Operators of Bounded Locally Optimal Controls for Dynamic Systems

Vladimir N. Kozlov and Artem A. Efremov(✉)

Peter the Great St. Petersburg Polytechnic University, Saint Petersburg, Russia
eartm@mail.ru

Abstract. The problems of locally linear and quadratic optimal stabilization in finite-dimensional and functional spaces based on the projection method were studied in a number of papers [1, 2, 4, 5], as well as in a series of other studies. In this paper, the problem of quadratic locally optimal program stabilization in functional space is formulated, from which follows the problem of quadratic locally optimal stabilization of the equilibrium state of a dynamical system.

Keywords: Locally optimal controls · Finite-dimensional optimization projectors

1 Formulation of the Problem

Let the dynamics of a controlled object in the space of vector functions $\mathbf{R}^n(L^2[0,\ T])$ be represented by a linear differential operator

$$x'(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0, \tag{1}$$

where the vector functions class is given by

$$x(t) \in \mathbf{R}^n\left(L^2[0,\ T]\right), \quad u(t) \in \mathbf{R}^m\left(L^2[0,\ T]\right).$$

The numeric matrices of the control object (1) belong to the spaces

$$A \in \mathbf{R}^{n \times n}, \quad B \in \mathbf{R}^{n \times m}.$$

For example, let's suppose that for the matrices the rank control criterion according to R. Kalman is satisfied, guaranteeing the existence of a nonempty set of controls, stabilizing the studied control object. In other words, these matrices form an asymptotically stabilized pair.

It is required to synthesize the vector of optimal controls in a given class of functions for optimal stabilization of the equilibrium position of this system in terms of minimum of a given local functional

$$\varphi = \|z(t)\|^2_{L^2[0,\ T]} \in \mathbf{R}, \tag{2}$$

© Springer Nature Switzerland AG 2020
D. G. Arseniev et al. (Eds.): CPS&C 2019, LNNS 95, pp. 140–145, 2020.
https://doi.org/10.1007/978-3-030-34983-7_14

which is given on a generalized vector

$$z(t) = [x(t)|u(t)]^T \in \mathbf{R}^{n+m}\left(L^2[0,\ T]\right),$$

which includes the coordinate vector of states and controls of the model of a dynamic object of type (1).

2 General Formulation of Problems for the Synthesis of Locally Optimal Controls

For the synthesis of stabilizing controls, the inverse operator for the operator (1) will be used in the form of the Cauchy integral operator, which determines the predictions of the optimized state coordinates and controls based on the solution of the problem for operator (1) in the form

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{3}$$

The problem of synthesis is formulated based on the minimization of the functional

$$\varphi = \|z(t) - C_z(t)\|^2_{\mathbf{R}^n(L^2[0,\ T])} = \int_0^T [z(t) - C_z(t)]^T [z(t) - C_z(t)]dt \tag{4}$$

with constraints like equalities and inequalities

$$Az(t) = b(t), \quad \|z(t)\|^2_{\mathbf{R}^n(L^2[0,\ T])} \le r^2, \tag{5}$$

which in $\mathbf{R}^n(L^2[0,\ T])$ are determined by the intersection of the linear manifold and a ball. Moreover, in the space of generalized vectors

$$z(t) = [x(t)|u(t)]^T \in \mathbf{R}^{n+m}\left(L^2[0,\ T]\right)$$

linear manifold is defined by the following relation

$$Az(t) = A\left[x(t)| - \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right] = e^{At}x^0 = b(t) \tag{6a}$$

of the finite-dimensional space of vector-functions.

The ball of constraints-inequalities of this space is determined by the quadratic inequality

$$\|z(t)\|^2_{\mathbf{R}^n(L^2[0,\ T])} = \int_0^T z^T(\tau)z(\tau)d\tau \le r^2 < \infty. \tag{6b}$$

Then the quadratic quality functional defined on the generalized vectors for the problem of locally optimal program stabilization will take form

$$\varphi = \|z(t) - C_z(t)\|^2_{\mathbf{R}^{n+m}(L^2[0,\ T])} = \left\|[x(t)|u(t)]^T - C_{xu}(t)\right\|^2_{\mathbf{R}^{n+m}(L^2[0,\ T])}, \tag{7a}$$

where $C_z(t) = C_{xu}(t) \in \mathbf{R}^n(L^2[0,\ T])$ is the target stabilized generalized vector for the problem of locally optimal program stabilization. If the vector-function of program action satisfies the condition

$$C_z(t) = C_{xu}(t) = 0_{n+m} \in \mathbf{R}^{n+m}\left(L^2[0,\ T]\right), \tag{7b}$$

then the task (3) transforms from the "quadratic grogram stabilization problem" to the "problem of optimal stabilization of the zero equilibrium position" of the dynamic object (1) for a system with feedback.

Thus, on the basis of relations (3)–(7a, 7b), we can formulate two main tasks for calculating controls:

- The task of synthesis of controls for a locally optimal system of program stabilization in discrete-continuous time, which can be constructed as finite-dimensional (approximating) optimization problems, and for this version, this problem is formulated based on relations of type (3)–(7a, 7b) and is solved further in Sect. 3.
- The tasks of calculating the controls for local or interval optimal stabilization of the zero position or a given program vector $C_z(t)$, which are also constructed on the basis of relations (3)–(7a, 7b) in an infinite-dimensional space to stabilize the zero equilibrium position, analyzed in Sect. 4.

Based on the objectives of the problem of locally optimal stabilization, a synthesis problem is formulated: to calculate the optimal generalized vector of the type $z(t)$, determined by the equality (2), defined on the object's trajectories (3) with the constraints (4) which have the form (6a, 6b) in $\mathbf{R}^{n+m}(L^2[0,\ T])$.

3 Calculation of Optimal Controls Based on Finite-Dimensional Optimization Projectors

Optimal controls are calculated based on the object's operator, which for piecewise constant controls is

$$u(kh) = \sum_{s=0}^{kh} u(t_s), \tag{8}$$

and taking into account the additive property (3), takes form

$$\begin{aligned} x(t_k) = x(kh) &= e^{Akh}x^0 + \int_0^{kh} e^{A(kh-\tau)}d\tau\, B\sum_{s=0}^{kh} u(t_s) \\ &= e^{At}x^0 + A^{-1}\left(e^{Akh} - E_n\right)B\sum_{s=0}^{kh} u(t_s). \end{aligned} \tag{9}$$

As a result, the synthesis problem as a problem of finite-dimensional conditional minimization has the form: to calculate the countable set of controls in the form of piecewise constant vector functions $u_*(ph)$, which provide for a minimum of the functional

$$u_*(ph) = T_u \arg\min\left\{ \begin{array}{l} \|z(ph) - C_z(ph)\|^2_{\mathbf{R}^n(L^2[0,\,T])}\Big| Az(ph) = b(z(ph)) \\ x(ph) - A^{-1}\left(e^{Aph} - E_n\right)B\sum\limits_{s=0}^{ph} u(t_s) = e^{Aph}x^0, \quad \|z(ph)\| < r^2 \end{array} \right\}, \tag{10}$$

in a discrete prediction interval $[0,\ p] \in N$, where the "filtering" matrix $T_u \in \mathbf{R}^{m\times(n+m)}$ "selects" from the generalized vector $z(ph)$ of controls $u_*(ph)$.

The finite-dimensional conditional minimization problem (10) for stabilizing the vector of program actions can be solved on the basis of reduction to the extremum problem and projection operators of finite-dimensional minimization of type (12a) and (12b), considered in [3, 4].

As noted above, quasi-analytic optimization operators, delivering solutions in the form of finite relations, will be used to calculate the optimal controls. The minimization operator for solving a finite-dimensional non-classical extremum problem has the form:

$$\begin{aligned} x_* = T_u \arg\min\Big\{ &\varphi(x) = \|x - C\|_2^2 | Ax = b,\ A \in \mathbf{R}^{m\times n},\ \operatorname{rang} A = m, \\ &x^T x \le r^2 \Big\} \in \mathbf{R}^n, \end{aligned} \tag{11}$$

where the vector of program control actions is separated from zero, i.e. satisfies the inequality

$$0 < \delta_c^2 \le \|C\|_2^2 \le \bar{\delta}_c^2 \le r^2 - r_c^2,$$

separating systems resources for program and stabilizing components of the control. In problem (10), the quality functional $\|x - C\|_2^2 = (x - C)^T(x - C) \in \mathbf{R}$ is given by the square of the Euclidean norm, and the admissible set is determined by a non-empty intersection of the linear manifold and the ball approximating a parallelepiped (ball) [3, 4].

The solution of the optimization problem (11) due to the "principle of boundary Lagrangian extremes" and "narrowing of the admissible region" is represented by a convex linear combination of patterns for regularized orthogonal projectors [1–5]

$$x_* = (1-\theta_*)x_3(\eta_+) + \theta_* x_3(\eta_-) = P^+ b(t) + (1-2\theta_*)P^0 C(t)\eta. \qquad (12a)$$

Lagrange vectors $x_3(\eta_+)$ and $x_3(\eta_-)$ in (12a), belonging to the intersection of the linear manifold (subspace) and the sphere as the boundary of the ball in (11), are defined by orthogonal projectors

$$x_{3\pm} = x_3(\pm\eta) = P^+ b(t) \pm P^0 C(t)\ \rho^{-1}\sqrt{\alpha/\rho}, \qquad (12b)$$

where operators have the form

$$P^+ = A^T\left(AA^T\right)^{-1};\ P^0 = E_n - P^+ A;\ \eta = \sqrt{\alpha/\rho};$$

$$\rho = \left\|P^0 C\right\|_2 \alpha = r^2 - \|P^+ b\|_2^2;\ \rho = \left\|P^0 C\right\|_2^2.$$

Then, as shown in [4, 5, 6], for the optimal solutions (12a) and (12b), it is necessary and sufficient that the optimal parameter in (12a, 12b) is given by the equality

$$\theta_* = P(\theta_*) = 0.5(|\theta_0| - |\theta_0 - 1| + 1) \in [0,\ 1], \qquad (12c)$$

where

$$\theta_0 = 0.5\left(1-\eta^{-1}\right);\ \eta^{-1} = \sigma = \sqrt{\rho/\alpha};\ \alpha = r^2 - b^T\left(AA^T\right)^{-1} b;\ \rho = C^T P^0 C.$$

The operators formulated in the statement are further used for the projection-operator representation of the problems of optimal stabilization of the equilibrium position of program assignment studied in this paper.

4 On the Calculation of Controls for the Problem of Calculating Locally Optimal Stabilizing Equilibrium Controls

In this case, based on relations (3)–(7a, 7b), it is necessary to use the Lagrange function, which for program optimal stabilization has the form [5, 6]

$$L = \|z(t)\|_2^2 + \lambda_0\left(x(t)| - \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau - e^{At}x^0\right) + \lambda\left(\|z(t)\|_2^2 - r^2\right). \qquad (13)$$

For the problem of locally optimal stabilization of the equilibrium position, the quality functional follows from (11) for $C(t) = 0$. As shown in [4, 5], the optimal control vector is calculated by the operator following from (12a), (12b), having the form

$$x_* = P^+ b(t), \tag{14}$$

since the second term in Eq. (12b) is zero due to $C(t) = 0$.

5 Conclusion

Thus, the principles of calculating optimal controls in the "state-control" spaces are formulated for objects with models like (1) or (3) for problems of optimal stabilization of the equilibrium position and problems of optimal stabilization of program actions. In contrast to the classical optimization methods, the optimal controls are calculated on the basis of solving mathematical programming problems, which are represented in quasi-analytical projection projectors that allow performing a qualitative analysis of the dynamics of systems with feedback.

The research results are used to optimize the frequency and active power control systems of energy associations, to control the transfer of hydrocarbons along the lines of main pipeline networks, as well as in studying the processes of multilayer thermal conductivity in solid multilayer objects.

References

1. Kozlov, V.N.: Method of non-linear operators in automated design of dynamic systems, 166 p. Leningrad state university named after A.A. Zhdanov, Leningrad (1986). (in Russian)
2. Kozlov, V.N., Kupriyanov, V.E., Zaborovskiy, V.S.: Computational methods for the synthesis of automatic control systems, 220 p. Leningrad state university named after A.A. Zhdanov, Leningrad (1989). (in Russian)
3. Kozlov, V.N.: The method of minimization of linear functionals based on compakt sets. In: Proceedings of the 12th International Workshop on Computer Science and Information Technologies (CSIT 2010), Moscow – St. Petersburg, vol. 2, pp. 157–159 (2010). (in Russian)
4. Kozlov, V.N.: Smooth systems, operators of optimization and stability of energy systems, 177 p. St. Petersburg Polytechnic University, St. Petersburg (2012). (in Russian)
5. Kozlov, V.N.: A projection method for optimizing optimal limited controls of dynamic systems, 190 p. Publishing and Printing Association of Higher Education Institutions St. Petersburg (2018). (in Russian)
6. Kozlov, V.N., Efremov, A.A.: Introduction to functional analysis, 79 p. Publishing and Printing Association of Higher Education Institutions, St. Petersburg (2018). (in Russian)