following Pielou's difference equation with variable delays

$$N(n+1) = N(n)\prod_{k=1}^{m}\left[\frac{\alpha_k}{1+\beta_k N\big(h_k(n)\big)}\right]^{p_k(n)} \quad \text{for } n \geq n_0, \tag{17}$$

where $\{h_k(n)\}$ satisfies (A2), $\alpha_k \in (1, \infty)$, $\beta_k \in \mathbb{R}^+$ and $\{p_k(n)\} \subset \mathbb{R}_0^+$ for $k = 1, 2, \ldots, m$ (see [8, p. 22]). One can show that if $N(n) \geq 0$ for $n < n_0$ and $N(n_0) > 0$, Eq. (17) has a unique positive solution.

In the case of a single delay term, (17) includes the so-called logistic equation

$$N(n+1) - N(n) = \gamma N(n+1)\left(1 - \frac{N\big(h(n)\big)}{K}\right) \quad \text{for } n \geq n_0,$$

where $\{h(n)\}$ satisfies (A2), $\gamma \in (0, 1)$ and $K \in \mathbb{R}^+$.

Let us suppose that there exists $K \in \mathbb{R}^+$ such that

$$\alpha_k - 1 = K\beta_k \quad \text{for } k = 1, 2, \ldots, m. \tag{18}$$

If we let

$$x(n) := \ln\left[\frac{N(n)}{K}\right] \quad \text{for } n \geq n_0, \tag{19}$$

then (17) takes the form

$$x(n+1) - x(n) + \sum_{k=1}^{m} p_k(n)\ln\Big[1 + \gamma_k\big(\mathrm{e}^{x(h_k(n))} - 1\big)\Big] = 0 \quad \text{for } n \geq n_0, \tag{20}$$

where $\gamma_k := 1 - \frac{1}{\alpha_k} \in (0, 1)$ for $k = 1, 2, \ldots, m$.

We therefore showed the equivalence between oscillation of all solutions of nonlinear equation (17) about K and oscillation of nonlinear equation (20) about zero.

Note that for $k = 1, 2, \ldots, m$, the function $f_k(u) := \frac{1}{\gamma_k}\ln[1 + \gamma_k(\mathrm{e}^u - 1)]$ for $u \in \mathbb{R}$ satisfies $0 \geq f_k(u) \geq u$ for $x \leq 0$, i.e., for $k = 1, 2, \ldots, m$, f_k fulfills (A5) with (9) and any $\delta \in \mathbb{R}^+$. In view of our discussion in Sect. 3, we associate (17) with the linear equation

$$x(n+1) - x(n) + \sum_{k=1}^{m}\gamma_k p_k(n) x\big(h_k(n)\big) = 0 \quad \text{for } n \geq n_0. \tag{21}$$

Thus, we obtain some explicit oscillation and nonoscillation tests for (17).

Proposition 9 *Assume that* $\{h_k(n)\}$ *satisfies (A2),* $\alpha_k \in (1, \infty)$*,* $\beta_k \in \mathbb{R}^+$*,* $\{p_k(n)\} \subset \mathbb{R}_0^+$ *for* $k = 1, 2, \ldots, m$*. Assume further that there exists* $K \in \mathbb{R}^+$ *such that* (18) *holds.*

(i) If there exists $\theta \in (0, 1)$ *such that the linear equation*

$$x(n+1) - x(n) + \theta \sum_{k=1}^{m} \gamma_k p_k(n) x\big(h_k(n)\big) = 0 \quad \text{for } n \geq n_0 \tag{22}$$

is oscillatory then (17) *is oscillatory about* K*.*

(ii) If (21) *is nonoscillatory then* (17) *is nonoscillatory about* K*.*

Corollary 2 *Assume that* $\{h_k(n)\}$ *satisfies (A2),* $\alpha_k \in (1, \infty)$*,* $\beta_k \in \mathbb{R}^+$*,* $\{p_k(n)\} \subset \mathbb{R}_0^+$ *for* $k = 1, 2, \ldots, m$*. Assume further that there exists* $K \in \mathbb{R}^+$ *such that* (18) *holds.*

(i) If

$$\liminf_{n\to\infty} \inf_{\lambda\in(0,1)} \left\{ \sum_{k=1}^{m} \frac{\gamma_k p_k(n)}{\lambda(1-\lambda)^{n-h_k(n)}} \right\} > 1 \tag{23}$$

then (17) *is oscillatory about* K*.*

(ii) If there exist $\lambda_0 \in (0, 1)$ *and* $n_1 \geq n_0$ *such that*

$$\sum_{k=1}^{m} \frac{\gamma_k p_k(n)}{\lambda_0(1-\lambda_0)^{n-h_k(n)}} \leq 1 \quad \text{for } n \geq n_1 \tag{24}$$

then (17) *is nonoscillatory about* K*.*

Proof (i) From (23), there exists $\theta \in (0, 1)$ such that

$$\liminf_{n\to\infty} \inf_{\lambda\in(0,1)} \left\{ \theta \sum_{k=1}^{m} \frac{\gamma_k p_k(n)}{\lambda(1-\lambda)^{n-h_k(n)}} \right\} > 1. \tag{25}$$

Due to Proposition 3, (25) implies that (22) is oscillatory. An application of Proposition 9 completes the proof.

(ii) The proof follows from Propositions 4 and 9.

The following result for autonomous equations follows from Proposition 7.

Proposition 10 *Assume that* $\tau_k \in \mathbb{N}_0$*,* $\alpha_k \in (1, \infty)$*,* $\beta_k, p_k \in \mathbb{R}^+$*, and there exists* $K \in \mathbb{R}^+$ *such that* (18) *holds. The equation*

$$N(n+1) = N(n) \prod_{k=1}^{m} \left[\frac{\alpha_k}{1 + \beta_k N(n-\tau_k)} \right]^{p_k} \quad \text{for } n \geq n_0$$

is oscillatory about K *if and only if the characteristic equation*

$$\lambda - 1 + \sum_{k=1}^{m} \gamma_k p_k \lambda^{-\tau_k} = 0, \tag{26}$$

where $\gamma_k := 1 - \frac{1}{\alpha_k}$ *for* $k = 1, 2, \ldots, m$*, has no positive roots.*

4.2 Generalized Ricker Model with Variable Arguments

Next, consider Ricker's stock and recruitment model with variable delays

$$N(n+1) = N(n) \exp\left\{ \sum_{k=1}^{m} p_k(n) \left(1 - \left[\frac{N\big(h_k(n)\big)}{K} \right]^{\gamma_k} \right) \right\} \quad \text{for } n \geq n_0, \tag{27}$$

where all $\{h_k(n)\}$ satisfy (A2), $\{p_k(n)\} \subset \mathbb{R}_0^+$, $\gamma_k \in \mathbb{R}^+$ and $K \in \mathbb{R}^+$ (see [1, p. 91]). Substitution (19) transforms (27) into

$$x(n+1) - x(n) + \sum_{k=1}^{m} p_k(n) \big[\mathrm{e}^{\gamma_k x(h_k(n))} - 1 \big] = 0 \quad \text{for } n \geq n_0. \tag{28}$$

This implies the equivalence of oscillation of nonlinear equation (27) about K to oscillation of (28) about zero.

Note that for $k = 1, 2, \ldots, m$, the function $f_k(u) := \dfrac{1}{\gamma_k}(\mathrm{e}^{\gamma_k u} - 1)$ for $u \in \mathbb{R}$ satisfies $0 \geq f_k(u) \geq u$ for $u \leq 0$, i.e., for $k = 1, 2, \ldots, m$, f_k fulfills (A5) with (9) and any $\delta \in \mathbb{R}^+$. We associate linear equation (21) with (27), see Sect. 3. Since (27) is associated with the same equation as Pielou's equation (17), we can give the following results without a proof.

Proposition 11 *Assume that* $K \in \mathbb{R}^+$*,* $\{h_k(n)\}$ *satisfies (A2),* $\{p_k(n)\} \subset \mathbb{R}_0^+$ *and* $\gamma_k \in \mathbb{R}^+$ *for* $k = 1, 2, \ldots, m$*.*

(i) If there exists $\theta \in (0, 1)$ *such that* (22) *is oscillatory then* (27) *is oscillatory about* K*.*
(ii) If (21) *is nonoscillatory then* (27) *is nonoscillatory about* K*.*

Corollary 3 *Assume that* $K \in \mathbb{R}^+$*,* $\{h_k(n)\}$ *satisfies (A2),* $\{p_k(n)\} \subset \mathbb{R}_0^+$ *and* $\gamma_k \in \mathbb{R}^+$ *for* $k = 1, 2, \ldots, m$*.*

(i) If (23) *holds then* (27) *is oscillatory about* K*.*
(ii) If there exists $\lambda_0 \in (0, 1)$ *such that* (24) *holds then* (27) *is nonoscillatory about* K*.*

Proposition 12 *Assume that* $K \in \mathbb{R}^+$, $\tau_k \in \mathbb{N}_0$ *and* $p_k, \gamma_k \in \mathbb{R}^+$ *for* $k = 1, 2, \ldots, m$. *The equation*

$$N(n+1) = N(n)\exp\left\{\sum_{k=1}^{m} p_k\left(1 - \left[\frac{N(n-\tau_k)}{K}\right]^{\gamma_k}\right)\right\} \quad \text{for } n \geq n_0$$

is oscillatory about K *if and only if the characteristic equation* (26) *has no positive roots.*

4.3 Lasota–Wazewska Equation

Finally, consider the discrete retarded Lasota–Wazewska equation for the survival of red-blood cells (see [13])

$$N(n+1) - N(n) = -p(n)N(n) + q(n)\mathrm{e}^{-\gamma N(h(n))} \quad \text{for } n \geq n_0, \tag{29}$$

where $\{h(n)\}$ satisfies (A2), $\{p(n)\} \subset [0, 1)$ describes probability of cell death at each step, $\{q(n)\} \subset \mathbb{R}_0^+$ and $\gamma \in \mathbb{R}^+$ are production parameters such that $p(n) = Kq(n)$ for some $K \in \mathbb{R}^+$ and $n = n_0, n_0 + 1, \ldots$. We will suppose that $\{p(n)\}$ or $\{q(n)\}$ does not vanish eventually. Then, there exists a unique number $N^* \in \mathbb{R}^+$ such that

$$KN^* = \mathrm{e}^{-\gamma N^*},$$

which is called the equilibrium of (29). By applying the change of variables

$$x(n) := \gamma\big[N(n) - N^*\big] \quad \text{for } n \geq n_0,$$

we transform (29) into another nonlinear equation

$$x(n+1) - x(n) + p(n)x(n) + \gamma N^* p(n)\Big[1 - \mathrm{e}^{-x(h(n))}\Big] = 0 \quad \text{for } n \geq n_0. \tag{30}$$

Denote $r_1(n) := p(n)$, $f_1(u) := u$, $h_1(n) := n$, $r_2(n) := \gamma N^* p(n)$, $f_2(u) := 1 - \mathrm{e}^{-u}$, $h_2(n) := h(n)$. Obviously f_1 and f_2 satisfy (A4) and (A5).

Therefore oscillation of nonlinear equation (29) about N^* is equivalent to oscillation of (30) about zero.

Proposition 13 *Let* $\{h(n)\} \subset \mathbb{Z}$, $h(n) \leq n$ *for all* $n \geq n_0$, $\lim_{n\to\infty} h(n) = \infty$, $\{p(n)\} \in [0, 1)$, $\{q(n)\} \subset \mathbb{R}_0^+$, $\gamma \in \mathbb{R}^+$ *and there exist* $K \in \mathbb{R}^+$ *such that* $p(n) = Kq(n)$ *for* $n = n_0, n_0 + 1, \ldots$.

(i) If there exists $\theta \in (0, 1)$ *such that the linear equation*

$$x(n+1) - x(n) + \theta p(n)x(n) + \theta\gamma N^* p(n)x\big(h(n)\big) = 0 \quad \text{for } n \geq n_0 \tag{31}$$

is oscillatory then (29) *is oscillatory about* N^*.

(ii) If the linear equation

$$x(n+1) - x(n) + p(n)x(n) + \gamma N^* p(n)x\big(h(n)\big) = 0 \quad \textit{for } n \geq n_0$$

is nonoscillatory, Eq. (29) *is nonoscillatory about* N^*.

Corollary 4 *Let* $\{h(n)\} \subset \mathbb{Z}$, $h(n) \leq n$ *for all* $n \geq n_0$, $\lim_{n\to\infty} h(n) = \infty$, $\{p(n)\} \in [0, 1)$, $\{q(n)\} \subset \mathbb{R}_0^+$ *and* $\gamma \in \mathbb{R}^+$. *Assume further that there exists* $K \in \mathbb{R}^+$ *such that* $p(n) = Kq(n)$ *for* $n = n_0, n_0 + 1, \ldots$.

(i) If

$$\liminf_{n\to\infty} \inf_{\lambda\in(0,1)} \left\{ \frac{\gamma N^* p(n)}{\lambda(1-\lambda)^{n-h(n)} \prod_{j=h(n)}^{n} \big(1 - p(j)\big)} \right\} > 1, \tag{32}$$

then (29) *is oscillatory about* N^*.

(ii) If there exists $\lambda_0 \in (0, 1)$ *such that*

$$\frac{\gamma N^* p(n)}{\lambda_0(1-\lambda_0)^{n-h(n)} \prod_{j=h(n)}^{n} \big(1 - p(j)\big)} \leq 1 \quad \textit{for all large } n,$$

then (29) *is nonoscillatory about* N^*.

Proof (i) From (32), there exists $\theta \in (0, 1)$ such that

$$\liminf_{n\to\infty} \inf_{\lambda\in(0,1)} \left\{ \frac{\theta\gamma N^* p(n)}{\lambda(1-\lambda)^{n-h(n)} \prod_{j=h(n)}^{n} \big(1 - \theta p(j)\big)} \right\} > 1. \tag{33}$$

Note that (31) transforms into

$$y(n+1) - y(n) + \frac{\theta\gamma N^* p(n)}{\prod_{j=h(n)}^{n} \big(1 - \theta p(j)\big)} y\big(h(n)\big) = 0 \quad \text{for } n \geq n_0 \tag{34}$$

by the sign-preserving substitution

$$y(n) := \frac{x(n)}{\prod_{j=n_0}^{n-1} \big(1 - \theta p(j)\big)} \quad \text{for } n \geq n_0.$$

Due to Proposition 3, (33) yields that (34) (and hence (31)) is oscillatory. Therefore, an application of Proposition 13 completes the proof.

(ii) The proof follows from Propositions 4 and 13.

Theorem 4 ([5, Theorem 1]) *Assume that* $p \in [0, 1)$, $q \in \mathbb{R}_0^+$, $\gamma \in \mathbb{R}^+$ *and* $\tau \in \mathbb{N}_0$. *The equation*

$$N(n+1) - N(n) = -pN(n) + q\mathrm{e}^{-\gamma N(n-\tau)} \quad \textit{for } n \geq n_0 \tag{35}$$

is oscillatory about N^* if and only if

$$p\gamma N^*(\tau+1)^{\tau+1} > (1-p)^{\tau+1}\tau^{\tau}.$$

Proof For (35), linearized Eq. (30) has the form

$$x(n+1) - x(n) + px(n) + p\gamma N^*\Big[1 - \mathrm{e}^{-x(n-\tau)}\Big] = 0 \quad \text{for } n \geq n_0$$

for which Proposition 8 applies, which concludes the proof.

5 Final Comments

In the present paper, we have reviewed some known results and mistakes connected to linearized oscillation of difference equations. Sufficient linearization results are obtained for equations with variable coefficients and delays. They are illustrated with examples and applications to discrete delay models of population dynamics. Let us note that Proposition 12 solves [6, Problems 1–3 of Exercise 7.3]. Theorem 4, obtained here as an illustration of the main linearization method, is the main result of [5].

It is well known that the properties of difference equation with constant and variable delays and variable coefficients are usually essentially different when delays are unbounded. It would be interesting to consider linearization in the case of pantograph-type difference equations. In particular, it is possible to explore models studied in the present paper:

$$N(n+1) = N(n)\prod_{k=1}^{m}\left[\frac{\alpha_k}{1+\beta_k N(\lfloor \frac{n}{\tau_k}\rfloor)}\right]^{\frac{p_k}{n}} \quad \text{for } n \geq n_0,$$

$$N(n+1) = N(n)\exp\left\{\sum_{k=1}^{m}\frac{p_k}{n}\left(1 - \left[\frac{N(\lfloor \frac{n}{\tau_k}\rfloor)}{K}\right]^{\gamma_k}\right)\right\} \quad \text{for } n \geq n_0,$$

$$N(n+1) = N(n)\exp\left\{\sum_{k=1}^{m}\frac{p_k}{n}\left(1 - \left[\frac{N(\lfloor \frac{n}{\tau_k}\rfloor)}{K}\right]^{\gamma_k}\right)\right\} \quad \text{for } n \geq n_0,$$

where $n_0 \in \mathbb{N}$ and $\lfloor\cdot\rfloor$ is the floor function, i.e., $\lfloor u\rfloor$ is the greatest integer not exceeding $u \in \mathbb{R}$. We expect that the result on monotonicity of oscillation properties on time scales [4] can be applied to connect pantograph differential and difference equations.

In addition, careful treatment of the critical case known for differential equations [2] will also be interesting for difference equations. As mentioned in the introduction, it is the area of parameters where the discrepancy between the properties of linearized and original equations is observed.

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