Chapter 3
On the Energy Decay of a Nonhomogeneous Hybrid System of Elasticity

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Abstract In this paper, we study the boundary stabilizing feedback control problem of well-known Scole model that has nonhomogeneous spatial parameters. By using an abstract result of Riesz basis, we show that the closed-loop system is a Riesz spectral system. The asymptotic distribution of eigenvalues, the spectrum-determinded growth condition and the exponential stability are concluded.

Keywords Euler-Bernoulli beam · Boundary control · Stabilization · Riesz basis

3.1 Introduction

The boundary and internal control problem of flexible structure has recently attracted much attention with the rapid development of high technology such as space science and flexible robots. In this paper, we study the boundary feedback stabilization of the nonuniform Scole model. Consisting of an elastic beam, linked to a rigid antenna, this dynamical system is governed by the nonuniform Euler–Bernoulli equation for the vibration of the elastic beam and the Newton–Euler rigid body equation for the oscillation of the antenna. The nonuniform Scole model in the case of a hinged (or "pinned") beam, correspond to the following hybrid system:

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$$\begin{cases} \rho(x)y_{tt}(x,t) + (EI(x)y_{xx}(x,t))_{xx} = 0\,, & 0 < x < 1,\ t > 0, \\ y_x(0,t) = 0,\ (y + (EI(.)y_{xx})_x + ay_t)(0,t) = 0, & t > 0, \\ (my_{tt} - (EI(.)y_{xx})_x)(1,t) = -by_t(1,t) & t > 0, \\ (Jy_{xtt} + EI(.)y_{xx})(1,t) = -cy_{xt}(1,t), & t > 0, \\ y(x,0) = y_0(x)\ \ ,\ y_t(x,0) = y_1(x)\,, & 0 < x < 1, \end{cases} \tag{3.1}$$

where y represents the transversal displacement of the beam, x denotes the position, and t denotes the time. $\rho(x)$ is the mass density of the beam and $EI(x)$ is its flexural rigidity. m is the mass of the antenna and J is its moment of inertia. a, b, and c, are constants feedback gains.

For further description of the physical structure of the system, we refer to Littman–Markus [5]. Furthermore, the coefficients are supposed to be variable because it is common in engineering, to adopt problems with nonhomogeneous materials such as smart materials [4]. Notice that the boundary feedbacks can be realized by means of passive mechanical systems of springs-dampers similar to those used in [1]. The stabilization problem of system (3.1) has been the subject of many studies. When the coefficients ρ, EI are supposed to be constants, Rao in [9] establish the uniform energy decay by using energy multiplier method [6]. It seems to be difficult to extend this method to the nonuniform case. In this paper, we extend the results obtained in [9] to variable coefficients. By using the Riesz basis approach, we show that the generalized eigenfunctions of the system form a Riesz basis for the state Hilbert space. As a consequence, the asymptotic expressions of eigenvalues together with exponential stability are obtained.

The rest of this paper is organized as follows. In Sect. 3.2, the well-posedness and the asymptotic stability of the closed-loop system are established. Section 3.3 is devoted to the asymptotic analysis for the eigenpairs of the closed-loop system. Finally, in Sect. 3.4, we prove the Riesz basis property, the spectrum determined growth condition and the optimal decay rate.

Throughout this paper, we assume that

$$(EI(.),\ \rho(.)) \in [C^4(0,1)]^2,\ \rho,\ EI > 0,\ m,\ J > 0, \tag{3.2}$$

and the constants a, b, and c satisfy the dissipation condition

$$a > 0,\ b \geq 0,\ c > 0. \tag{3.3}$$

3.2 Well-Posedness and Asymptotic Stability

We consider system (3.1) on the following complex Hilbert space:

$$\mathbb{H} = \mathbb{V} \times L^2(0,1) \times \mathbb{C}^2, \tag{3.4}$$

where

$$\mathbb{V} = \{f \in H^2(0,1)/f'(0) = 0\}, \tag{3.5}$$

equipped with the inner product defined as
$\forall (F = (f_1, g_1, \zeta_1, \delta_1), G = (f_2, g_2, \zeta_2, \delta_2)) \in \mathbb{H}^2$

$$\begin{aligned}(F, G)_{\mathbb{H}} = &\int_0^1 (\rho(x) g_1(x)\overline{g_2(x)} + EI(x) f_1''(x)\overline{f_2''(x)})dx + f_1(0)\overline{f_2(0)} \\ &+ \frac{1}{m}\zeta_1\overline{\zeta_2} + \frac{1}{J}\delta_1\overline{\delta_2}.\end{aligned} \tag{3.6}$$

Then, we define an operator as follows: $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{H} \to \mathbb{H}$

$$\begin{cases} D(\mathbb{A}) = \{(f, g, \zeta, \delta) \in (H^4(0,1) \cap \mathbb{V}) \times \mathbb{V} \times \mathbb{C}^2 / f(0) + (EI(.) f'')'(0) + a g(0) = 0, \\ \qquad \zeta = m g(1), \delta = J g'(1)\} \\ \mathbb{A}(f, g, \zeta, \delta) = \left(g, \dfrac{-(EI(.) f''(.))''}{\rho(.)}, (EI(.) f'')'(1) - b g(1), -(EI(1) f''(1) + c g'(1))\right), \end{cases} \tag{3.7}$$

with the initial condition $Y_0 = \left(y_0, y_1, m y_1(1), J y_1'(1)\right)$, the system (3.1) can be written as an evolutionary equation in $\mathbb{H}$:

$$\begin{cases} \dfrac{dY(t)}{dt} = \mathbb{A} Y(t), \\ Y(t) = (y(., t), y_t(., t), m y_t(1, t), J y_{xt}(1, t)), \; Y(0) = Y_0. \end{cases} \tag{3.8}$$

We have the following Lemma

Lemma 3.1 *Let the operator $\mathbb{A}$ defined by (3.7). Then $\mathbb{A}$ is a densely defined, closed dissipative operator in $\mathbb{H}$, and $\mathbb{A}^{-1}$ exists and is compact on $\mathbb{H}$. Moreover, $\mathbb{A}$ generates a C_0 semigroup of contractions $e^{\mathbb{A}t}$ on $\mathbb{H}$ and the spectrum $\sigma(\mathbb{A})$ of $\mathbb{A}$ consists only of the isolated eigenvalues.*

Proof Let $(f, g, \zeta, \delta) \in D(\mathbb{A})$, then we have

$$Re(\mathbb{A}Y, Y)_{\mathbb{H}} = -a\,|g(0)|^2 - b\,|g(1)|^2 - c\,\left|g'(1)\right|^2. \tag{3.9}$$

Thus $\mathbb{A}$ is dissipative in $\mathbb{H}$. Next, we show that $\mathbb{A}^{-1}$ exists. Let $(u, v, \omega, \xi) \in \mathbb{H}$, we will find $(f, g, \zeta, \delta) \in D(\mathbb{A})$ such that

$$\mathbb{A}(f, g, \zeta, \delta) = (u, v, \omega, \xi) \in \mathbb{H},$$

which yields

$$\begin{cases} g = u, \zeta = mg(1) = mu(1), \delta = Jg'(1) = Ju'(1), \\ (EI(.)f'')''(x) = -\rho(x)v(x), \\ f'(0) = 0, f(0) + (EI(.)f'')'(0) + au(0) = 0, \\ (EI(.)f'')'(1) - bu(1) = \omega, \\ -(EI(1)f''(1) + cu'(1)) = \xi. \end{cases}$$

After a simple calculation, we show that

$$f(x) = f(0) - \int_0^x \int_0^y drdy \left[\frac{\beta(1-r)+\alpha}{EI(r)} + \frac{1}{EI(r)} \int_r^1 \int_s^1 \rho(x)v(x)dtds \right],$$

where

$$\begin{cases} f(0) = -(\beta + au(0) + \int_0^1 \int_r^1 \rho(x)v(x)dsdr), \\ \alpha = \xi + cu'(1), \ \beta = \omega + bu(1). \end{cases}$$

Thus, $\mathbb{A}^{-1}$ exists and is bounded in $\mathbb{H}$. Furthermore, the Sobolev embedding theorem, implies that $\mathbb{A}^{-1}$ is compact on $\mathbb{H}$ and the Lumer–Phillips theorem [8] can be applied to conclude that $\mathbb{A}$ generates a C_0 semigroup of contractions $e^{\mathbb{A}t}$ in $\mathbb{H}$. The Lemma is proved. □

Now, we turn our attention to the asymptotic stability of the system.

Lemma 3.2 *Let* $\mathbb{A}$ *be the operator defined by (3.7). Then* $\Re e(\mathbb{A}) < 0$ *and hence the system (3.1) is asymptotically stable.*

Proof It suffices to show that $\{i\gamma, \gamma \in \mathbb{R}\} \subset \rho(\mathbb{A})$. Assume that this is false. This together with Lemma 3.1 implies that there exists nonzero $\gamma \in \mathbb{R}$ such that $i\gamma \in \sigma(\mathbb{A})$, where $\sigma(\mathbb{A})$ is the point spectrum, i.e., there exists $\phi = (f, g, \zeta, \delta) \in D(\mathbb{A})$ satisfying without loss of generality, the conditions $\|\phi\|_{\mathbb{H}} = 1$ and $(i\gamma - \mathbb{A})\phi = 0$ i.e.,

$$\begin{cases} (EI(.)f'')''(x) - \gamma^2\rho(x)f(x) = 0, \\ f'(0) = 0, \ -(EI(.)f'')'(0) = (1 + i\gamma a)f(0), \\ (EI(.)f'')'(1) = (-\gamma^2 m + i\gamma b)f(1), \\ -EI(1)f''(1) = (-\gamma^2 J + i\gamma c)f'(1), \\ g = i\gamma f, \zeta = i\gamma m f(1), \delta = i\gamma J f'(1). \end{cases} \quad (3.10)$$

Using (3.9), we obtain $g'(1) = f'(1) = 0$ and $f(0) = 0$, which further implies by means of (3.10) that $f''(1) = 0$ and the system (3.10) yields

$$\begin{cases} (EI(.)f'')''(x) - \gamma^2\rho(x)f(x) = 0, \\ f(0) = f'(0) = (EI(.)f'')'(0) = 0, \\ f'(1) = f''(1) = 0, \\ (EI(.)f'')'(1) = (-\gamma^2 m + i\gamma b)f(1). \end{cases} \tag{3.11}$$

1. If $b > 0$, then from (3.9), $g(1) = f(1) = 0$, by means of (3.11), we have

$$(EI(.)f'')'(1) = 0$$

and the system (3.11) yields

$$\begin{cases} (EI(.)f'')''(x) - \gamma^2\rho(x)f(x) = 0, \\ f(0) = f'(0) = (EI(.)f'')'(0) = 0, \\ f(1) = (EI(.)f'')'(1) = 0, \\ EI(1)f''(1) = 0. \end{cases} \tag{3.12}$$

It has been proved in [3] that the above system has only the trivial solution, i.e., $f = 0$. Then $\phi = 0$, which contradict the first that $\|\phi\|_{\mathbb{H}} = 1$.
2. If $b = 0$. First, assume that

$$f(1) > 0 \text{ (the negative case is similar)},$$

which implies by the last boundary condition in (3.11) that

$$(EIf'')'(1) < 0.$$

Let $[c, 1]$ be a subspace of $[0, 1]$ so that $f(x) > 0$ for each $x \in (c, 1]$, $f(c) = 0$. Then,

$$(EI(.)f'')''(x) > 0, \text{ for any } x \in (c, 1].$$

Hence, $(EI(.)f'')'$ is increasing in $(c, 1]$. Since

$$(EI(.)f'')'(1) < 0,$$

we have

$$(EI(.)f'')'(x) < 0, \text{ for any } x \in (c, 1].$$

It follows that $EI(x)f''(x)$ is decreasing in $(c, 1]$. Since

$$EI(1)f''(1) = 0,$$

we have

$$f''(x) > 0, \text{ for any } x \in (c, 1).$$

So, $f'(x)$ is increasing in $(c, 1)$. Since $f'(1) = 0$, we have

$$f'(x) < 0, \text{ for any } x \in (c, 1).$$

Hence, $f(x)$ is decreasing in $(c, 1)$, and so,

$$f(c) > f(1) > 0,$$

contradicts the assumption that $f(c) = 0$. Therefore, $f(1) = 0$. Now, (3.11) implies that f satisfies system (3.12). We can conclude as in 1. The Lemma 3.2 (in the end of proof of Lemma 3.2) is proved. □

3.3 Asymptotic Expressions of Eigenfrequencies

Note that

$$\mathbb{A}\phi = \lambda\phi, \phi = (f, g, \zeta, \delta), \tag{3.13}$$

yields

$$\begin{cases} (EI(.)f'')''(x) + \lambda^2\rho(x)f(x) = 0,\ 0 < x < 1, \\ f'(0) = 0,\ f(0) + (EI(.)f'')'(0) + ag(0) = 0, \\ (EI(.)f'')'(1) = (\lambda^2 m + \lambda b)f(1), \\ -EI(1)f''(1) = (\lambda^2 J + \lambda c)f'(1), \\ g(x) = \lambda f(x), \zeta = mg(1), \delta = Jg'(1). \end{cases} \tag{3.14}$$

Writing (3.14) in the standard form of a linear differential operator with homogeneous boundary conditions, we obtain

$$\begin{cases} f^{(4)}(x) + \dfrac{2EI'(x)}{EI(x)} f'''(x) + \dfrac{EI''(x)}{EI(x)} f''(x) + \lambda^2 \dfrac{EI(x)}{\rho(x)} f(x) = 0, \;\; 0 < x < 1, \\ f'(0) = 0, \lambda f(0) + a_1 f'''(0) + a_2 f''(0) + a_3 f(0) = 0, \\ \lambda^2 f(1) + a_4 \lambda f(1) - a_5 f'''(1) - a_6 f''(1) = 0, \\ \lambda^2 f'(1) + a_7 \lambda f'(1) + a_8 f''(1) = 0, \end{cases} \tag{3.15}$$

where

$$\begin{cases} a_1 = \dfrac{EI(0)}{a}, \;\; a_2 = \dfrac{EI'(0)}{a}, \;\; a_3 = \dfrac{1}{a}, a_4 = \dfrac{b}{m}, \\ a_5 = \dfrac{EI(1)}{m}, \;\; a_6 = \dfrac{EI'(1)}{m}, \;\; a_7 = \dfrac{c}{m}, \;\; a_8 = \dfrac{EI(1)}{J}. \end{cases} \tag{3.16}$$

In order to simplify the computations, we introduce a spatial scale transformation in x.

$$\Phi(z) = f(x), \;\; z = z(x) = \frac{1}{p} \int_0^x \left(\frac{\rho(s)}{EI(s)} \right)^{1/4} ds, \;\; p = \int_0^1 \left(\frac{\rho(s)}{EI(s)} \right)^{1/4} ds, \tag{3.17}$$

then Φ satisfies the following system:

$$\begin{cases} \Phi^{(4)}(z) + a(z)\Phi'''(z) + b(z)\Phi''(z) + c(z)\Phi'(z) + \lambda^2 p^4 \Phi(z) = 0, \\ \Phi'(0) = 0, \lambda\Phi(0) + b_1\Phi'''(0) + b_2\Phi''(0) + b_3\Phi'(0) + a_3\Phi(0) = 0, \\ \lambda^2\Phi(1) + a_4\lambda\Phi(1) - b_4\Phi'''(1) - b_5\Phi''(1) - b_6\Phi'(1) = 0, \\ \lambda^2\Phi'(1) + a_7\lambda\Phi'(1) + b_7\Phi''(1) + b_8\Phi'(1) = 0, \end{cases} \tag{3.18}$$

where $a(z)$, $b(z)$, and $c(z)$ are the smooth functions defined by

$$\begin{cases} a(z) = \dfrac{6z''}{z'^2} + \dfrac{2EI'(x)}{z'EI(x)}, \\ b(z) = \dfrac{3z''^2}{z'^4} + \dfrac{6z''EI'(x)}{z'^3 EI(x)} + \dfrac{EI''(x)}{z'^2 EI(x)} + \dfrac{4z'''}{z'^3}, \\ c(z) = \dfrac{z''''}{z'^4} + \dfrac{2z'''EI'(x)}{z'^4 EI(x)} + \dfrac{z''EI''(x)}{z'^4 EI(x)}, \end{cases} \tag{3.19}$$

and

$$\begin{cases} b_1 = a_1 z'^3(0),\ b_2 = 3a_1 z'(0)z''(0) + a_2 z'^2(0), \\ b_3 = a_1 z'''(0) + a_2 z''(0),\ b_4 = a_5 z'^3(1), \\ b_5 = 3a_5 z'(1)z''(1) + a_6 z'^2(1),\ b_6 = a_5 z'^3(1) + a_6 z''(1), \\ b_7 = a_8 z'(1),\ b_8 = \dfrac{a_8 z''(1)}{z'(1)}. \end{cases} \tag{3.20}$$

Equation (3.18) can be simplified by applying another invertible transformation

$$\varphi(z) = e^{1/4 \int_0^z a(s)ds}\ \Phi(z), \tag{3.21}$$

which allows one to cancel the term $a(z)\Phi'''(z)$ in (3.18); hence, φ satisfies the following equivalent eigenvalue problem:

$$\begin{cases} \varphi^{(4)}(z) + a_1(z)\varphi''(z) + a_2(z)\varphi'(z) + a_3(z)\varphi(z) + \lambda^2 p^4 \varphi(z) = 0, \\ \varphi'(0) - \dfrac{a(0)}{4}\varphi(0) = 0,\ \lambda\varphi(0) + b_1\varphi'''(0) + F_1(\varphi(0), \varphi'(0), \varphi''(0)) = 0, \\ \lambda^2\varphi(1) + a_4\lambda\varphi(1) - b_4\varphi'''(1) + F_2(\varphi(1), \varphi'(1), \varphi''(1)) = 0, \\ \lambda^2(\varphi'(1) - \dfrac{a(1)}{4}\varphi(1)) + a_7\lambda(\varphi'(1) - \dfrac{a(1)}{4}\varphi(1)) + F_3(\varphi(1), \varphi'(1), \varphi''(1)) = 0, \end{cases} \tag{3.22}$$

where $a_1(z)$, $a_2(z)$ and $a_3(z)$ are the smooth functions defined by

$$\begin{cases} a_1(z) = -\dfrac{3a'(z)}{2} - \dfrac{3a^2(z)}{8} + b(z), \\ a_2(z) = \dfrac{a^3(z)}{8} - a''(z) - \dfrac{a(z)b(z)}{2} + c(z), \\ a_3(z) = \dfrac{3a'^2(z)}{16} - \dfrac{a'''(z)}{4} - \dfrac{3a^4(z)}{256} + \dfrac{3a^2(z)a'(z)}{32} \\ \qquad\qquad + b(z)(\dfrac{a^2(z)}{16} - \dfrac{a'(z)}{4}) - \dfrac{a(z)c(z)}{4}, \end{cases} \tag{3.23}$$

and $F_1(x_1, x_2, x_3)$, $F_2(x_1, x_2, x_3)$, and $F_3(x_1, x_2, x_3)$ are linear combinations of x_1, x_2, and x_3.

To estimate asymptotically the solutions to the eigenvalue problem (3.22), we proceed as in [7]. First due to Lemma 3.2 and the fact that eigenvalues of $\mathbb{A}$ are symmetric with respect to the real axis, we only need to consider those $\lambda \in \sigma(\mathbb{A})$ that satisfy $\dfrac{\pi}{2} \leq \arg\lambda \leq \pi$, which we assume in the sequel. Next, we set $\lambda = \tau^2$ and hence

$$\frac{\pi}{4} \leq \arg \tau \leq \frac{\pi}{2}.$$

Now, let us choose ω_j, $j = 1, 2, 3, 4$ as follows:

$$\omega_1 = \frac{-1+i}{\sqrt{2}},\ \omega_2 = \frac{1+i}{\sqrt{2}},\ \omega_3 = -\omega_2,\ \omega_4 = -\omega_1,$$

consequently, we have for $\tau \in S = \left\{\tau / \frac{\pi}{4} \leq \arg \tau \leq \frac{\pi}{2}\right\}$

$$\begin{cases} Re(\tau\omega_1) = - \mid \tau \mid \sin(\arg \tau + \frac{\pi}{4}) \leq -\frac{\sqrt{2} \mid \tau \mid}{2} < 0, \\ Re(\tau\omega_2) = \mid \tau \mid \cos(\arg \tau + \frac{\pi}{4}) \leq 0. \end{cases} \tag{3.24}$$

In order to analyze the asymptotic distribution of eigenpairs for (3.22), we need the following result [10].

Lemma 3.3 *For $\mid \tau \mid$ large enough and $\tau \in S$, there are four linearly independent asymptotic fundamental solutions φ_j, $j = 1, 2, 3, 4$, to*

$$\varphi^{(4)}(z) + a_1(z)\varphi''(z) + a_2(z)\varphi'(z) + a_3(z)\varphi(z) + \tau^4 p^4 \varphi(z) = 0, \tag{3.25}$$

such that

$$\begin{cases} \varphi_j(z,\tau) = e^{\tau\omega_j z}\left(1 + \frac{\varphi_{j1}(z)}{\tau} + O\left(\tau^{-2}\right)\right), \\ \varphi_j'(z,\tau) = \tau\omega_j e^{\tau\omega_j z}\left(1 + \frac{\varphi_{j1}(z)}{\tau} + O\left(\tau^{-2}\right)\right), \\ \varphi_j''(z,\tau) = (\tau\omega_j)^2 e^{\tau\omega_j z}\left(1 + \frac{\varphi_{j1}(z)}{\tau} + O\left(\tau^{-2}\right)\right), \\ \varphi_j'''(z,\tau) = \left(\tau\omega_j\right)^3 e^{\tau\omega_j z}\left(1 + \frac{\varphi_{j1}(z)}{\tau} + O\left(\tau^{-2}\right)\right), \end{cases} \tag{3.26}$$

where

$$\varphi_{j1}(z) = -\frac{1}{4\omega_j}\int_0^z a_1(s)ds.$$

Hence, for $j = 1, 2, 3, 4$,

$$\varphi_{j1}(0) = 0,\ \varphi_{j1}(1) = -\frac{1}{4\omega_j}\int_0^1 a_1(s)ds = \frac{1}{\omega_j}\mu,\ \mu = -\frac{1}{4}\int_0^1 a_1(s)ds.$$

For convenience, we introduce the notation $[r]_j = r + O\left(\tau^{-j}\right)$ for $j = 1, 2$. From Lemma 3.3, one can write the asymptotic solution of (3.22) as follows:

$$\varphi(z) = d_1\varphi_1(z) + d_2\varphi_2(z) + d_3\varphi_3(z) + d_4\varphi_4(z), \tag{3.27}$$

where $\varphi_j,\ \ j = 1, 2, 3, 4$ are defined by Lemma 3.3 and $d_j,\ \ j = 1, 2, 3, 4$ are chosen so that φ satisfy the boundary conditions of (3.22). Note that $\lambda = \tau^2 \neq 0$, is the eigenvalue of (3.22) if and only if τ satisfies the characteristic determinant

$$\Delta(\tau) = \begin{vmatrix} \tau\omega_1[1 - \frac{a_0}{\tau\omega_1}]_2 & \tau\omega_2[1 - \frac{a_0}{\tau\omega_2}]_2 & \tau\omega_3[1 - \frac{a_0}{\tau\omega_3}]_2 & \tau\omega_4[1 - \frac{a_0}{\tau\omega_4}]_2 \\ \tau^3\omega_1^3[b_1 - \frac{\omega_1}{\tau}]_2 & \tau^3\omega_2^3[b_1 - \frac{\omega_2}{\tau}]_2 & \tau^4\omega_3^3[b_1 - \frac{\omega_3}{\tau}]_2 & \tau^3\omega_4^3[b_1 - \frac{\omega_4}{\tau}]_2 \\ \tau^4[1 + \frac{\mu + b_4}{\tau\omega_1}]_2 e^{\tau\omega_1} & \tau^4[1 + \frac{\mu + b_4}{\tau\omega_2}]_2 e^{\tau\omega_2} & \tau^4[1 + \frac{\mu + b_4}{\tau\omega_3}]_2 e^{\tau\omega_3} & \tau^4[1 + \frac{\mu + b_4}{\tau\omega_4}]_2 e^{\tau\omega_4} \\ \tau^5[\omega_1 + \frac{b_9}{\tau}]_2 e^{\tau\omega_1} & \tau^5[\omega_2 + \frac{b_9}{\tau}]_2 e^{\tau\omega_2} & \tau^5[\omega_3 + \frac{b_9}{\tau}]_2 e^{\tau\omega_3} & \tau^5[\omega_4 + \frac{b_9}{\tau}]_2 e^{\tau\omega_4} \end{vmatrix}, \tag{3.28}$$

where

$$a_0 = \frac{a(0)}{4},\ \ b_9 = \mu - \frac{a(1)}{4}.$$

Noting that from (3.24)

$$|e^{\tau\omega_2}| \leq 1, |e^{\tau\omega_1}| = O(e^{-q|\tau|}) \text{ as } |\tau| \to +\infty,$$

for some constant $q > 0$, then each element of the matrix in (3.28) is bounded, we may rewrite (3.28) as

$$\tau^{-13}e^{\tau(\omega_1+\omega_2)}\Delta(\tau)=\begin{vmatrix}\omega_1(1-\frac{a_0}{\tau\omega_1}) & \omega_2(1-\frac{a_0}{\tau\omega_2}) & -\omega_2(1+\frac{a_0}{\tau\omega_2})e^{\tau\omega_2} & 0\\ \omega_1^3(b_1-\frac{\omega_1}{\tau}) & \omega_2^3(b_1-\frac{\omega_2}{\tau}) & -\omega_2^3(b_1+\frac{\omega_2}{\tau})e^{\tau\omega_2} & 0\\ 0 & (1+\frac{\mu+b_4}{\tau\omega_2})e^{\tau\omega_2} & (1-\frac{\mu+b_4}{\tau\omega_2}) & (1-\frac{\mu+b_4}{\tau\omega_1})\\ 0 & (\omega_2+\frac{b_9}{\tau})e^{\tau\omega_2} & (-\omega_2+\frac{b_9}{\tau}) & (-\omega_1+\frac{b_9}{\tau})\end{vmatrix}+O\left(\tau^{-2}\right). \tag{3.29}$$

A direct calculation gives

$$\begin{aligned}&\tau^{-13}e^{\tau(\omega_1+\omega_2)}\Delta(\tau)\\ &\quad=b_1(\omega_1\omega_2^3-\omega_1^3\omega_2)\Big[\omega_2-\omega_1+(\mu+b_4)\left(\omega_1\omega_2^{-1}-\omega_2\omega_1^{-1}\right)\tau^{-1}\Big]\\ &\qquad+(\omega_2-\omega_1)(\omega_1^3-\omega_2^3)(\omega_1\omega_2+a_0b_1)\tau^{-1}+e^{2\tau\omega_2}\Big\{b_1(\omega_1^3\omega_2-\omega_1\omega_2^3)\\ &\qquad\times\Big[\omega_1+\omega_2+(\mu+b_4)(\omega_2\omega_1^{-1}-\omega_1\omega_2^{-1})\tau^{-1}\Big]+(\omega_1+\omega_2)(\omega_1^3+\omega_2^3)\\ &\qquad\times(b_1a_0-\omega_1\omega_2)\tau^{-1}\Big\}+O\left(\tau^{-2}\right).\end{aligned}$$

A straightforward simplification will arrive at the following result.

Theorem 3.1 *Let $\lambda=\tau^2$ where $\tau\in S$.*

1. *The characteristic determinant $\Delta(\tau)$ of the eigenfunction problem (3.22) has the following asymptotic expression in the sector S*

$$\begin{aligned}\tau^{-13}e^{\tau(\omega_1+\omega_2)}\Delta(\tau)&=2\left[-\sqrt{2}ib_1+(b_{10}-1)\tau^{-1}\right]\\ &\quad+2e^{2\tau\omega_2}\left[-\sqrt{2}b_1+(-b_{10}-1)\tau^{-1}\right]+O\left(\tau^{-2}\right),\end{aligned}\tag{3.30}$$

   where $b_{10}=b_1[2(\mu+b_4)+a_0]$.

2. *Let $\sigma(\mathbb{A})=\{\lambda_n,\overline{\lambda_n},n\in\mathbb{N}\}$, be the eigenvalues of $\mathbb{A}$, then for $k=n-\dfrac{1}{4}$ and $\tau_n\in S$, the following asymptotic expression holds*

$$\tau_n = \frac{1}{\omega_2} k\pi i - \frac{1}{2\sqrt{2}b_1}\left[\frac{(1-i)b_{10}+(1+i)}{k\pi i}\right] + O\left(n^{-2}\right)$$
$$\lambda_n = -\frac{1}{b_1} + (k\pi)^2 i + \frac{b_{10}i}{b_1} + O\left(n^{-1}\right), \tag{3.31}$$

for sufficiently large positive integer n. Moreover, by (3.16) and (3.20), we obtain

$$\lim_{n\to+\infty} Re(\lambda_n) = -\frac{1}{b_1} = -\frac{a}{EI(0)z'^3(0)} < 0. \tag{3.32}$$

3. λ_n is geometrically simple when n is large enough.

Proof Note that $\lambda = \tau^2 \in \sigma(\mathbb{A})$, where $\tau \in S$ if and only if

$$-\sqrt{2}ib_1e^{-\tau\omega_2} - \sqrt{2}b_1e^{\tau\omega_2} + \frac{(b_{10}-1)}{\tau}e^{-\tau\omega_2} - \frac{(b_{10}+1)}{\tau}e^{\tau\omega_2} + O\left(\tau^{-2}\right) = 0, \tag{3.33}$$

which can be written as

$$-\sqrt{2}ib_1e^{-\tau\omega_2} - \sqrt{2}b_1e^{\tau\omega_2} + O\left(\tau^{-1}\right) = 0. \tag{3.34}$$

Obviously, the equation

$$ie^{-\tau\omega_2} + e^{\tau\omega_2} = 0, \tag{3.35}$$

has solutions

$$\widetilde{\tau_n} = \frac{1}{\omega_2}k\pi i,\ n \in \mathbb{N}, \quad k = n - \frac{1}{4}. \tag{3.36}$$

Applying Rouche's theorem to (3.34), we obtain

$$\tau_n = \widetilde{\tau_n} + \alpha_n = \frac{1}{\omega_2}k\pi i + \alpha_n, \alpha_n = O\left(n^{-1}\right), n = N, N+1\dots, \tag{3.37}$$

where N is large positive integer.

Substituting τ_n into (3.33) and using the fact that $e^{\widetilde{\tau_n}\omega_2} = -ie^{-\widetilde{\tau_n}\omega_2}$, we obtain

$$-\sqrt{2}ib_1e^{-\alpha_n\omega_2} + \sqrt{2}ib_1e^{\alpha_n\omega_2} + (b_{10}-1)\widetilde{\tau}_n^{-1}e^{-\alpha_n\omega_2} + i(b_{10}+1)\widetilde{\tau}_n^{-1}e^{\alpha_n\omega_2} + O\left(\widetilde{\tau}_n^{-2}\right) = 0.$$

On the other hand, expanding the exponential function according to its Taylor series, we obtain

$$\alpha_n = -\frac{1}{2\sqrt{2}b_1}\left[\frac{b_{10}(1-i)+(1+i)}{k\pi i}\right] + O\left(n^{-2}\right). \tag{3.38}$$

Substituting this estimate in (3.37), we have,

$$\tau_n = \frac{k\pi i}{\omega_2} - \frac{1}{2\sqrt{2}b_1}\left[\frac{b_{10}(1-i)+(1+i)}{k\pi i}\right] + O\left(n^{-2}\right), n = N, N+1, \ldots. \tag{3.39}$$

Finally, recall that $\lambda_n = \tau_n^2$, $\omega_2 = \frac{1+i}{\sqrt{2}}$, $\omega_2^2 = i$, and hence the last estimate yields

$$\lambda_n = -\frac{1}{b_1} + (k\pi)^2 i + \frac{b_{10} i}{b_1} + O\left(n^{-1}\right), n = N, N+1, \ldots,$$

where N is sufficiently large.

Since the matrix in (3.29) has rank 3 for each sufficiently large n, there is only one linearly independent solution φ_n to (3.22) for $\tau = \tau_n$. Hence, each λ_n is geometrically simple for n sufficiently large. The theorem is proved. □

Theorem 3.2 *Let $\lambda_n = \tau_n^2$ where $\tau_n \in S$ is given by (3.31). Then the corresponding eigenfunction $\{\phi_n = (f_n, \lambda_n f_n, \zeta_n, \delta_n), \overline{\phi_n} = (\overline{f_n}, \overline{\lambda_n}\ \overline{f_n}, \overline{\zeta_n}, \overline{\delta_n})\}$ has the following asymptotic:*

$$\lambda_n f_n(x) = e^{-1/4\int_0^z a(s)ds}\left[\sqrt{2}cos(n-1/4)\pi z - (-1)^n e^{-(n-1/4)\pi(1-z)} + O\left(n^{-1}\right)\right], \tag{3.40}$$

$$f_n''(x) = \frac{1}{p^2}\left(\frac{\rho(x)}{EI(x)}\right)^{1/2} e^{-1/4\int_0^z a(s)ds}\left[\sqrt{2}icos(n-1/4)\pi z + i(-1)^n \times e^{-(n-1/4)\pi(1-z)} + O\left(n^{-1}\right)\right], \tag{3.41}$$

$$\zeta_n = O\left(n^{-1}\right), \delta_n = O\left(n^{-2}\right). \tag{3.42}$$

Proof From (3.25), (3.26), (3.28), and a simple fact of linear algebra, the eigenfunction φ_n corresponding to λ_n is given by

$$\varphi_n(z) = \begin{vmatrix} \tau_n\omega_1[1]_1 & \tau_n\omega_2[1]_1 & \tau_n\omega_3[1]_1 & \tau_n\omega_4[1]_1 \\ e^{\tau_n\omega_1 z}[1]_1 & e^{\tau_n\omega_2 z}[1]_1 & e^{\tau_n\omega_3 z}[1]_1 & e^{\tau_n\omega_4 z}[1]_1 \\ \tau_n^4 e^{\tau_n\omega_1}[1]_1 & \tau_n^4 e^{\tau_n\omega_2}[1]_1 & \tau_n^4 e^{\tau_n\omega_3}[1]_1 & \tau_n^4 e^{\tau_n\omega_4}[1]_1 \\ \tau_n^5\omega_1 e^{\tau_n\omega_1}[1]_1 & \tau_n^5\omega_2 e^{\tau_n\omega_2}[1]_1 & \tau_n^5\omega_3 e^{\tau_n\omega_3}[1]_1 & \tau_n^5\omega_4 e^{\tau_n\omega_4}[1]_1 \end{vmatrix}, \tag{3.43}$$

then

$$\omega_2^2 e^{\tau_n(\omega_1+\omega_2)}\varphi_n(z) = \tau_n^{10}\begin{vmatrix} -[1]_1 & i[1]_1 & -ie^{\tau_n\omega_2}[1]_1 & e^{\tau_n\omega_1}[1]_1 \\ e^{\tau_n\omega_1 z}[1]_1 & e^{\tau_n\omega_2 z}[1]_1 & e^{\tau_n\omega_2(1-z)}[1]_1 & e^{\tau_n\omega_1(1-z)}[1]_1 \\ e^{\tau_n\omega_1}[1]_1 & e^{\tau_n\omega_2}[1]_1 & [1]_1 & [1]_1 \\ -e^{\tau_n\omega_1}[1]_1 & ie^{\tau_n\omega_2}[1]_1 & -i[1]_1 & -[1]_1 \end{vmatrix}$$

$$= \tau_n^{10}\begin{vmatrix} -1 & i & -ie^{\tau_n\omega_2} & 0 \\ e^{\tau_n\omega_1 z} & e^{\tau_n\omega_2 z} & e^{\tau_n\omega_2(1-z)} & e^{\tau_n\omega_1(1-z)} \\ 0 & e^{\tau_n\omega_2} & 1 & 1 \\ 0 & ie^{\tau_n\omega_2} & -i & 1 \end{vmatrix} + O\left(\tau_n^{-1}\right)$$

$$= \tau_n^{10}\Big\{ -(1+i)e^{\tau_n\omega_2 z} + (1-i)e^{\tau_n\omega_2(1-z)}e^{\tau_n\omega_2} + 2ie^{\tau_n\omega_1(1-z)}e^{\tau_n\omega_2}$$
$$+ (1-i)e^{\tau_n\omega_1 z} - (i+1)e^{\tau_n\omega_1 z}e^{2\tau_n\omega_2}\Big\} + O\left(\tau_n^{-1}\right).$$

It follows from (3.35) that $e^{2\tau_n\omega_2} = -i + O\left(\tau_n^{-1}\right)$, and hence the last estimate yields

$$\frac{-\omega_2^2 e^{\tau_n(\omega_1+\omega_2)}\varphi_n(z)}{(1+i)\tau_n^{10}}$$
$$= \left\{e^{\tau_n\omega_2 z} + ie^{\tau_n\omega_2(1-z)}e^{\tau_n\omega_2} - (i+1)e^{\tau_n\omega_1(1-z)}e^{\tau_n\omega_2}\right\} + O\left(n^{-1}\right).$$

Similarly

$$\frac{-\omega_2^2 e^{\tau_n(\omega_1+\omega_2)}\varphi_n''(z)}{(1+i)\tau_n^{12}}$$
$$= \begin{vmatrix} -[1]_1 & i[1]_1 & -ie^{\tau_n\omega_2}[1]_1 & e^{\tau_n\omega_1}[1]_1 \\ -ie^{\tau_n\omega_1 z}[1]_1 & ie^{\tau_n\omega_2 z}[1]_1 & ie^{\tau_n\omega_2(1-z)}[1]_1 & -ie^{\tau_n\omega_1(1-z)}[1]_1 \\ e^{\tau_n\omega_1}[1]_1 & e^{\tau_n\omega_2}[1]_1 & [1]_1 & [1]_1 \\ -e^{\tau_n\omega_1}[1]_1 & ie^{\tau_n\omega_2}[1]_1 & -i[1]_1 & -[1]_1 \end{vmatrix}$$

$$= \left\{ie^{\tau_n\omega_2 z} - e^{\tau_n\omega_2(1-z)}e^{\tau_n\omega_2 z} + (i-1)e^{\tau_n\omega_1(1-z)}e^{\tau_n\omega_2 z}\right\} + O\left(n^{-1}\right).$$

Moreover,

$$\frac{-\omega_2^2 e^{\tau_n(\omega_1+\omega_2)}\varphi_n'(z)}{(i+1)\tau_n^{11}} = \frac{1}{\sqrt{2}}\Big\{(i+1)e^{\tau_n\omega_2 z} + (1-i)e^{\tau_n\omega_2(1-z)}e^{\tau_n\omega_2}$$
$$- 2e^{\tau_n\omega_2}e^{\tau_n\omega_1(1-z)}\Big\} + O\left(n^{-1}\right).$$

We note from (3.31) that

$$\begin{cases} e^{\tau_n \omega_2} = e^{i(n-1/4)\pi} + O\left(n^{-1}\right) = \dfrac{(1-i)(-1)^n}{\sqrt{2}} + O\left(n^{-1}\right), \\ e^{\tau_n \omega_2 z} = e^{i(n-1/4)\pi z} + O\left(n^{-1}\right), \\ e^{\tau_n \omega_1 z} = e^{-(n-1/4)\pi z} + O\left(n^{-1}\right). \end{cases} \tag{3.44}$$

By setting

$$f_n(x) = \frac{-\omega_2^2 e^{-1/4\int_0^z a(s)ds} e^{\tau_n(\omega_1+\omega_2)} \varphi_n(z)}{(1+i)\tau_n^{12}}, \tag{3.45}$$

the expression (3.40) can then be concluded. Furthermore,

$$\frac{-\omega_2^2 e^{\tau_n(\omega_1+\omega_2)} \varphi_n'(z)}{(i+1)\tau_n^{12}} = O\left(n^{-1}\right),$$

then

$$f_n''(x) = \frac{\left(\dfrac{\rho(x)}{EI(x)}\right)^{1/2} \omega_2^2 e^{-1/4\int_0^z a(s)ds} e^{\tau_n(\omega_1+\omega_2)} \varphi_n''(z)}{p^2(1+i)\tau_n^{12}}, \tag{3.46}$$

then the expression (3.41) is obtained. Also, from (3.40), we have

$$\begin{aligned} \zeta_n = m\lambda_n f_n(1) &= me^{-1/4\int_0^1 a(s)ds}\left[\sqrt{2}cos(n-1/4)\pi - (-1)^n + O\left(n^{-1}\right)\right] \\ &= O\left(n^{-1}\right), \end{aligned} \tag{3.47}$$

also, we have from the boundary condition of (3.14), $\lambda_n J f_n'(1) = -\dfrac{EI(1) f_n''(1)}{\lambda_n J} - cf_n'(1)$, we obtain $\delta_n = J\lambda_n f_n'(1) = O\left(n^{-2}\right)$. The theorem is proved. □

3.4 Riesz Basis Property

Definition 3.1 Let $\mathbb{A}$ be a closed-loop operator in a Hilbert space $\mathbb{H}$. A nonzero element $x \neq 0 \in \mathbb{H}$ is called a generalized eigenvector of $\mathbb{A}$ corresponding to an eigenvalue λ (with finite algebraic multiplicity) of $\mathbb{A}$ if there exists a nonnegative integer n such that $(\lambda - \mathbb{A})^n x = 0$.

Definition 3.2 A sequence $(x_n)_{n\geq 1}$ in $\mathbb{H}$ is called a Riesz basis for $\mathbb{H}$ if there exists an orthonormal basis $(z_n)_{n\geq 1}$ in $\mathbb{H}$ and a linear bounded invertible $T \in \mathcal{L}(\mathbb{H})$ such that $Tx_n = z_n$ for any $n \in \mathbb{N}^*$.

Theorem 3.3 (See [2]) *Let $(\lambda_n)_{n\geq 1} \subset \sigma(\mathbb{A})$ be the spectrum of $\mathbb{A}$. Assume that each λ_n has a finite algebraic multiplicity m_n and $m_n = 1$ as $n > N$ for some integer N, then there is a sequence of linearly independent generalized eigenvectors $\{x_n\}_1^{m_n}$ corresponding to λ_n. If $\left\{\{x_n\}_1^{m_n}\right\}_{n\geq 1}$ forms a Riesz basis for $\mathbb{H}$, then $\mathbb{A}$ generates a C_0 semigroup $e^{\mathbb{A}t}$ which can be represented as*

$$e^{\mathbb{A}t}x = \sum_{n=1}^{+\infty} e^{\lambda_n t} \sum_{i=1}^{m_n} a_{ni} \sum_{j=1}^{m_n} f_{nj}(t) x_{nj},$$

for any $x = \sum_{n=1}^{+\infty} \sum_{i=1}^{m_n} a_{ni} x_{ni} \in \mathbb{H}$ where $f_{nj}(t)$ is a polynomial of t with order less than m_n. In particular, if $a^* < Re\lambda < b^*$ for some real numbers a^* and b^*, then $\mathbb{A}$ generates a C_0 group on $\mathbb{H}$. Moreover, the spectrum-determined growth condition holds $e^{\mathbb{A}t}$: $\omega(\mathbb{A}) = S(\mathbb{A})$, where

$$\omega(\mathbb{A}) = \lim_{t\to+\infty} \frac{1}{t} \, || \, e^{\mathbb{A}t} || \ \text{is the growth order of } e^{\mathbb{A}t} \text{ and } S(\mathbb{A}) = \sup\{Re\lambda / \ \lambda \in \sigma(\mathbb{A})\}$$

is the spectral bound of $\mathbb{A}$.

In order to remove the requirement of the estimation of the low eigenpairs of the system, a corollary of Bari's theorem is recently reported in [2], which provides a much less demanding approach in generating a Riesz basis for general discrete operators in the Hilbert spaces. The result is cited here.

Theorem 3.4 (See [2]) *Let $\mathbb{A}$ be a densely defined discrete operator, that is, $(\lambda - \mathbb{A})^{-1}$ is compact for some λ in a Hilbert space $\mathbb{H}$. Let $\{z_n\}_1^{+\infty}$ be a Riesz basis for $\mathbb{H}$. If there are an $N \geq 0$ and a sequence of a generalized eigenvectors $\{x_n\}_{N+1}^{+\infty}$ of $\mathbb{A}$ such that*

$$\sum_{n=N+1}^{+\infty} \|x_n - z_n\|^2 < +\infty,$$

then

1. *There are an $M > N$ and generalized eigenvectors $\left\{x_{n_0}\right\}_1^M \cup \{x_n\}_{M+1}^{+\infty}$ form a Riesz basis for $\mathbb{H}$.*
2. *Consequently, let $\left\{x_{n_0}\right\}_1^M \cup \{x_n\}_{M+1}^{+\infty}$ correspond to eigenvalues $\{\sigma_n\}_1^{+\infty}$ of $\mathbb{A}$, then $\sigma(\mathbb{A}) = \{\sigma_n\}_1^{+\infty}$ where σ_n is counted according to its algebraic multiplicity.*
3. *If there is an $M_0 > 0$ such that $\sigma_n \neq \sigma_m$ for all $m, n \geq M_0$, then there is an $N_0 > M_0$ such that all $\sigma_n, n > N_0$ are algebraically simple.*

In order to apply Theorem 3.4 to the operator $\mathbb{A}$ when we consider $\{x_n\}$ in Theorem 3.4 as the eigenfunctions of $\mathbb{A}$, we need a referring Riesz basis $\{z_n\}_1^{+\infty}$ as well. For the system (3.1), this is accomplished by collecting (approximately) normalized eigenfunctions of the following free conservative system:

$$\begin{cases} \rho(x)y_{tt}(x,t) + (EI(.)y_{xx})_{xx}(x,t) = 0, \ 0 < x < 1, \ t > 0, \\ y_x(0,t) = y(0,t) + (EI(.)y_{xx})_x(0,t) = 0, \qquad t > 0, \\ (my_{tt} - (EI(.)y_{xx})_x)(1,t) = 0, \qquad t > 0, \\ (Jy_{xtt} + EI(.)y_{xx})(1,t) = 0, \qquad t > 0, \\ y(x,0) = y_0(x), \ y_t(x,0) = y_1(x), \qquad 0 < x < 1. \end{cases} \tag{3.48}$$

The system operator $\mathbb{A}_0$ associated with (3.48) is nothing but the operator $\mathbb{A}$ with $b = c = \dfrac{1}{a} = 0$.

$$\begin{cases} \mathbb{A}_0(f,g,\zeta,\delta) = (g, -\dfrac{1}{\rho(.)}(EI(.)f'')'', (EI(.)f'')'(1), -EI(1)f''(1)), \\ D(\mathbb{A}_0) = \{(f,g,\zeta,\delta) \in (H^4(0,1) \cap \mathbb{V}) \times \mathbb{V} \times \mathbb{C}^2 / f(0) + (EI(.)y_{xx})_x(0) = 0, \\ \zeta = Jg'(1), \ \delta = mg(1)\}. \end{cases} \tag{3.49}$$

$\mathbb{A}_0$ is skew-adjoint with compact resolvent in $\mathbb{H}$. It is seen that all the analyses in the previous sections for the operator $\mathbb{A}$ are still true for the operator $\mathbb{A}_0$. Therefore, we have the following counterpart of Theorem 3.2 for the operator $\mathbb{A}_0$:

Lemma 3.4 *Each eigenvalue υ_{n_0} of $\mathbb{A}_0$ with sufficiently large module is geometrically simple hence algebraically simple.*

The eigenfunctions $\overrightarrow{\Psi_{n_0}} = (f_{n_0}, \upsilon_{n_0} f_{n_0}, m\upsilon_{n_0} f_{n_0}(1), J\upsilon_{n_0} f'_{n_0}(1))$ of υ_{n_0} have the following asymptotic expressions:

$$\upsilon_{n_0} f_{n_0}(x) = e^{-1/4\int_0^z a(s)ds}\left[\sqrt{2}cos(n-1/4)\pi z - (-1)^n e^{-(n-1/4)\pi(1-z)} + O\left(n^{-1}\right)\right], \tag{3.50}$$

$$f''_{n_0}(x) = \frac{1}{p^2}\left(\frac{\rho(x)}{EI(x)}\right)^{1/2} e^{-1/4\int_0^z a(s)ds}\left[\sqrt{2}icos(n-1/4)\pi z + i(-1)^n \times e^{-(n-1/4)\pi(1-z)} + O\left(n^{-1}\right)\right], \tag{3.51}$$

$$\zeta_{n_0} = O\left(n^{-1}\right), \delta_{n_0} = O\left(n^{-2}\right), \tag{3.52}$$

where all $(\upsilon_{n_0}, \overline{\upsilon_{n_0}})$, but possibly a finite number of other eigenvalues, are composed of all the eigenvalues of $\mathbb{A}_0$.

The eigenfunctions $\overrightarrow{\Psi_{n_0}} = (f_{n_0}, \upsilon_{n_0} f_{n_0}, m\upsilon_{n_0} f_{n_0}(1), J\upsilon_{n_0} f'_{n_0}(1))$ are normalized approximately.

From a well-known result in functional analysis, we know that the eigenfunctions of $\mathbb{A}_0$ form an orthogonal basis for $\mathbb{H}$, particularly, all $\overrightarrow{\Psi_{n_0}}$ and their conjugates form an (orthogonal) Riesz basis for $\mathbb{H}$.

Then there exists a positive integer large enough N such that

$$\sum_{n=N+1}^{+\infty} \left\| \overrightarrow{\Phi}_n - \overrightarrow{\Psi}_{n_0} \right\|_{\mathbb{H}}^2 = \sum_{n=N+1}^{+\infty} O(n^{-2}) < +\infty. \tag{3.53}$$

The same result is verified for their conjugates. We can now apply Theorem 3.4 to obtain the main results of the present paper.

Theorem 3.5 *Let the operator be $\mathbb{A}$ defined by (3.7).*

1. *There is a sequence of generalized functions properly normalized of $\mathbb{A}$ which forms a Riesz basis of the Hilbert space $\mathbb{H}$.*
2. *The eigenvalues of $\mathbb{A}$ have the asymptotic behavior (3.31).*
3. *All $\lambda \in \sigma(\mathbb{A})$ with sufficiently large modulus are algebraically simple. Therefore, $\mathbb{A}$ generates a C_0 semigroup on $\mathbb{H}$. Moreover, for the semigroup $e^{\mathbb{A}t}$ generated by $\mathbb{A}$, the spectrum-determined growth condition holds.*

As a consequence of Theorem 3.5, we have a stability result for system (3.1).

Corollary 3.1 *The system (3.1) is exponentially stable for any* $a > 0$, $b \geq 0$, *and* $c > 0$.

Proof Theorem 3.5 ensures the spectrum-determined growth condition: $\omega(\mathbb{A}) = sup\{Re\lambda : \lambda \in \sigma(\mathbb{A})\}$, Lemma 3.2 (in the proof of Corollary 3.1), say that $Re\lambda < 0$ provided $\lambda \in \sigma(\mathbb{A})$ and Theorem 3.1 shows that imaginary axis is not an asymptote of $\sigma(\mathbb{A})$. Therefore $sup\{Re\lambda : \lambda \in \sigma(\mathbb{A})\} < 0$. □

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